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Kristoffer Varholm

On steady water waves with stagnation points

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Department of Mathematical Sciences

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Thesis for the degree of Philosophiae Doctor

Trondheim, September 2019

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PREFACE

The submission of this thesis is in partial fulfillment with the requirements for obtaining the degree of Philosophiae Doctor in Mathematics at the Norwegian University of Science and Technology. With a slight wink, I allow myself to open with a quote:

> *If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.*
>
> — *John Von Neumann*

As this text is the culmination of many years of work, which were not performed in isolation, there are several people whom I would like to thank. At the top of this list is my main advisor, Professor Mats Ehrnström. His support and insight has been invaluable to me over the years. He is also the reason why I have been able to travel so extensively — enabling me to interact with the greater mathematical community. This is how I first met my co-advisor Professor Samuel Walsh at the University of Missouri, only a few short months after becoming a graduate student. I am grateful for his hospitality then, and during my more recent extended stay two years ago. The same can be said for Professor Erik Wahlén at Lund University, who graciously invited me to come visit.

I would also like to thank the members and alumni of Mats' research group, whom I have come to see as not only co-workers, but friends. There are too many of them to mention them all by name in this preface, but they know who they are.

Last, but certainly not least, I would like to thank my parents for their incredible patience, and my brother for late-night gaming sessions and conversations — my escapism from everyday tedium.

Trondheim, July 1, 2019,

Kristoffer Varholm

CONTENTS

INTRODUCTION

Derivation of the equations of motion

Let us for a moment play physicists, as mathematicians quite rarely think about where the equations they solve come from. We are lucky that "our" equations are so tightly connected with the everyday physical world, and are easy to derive from first principles without any deep knowledge of physics. The derivation will naturally be mostly formal, with any regularity issues tacitly smoothed over. Various versions of these computations can be found in classical literature like [44, 63], or more modern books like [13].

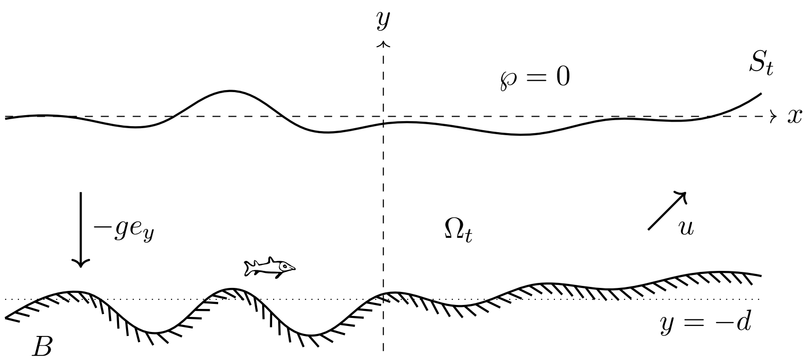

Figure 1: Sketch of Ω_t, with a guest.[1]

We will use the notation

$$z = (x, y) \in \mathbb{R}^2 \times \mathbb{R},$$

where the two first components are horizontal and the final is vertical, with the benefit being a seamless notational transition to two dimensions at a later time. The time-dependent fluid domain will be of the form

$$\Omega_t = \Big\{(x, y) \in \mathbb{R}^3 : -d + \beta(x) < y < \eta(x, t)\Big\}$$

[1]The fish is from the more than 3,000-year-old Minoan Phaistos Disk.

for each $t \geq 0$; where β is a fixed bottom topography, η is the free surface, and $d > 0$ is the mean unperturbed fluid depth. See Figure 1 for a two-dimensional sketch of the domain. We will write

$$B \coloneqq \{(x, -d + \beta(x)) : x \in \mathbb{R}^2\}, \quad \text{and} \quad S_t \coloneqq \{(x, \eta(x,t)) : x \in \mathbb{R}^2\}$$

for the two components of $\partial\Omega_t$, which are assumed to maintain a positive separation.

The incompressible Euler equations

Imagine a fixed immersed region $U \subset \Omega_t$, with the fluid taken to be inviscid, incompressible, but stratified[2]. The fluid particles are assumed move according to a velocity field u, and have a well-defined mass density $\rho > 0$, both of which depend on time and space. The total mass and momentum in the region, as long as U remains inside Ω_t, are then given by the integrals

$$m = \int_U \rho \, dz \qquad \text{and} \qquad P = \int_U \rho u \, dz,$$

respectively.

The only way that mass can enter or leave the region U is by the flux of fluid particles through its surface. Hence

$$\dot{m} = \int_U \partial_t \rho \, dz = -\overbrace{\int_{\partial U} \rho u \cdot n \, d\sigma}^{\text{Mass flux}}$$

holds, where σ is the surface measure on ∂U, and n is its outward-pointing unit normal. By applying the divergence theorem to the final integral, we obtain the identity

$$\int_U (\partial_t \rho + \nabla \cdot (\rho u)) \, dz = 0. \tag{1}$$

At the same time, since the fluid is assumed to be incompressible, we must have

$$\overbrace{\int_{\partial U} u \cdot n \, d\sigma}^{\text{Particle flux}} = \int_U \nabla \cdot u \, dz = 0 \tag{2}$$

[2]This may sound strange to the uninitiated, but effects like salinity or temperature gradients can easily cause stratification, even if the fluid is incompressible. The next time you boil water on your stove, look at the "mirage"-like effect at the bottom of the pot [18].

in order for the total number of particles in U to remain constant.

Suppose further that the only body forces acting on the fluid are due to its pressure $\wp$, and the force density $-\rho g e_y$ set up by the gravitational field. Here, $g > 0$ is the acceleration due to gravity. Like mass, momentum is carried by the particles, but it is also affected by the forces on those particles: Indeed, Newton's second law states that

$$\dot{P} = \int_U \partial_t(\rho u)\, dz = -\overbrace{\int_{\partial U} (\rho u) u \cdot n \, d\sigma}^{\text{Momentum flux}} - \int_{\partial U} \wp n \, d\sigma - \int_U \rho g e_y \, dz$$

in this setting. Since $u(u \cdot n) = (u \otimes u)n$, this leads to a third integral identity

$$\int_U \left(\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla \wp + \rho g e_y\right) dz = 0, \tag{3}$$

through the same procedure as above.

We need now only appeal to the arbitrariness of U to be able to conclude from (1)–(3) that the *incompressible Euler equations*

$$\begin{aligned} \partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla \wp + \rho g e_y &= 0, \\ \partial_t \rho + \nabla \cdot (\rho u) &= 0, \\ \nabla \cdot u &= 0 \end{aligned} \tag{4}$$

hold throughout Ω_t, for every $t > 0$. By manipulating (4), we furthermore see that they are equivalent to the somewhat simpler-looking system

$$\begin{aligned} \partial_t u + \nabla \cdot (u \otimes u) + \frac{1}{\rho} \nabla \wp + g e_y &= 0, \\ \partial_t \rho + u \cdot \nabla \rho &= 0, \\ \nabla \cdot u &= 0 \end{aligned} \tag{5}$$

in Ω_t.

Their eponym[3] introduced these equations more than 250 years ago in his "Principes généraux du mouvement des fluides" [31], not too long after the very first examples of partial differential equations appeared [12]. From our derivation, we see that the equations in (5) represent conservation of momentum, conservation of mass, and incompressibility, respectively.

Observe that the second equation in (5) simply says that the density is transported by the velocity field. In particular, if the density is initially

[3] A person after which something is named.

uniform, it will remain so for all time. In this event, (5) becomes simply

$$\begin{aligned} \partial_t u + \nabla \cdot (u \otimes u) + \nabla p + g e_y &= 0, \\ \nabla \cdot u &= 0, \end{aligned} \tag{6}$$

where $p := \wp/\rho$, which may be the more familiar form of the equations.

Kinematic boundary conditions

We need different kinematic boundary conditions on the bottom B and the free surface S_t, but they are just two facets of the same condition: Namely, that there can be no flux of particles *through* a boundary (otherwise, it would not be a boundary).

At the water bottom

The kinematic boundary condition at the bed is simply the demand

$$u \cdot \beta^{\perp} = 0 \qquad \text{on } B, \tag{7}$$

that the velocity field be tangential there. We have here used the notation

$$\beta^{\perp} := (-\nabla\beta, 1)$$

for the non-normalized normal vector in terms of the topography β.

At the free surface

The velocity field still needs to be tangential to the surface, but now we really have to take into account that this occurs in *spacetime*. This is because it is the fluid particles that actually create the surface. After some pondering, we see that this entails the requirement of

$$(u, 1) \cdot (\eta^{\perp}, -\eta_t) = 0,$$

or

$$\partial_t \eta = u \cdot \eta^{\perp} \qquad \text{on } S_t. \tag{8}$$

The dynamic boundary condition

Surface tension is a cohesive, tangential force acting equally between all particles on S_t, and is the sole mechanism that can maintain a pressure

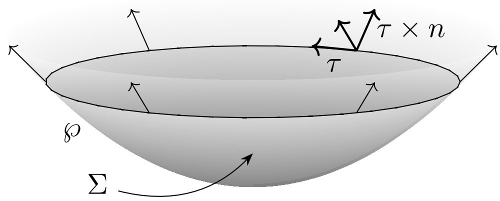

Figure 2: The pressure difference is balanced by the surface tension, which is caused by the rest of S_t pulling on Σ.

difference across S_t. Its effect is that the surface "wants" to be flatter, and its influence is the greatest in waves at small scales.

Let us look at some (nice) part of S_t, which we will call Σ. Then we find that we should have

$$\alpha \int_{\partial\Sigma} \tau \times n \, d\ell + \int_{\Sigma} \wp n \, d\sigma = 0$$

in order for the forces on Σ to balance, with $\alpha \geq 0$ known as the coefficient of surface tension (force per unit length), τ the positively oriented unit tangent of $\partial\Sigma$, and ℓ its arc measure.

For any constant vector $\xi \in \mathbb{R}^3$, one has $(\tau \times n) \cdot \xi = (n \times \xi) \cdot \tau$, and therefore

$$\xi \cdot \int_{\partial\Sigma} \tau \times n \, d\ell = \int_{\Sigma} (\nabla \times (n \times \xi)) \cdot n \, d\ell$$

by Stokes' theorem. As one may verify that

$$(\nabla \times (n \times \xi)) \cdot n = ((\xi \cdot \nabla)n - \xi(\nabla \cdot n)) \cdot n = \xi \cdot \left(\frac{1}{2}\nabla(|n|^2) - n(\nabla \cdot n)\right),$$

where $|n| \equiv 1$ by definition, we therefore obtain

$$\xi \cdot \int_{\Sigma} (\wp - \alpha \nabla \cdot n) n \, d\sigma = 0,$$

whence

$$\wp = \alpha \nabla \cdot n \qquad \text{on } S_t, \tag{9}$$

which is called the dynamic boundary condition. More specifically, (9) is known as the Young–Laplace equation, as it originates from [24, 64]. Note that

$$\nabla \cdot n = \nabla \cdot \left(\frac{\eta^\perp}{|\eta^\perp|}\right) = -\nabla \cdot \left(\frac{\nabla \eta}{\langle \nabla \eta \rangle}\right)$$

measures the mean curvature of the surface in terms of the profile η, using the convenient Japanese bracket $\langle w \rangle := \sqrt{1 + |w|^2}$.

The water-wave problem

Collectively, the initial value problem for the incompressible Euler equations (5) or (6) (but more commonly the latter) on Ω_t with boundary conditions (7)–(9), is known as the *water-wave problem.* Its most striking feature is the presence of a free boundary S_t, which means that the fluid domain is an a priori unknown.

The *steady water-wave problem* concerns traveling-wave solutions of the water-wave problem. That is, solutions of the form

$$\begin{aligned} \eta(z,t) &= \breve{\eta}(x - ct, y) \\ u(z,t) &= \breve{u}(x - ct, y) \\ \rho(z,t) &= \breve{\rho}(x - ct, y) \end{aligned}$$

for all $(z,t) \in \Omega_0 \times \mathbb{R}$, where $\breve{\eta}$, $\breve{u}$, and $\breve{\rho}$ depend only on space and $c \in \mathbb{R}^2 \setminus \{0\}$ is a fixed wave speed. For such solutions to exist, the bottom topography β cannot vary in the direction of motion.

Of course, one's favorite direction for c can readily be picked by a simple rotation about the y-axis. The observant reader may also have noticed that we made no mention of the pressure $\wp$. The reason for this is that we will see that it can be eliminated, and therefore only "tags along" with the rest of the variables. Furthermore, we say that a steady wave is two dimensional if its only dependence on x is in the direction of motion. In this case, one may as well take $x \in \mathbb{R}$, and necessarily $\beta \equiv 0$. Note that treating infinite depth ($d = \infty$ and $\beta = 0$) requires only the modification that (7) holds in the sense of limits.

Finally, to fix terminology; we say that steady waves are *solitary* if they are localized in space, while the modifier word *capillary* is commonly used to indicate that surface tension is present in (9).

Various matters of importance

Vorticity and its evolution

The vector field

$$\omega := \nabla \times u \tag{10}$$

is known as the *vorticity* in the context of fluids, and is something we will become intimately familiar with over the course of this thesis. By Stokes' theorem, this quantity measures the circulation density of the fluid (its local tendency to rotate). If we take the curl of the equation corresponding to conservation of momentum in (5), we eventually find the evolution equation

$$\partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u + \frac{1}{\rho^2}\nabla\rho \times \nabla\wp \tag{11}$$

for the vorticity, aptly named the *vorticity equation*. Most importantly, if the fluid is *homogeneous* (ρ constant) and it starts out *irrotational* ($\omega \equiv 0$), it remains irrotational for all time.

Velocity potentials and stream functions

Suppose that the fluid under consideration is homogeneous. By our observation immediately after (11), we may then write

$$u = \nabla\varphi \tag{12}$$

when the initial velocity field is irrotational. The function φ is known as a *velocity potential*, and

$$\nabla \cdot u = \Delta\varphi = 0$$

by incompressibility; that is, φ is harmonic. Moreover, the kinematic boundary conditions (7) and (8) correspond to

$$\begin{aligned} \partial_t \eta &= \partial^\perp \varphi \quad \text{on } S_t \\ 0 &= \partial^\perp \varphi \quad \text{on } B, \end{aligned} \tag{13}$$

where $\partial^\perp := f^\perp \cdot \nabla$ is the non-normalized *normal* derivative on the graph of a function f.

There is no direct analogue to (12) for rotational fluids in three dimensions, but by virtue of incompressibility, we may write

$$u = \nabla^\perp \psi,$$

where $\nabla^\perp := (-\partial_y, \partial_x)$ is the skew gradient, for *two*-dimensional flows. The function ψ is known as a *stream function*, and we find that

$$\omega = \nabla^\perp \cdot \nabla^\perp \psi = \Delta\psi, \tag{14}$$

with the vorticity ω now viewed as a scalar quantity. (Note that the operator $\nabla^\perp\cdot$ yields the third component of $\nabla\times$ when a trivial dependence on a second horizontal variable is added.) The stream function is therefore harmonic only for irrotational fluids. As opposed to (13), we find

$$\begin{aligned} \partial_t \eta &= \partial^\top \psi \quad \text{on } S_t \\ 0 &= \partial^\top \psi \quad \text{on } B, \end{aligned} \tag{15}$$

for the stream function, where $\partial^\top := f^\top \cdot \nabla$ is the non-normalized *tangential* derivative in terms of the non-normalized tangential vector $f^\top := (1, \partial_x f)$.

For the steady water-wave problem, we remark that one often uses the *relative* potentials. These correspond to the relative velocity $u - (c, 0)$ instead of u, and cause c to drop out of the equations entirely. Moreover, (13) simplifies to vanishing normal derivatives, and (15) to the stream function being *constant* on both the surface S_0 and the bed B.

The Bernoulli equation

If we insert (12) into the first part of (6) and rearrange things slightly, we find that it can be written as

$$\nabla\Big(\varphi_t + \frac{1}{2}|u|^2 + p + gy\Big) = 0,$$

whereby we can *normalize* φ by a constant such that the *Bernoulli equation*

$$\varphi_t + \frac{1}{2}|u|^2 + p + gy = 0 \tag{16}$$

holds in Ω_t. This is a profoundly useful identity for homogeneous, irrotational flow. In *two*-dimensional *steady* flow, (16) takes the form

$$\frac{1}{2}|u - ce_x|^2 + p + gy = C$$

for some constant C, even when the vorticity is nonzero.

Hamiltonian formulation

Equations (13) and (16) form the basis for a Hamiltonian formulation of the water-wave problem for irrotational, homogeneous fluids; which is known as the Zakharov–Craig–Sulem formulation. This widely used formulation was first introduced by Zakharov in the paper [65], and later put into a more rigorous mathematical framework by Craig and Sulem in [20, 21].

The starting point is the introduction of the nonlocal *Dirichlet–Neumann operator* $G(\eta, \beta)$, which maps Dirichlet-data on the surface to corresponding Neumann-data (at a fixed time). If we consider the formal boundary value problem

$$\begin{aligned} \Delta\varphi &= 0 && \text{in } \Omega, \\ \varphi &= \phi && \text{on } S, \\ \partial^\perp \varphi &= 0 && \text{on } B, \end{aligned}$$

then the Dirichlet–Neumann operator is defined by

$$G(\eta, \beta)\phi \coloneqq (\partial^\perp \varphi)|_S,$$

where we have made the identification $\phi(x) = \varphi(x, \eta(x))$. This definition can be made rigorous in the setting of Sobolev spaces, see the book of Lannes [46] for a plethora of results concerning this operator.

Zakharov's observation was that the water-wave problem for an irrotational, homogeneous fluid can be written as the scalar evolution equation

$$\begin{aligned} \partial_t \eta &= G(\eta, \beta)\phi, \\ \partial_t \phi &= \frac{1}{2}\left(\left(\frac{G(\eta, \beta)\phi + \nabla\phi \cdot \nabla\eta}{\langle \nabla\eta \rangle}\right)^2 - |\nabla\phi|^2\right) - g\eta + \alpha\nabla \cdot \left(\frac{\nabla\eta}{\langle\nabla\eta\rangle}\right), \end{aligned} \tag{17}$$

entirely in terms of the surface variables η and ϕ. The evolution of η comes directly from (13), while the equation for ϕ is obtained from plentiful applications of the chain rule to the trace of (16) on S_t, together with (9).

The system in (17) even has the canonical Hamiltonian structure

$$\partial_t(\eta, \phi) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla E(\eta, \phi), \tag{18}$$

in terms of the energy

$$E(\eta, \phi) \coloneqq \frac{1}{2}\int_{\mathbb{R}^2} \phi G(\eta, \beta)\phi \, dx + \frac{g}{2}\int_{\mathbb{R}^2} \eta^2 \, dx + \alpha \int_{\mathbb{R}^2} (\langle\nabla\eta\rangle - 1)\, dx, \tag{19}$$

which makes the Zakharov–Craig–Sulem formulation exceptionally useful. The terms in the energy have the interpretation of being the kinetic energy, and the potential energies corresponding to gravity and surface tension, respectively. There are also various, but finitely many, additional conserved quantities, such as

$$P(\eta,\phi) := \int_{\mathbb{R}^2} \eta \nabla\phi \, dx, \qquad \text{(Horizontal momentum)}$$

$$m(\eta) := \int_{\mathbb{R}^2} \eta \, dx, \qquad \text{(Excess mass)}$$

with yet more described in [6]. An oft-used fact is that steady waves appear as critical points of the energy, with the constraint of fixed momentum. The wave speed c then appears naturally as a Lagrange multiplier.

Previous research

The purpose of this section is to list some of the earlier research regarding water waves. The field is so broad that we are only able to cover the small subset of results that are most relevant to us. In particular, we will focus on the two-dimensional case; and then mainly periodic gravity waves and solitary capillary-gravity waves, all propagating in a homogeneous fluid. The reader may also wish to consult each of the introductions to the papers that are included in this thesis, containing manifold references in addition to those found here.

The Cauchy problem

While this thesis does *not* concern the Cauchy problem for (17), we do touch on the topic of well-posedness in our third paper. We therefore find it appropriate to outline the current state of knowledge in this area. A good starting resource for this purpose is the monograph [46] by Lannes, but there have been quite significant developments since its publication.

For irrotational waves, there have been several proofs [3, 36, 38] of *global* well-posedness of infinite-depth gravity waves for small data (in appropriate Sobolev spaces). *Local* well-posedness is known on finite depth, even when the bottom is not flat [45], and a very recent paper [62] claims that this can be made global with a flat bed. The problem is also known to be locally well posed with surface tension [2].

Significantly less is known about the water-wave problem *with* vorticity. There are general results conerning local well-posedness [19], but the first long-time result is the paper [37] on cubic lifespan for with constant vorticity on infinite depth. There is also a recent preprint for waves with a so-called point vortex [57], which we will briefly touch on again below.

Irrotational steady waves

This part would not be complete without a cursory mention of Stokes' conjecture on Stokes waves, although we cannot do the tale justice here. Such waves are the simplest form of steady periodic gravity waves on (mainly) infinite depth; being symmetric, and monotone between their single crest and trough in each period. Loosely stated, the conjecture [56] says that there is a *highest* Stokes' wave, which is convex between sharp crests with a *stagnation point* and angles of precisely 120°; see Figure 3. Here, "highest" essentially means that this wave can be reached as a limit of smooth waves.

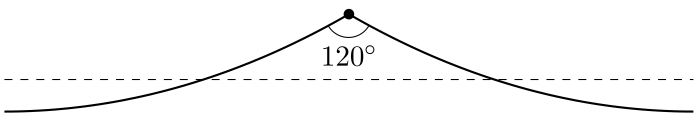

Figure 3: Approximately the surface profile of a highest Stokes' wave, based on [53]. The stagnation point at the crest corresponds to a fluid particle that is stationary in the steady frame.

The conjecture was settled in the affirmative in a series of papers over the course of the 20th century, starting with small waves in [48] and later large [43], culminating with [4, 51, 52] (but this list is far from exhaustive). The proofs rely on global bifurcation theory in cones [23] applied to the nonlocal Nekrasov equation [50], which is a one-dimensional formulation based on traces of conformal variables. The idea being that the relative potentials form a conformal map

$$(x, y) \mapsto \varphi(x, y) + i\psi(x, y), \tag{20}$$

known as the *hodograph* transform, onto a rectangle in the complex plane. More recent results typically use the alternative Babenko equation [5] instead, which is also based on conformal variables. Many more details can be found in e.g. [11, 58], or the more broad survey [33].

Apart from the Stokes waves, which act as a stepping stone to *rotational* periodic gravity waves, we are particularly interested in solitary capillary-gravity waves. One result we want to single out is Mielke's *conditional orbital stability* result for finite-depth solitary capillary-gravity waves [49]. Orbitally stable means that small perturbations of the wave remain close to its orbit, typically with respect to the natural norm for the energy in (19). The result is conditional both in the sense that the solution must exist, *and* must be bounded in a more regular norm.

The gist of the paper of Mielke is that it required modifying the seminal stability theory of Grillakis, Shatah & Strauss [32]. This theory applies to many abstract Hamiltonian systems of the form

$$\partial_t u = J\nabla E(u), \tag{21}$$

with J skew-symmetric, but does not quite apply to (18): There is a mismatch between the natural space for the energy in (19) and the much smaller space where (18) is well posed. In [32], these spaces are assumed to be one and the same, so this mismatch had to be dealt with. Mielke's paper was later followed by the existence result [8], using a variational approach that also independently establishes stability at the same time. This was more recently extended to infinite depth over several papers by Buffoni, Groves & Wahlén [9, 10, 34, 35].

Rotational steady waves

Although a highest Stokes' wave has a stagnation point at its crest, irrotational waves cannot have *interior* stagnation points; owing to the maximum principle for the Laplace equation. The necessary first step towards steady waves with more "interesting" behavior, with one or more stagnation points, is therefore to allow nonzero vorticity, and in the process losing the velocity potential. Tied in with stagnation points are also *critical layers*, which are regions of closed *streamlines*. A streamline is simply an integral curve for the relative velocity field, and therefore a subset of a level curve for the relative stream function.

In [25], Dubreil-Jacotin established the existence of small-amplitude gravity waves, with a quite general *vorticity distribution*. This is a function $\gamma\colon \mathbb{R} \to \mathbb{R}$ such that

$$\Delta\psi + \gamma(\psi) = 0 \tag{22}$$

on the fluid domain, with ψ denoting the relative stream function. That is, by (14), the vorticity distribution relates the vorticity to the value of the value of ψ on each streamline. Under the assumption that $\psi_y > 0$ in the fluid, Dubreil-Jacotin employs the *semi*-hodograph transform

$$(x, y) \mapsto (x, \psi(x, y))$$

as a replacement for the hodograph transform (20) used for irrotational waves. With this hypothesis, the vorticity distribution always exists. Much later, this framework was expanded upon to prove the existence of the first *large*-amplitude waves with vorticity in the highly influential paper [14].

As we alluded to, the waves constructed in [14, 25] do not exhibit stagnation, as this is immediately precluded by the semi-hodograph transform. A different approach to fixing the domain must therefore be utilized. Wahlén used the so-called naive flattening transform $(x, y) \mapsto (x, y/(d + \eta(x)))$ in his existence proof for small-amplitude waves with constant vorticity and a critical layer [61], on finite depth. This followed the earlier linear result in [29], which was the first investigation into waves with interior stagnation. An alternative, and perhaps more elegant, method was used for the result of Wahlén in [17]. This framework, which has the potential to allow overhanging waves, was developed further, and more recently used in a proof of *large* amplitude waves with stagnation [16], also with constant vorticity.

Even if the framework of [14, 16] is refined, it is also decidedly tailored for constant vorticity. Following [61], the naive transform was used also for small-amplitude waves with affine [27] and more general [41] vorticity distributions. Going from just constant to affine vorticity allows for an *arbitrary* number of critical layers [26], and multimodal [30] waves. A pair of recent papers [40, 42] concern solitary waves admitting critical layers, using spatial dynamics. A multitude of questions still remain in this area, and we shall in particular address the entirely open problem of global bifurcation for stagnant waves with vorticity distributions other than constant.

A separate avenue for constructing stagnant waves has spawned from [55], treating capillary-gravity waves with *compactly supported* vorticity. The paper establishes the existence of waves where the vorticity ω is either a *Dirac measure* supported in the fluid (point vortex), or a regular distribution with small compact support (vortex patch), on infinite depth. Specifically, the authors construct small- and large-amplitude periodic waves with a point vortex; and small-amplitude and vorticity solitary waves with either a point vortex or vortex patch. The waves with point vortices are constructed by applying the implicit function theorem to a modified, *stationary* version of the Hamiltonian formulation in (17). We recall that, in its original form, this formulation only applies to irrotational waves.

An interesting feature of the waves from [55], is that their wave speed is constrained. As opposed to the papers discussed above, this speed is forced to take a specific value in order for the point vortex or vortex patch to remain stationary with respect to the wave. As a consequence, the small-amplitude waves constructed are also *slow*. For the point vortex, a challenge is that both the steady water-wave problem and the evolution equation (11) for the vorticity must be understood in an appropriate weak sense, in particular disallowing self-propagation of the vortex.

We will investigate several unresolved problems pertaining to waves with point vortices. The first is the natural question of existence of the analog of the waves from [55] on *finite* depth. One may also inquire about the existence of waves with *multiple* vortices. Finally, we address the pressing issue of stability in the presence of a point vortex: It is highly desirable for a solitary wave to be stable, as these waves are the ones found in nature. This requires developing the ideas of the aforementioned paper of Mielke [49] even further.

THE WORKS CONTAINED IN THIS THESIS

Paper 1: Solitary gravity-capillary water waves with point vortices

Kristoffer Varholm

We take some of the ideas of [55], apply them to the *finite* depth, and develop them further. The main results of the paper are Theorems 1.12 and 1.20, which construct curves of small amplitude, vorticity and speed solitary capillary-gravity waves with one or more point vortices, respectively. The inclusion of more than one point vortex is novel, as [55] exclusively treats waves with a single point vortex.

A new phenomenon on finite depth is that the waves behave very differently depending on whether the point vortex is situated below or above half the unperturbed fluid depth. Not only do the resulting waves switch the direction they propagate as the point vortex passes the midpoint (for a fixed vorticity); the critical layer also connects with whichever is closer of the surface and the flat bed. See Figure 1.2.

The paper can afford to go into significantly more detail than [55] due to its focus on point vortices, providing higher-order terms in Theorem 1.12, and series expansions and asymptotics for the leading order surface profile (Proposition 1.15 and Theorem 1.17). We also analyze the particularly delicate case where the point vortex is precisely halfway in the fluid column at the bifurcation point, in Proposition 1.19. Finally, in Section 6 of the paper, we provide several explicit expressions for the waves constructed in [55].

This paper started out from the author's Master's thesis [59], but has seen substantial changes since then. This includes entirely new results, such as higher order terms and Proposition 1.19.

Paper 2: Traveling gravity water waves with critical layers
Ailo Aasen, Kristoffer Varholm

We take the same setting as [27]; that is, steady periodic gravity waves on finite depth, with an *affine* vorticity distribution (recall (22)). The main difference is that our paper applies bifurcation theory [22, 39] with respect to a different parameter than in [27]. Specifically, the trivial streams for the affine vorticity $\gamma(t) = \alpha t$ with $\alpha > 0$, and unperturbed depth $d = 1$, take the oscillating form

$$\psi_0(y; \mu, \lambda) \coloneqq \mu \cos(\alpha^{1/2} y + \lambda)$$

for two parameters $(\mu, \lambda) \in \mathbb{R}^2$. While [27] uses the parameter μ for one-dimensional bifurcation and (μ, α) for two-dimensional bifurcation, we use λ and (λ, α) in this paper, resulting in new bifurcation curves and sheets. The transversality condition

$$\cot(\lambda) \neq -\frac{\mu^2 \alpha^{3/2}}{2}$$

appears with this new choice of parameters, while no such condition was needed in [27].

Novel contributions include a fairly complete treatment of the geometry of the kernel equation for the linearized problem (Lemmas 2.11 and 2.14, and Theorem 2.12), including construction of *arbitrarily* large finite kernels in Theorem 2.18; addressing a question posed in [27, 28]. We also prove analyticity of the solutions (Theorem 2.5), give local descriptions of the solution set (Theorems 2.31 and 2.39), and compute derivatives of the solution curves at the bifurcation point in specially crafted cases (Theorem 2.30 and Proposition 2.38).

This paper, like Paper 1, also originated from a Master's thesis [1], but very little of the original thesis remains in the finished article. In particular, everything listed in the previous paragraph on novel contributions is either entirely new or heavily improved.

Paper 3: On the stability of solitary water waves with a point vortex
Kristoffer Varholm, Erik Wahlén, Samuel Walsh

We prove conditional orbital stability of solitary capillary-gravity waves with a point vortex on infinite depth, originally constructed in [55]. The starting point is to take the canonical Hamiltonian formulation of [54], which extends (18), and make it rigorous (Theorem 3.29); in the process making it non-canonical, with a state-dependent Poisson map. In turn, we make several nontrivial generalizations to the abstract stability theory due to Grillakis, Shatah & Strauss [32], to account for various deficits that makes their theory not apply to the problem at hand. A full description of our theory does not fit in this introduction; but in essence it allows for:

- *State-dependent* Poisson maps of the form $u \mapsto J(u) = B(u)\hat{J}$, where B is bounded and $\hat{J}$ is closed, cf. (21).
- Poisson maps that are *not* surjective, but merely have dense range
- *Affine* symmetry groups, as opposed to only linear (necessary to describe the evolution of a point vortex).
- *Three* different spaces of varying roughness: One where well-posedness is expected, one where the energy is regular, and finally the natural energy space.

The main abstract results are Theorem 3.11 and Corollary 3.15 for stability, and Theorem 3.14 for instability.

We provide two different applications of the theory. One, in the spirit of Mielke [49], is the titular application to solitary capillary-gravity waves with a point vortex. Theorem 3.33 establishes the conditional orbital stability of such waves.[4] The second concerns the stability, or instability, of solitary waves for nonlinear dispersive model equations of the form

$$\partial_t u = \partial_x(|\partial_x|^\alpha u - u^p),$$

encompassing well-known equations like Korteweg–de Vries ($p = 2$ and $\alpha = 2$) and Benjamin–Ono ($p = 2$ and $\alpha = 1$). Our abstract theory results in a new proof of the stability theory of Bona, Souganidis & Strauss [7], but in more generality and with fewer assumptions. Our main result for this application is Theorem 3.42, which in particular furnishes a novel result for conditional instability when $p = 2$ and $\alpha \in (1/3, 1/2)$; fractional Korteweg–de Vries with very weak dispersion.

Finally, we would like to mention that our abstract theory has already seen use in an *instability*-result for finite dipoles in the preprint [47]. Furthermore, we need to address the well-posedness result [57], mentioned in the previous section. This paper concerns *gravity* waves, and therefore does

[4]It is expected that a similar result holds for the finite-depth waves with a single point vortex from [60], but we did not consider these.

not apply in our setting, but does bode well for the prospect of a future result that includes surface tension.

Paper 4: Global bifurcation of waves with multiple critical layers
Kristoffer Varholm

The final paper concerns steady periodic periodic gravity waves on finite depth, with a *real analytic* vorticity distribution. We take the framework of [27, 61], which was also used in Paper 2, and develop it further (see below). This is done in order to be able to construct *large*-amplitude waves with *any* number of critical layers, using the analytic global bifurcation theory described in [11]. As we mentioned prior to this section of the introduction, the *only* previous global bifurcation result for steady gravity waves with a critical layer is [15]. While a groundbreaking paper, the formulation used is highly specialized to the case of constant vorticity, and admits at most one critical layer.

Our main results are Theorems 4.17 and 4.18, which establish the existence of small curves of solutions (a similar local result can be found in [41]), and extends them to global solution curves, respectively. *Any* real analytic vorticity distribution, as in (22), with bounded derivative is admissible. The proof of the global bifurcation theorem requires both the generalization of the so-called $\mathcal{T}$-isomorphism from [27] to nontrivial solutions (Theorem 4.5), which is used to extend the Fredholm property to the solution set at large, and a proof of compactness using a near-surface semi-hodograph transform (Proposition 4.22). This latter approach is a new feature in the theory of steady water waves.

Another aspect of the work is a new description of the kernel of the linearized equation (in Section 3), using the Prüfer angle, which includes the kernel equation of [27] as a special case; see in particular Theorem 4.9 and Proposition 4.15. We also construct exactly one-dimensional kernels in Proposition 4.11, suitable for one-dimensional bifurcation, when the vorticity distribution is sufficiently close to affine.

Appendix A: Periodic point vortices on finite depth

Finally, we provide a closed-form expression for the stream function for periodic waves with a point vortex on finite depth, in terms of so-called Weierstrass functions. To our knowledge, this does not appear earlier in the literature. These expressions could be used to treat periodic waves (as opposed to solitary) in the setting of Paper 1.

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PAPER 1

SOLITARY GRAVITY-CAPILLARY WATER WAVES WITH POINT VORTICES

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Abstract. We construct small-amplitude solitary traveling gravity-capillary water waves with a finite number of point vortices along a vertical line, on finite depth. This is done using a local bifurcation argument. The properties of the resulting waves are also examined: We find that they depend significantly on the position of the point vortices in the water column.

1 Introduction

The steady water-wave problem concerns two-dimensional water waves propagating with constant velocity and without change of shape. Historically, the focus has mainly been on irrotational waves, which are waves where the vorticity[1]

$$\omega := \nabla \times w = v_x - u_y$$

of the velocity field $w = (u, v)$ is identically zero. One reason for this is Kelvin's circulation theorem [29, 31], which says that a flow which is initially irrotational will remain so for all time, as long as it is only affected by conservative body forces (e.g. gravity). Another reason is mathematical, as the velocity field can then be written as the gradient of a harmonic function; the velocity potential. This enables the use of powerful tools from complex- and harmonic analysis, and the problem can be reduced to one on the boundary in a number of different ways [1, 38]. An important class of such waves are the Stokes waves, which are periodic waves that rise and fall exactly once every minimal period. The Stokes conjecture on the nature of the so-called Stokes wave of greatest height fueled research on waves throughout the 20th century, and would not be fully resolved until 2004 (see the survey [43] and [39], which settled the convexity of this wave).

[1] Informally, the vorticity describes (twice) the velocity at which an infinitesimal paddle wheel placed in the fluid will rotate.

More recently, however, there has been renewed interest in rotational waves. There are several situations where such waves are appropriate, as effects like wind, temperature or salinity gradients can all induce rotation [37]. Rotational waves can be markedly different from irrotational waves: For instance, in rotational waves it is possible to have internal stagnation points and critical layers of closed streamlines known as cat's eye vortices [15].

The first result on rotational waves came surprisingly early, in the beginning of the 1800s with [22] (for a more modern exposition, see [3]). There, Gerstner gave the first, and still the only known, explicit (nontrivial) gravity-wave solution to the Euler equations on infinite depth. Although significant because it is an exact solution, it is viewed as more of a mathematical curiosity, even today (see [4, Chapter 4.3]). Much later, in [14], came the first existence result for small-amplitude waves with quite general vorticity distributions. A vorticity distribution is a function $\gamma\colon \mathbb{R} \to \mathbb{R}$ such that

$$\Delta\psi = \gamma(\psi),$$

where ψ is the relative stream function (which, unlike the velocity potential, is still available for rotational waves, but is not harmonic). A sufficient, but not necessary, condition for such a vorticity distribution to exist is that the wave has no stagnation points. Several improvements have been made to the existence result of Dubreil-Jacotin, but it was not until the pioneering article [5] that large waves were constructed, using global bifurcation theory. This article sparked mathematical research into rotational waves.

The use of a semi-hodograph transform in [14] and [5], and the corresponding deep-water result in [27], means that the resulting waves cannot exhibit critical layers. Since then, small-amplitude waves with constant vorticity and a critical layer have been constructed in [47], and later in [8] with a different approach that allows for waves with overhanging surface profiles (there is numerical evidence for the existence of such waves, e.g. [44], but this is still an open problem). A reasonable next step is that of waves with an affine vorticity distribution, whose existence was shown in [16, 19]. Spurred by the above results there has also been interest in studying the properties and dynamics of these waves below the surface [15, 47]. This had been done for linear waves in [18]. Several other avenues have also been considered: We mention heterogeneous waves both with [25, 49] and without [20, 48] surface tension, waves with discontinuous vorticity [6], a variational approach [2] and Hamiltonian formulation with center manifold reduction [24]. Existence of large amplitude waves with constant vorticity and a critical layer was established in [36], in the presence of capillary effects.

There is also a forthcoming result for pure gravity waves [7], using an entirely different approach.

Common for all the previously mentioned works on rotational waves is the feature that the vorticity is supported on the entire fluid domain (due to the assumption of the existence of a vorticity distribution). Recently, gravity-capillary waves with compactly supported vorticity were constructed in [41], on infinite depth. This includes small- and large-amplitude periodic waves with a point vortex, and small-amplitude solitary waves with either a point vortex or vortex patch. By a point vortex we mean that the vorticity is given by a δ-function, while we use vortex patch to mean that the vorticity is locally integrable and compactly supported. The waves with a point vortex are the simplest form of waves with compactly supported vorticity, and are in a sense “almost irrotational”.

In this paper, which is based on [45], we extend the existence result for solitary small-amplitude waves with a point vortex to finite depth, and also give both qualitative and quantitative properties for these waves. The main approach to showing existence follows that of [41], but we also treat the natural generalization of waves with several point vortices along a vertical line, and show existence for all but exceptional configurations of vortices. Finally, by finding an explicit expression for the rotational part of the stream function, we give some explicit expressions for the small-amplitude periodic waves with a point vortex on infinite depth that were constructed in [41].

An outline of the paper is as follows: In Section 2 we formulate the problem, and in Section 3 we give the functional-analytic setting for this formulation. Then, in Section 4 we prove existence of small solutions, and give some properties for these. Section 5 treats the extension to several point vortices. The final section, Section 6, contains the explicit expressions for periodic waves on infinite depth.

2 Formulation

Under the assumption of inviscid (absence of viscosity) and incompressible (constant fluid density) flow, the governing equations of motion are the so-called incompressible Euler equations. For describing water waves on the open sea, these are realistic assumptions [29, 31], and standard. We will further assume two-dimensional flow under the influence of gravity, where the Cartesian coordinates (x, y) describe the horizontal and vertical direction, respectively. Then the equations read

$$\begin{aligned} w_t + (w \cdot \nabla)w &= -\nabla p - g e_2, && \text{(Conservation of momentum)} \\ \nabla \cdot w &= 0, && \text{(Conservation of mass)} \end{aligned} \tag{1.1}$$

where $w = (u, v)$ is the velocity of the fluid, p is the pressure distribution and $-ge_2 = (0, -g)$ is the constant gravitational acceleration[2].

For convenience we place, at time t, the flat bottom at

$$\{(x, y) \in \mathbb{R}^2 : y = -h\}$$

and the surface at

$$\{(x, y) \in \mathbb{R}^2 : y = \eta(x, t)\},$$

where η describes the deviation of the free boundary. We assume that $\eta(\cdot, t)$ is bounded, continuous and strictly bounded below by $-h$. It should be emphasized that, due to the free boundary assumption, the function η is a priori unknown; determining it is part of the problem.

In addition to Equation (1.1), we require boundary conditions to match our domain. In order to model the bottom being impermeable, we will demand that

$$v|_{y=-h} = 0 \qquad \text{(Kinematic boundary condition at bottom)},$$

with which we mean that $v(x, -h, t) = 0$ for all x and t. Next, we impose the condition

$$u\eta_x + \eta_t = v \qquad \text{(Kinematic boundary condition at surface)}$$

at the surface. This equation is what connects the free boundary to the fluid, and is equivalent to demanding that particles at the surface will remain there. We also require that

$$p|_{y=\eta} = -\alpha^2\kappa(\eta), \qquad \text{(Dynamic boundary condition)} \tag{1.2}$$

where $\alpha^2 > 0$ describes the surface tension and κ is the nonlinear differential operator

$$\kappa(\eta) := \left(\frac{\eta_x}{\langle\eta_x\rangle}\right)' = \frac{\eta_{xx}}{\langle\eta_x\rangle^3},$$

yielding the curvature of the surface. The symbol $\langle\cdot\rangle$ denotes the Japanese bracket defined through $x \mapsto (1 + |x|^2)^{1/2}$. Equation (1.2) is known to physicists as the Young–Laplace equation, and states that the pressure difference across a fluid interface (in this case water/air) is proportional to its curvature.

Note that in the lower limit $\alpha^2 = 0$, the dynamic boundary condition in Equation (1.2) corresponds to the assumption of constant pressure on the surface, but we will require that α^2 be strictly positive. The proof of Theorem 1.12, for example, relies upon the assumption that $\alpha^2 > 0$.

[2]The constant g is approximately $9.8\,\mathrm{m/s^2}$, varying by less than $0.4\,\%$ on the Earth's suface (see [26]).

The vorticity equation

By taking the curl of Equation (1.1), one obtains after some simple calculations that

$$\omega_t + \nabla \cdot (\omega w) = 0, \tag{1.3}$$

which states that the vorticity ω is transported by the vector field w. Due to this, it is natural to expect that if the vorticity consists of a point vortex at some time, then it will remain a point vortex at all future times, and be transported with the flow. It should be emphasized that, for now, this is not justified by Equation (1.3); the multiplication of ω with w is not well defined, as w will not be smooth at the point vortex. Thus, we will have need of a weaker form of the equation. We remind the reader of the fundamental solution of the Poisson equation.

Proposition 1.1 (Newtonian potential)**.** *The distribution* $\Gamma \in L^2_{\text{loc}}(\mathbb{R}^2)$ *defined by*

$$\Gamma(x, y) \coloneqq \frac{1}{4\pi} \log(x^2 + y^2)$$

satisfies

$$\nabla^\perp \Gamma(x, y) \coloneqq (-\Gamma_y, \Gamma_x)(x, y) = \frac{1}{2\pi} \frac{(-y, x)}{x^2 + y^2}$$

and

$$\Delta \Gamma = \nabla \times \nabla^\perp \Gamma = \delta.$$

If ω is of the form

$$\omega(t) = \delta_{(x_0(t), y_0(t))},$$

then we deduce from Proposition 1.1 that w is of the form

$$w(x, y, t) = \frac{1}{2\pi} \frac{(y_0(t) - y, x - x_0(t))}{(x - x_0(t))^2 + (y - y_0(t))^2} + \hat{w}(x, y, t),$$

where $\hat{w}$ satisfies $\nabla \cdot \hat{w} = 0$ and $\nabla \times \hat{w} = 0$, and is therefore smooth in space (see the discussion before Equation (1.10)). As the first term, which we may think of as the part of w generated by the point vortex, is singular, divergence free and odd around $(x_0(t), y_0(t))$, it is not unreasonable to think that the dynamics of the point vortex should depend only on $\hat{w}$. In other words, that the path $t \mapsto (x_0(t), y_0(t))$ along which the point vortex moves should satisfy

$$(\dot{x}_0, \dot{y}_0) = \hat{w}. \tag{1.4}$$

This can indeed be made rigorous. In [32, Theorems 4.1 and 4.2] it is proved that if one considers initial data consisting of a vortex patch converging in the sense of distributions to a point vortex, then the weak solutions of the vorticity transport equation converge to a moving point vortex in an appropriate sense. Moreover, the position of this point vortex satisfies Equation (1.4). Thus, we will allow for point vortices, as long as they are propagated in the fluid as in Equation (1.4).

Traveling waves

We now assume that there are functions $\tilde{w}, \tilde{p}, \tilde{\eta}$, depending only on space, and a constant velocity $c \in \mathbb{R}$ such that

$$\begin{aligned} w(x,y,t) &= \tilde{w}(x-ct,y), \\ p(x,y,t) &= \tilde{p}(x-ct,y), \\ \eta(x,t) &= \tilde{\eta}(x-ct) \end{aligned}$$

for all relevant x, y and t. Positive and negative c then correspond to waves moving in the positive and negative x-directions, respectively. In the new *steady* variables $(\tilde{x}, \tilde{y}) = (x-ct, y)$, after dropping the tildes, our equations read

$$\begin{aligned} (w\cdot\nabla)w - cw_x &= -\nabla p - ge_2, && \text{(Conservation of momentum)} \\ \nabla\cdot w &= 0, && \text{(Conservation of mass)} \end{aligned} \tag{1.5}$$

with boundary conditions

$$\begin{aligned} v &= 0, && \text{at } y=-h, && \text{(Kinematic)} && (1.6) \\ (u-c)\eta' &= v, && \text{at } y=\eta(x), && \text{(Kinematic)} && (1.7) \\ p &= -\alpha^2\kappa(\eta), && && \text{(Dynamic)} && (1.8) \end{aligned}$$

on the now time-independent domain

$$\Omega(\eta) \coloneqq \{(x,y)\in\mathbb{R}^2 : -h < x < \eta(x)\}.$$

We call the problem of finding w, p and η such that these equations are satisfied the *steady water-wave problem.* Note also that the vorticity equation given in Equation (1.4) reduces to

$$(c,0) = \hat{w}(x_0,y_0) \tag{1.9}$$

for a point vortex centered at $(x_0,y_0)\in\Omega(\eta)$.

The Zakharov–Craig–Sulem formulation

It turns out that it is possible to reduce the water-wave problem to an entirely one-dimensional one on the surface in a clever way. This is known as the Zakharov–Craig–Sulem formulation, and was first introduced by Zakharov in [50], and then later put on a firmer mathematical basis in [10, 11]. The original formulation relies on the fluid being irrotational, but it is in fact sufficient that this holds near the surface. This is where the compact support of the vorticity comes in.

Suppose that we have solved the steady water wave problem for some w, p, η. It is then convenient to split the velocity w as

$$w = \hat{w} + W,$$

where $\hat{w}$ is irrotational, that is, $\nabla \times \hat{w} = 0$ and $\nabla \times W = \omega$. We also assume that both $\hat{w}$ and W are divergence free. For us, W will be known.

Although we will allow for ω to be a singular distribution, the vector field $\hat{w} = (\hat{u}, \hat{v})$ will be assumed to at least be in the Sobolev space $H^1(\Omega(\eta))^2$, and $W = (U, V)$ to be at least $L^1_{\text{loc}}(\Omega(\eta))$. By the assumption of $\nabla \cdot \hat{w} = \nabla \cdot W = 0$, the differentials

$$\hat{v}\, dx - \hat{u}\, dy, \quad V\, dx - U\, dy$$

on $\Omega(\eta)$ are closed. Hence, as $\Omega(\eta)$ is simply connected, these differentials are exact by generalizations of the Poincaré lemma [33, Theorems 2.1 and 3.1]. Thus, there are stream functions $\hat{\psi} \in H^2_{\text{loc}}(\Omega(\eta)), \Psi \in L^1_{\text{loc}}(\Omega(\eta))$ (by [13, Corollary 2.1]), determined uniquely modulo constants by $\hat{w}$ and W, such that

$$\hat{w} = \nabla^\perp \hat{\psi}, \quad W = \nabla^\perp \Psi.$$

Moreover, by the assumed curl of these vector fields, the function $\hat{\psi}$ is harmonic, and Ψ satisfies $\Delta \Psi = \omega$. In particular, $\hat{\psi}$ is smooth, and so is W outside the support of ω.

By the above, we thus have that

$$w = \nabla^\perp(\hat{\psi} + \Psi) \tag{1.10}$$

holds for the velocity field w. Suppose now that W and Ψ are chosen such that $\Psi = 0$ at the bottom. Then the boundary condition in Equation (1.6) translates to $\hat{\psi}$ being constant along the bottom. Since $\hat{\psi}$ is unique modulo constants, we may as well take this condition to be

$$\hat{\psi}\Big|_{y=-h} = 0$$

instead.

We will now apply the assumption that $\operatorname{supp}\omega \subseteq \Omega(\eta)$ is compact. This means that a velocity potential for w exists on any simply connected domain not containing $\operatorname{supp}\omega$. Equation (1.5) can then in turn be used to show that

$$\nabla\left(-cu + \frac{1}{2}|w|^2 + p + gy\right) = 0$$

holds on any such domain. Hence, in particular, we obtain the surface Bernoulli equation

$$c(\hat{\psi}_y + \Psi_y) + \frac{1}{2}|\nabla\hat{\psi} + \nabla\Psi|^2 - \alpha^2\kappa(\eta) + g\eta = C, \quad \text{at } y = \eta(x), \tag{1.11}$$

where C is a real constant. Here we have inserted the boundary condition for the pressure at the surface, Equation (1.8). We will set $C = 0$, since we are looking for localized waves. This can be seen by letting $|x| \to \infty$ in Equation (1.11). We now need the following formal definitions to proceed, which will be specified later on.

Definition 1.2 (Harmonic extension operator)**.** Given η, we define the harmonic extension operator $H(\eta)$ as the operator mapping a function $\zeta\colon \mathbb{R} \to \mathbb{R}$ to the harmonic function $\hat{\psi}\colon \Omega(\eta) \to \mathbb{R}$ satisfying

$$\begin{aligned} \hat{\psi}(\cdot, \eta(\cdot)) &= \zeta, \\ \hat{\psi}(\cdot, -h) &= 0. \end{aligned}$$

Definition 1.3 (Dirichlet-to-Neumann operator)**.** Given η, we define the Dirichlet-to-Neumann operator $G(\eta)$ as the operator mapping Dirichlet data to non-normalized Neumann data; that is, the operator defined by

$$G(\eta)\zeta \coloneqq (-\eta', 1) \cdot \nabla[H(\eta)\zeta]|_{y=\eta}$$

for functions $\zeta\colon \mathbb{R} \to \mathbb{R}$.

With Definition 1.2 in mind, define $\zeta\colon \mathbb{R} \to \mathbb{R}$ by

$$\zeta \coloneqq \hat{\psi}(\cdot, \eta(\cdot)), \tag{1.12}$$

that is, the trace of $\hat{\psi}$ on the surface. By our assumptions, then, we have

$$\hat{\psi} = H(\eta)\zeta,$$

and we will use this to reformulate Equation (1.11) in a way that only involves ζ and Ψ. Note that

$$\begin{aligned} \zeta' &= \hat{\psi}_x + \eta' \hat{\psi}_y, \\ G(\eta)\zeta &= -\eta' \hat{\psi}_x + \hat{\psi}_y, \end{aligned}$$

where the right-hand side is evaluated at $y = \eta(x)$. Inverting these relations and inserting them in Equation (1.11) yields

$$\begin{aligned} c&\left[\frac{\eta'\zeta' + G(\eta)\zeta}{\langle \eta' \rangle^2} + \Psi_y\right] \\ &+ \frac{(\zeta' + (1, \eta') \cdot \nabla \Psi)^2 + (G(\eta)\zeta + (-\eta', 1) \cdot \nabla \Psi)^2}{2\langle \eta' \rangle^2} \\ &\qquad + g\eta - \alpha^2 \kappa(\eta) = 0, \end{aligned} \tag{1.13}$$

and in a similar fashion, one obtains

$$c\eta' + \zeta' + (1, \eta') \cdot \nabla \Psi = 0 \tag{1.14}$$

from the kinematic boundary condition in Equation (1.7). Equation (1.14) can be integrated once to yield

$$c\eta + \zeta + \Psi = 0, \tag{1.15}$$

where we have assumed decay at infinity. We emphasize that the function Ψ and its derivatives are evaluated at $y = \eta(x)$ in Equations (1.13) to (1.15), which we suppress for readability. Equations (1.13) and (1.15) form the Zakharov–Craig–Sulem formulation, which we will combine with a suitable vorticity equation. One may note that the pressure, p, has been eliminated from the formulation entirely.

Remark 1.4. Equation (1.13) is slightly different than the equation used in [41]. The equation in [41] can be obtained by inserting the kinematic boundary condition, Equation (1.14), into Equation (1.13).

3 Functional-analytic setting

We now focus on proving the existence of a family of small amplitude and small velocity traveling waves with vorticity consisting of a point vortex situated on the y-axis. In other words, solutions with vorticity of the form

$$\omega = \varepsilon \delta_\theta,$$

where $0 < |\varepsilon| \ll 1$, $\theta \in (0,1)$ and where we have defined

$$\delta_\theta := \delta_{(0,-(1-\theta)h)}.$$

The constant θ then corresponds to the relative position of the point vortex above the bottom, and the parameter ε, describing the strength of the vortex, will be used as the bifurcation parameter.

We will from here on always assume that η is such that $\eta(0) > -(1-\theta)h$, which prevents the surface from touching the point vortex. Furthermore, we will also assume that $\eta(0) < (1-\theta)h$. The reason for this is purely technical (as we will see after Proposition 1.5). For the purpose of accounting for these assumptions, define the set

$$\Lambda_\theta := \{\eta \in BC(\mathbb{R}) : \eta > -h, |\eta(0)| < (1-\theta)h\}, \tag{1.16}$$

whose intersection $\Lambda_\theta \cap H^s(\mathbb{R})$ is open in $H^s(\mathbb{R})$ for any $s > 1/2$, by the Sobolev embedding $H^s(\mathbb{R}) \hookrightarrow BC(\mathbb{R})$.

The next proposition describes the stream function that we will use for the rotational part of the velocity; the counterpart of the function $\mathbf{G}$ in [41]. While we could use a similar stream function on finite depth, it is more beneficial to work with one that is tailored for finite depth.

Proposition 1.5 (Stream function)**.** *Let $\eta \in \Lambda_\theta$ and define $\Phi\colon \Omega(\eta) \to \mathbb{R}$ by*

$$\Phi(x,y) := \frac{1}{4\pi} \log\left(\frac{\cosh(\pi x/h) + \cos(\pi(y/h - \theta))}{\cosh(\pi x/h) + \cos(\pi(y/h + \theta))}\right).$$

Then Φ defines a regular distribution, and

$$\begin{aligned} \Delta\Phi &= \delta_\theta, \\ \Phi|_{y=0} &= 0, \\ \Phi|_{y=-h} &= 0. \end{aligned} \tag{1.17}$$

Moreover, the function $(x,y) \mapsto \Phi(x,y) - \Gamma(x, y + (1-\theta)h)$ is harmonic and satisfies

$$\nabla^\perp \left(\Phi - \Gamma(x, y + (1-\theta)h)\right)(0, -(1-\theta)h) = \left(\frac{1}{4h}\cot(\pi\theta), 0\right), \tag{1.18}$$

where Γ is the Newtonian potential introduced in Proposition 1.1.

Proof. We will apply Theorem 1.29 (see Section A) to prove this result, and thus need a bijective conformal map from the strip $\mathbb{R} \times (-h, 0) \subseteq \mathbb{C}$ to the unit disk $\mathbb{D}$, mapping the point $-i(1-\theta)h$ to the origin. This is done in three steps:

$$\begin{tikzcd}
\mathbb{R}\times(-h,0) \arrow[d, "(z+ih)/h"] \arrow[r, bend left, "f"] & \mathbb{D} \\
\mathbb{R}\times(0,1) \arrow[r, "\exp(\pi z)"] & \mathbb{R}\times(0,\infty) \arrow[u, "\frac{z-\exp(i\pi\theta)}{z-\exp(-i\pi\theta)}"']
\end{tikzcd}$$

The conformal map for each individual step is well known from elementary complex analysis, see for instance [21, II.7 and p. 60]. Hence

$$f(z) \coloneqq \frac{e^{\pi(z+ih)/h} - e^{i\pi\theta}}{e^{\pi(z+ih)/h} - e^{-i\pi\theta}}$$

defines the desired map from the strip to the unit disk. By the aforementioned theorem, then,

$$\begin{aligned}
\Phi(x,y) &\coloneqq \frac{1}{2\pi}\log(|f(x+iy)|) \\
&= \frac{1}{4\pi}\log\left(\frac{\cosh(\pi x/h)+\cos(\pi(y/h-\theta))}{\cosh(\pi x/h)+\cos(\pi(y/h+\theta))}\right)
\end{aligned}$$

solves Equation (1.17) in $\mathbb{R}\times(-h,0)$. Because f extends to a meromorphic function on $\mathbb{C}$, it is immediate that Φ also solves Equation (1.17) in $\Omega(\eta)$ (recall that $\eta\in\Lambda_\theta$).

Finally, we have

$$\begin{aligned}
\nabla^\perp\left(\Phi-\Gamma(x,y+(1-\theta)h)\right)(0,\theta) &= \frac{i}{4\pi}\overline{\left(\frac{f''(i\theta)}{f'(i\theta)}\right)} \\
&= \left(\frac{1}{4h}\cot(\pi\theta),0\right)
\end{aligned}$$

by the final part of the same theorem, which will be important for the asymptotic velocity of the traveling waves that we shall obtain in Theorem 1.12. $\square$

Note that the stream function Φ introduces a "mirror vortex" at $(0,(1-\theta)h)$, and moreover is $2h$-periodic in the y-direction. This is the reason for the limitation on the height of the surface profiles in the set Λ_θ defined in Equation (1.16).

The next proposition is crucial, because the traces of Φ and its derivatives on the surface enter in the Zakharov–Sulem–Craig formulation of the problem. Having an explicit expression for Φ enables us to prove the proposition in a quite direct way.

Proposition 1.6. *Suppose that $\eta \in H^s(\mathbb{R}) \cap \Lambda_\theta$, where $s > \frac{1}{2}$. Then*

$$\begin{aligned} \Phi(\cdot, \eta(\cdot)) &\in H^s(\mathbb{R}), \\ \nabla^\perp \Phi(\cdot, \eta(\cdot)) &\in H^s(\mathbb{R})^2. \end{aligned}$$

Moreover, the dependence on η is analytic.

Proof. We will only treat Φ, as the argument for the derivative is similar. Observe that it is sufficient to consider the function defined by

$$x \mapsto \log(1 + \cos(\eta(x) - \theta)\,\mathrm{sech}(x)) \tag{1.19}$$

for $\eta \in H^s(\mathbb{R})$ such that $|\eta(0)| < \pi - \theta$, where $\theta \in (0, \pi)$. Since

$$\cos(\eta(x) - \theta)\,\mathrm{sech}(x) = (\cos(\eta(x) - \theta) - \cos(\theta))\,\mathrm{sech}(x) + \cos(\theta)\,\mathrm{sech}(x)$$

it follows by [40, Theorem 4 of 5.5.3], sech $\in H^s(\mathbb{R})$ and $H^s(\mathbb{R})$ being an algebra that the function in Equation (1.19) lies in $H^s(\mathbb{R})$ and that the dependence on η is analytic. Another application of the result in [40] then yields the desired result. □

As we have seen, because of the reliance on the stream function and the operators $H(\eta)$ and $G(\eta)$, a central problem is the solution of the Laplace equation,

$$\begin{aligned} \Delta\hat{\psi} = 0 \quad \text{in } \Omega(\eta), \\ \hat{\psi}\Big|_{y=\eta} = \zeta, \quad \hat{\psi}\Big|_{y=-h} = 0 \end{aligned} \tag{1.20}$$

on the fluid domain $\Omega(\eta)$, given η and ζ. We have the following theorem, which is adapted from Corollary 2.44 in [30], and which establishes both existence and uniqueness to Equation (1.20) in suitable Sobolev spaces. Functions on the surface will be identified with functions on the real line as in Equation (1.12).

Theorem 1.7 (Well-posedness of the Laplace equation [30])**.** *Suppose that $\eta \in H^s(\mathbb{R}) \cap \Lambda_\theta$ for some $s > 3/2$, and that $\zeta \in H^{3/2}(\mathbb{R})$. Then Equation* (1.20) *has a unique solution in $H^2(\Omega(\eta))$.*

Remark 1.8. While the natural setting for the velocity potential or the stream function on infinite depth is that of homogeneous Sobolev spaces, used in both [41] and [30], this is not the case for the stream function on finite depth. Because we require $\hat{\psi}$ to be equal to a constant at the bottom, it must necessarily be the case that $\hat{\psi}$ tends to the same constant at infinity.

Otherwise, because of the finite depth, $\hat{\psi}_y$ would not decay at infinity (in the sense that $\lim_{|(x,y)|\to\infty} \psi_y(x,y) = 0$), and therefore not describe a localized wave[3].

Theorem 1.7 enables us to define the harmonic extension operator described in Definition 1.2 as an operator $H^{3/2}(\mathbb{R}) \to H^2(\Omega(\eta))$, and using this, defining the Dirichlet-to-Neumann operator. We refer the reader to [30], which is a rich source of results for these operators also in a more general setting. The results there are proved for the velocity potential, but should be adaptable for the stream function.[4] The below theorem is an amalgamation of parts from Corollary 2.40 and Theorems 3.15 and A.11 in [30]. See also [42].

Theorem 1.9 (Boundary operators [30]). *Let $s > 3/2$ and suppose that $\eta \in H^s(\mathbb{R}) \cap \Lambda_\theta$. Then the harmonic extension operator $H(\eta)$ and the Dirichlet-to-Neumann operator $G(\eta)$ are members of $B(H^{3/2}(\mathbb{R}), H^2(\Omega(\eta))$ and $B(H^s(\mathbb{R}), H^{s-1}(\mathbb{R}))$, respectively. The norms of these operators are uniformly bounded on subsets of $H^s(\mathbb{R}) \cap \Lambda_\theta$ that are bounded in the norm on $H^s(\mathbb{R})$. Moreover, the map $G(\cdot)\zeta$ is analytic for fixed $\zeta \in H^s(\mathbb{R})$.*

In the same setting as in Theorem 1.9, the curvature of the surface is well defined:

Proposition 1.10 (Curvature). *The curvature operator κ is well defined as an operator $H^s(\mathbb{R}) \to H^{s-2}(\mathbb{R})$ for any $s > 3/2$. Moreover, the map is analytic.*

Proof. Observe that the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x\langle x\rangle^{-1}$ is smooth and satisfies $f(0) = 0$. As $s - 1 > \frac{1}{2}$, the result [40, Theorem 4 of 5.5.3] ensures that $f(\eta') \in H^{s-1}(\mathbb{R})$. Since f is also analytic, κ is analytic by the same result. □

There is one thing we have not yet looked at, namely the vorticity equation Equation (1.9). Recalling Equation (1.10), we will consider velocity fields of the form

$$w = \nabla^\perp \left(H(\eta)\zeta + \varepsilon\Phi \right). \tag{1.21}$$

[3] One could say that such a wave is localized if the limit exists and is different from zero, but this does not yield any new waves (only a change in the frame of reference).

[4] Compare with Theorems 3.49 and A.13 in [30] for the case of infinite depth, where the boundary conditions for the Laplace equation for the stream function and velocity potential coincide.

From Proposition 1.1 we know that the part of the stream function that is generated by the point vortex at $(0, -(1-\theta)h)$ is given by the Newtonian potential

$$\varepsilon\Gamma(x, y + (1-\theta)h),$$

whence Equation (1.21) reduces to

$$(c, 0) = \nabla^{\perp}[H(\eta)\zeta](0, -(1-\theta)h) + \varepsilon\left(\frac{1}{4h}\cot(\pi\theta), 0\right),$$

by Equation (1.18) in Proposition 1.5.

In particular, this means that any solution necessarily must satisfy

$$[H(\eta)\zeta]_x(0, -(1-\theta)h) = 0.$$

For simplicity, we choose to look for η, ζ in appropriately chosen subspaces of $H^s(\mathbb{R})$, such that this condition is automatically satisfied. Specifically, define

$$H^s_{\text{even}}(\mathbb{R}) := \{f \in H^s(\mathbb{R}) : f \text{ is even}\},$$

which is closed in $H^s(\mathbb{R})$, and therefore a Hilbert space in the inherited norm. We mention that it is still an open question whether asymmetric traveling waves exist. However, it is known that, in many situations, certain properties imply symmetry (see for instance [9, 28]). Furthermore, under suitable assumptions, all symmetric waves are traveling waves [17].

Assume now that $\eta \in H^s_{\text{even}}(\mathbb{R}) \cap \Lambda_\theta$, with $s > 3/2$, and that $\zeta \in H^{3/2}_{\text{even}}(\mathbb{R})$. Then it must necessarily be the case that $H(\eta)\zeta$ is even in x. Hence $[H(\eta)\zeta]_x$ vanishes along the y-axis and so the vorticity equation reduces further to

$$c = c_1\varepsilon - [H(\eta)\zeta]_y(0, -(1-\theta)h), \quad \text{where } c_1 := \frac{1}{4h}\cot(\pi\theta). \tag{1.22}$$

Observe also that the Dirichlet-to-Neumann operator $G(\eta)$ is well defined as an operator $H^s_{\text{even}}(\mathbb{R}) \to H^{s-1}_{\text{even}}(\mathbb{R})$ for $\eta \in H^s_{\text{even}} \cap \Lambda_\theta$ and $s > 3/2$, and that κ can be viewed as an operator $H^s_{\text{even}}(\mathbb{R}) \to H^{s-2}_{\text{even}}(\mathbb{R})$.

Remark 1.11. One has to be careful with claims about the solution set when $\varepsilon = 0$. Equation (1.22) of course actually only needs to be satisfied if $\varepsilon \neq 0$. This means that if we impose Equation (1.22), then we lose the trivial set of solutions

$$(\eta, \zeta, c, \varepsilon) \in \{0\} \times \{0\} \times \mathbb{R} \times \{0\}$$

for the other equations, except for the point $(0, 0, 0, 0)$. This should be kept in mind in any claims of uniqueness.

For convenience, define now the spaces

$$
\begin{aligned}
X^s &:= H^s_{\text{even}}(\mathbb{R}) \times H^s_{\text{even}}(\mathbb{R}) \times \mathbb{R}, \\
Y^s &:= H^{s-2}_{\text{even}}(\mathbb{R}) \times H^s_{\text{even}}(\mathbb{R}) \times \mathbb{R}
\end{aligned}
$$

and the set

$$
U^s_\theta := \{(\eta, \zeta, c) \in X^s : \eta \in \Lambda_\theta\} \subseteq X^s,
$$

which accounts for the limitations on η.

We proceed to introduce three maps that together will form the basis for our argument. For $s > 3/2$ we define $F_1 \colon U^s_\theta \times \mathbb{R} \to H^{s-2}_{\text{even}}(\mathbb{R})$ by

$$
\begin{aligned}
F_1(\eta, \zeta, c, \varepsilon) &= c\left[\frac{\eta'\zeta' + G(\eta)\zeta}{\langle \eta' \rangle^2} + \varepsilon \Phi_y\right] \\
&\quad + \frac{(\zeta' + \varepsilon(1, \eta') \cdot \nabla\Phi)^2 + (G(\eta)\zeta + \varepsilon(-\eta', 1) \cdot \nabla\Phi)^2}{2\langle \eta' \rangle^2} + g\eta - \alpha^2 \kappa(\eta),
\end{aligned}
$$

the map $F_2 \colon U^s_\theta \times \mathbb{R} \to H^s_{\text{even}}(\mathbb{R})$ by

$$
F_2(\eta, \zeta, c, \varepsilon) = c\eta + \zeta + \varepsilon\Phi,
$$

and finally the map $F_3 \colon U^s_\theta \times \mathbb{R} \to \mathbb{R}$ by

$$
F_3(\eta, \zeta, c, \varepsilon) = c - c_1\varepsilon + [H(\eta)\zeta]_y(0, -(1-\theta)h). \tag{1.23}
$$

In all of these definitions, we really mean the traces of Φ and its derivatives on the surface. The pointwise evaluation in the second term of Equation (1.23) is allowed because $H(\eta)\zeta$ is harmonic. It should be clear that all three maps F_1, F_2, F_3 are smooth.

We can now define $F \colon U^s_\theta \times \mathbb{R} \to Y^s$ by

$$
F := (F_1, F_2, F_3),
$$

and our task will then be to find solutions of the equation

$$
F(\eta, \zeta, c, \varepsilon) = 0. \tag{1.24}
$$

One may immediately note that $F(0,0,0,0) = 0$, so that the origin is a trivial solution. It will turn out that in a small neighborhood of the origin in $X^s \times \mathbb{R}$, there is a unique curve of nontrivial solutions parametrized by the vortex strength parameter ε.

4 Local bifurcation

We can now finally state and prove the following theorem, establishing the existence of small, localized, traveling wave solutions with a point vortex. For this, we will use an implicit function theorem argument on F. Note that while we do not apply Crandall–Rabinowitz theorem [12], the situation is very much in the spirit of that theorem. We bifurcate from the family of trivial waves described in the remark after Equation (1.22) by introducing the vorticity equation.

Theorem 1.12 (Traveling waves with a point vortex)**.** *Let $s > 3/2$ and $\theta \in (0,1)$. Then there exists an open interval $I \ni 0$ and a C^∞-curve*

$$\begin{array}{ccc} I & \to & (H^s_{even}(\mathbb{R}) \cap \Lambda_\theta) \times H^s_{even}(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \\ \varepsilon & \mapsto & (\eta(\varepsilon), \zeta(\varepsilon), c(\varepsilon), \varepsilon) \end{array}$$

of solutions to the Zakharov–Craig–Sulem formulation, Equation (1.24)*, for a point vortex of strength ε situated at $(0, -(1-\theta)h)$. The solutions fulfil*

$$\begin{aligned} \eta(\varepsilon) &= \eta_2 \varepsilon^2 + O(\varepsilon^4), \\ \zeta(\varepsilon) &= \zeta_3 \varepsilon^3 + O(\varepsilon^4), \\ c(\varepsilon) &= c_1 \varepsilon + c_3 \varepsilon^3 + O(\varepsilon^4), \end{aligned} \tag{1.25}$$

in their respective spaces as $\varepsilon \to 0$, where $\eta_2 \in H^s_{even}(\mathbb{R})$ is defined by

$$\eta_2 \coloneqq -(g - \alpha^2 \partial_x^2)^{-1} \chi, \quad \chi \coloneqq c_1 \Phi_y(\cdot, 0) + \frac{1}{2} \Phi_y(\cdot, 0)^2,$$

and where

$$\begin{aligned} \zeta_3 &\coloneqq -\eta_2 (c_1 + \Phi_y(\cdot, 0)), \\ c_3 &\coloneqq -[H(0)\zeta_3]_y (0, -(1-\theta)h), \end{aligned}$$

with c_1 as in Equation (1.22)*, Φ as in Proposition 1.5 and H as in Definition 1.2.*

Moreover, there is a neighborhood of the origin in $U^s_\theta \times \mathbb{R}$ such that this curve describes all solutions to $F(\eta, \zeta, c, \varepsilon) = 0$ in that neighborhood.

Proof. As remarked at the end of Section 3, the origin is a trivial solution. In order to apply the implicit function theorem, we require the first partial derivatives of F at this point. A direct calculation yields

$$D_X F(0,0,0,0) = \begin{bmatrix} g - \alpha^2 \partial_x^2 & 0 & 0 \\ 0 & I_{H^s_{\mathrm{even}}(\mathbb{R})} & 0 \\ 0 & [H(0)\cdot]_y (0, -(1-\theta)h) & 1 \end{bmatrix}, \tag{1.26}$$

where the subscript X denotes the partial derivative with respect to the variable (η, ζ, c) in X^s.

Now, every operator on the diagonal of $D_X F(0,0,0,0)$ is an isomorphism. Indeed, the operator

$$[g - \alpha^2 \partial_x^2] \colon H^s_{\mathrm{even}}(\mathbb{R}) \to H^{s-2}_{\mathrm{even}}(\mathbb{R})$$

corresponds to the Fourier multiplier $g + \alpha^2 \xi^2$. Since $g, \alpha^2 > 0$, this operator is invertible, with inverse corresponding to the multiplier $(g + \alpha^2 \xi^2)^{-1}$. The other two operators on the diagonal are identity operators, and therefore trivially invertible. Hence $D_X F(0,0,0,0) \in B(X^s, Y^s)$ is also an isomorphism.

Thus we can use the implicit function theorem to conclude that there is an open interval I containing zero, an open set $V \subseteq U^s_\theta$ containing $(0,0,0)$, and a map $f \in C^\infty(I, V)$ such that for $(\eta, \zeta, c, \varepsilon) \in V \times I$, we have

$$F(\eta, \zeta, c, \varepsilon) = 0 \iff (\eta, \zeta, c) = f(\varepsilon).$$

Furthermore, we obtain

$$Df(0) = -D_X F(0,0,c_1,0)^{-1} D_\varepsilon F(0,0,c_1,0) = \begin{bmatrix} 0 \\ 0 \\ c_1 \end{bmatrix},$$

which yields the first order terms in Equation (1.25). The higher-order terms can be obtained by inserting expansions for $\eta(\varepsilon), \zeta(\varepsilon)$ and $c(\varepsilon)$ into the equation $F(\eta(\varepsilon), \zeta(\varepsilon), c(\varepsilon), \epsilon) = 0$. This concludes the proof of the theorem. □

Remark 1.13. Because Theorem 1.12 holds for any $s > 3/2$, we can get arbitrarily high regularity on the solutions, by possibly making the interval I smaller. We have not been able to conclude that they are smooth, however, since the interval could possibly shrink to a point as $s \to \infty$.

Observe that, because c_1 changes sign at $\theta = 1/2$, the direction in which the waves obtained in Theorem 1.12 will travel (for small ε) depends on where the point vortex is in relation to the line $y = -h/2$. This does not come into play for waves on infinite depth. Note that if $\theta = 1 - 1/h$ (when $h > 1$) then

$$c_1 = -\frac{1}{4\pi} + O(1/h^2)$$

as $h \to \infty$, which is in agreement with what was found in [41] for a point vortex situated at $(0, -1)$ on infinite depth.

Since c_1 vanishes when $\theta = 1/2$, also the next term in the expansion for $c(\varepsilon)$ is of interest. We gave an expression for c_3 in Theorem 1.12, but have not determined its sign yet. We will treat the sign of c_3 after Theorem 1.17, which establishes some properties of the function η_2.

Written out, we have

$$\chi(x) = \frac{1}{8h^2} \frac{1 + \cos(\pi\theta)\cosh(\pi x/h)}{(\cosh(\pi x/h) + \cos(\pi\theta))^2}, \tag{1.27}$$

for the function χ defined in Theorem 1.12. We will have use for the fact that χ has an elementary antiderivative $\chi^\sharp$ and a double antiderivative $\chi^{\sharp\sharp}$ given by

$$\begin{aligned} \chi^\sharp(x) &= \frac{1}{8\pi h} \frac{\sinh(\pi x/h)}{\cosh(\pi x/h) + \cos(\pi\theta)}, \\ \chi^{\sharp\sharp}(x) &= \frac{1}{8\pi^2} \log(\cosh(\pi x/h) + \cos(\pi\theta)), \end{aligned} \tag{1.28}$$

respectively. While there in general seems to be no nice closed form of the leading order surface profile

$$\eta_2 = -(g - \alpha^2\partial_x^2)^{-1}\chi$$

obtained in Theorem 1.12, we can still give some of its properties. An immediate one is that η_2 is smooth. In Proposition 1.15 we give a series expansion for η_2 in powers of $e^{-\pi|x|/h}$. Furthermore, perhaps more surprisingly, we *can* find an explicit expression for η_2 in terms of elementary functions whenever

$$m := \frac{\sqrt{g}h}{\pi\alpha}. \tag{1.29}$$

is a natural number. If $m \in \mathbb{N}$, then $e^{\pm\sqrt{g}x/\alpha} = e^{\pm m\pi x/h}$ are integral powers of $e^{\pm\pi x/h}$, which would appear on the right side of Equation (1.27) if we had written out $\cosh(\pi x/h)$ and $\sinh(\pi x/h)$. Since $x \mapsto e^{\pm\sqrt{g}x/\alpha}$ spans the kernel of $g - \alpha^2\partial_x^2$, this explains integral values of m being special.

Before we state Proposition 1.15 and Theorem 1.17, we need a lemma to simplify some expressions.

Lemma 1.14. *For $m \in (0,\infty) \setminus \mathbb{N}$ and $\theta \in (0,1)$, we have*

$$\frac{1}{m} + 2m\sum_{k=1}^{\infty}(-1)^k \frac{\cos(k\pi\theta)}{m^2 - k^2} = \pi\frac{\cos(m\pi\theta)}{\sin(m\pi)}, \tag{1.30}$$

which is equal to

$$\int_0^\infty y^{m-1} \frac{\cos(\pi\theta)y + 1}{y^2 + 2\cos(\pi\theta)y + 1}\, dy \tag{1.31}$$

whenever $m \in (0,1)$. Furthermore, for $m \in \mathbb{N}$,

$$\frac{1}{m} + 2m \sum_{\substack{k=1 \\ k \neq m}}^{\infty} (-1)^k \frac{\cos(k\pi\theta)}{m^2 - k^2} = -(-1)^m \left(\frac{\cos(m\pi\theta)}{2m} + \pi\theta \sin(m\pi\theta) \right). \tag{1.32}$$

Proof. Both sides of Equation (1.30) define meromorphic functions in m on $\mathbb{C}$ with simple poles in the points $\mathbb{Z} \times \{0\}$. Moreover, they are both equal to the integral in Equation (1.31) when $m \in (0,1)$, which can be seen by calculating the integral with both the residue theorem (around a keyhole contour) and a Laurent series expansion of the integrand. Since the interval consists of non-isolated points, we have equality on all of $\mathbb{C}$. Finally, Equation (1.32) follows from Equation (1.30) by taking limits. □

Proposition 1.15 (Expansion for η_2)**.** *If the number m in Equation* (1.29) *satisfies $m \in (0,\infty) \setminus \mathbb{N}$, then the leading order term of the surface profile from Theorem 1.12 is given by*

$$\eta_2(x) = \frac{1}{8\pi^2\alpha^2} \Bigg[\log(1 + 2\cos(\pi\theta) e^{-\pi|x|/h} + e^{-2\pi|x|/h}) - \pi \frac{\cos(m\pi\theta)}{\sin(m\pi)} e^{-\sqrt{g}|x|/\alpha} + 2m^2 \sum_{k=1}^{\infty} (-1)^k \frac{\cos(k\pi\theta)}{k(m^2-k^2)} e^{-k\pi|x|/h} \Bigg],$$

while if $m \in \mathbb{N}$, then

$$\begin{aligned} \eta_2(x) = \frac{1}{8\pi^2\alpha^2} \Bigg[&\log(1 + 2\cos(\pi\theta) e^{-\pi|x|/h} + e^{-2\pi|x|/h}) \\ &+ (-1)^m \left(\frac{3\cos(m\pi\theta)}{2m} + \pi\theta \sin(m\pi\theta) + \cos(m\pi\theta) \frac{\pi|x|}{h} \right) e^{-\sqrt{g}|x|/\alpha} \\ &+ 2m^2 \sum_{\substack{k=1 \\ k\neq m}}^{\infty} (-1)^k \frac{\cos(k\pi\theta)}{k(m^2-k^2)} e^{-k\pi|x|/h} \Bigg]. \end{aligned}$$

These series converge uniformly, and, excluding the origin, so do the series for the first derivative. Moreover, when $m \in \mathbb{N}$, the function η_2 is given explicitly in terms of elementary functions by

$$\eta_2(x) = \frac{1}{8\pi^2\alpha^2} \Bigg[\frac{1}{m} + 2 \sum_{k=1}^{m-1} (-1)^{m-k} \frac{\cos((m-k)\pi\theta)}{k} \cosh((m-k)\pi x/h) + r(e^{\pi x/h}) + r(e^{-\pi x/h}) \Bigg],$$

where $r\colon (0,\infty) \to \mathbb{R}$ *is defined by*

$$\begin{aligned} r(x) &:= \frac{1}{2}(-1)^m \cos(m\pi\theta)x^{-m}\log(1+2\cos(\pi\theta)x+x^2) \\ &\quad + (-1)^m \sin(m\pi\theta)x^{-m}(\arctan(\cot(\pi\theta)+\csc(\pi\theta)x) - \pi(1/2-\theta)). \end{aligned}$$

Proof. It follows from

$$\mathscr{F}(e^{-a|\cdot|})(\xi) = \sqrt{\frac{2}{\pi}}\frac{a}{a^2+\xi^2}, \quad a>0,$$

and the definition of η_2, that we may write η_2 as the convolution

$$\begin{aligned} \eta_2(x) &= -\frac{1}{2\alpha\sqrt{g}}(e^{-\sqrt{g}|\cdot|/\alpha} * \chi)(x) \\ &= -\frac{1}{2\alpha\sqrt{g}}\left(J(x,\chi)+J(-x,\chi)\right), \end{aligned} \tag{1.33}$$

where

$$J(x,\chi) := e^{-\sqrt{g}x/\alpha}\int_{-\infty}^{x} e^{\sqrt{g}y/\alpha}\chi(y)\,dy.$$

Equivalently

$$\eta_2(x) = \frac{1}{2\alpha^2}\Big(J(x,\chi^\sharp)+J(-x,\chi^\sharp)\Big) \tag{1.34}$$

$$= \frac{1}{\alpha^2}\chi^{\sharp\sharp}(x) - \frac{\sqrt{g}}{2\alpha^3}\Big(J(x,\chi^{\sharp\sharp})+J(-x,\chi^{\sharp\sharp})\Big) \tag{1.35}$$

through integration by parts, where $\chi^\sharp$ and $\chi^{\sharp\sharp}$ are the antiderivatives defined in Equation (1.28).

We first use Equation (1.34) to obtain an explicit expression for η_2 when $m\in\mathbb{N}$. By using the substitution $x\mapsto e^{\pi x/h}$, we find that

$$J(x,\chi^\sharp) = \frac{1}{8\pi^2}f_1(e^{\pi x/h}), \quad f_1(x) := x^{-m}\int_0^x z^{m-1}\frac{z^2-1}{z^2+2\cos(\pi\theta)z+1}\,dz.$$

The fraction in the integrand in the definition of f_1 has partial fraction decomposition

$$\frac{z^2-1}{z^2+2\cos(\pi\theta)z+1} = 1 - \frac{e^{i\pi\theta}}{z+e^{i\pi\theta}} - \frac{e^{-i\pi\theta}}{z+e^{-i\pi\theta}},$$

and since

$$z^{m-1}\frac{a}{z+a} = -\frac{(-a)^m}{z+a} - \sum_{k=0}^{m-2}(-a)^{m-k-1}z^k, \quad a\in\mathbb{C}, z\neq -a,$$

this means that

$$f_1(x) = \frac{1}{m} + (-1)^m x^{-m}\Big[e^{im\pi\theta}\log\Big(x+e^{i\pi\theta}\Big) + e^{-im\pi\theta}\log\Big(x+e^{-i\pi\theta}\Big) \\ + 2\pi\theta\sin(m\pi\theta)\Big] + 2\sum_{k=1}^{m-1}\frac{(-1)^k\cos(k\pi\theta)}{m-k}x^{-k},$$

where $\log(\cdot)$ denotes the principal branch of the logarithm. The result now follows by using the identity

$$\log(x+e^{i\pi\theta}) = \frac{1}{2}\log(1+2\cos(\pi\theta)x+x^2) \\ - i(\arctan(\cot(\pi\theta)+\csc(\pi\theta)x) - \pi/2),$$

valid for all $x \in \mathbb{R}$.

For the series representation of η_2, we use Equation (1.35), because this leads to a series that converges more rapidly. We will assume that $m \in (0,\infty) \setminus \mathbb{N}$; the case for $m \in \mathbb{N}$ is similar, except that one needs to use Equation (1.32) instead of Equation (1.30). We use the same substitution as before to arrive at

$$J(x,\chi^{\sharp\sharp}) = \frac{\alpha}{8\pi^2\sqrt{g}}f_2(e^{\pi x/h}),$$

where $f_2\colon (0,\infty) \to \mathbb{R}$ is defined by

$$f_2(x) \coloneqq m x^{-m}\int_0^x z^{m-1}\log((z^{-1}+z)/2+\cos(\pi\theta))\,dz.$$

One may check that one has

$$\log((z^{-1}+z)/2+\cos(\pi\theta)) = -\log(2) - \log(z) - 2\sum_{k=1}^{\infty}(-1)^k\frac{\cos(k\pi\theta)}{k}z^k$$

for $z \in (0,1)$ and

$$\log((z^{-1}+z)/2+\cos(\pi\theta)) = -\log(2) + \log(z) - 2\sum_{k=1}^{\infty}(-1)^k\frac{\cos(k\pi\theta)}{k}z^{-k}$$

for $z \in (1,\infty)$.

It then follows by termwise integration that

$$f_2(x) = \frac{1}{m} - \log(2) - \log(x) - 2m\sum_{k=1}^{\infty}(-1)^k\frac{\cos(k\pi\theta)}{k(m+k)}x^k$$

on $(0, 1]$ (the endpoint is Abel's theorem [34, Theorem 17.14]), and that

$$
\begin{aligned}
f_2(x) &= f_2(1)x^{-m} + mx^{-m} \int_1^x z^{m-1} \log((z + z^{-1})/2 + \cos(\pi\theta))\, dz \\
&= -\frac{1}{m} - \log(2) + \log(x) - 2m \sum_{k=1}^{\infty} (-1)^k \frac{\cos(k\pi\theta)}{k(m-k)} x^{-k} \\
&\qquad - \left(\frac{2}{m} + 4m \sum_{k=1}^{\infty} (-1)^k \frac{\cos(k\pi\theta)}{m^2 - k^2} \right) x^{-m}.
\end{aligned}
$$

for $x \in [1, \infty)$. Employing Equation (1.35), we find that η_2 is given by

$$
\begin{aligned}
\eta_2(x) = \frac{1}{8\pi^2\alpha^2} \Bigg[&\log(\cosh(\pi x/h) + \cos(\pi\theta)) - \pi|x|/h + \log(2) \\
&- \left(\frac{1}{m} + 2m \sum_{k=1}^{\infty} (-1)^k \frac{\cos(k\pi\theta)}{m^2 - k^2} \right) e^{-\sqrt{g}|x|/\alpha} \\
&+ 2m^2 \sum_{k=1}^{\infty} (-1)^k \frac{\cos(k\pi\theta)}{k(m^2 - k^2)} e^{-k\pi|x|/h} \Bigg]
\end{aligned}
$$

for all $x \in \mathbb{R}$, by using that η_2 is even and observing that for $x \geq 0$ we have $e^{\pi x/h} \in [1, \infty)$ and $e^{-\pi x/h} \in (0, 1]$. If we now apply Equation (1.30) from Lemma 1.14 in order to get a closed-form expression for the coefficient in front of $e^{-\sqrt{g}|x|/\alpha}$, we arrive at the desired expansion. □

Remark 1.16. The only obstacle to convergence of the series given in Proposition 1.15 is the origin; thanks to the exponential factor $e^{-k\pi|x|/h}$, the convergence is rapid away from the origin. It should also be noted that, while Equation (1.27) seems to suggest that η_2 should be expandable in a series of powers of $\operatorname{sech}(\pi x/h)$ by equating coefficients in the differential equation defining it, this seems to lead to a series that does not converge. We have kept the series expansion for η_2 also when $m \in \mathbb{N}$, because the expression in terms of elementary functions is unwieldy, and prone to numerical errors even for small values of m.

The expressions found in Proposition 1.15 have well defined pointwise limits as $\theta \uparrow 1$ (for $x \neq 0$) and $\theta \downarrow 0$. In particular, when $m = 1$ these are given by

$$
\begin{aligned}
\lim_{\theta \downarrow 0} \eta_2(x) &= \frac{1}{8\pi^2\alpha^2} [1 - e^{\pi x/h} \log(1 + e^{-\pi x/h}) - e^{-\pi x/h} \log(1 + e^{\pi x/h})] \\
\lim_{\theta \uparrow 1} \eta_2(x) &= \frac{1}{8\pi^2\alpha^2} [1 + e^{\pi x/h} \log|1 - e^{-\pi x/h}| + e^{-\pi x/h} \log|1 - e^{\pi x/h}|],
\end{aligned}
$$

which can can be seen as graphs drawn with thicker lines in Figure 1.1, together with η_2 for various values of the parameter θ.

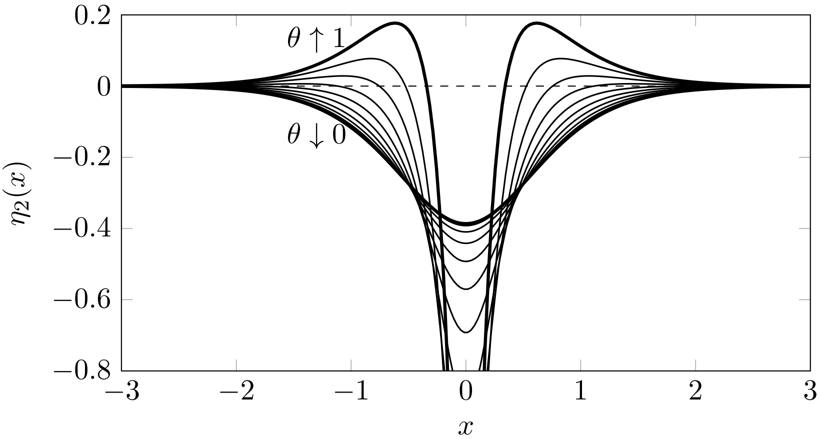


Figure 1.1: The leading order term η_2 in $\eta(\varepsilon)$, with $h = 1, \alpha^2 = 1/(8\pi^2), m = 1$. The values of θ shown are $\theta = 0.1, 0.2, \ldots, 0.9$, together with the thicker lower and upper limits $\theta \downarrow 0$ and $\theta \uparrow 1$.

We see from Figure 1.1 that one gets a depression at the origin, which becomes more pronounced the closer the point vortex is situated to the surface. The profile when the point vortex is close to the surface is very similar to the profile for the infinite depth case, found in [41]. However, a feature which is not seen on infinite depth is that there is a significant difference between the case $\theta \leq 1/2$ and the case $\theta > 1/2$ (in addition to the changing sign of c_1). For $\theta \leq 1/2$ there is a single trough at the origin, and η_2 is everywhere strictly negative. When $\theta > 1/2$ one in addition gets crests on either side of the origin. As we can see from Figure 1.1, the positions of these crests depend on the position of the point vortex.

Some of what we have just discussed is not limited to the specific choice of constants that are used in Figure 1.1, and for which Proposition 1.15 yields an explicit expression for η_2. We will see that $m = 1$ plays a special role in the asymptotic behavior of η_2, however. More precisely, we have the following theorem:

Theorem 1.17 (Properties of η_2)**.** *The leading order surface term η_2 always satisfies $\eta_2(0) < 0$ and $\eta_2''(0) > 0$, meaning that the origin is a depression. When $\theta \leq 1/2$, the function η_2 is everywhere negative, and strictly increasing* *on* $[0, \infty)$. *For* $\theta > 1/2$, *we have two cases, depending on the number* m *defined in Equation* (1.29)*:*

(i) If $m > 1/(2\theta)$, *then* $\eta_2(x)$ *is positive for sufficiently large* $|x|$. *In particular,* η_2 *has crests on either side of the origin.*

(ii) If $m \leq 1/(2\theta)$, *then* $\eta_2(x)$ *is negative for sufficiently large* $|x|$.

Furthermore, η_2 *has the following asymptotic properties for any* $\theta \in (0, 1)$*:*

(i) For $m > 1$

$$\lim_{x\to\infty} \eta_2(x) e^{\pi x/h} = -\frac{2}{m^2-1} \frac{\cos(\pi\theta)}{8\pi^2\alpha^2}. \tag{1.36}$$

(ii) If $m = 1$, *then*

$$\lim_{x\to\infty} \eta_2(x) \frac{e^{\pi x/h}}{\pi x/h} = -\frac{\cos(\pi\theta)}{8\pi^2\alpha^2}.$$

(iii) For $m < 1$

$$\lim_{x\to\infty} \eta_2(x) e^{\sqrt{g}x/\alpha} = -\frac{\pi}{\sin(m\pi)} \frac{\cos(m\pi\theta)}{8\pi^2\alpha^2}. \tag{1.37}$$

Proof. We first prove that $\eta_2(0) < 0$ and $\eta_2''(0) > 0$, which holds for all values of m and θ. By inserting $x = 0$ in Equation (1.33), and using the evenness of χ, we find

$$\begin{aligned}
\eta_2(0) &= -\frac{1}{\alpha\sqrt{g}} \int_0^\infty e^{-\sqrt{g}y/\alpha} \chi(y)\, dy \\
&= -\frac{1}{\alpha^2} \int_0^\infty \underbrace{e^{-\sqrt{g}y/\alpha} \chi^\sharp(y)}_{>0 \text{ on } (0,\infty)}\, dy < 0,
\end{aligned}$$

where the second equality follows from integration by parts, and the function $\chi^\sharp$ was defined in Equation (1.28). Since $\eta_2 = -(g - \alpha^2\partial_x^2)^{-1}\chi$, we also have

$$\begin{aligned}
\eta_2''(0) &= \frac{1}{\alpha^2}(g\eta_2(0) + \chi(0)) \\
&= \frac{\sqrt{g}}{2\alpha^3} \int_{-\infty}^\infty e^{-\sqrt{g}|y|/\alpha} (\chi(0) - \chi(y))\, dy \\
&> 0,
\end{aligned}$$

as χ achieves a global maximum at the origin.

Suppose now that $\theta \leq 1/2$. Like in Proposition 1.15, we use the fact that η_2 may be written as the convolution

$$\eta_2 = -\frac{1}{2\alpha\sqrt{g}}(e^{-\sqrt{g}|\cdot|/\alpha} * \chi), \tag{1.38}$$

which shows that η_2 is strictly negative, since χ is strictly positive when $\theta \leq 1/2$. Moreover, some manipulations of the above formula shows that we may write the derivative of η_2 as

$$\begin{aligned}\eta_2'(x) = -\frac{1}{\alpha\sqrt{g}}\Bigg[\sinh\left(\frac{\sqrt{g}}{\alpha}x\right)\int_x^\infty e^{-\sqrt{g}y/\alpha}\chi'(y)\,dy \\ + e^{-\sqrt{g}x/\alpha}\int_0^x \sinh\left(\frac{\sqrt{g}}{\alpha}y\right)\chi'(y)\,dy\Bigg],\end{aligned}$$

where we have used the fact that χ is even. One may check that χ' is strictly negative for $x > 0$ when $\theta \leq 1/2$. This shows that η_2' is strictly positive for $x > 0$, and so η_2 is strictly increasing on $[0, \infty)$ by the mean value theorem.

Before we consider the case $\theta > 1/2$, we prove the asymptotic properties for η_2 listed in Equations (1.36) to (1.37). These follow by multiplying each side in Equation (1.33) with the appropriate factor and taking limits. For instance, suppose that $m > 1$, meaning that $\sqrt{g}/\alpha > \pi/h$. For the integral in

$$e^{\pi x/h}\left(e^{-\sqrt{g}x/\alpha}\int_{-\infty}^x e^{\sqrt{g}y/\alpha}\chi(y)\,dy\right) = \frac{\int_{-\infty}^x e^{\sqrt{g}y/\alpha}\chi(y)\,dy}{e^{(\sqrt{g}/\alpha-\pi/h)x}}$$

there are two possibilities: If $\theta = 1/2$, then it is possible that the integrand is integrable on the entire real line, meaning that the limit as $x \to \infty$ is zero; otherwise, the integral tends to $\pm\infty$, and so

$$\begin{aligned}\lim_{x\to\infty}\frac{\int_{-\infty}^x e^{\sqrt{g}y/\alpha}\chi(y)\,dy}{e^{(\sqrt{g}/\alpha-\pi/h)x}} &= \lim_{x\to\infty}\frac{e^{\sqrt{g}x/\alpha}\chi(x)}{(\sqrt{g}/\alpha-\pi/h)e^{(\sqrt{g}/\alpha-\pi/h)x}} \\ &= \frac{1}{\sqrt{g}/\alpha-\pi/h}\frac{\cos(\pi\theta)}{4h^2}\end{aligned}$$

by L'Hôpital's rule. The other limits can be treated in a similar way, with one exception:

The procedure will show that when $m < 1$, we have

$$
\begin{aligned}
\lim_{x\to\infty} \eta_2(x) e^{\sqrt{g}x/\alpha} &= -\frac{1}{2\alpha\sqrt{g}} \int_{-\infty}^{\infty} e^{\sqrt{g}y/\alpha} \chi(y)\, dy \\
&= -\frac{1}{8\pi^2\alpha^2 m} \int_0^{\infty} y^m \frac{\cos(\pi\theta)y^2 + 2y + \cos(\pi\theta)}{(y^2 + 2\cos(\pi\theta)y + 1)^2}\, dy \\
&= -\frac{1}{8\pi^2\alpha^2} \int_0^{\infty} y^{m-1} \frac{\cos(\pi\theta)y + 1}{y^2 + 2\cos(\pi\theta)y + 1}\, dy
\end{aligned}
$$

where the second and third equality follows from the substitution $y \mapsto e^{\pi y/h}$ and an integration by parts, respectively. The result now follows since the integral on the final line is equal to the right-hand side of Equation (1.30) by Lemma 1.14.

Finally, we consider the case of $\theta > 1/2$, which is harder to describe completely, as the integrand in the convolution in Equation (1.38) changes sign. Observe that the claims on the sign of $\eta_2(x)$ for sufficiently large x follows for $m \neq 1/(2\theta)$ from the limits in Equations (1.36) to (1.37). An additional argument is needed for the edge case $m = 1/(2\theta)$, because the limit in Equation (1.37) vanishes. It turns out that Equation (1.36) also holds in the special case $m = 1/(2\theta)$, which can be shown with the same method we used to show the other limits. Hence η_2 is negative for sufficiently large x when $m = 1/(2\theta)$, which exhausts the values of m. $\square$

Remark 1.18. It is likely that η_2 has similar properties to those for the case $\theta \leq 1/2$ when $\theta > 1/2$ and $m \leq 1/(2\theta)$, but we have not been able to prove this.

We are now in a position where we can give the sign of c_3 in the expansion in Theorem 1.12 for $\theta \leq 1/2$.

Proposition 1.19 (Sign of c_3)**.** *The constant c_3 in Equation* (1.25) *is negative when $\theta \leq 1/2$. In particular, if $\theta = 1/2$ and ε is sufficiently small, the waves obtained in Theorem 1.12 are left-moving when $\varepsilon > 0$ and right-moving when $\varepsilon < 0$.*

Proof. Recall the definition of ζ_3 in Equation (1.25). From Theorem 1.17 we know that η_2 is negative, and strictly increasing on $[0, \infty)$. Furthermore, the factor $c_1 + \Phi_y(\cdot, 0)$ is positive and strictly decreasing on the same interval. It follows that also ζ_3 is positive and strictly decreasing on $[0, \infty)$.

The harmonic function $H(0)\zeta_3$ on $\mathbb{R} \times (-h, 0)$ assumes the value 0 at the bottom of the domain and $\zeta_3 > 0$ at the top of the domain. By the maximum principle, it is positive on the entire domain. Thus we may use the Hopf boundary point lemma (see [23, Lemma 3.4]) in order to conclude that $[H(0)\zeta_3]_y(0,-h) > 0$. The result will therefore follow if we can show that $[H(0)\zeta_3]_y$ is increasing along the y-axis. We will do this by looking at $[H(0)\zeta_3]_x$ on $(0,\infty) \times (-h,0)$. Because of its values on the boundary, it is negative in the interior. Another application of the Hopf boundary point lemma implies that $[H(0)\zeta_3]_{xx}$ is negative on the y-axis (except at the point $(0,-h)$, where it vanishes). Since $[H(0)\zeta_3]_{yy} = -[H(0)\zeta_3]_{xx}$ by the harmonicity of $H(0)\zeta_3$, this concludes the proof. □

We finish our exposition on a single point vortex with a short discussion on the streamlines of waves obtained in Theorem 1.12. Observe that if $(x(t), y(t))$ denotes the position of a fluid particle at time t, then

$$(\dot{x}(t), \dot{y}(t)) = w(x(t), y(t), t), \tag{1.39}$$

before the new variables in Section 2. After introducing the steady variables, Equation (1.39) becomes

$$(\dot{x}(t), \dot{y}(t)) = w(x(t), y(t)) - (c, 0), \tag{1.40}$$

meaning that if we only keep the first order terms for w and c from Theorem 1.12, we obtain (keeping the same notation for the paths)

$$(\dot{x}(t), \dot{y}(t)) = \varepsilon \nabla^{\perp} (\Phi + c_1 y) (x(t), y(t)). \tag{1.41}$$

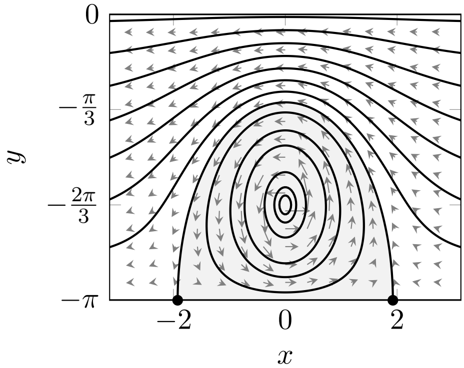
We have used this to obtain Figure 1.2, which shows streamlines in the steady frame moving with the wave. The portraits corresponding to θ and $1-\theta$ can be obtained from each other by a 180° rotation. When $\theta = 1/2$, all the streamlines are closed (not shown), so we will focus on the case $\theta \neq 1/2$. The lines $y = -h$ and $y = 0$ are nullclines for the system in Equation (1.41), and the points (x, y) with

$$\begin{aligned} x &= \pm h/\pi \operatorname{arcosh}(|2\sin(\pi\theta)\tan(\pi\theta) + \cos(\pi\theta)|), \\ y &= \begin{cases} -h & \theta < 1/2 \\ 0 & \theta > 1/2 \end{cases} \end{aligned} \tag{1.42}$$

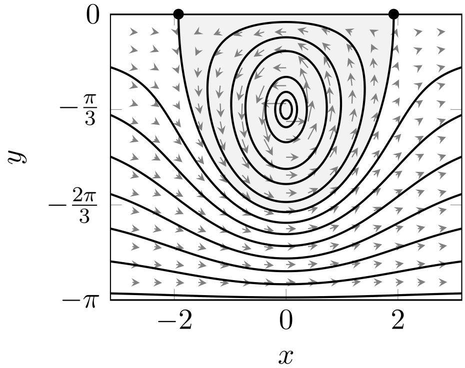
are equilibrium points, corresponding to stagnation points. One may check that

$$h/\pi \operatorname{arcosh}(2\sin(\pi\theta)\tan(\pi\theta) + \cos(\pi\theta)) = \sqrt{3}h\theta + O(\theta^5)$$

as $\theta \downarrow 0$, meaning that the distance between the equilibria is very close to linear in θ for small θ (a corresponding statement holds for $1-\theta$ small).



(a) $\theta = 1/3$



(b) $\theta = 2/3$

Figure 1.2: Streamlines in the frame of reference traveling with the wave, for $h = \pi$ and $\varepsilon > 0$. The wave corresponding to $\theta = 1/3$ propagates to the right, while the wave corresponding to $\theta = 2/3$ propagates to the left. The arrows illustrating the vector field on the right hand side of Equation (1.41) have been scaled here for visibility, and only their direction is quantitatively accurate.

They go off to infinity as $\theta \to 1/2$ from either side. The heteroclinic orbit (which can be expressed explicitly in terms of arcosh) connecting the two equilibrium points described in Equation (1.42) encloses a critical layer containing closed streamlines. Outside this region the particles always move in the same direction with respect to the steady frame. This direction is either to the left or right depending on the sign of $\cot(\pi\theta)$ and ε.

We also mention that on infinite depth, the streamlines always look like those in Figure 1.2b. If the point vortex is situated at $(0, -d)$, the equilibrium points at the surface will be at $(\pm\sqrt{3}d, 0)$, and the points on the heteroclinic orbit between these satisfy

$$x^2 + (y + d)^2 = 2dy(1 + \coth(y/(2d))),$$

which is close to half an ellipse centered at $(0, -d)$ with semiaxes $\sqrt{3}d$ and $\sim 2.0873d$. The equilibrium points in Equation (1.42) converge to those on infinite depth as $h \to \infty$ if d is held fixed.

Because only the first order terms in ε have been kept in Equation (1.41), we do not make any claim about the accuracy of the phase portraits in Figure 1.2 for the full system in Equation (1.40). That would require further and more thorough analysis, in particular for the case $\theta = 1/2$. Still, the phase portraits can give some indication as to how these waves look beneath the surface. One feature will remain the same for Equation (1.40): Because of the singularity of Φ at $(0, -(1-\theta)h)$, the streamlines will always remain closed sufficiently close to the point vortex.

5 Several point vortices

We aim to extend the existence result for traveling waves with a single point vortex in Theorem 1.12 to a finite number of point vortices on the y-axis. As opposed to the single vortex case, where we could choose θ freely, there will be limitations on the positions that the point vortices can occupy. We will return to this. Suppose that

$$1 > \theta_1 > \theta_2 > \cdots > \theta_n > 0,$$

and that we wish to establish the existence of a traveling wave with point vortex at the points

$$(0, -(1-\theta_1)h), \ldots, (0, -(1-\theta_n)h),$$

the situation being otherwise similar to that of a single point vortex. The admissible surface profiles are those in Λ_{θ_1}, as the uppermost point vortex is the most restrictive.

For $\eta \in \Lambda_{\theta_1}$ and $\gamma = (\gamma^1, \ldots, \gamma^n) \in \mathbb{R}^n$ we may define

$$\Phi^\gamma := \sum_{j=1}^{n} \gamma^j \Phi^j, \tag{1.43}$$

where

$$\Phi^j(x,y) := \frac{1}{4\pi} \log\left(\frac{\cosh(\pi x/h) + \cos(\pi(y/h - \theta_j))}{\cosh(\pi x/h) + \cos(\pi(y/h + \theta_j))}\right), \quad j = 1, \ldots, n,$$

in $\Omega(\eta)$. We will seek solutions of the form

$$w = \nabla^\perp[H(\eta)\zeta + \Phi^\gamma],$$

cf. Equation (1.21) for a single point vortex.

The main difference from the single point vortex case is of course the vorticity equation, Equation (1.9), which needs to be imposed for each of the point vortices. For the ith point vortex, the vorticity equation reduces to

$$\begin{aligned}(c,0) &= \nabla^\perp[H(\eta)\zeta](0, -(1-\theta_i)h) \\ &\quad + \frac{1}{4h}\left(\gamma^i \cot(\pi\theta_i) + \sum_{\substack{j=1 \\ j\neq i}}^{n} \gamma^j \left[\cot\left(\pi\frac{\theta_i+\theta_j}{2}\right) - \cot\left(\pi\frac{\theta_i-\theta_j}{2}\right)\right], 0\right),\end{aligned}$$

which, if we assume that η and ζ are even (see the discussion before Equation (1.22)), can be written more succinctly as

$$c\mathbf{1} = -([H(\eta)\zeta]_y(0, -(1-\theta_i)h))_{i=1}^n + \Theta\gamma. \tag{1.44}$$

Here, we have defined $\mathbf{1} \coloneqq (1, \ldots, 1) \in \mathbb{R}^n$ and the matrix $\Theta \in \mathbb{R}^{n\times n}$ by

$$\Theta_{i,j} = \begin{cases} \dfrac{1}{4h}\cot(\pi\theta_i) & i = j \\ \dfrac{1}{4h}\left(\cot\left(\pi\dfrac{\theta_i+\theta_j}{2}\right) - \cot\left(\pi\dfrac{\theta_i-\theta_j}{2}\right)\right) & i \neq j \end{cases} \tag{1.45}$$

for $1 \le i, j \le n$.

As opposed to for one vortex, it is now more natural to use the wave velocity c as the bifurcation parameter. We will therefore write ε instead of c in order to have notation that is more consistent with the one vortex case. The idea is to use the vortex strengths γ in order to balance Equation (1.44), which is possible when Θ is invertible. It should be emphasized that this is almost always the case (Theorem 1.25), but that there always are configurations of n point vortices that yield singular Θ (Proposition 1.26). We have already seen such a configuration, albeit a trivial one: For the case $n = 1$ one has $\Theta = 0$ when $\theta = 1/2$.

We make the necessary redefinitions

$$\begin{aligned} X^s &\coloneqq H^s_{\text{even}}(\mathbb{R}) \times H^s_{\text{even}}(\mathbb{R}) \times \mathbb{R}^n, \\ Y^s &\coloneqq H^{s-2}_{\text{even}}(\mathbb{R}) \times H^s_{\text{even}}(\mathbb{R}) \times \mathbb{R}^n, \\ U^s_{\theta_1} &\coloneqq \{(\eta, \zeta, \gamma) \in X^s : \eta \in \Lambda_{\theta_1}\}, \end{aligned}$$

and proceed to define, for $s > 3/2$, the map $F_1\colon U^s_{\theta_1} \times \mathbb{R} \to H^{s-2}_{\text{even}}(\mathbb{R})$ by

$$\begin{aligned} F_1(\eta, \zeta, \gamma, \varepsilon) &= \varepsilon\left[\frac{\eta'\zeta' + G(\eta)\zeta}{\langle\eta'\rangle^2} + \Phi^\gamma_y\right] \\ &\quad + \frac{(\zeta' + (1, \eta')\cdot\nabla\Phi^\gamma)^2 + (G(\eta)\zeta + (-\eta', 1)\cdot\nabla\Phi^\gamma)^2}{2\langle\eta'\rangle^2} + g\eta - \alpha^2\kappa(\eta), \end{aligned}$$

the map $F_2\colon U^s_{\theta_1} \times \mathbb{R} \to H^s_{\text{even}}(\mathbb{R})$ by

$$F_2(\eta, \zeta, \gamma, \varepsilon) \coloneqq \varepsilon\eta + \zeta + \Phi^\gamma,$$

and finally the map $F_3\colon U^s_{\theta_1} \times \mathbb{R} \to \mathbb{R}^n$ by

$$F_3(\eta, \zeta, \gamma, \varepsilon) \coloneqq \Theta\gamma - \varepsilon\mathbf{1} - ([H(\eta)\zeta]_y(0, -(1-\theta_i)h))_{i=1}^n.$$

In all of these definitions, the function Φ^γ and its derivatives are evaluated at $(x, \eta(x))$, which is suppressed for readability.

We now define $F \coloneqq (F_1, F_2, F_3) \colon U^s_\theta \to Y^s$, and seek solutions of the equation

$$F(\eta, \zeta, \gamma, \varepsilon) = 0, \tag{1.46}$$

which has the origin as a trivial solution. We are led to the following analog of Theorem 1.12 for several point vortices, establishing the existence of a family of small, localized solutions, assuming that Θ is nonsingular. The resulting waves have one critical layer for each point vortex, assuming that no component of γ vanishes.

Theorem 1.20 (Traveling waves with several point vortices)**.** *Let $s > 3/2$, and let $1 > \theta_1 > \theta_2 > \cdots > \theta_n > 0$. Suppose that the matrix Θ defined in Equation* (1.45) *is invertible. Then there exists an open interval $I \ni 0$ and a C^∞-curve*

$$\begin{array}{ccc} I & \to & (H^s_{even}(\mathbb{R}) \cap \Lambda_{\theta_1}) \times H^s_{even}(\mathbb{R}) \times \mathbb{R}^n \times \mathbb{R} \\ \varepsilon & \mapsto & (\eta(\varepsilon), \zeta(\varepsilon), \gamma(\varepsilon), \varepsilon) \end{array}$$

of solutions with velocity $c = \varepsilon$ to the Zakharov–Craig–Sulem formulation, Equation (1.46)*, for point vortices of strengths $\gamma^1(\varepsilon), \ldots, \gamma^n(\varepsilon)$ situated at*

$$(0, -(1-\theta_1)h), \ldots, (0, -(1-\theta_n)h).$$

The solutions fulfil

$$\begin{aligned} \eta(\varepsilon) &= \eta_2 \varepsilon^2 + O(\varepsilon^4), \\ \zeta(\varepsilon) &= \zeta_3 \varepsilon^3 + O(\varepsilon^4), \\ \gamma(\varepsilon) &= \gamma_1 \varepsilon + \gamma_3 \varepsilon^3 + O(\varepsilon^4), \end{aligned} \tag{1.47}$$

in their respective spaces as $\varepsilon \to 0$, where $\gamma_1 \coloneqq \Theta^{-1}\mathbf{1}$, the function $\eta_2 \in H^s_{even}(\mathbb{R})$ is defined by

$$\eta_2 \coloneqq -(g - \alpha^2 \partial_x^2)^{-1} \chi, \quad \chi \coloneqq \Phi^{\gamma_1}_y(\cdot, 0) + \frac{1}{2} \Phi^{\gamma_1}_y(\cdot, 0)^2,$$

and where

$$\begin{aligned} \zeta_3 &= -\eta_2 (1 + \Phi^{\gamma_1}_y(\cdot, 0)), \\ \gamma_3 &= \Theta^{-1}([H(0)\zeta_3]_y(0, -(1-\theta_i)h))_{i=1}^n, \end{aligned}$$

with Φ^{γ_1} as in Equation (1.43) *and H as in Definition 1.2.*

Moreover, there is a neighborhood of the origin in $U^s_\theta \times \mathbb{R}$ such that this curve describes all solutions to $F(\eta, \zeta, \gamma, \varepsilon) = 0$ in that neighborhood.

Proof. As for a single point vortex, we wish to apply the implicit function theorem at the origin. We find the derivative

$$D_X F(0,0,0,0) = \begin{bmatrix} g - \alpha^2 \partial_x^2 & 0 & 0 \\ 0 & I_{H^s_{\text{even}}(\mathbb{R})} & 0 \\ 0 & -([H(0)\cdot]_y(0, -(1-\theta_i)h))_{i=1}^n & \Theta \end{bmatrix},$$

where $([H(0)\cdot]_y(0,-(1-\theta_i)h))_{i=1}^n$ means the operator $H^s_{\text{even}}(\mathbb{R}) \to \mathbb{R}^n$ defined by

$$\zeta \mapsto ([H(0)\zeta]_y(0, -(1-\theta_i)h))_{i=1}^n.$$

Recalling that $g - \alpha^2 \partial_x^2$ and Θ are invertible by the discussion after Equation (1.26) and by assumption, respectively, $D_X F(0,0,0,0)$ is an isomorphism.

Hence we can use the implicit function theorem to deduce the existence of an open interval I around zero, an open set $V \subseteq U^s_{\theta_1}$ containing the origin, and a map $f \in C^\infty(I, V)$ such that for $(\eta, \zeta, \gamma, \varepsilon) \in V \times I$, we have

$$F(\eta, \zeta, \gamma, \varepsilon) = 0 \iff (\eta, \zeta, \gamma) = f(\varepsilon).$$

The terms in the expansion in Equation (1.47) can be obtained as in the proof of Theorem 1.12. □

Remark 1.21. It is worth mentioning that on infinite depth, the matrix Θ is always invertible. A corresponding existence theorem for infinite depth would thus hold for any configuration.

Remark 1.22. An extension of the existence result in Theorem 1.20 to point vortices that are not all on the same vertical line would require a different argument than the one we have used. The main issue is that assuming η and ζ to be even is then no longer sufficient to satisfy the vertical component of the vorticity equation, like we did to obtain Equation (1.44).

One may note that the sign reversal of the wave velocity about the midpoint $\theta = 1/2$ that we saw with the single point vortex, can be seen also for several point vortices, albeit in a different manner. If the matrix Θ corresponds to $1 > \theta_1 > \cdots > \theta_n > 0$, and we reflect the vortices across the line $y = -h/2$ by considering $\vartheta_i := 1 - \theta_i, 1 \leq i \leq n$ instead (without reordering them), then the new matrix is $-\Theta$. This causes a swap of sign on the leading order vortex strengths, $\gamma_1 = \Theta^{-1}\mathbf{1}$.

We have pointed out that the matrix Θ is not invertible for all configurations of point vortices, and gave the trivial example of $\theta = 1/2$ for a single point vortex. This example, together with Theorem 1.12, also shows that invertibility of Θ is not a necessary condition for the existence of a traveling wave with point vortices in those points. See also Remark 1.24.

The only case for multiple point vortices on the y-axis where we can feasibly describe the admissible positions directly is for $n = 2$. In fact, we give a complete description of when Θ is invertible in Proposition 1.23; see also Figure 1.3, which presents this result graphically. One may observe that the midpoint between the bottom and surface plays a role also here.

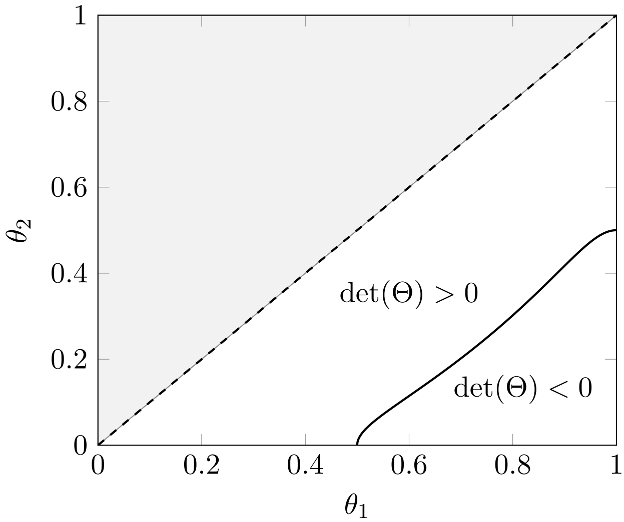


Figure 1.3: The determinant of Θ for the case $n = 2$ as a function of (θ_1, θ_2). The determinant vanishes along the solid black curve, which is given explicitly as a parametrization in Proposition 1.23. (In the figure, the level curve for $\det(\Theta) = 0$ is computed numerically.)

Proposition 1.23 (Θ for $n = 2$). *For two point vortices, we have the following:*

(i) If $\theta_1 \leq 1/2$, then Θ is invertible for all $\theta_2 \in (0, \theta_1)$.

(ii) If $\theta_1 > 1/2$, then Θ is invertible for all $\theta_2 \in (0, \theta_1)$ except for exactly one value, $0 < \hat{\theta}_2(\theta_1) < 1/2$. The graph of $\hat{\theta}_2 \colon (1/2, 1) \to (0, 1/2)$ is described by the curve

$$\begin{array}{ccc} (\pi/4, 3\pi/4) & \to & (1/2, 1) \times (0, 1/2) \\ t & \mapsto & (t + f(t), t - f(t))/\pi \end{array}$$

where $f\colon (\pi/4, 3\pi/4) \to \mathbb{R}$ *is defined by*

$$f(x) := \operatorname{arccot}\left(\sqrt{\frac{1}{2}\left(\cot(x)^2 + \sqrt{4 - 3\cot(x)^4}\right)}\right).$$

Proof. It is useful to write the determinant of Θ as

$$\det(\Theta) = \frac{1}{16h^2}\left[\cot(\pi\theta_1)\cot(\pi\theta_2) + \frac{4\sin(\pi\theta_1)\sin(\pi\theta_2)}{(\cos(\pi\theta_2) - \cos(\pi\theta_1))^2}\right].$$

One immediately observes that the second term inside the parentheses is always strictly positive. If $\theta_1 \leq 1/2$, then one has in addition that the first term is nonnegative for any $\theta_2 \in (0, \theta_1) \subseteq (0, 1/2)$. This proves the first part of the proposition.

For the second part, let us first prove that there is exactly one value of θ_2 for each $\theta_1 \in (1/2, 1)$ that makes Θ singular, and that this value lies in the interval $(0, 1/2)$. For fixed $\theta_1 \in (1/2, 1)$ the determinant is strictly increasing in θ_2, and tends to $-\infty$ as $\theta_2 \downarrow 0$, and to ∞ as $\theta_2 \uparrow \theta_1$. Hence it vanishes at exactly one value of θ_2, say $\hat{\theta}_2(\theta_1)$. Because the determinant is positive when $\theta_2 = 1/2$, this value must necessarily lie in the interval $(0, 1/2)$.

We now move on to the parametrization of the graph of the map $\hat{\theta}_2\colon (1/2, 1) \to (0, 1/2)$. One may note from Figure 1.3 that there is symmetry across the diagonal line

$$\{(\theta_1, \theta_2) \in (0,1)^2 : \theta_1 > \theta_2, \theta_1 + \theta_2 = 1\},$$

which suggests making a change of variables. By letting

$$\phi_1 := \pi\frac{\theta_1 + \theta_2}{2}, \quad \phi_2 := \pi\frac{\theta_1 - \theta_2}{2}, \tag{1.48}$$

we can write the determinant in the form

$$\det(\Theta) = \frac{1}{16h^2}\left[\frac{\cot(\phi_1)^2\cot(\phi_2)^2 - 1}{\cot(\phi_2)^2 - \cot(\phi_1)^2} + \cot(\phi_2)^2 - \cot(\phi_1)^2\right],$$

which leads us to solve the quadratic equation

$$x^2 - ax + a^2 - 1 = 0, \quad a := \cot^2(\phi_1), \quad x := \cot^2(\phi_2)$$

for x, given a. Doing this yields the parametrization, by using ϕ_1 as the parameter (some care has to be taken to ensure that one picks the right branches of the functions involved) and going back to the original variables by inverting Equation (1.48). □

Remark 1.24. By employing the parametrization of the graph of $\hat{\theta}_2$ provided by Proposition 1.23, one can show that the each column of Θ is linearly independent from $\mathbf{1}$ when $\det(\Theta) = 0$. This implies that an argument similar to that of Theorem 1.20 can be performed, by using the vortex strength γ^1 as the bifurcation parameter, instead of c. Thus it is possible to show existence for any configuration when $n = 2$. An extension of this argument to $n > 2$ is harder, because it requires the rank of Θ to be $n - 1$.

While the set of configurations that make $\det(\Theta)$ vanish is hard to describe in general when $n > 2$, some observations can be made. Of course, if $n \geq 2$, and as long as the derivative of $\det(\Theta)$ with respect to the variable $(\theta_1, \ldots, \theta_n)$ does not vanish at a point where $\det(\Theta) = 0$, the zero set of $\det(\Theta)$ is locally a smooth manifold of dimension $n - 1$ around that point by the implicit function theorem. When $n = 2$, the zero set is actually the graph of a smooth function in θ_1 by Proposition 1.23, and numerical evidence suggests that the zero set is the graph of a smooth function in (θ_1, θ_2) when $n = 3$. Actually checking that the derivative does not vanish is hard, but we have the following theorem:

Theorem 1.25. *The subset of configurations of point vortices in*

$$\{(\theta_1, \ldots, \theta_n) \in (0,1)^n : 1 > \theta_1 > \theta_2 > \cdots > \theta_n > 0\}$$

such that Θ is not invertible has measure zero.

Proof. Each entry in Θ is analytic in each θ_i for $\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n$ fixed. It follows that $\det(\Theta)$ also has this property, when viewed as a function

$$U \coloneqq \{(\theta_1, \ldots, \theta_n) \in (0,1)^n : 1 > \theta_1 > \theta_2 > \cdots > \theta_n > 0\} \to \mathbb{R}.$$

We first verify that $\det(\Theta)$ does not vanish identically on U. To that end, fix $1/2 > \tilde{\theta}_1 > \tilde{\theta}_2 > \cdots > \tilde{\theta}_n > 0$ and consider $\theta_1 = \varepsilon\tilde{\theta}_1, \ldots \theta_n = \varepsilon\tilde{\theta}_n$ for $1 > \varepsilon > 0$. The purpose of the upper bound of $1/2$ is to make sure that $\tan(\pi\theta_i)$ is well defined for all $1 \leq i \leq n$. Observe now that if we let $T \coloneqq \operatorname{diag}(\tan(\pi\theta_k))_{k=1}^n$, then

$$4h[T\Theta]_{i,j} = \begin{cases} 1 & i = j \\ \tan(\pi\theta_i)\left(\cot\left(\pi\frac{\theta_i+\theta_j}{2}\right) - \cot\left(\pi\frac{\theta_i-\theta_j}{2}\right)\right) & i \neq j \end{cases},$$

where

$$\lim_{\varepsilon \downarrow 0} \tan(\varepsilon\pi\tilde{\theta}_i)\left(\cot\left(\varepsilon\pi\frac{\tilde{\theta}_i + \tilde{\theta}_j}{2}\right) - \cot\left(\varepsilon\pi\frac{\tilde{\theta}_i - \tilde{\theta}_j}{2}\right)\right) = -\frac{4\tilde{\theta}_i\tilde{\theta}_j}{\tilde{\theta}_i^2 - \tilde{\theta}_j^2}$$

for $i \neq j$. It follows that $\operatorname{diag}(\tan(\pi\theta_k))_{k=1}^n \Theta$ has a limit in $B(\mathbb{R}^n)$ as $\varepsilon \downarrow 0$, and that this limit is

$$\lim_{\varepsilon \downarrow 0} \operatorname{diag}(\tan(\pi\theta_k))_{k=1}^n \Theta = \frac{1}{4h}(I_{\mathbb{R}^n} - B), \tag{1.49}$$

where we have defined $B \in \mathbb{R}^{n \times n}$ by

$$B_{i,j} := \begin{cases} 0 & i = j \\ 4\tilde{\theta}_i \tilde{\theta}_j (\tilde{\theta}_i^2 - \tilde{\theta}_j^2)^{-1} & i \neq j \end{cases}. \tag{1.50}$$

In particular, B is skew-symmetric, which implies that $I_{\mathbb{R}^n} - B$ is invertible. Since the set of invertible operators is open, so is the matrix $\operatorname{diag}(\tan(\pi\theta_k))_{k=1}^n \Theta$ for sufficiently small ε, which in turn means that Θ is invertible for such ε.

Finally, the set U is connected. Hence, since we know that $\det(\Theta)$ is analytic in each variable and does not vanish identically, we infer[5] that the subset of U on which $\det(\Theta)$ vanishes has measure zero. □

In general we cannot do better than Theorem 1.25, in the sense that for any $n \geq 1$ there will always be a configuration of n point vortices that makes $\det(\Theta)$ vanish.

Proposition 1.26. *There are always configurations of point vortices in*

$$\{(\theta_1, \ldots, \theta_n) \in (0,1)^n : 1 > \theta_1 > \theta_2 > \cdots > \theta_n > 0\}$$

where Θ is singular.

Proof. The matrix appearing on the right-hand side of Equation (1.49) in the proof of Theorem 1.25 has a positive determinant. Indeed, the matrix B defined in Equation (1.50) is skew-symmetric, so its spectrum is purely imaginary. Moreover, since the matrix is real, the eigenvalues are either zero or appear in complex conjugate pairs.

Say that the first m eigenvalues $\lambda_1, \ldots, \lambda_m$ of B are zero and that

$$\lambda_{m+2j-1} = \overline{\lambda_{m+2j}} = i\mu_j, \quad j = 1, \ldots, (n-m)/2,$$

where the μ_j are real. Then it follows that

$$\begin{aligned} \det\left(\frac{1}{4h}(I_{\mathbb{R}^n} - B)\right) &= \frac{1}{(4h)^n} \det(I_{\mathbb{R}^n} - B) \\ &= \frac{1}{(4h)^n}(1 + \mu_1^2)(1 + \mu_2^2) \cdots (1 + \mu_{(n-m)/2}^2), \end{aligned}$$

[5] This follows by induction on the dimension, by using the well known result in one dimension.

because the determinant of a matrix is equal to the product of its eigenvalues (taking algebraic multiplicity into account). By Equation (1.49) we then have

$$\det(\Theta) \prod_{k=1}^{n} \tan(\pi\theta_k) > 0 \tag{1.51}$$

for small $\varepsilon > 0$ (as in the proof of Theorem 1.25) by continuity of the determinant. Since all the tangents are also positive, this implies that $\det(\Theta) > 0$ for small $\varepsilon > 0$.

It remains to exhibit a configuration where $\det(\Theta) < 0$. To that end, fix $\frac{1}{2} > \tilde{\theta}_1 > \tilde{\theta}_2 > \tilde{\theta}_3 > \cdots > \tilde{\theta}_n > 0$ and consider $\theta_1 = 1-\varepsilon\tilde{\theta}_1, \theta_2 = \varepsilon\tilde{\theta}_2, \ldots \theta_n = \varepsilon\tilde{\theta}_n$ for $1 > \varepsilon > 0$. Proceeding as in the proof of Theorem 1.25 we find

$$\lim_{\varepsilon\downarrow 0} \operatorname{diag}(\tan(\pi\theta_k))_{k=1}^{n}\Theta = \frac{1}{4h}(I_{\mathbb{R}^n} - \tilde{B}), \tag{1.52}$$

where we have defined $\tilde{B} \in \mathbb{R}^{n\times n}$ by

$$\tilde{B}_{i,j} := \begin{cases} 0 & i = j \text{ or } i = 1 \text{ or } j = 1, \\ 4\tilde{\theta}_i\tilde{\theta}_j(\tilde{\theta}_i^2 - \tilde{\theta}_j^2)^{-1} & \text{otherwise.} \end{cases}$$

This matrix is still skew-symmetric like B, and so the right-hand side of Equation (1.52) has a positive determinant, as before. Hence Equation (1.51) holds for small ε. However, now $\tan(\pi\theta_1)$ is negative and the rest of the tangents are positive, meaning that $\det(\Theta)$ must be negative. □

6 Explicit expressions for infinite depth

In this section we give some explicit expressions for periodic waves with a point vortex on infinite depth, constructed in [41]. We will adopt the notation and conventions used there. The fluid domain for the trivial surface is $\mathbb{R} \times (-\infty, 1)$ and the waves have period $2\pi L$. The stream function for the rotational part is denoted by $\mathbf{G}$.

Proposition 1.27 (Stream function)**.** *The stream function for the rotational part is given by*

$$\mathbf{G}(x, y) = \frac{1}{4\pi} \log\left(\frac{\cos(x/L) - \cosh(y/L)}{\cos(x/L) - \cosh((y-2)/L)}\right).$$

Proof. We wish to find the stream function $\mathbf{G}\colon \mathbb{R} \times (-\infty, 1) \to \mathbb{R}$ corresponding to equally spaced point vortices of unit strength at the points

$2\pi L\mathbb{Z} \times \{0\} \subseteq \mathbb{R}^2$, and which is such that this stream function vanishes at the surface, $\mathbb{R} \times \{1\}$. By symmetry, it must be the case that $\mathbf{G}_x$ vanishes on $\pi L(1+2\mathbb{Z}) \times (-\infty, 1)$. This leads us to the boundary value problem

$$\Delta \mathbf{G} = \delta, \quad \mathbf{G}|_{y=0} = 0, \quad \mathbf{G}_x|_{x=\pm\pi L} = 0,$$

on $(-\pi L, \pi L) \times (-\infty, 1)$. This equation can be dealt with using Theorem 1.30 in Section A.

In order to apply Theorem 1.30 we require a conformal map satisfying the requirements in the theorem statement. One may check (see [45, Sections 7.1 and 7.2]) that

$$f(z) := \frac{\tanh(1/(2L)) - \tanh((1+iz)/(2L))}{\tanh(1/(2L)) + \tanh((1+iz)/(2L))} \tag{1.53}$$

defines a bijective conformal map from the half strip $(-\pi L, \pi L) \times (-\infty, 1)$ onto the slit unit disk $\mathbb{D} \setminus ((0, \exp(-1/L)) \times \{0\})$, and which is such that

(i) The origin is fixed.

(ii) The surface is mapped to the unit circle.

(iii) The sides $\{\pm\pi L\} \times (-\infty, 1)$ are mapped to the slit.

The result now follows by taking the logarithm of the modulus of the map f in Equation (1.53). □

By using Proposition 1.27, we can obtain an explicit expression for the leading order wave velocity c_1, and a Fourier series for the leading order surface profile η_*:

Proposition 1.28 (c_1 and η_*)**.** *The leading-order wave velocity c_1 and surface profile η_* are given by*

$$c_1 = -\frac{1}{4\pi L} \coth(1/L),$$
$$\eta_* = -\frac{1}{4\pi^2} \sum_{n=1}^{\infty} \frac{n}{gL^2 + \alpha^2 n^2} e^{-n/L} \cos(nx/L),$$

respectively.

Proof. Recall how the wave velocity appeared on the right-hand side of Equation (1.18). By using the final part of Theorem 1.29, we find

$$c_1 = \frac{i}{4\pi} \overline{\left(\frac{f''(0)}{f'(0)}\right)} = -\frac{1}{4\pi L} \coth(1/L),$$

where f is the conformal map introduced in Equation (1.53) in the proof of Proposition 1.27.

We now move to the surface profile. From [41] we know that

$$\eta_* = -(g - \alpha^2\partial_x^2)^{-1}\left(\chi - \frac{1}{2\pi L}\int_{-\pi L}^{\pi L}\chi\, d\mu\right), \tag{1.54}$$

where χ is defined by

$$\chi(x) := c_1\mathbf{G}_y(x,1) + \frac{1}{2}\mathbf{G}_y(x,1)^2.$$

Written out, we have

$$\chi(x) = \frac{1}{8\pi^2 L^2}\frac{\cosh(1/L)\cos(x/L) - 1}{(\cos(x/L) - \cosh(1/L))^2}$$

with the elementary antiderivative

$$\chi^\sharp(x) = -\frac{1}{8\pi^2 L}\frac{\sin(x/L)}{\cos(x/L) - \cosh(1/L)}.$$

In particular, this means that

$$\int_{-\pi L}^{\pi L}\chi\, d\mu = \chi^\sharp(\pi L) - \chi^\sharp(-\pi L) = 0,$$

so that Equation (1.54) reduces to

$$\eta_* = -(g - \alpha^2\partial_x^2)^{-1}\chi. \tag{1.55}$$

In order to find the Fourier series for η_*, we require the Fourier series of χ. We may write

$$\chi^\sharp(x) = \frac{1}{i}\left[1 + \frac{e^{-ix/L-1/L}}{1 - e^{-ix/L-1/L}} - \frac{1}{1 - e^{ix/L-1/L}}\right],$$

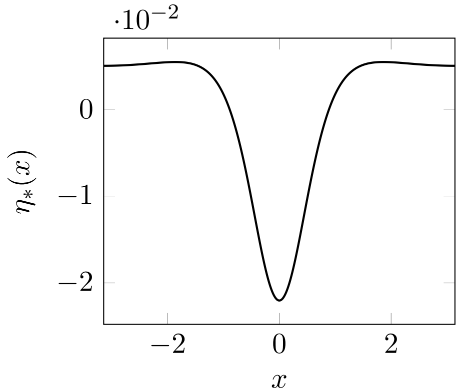
which, by expanding into geometric series, means that

$$\chi^\sharp(x) = i\sum_{n=-\infty}^{\infty}\operatorname{sgn}(n)e^{-|n|/L}e^{inx/L}.$$

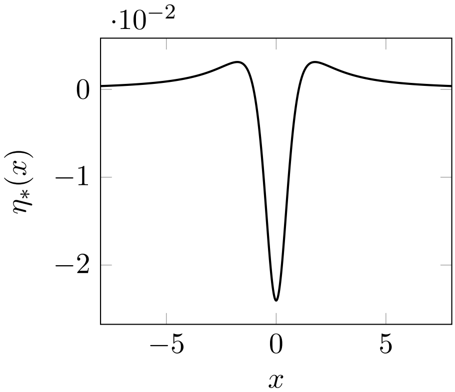
Hence, by termwise differentiation, we obtain

$$\chi(x) = \frac{1}{4\pi^2 L^2}\sum_{n=1}^{\infty} ne^{-n/L}\cos(nx/L),$$

which, combined with Equation (1.55), yields the result. □



(a) $L = 1$ (Whole period)



(b) $L = 100$ (Only part of a period)

Figure 1.4: The leading order surface profile term, η_*, when $g = 1, \alpha^2 = 0.01$, cf. [41, Figure 1].

One may note that

$$c_1 = -\frac{1}{4\pi} + O(1/L^2)$$

as $L \to \infty$, which agrees with the speed of the solitary waves on infinite depth. When L is large, the surface profile is very similar to the surface in the localized case, see Figure 1.4b. At the other extreme, the first terms in the Fourier series will dominate.

A Green's functions

In this appendix, we provide two theorems that are used in order to get exact expressions for the rotational part of the stream function. Except for the final part, Theorem 1.29 is a standard result [35, p. 166]. Theorem 1.30 is a less well known extension of Theorem 1.29.

Theorem 1.29 (Green's functions in $\mathbb{R}^2$)**.** *Suppose that $\Omega \subsetneq \mathbb{R}^2$ is a simply connected domain and that $z_0 \in \Omega$. Furthermore, suppose that $f\colon \Omega \to \mathbb{D}$ is a bijective conformal map onto the open unit disk, extending continuously to a function $\overline{\Omega} \to \overline{\mathbb{D}}$ and satisfying $f(z_0) = 0$. Then the function $\varphi\colon \Omega \to \mathbb{R}$ defined by*

$$\varphi(z) := \frac{1}{2\pi} \log(|f(z)|)$$

is in $L^1_{\mathrm{loc}}(\Omega)$*, extends continuously to the boundary of* Ω*, and satisfies*

$$\begin{aligned}\Delta\varphi &= \delta_{z_0},\\ \varphi|_{\partial\Omega} &= 0.\end{aligned}$$

Furthermore, the harmonic function h *defined by*

$$h(z) \coloneqq \varphi(z) - \frac{1}{2\pi}\log\left(|z-z_0|\right)$$

satisfies

$$\nabla h(z_0) = \frac{1}{4\pi}\overline{\left(\frac{f''(z_0)}{f'(z_0)}\right)}$$

after identifying $\mathbb{R}^2$ *and* $\mathbb{C}$ *via* $(x,y)\mapsto x+iy$.

Proof. We first check the boundary values of the function φ. By assumption, f extends continuously to $\partial\Omega$, and every point on $\partial\Omega$ must necessarily be mapped to the unit circle. It is thus immediate that φ also extends continuosly to the boundary, and moreover, vanishes there.

Identify now $\mathbb{R}^2$ and $\mathbb{C}$. Observe that since $f(z_0)=0$, we have

$$f(z) = g(z)(z-z_0), \quad z\in\Omega$$

for some holomorphic function g, where $|g|>0$. Indeed, we must have $g(z_0)=f'(z_0)\neq 0$ because f is injective, and the injectivity of f also ensures that there can be no other roots. Thus

$$\varphi(z) = \frac{1}{2\pi}\log\left(|z-z_0|\right) + h(z),$$

where

$$h(z) \coloneqq \frac{1}{2\pi}\operatorname{Re}\log\left(g(z)\right)$$

is harmonic by $|g|>0$ and the Cauchy-Riemann equations. Hence, by Proposition 1.1, the function φ is L^1_{loc} and satisfies

$$\Delta\varphi = \delta_{z_0}.$$

The last assertion follows by observing that one must necessarily have $g'(z_0) = \frac{1}{2}f''(z_0)$, meaning that

$$\left(\frac{1}{2\pi}\log(g(\cdot))\right)'(z_0) = \frac{1}{2\pi}\frac{g'(z_0)}{g(z_0)} = \frac{1}{4\pi}\frac{f''(z_0)}{f'(z_0)},$$

whence we deduce from the Cauchy-Riemann equations that

$$\nabla h(z_0) = \frac{1}{4\pi}\overline{\left(\frac{f''(z_0)}{f'(z_0)}\right)}. \qquad \square$$

Theorem 1.30 (Green's functions in $\mathbb{R}^2$, mixed)**.** *Suppose that $\Omega \subsetneq \mathbb{R}^2$ is a simply connected domain and that $z_0 \in \Omega$. Furthermore, assume that $\partial\Omega = \Gamma_D \sqcup \Gamma_N$, where Γ_N is C^1 and open in $\partial\Omega$. Finally, suppose that $f\colon \Omega \to \mathbb{D} \setminus ((-1,-a] \times \{0\})$, where $a > 0$, is a bijective conformal map of Ω onto the unit disk with a slit, satisfying $f(z_0) = 0$ and extending continuously to the boundary. This map should send Γ_D to the unit circle and Γ_N to the interval $(-1, a] \times \{0\}$, and should extend analytically across Γ_N (when viewed as a map on $\mathbb{C}$). Then the function $\varphi\colon \Omega \to \mathbb{R}$ defined by*

$$\varphi(z) \coloneqq \frac{1}{2\pi} \log\left(|f(z)|\right)$$

is in $L^1_{\mathrm{loc}}(\Omega)$, extends continuously to the boundary and satisfies

$$\begin{aligned} \Delta\varphi &= \delta_{z_0}, \\ \varphi|_{\Gamma_D} &= 0, \\ \partial_n\varphi|_{\Gamma_N} &= 0, \end{aligned}$$

where ∂_n denotes the normal derivative.

Proof. The only change from Theorem 1.29 is checking that the normal derivative vanishes on Γ_N. This follows by using conformality. □

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PAPER 2

TRAVELING GRAVITY WATER WAVES WITH CRITICAL LAYERS

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ABSTRACT. We establish the existence of small-amplitude uni- and bimodal steady periodic gravity waves with an affine vorticity distribution, using a bifurcation argument that differs slightly from earlier theory. The solutions describe waves with critical layers and an arbitrary number of crests and troughs in each minimal period. An important part of the analysis is a fairly complete description of the local geometry of the so-called kernel equation, and of the small-amplitude solutions. Finally, we investigate the asymptotic behavior of the bifurcating solutions.

1 INTRODUCTION

Up until fairly recently, most authors working with steady water waves have made the assumption that the vorticity

$$\omega := v_x - u_y \tag{2.1}$$

of the velocity field (u, v) vanishes identically. Such waves are known as irrotational, as opposed to rotational waves where ω is allowed to be nonzero. Rotational waves can exhibit more exotic behavior than irrotational ones, including interior stagnation points and critical layers of closed streamlines [11]. Stagnation points correspond to fluid particles that are stationary with respect to the wave, and for irrotational flows this can only occur at a sharp crest [35].

Irrotational waves are mathematically simpler to work with than rotational ones, due to the existence of the velocity potential. The velocity potential is the harmonic conjugate of the stream function, thus enabling the use of tools such as complex analysis, which are typically not available with nonzero vorticity. The survey [32] treats the theory of Stokes waves—an important class of irrotational waves—and the results on the so-called Stokes conjecture for such waves. This conjecture was not fully settled until the appearance of the paper [28].

Although rotational waves were considered intractable for mathematical analysis, they have long been important in more applied fields because rotational waves are not uncommon in nature: There are many physical effects that can induce rotation in waves, such as wind and thermal or salinity gradients [26], and rotational waves are also important in wave-current interactions [31].

The first, and still the only known, explicit example of a nontrivial traveling gravity water wave solution to the Euler equations was given in [15] (see also [3] for a more modern treatment) and is rotational; a fact which was only later pointed out by Stokes. Much later came the first existence result for small-amplitude waves with general vorticity distributions [10]. It was not, however, before the pioneering article [5] that large-amplitude waves were constructed, using an extension of the global bifurcation theory of Rabinowitz [17, 29], leading to renewed interest in rotational waves. A corresponding result on deep water, where the lack of compactness is an obstacle, was established in [19].

Due to the methods used, neither the waves in [10] nor those in [5] exhibit stagnation. The first waves with a critical layer were constructed in [36], having constant vorticity. A different approach was used in [8], allowing for wave profiles with overhang (for which existence is still an open question, with some numerical evidence in the affirmative [33]). The method of proof for the existence of nontrivial rotational waves is typically bifurcation from parallel flows with a prescribed vorticity distribution. Such parallel flows are described in great detail in [23].

Other authors have looked at waves with density stratification [14, 18, 37], waves with compactly supported vorticity [30, 34], waves with discontinuous vorticity [6], and waves with a general vorticity distribution and stagnation [24]. An upcoming result also establishes the existence of large-amplitude gravity water waves with a critical layer [7]. This was done in the presence of capillary effects in [25], using an entirely different formulation.

Of particular interest to us are [12, 13], which cover small-amplitude waves with an affine vorticity distribution. This is the natural step up from the constant vorticity considered in [36], and the resulting waves can have an arbitrary number of critical layers [11].

In this paper, which builds upon [1], we consider the same setting as in [12]. Small-amplitude solutions with an affine vorticity distribution are found by bifurcating from trivial solutions that depend naturally on three parameters. By using other choices for the bifurcation parameters in our argument, we obtain solution curves and sheets that, in general, do not coincide with those found in [12]. We are led to examine the asymptotic behavior of the bifurcating solutions; in particular for carefully chosen special cases. A complicating factor for our choice of bifurcation parameters is that they require an additional condition on the parameters. This condition can be interpreted as a nondegeneracy condition for the equation governing the dimension of the linearized problem.

Another novel aspect of this work is a fairly complete description of the local geometry of the kernel of the operator appearing in the linearization. This is used to describe the geometry of the solution set near any trivial solution where the linear problem is one-dimensional, and for a class of trivial solutions with a two-dimensional linearized problem. We also show, by explicit construction, that the dimension of the linear problem can become arbitrary large for certain wavenumbers. This opens up the possibility for waves with arbitrarily many modes. Finally, we prove a regularity result, showing that the solutions we find are real analytic.

The outline of the paper is as follows: In Section 2 we formulate the problem and describe the setting in which we will work. Next, Section 3 focuses on the kernel of the linearized operator. Section 4 contains the bifurcation result for a one-dimensional kernel and gives the properties of the resulting bifurcation curves, while the final section, Section 5, covers two-dimensional bifurcation. Some useful derivatives are listed in Section A.

2 The governing equations

We consider pure gravity waves. The fluid motion is assumed to be incompressible and two-dimensional, with the coordinate system oriented so that the x- and y-axes are horizontal and vertical, respectively. The fluid domain is bounded below by a flat bottom, and above by a free surface. Within this setting, our aim is to construct solutions of the steady water-wave problem; that is, to find a surface profile η and a velocity field (u, v), defined in the fluid domain

$$\Omega_\eta \coloneqq \Big\{(x,y) \in \mathbb{R}^2 : 0 < y < d + \eta(x)\Big\},$$

where d is the depth of the undisturbed fluid, satisfying the Euler equations

$$u_x + v_y = 0, \tag{2.2a}$$

$$(u-c)u_x + vu_y = -p_x, \tag{2.2b}$$

$$(u-c)v_x + vv_y = -p_y - g \tag{2.2c}$$

in Ω_η. The surface profile is assumed to satisfy $\eta > -d$, so that the bottom is not exposed to air. In (2.2b) and (2.2c) the quantity p is the pressure, g is the gravitational acceleration, and c is the constant velocity at which the wave travels.

In addition to the equations in (2.2), we impose the boundary conditions

$$
\begin{aligned}
v &= 0 && \text{at } y = 0, && \text{(2.3a)}\\
v &= (u-c)\eta_x && && \text{(2.3b)}\\
p &= 0 && \text{at } y = d + \eta(x). && \text{(2.3c)}
\end{aligned}
$$

The first two boundary conditions are known as kinematic boundary conditions, and state that there is no flux through the surface or bottom. The dynamic boundary condition in (2.3c) ensures that there is no jump in pressure across the free surface.

We will be searching for periodic waves only, and so we introduce the wavenumber $\kappa > 0$, and stipulate that all functions above be $2\pi/\kappa$-periodic in the horizontal variable.

Stream function formulation

We now reformulate the water wave problem (2.2)–(2.3) in terms of a potential ψ, called the *relative stream function*. From incompressibility (2.2a), together with Ω_η being simply connected, we know that there exists a function $\psi\colon \Omega_\eta \to \mathbb{R}$ satisfying

$$\psi_x = -v, \qquad \psi_y = u - c.$$

This function is uniquely determined by (u, v), up to a constant.

The kinematic boundary condition (2.3a) is equivalent to $\psi_x = 0$ at $y = 0$, and so ψ is constant on the bottom. Similarly, we can use (2.3b) to deduce that ψ is constant also on the surface. Next, (2.2b) and (2.2c) can be used to show that

$$\{\psi, \Delta\psi\} = 0,$$

where $\{\cdot,\cdot\}$ is the Poisson bracket defined by

$$\{f, g\} \coloneqq f_y g_x - f_x g_y.$$

Furthermore, by also using the boundary conditions in (2.3), one can infer that the surface Bernoulli equation

$$\tfrac{1}{2}|\nabla\psi|^2 + g\eta = Q \qquad \text{on } y = d + \eta(x)$$

holds for some $Q \in \mathbb{R}$.

In terms of the stream function, the vorticity is given by

$$\omega = -\Delta\psi,$$

which follows directly from its definition in (2.1). Observe also that ψ is $2\pi/\kappa$-periodic in the horizontal variable. To see this, note that $(x, y) \mapsto \psi(x + 2\pi/\kappa, y)$ is also a stream function, taking the same constant values as ψ on the boundary. By uniqueness, they must be identical.

The motivation for introducing the stream function is that, for a prescribed vorticity, the preceding equations are in fact equivalent to the steady water-wave problem. A precise statement, taken from [11], can be found in Proposition 2.1 below. We will use a subscript κ to denote $2\pi/\kappa$-periodicity in the horizontal variable.

Proposition 2.1 (Stream function [11])**.** *For $\eta \in C^3_\kappa(\mathbb{R})$, $u, v \in C^2_\kappa(\overline{\Omega}_\eta)$ and a prescribed vorticity $\omega \in C^1_\kappa(\overline{\Omega}_\eta)$, the steady water-wave problem* (2.2)–(2.3) *is equivalent to the stream function formulation*

$$\begin{aligned} \Delta\psi &= -\omega && \\ \{\psi, \Delta\psi\} &= 0 && \text{in } \Omega_\eta, \\ \psi &= m_0 && \text{at } y = 0, \\ \psi &= m_1 && \\ \tfrac{1}{2}|\nabla\psi|^2 + g\eta &= Q && \text{at } y = d + \eta(x), \end{aligned} \tag{2.4a}$$

for $\psi \in C^3_\kappa(\overline{\Omega}_\eta)$ and constants m_0, m_1 and Q.

The vorticity distribution

As long as the fluid velocity does not exceed the wave velocity, so there is no stagnation, the vorticity at a point only depends on the value of the stream function at that point. This dependency is described by what is known as the vorticity distribution.

Lemma 2.2 (Vorticity distribution [5])**.** *Suppose that $u < c$. Then there exists a function γ such that $\omega = \gamma(\psi)$ in Ω_η.*

A notable consequence of the existence of a vorticity distribution is that Equation (2.4a) is trivially satisfied, because

$$\{\psi, \Delta\psi\} = \psi_y(-\gamma(\psi))_x - \psi_x(-\gamma(\psi))_y = 0$$

by the chain rule. Observe also that the condition in Lemma 2.2 is sufficient, but not necessary. By *assuming* the existence of a vorticity distribution, we will still obtain solutions of the water-wave problem, even if $u < c$ is not satisfied. In fact, the solutions that we will find can exhibit stagnation and critical layers. The introduction of the vorticity distribution is standard for rotational waves, and was used already in [10].

We shall consider the case where γ is affine. By making a shift of ψ, it is sufficient to consider the case of linear γ. After scaling to unit depth and scaling away the gravitational acceleration, the stream function formulation (2.4) reduces to

$$\begin{aligned} \Delta\psi &= \alpha\psi && \text{in } \Omega_\eta, && (2.5a)\\ \tfrac{1}{2}|\nabla\psi|^2 + \eta &= Q && \text{on } S, && (2.5b)\\ \psi &= m_0 && \text{on } B, && (2.5c)\\ \psi &= m_1 && \text{on } S, && (2.5d) \end{aligned}$$

where we have introduced the bottom $B \coloneqq \{(x,y) : y = 0\}$ and the surface $S \coloneqq \{(x,y) : y = 1 + \eta(x)\}$. The parameter α in (2.5a) controls the vorticity, and will be assumed to be *negative*. For positive α, one-dimensional—but not higher-dimensional—bifurcation is possible. More discussion on this can be found in [12].

Observe now that the system (2.5) makes sense also in less regular function spaces than those specified in Proposition 2.1, and we will therefore allow for less regular (but still classical) solutions. More precisely, we will search for solutions

$$\eta \in C^{2,\beta}_{\kappa,\mathrm{e}}(\mathbb{R}) \qquad \text{and} \qquad \psi \in C^{2,\beta}_{\kappa,\mathrm{e}}(\overline{\Omega}_\eta),$$

where $\beta \in (0,1)$, and the subscript e signifies the subspace of functions which are even in the horizontal variable. The motivation for working in these Hölder spaces is that Theorem 2.19 then holds.

Remark 2.3 (Regularity). Due to Equation (2.5a) and elliptic regularity for the differential operator $\alpha - \Delta$, the stream function ψ is analytic in Ω_η. In fact, we show in Theorem 2.5 that this is true even up to the boundary.

Trivial solutions and flattening

The solutions of (2.5) that we shall construct will be small perturbations of steady flows that are parallel to the bottom. These parallel flows are the trivial solutions of (2.5), in the sense that $\eta = 0$ and the stream function

ψ only depends on y. By integrating Equation (2.5a), we arrive at trivial solutions of the form

$$\psi_0(y,\Lambda) \coloneqq \mu\cos\bigl(|\alpha|^{1/2}(y-1)+\lambda\bigr), \qquad \Lambda = (\mu,\alpha,\lambda) \in \mathbb{R}^3, \tag{2.6}$$

with corresponding $Q(\Lambda)$, $m_0(\Lambda)$ and $m_1(\Lambda)$ determined from (2.5b)–(2.5d) as

$$Q(\Lambda) = \frac{\mu^2|\alpha|\sin^2(\lambda)}{2}, \quad \begin{aligned} m_0(\Lambda) &= \mu\cos(\lambda - |\alpha|^{1/2}), \\ m_1(\Lambda) &= \mu\cos(\lambda). \end{aligned} \tag{2.7}$$

Our goal is to find nontrivial solutions of (2.5) for certain values of Λ, corresponding to these particular values of Q, m_0 and m_1. For technical reasons which we will elucidate later in Remark 2.20, it is assumed that

$$\psi_{0y}(1) = -\mu|\alpha|^{1/2}\sin(\lambda) \neq 0. \tag{2.8}$$

As in (2.8), we will often omit the dependence on Λ in our notation.

The main difficulty with the system (2.5) is that it is a free-boundary problem, which entails that the domain is a priori unknown. There are several ways of fixing the domain. Here, we will use the "naive" flattening transform

$$G\colon (x,y) \mapsto \left(x, \frac{y}{1+\eta(x)}\right),$$

giving a bijection from the sets Ω_η, B and S onto

$$\hat{\Omega} = \{(x,s) : s \in [0,1]\}, \quad \hat{B} \coloneqq \{(x,s) : s = 0\} \quad \text{and} \quad \hat{S} \coloneqq \{(x,s) : s = 1\},$$

respectively. Using that $\eta \in C^{2,\beta}_{\kappa,\mathrm{e}}(\mathbb{R})$, we find that the map G is a $C^{2,\beta}$-diffeomorphism, with inverse given by

$$G^{-1}(x,s) = \bigl(x, (1+\eta(x))s\bigr).$$

If we define $\hat{\psi}$ on $\hat{\Omega}$ by $\hat{\psi} \coloneqq \psi \circ G^{-1}$, then (2.5b) and (2.5a) become

$$\begin{aligned} \left(\partial_x - \frac{s\eta_x}{1+\eta}\partial_s\right)^2\hat{\psi} + \frac{\hat{\psi}_{ss}}{(1+\eta)^2} &= \alpha\hat{\psi} \quad \text{in } \hat{\Omega}, \\ \frac{(1+\eta_x^2)\hat{\psi}_s^2}{2(1+\eta)} + \eta &= Q \qquad \text{on } \hat{S}, \end{aligned} \tag{2.9}$$

in the new flattened variables, for which we have the following:

Lemma 2.4 (Equivalence [12]). *For functions $\eta \in C^{2,\beta}_{\kappa,\mathrm{e}}(\mathbb{R})$ and $\psi \in C^{2,\beta}_{\kappa,\mathrm{e}}(\overline{\Omega}_\eta)$, the stream function formulation* (2.5) *is equivalent to the transformed problem in* (2.9) *for $\eta \in C^{2,\beta}_{\kappa,\mathrm{e}}(\mathbb{R})$ and*

$$\{\hat{\psi} \in C^{2,\beta}_{\kappa,\mathrm{e}}(\overline{\hat{\Omega}}) : \hat{\psi}|_{s=0} = m_0,\ \hat{\psi}|_{s=1} = m_1\}.$$

Moreover, in this setting, a pair $(\eta, \hat{\psi}) = (0, \hat{\psi}(s))$ solves (2.9) *if and only if $\hat{\psi} = \psi_0$.*

With the trivial solutions found and the flattening transform introduced, we now elaborate on Remark 2.3. Any solution which is sufficiently close to a trivial solution is in fact analytic, as long as (2.8) holds. The precise statement can be found in Theorem 2.5 below.

Theorem 2.5 (Regularity). *Suppose that a solution (η, ψ) of the problem* (2.5) *in $C^1(\mathbb{R}) \times C^2(\mathbb{R})$ is such that the normal derivative $\partial_n \psi$ of the stream function vanishes at no point on the surface. Then we have the following:*

(i) The surface profile η is analytic.

(ii) The stream function ψ extends to an analytic function on an open set containing $\overline{\Omega}_\eta$.

The assumption on $\partial_n \psi$ holds when Λ satisfies (2.8) *and $\hat{\psi}$ is sufficiently close to $\psi_0(\cdot, \Lambda)$ in $C^2(\overline{\hat{\Omega}})$.*

Proof. We start by showing that η is analytic. For this, we will use the approach taken in [4], which is to apply [22, Theorem 3.2]. In [4] this was done under the assumption of no stagnation, but it is sufficient to assume that stagnation does not occur on the surface. This corresponds to the Shapiro–Lopatinskiĭ condition for a certain elliptic system.

Let Ω_η^+ be the component of $\mathbb{R}^2 \setminus S$ that does not contain Ω_η. Proceed to define the function $u\colon \Omega_\eta \cup S \cup \Omega_\eta^+ \to \mathbb{R}$ by

$$u(x, y) \coloneqq \begin{cases} 0 & (x, y) \in S \cup \Omega_\eta^+, \\ \psi(x, y) - m_1 & (x, y) \in \Omega_\eta \cup S, \end{cases}$$

and the differential operator L by $L \coloneqq \alpha - \Delta$. Observe that Equations (2.5b) and (2.5d) imply that

$$f(y, \partial_n \psi) \coloneqq \frac{1}{2}(\partial_n \psi)^2 + y - 1 - Q = 0$$

on S. All the assumptions of [22, Theorem 3.2] are now satisfied, with $G \coloneqq L$ and $F(u) \coloneqq Lu + \alpha m_1$ (see the remark immediately after the theorem). We conclude that η is analytic.

Note now that the differential operator L is strongly elliptic in the sense of [27, Equation (1.7)]. Equipped with the fact that η is analytic, we can use [27, Theorem A] to conclude that ψ extends to an analytic function on an open set containing $\overline{\Omega}_\eta$.

The final part of the theorem follows because

$$\partial_n \psi = \sqrt{1 + (\eta')^2}\psi_y(\cdot, \eta) = \frac{\sqrt{1 + (\eta')^2}}{1 + \eta}\hat{\psi}_s(\cdot, 1),$$

where $\hat{\psi}_s(\cdot, 1)$ is bounded away from 0 as long as $\hat{\psi}$ is sufficiently close to ψ_0 in $C^2(\overline{\hat{\Omega}})$, due to the assumption that Equation (2.8) holds. □

Remark 2.6. Theorem 2.5 is a local result at heart. It is clear from the proof that if $\partial_n \psi(x_0, \eta(x_0)) \neq 0$, then η is analytic in a neighborhood of x_0. This, in turn, implies that ψ extends analytically across the surface near the point $(x_0, \eta(x_0))$.

Remark 2.7. Recall that the stream function ψ is analytic on Ω_η, regardless of whether the condition on $\partial_n \psi$ on the surface in Theorem 2.5 is satisfied. It is worth noting that this implies, through the implicit function theorem, that the streamlines are analytic curves away from stagnation points.

The linearized problem

In order to linearize Equation (2.9) around a trivial solution ψ_0, we write $\hat{\psi} = \psi_0 + \hat{\phi}$, and introduce the spaces

$$X = X_1 \times X_2 \coloneqq C^{2,\beta}_{\kappa,\mathrm{e}}(\mathbb{R}) \times \{\hat{\phi} \in C^{2,\beta}_{\kappa,\mathrm{e}}(\overline{\hat{\Omega}}) : \hat{\phi}|_{s=0} = \hat{\phi}|_{s=1} = 0\}$$

and

$$Y = Y_1 \times Y_2 \coloneqq C^{1,\beta}_{\kappa,\mathrm{e}}(\mathbb{R}) \times C^{\beta}_{\kappa,\mathrm{e}}(\overline{\hat{\Omega}}).$$

We will write $w = (\eta, \hat{\phi})$ for elements of X. To capture our assumptions, it is convenient to define the sets

$$\mathcal{O} \coloneqq \{w \in X : \min \eta > -1\},$$

and, to enforce that $\alpha < 0$ and (2.8) hold,

$$\mathcal{U} \coloneqq \Big\{(\mu, \alpha, \lambda) \in \mathbb{R}^3 : \mu \neq 0,\ \alpha < 0,\ 0 < \lambda < \pi\Big\}.$$

We now define the map $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)\colon \mathcal{O} \times \mathcal{U} \to Y$ by

$$\mathcal{F}_1(w, \Lambda) \coloneqq \frac{(1+\eta_x^2)(\psi_{0s} + \hat{\phi}_s)^2}{2(1+\eta)^2} + \eta - Q(\Lambda), \tag{2.10a}$$

$$\mathcal{F}_2(w, \Lambda) \coloneqq \left(\partial_x - \frac{s\eta_x}{1+\eta}\partial_s\right)^2 (\psi_0 + \hat{\phi}) + \frac{\psi_{0ss} + \hat{\phi}_{ss}}{(1+\eta)^2} - \alpha(\psi_0 + \hat{\phi}),$$

where ψ_0 is as in Equation (2.6) and $Q(\Lambda)$ is given in (2.7). In (2.10a), it is understood that the functions ψ_{0s} and $\hat{\phi}_s$ are evaluated at $s = 1$. It is clear that $\mathcal{F}$ is well defined and smooth as a map $\mathcal{O} \times \mathcal{U} \to Y$. We wish to solve the equation

$$\mathcal{F}(w, \Lambda) = 0. \tag{2.11}$$

We obtain the linearized problem by taking the partial derivative of $\mathcal{F}$ with respect to w at the point $(0, \Lambda)$. This yields

$$D_w\mathcal{F}_1(0, \Lambda)w = \psi_{0s}\hat{\phi}_s - \psi_{0s}^2\eta + \eta, \tag{2.12a}$$

$$D_w\mathcal{F}_2(0, \Lambda)w = (\Delta - \alpha)\hat{\phi} - s\psi_{0s}\eta_{xx} - 2\psi_{0ss}\eta,$$

where it again is understood that the functions are evaluated at $s = 1$ in (2.12a). By introducing an isomorphism, in Proposition 2.8 below, we can transform $D_w\mathcal{F}$ into a simpler elliptic operator. For this purpose, define

$$\tilde{X}_2 \coloneqq \{\phi \in C^{2,\beta}_{\kappa,\mathrm{e}}(\overline{\hat{\Omega}}) : \phi|_{s=0} = 0\}, \quad \tilde{X} \coloneqq X_1 \times \tilde{X}_2,$$

where we have the inclusion $X \subset \tilde{X} \subset Y$. We will typically use the letter ϕ for elements of $\tilde{X}_2$.

Proposition 2.8 (The $\mathcal{T}$ isomorphism [12])**.** *The bounded linear operator $\mathcal{T}(\Lambda) : \tilde{X}_2 \to X$ defined by*

$$\mathcal{T}(\Lambda)\phi = (\eta_\phi, \hat{\phi}) \coloneqq \left(-\frac{\phi|_{s=1}}{\psi_{0s}(1)}, \phi - \frac{s\psi_{0s}}{\psi_{0s}(1)}\phi|_{s=1}\right)$$

is an isomorphism of Banach spaces, and the operator

$$\mathcal{L}(\Lambda) = (\mathcal{L}_1(\Lambda), \mathcal{L}_2(\Lambda)) \coloneqq D_w\mathcal{F}(0, \Lambda)\mathcal{T}(\Lambda) : \tilde{X}_2 \to Y$$

satisfies

$$\mathcal{L}(\Lambda)\phi = \left(\left[\psi_{0s}\phi_s - \left(\psi_{0ss} + \frac{1}{\psi_{0s}}\right)\phi\right]_{s=1}, (\Delta - \alpha)\phi\right). \tag{2.13}$$

Proof. That $\mathcal{T}$ is well defined and an isomorphism is almost immediate. The expression for $\mathcal{L}(\Lambda)$ in (2.13) follows by direct computation. □

3 The kernel and dimensional reduction

Introduce the complex parameter

$$\theta_n = \theta(n,\alpha) := \sqrt{\alpha + n^2\kappa^2} = \begin{cases} \sqrt{n^2\kappa^2 - |\alpha|}, & n \geq |\alpha|^{1/2}/\kappa, \\ i\sqrt{|\alpha| - n^2\kappa^2} & n < |\alpha|^{1/2}/\kappa, \end{cases}$$

for nonnegative integers n. This parameter will appear in functions of the form $\cosh(\theta_n s)$ and $\sinh(\theta_n s)/\theta_n$, which are always real-valued. We record that

$$\cosh(\theta_n s) = \begin{cases} \cosh(|\theta_n| s), & n \geq |\alpha|^{1/2}/\kappa, \\ \cos(|\theta_n| s), & n < |\alpha|^{1/2}/\kappa, \end{cases}$$
$$\frac{\sinh(\theta_n s)}{\theta_n} = \begin{cases} \sinh(|\theta_n| s)/|\theta_n|, & n \geq |\alpha|^{1/2}/\kappa, \\ \sin(|\theta_n| s)/|\theta_n|, & n < |\alpha|^{1/2}/\kappa. \end{cases}$$

In the event that $\theta_n = 0$, we will interpret expressions with θ_n as extended by continuity. In particular, $\sinh(\theta_n s)/\theta_n$ is interpreted as s.

We now describe the kernel of $\mathcal{L}(\Lambda)$, which is directly related to the kernel of $D_w\mathcal{F}(0,\lambda)$ through $\mathcal{T}(\Lambda)$, in terms of the above functions. The following proposition is stated, but not proved, in [12]. We include its proof because it is instructive.

Proposition 2.9 (Kernel of $\mathcal{L}(\Lambda)$ [12])**.** *Let $\Lambda \in \mathcal{U}$. A basis for* $\ker \mathcal{L}(\Lambda)$ *is then given by $\{\phi_n\}_{n\in M}$, where*

$$\phi_n(x,s) := \cos(n\kappa x)\frac{\sinh(\theta_n s)}{\theta_n} \tag{2.14}$$

and $M = M(\Lambda)$ is the finite set of all $n \in \mathbb{N}_0$ satisfying the kernel equation

$$l(n,\alpha) = r(\Lambda), \tag{2.15}$$

where

$$\begin{aligned} l(n,\alpha) &:= \theta_n \coth(\theta_n), \\ r(\Lambda) &:= \frac{1}{\mu^2|\alpha|\sin^2(\lambda)} + |\alpha|^{1/2}\cot(\lambda). \end{aligned} \tag{2.16}$$

Proof. Suppose that $\phi \in \ker \mathcal{L}(\Lambda)$, and expand it in a Fourier series

$$\phi(x,s) = \sum_{n=0}^{\infty} a_n(s)\cos(n\kappa x).$$

From $\mathcal{L}_2(\Lambda)\phi = 0$, we deduce that the coefficients satisfy

$$a_n''(s) - \theta_n^2 a_n(s) = 0, \quad s \in (0,1), \tag{2.17}$$

while $\phi|_{s=0} = 0$ and $\mathcal{L}_1(\Lambda)\phi = 0$ yield the boundary conditions

$$a_n(0) = 0, \tag{2.18a}$$

$$\psi_{0s}(1)a_n'(1) - \left(\psi_{0ss}(1) + \frac{1}{\psi_{0s}(1)}\right)a_n(1) = 0, \tag{2.18b}$$

for all $n \geq 0$.

The general solution of (2.17) with the boundary condition (2.18a) is

$$a_n(s) = B_n \frac{\sinh(\theta_n s)}{\theta_n}, \quad B_n \in \mathbb{R},\, n \geq 0,$$

for which the Robin condition (2.18b) reduces to

$$\left(\psi_{0s}(1)\cosh(\theta_n) - \left(\psi_{0ss}(1) + \frac{1}{\psi_{0s}(1)}\right)\frac{\sinh(\theta_n)}{\theta_n}\right)B_n = 0.$$

Hence, if B_n (and thus a_n) is nonzero, then

$$\psi_{0s}(1)\cosh(\theta_n) - \left(\psi_{0ss}(1) + \frac{1}{\psi_{0s}(1)}\right)\frac{\sinh(\theta_n)}{\theta_n} = 0 \tag{2.19}$$

must hold. Observe that Equation (2.19) implies that $\sinh(\theta_n)/\theta_n \neq 0$; otherwise we would have $\cosh(\theta_n) = \sinh(\theta_n) = 0$, and therefore $\exp(\theta_n) = 0$. Thus, by inserting the definition (2.6) of ψ_0 into Equation (2.19), we arrive at (2.15). This condition is also sufficient for ϕ_n to lie in the kernel.

The set M of $n \in \mathbb{N}_0$ such that (2.15) holds is finite, because the function $l(\cdot, \alpha)$ is strictly increasing as soon as $n \geq |\alpha|^{1/2}/\kappa$. $\square$

Remark 2.10. In order to get nontrivial solutions, Λ should be chosen such that $0 \notin M(\Lambda)$. The function ϕ_0, see (2.14), does not depend on x.

The next lemma, inspired by [20, Theorem IV.5.17], serves to show that the set-valued map $M\colon \mathcal{U} \to 2^{\mathbb{N}_0}$ defined in Proposition 2.9 is upper semicontinuous. This implies that no new solutions of the kernel equation (2.15) can appear if Λ is perturbed slightly.

Lemma 2.11 (Upper semicontinuity)**.** *Let* $\Lambda^* \in \mathcal{U}$. *Then*

$$M(\Lambda) \subset M(\Lambda^*)$$

for all Λ *in a neighborhood of* Λ^*.

Proof. Suppose that this is not the case. Then we can construct a sequence $(\Lambda_i)_{i\in\mathbb{N}}$ converging to Λ^*, and a corresponding sequence $(n_i)_{i\in\mathbb{N}}$ such that $n_i \notin M(\Lambda^*)$ and $l(n_i, \alpha_i) = r(\Lambda_i)$ for all $i \in \mathbb{N}$. By the continuity of r at Λ^*, the sequence $(r(\Lambda_i))_{i\in\mathbb{N}}$, and therefore $(l(n_i, \alpha_i))_{i\in\mathbb{N}}$, is bounded. This implies that $(n_i)_{i\in\mathbb{N}}$ is bounded, so we may assume that it is constant. Thus there is an $n \notin M(\Lambda^*)$ such that $l(n, \alpha_i) = r(\Lambda_i)$ for all $i \in \mathbb{N}$. The boundedness of the sequences now ensures that $l(n, \cdot)$ is well-defined and continuous at α^*. We conclude that $l(n, \alpha^*) = r(\Lambda^*)$, which contradicts $n \notin M(\Lambda^*)$. $\square$

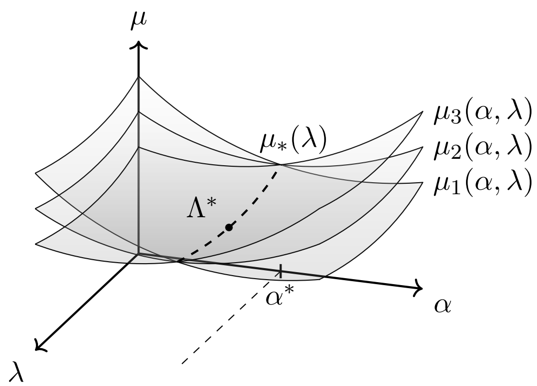

Figure 2.1: An illustration of Theorem 2.12 when $|M(\Lambda^*)| = 3$. A full description of the kernel can be given near Λ^*.

We can now use Lemma 2.11 to give a local description of the structure of the kernel equation. This will be useful when describing the solution set of (2.11). See also Figure 2.1.

Theorem 2.12 (Local description)**.** *Suppose that* $M(\Lambda^*) = \{n_1, \ldots, n_N\}$. *Then we may define*

$$\mu_i(\alpha, \lambda) := \frac{\operatorname{sgn}(\mu^*)}{|\alpha|^{1/2}\sin(\lambda)(l(n_i, \alpha) - |\alpha|^{1/2}\cot(\lambda))^{1/2}}, \quad 1 \le i \le N, \quad (2.20)$$

on a neighborhood of (α^*, λ^*),

$$\mu_*(\lambda) := \mu_1(\alpha^*, \lambda) \quad (= \cdots = \mu_N(\alpha^*, \lambda))$$

on a neighborhood of λ^*, and we have

$$M(\Lambda) = \begin{cases} M(\Lambda^*) & \alpha = \alpha^*, \mu = \mu_*(\lambda), \\ \{n_i\} & \alpha \neq \alpha^*, \mu = \mu_i(\alpha, \lambda), \\ \varnothing & \textit{otherwise}, \end{cases}$$

for all Λ in a neighborhood of Λ^*.

Proof. By Lemma 2.11 there is a neighborhood of Λ^* in which $M(\Lambda)$ is the set of $n_i \in M(\Lambda^*)$ for which $l(n_i, \alpha) = r(\Lambda)$. Observe now that Λ sufficiently close to Λ^* we have $l(n_i, \alpha) = r(\Lambda)$ if and only if $\mu = \mu_i(\alpha, \lambda)$, where μ_i is as in (2.20). Moreover, if $i \neq j$ then $l(n_i, \cdot) - l(n_j, \cdot)$ is a nonzero analytic function on a neighborhood of α^*. It follows that we may choose the neighborhood of Λ^* in such a way that the only intersection of the graphs of the μ_i occurs when $\alpha = \alpha^*$. □

The bifurcation results in Sections 4 and 5 are valid under the assumption that $\ker \mathcal{L}(\Lambda)$ is respectively one- and two-dimensional. Lemma 2.13 below is a general result on the kernel equation (2.15) from [12], which in particular shows that it is indeed possible to choose $\Lambda \in \mathcal{U}$ such that the dimension of the kernel is one or two.

Lemma 2.13 (Kernel equation [12])**.**

(i) For every α and any n for which $l(n, \alpha)$ is well-defined there are μ and λ such that $n \in M(\Lambda)$.

(ii) Suppose that $\lambda \in [\pi/2, \pi)$ and that $n_1, n_2 \in \mathbb{N}_0$ satisfy

$$n_2^2 \geq n_1^2 + \left(\frac{3\pi}{2\kappa}\right)^2$$

Then there are α and μ such that $n_1, n_2 \in M(\Lambda)$ and any other solution of (2.15) *must be smaller than n_1.*

It is, however, the case that higher-dimensional kernels are, in a sense, rare[1]:

[1] A variation of this was pointed out already in [12]; however, not taking into account the points where l is not well-defined. We slightly improve upon the result here. Both J and its closure are small.

Lemma 2.14. *Let J be set of all values of α for which there exist λ and μ such that $|M(\Lambda)| \geq 2$. Then the limit points of J are contained in the set*

$$\{-(m_1^2\kappa^2 + m_2^2\pi^2) : m_1 \in \mathbb{N}_0, m_2 \in \mathbb{N}\}. \tag{2.21}$$

In particular, J consists of isolated points, except possibly those that lie in the set defined in (2.21)*, and has countable closure.*

Proof. The set defined in (2.21) consists precisely of the values of α for which there is at least one $n \in \mathbb{N}_0$ such that $l(n, \alpha)$ is not well-defined. Let α be such that it is *not* in this set.

Suppose, to the contrary, that there is a sequence $(\alpha_i)_{i\in\mathbb{N}}$ satisfying $\alpha_i \neq \alpha$ for all $i \in \mathbb{N}$ and which converges to α, with corresponding sequences $(n_{1,i})_{i\in\mathbb{N}}$ and $(n_{2,i})_{i\in\mathbb{N}}$ such that $n_{1,i} < n_{2,i}$ and

$$l(n_{1,i}, \alpha_i) = l(n_{2,i}, \alpha_i)$$

for all $i \in \mathbb{N}$. We must necessarily have

$$n_{1,i} \leq |\alpha_i|^{1/2}/\kappa$$

for all $i \in \mathbb{N}$, and so the sequence $(n_{1,i})_{i\in\mathbb{N}}$ is bounded. The continuity of $l(n, \cdot)$ at α for each $n \in \mathbb{N}_0$ now implies that $(l(n_{1,i}, \alpha_i))_{i\in\mathbb{N}}$, and therefore also $(l(n_{2,i}, \alpha_i))_{i\in\mathbb{N}}$, is bounded. This, in turn, implies that $(n_{2,i})_{i\in\mathbb{N}}$ is bounded.

By going to a subsequence, we may assume that both $(n_{1,i})_{i\in\mathbb{N}}$ and $(n_{2,i})_{i\in\mathbb{N}}$ are constant. Thus there are $n_1 < n_2 \in \mathbb{N}_0$ such that $l(n_1, \alpha) = l(n_2, \alpha)$ and

$$l(n_1, \alpha_i) = l(n_2, \alpha_i)$$

for all $i \in \mathbb{N}$. But this is impossible, because $l(n_2, \cdot) - l(n_1, \cdot)$ is a nonconstant analytic function in a neighborhood of α. □

For later use, we give some explicit examples of one- and two-dimensional kernels for $\mathcal{L}(\Lambda)$. All satisfy $r(\Lambda) = 1$, and the two-dimensional examples have been chosen such that $\theta(n_2, \alpha) = 0$. To simplify the parameters involved, we choose specific values of κ.

Example 2.15 (Explicit kernels)**.** Let σ be the smallest positive solution of the equation $x \cot(x) = 1$.

(i) When $\kappa = 1$, $\mu = 1$, $\alpha = -1$ and $\lambda = \pi/2$, the kernel is one-dimensional, being spanned by $\phi_1(x, s) = \cos(x)s$.

(ii) Let $\kappa = \sigma/\sqrt{3}$. When $\mu = 1/(2\kappa)$, $\alpha = -4\kappa^2$ and $\lambda = \pi/2$, the kernel is two-dimensional, with $M = \{1, 2\}$.

(iii) Let $\kappa = \sigma/\sqrt{5}$. When $\mu = 1/(3\kappa)$, $\alpha = -9\kappa^2$ and $\lambda = \pi/2$, the kernel is two-dimensional, with $M = \{2, 3\}$.

Arbitrarily large kernels

We now address a question that was raised in [12]: Do there exist $\Lambda \in \mathcal{U}$ such that $\ker D_w\mathcal{F}(0, \Lambda)$ is at least three-dimensional? By also letting the wave number κ vary, this question was answered in the affirmative for dimension three in [13]. In essence, their result says that many two-dimensional kernels can be modified in order to yield a three-dimensional kernel. Here, we use a different approach to find kernels of arbitrary dimension for any κ in a set K that is dense in $(0, \infty)$.

For any $\alpha < 0$ and $\lambda \in (\pi/2, \pi)$, we can obtain $r(\Lambda) = 0$ by choosing μ to satisfy

$$\mu^2 = -\frac{2}{|\alpha|^{3/2}\sin(2\lambda)},$$

which reduces the kernel equation (2.15) to finding $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$ such that

$$\sqrt{|\alpha| - (n\kappa)^2} = \left(m - \frac{1}{2}\right)\pi,$$

is satisfied. This can be written in the form

$$\left(2n\frac{\kappa}{\pi}\right)^2 + (2m-1)^2 = \frac{4|\alpha|}{\pi^2}. \tag{2.22}$$

We first consider the case $\kappa = \pi$.

Lemma 2.16 (Arbitrary kernel with $\kappa = \pi$)**.** *For $\kappa = \pi$ and any $N \in \mathbb{N}$, there exist $\Lambda \in \mathcal{U}$ such that $|M(\Lambda)| = N$ and $0 \notin M(\Lambda)$.*

Proof. When $\kappa = \pi$ and $\alpha = -\pi^2 H/4$ for an odd number $H \in \mathbb{N}$, (2.22) becomes the Diophantine equation

$$(2n)^2 + (2m-1)^2 = H. \tag{2.23}$$

The size of the kernel then corresponds to the number of representations of H as the sum of two squares. As long as H is not a square number, any such representation has $n \neq 0$ (see Remark 2.10).

In order to conclude, we therefore need to find an odd non-square number H such that H has exactly N representations as a sum of squares. By [16, Theorem 3 in Chapter 2], this is the case for instance when $H = p^{2N-1}$ for a prime $p \in 4\mathbb{N}+1$. $\square$

Some examples of Lemma 2.16 are listed below, for various choices of H in (2.23). The values of H used in second and third example, which are the smallest possible, are not in the form p^{2N-1}. They can easily be deduced by using the general formula given in [16].

Example 2.17 (Higher-dimensional kernels)**.**

(i) The choice $H = 5^{2\cdot 3-1} = 3125$ yields a three-dimensional kernel, with $M = \{5, 19, 25\}$. However, this is not the smallest example:

(ii) Since $325 = 6^2 + 17^2 = 10^2 + 15^2 = 18^2 + 1^2$ (with no other representations), the choice $H = 325$ yields a three-dimensional kernel, with $M = \{3, 5, 9\}$.

(iii) Since $1105 = 4^2 + 33^2 = 12^2 + 31^2 = 24^2 + 23^2 = 32^2 + 9^2$, the choice $H = 1105$ yields a four-dimensional kernel, with $M = \{2, 6, 12, 16\}$.

Let $\mathbb{Q}_{\mathrm{o}}^+$ denote the set of positive rational numbers with odd numerators when reduced to lowest terms. We can then generalize Lemma 2.16 in the following way:

Theorem 2.18 (Arbitrary kernel)**.** *For $\kappa \in \pi\mathbb{Q}_{\mathrm{o}}^+$ and any $N \in \mathbb{N}$, there exist $\Lambda \in \mathcal{U}$ such that $|M(\Lambda)| = N$ and $0 \notin M(\Lambda)$.*

Proof. Write $\kappa = \pi r/s$, with r and s coprime. When $\alpha = -\pi^2 r^2 H/4$, (2.22) becomes

$$r^2(2n)^2 + s^2(2m-1)^2 = r^2 s^2 H. \tag{2.24}$$

Choosing $H = p^{2N-1}$ for a prime $p \in 4\mathbb{N}+1$, we know that (2.23) has exactly N solutions $(\tilde{m}_j, \tilde{n}_j)$ in $\mathbb{N}^2$. The pairs $(m_j, n_j) \in \mathbb{N}^2$ defined by

$$2m_j - 1 = r(2\tilde{m}_j - 1), \quad n_j = s\tilde{n}_j$$

then solve Equation (2.24).

Moreover, these are the only solutions: Suppose that (m, n) solves Equation (2.24). Then $r \mid (2m-1)$ and $s \mid 2n$, by coprimality of r and s. It follows that $2m - 1 = r(2\tilde{m} - 1)$ and $2n = s\hat{n}$, where $\tilde{m}$ and $\hat{n}$ solve $\hat{n}^2 + (2\tilde{m} - 1)^2 = H$. Since H is odd, $\hat{n} = 2\tilde{n}$. Uniqueness in Equation (2.23) now yields the result. $\square$

Although we provide kernels of arbitrary dimension in Theorem 2.18, the corresponding triples Λ satisfy $r(\Lambda) = 0$, unlike the three-dimensional kernels obtained in [13]. In particular, this means that the two-dimensional bifurcation result in Theorem 2.34 does not apply for the kernels from Theorem 2.18 with $N = 2$. We remark that an obstacle for higher-dimensional bifurcation is that there are only four parameters to work with, namely Λ and κ. This may be remedied by for instance including surface tension.

An application of Theorem 2.18 is one-dimensional bifurcation with several different wave numbers for fixed Λ. If the set $M(\Lambda) = \{n_1, \ldots, n_N\}$ is such $n_i \nmid n_j$ for all $i \neq j$, we can make restrictions to each $X^{(n_i)}$ and then apply Theorem 2.24. This will yield N different solution curves. Two examples for which the condition on M is fulfilled are $H = 725$ and $H = 3145$, corresponding to $M = \{5, 7, 13\}$ and $M = \{18, 24, 26, 28\}$, respectively.

Lyapunov–Schmidt reduction

Before we can reduce (2.11) to a finite-dimensional problem by applying the Lyapunov–Schmidt reduction, we need the following result from elliptic theory.

Theorem 2.19 (Fredholm property [12])**.** *The operator $\mathcal{L}(\Lambda)$ is Fredholm for each $\Lambda \in \mathcal{U}$, with index 0. The range of $\mathcal{L}(\Lambda)$ is the orthogonal complement of*

$$Z := \big\{(\eta_\phi, \phi) : \phi \in \ker \mathcal{L}(\Lambda)\big\} \subset \tilde{X} \subset Y$$

in Y with respect to the inner product

$$\langle w_1, w_2 \rangle_Y = \int_0^{2\pi/\kappa} \eta_1 \eta_2 \, dx + \int_0^1 \int_0^{2\pi/\kappa} \hat{\phi}_1 \hat{\phi}_2 \, dx \, ds, \quad w_j \in Y. \tag{2.25}$$

Let $\tilde{w}_n := (\eta_{\phi_n}, \phi_n)$ for $n \in M(\Lambda)$, where ϕ_n and $M(\Lambda)$ are as in Proposition 2.9. Then the projection $\Pi_Z : Y \to Z$ onto Z along $\operatorname{ran} \mathcal{L}(\Lambda)$ is given by

$$\Pi_Z w = \sum_{n \in M(\Lambda)} \frac{\langle w, \tilde{w}_n \rangle_Y}{\|\tilde{w}_n\|_Y^2} \tilde{w}_n. \tag{2.26}$$

Remark 2.20. That $D_w \mathcal{F}(0, \Lambda)$ be Fredholm is the main reason for making the assumption (2.8). When we have equality in (2.8), (2.12a) reduces to $D_w \mathcal{F}_1(0, \Lambda) w = \eta$, whence $\operatorname{ran} D_w \mathcal{F}_1(0, \Lambda) = X_1$. The operator $D_w \mathcal{F}(0, \Lambda)$ then cannot be Fredholm, since X_1 is not closed in Y_1.

Let $\Lambda^* \in \mathcal{U}$ be a triple $(\mu^*, \alpha^*, \lambda^*)$ such that $N \coloneqq |M(\Lambda^*)| \geq 1$. Then Proposition 2.9 says that the pairs

$$w_n^* \coloneqq \mathcal{T}(\Lambda^*)\phi_n^* \in X, \quad n \in M(\Lambda^*),$$

span the kernel of $D_w\mathcal{F}(0, \Lambda^*)$. Since the kernel is finite-dimensional, there exists a closed subspace $X_0 \subset X$ such that

$$X = \ker D_w\mathcal{F}(0, \Lambda^*) \oplus X_0.$$

By Theorem 2.19, we can also decompose Y into the direct sum

$$Y = Z \oplus \operatorname{ran} \mathcal{L}(\Lambda^*),$$

where $Z \coloneqq \operatorname{span}\{\tilde{w}_n^*\}_{n \in M(\Lambda^*)}$, since these are orthogonal complements in the inner product (2.25) on Y. Applying the Lyapunov–Schmidt reduction (see e.g. Kielhöfer [21]) for these decompositions of X and Y, we obtain the following lemma.

Lemma 2.21 (Lyapunov–Schmidt)**.** *There exist open neighborhoods $\mathcal{N}$ of 0 in $\ker D_w\mathcal{F}(0, \Lambda^*)$, $\mathcal{M}$ of 0 in X_0, and $\mathcal{U}'$ of Λ^* in $\mathcal{U}$, and a uniquely determined function $\psi : \mathcal{N} \times \mathcal{U}' \to \mathcal{M}$ such that*

$$\mathcal{F}(w, \Lambda) = 0 \qquad \textit{for } w \in \mathcal{N} + \mathcal{M},\ \Lambda \in \mathcal{U}',$$

if and only if $w = w^* + \psi(w^*, \Lambda)$ and $w^* = \sum_{n \in M(\Lambda^*)} t_n w_n^* \in \mathcal{N}$ solves the finite-dimensional problem

$$\Phi(t, \Lambda) = 0 \qquad \textit{for } t \in \mathcal{V},\ \Lambda \in \mathcal{U}',$$

where

$$\Phi(t, \Lambda) \coloneqq \Pi_Z \mathcal{F}(w, \Lambda) \quad \textit{and} \quad \mathcal{V} \coloneqq \Big\{(t_n)_{n \in M(\Lambda^*)} \in \mathbb{R}^N : w^* \in \mathcal{N}\Big\}.$$

The function ψ is smooth, and satisfies $\psi(0, \Lambda) = 0$ for all $\Lambda \in \mathcal{U}'$, and $D_w\psi(0, \Lambda^*) = 0$.

4 One-dimensional bifurcation

We are now in a position to show that a curve of nontrivial solutions of (2.11) bifurcates from each point $(0, \Lambda^*) \in X \times \mathcal{U}$ where the kernel of $D_w\mathcal{F}(0, \Lambda^*)$ is one-dimensional, given that Λ^* satisfies an additional technical condition. This condition comes from Lemma 2.22 below.

Lemma 2.22 (Orthogonality)**.** *Suppose that $n \in M(\Lambda)$, so that the function ϕ_n given by (2.14) lies in* $\ker \mathcal{L}(\Lambda)$*. Then, if $\tilde{w}_n := (\eta_{\phi_n}, \phi_n)$ is the corresponding basis function of Z, we have*

$$\langle D_\lambda \mathcal{L}(\Lambda)\phi_n, \tilde{w}_n \rangle_Y = A\left(\frac{\sinh(\theta_n)}{\theta_n}\right)^2, \tag{2.27}$$

where

$$A := -\frac{2\pi}{\kappa \psi_{0s}(1)^2}\left[\cot(\lambda) + \frac{\mu^2 |\alpha|^{3/2}}{2}\right]$$

does not depend on n. In particular,

$$\langle D_\lambda \mathcal{L}(\Lambda)\phi_n, \tilde{w}_n \rangle_Y = 0 \quad \textit{if and only if} \quad \cot(\lambda) = -\frac{\mu^2|\alpha|^{3/2}}{2}.$$

Proof. Recalling (2.13), we find the derivative

$$D_\lambda \mathcal{L}(\Lambda)\phi = \left(\psi_{0s\lambda}(1)\phi_s|_{s=1} - \left(\psi_{0ss\lambda}(1) - \frac{\psi_{0s\lambda}(1)}{\psi_{0s}(1)^2}\right)\phi|_{s=1}, 0\right).$$

Using that $\phi_n(x,s) = \cos(n\kappa x)\sinh(\theta_n s)/\theta_n$, we get

$$D_\lambda \mathcal{L}_1(\Lambda)\phi_n = \tilde{A}\frac{\sinh(\theta_n)}{\theta_n}\cos(n\kappa x),$$

where

$$\begin{aligned}\tilde{A} &:= \psi_{0s\lambda}(1) l(n,\alpha) - \left(\psi_{0ss\lambda}(1) - \frac{\psi_{0s\lambda}(1)}{\psi_{0s}(1)^2}\right)\\ &= \frac{2}{\psi_{0s}(1)}\left(\cot(\lambda) + \frac{\mu^2|\alpha|^{3/2}}{2}\right),\end{aligned}$$

by the kernel equation (2.15) and the definition of ψ_0.

Since $\tilde{w}_n = (\eta_{\phi_n}, \phi_n)$, with

$$\eta_{\phi_n}(x) = -\frac{\phi_n(x,1)}{\psi_{0s}(1)} = -\frac{1}{\psi_{0s}(1)}\frac{\sinh(\theta_n)}{\theta_n}\cos(n\kappa x),$$

we now find

$$\begin{aligned}\langle D_\lambda \mathcal{L}(\Lambda)\phi_n, \tilde{w}_n \rangle_Y &= \int_0^{2\pi/\kappa} \eta_{\phi_n} D_\lambda \mathcal{L}_1(\Lambda)\phi_n \, dx\\ &= -\frac{\pi \tilde{A}}{\kappa \psi_{0s}(1)}\left(\frac{\sinh(\theta_n)}{\theta_n}\right)^2,\end{aligned}$$

which is (2.27) with $A = -\pi\tilde{A}/(\kappa\psi_{0s}(1))$. ∎

We will refer to

$$\cot(\lambda) \neq -\frac{\mu^2|\alpha|^{3/2}}{2} \tag{2.28}$$

as *the transversality condition*, because it corresponds to transversality in the Crandall–Rabinowitz theorem (see [9] or [21]). Note that all the examples we provided in Example 2.15 satisfy this condition. It is straightforward to check that the transversality condition fails at $\Lambda^* \in \mathcal{U}$ precisely when $\mu_*'(\lambda^*) = 0$ in Theorem 2.12. This means that we can obtain the following by moving slightly along the graph of μ_* (see Figure 2.1).

Lemma 2.23. *Suppose that $\Lambda^* = (\mu^*, \alpha^*, \lambda^*) \in \mathcal{U}$ is such that the transversality condition* (2.28) *fails. Then there are $\mu, \lambda \in \mathbb{R}$ with $\Lambda = (\mu, \alpha^*, \lambda) \in \mathcal{U}$ such that the transversality condition holds and $M(\Lambda) = M(\Lambda^*)$. The triple Λ can be chosen arbitrarily close to Λ^*.*

The one-dimensional bifurcation result is an application of the Crandall–Rabinowitz bifurcation theorem. To clarify the proof of the two-dimensional bifurcation in the next section, we will nonetheless spell out the details of the proof.

Theorem 2.24 (One-dimensional bifurcation)**.** *Suppose that $\Lambda^* \in \mathcal{U}$ is such that $M(\Lambda^*) = \{n\}$ with $n \in \mathbb{N}$, and therefore that*

$$\ker D_w\mathcal{F}(0, \Lambda^*) = \operatorname{span}\{w^*\},$$

where $w^* = \mathcal{T}(\Lambda^*)\phi^*$, with $\phi^* := \phi_n$ as in Proposition 2.9. If the transversality condition (2.28) *holds, there exists a smooth curve $\{(\overline{w}(t), \overline{\lambda}(t)) : 0 < |t| < \varepsilon\}$ of nontrivial small-amplitude solutions to*

$$\mathcal{F}(w, \mu^*, \alpha^*, \lambda) = 0, \tag{2.29}$$

in $\mathcal{O} \times (0, \pi)$, passing through $(\overline{w}(0), \overline{\lambda}(0)) = (0, \lambda^*)$, with

$$\overline{w}(t) = tw^* + O(t^2) \quad \text{in } X \text{ as } t \to 0. \tag{2.30}$$

These are all the nontrivial solutions of (2.29) *in a neighborhood of $(0, \lambda^*)$ in $\mathcal{O} \times (0, \pi)$.*

Proof. Using Lemma 2.21, we know that there exists a neighborhood of $(0, \lambda^*)$ in $\mathcal{O} \times (0, \pi)$ for which the equation $\mathcal{F}(w, \mu^*, \alpha^*, \lambda) = 0$ is equivalent to $\Phi(t, \mu^*, \alpha^*, \lambda) = 0$, where $t \in \mathbb{R}$. From the same lemma we also have the identity $\Phi(0, \Lambda) = 0$, and hence we can write

$$\Phi(t, \Lambda) = \int_0^1 \partial_z\big(\Phi(tz, \Lambda)\big)\, dz = t\Psi(t, \Lambda),$$

where

$$\Psi(t,\Lambda) := \int_0^1 \Phi_t(tz,\Lambda)\,dz \tag{2.31}$$

is smooth. For nontrivial solutions ($t \neq 0$), the equations $\Phi = 0$ and $\Psi = 0$ are equivalent, whence we need only concern ourselves with the latter equation.

We want to apply the implicit function theorem to Ψ, which requires that $\Psi(0,\Lambda^*) = 0$ and $\Psi_\lambda(0,\Lambda^*) \neq 0$ (recall that Z is one-dimensional). Now, from (2.31), we find

$$\begin{aligned}\Psi(0,\Lambda^*) &= \Phi_t(0,\Lambda^*),\\ \Psi_\lambda(0,\Lambda^*) &= \Phi_{t\lambda}(0,\Lambda^*),\end{aligned}$$

so these are the derivatives of Φ we need to compute. By the definition of Φ,

$$\Phi_t(t,\Lambda) = \Pi_Z D_w\mathcal{F}(tw^* + \psi(tw^*,\Lambda),\Lambda)(w^* + D_w\psi(tw^*,\Lambda)w^*), \tag{2.32}$$

and so by evaluating in $t = 0$, and using the properties of ψ listed in Lemma 2.21, we have

$$\Phi_t(0,\Lambda) = \Pi_Z D_w\mathcal{F}(0,\Lambda)(w^* + D_w\psi(0,\Lambda)w^*),$$

which also yields

$$\begin{aligned}\Phi_{t\lambda}(0,\Lambda) = \Pi_Z D_{w\lambda}\mathcal{F}(0,\Lambda)(w^* + D_w\psi(tw^*,\Lambda))&\\ + \Pi_Z D_w\mathcal{F}(0,\Lambda)D_{w\lambda}\psi(w^*,\Lambda).&\end{aligned}$$

We now obtain

$$\Psi(0,\Lambda^*) = \Phi_t(0,\Lambda^*) = \Pi_Z D_w\mathcal{F}(0,\Lambda^*)w^* = 0,$$

because $D_w\psi(0,\Lambda^*) = 0$ by the last part of Lemma 2.21, and because Π_Z projects along $\operatorname{ran} D_w\mathcal{F}(0,\Lambda^*)$. Similarly,

$$\Phi_{t\lambda}(0,\Lambda^*) = \Pi_Z D_{w\lambda}\mathcal{F}(0,\Lambda^*)w^*. \tag{2.33}$$

Note that $D_w\mathcal{F}(0,\Lambda)w^* = \mathcal{L}(\Lambda)\mathcal{T}(\Lambda)^{-1}w^*$, and hence

$$D_{w\lambda}\mathcal{F}(0,\Lambda^*)w^* = D_\lambda\mathcal{L}(\Lambda^*)\phi^* - D_w\mathcal{F}(0,\Lambda^*)\partial_\lambda\mathcal{T}(\Lambda^*)\phi^*,$$

which implies that Equation (2.33) can be written

$$\Phi_{t\lambda}(0,\Lambda^*) = \Pi_Z D_\lambda\mathcal{L}(\Lambda^*)\phi^*,$$

again using that Π_Z projects along $\operatorname{ran} D_w\mathcal{F}(0,\Lambda^*)$. We can now use Lemma 2.22 to deduce that $\Psi_\lambda(0,\Lambda^*) = \Phi_{t\lambda}(0,\Lambda^*) \neq 0$, due to the assumption of transversality.

Finally, since $\Psi(0,\Lambda^*) = 0$ and $\Psi_\lambda(0,\Lambda^*) \neq 0$, we can invoke the implicit function theorem to deduce that there exists an $\varepsilon > 0$ and a smooth function $\overline{\lambda}\colon (-\varepsilon,\varepsilon) \to \mathbb{R}$ with $\overline{\lambda}(0) = \lambda^*$ such that $\Psi(t,\mu^*,\alpha^*,\overline{\lambda}(t)) \equiv 0$. Moreover, the curve $\{(t,\overline{\lambda}(t)) : |t| < \varepsilon\}$ describes all solutions to $\Psi(t,\mu^*,\alpha^*,\lambda) = 0$ in a neighborhood of $(0,\lambda^*)$. The corresponding solution curve to $\mathcal{F}(w,\mu^*,\alpha^*,\lambda) = 0$ is $\{(\overline{w}(t),\overline{\lambda}(t)) : |t| < \varepsilon\}$, where $\overline{w}(t) := tw^* + \psi(tw^*,\mu^*,\alpha^*,\overline{\lambda}(t))$. It follows that

$$\dot{\overline{w}}(t) = w^* + D_w\psi(tw^*,\mu^*,\alpha^*,\overline{\lambda}(t))w^* + D_\lambda\psi(tw^*,\mu^*,\alpha^*,\overline{\lambda}(t))\dot{\overline{\lambda}}(t),$$

and we can conclude, once again using the properties of the function ψ given in Lemma 2.21, that $\overline{w}(0) = 0$ and $\dot{\overline{w}}(0) = w^*$. Consequently, we obtain (2.30). ☐

If ε is sufficiently small, the waves obtained from Theorem 2.24 are Stokes waves. This can be seen from the asymptotic formula in (2.30).

Properties of the bifurcation curve

The one-dimensional bifurcation result in Theorem 2.24 is analogous to [12, Theorem 4.6], which uses μ instead of λ as the bifurcation parameter. Other than the parameters, the main difference between the theorems is the addition of the transversality condition (2.28) for bifurcation with respect to λ. Here, we will investigate the properties of the solution curves more closely.

The motivation is to understand the solution set of (2.11) better, and in particular to rule out the possibility that the solution curve found here coincides with the one from [12]. The only way this can occur is if $\overline{\lambda}(t)$ and $\overline{\mu}(t)$, in the notation of [12], are constant along the curves. (If they were constant, we would obtain the same solutions by uniqueness in Theorem 2.24.) Proposition 2.25 shows that we need to consider at least second-order properties of the bifurcation curve in order to achieve this.

Proposition 2.25 (First derivative of $\overline{\lambda}$)**.** *Under the hypothesis of Theorem 2.24, the function $\overline{\lambda}$ satisfies*

$$\dot{\overline{\lambda}}(0) = 0,$$

and so the bifurcation parameter is constant to the first order along the bifurcation curve.

Proof. We adopt the notation used in the proof of Theorem 2.24. Differentiation of the identity $\Psi(t, \mu^*, \alpha^*, \overline{\lambda}(t)) = 0$, and evaluation at $t = 0$, yields the equation

$$\Psi_t(0, \Lambda^*) + \Psi_\lambda(0, \Lambda^*)\dot{\overline{\lambda}}(0) = 0 \tag{2.34}$$

for the derivative of $\overline{\lambda}$ at the origin. From (2.33) and the discussion immediately after, we know that

$$\Psi_\lambda(0, \Lambda^*) = \Pi_Z D_\lambda \mathcal{L}(\Lambda^*)\phi^* \neq 0, \tag{2.35}$$

which means that (2.34) uniquely determines $\dot{\overline{\lambda}}(0)$. However, we still need to compute $\Psi_t(0, \Lambda^*)$.

From (2.31), we obtain

$$\Psi_t(0, \Lambda^*) = \frac{1}{2}\Phi_{tt}(0, \Lambda^*),$$

and differentiation in Equation (2.32) leads to

$$\begin{aligned}\Phi_{tt}(t, \Lambda) = \Pi_Z D_w^2 \mathcal{F}(tw^* + \psi(tw^*, \Lambda), \Lambda)(w^* + D_w\psi(tw^*, \Lambda)w^*)^2 \\ + \Pi_Z D_w \mathcal{F}(tw^* + \psi(tw^*, \Lambda), \Lambda) D_w^2 \psi(tw^*, \Lambda)(w^*)^2.\end{aligned}$$

Hence, by using the properties of ψ given in Lemma 2.21, and using that Π_Z projects along the range of $D_w\mathcal{F}(0, \Lambda^*)$, we find

$$\Psi_t(0, \Lambda^*) = \frac{1}{2}\Pi_Z D_w^2 \mathcal{F}(0, \Lambda^*)(w^*)^2. \tag{2.36}$$

Using Equations (2.34) to (2.36) and the formula (2.26) for Π_Z given in Theorem 2.19, we now find

$$\dot{\overline{\lambda}}(0) = -\frac{\langle D_w^2 \mathcal{F}(0, \Lambda^*)(w^*)^2, \tilde{w}^*\rangle_Y}{2\langle D_\lambda \mathcal{L}(\Lambda^*)\phi^*, \tilde{w}^*\rangle_Y}, \tag{2.37}$$

and so it is sufficient to show that the numerator,

$$\int_0^{2\pi/\kappa} \eta^* D_w^2 \mathcal{F}_1(0, \Lambda)(w^*)^2\, dx + \int_0^1 \int_0^{2\pi/\kappa} \phi^* D_w^2 \mathcal{F}_2(0, \Lambda)(w^*)^2\, dx\, ds, \tag{2.38}$$

vanishes. Since $w^* = \mathcal{T}(\Lambda^*)\phi^*$ with ϕ^* being a separable function of x and s, so is η^*. Moreover, we see from Equation (2.14) that their x-dependence is through $\cos(n\kappa x)$. Thus each term in $D_w^2\mathcal{F}_1(0, \Lambda^*)(w^*)^2$ and $D_w^2\mathcal{F}_1(0, \Lambda^*)(w^*)^2$ has an x-dependence of the form $\sin^a(n\kappa x)\cos^b(n\kappa x)$ with $a + b = 2$ (see the derivatives listed in Section A). It follows that we will be integrating terms whose x-dependence is $\sin^a(n\kappa x)\cos^b(n\kappa x)$ with $a + b = 3$ in (2.38), and therefore that the numerator in (2.37) vanishes. □

Remark 2.26. Equation (2.37) also holds if μ is substituted for λ. Since the proof of Proposition 2.25 only depends on the fact that the numerator in (2.37) vanishes, we can conclude that one also has $\dot{\overline{\mu}}(0) = 0$ when using μ as the bifurcation parameter.

To consider the question of second-order behavior of the solution curve, let us return to the expression for $\dot{\overline{w}}$ we found in the one-dimensional bifurcation result Theorem 2.24, namely

$$\dot{\overline{w}}(t) = w^* + D_w\psi w^* + \dot{\overline{\lambda}}(t)\psi_\lambda, \tag{2.39}$$

where $D_w\psi$ and ψ_λ are evaluated at $(tw^*, \mu^*, \alpha^*, \overline{\lambda}(t))$. Taking another derivative in (2.39) yields

$$\ddot{\overline{w}}(t) = D_w^2\psi(w^*)^2 + 2\dot{\overline{\lambda}}(t)D_{w\lambda}\psi w^* + \dot{\overline{\lambda}}(t)^2\psi_{\lambda\lambda} + \ddot{\overline{\lambda}}(t)\psi_\lambda. \tag{2.40}$$

This simplifies significantly at $t = 0$. An expression for $\ddot{\overline{\lambda}}(0)$ can also be found, akin to how (2.37) was derived. The details are omitted, see for instance [21, Section I.6].

Proposition 2.27 (Second derivatives)**.** *Under the hypothesis of Theorem 2.24, we have*

$$\ddot{\overline{w}}(0) = D_w^2\psi(0, \Lambda^*)(w^*)^2.$$

and

$$\ddot{\overline{\lambda}}(0) = -\frac{\langle D_w^3\mathcal{F}(0, \Lambda^*)(w^*)^3 + 3D_w^2\mathcal{F}(0, \Lambda^*)(w^*, D_w^2\psi(0, \Lambda^*)(w^*)^2), \tilde{w}^*\rangle_Y}{3\langle D_\lambda\mathcal{L}(\Lambda^*)\phi^*, \tilde{w}^*\rangle_Y}$$

for the solution curve $(\overline{w}(t), \overline{\lambda}(t))$*.*

Proof for $\ddot{\overline{w}}(0)$*.* In view of Lemma 2.21, we have that $\psi(0, \Lambda) = 0$ for all Λ in an open neighborhood of Λ^*, and thus $\psi_\lambda(0, \Lambda^*)$ is zero. Finally, the second and third terms in (2.40) vanish due to Proposition 2.25. □

Remark 2.28. The expression for $\ddot{\overline{\mu}}(0)$ can be obtained by simply substituting μ for λ in the expression for $\ddot{\overline{\lambda}}(0)$. From this, it follows that

$$\ddot{\overline{\lambda}}(0) = \frac{\langle D_\mu\mathcal{L}(\Lambda^*)\phi^*, \tilde{w}^*\rangle_Y}{\langle D_\lambda\mathcal{L}(\Lambda^*)\phi^*, \tilde{w}^*\rangle_Y}\ddot{\overline{\mu}}(0) = \frac{\ddot{\overline{\mu}}(0)}{\mu(\cot(\lambda) + \mu^2|\alpha|^{3/2}/2)}.$$

In particular, this implies that $\ddot{\overline{\lambda}}(0)$ and $\ddot{\overline{\mu}}(0)$ must have either the same or opposite sign, depending on the sign of μ and which "side" of the transversality condition (2.28) Λ^* is on.

We now give a more transparent description of $D_w^2\psi(0,\Lambda^*)(w^*)^2$, which Proposition 2.27 shows is required for computing both $\dddot{w}(0)$ and $\ddot{\lambda}(0)$.

Lemma 2.29 (Description of $D_w^2\psi(0,\Lambda^*)(w^*)^2$). *Write*

$$\begin{aligned} D_w^2\mathcal{F}_1(0,\Lambda^*)(w^*)^2 &= c_0 + c_2\cos(2n\kappa x),\\ D_w^2\mathcal{F}_2(0,\Lambda^*)(w^*)^2 &= b_0(s) + b_2(s)\cos(2n\kappa x), \end{aligned} \tag{2.41}$$

and let $\zeta \in C^{2,\beta}_{\kappa,\mathrm{e}}(\overline{\hat\Omega})$ *be such that*

$$\zeta(x,s) := a_0(s) + a_2(s)\cos(2n\kappa x),$$

where the coefficients a_0 *and* a_2 *solve the boundary value problems*

$$\begin{gathered} a_j''(s) - \theta_{jn}^2 a_j(s) = -b_j(s),\\ a_j(0) = 0, \quad \psi_{0s}(1)a_j'(1) - \left(\psi_{0ss}(1) + \frac{1}{\psi_{0s}(1)}\right)a_j(1) = -c_j, \end{gathered} \tag{2.42}$$

for $j = 0,2$. *Then*

$$D_w^2\psi(0,\Lambda^*)(w^*)^2 = \mathcal{T}(\Lambda^*)\zeta.$$

Proof. That $D_w^2\mathcal{F}(0,\Lambda^*)(w^*)^2$ can always be written as in (2.41) can be deduced from the expressions for the derivatives of $\mathcal{F}$ listed in Section A.

The function ψ satisfies the identity

$$(I-\Pi_Z)\mathcal{F}(tw^* + \psi(tw^*,\Lambda^*),\Lambda^*) = 0$$

for sufficiently small t. If we take two derivatives of this equation and evaluate at $t = 0$ we obtain the equation

$$(I-\Pi_Z)(D_w^2\mathcal{F}(0,\Lambda^*)(w^*)^2 + D_w\mathcal{F}(0,\Lambda^*)D_w^2\psi(0,\Lambda^*)(w^*)^2) = 0$$

for $D_w^2\psi(0,\Lambda^*)(w^*)^2$. Since we established in the proof of Proposition 2.25 that $D_w^2\mathcal{F}(0,\Lambda^*)(w^*)^2$ lies in the range of $D_w\mathcal{F}(0,\Lambda^*)$, and since Π_Z projects along $\operatorname{ran} D_w\mathcal{F}(0,\Lambda^*)$, this implies that

$$D_w\mathcal{F}(0,\Lambda^*)D_w^2\psi(0,\Lambda^*)(w^*)^2 = -D_w^2\mathcal{F}(0,\Lambda^*)(w^*)^2, \tag{2.43}$$

which uniquely determines $D_w^2\psi(0,\Lambda^*)(w^*)^2$.

If we now introduce the function $\zeta \in \tilde{X}_2$ by

$$\mathcal{T}(\Lambda^*)\zeta := D_w^2\psi(0,\Lambda^*)(w^*)^2,$$

then Equation (2.43) can be written

$$\mathcal{L}(\Lambda^*)\zeta = -D_w^2\mathcal{F}(0,\Lambda^*)(w^*)^2.$$

Utilizing (2.13), this proves the lemma. □

Due to the form of w^* and the expressions for the derivatives of $\mathcal{F}$ in Section A, we know that the coefficients a_0 and a_2 in Lemma 2.29 are polynomials in s, $\sinh(\theta_{jn}s)/\theta_{jn}$ and $\cosh(\theta_{jn}s)$ for $j = 0, 1, 2$. They can, with some effort, be computed explicitly using a computer algebra system. However, the general expressions are much too long to perform any useful analysis of the second derivatives. We will therefore content ourselves with presenting the result for the first special case of Example 2.15, which was constructed specifically to make ϕ^* and ψ_0 as simple as possible. This, in turn, yields particularly simple a_0 and a_2.

Theorem 2.30 (Special case). *When $\kappa = 1$ and $\Lambda^* = (1, -1, \pi/2)$, the functions a_0 and a_2 are given by*

$$\begin{aligned} a_0(s) &= s + \frac{1}{2}s^2 \sin(s-1) + \frac{3\sin(s)}{2(\cos(1) - \sin(1))}, \\ a_2(s) &= s + \frac{1}{2}s^2 \sin(s-1) + \frac{\sinh(\sqrt{3}s)}{2(\sqrt{3}\cosh(\sqrt{3}) - \sinh(\sqrt{3}))}, \end{aligned} \tag{2.44}$$

respectively. This yields

$$\ddot{\overline{\lambda}}(0) = \frac{3}{2} + 3a_0(1) + \frac{1}{2}a_2(1) < 0. \tag{2.45}$$

In particular, $\overline{\lambda}$ is not constant along the bifurcation curve, which therefore does not coincide with the one found in [12].

Proof. The kernel of $\mathcal{L}(\Lambda^*)$ is spanned by $\cos(x)s$, and moreover $\psi_0(s) = -\sin(s-1)$. One may check that for this special case, the coefficients in Equation (2.41) are given by

$$\begin{aligned} c_0 &= 2, \qquad & b_0(s) &= -s - 2s\cos(s-1) - \sin(s-1), \\ c_2 &= 1, \qquad & b_2(s) &= 3s - 2s\cos(s-1) + (2s^2 - 1)\sin(s-1). \end{aligned}$$

It follows by direct verification that the functions a_0 and a_2 in (2.44) solve the boundary value problems in (2.42). Finally, a long (but direct) computation from the expression for $\ddot{\overline{\lambda}}(0)$ in Proposition 2.27 yields (2.45). □

For the same special case as in Theorem 2.30, we can consider bifurcation from other points on the graph of the associated function μ_* that was introduced in Theorem 2.12. One may verify that the numerator in the expression for $\ddot{\overline{\lambda}}(0)$ in Proposition 2.27 is negative on the entire graph of μ_*. This means that, locally, the solution set of the function Ψ from the proof of

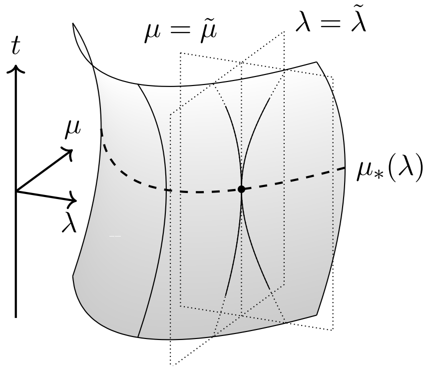

Figure 2.2: The solution curves emanating from the graph of μ_*, making up the solution set of $\Psi = 0$ when α is fixed. The specific point used in Theorem 2.30 can be found to the left of where the transversality condition fails.

Theorem 2.24 looks qualitatively like the surface shown in Figure 2.2, when α is fixed. Recall that the transversality condition corresponds to $\mu_*' \neq 0$, and observe that $\ddot{\lambda}(0)$ changes sign when this condition fails.

We remark that for some other choices of κ and Λ the numerator *does* change sign on the graph of μ_*. It follows that Figure 2.2 does not, in general, tell the whole story.

Local description

Using [12, Theorem 4.6], or Theorem 2.24 when the transversality condition (2.28) is fulfilled, we can describe all solutions of Equation (2.11) in a neighborhood of any $(0, \Lambda^*)$ in $X \times \mathcal{U}$ for which $|M(\Lambda^*)| = 1$.

Suppose that we have such a point, and that $M(\Lambda^*) = \{n\}$. Then Theorem 2.12 tells us that there is a neighborhood of Λ^* in which

$$M(\Lambda) = \begin{cases} \{n\} & \mu = \mu_1(\alpha, \lambda), \\ \varnothing & \text{otherwise.} \end{cases}$$

This allows us to invoke [12, Theorem 4.6] on each point on the graph of μ_1, obtaining a family of solution curves. These are, in fact, all the nontrivial solutions near $(0, \Lambda^*)$:

Theorem 2.31 (Local description)**.** *The above family $\mathcal{S}$ of solution curves bifurcating from points $(0, \mu_1(\alpha, \lambda), \alpha, \lambda)$ for (α, λ) in a neighborhood of*

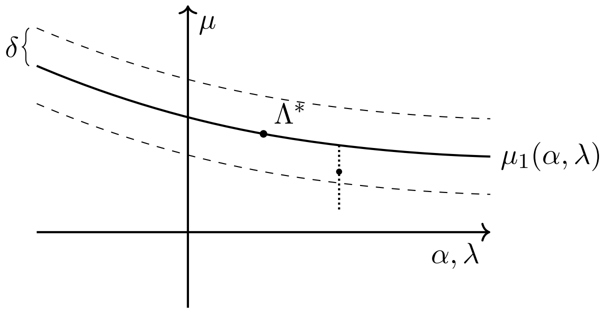

Figure 2.3: All nontrivial solutions near $(0, \Lambda^*)$ can be found by bifurcation from points on the graph of μ_1.

(α^*, λ^*) *contains all nontrivial solutions of Equation* (2.11) *in a neighborhood of* $(0, \Lambda^*)$ *in* $X \times \mathcal{U}$.

Proof. For each $\Lambda = (\mu_1(\alpha, \lambda), \alpha, \lambda)$ we have uniqueness in a set

$$U(\Lambda) := \{(w, \mu', \alpha, \lambda) \in X \times \mathcal{U} : \|w\| \vee |\mu' - \mu_1(\alpha, \lambda)| < \delta(\Lambda)\},$$

in the sense that all nontrivial solutions in $U(\Lambda)$ are given by the solution curve obtained in [12, Theorem 4.6]. Due to the regularity of the problem, and by possibly shrinking the neighborhood of (α^*, λ^*), we can assume that $\delta(\Lambda)$ is constant. It is then clear that the family $\mathcal{S}$ yields all the nontrivial solutions in the open neighborhood

$$U := \bigcup_{\mu = \mu_1(\alpha, \lambda)} U(\Lambda)$$

of $(0, \Lambda^*)$, see Figure 2.3. □

Remark 2.32. We mentioned above that the same procedure can be performed using Theorem 2.24 instead of [12, Theorem 4.6] when the transversality condition is fulfilled. The implication is that, locally, the same solutions can be found through bifurcation with either μ or λ. It is not clear whether this is still the case for possible global solution curves.

5 Two-dimensional bifurcation

For two-dimensional bifurcation we will use α as the second bifurcation parameter. We therefore need the following analogue of Lemma 2.22 for the parameter α.

Lemma 2.33. *Suppose that $\phi_n \in \ker \mathcal{L}(\Lambda)$, where ϕ_n is as defined in* (2.14)*. Then, if $\tilde{w}_n = (\eta_{\phi_n}, \phi_n)$ is the corresponding basis function of Z, we have*

$$\langle D_\alpha \mathcal{L}(\Lambda)\phi_n, \tilde{w}_n \rangle_Y = B\left(\frac{\sinh(\theta_n)}{\theta_n}\right)^2 + f(\theta_{n_j}),$$

where

$$B := \frac{\pi}{\kappa}\left[\frac{1}{\mu^2|\alpha|^2 \sin^2(\lambda)} - \frac{\cot(\lambda)}{2|\alpha|^{1/2}}\right]$$

and f is defined by

$$f(t) := \frac{\pi}{\kappa}\begin{cases} \dfrac{t - \cosh(t)\sinh(t)}{2t^3} & t \neq 0, \\ -\dfrac{1}{3} & t = 0. \end{cases}$$

Suppose that $M(\Lambda^*) = \{n_1, n_2\}$, where $n_1 < n_2$, and that Λ^* satisfies the transversality condition (2.28) (which also appears for two-dimensional bifurcation). Denote the subspace of X consisting of functions that have wavenumber $n\kappa$ in the horizontal variable by $X^{(n)}$. Then $\ker \mathcal{L}(\Lambda^*)|_{X^{(n_2)}} = \operatorname{span}\{\phi^*_{n_2}\}$. Using the local description from Section 4, we obtain the set $\mathcal{S}^{(n_2)}$ of all nontrivial solutions of (2.11) in a neighborhood of $(0, \Lambda^*)$ in $X^{(n_2)} \times \mathcal{U}$. Similarly, under the condition that $n_1 \nmid n_2$, the kernel of $\mathcal{L}(\Lambda)|_{X^{(n_1)}}$ is spanned by $\phi^*_{n_1}$, and we obtain the set $\mathcal{S}^{(n_1)}$ of all nontrivial solutions in a neighborhood of $(0, \Lambda^*)$ in $X^{(n_1)} \times \mathcal{U}$. The next result describes bimodal solutions near $(0, \Lambda^*)$, which are neither in $\mathcal{S}^{(n_1)}$ nor in $\mathcal{S}^{(n_2)}$.

As mentioned before Lemma 2.33, we will use α as the second bifurcation parameter. Hence, we will look for solutions of the equation

$$\mathcal{F}(w, \mu^*, \alpha, \lambda) = 0 \tag{2.46}$$

for $w \in \mathcal{O}$ and $(\mu^*, \alpha, \lambda) \in \mathcal{U}$. Let therefore, for $j \in \{1, 2\}$, the set $\mathcal{S}^{(n_j)}_{\mu^*}$ consist of all (w, α, λ) such that $(w, \mu^*, \alpha, \lambda) \in \mathcal{S}^{(n_j)}$.

Theorem 2.34 (Two-dimensional bifurcation)**.** *Suppose that $\Lambda^* \in \mathcal{U}$ is such that the transversality condition* (2.28) *holds, and that*

$$\ker D_w \mathcal{F}(0, \Lambda^*) = \operatorname{span}\{w_1^*, w_2^*\}, \quad w_j^* = \mathcal{T}(\Lambda^*)\phi_j^*,$$

with $1 \leq n_1 < n_2$ and $\phi^*_j := \phi^*_{n_j}$ as in Proposition 2.9. Furthermore, suppose that either $r(\Lambda^*) \notin \{0, 1\}$ or $\theta(n_2, \alpha^*) = 0$ (in which case $r(\Lambda^*) = 1$).

(i) If $n_1 \nmid n_2$*, there exists a smooth family of nontrivial small-amplitude solutions*

$$\mathcal{S}_{\mu^*} := \Big\{ \Big(\overline{w}(t_1,t_2), \overline{\alpha}(t_1,t_2), \overline{\lambda}(t_1,t_2) \Big) : 0 < |(t_1,t_2)| < \varepsilon \Big\}$$

of (2.46) *in* $\mathcal{O} \times (-\infty, 0) \times (0, \pi)$*, passing through* $(0, \alpha^*, \lambda^*)$ *when* $(t_1, t_2) = 0$*, with*

$$\overline{w}(t_1,t_2) = t_1 w_1^* + t_2 w_2^* + O(|(t_1,t_2)|^2) \quad \textit{in } X \textit{ as } (t_1,t_2) \to 0. \tag{2.47}$$

In a neighborhood of $(0, \alpha^*, \lambda^*)$ *in* $\mathcal{O} \times (-\infty, 0) \times (0, \pi)$*, the union* $\mathcal{S}_{\mu^*} \cup \mathcal{S}_{\mu^*}^{(n_1)} \cup \mathcal{S}_{\mu^*}^{(n_2)}$ *captures all nontrivial solutions of* (2.46)*.*

(ii) Let $0 < \delta < 1$*. If* $n_1 \mid n_2$*, there exists a smooth family of nontrivial small-amplitude solutions*

$$\mathcal{S}^{\delta}_{\mu^*} := \Big\{ \Big(\overline{w}(r,v), \overline{\alpha}(r,v), \overline{\lambda}(r,v) \Big) : 0 < r < \varepsilon,\ |\sin(v)| > \delta \Big\}$$

of (2.46) *in* $\mathcal{O} \times (-\infty, 0) \times (0, \pi)$*, passing through* $(0, \alpha^*, \lambda^*)$ *when* $r = 0$*, with*

$$\overline{w}(r,v) = r\cos(v) w_1^* + r\sin(v) w_2^* + O(r^2) \quad \textit{in } X \textit{ as } r \to 0.$$

In a neighborhood of $(0, \alpha^*, \lambda^*)$ *in* $\mathcal{O} \times (-\infty, 0) \times (0, \pi)$*, the set* $\mathcal{S}^{\delta}_{\mu^*} \cup \mathcal{S}^{(n_2)}_{\mu^*}$ *contains all nontrivial solutions of* (2.46) *such that* $|\sin(v)| > \delta$ *in their projection* $r\cos(v) w_1^* + r\sin(v) w_2^*$ *on* $\ker D_w \mathcal{F}(0, \Lambda^*)$ *along* X_0*.*

Proof. The Lyapunov–Schmidt reduction in Lemma 2.21 tells us that there is a neighborhood of $(0, \Lambda^*)$ in $X \times \mathcal{U}$ in which (2.11) is equivalent to the equation $\Phi(t_1, t_2, \Lambda) = 0$, where

$$\Phi(t_1,t_2,\Lambda) := \Pi_Z \mathcal{F}(t_1 w_1^* + t_2 w_2^* + \psi(t_1 w_1^* + t_2 w_2^*, \Lambda), \Lambda). \tag{2.48}$$

Recall that $Z = \operatorname{span}\{\tilde{w}_1^*, \tilde{w}_2^*\}$, where $\tilde{w}_j := (\eta_{\phi_j^*}, \phi_j^*)$. If we let Π_j denote the projection onto the span of $\tilde{w}_j^*$ along the image of $\mathcal{L}(\Lambda^*)$, then $\Pi_Z = \Pi_1 + \Pi_2$. Defining $\Phi_j := \Pi_j \Phi$ for $j = 1, 2$, the equation $\Phi(t_1, t_2, \Lambda) = 0$ can then be rewritten as the system of equations

$$\Phi_1(t_1,t_2,\Lambda) = 0, \qquad \Phi_2(t_1,t_2,\Lambda) = 0. \tag{2.49}$$

Let us first consider case *(i)*, where $n_1 \nmid n_2$. We claim that

$$\Phi_1(0,t_2,\Lambda) = 0, \qquad \Phi_2(t_1,0,\Lambda) = 0, \tag{2.50}$$

for all t_1, t_2 and Λ. We will show the first identity; the proof of the second being similar. Using that $\ker D_w\mathcal{F}(0,\Lambda^*)|_{X^{(n_2)}} = \operatorname{span}\{w_2^*\}$, an application of the Lyapunov–Schmidt reduction in $X^{(n_2)}$ yields a function $\tilde{\psi}$ mapping into $X_0^{(n_2)}$ satisfying

$$(I-\Pi_2)\mathcal{F}(t_2w_2^*+\tilde{\psi}(t_2w_2^*,\Lambda),\Lambda)=0, \tag{2.51}$$

for all t_2 in a neighborhood of $0\in\mathbb{R}$. Due to the $2\pi/(n_2\kappa)$-periodicity of $\mathcal{F}(tw_2^*+\tilde{\psi}(tw_2^*,\Lambda),\Lambda)$, (2.51) holds with Π_2 replaced by Π_Z, whence uniqueness of ψ yields $\psi(t_2w_2^*)=\tilde{\psi}(t_2w_2^*)$. The first identity in Equation (2.50) then follows by definition of Φ_1.

Following the proof of Theorem 2.24, we now introduce the function Ψ_1 by

$$\Psi_1(t_1,t_2,\Lambda) \coloneqq \int_0^1 \Phi_{1t_1}(zt_1,t_2,\Lambda)\,dz, \tag{2.52}$$

and similarly the function Ψ_2. Then $\Phi_j = t_j\Psi_j$ by (2.50), and we get different cases for (2.49) depending on the values of t_1 and t_2. If both t_1 and t_2 vanish, we get trivial solutions. When $t_2=0$ but $t_1\neq 0$, we know from (2.50) that the system reduces to $\Phi_1(t_1,0,\Lambda)=0$, the solutions of which correspond to $\mathcal{S}^{(n_1)}$, while when $t_1=0$ but $t_2\neq 0$ we get solutions of (2.11) that lie in $\mathcal{S}^{(n_2)}$. The remaining case is where both t_1 and t_2 are allowed to be nonzero, which amounts to investigating the solutions of

$$\Psi_1(t_1,t_2,\Lambda)=0, \qquad \Psi_2(t_1,t_2,\Lambda)=0$$

in a neighborhood of $(0,0,\Lambda^*)$. We will look at those solutions with $\mu=\mu^*$.

For this, we intend to use the implicit function theorem. Note that from (2.52) we get $\Psi_j(0,0,\Lambda^*)=\Pi_j\Phi_{t_j}(0,0,\Lambda^*)$, and that by definition (2.48) of Φ we have

$$\Phi_{t_j}(t_1,t_2,\Lambda^*) = \Pi_Z D_w\mathcal{F}(w+\psi(w,\Lambda^*),\Lambda^*)(w_j^*+D_w\psi(w,\Lambda^*)w_j^*),$$

where $w\coloneqq t_1w_1^*+t_2w_2^*$. Using the properties of ψ and Theorem 2.19, we can therefore conclude that $\Psi_j(0,0,\Lambda^*)=0$. It remains to show that the derivative $D_{(\alpha,\lambda)}(\Psi_1,\Psi_2)(0,0,\Lambda^*)$ is invertible.

Using analogous computations to those after Equation (2.33) in the proof of Theorem 2.24, we have

$$\Psi_{j\beta}(0,0,\Lambda^*) = \Pi_j D_\beta\mathcal{L}(\Lambda^*)\phi_j = \frac{\langle D_\beta\mathcal{L}(\Lambda^*)\phi_j, \tilde{w}_j^*\rangle_Y}{\|\tilde{w}_j^*\|_Y^2}\tilde{w}_j^*$$

for $j = 1, 2$ and $\beta = \alpha, \lambda$. It follows that $D_{(\alpha,\lambda)}(\Psi_1, \Psi_2)(0, 0, \Lambda^*)$ is invertible if and only if the determinant

$$C := \begin{vmatrix} \langle D_\lambda \mathcal{L}(\Lambda^*)\phi_1^*, \tilde{w}_1^* \rangle_Y & \langle D_\lambda \mathcal{L}(\Lambda^*)\phi_2^*, \tilde{w}_2^* \rangle_Y \\ \langle D_\alpha \mathcal{L}(\Lambda^*)\phi_1^*, \tilde{w}_1^* \rangle_Y & \langle D_\alpha \mathcal{L}(\Lambda^*)\phi_2^*, \tilde{w}_2^* \rangle_Y \end{vmatrix} \tag{2.53}$$

is nonzero. The inner products appearing in this determinant have already been computed in Lemma 2.22 and Lemma 2.33. Using these, and elementary properties of determinants, we have

$$C = A\left(\left(\frac{\sinh(\theta_{n_1})}{\theta_{n_1}}\right)^2 f(\theta_{n_2}) - \left(\frac{\sinh(\theta_{n_2})}{\theta_{n_2}}\right)^2 f(\theta_{n_1})\right).$$

Observe that of θ_{n_1} and θ_{n_2}, only θ_{n_2} can vanish. Suppose for the moment that also $\theta_{n_2} \neq 0$. Then

$$\begin{aligned} f(\theta_{n_j}) &= \frac{\pi}{2\kappa\theta_{n_j}^2}\left(\frac{\sinh(\theta_{n_j})}{\theta_{n_j}}\right)^2\left(\left(\frac{\theta_{n_j}}{\sinh(\theta_{n_j})}\right)^2 - \theta_{n_j}\frac{\cosh(\theta_{n_j})}{\sinh(\theta_{n_j})}\right) \\ &= \frac{\pi}{2\kappa\theta_{n_j}^2}\left(\frac{\sinh(\theta_{n_j})}{\theta_{n_j}}\right)^2 (r(\Lambda^*)^2 - \theta_{n_j}^2 - r(\Lambda^*)), \end{aligned}$$

whence the determinant in Equation (2.53) can be written as

$$C = \frac{\pi A}{2\kappa}\left(\frac{\sinh(\theta_{n_1})}{\theta_{n_1}}\right)^2\left(\frac{\sinh(\theta_{n_2})}{\theta_{n_2}}\right)^2 r(\Lambda^*)(r(\Lambda^*) - 1)\left(\frac{1}{\theta_{n_2}^2} - \frac{1}{\theta_{n_1}^2}\right),$$

where A is nonzero due to the assumption of transversality. Hence, we immediately see that C is nonzero if and only if $r(\Lambda^*) \notin \{0, 1\}$. A similar computation shows that

$$C = \frac{\pi A}{6\kappa}\left(\frac{\sinh(\theta_{n_1})}{\theta_{n_1}}\right)^2 \neq 0$$

when $\theta_{n_2} = 0$. This concludes the proof of part (i).

Next, we move on to case *(ii)*, where $n_1 \mid n_2$. We still find that

$$\Phi_1(0, t_2, \Lambda) = 0$$

for all t_2 and Λ, and so we can introduce

$$\Psi_1(r, v, \Lambda) := \int_0^1 \Phi_{1t_1}(zr\cos(v), r\sin(v), \Lambda)\, dz$$

as before; only now written using the polar coordinates $(t_1, t_2) = re^{iv}$ (identifying $\mathbb{C}$ and $\mathbb{R}^2$). Then $\Phi_1 = t_1\Psi_1$. For Φ_2, the corresponding identity in (2.50) is no longer true in general, but we still have $\Phi_2(0, 0, \Lambda) = 0$. We therefore introduce Ψ_2 through

$$\Psi_2(r, v, \Lambda) := \int_0^1 \Big[\Phi_{2t_1}(zre^{iv}, \Lambda)\cos(v) + \Phi_{2t_2}(zre^{iv}, \Lambda)\sin(v)\Big] \mathrm{d}z,$$

which yields $\Phi_2 = r\Psi_2$.

Like for case *(i)*, the solutions of $\Phi(0, t_2, \Lambda) = 0$ near $(0, \Lambda^*)$ for $t_2 \neq 0$ correspond to solutions in $S^{(n_2)}$. When $t_1 \neq 0$, also $r \neq 0$, and so (2.49) is equivalent to the problem

$$\Psi_1(r, v, \Lambda) = 0, \qquad \Psi_2(r, v, \Lambda) = 0,$$

which we will now consider. Again, we will use the implicit function theorem to find solutions with $\mu = \mu^*$. Due to similar computations as those for case *(i)*, we have $\Psi_1(0, v, \Lambda^*) = \Psi_2(0, v, \Lambda^*) = 0$ and

$$\Psi_{1\beta}(0, v, \Lambda^*) = \frac{\langle D_\beta \mathcal{L}(\Lambda^*)\phi_1^*, \tilde{w}_1^*\rangle_Y}{\|\tilde{w}_1^*\|_Y^2}\tilde{w}_1^* \tag{2.54}$$

for all v and $\beta = \alpha, \lambda$. To find the derivatives of Ψ_2, note that

$$\Psi_{2\beta}(0, v, \Lambda^*) = \Phi_{2t_1\beta}(0, 0, \Lambda^*)\cos(v) + \Phi_{2t_2\beta}(0, 0, \Lambda^*)\sin(v),$$

and so

$$\begin{aligned} \Psi_{2\beta}(0, v, \Lambda^*) &= \frac{\langle D_\beta \mathcal{L}(\Lambda^*)(\cos(v)\phi_1^* + \sin(v)\phi_2^*), \tilde{w}_2^*\rangle_Y}{\|\tilde{w}_2^*\|_Y^2}\tilde{w}_2^* \\ &= \frac{\langle D_\beta \mathcal{L}(\Lambda^*)\phi_2^*, \tilde{w}_2^*\rangle_Y}{\|\tilde{w}_2^*\|_Y^2}\sin(v)\tilde{w}_2^*, \end{aligned} \tag{2.55}$$

where we have used that $\cos(n_1\kappa x)$ and $\cos(n_2\kappa x)$ are orthogonal in $L^2_\kappa(\mathbb{R})$.

From the preceding, we see that the derivative $D_{(\alpha,\lambda)}(\Psi_1, \Psi_2)(0, v, \Lambda^*)$ is invertible if and only if the determinant

$$\begin{aligned} \tilde{C} &:= \begin{vmatrix} \langle D_\lambda \mathcal{L}(\Lambda^*)\phi_1^*, \tilde{w}_1^*\rangle_Y & \langle D_\lambda \mathcal{L}(\Lambda^*)\phi_2^*, \tilde{w}_2^*\rangle_Y \sin(v) \\ \langle D_\alpha \mathcal{L}(\Lambda^*)\phi_1^*, \tilde{w}_1^*\rangle_Y & \langle D_\alpha \mathcal{L}(\Lambda^*)\phi_2^*, \tilde{w}_2^*\rangle_Y \sin(v) \end{vmatrix} \\ &= C\sin(v) \end{aligned} \tag{2.56}$$

is nonzero, where C is the determinant introduced in (2.53). We know that $C \neq 0$ under the assumptions of the theorem, so we can apply the implicit function theorem at $(0, v)$ if $\sin(v) \neq 0$. This can be done uniformly in v as long as $\sin(v)$ is bounded away from zero. □

Remark 2.35. In case (i), the surface profiles in $\mathcal{S}_{\mu^*} \setminus \cup_j \mathcal{S}_{\mu^*}^{(n_j)}$ have multiple crests and troughs in each minimal period, at least when (t_1, t_2) is sufficiently small. This follows from the asymptotic formula in (2.47).

Remark 2.36. Observe that the second special case listed in Example 2.15 has $n_1 \mid n_2$, while the third has $n_1 \nmid n_2$. They therefore fall into different cases in Theorem 2.34.

Properties of the bifurcation sheet

We will now present some properties of the sheets of solutions that were found in the two-dimensional bifurcation result, Theorem 2.34, following the lines of Section 4. The main purpose of this is to show that these sheets, found by bifurcating with respect to λ and α, do not, in general, coincide with the sheets found in [12, Theorem 4.8]. Like for one-dimensional bifurcation, Theorem 2.34 differs from the one in [12] by the use of λ instead of μ, and the addition of the transversality condition (2.28).

The first step towards showing that the sheets differ is Proposition 2.37, which is the two-dimensional counterpart of Proposition 2.25.

Proposition 2.37 (Gradients of $\overline{\alpha}$ and $\overline{\lambda}$)**.** *For the solution sheets obtained in Theorem 2.34, we have the following:*

- *In case (i), the solutions satisfy*

$$\nabla\overline{\alpha}(0,0) = \nabla\overline{\lambda}(0,0) = 0.$$

- *In case (ii), we have*

$$\overline{\alpha}_r(0,v) = \overline{\lambda}_r(0,v) = 0,$$

  as long as $n_2 \neq 2n_1$.

Proof. The proof for case *(i)* is a simpler variant of that for case *(ii)*, so we focus on the latter. By definition of $\overline{\alpha}$ and $\overline{\lambda}$, we have the identity

$$\Psi_j(r, v, \mu^*, \overline{\alpha}(r,v), \overline{\lambda}(r,v)) = 0$$

for $j = 1, 2$. Through taking derivatives with respect to r, this implies that

$$\Psi_{jr} + \Psi_{j\alpha}\overline{\alpha}_r(0,v) + \Psi_{j\lambda}\overline{\lambda}_r(0,v) = 0, \tag{2.57}$$

where the derivatives of Ψ_j are evaluated at $(0, v, \Lambda^*)$. This linear system of equations can be solved for $\overline{\alpha}_r(0,v)$ and $\overline{\lambda}_r(0,v)$ because the determinant $\tilde{C}$

in Equation (2.56) is nonzero. In order to show that the derivatives vanish, it is therefore sufficient (and necessary) to show that $\Psi_{jr}(0, v, \Lambda^*) = 0$ for $j = 1, 2$.

Using the definitions of Ψ_1 and Ψ_2, we find

$$\begin{aligned}
\Psi_{1r}(0, v, \Lambda^*) &= \frac{1}{2}\cos(v)\Phi_{1t_1t_1}(0, 0, \Lambda^*) + \sin(v)\Phi_{1t_1t_2}(0, 0, \Lambda^*), \\
\Psi_{2r}(0, v, \Lambda^*) &= \frac{1}{2}\cos^2(v)\Phi_{2t_1t_1}(0, 0, \Lambda^*) + \frac{1}{2}\sin^2(v)\Phi_{2t_2t_2}(0, 0, \Lambda^*) \\
&\qquad + \sin(v)\cos(v)\Phi_{2t_1t_2}(0, 0, \Lambda^*),
\end{aligned}$$

where

$$\begin{aligned}
\Phi_{lt_it_j}(0, 0, \Lambda^*) &= \Pi_l D_w^2 \mathcal{F}(0, \Lambda^*)(w_i^*, w_j^*) \\
&= \frac{\langle D_w^2 \mathcal{F}(0, \Lambda^*)(w_i^*, w_j^*), \tilde{w}_l^* \rangle_Y}{\|\tilde{w}_l^*\|_Y^2} \tilde{w}_l^*
\end{aligned} \tag{2.58}$$

for $i, j, l = 1, 2$. Using orthogonality in $L^2_\kappa(\mathbb{R})$, like in the proof Proposition 2.25, one can show that the derivatives in Equation (2.58) are zero, except possibly when $n_l = n_i + n_j$ or $n_l = |n_i - n_j|$. This is only the case when $n_2 = 2n_1$ and either $i = j = 1$ and $l = 2$ or $i \neq j$ and $l = 1$. □

We now show that $\overline{\alpha}_r(0, v)$ and $\overline{\lambda}_r(0, v)$ can indeed be nonzero when $n_2 = 2n_1$, which is not covered by Proposition 2.37, by considering the second special case listed in Example 2.15.

Proposition 2.38 (Special case)**.** *Let σ be the smallest positive solution of $x\cot(x) = 1$. When $\kappa = \sigma/\sqrt{3}$ and $\Lambda^* = (1/(2\kappa), -4\kappa^2, \pi/2)$, we have*

$$\begin{aligned}
\Psi_{1r}(0, v, \Lambda^*) &= \left(1 + \frac{1}{3}\kappa^2\right)\sin(v)\tilde{w}_1^*, \\
\Psi_{2r}(0, v, \Lambda^*) &= \left(\frac{1}{16} + \frac{1}{2}\cos^2(\sigma)\right)\cos^2(v)\tilde{w}_2^*,
\end{aligned}$$

and so by (2.57)*,* (2.54) *and* (2.55) *that*

$$\begin{bmatrix} \overline{\alpha}_r(0, v) \\ \overline{\lambda}_r(0, v) \end{bmatrix} = \mathbb{M} \begin{bmatrix} \left(1 + \frac{1}{3}\kappa^2\right)\sin(v) \\ \left(\frac{1}{16} + \frac{1}{2}\cos^2(\sigma)\right)\cos(v)\cot(v) \end{bmatrix},$$

for a nonsingular matrix $\mathbb{M}$ not depending on v. In particular, $\overline{\alpha}_r(0, v)$ and $\overline{\lambda}_r(0, v)$ are both nonzero, except possibly for isolated values of v.

Proposition 2.38 shows that the sheets obtained in Theorem 2.34 are, in general, not the same as those obtained in [12]—at least when $n_2 = 2n_1$.

Local description of solutions

We finish by using [12, Theorem 4.8] to prove a two-dimensional version of Theorem 2.31, describing all nontrivial solutions in a neighborhood of a point falling into case *(i)*. Let therefore $\Lambda^* \in \mathcal{U}$ be such that $M(\Lambda^*) = \{n_1, n_2\}$ with $n_1 < n_2$ and $n_1 \nmid n_2$, and such that either $r(\Lambda^*) \notin \{0,1\}$ or $\theta(n_2, \alpha^*) = 0$.

Proceeding as in Section 4, we use Theorem 2.12 to conclude that there is a neighborhood of Λ in which

$$M(\Lambda) = \begin{cases} \{n_1, n_2\} & \alpha = \alpha^*, \mu = \mu_*(\lambda), \\ \{n_i\} & \alpha \neq \alpha^*, \mu = \mu_i(\alpha, \lambda), \\ \varnothing & \text{otherwise.} \end{cases}$$

We may now apply [12, Theorem 4.8] to each point on the graph of μ_* (where $r(\Lambda) = r(\Lambda^*)$ and $\alpha = \alpha^*$), obtaining a family $\mathcal{S}$ of bifurcating solution sheets. In addition, one has the solutions in $\mathcal{S}^{(n_1)}$ and $\mathcal{S}^{(n_2)}$, which were described before Theorem 2.34. These are all the nontrivial solutions near $(0, \Lambda^*)$. We omit the proof, which is essentially the same as for Theorem 2.31.

Theorem 2.39 (Local description)**.** *The family $\mathcal{S}$ of solution sheets bifurcating from points $(0, \mu_*(\lambda), \alpha^*, \lambda)$ for λ in a neighborhood of λ^*, together with the families $\mathcal{S}^{(n_1)}$ and $\mathcal{S}^{(n_2)}$, constitutes all nontrivial solutions in a neighborhood of $(0, \Lambda^*)$ in $X \times \mathcal{U}$.*

Remark 2.40. We could alternatively have used Theorem 2.34 at points where the transversality condition is fulfilled. It follows that, locally, the same solutions can be found through bifurcation with either μ or λ.

A Derivatives of $\mathcal{F}$

The purpose of this appendix is simply to record the derivatives of $\mathcal{F}$ with respect to w at $(0, \Lambda)$, up to the third order. These are used to obtain derivatives of the bifurcation curves from Theorem 2.24 and the bifurcation sheets from Theorem 2.34.

We have

$$\begin{aligned} D_w \mathcal{F}_1(0, \Lambda) w &= (1 - \psi_{0s}^2)\eta + \psi_{0s}\hat{\phi}_s \\ D_w^2 \mathcal{F}_1(0, \Lambda) w^2 &= 3\psi_{0s}^2 \eta^2 + \psi_{0s}^2 \eta_x^2 - 4\psi_{0s}\eta\hat{\phi}_s + \hat{\phi}_s^2 \\ D_w^3 \mathcal{F}_1(0, \Lambda) w^3 &= -12\psi_{0s}^2 \eta^3 - 6\psi_{0s}^2 \eta\eta_x^2 + 18\psi_{0s}\eta^2\hat{\phi}_s + 6\psi_{0s}\eta_x^2\hat{\phi}_s - 6\eta\hat{\phi}_s^2 \end{aligned}$$

for $\mathcal{F}_1$, where we suppress the evaluation at $s = 1$, and the derivatives

$$\begin{aligned}
D_w\mathcal{F}_2(0,\Lambda)w &= -2\psi_{0ss}\eta - s\psi_{0s}\eta_{xx} + (\partial_x^2 + \partial_s^2 - \alpha)\hat{\phi} \\
D_w^2\mathcal{F}_2(0,\Lambda)w^2 &= 6\psi_{0ss}\eta^2 + 2s\psi_{0s}\eta\eta_{xx} + (4s\psi_{0s} + 2s^2\psi_{0ss})\eta_x^2 \\
&\quad - 4\eta\hat{\phi}_{ss} - 4s\eta_x\hat{\phi}_{xs} - 2s\eta_{xx}\hat{\phi}_s \\
D_w^3\mathcal{F}_2(0,\Lambda)w^3 &= -24\psi_{0ss}\eta^3 - 6s\psi_{0s}\eta^2\eta_{xx} - (24s\psi_{0s} + 12s^2\psi_{0ss})\eta\eta_x^2 \\
&\quad + 18\eta^2\hat{\phi}_{ss} + 12s\eta\eta_x\hat{\phi}_{xs} + 6s\eta\eta_{xx}\hat{\phi}_s + 12s\eta_x^2\hat{\phi}_s \\
&\quad + 6s^2\eta_x^2\hat{\phi}_{ss}
\end{aligned}$$

for $\mathcal{F}_2$.

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PAPER 3

ON THE STABILITY OF SOLITARY WATER WAVES WITH A POINT VORTEX

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Abstract. This paper investigates the stability of traveling wave solutions to the free boundary Euler equations with a submerged point vortex. We prove that sufficiently small-amplitude waves with small enough vortex strength are conditionally orbitally stable. In the process of obtaining this result, we develop a quite general stability/instability theory for bound state solutions of a large class of infinite-dimensional Hamiltonian systems in the presence of symmetry. This is in the spirit of the seminal work of Grillakis, Shatah, and Strauss [20], but with hypotheses that are relaxed in a number of ways necessary for the point vortex system, and for other hydrodynamical applications more broadly. In particular, we are able to allow the Poisson map to have merely dense range, as opposed to being surjective, and to be state-dependent.

As a second application of the general theory, we consider a family of nonlinear dispersive PDEs that includes the generalized KdV and Benjamin–Ono equations. The stability/instability of solitary waves for these systems has been studied extensively, notably by Bona, Souganidis, and Strauss [6], who used a modification of the GSS method. We provide a new, more direct proof of these results that follows as a straightforward consequence of our abstract theory. At the same time, we extend them to fractional order dispersive equations.

1 Introduction

The persistence of localized regions of vorticity is a remarkable feature of two-dimensional incompressible inviscid fluid motion. For instance, high Reynolds number flow over an immersed body may produce a wake of shed vortices outside of which the velocity field is largely irrotational. While the small-scale structure of these regions can be quite intricate, their large-scale movement is well predicted by the so-called Helmholtz–Kirchhoff point vortex model, so long as they remain sufficiently isolated. The stability of various configurations of point vortices in a fixed domain has therefore been the subject of extensive study since the early work of Poincaré [41]. In this paper, we are interested in point vortices carried by water waves. Unlike the fixed domain case, this will involve understanding the subtle dynamical implications of wave–vortex interactions. Our main results concern the orbital stability of small-amplitude solitary waves with a single point vortex.

To state things more precisely, by "water" we mean an incompressible and inviscid fluid occupying a time-dependent domain $\Omega_t \subset \mathbb{R}^2$. For simplicity, assume that at time $t \geq 0$, Ω_t consists of the (unbounded) region lying below the graph of a function $\eta = \eta(t, x_1)$, and above Ω_t is vacuum. This is a free boundary problem, in the sense that η is not prescribed, but evolves dynamically.

Let $v = v(t, \cdot) \colon \Omega_t \to \mathbb{R}^2$ denote the fluid velocity at time $t \geq 0$. The *vorticity* is defined to be the quantity

$$\omega := \nabla \times v = \partial_{x_1} v_2 - \partial_{x_2} v_1,$$

measuring the circulation density of the fluid. Mathematically, a *point vortex* describes the situation where $\omega = \epsilon \delta_{\bar{x}(t)}$, the Dirac measure supported at $\bar{x} = \bar{x}(t) \in \Omega_t$. We call ϵ the *vortex strength* and $\bar{x}$ the *vortex center*. It is fairly easy to see that this is not a valid measure-valued solution of the vorticity equation, as the advection term $v \cdot \nabla \omega$ has no distributional meaning. Instead, we ask only that the velocity field be a weak solution to the incompressible irrotational Euler equations away from the vortex center. That is,

$$\begin{cases} \partial_t v + \nabla \cdot (v \otimes v) = -\nabla p - g e_2 & \text{in } \Omega_t \setminus \{\bar{x}(t)\} \\ \omega = \epsilon \delta_{\bar{x}(t)} & \text{in } \Omega_t \\ \nabla \cdot v = 0 & \text{in } \Omega_t, \end{cases} \tag{3.1a}$$

with each of these holding in the sense of distributions. Here $p = p(t, \cdot) \colon \Omega_t \to \mathbb{R}$ is the pressure and $g > 0$ is the gravitational constant. We consider the finite *excess* energy case where $v(t) \in L^1_{\mathrm{loc}}(\Omega_t) \cap L^2(\Omega_t \setminus U_t)$ for every neighborhood $U_t \ni \bar{x}(t)$. The motion of the point vortex is taken to be governed by the Helmholtz–Kirchhoff model

$$\partial_t \bar{x} = \left(v - \frac{1}{2\pi} \epsilon \nabla^\perp \log |\cdot - \bar{x}| \right) \Big|_{\bar{x}}, \tag{3.1b}$$

where the subtracted term is the velocity field generated by the point vortex. Thus (3.1b) states that the vortex center does not self-advect, but rather is transported only by the irrotational part of the fluid velocity field.

Finally, the evolution of the free boundary is coupled to that of the fluid by the requirements that

$$\partial_t \eta = (-\partial_{x_1}\eta, 1) \cdot v, \qquad \text{and} \qquad p = -b\partial_{x_1}\left(\frac{\partial_{x_1}\eta}{\langle \partial_{x_1}\eta \rangle}\right) \qquad \text{on } S_t, \tag{3.1c}$$

where $S_t := \partial\Omega_t$ is the interface, and $b > 0$ is the coefficient of surface tension. The first of these is the kinematic condition, linking the surface to the velocity field. The second is the dynamic condition, which states that the pressure deviates from atmospheric pressure (normalized here to 0) in proportion to the signed curvature.

Point vortices have been studied in fluid mechanics for centuries. The specific model (3.1a)–(3.1b) was first proposed by Helmholtz [24] and Kirchhoff [27] for incompressible fluids in a fixed domain. Later, Marchioro and Pulvirenti (see [31] and [32, Chapter 4]) offered a rigorous justification by proving that (3.1a)–(3.1b) is the limiting equation governing the motion of vortex patch solutions of the Euler equations as the diameter of the patch approaches 0. Another derivation was given by Gallay [16], who showed that the system can be obtained as the vanishing viscosity limit for smooth solutions of the Navier–Stokes equation with increasingly concentrated vorticity. The recent work of Glass, Munnier, and Sueur [18] provides a second physical interpretation: they prove that the Helmholtz–Kirchhoff system governs irrotational incompressible inviscid flow around an immersed rigid body, with a fixed circulation around the body, in the limit where the body shrinks to a point in a certain way.

The primary objective in this paper is to study the stability of steady solutions of the water wave with a point vortex problem (3.1). An existence theory for waves of this type was given by Shatah, Walsh, and Zeng [43]. The analogous problem for capillary-gravity waves in finite-depth water was recently considered by Varholm [46], and for gravity waves by Ter-Krikorov [45] and Filippov [13, 14]. These are among the very few examples of exact steady water waves with localized vorticity currently available. Numerical studies of water waves with a point vortex have been carried out in [10–12], for example.

Stated informally, our main result is as follows. First, observe that in a neighborhood of S_t, the velocity field v can be decomposed as

$$v = \nabla\Phi + \epsilon\nabla\Theta,$$

where $\Phi(t,\cdot)$ is harmonic in Ω_t, and Θ is an explicit function depending on $\bar{x}$ that captures the contribution of the point vortex; see Section 5. The system (3.1) can then be reformulated as an equation for $u = (\eta, \varphi, \bar{x})$, where

$$\varphi = \varphi(t, x_1) \coloneqq \Phi(t, x_1, \eta(t, x_1)).$$

A *solitary wave* in this setting corresponds to a solution of the form

$$u(t, x_1) = (\eta^c(x_1 - ct), \varphi^c(x_1 - ct), \bar{x}^c + cte_1),$$

for some spatially localized $(\eta^c, \varphi^c, \bar{x}^c)$ and wave speed $c \in \mathbb{R}$.

Theorem 3.1 (Main result)**.** *Every symmetric solitary capillary-gravity water wave with a point vortex $(\eta^c, \varphi^c, \bar{x}^c)$ having (η^c, φ^c), c, and ϵ sufficiently small is* conditionally orbitally stable *in the following sense. For all $R > 0$ and $\rho > 0$, there exists $\rho_0 = \rho_0(R, \rho) > 0$ such that, if $(\eta, \varphi, \bar{x})$ is any solution defined on a time interval $[0, t_0)$, obeying a bound*

$$\sup_{t\in[0,t_0)} \left(\|\eta(t)\|_{H^{3+}} + \|\varphi(t)\|_{\dot{H}^{\frac{5}{2}+}\cap \dot{H}^{\frac{1}{2}}} + |\bar{x}_2(t)| \right) < R, \tag{3.2}$$

and having initial data satisfying

$$\|\eta(0) - \eta^c\|_{H^1} + \|\varphi(0) - \varphi^c\|_{\dot{H}^{\frac{1}{2}}} + |\bar{x}(0) - \bar{x}^c| < \rho_0,$$

then

$$\sup_{t\in[0,t_0)} \inf_{s\in\mathbb{R}} \Big(\|\eta(t, \cdot - s) - \eta^c\|_{H^1} + \|\varphi(t, \cdot - s) - \varphi^c\|_{\dot{H}^{\frac{1}{2}}} + |\bar{x}(t) + se_1 - \bar{x}^c| \Big) < \rho \tag{3.3}$$

A more precise version is given in Theorem 3.33. Several remarks are in order. Orbital here refers to the fact that we are controlling the distance to the family of translates of the steady wave; this is natural given the invariance of the problem. It is also important to note that ρ_0 above is independent of t_0, and hence the conclusion of Theorem 3.1 is much stronger than just continuity of the solution map at $(\eta^c, \varphi^c, \bar{x}^c)$. Indeed, for a global-in-time solution, this gives orbital stability in the classical sense. The norm occurring in (3.2) represents the lowest regularity in which a local well-posedness theory has been established for irrotational capillary-gravity waves [1]. On the other hand, the norm in (3.3) is associated to the physical energy for the system, which we will discuss shortly.

Our approach is to rewrite (3.1) as an infinite-dimensional Hamiltonian system of the general form

$$\frac{du}{dt} = J(u)DE(u),$$

with u appropriate Banach space. Here, E is a functional (the energy), and J is a state-dependent skew-adjoint operator (the Poisson map). A similar system was established formally by Rouhi and Wright [42]; we use a slightly different version, and give a rigorous proof in Section 5.

As the entire problem is invariant under translation, there is a conserved momentum functional $P = P(u)$. A natural strategy for analyzing the (orbital) stability of bound states in abstract Hamiltonian systems with symmetries is to use the *energy-momentum method* first introduced by Benjamin [5]. In brief, this involves constructing a Lyapunov functional using a carefully chosen combination of E and P. Actually carrying out this argument, however, can be quite challenging. Over three decades ago, Grillakis, Shatah, and Strauss [20] introduced a powerful machinery — now commonly referred to as the GSS method — that reduced these many difficulties down to discerning the convexity or concavity of a single scalar-valued quantity called the moment of instability.

Not surprisingly, this had an enormous impact on the field and generated a great deal of research activity. However, the hypotheses of GSS limit somewhat its applicability to infinite-dimensional Hamiltonians with more complicated structure. For instance, they require that J is surjective, and independent of the state u. But, recall that the Poisson map for KdV is ∂_x, which is not surjective in the natural class of spaces. In fact, for water waves with a point vortex (3.1), we will see that J is neither independent of state, nor surjective.

There is also a somewhat practical issue with the functional analytic setting. Consider for a moment the irrotational case. GSS supposes that the Cauchy problem is globally well-posed in the energy space. But, as remarked above, the local well-posedness of the gravity water wave problem with surface tension proved by Burq, Alazard, and Zuily in [1] takes $\eta(t) \in H^{3+}$ and $\varphi(t) \in \dot{H}^{5/2+} \cap \dot{H}^{1/2}$. On the other hand, the kinetic energy is given by the much rougher $\|v\|_{L^2}^2$ and the potential energy is equivalent to $\|\eta\|_{H^1}^2$. Moreover, writing the kinetic energy in terms of (η, φ) yields

$$\|v\|_{L^2(\Omega_t)}^2 = \frac{1}{2} \int_{\mathbb{R}} \varphi G(\eta)\varphi \, dx_1,$$

where $G(\eta)$ is the Dirichlet–Neumann operator; see the discussion in Section 5. For this to be smooth as a functional of (η, φ) in the Sobolev setting, one must have that $\eta \in H^{3/2+} \hookrightarrow W^{1,\infty}$. In effect, then, there are three levels of regularity: a rough space in which the physical energy is defined, an intermediate space where the energy functional is smooth, and a higher regularity space where we can hope to have well-posedness. This situation is exceedingly common in the analysis of quasilinear equations. Indeed, it is the natural by-product of so-called higher-order energy estimates, which are among the most basic and widespread tools in nonlinear PDE theory.

With that in mind, as one of the primary contributions of this paper, we introduce a new abstract stability and instability result in the spirit of GSS, but with relaxed assumptions, making it directly applicable to problems such as (3.1). Specifically, we allow for a large class of state-dependent Poisson maps $J = J(u)$, and essentially only require that J is injective with dense range. Moreover, the entire theory is formulated in a scale of Banach spaces, offering a simple way to accommodate a gap between the regularity of the energy space and the regularity necessary for well-posedness. Finally, in view of the point vortex problem, we allow the symmetry group to be merely affine.

There are a number of new assumptions and technical conditions, but the main conclusion is the same as GSS: stability or instability of the bound state hinges on the sign of a scalar quantity. Because of the mismatch in spaces, the results are conditional in the sense that they only hold on a time interval in which the solutions of the problem exist and their growth is controlled. Using this general theory, we are then able to address the question of stability of traveling water waves with a point vortex and prove Theorem 3.1. Finally, we also consider further applications of this same framework to KdV and related dispersive model equations.

One of the main inspirations for this paper is Mielke's work on conditional energetic stability of irrotational solitary waves on water of finite depth with strong surface tension [34], in which he also had to modify the GSS method to deal with the mismatch between well-posedness and energy spaces. While our basic strategy is the same, we make the additional effort of formulating a general theory which also deals with instability. On a technical level, the presence of the point vortex requires a number of non-trivial modifications. Mielke's work was followed by a series of papers proving the existence and conditional stability of different families of solitary water waves by a variational approach in which the waves are constructed using the direct method of the calculus of variations as minimizers of the energy subject to the constraint of fixed momentum. The stability of the set of minimizers then follows from classical arguments by Cazenave and Lions. In particular, Buffoni [7] considered solitary waves on finite depth with strong surface tension. He also obtained partial results in the case of finite depth and weak surface tension, as well as in the case of infinite depth [8, 9], which were later completed by Groves and Wahlén [21, 22]. More recently, Groves and Wahlén extended the method to solitary water waves with constant vorticity [23]. Similar to the present study, the Hamiltonian formulation is non-canonical in that case. It's likely that direct variational methods could also be used in the presence of point vortices.

Plan of the article

In Section 2, we give a detailed description of our results regarding conditional orbital stability and instability of bound states in abstract Hamiltonian systems with symmetry. Our main result on orbital stability is Theorem 3.11, which is proved in in Section 3. The unstable case is addressed in Theorem 3.14, whose proof is carried out in Section 4.

We return to the water wave with a point vortex problem in Section 5, where it is shown that (3.1) can be reformulated as an infinite-dimensional Hamiltonian system of the type covered by the general theory. In Section 6, we characterize the spectrum of the so-called linearized augmented Hamiltonian at a solitary wave, which is used to prove our main result: small-amplitude and small vorticity symmetric solitary capillary-gravity water waves with a point vortex are conditionally orbitally stable; see Theorem 3.33.

To demonstrate the broader implications of the general theory, we consider a large family of nonlinear dispersive PDEs in Section 7. These serve as approximate models for water waves, and include the Korteweg–de Vries equation and Benjamin–Ono equation. Because the corresponding J is not surjective between the relevant spaces, they lie outside the GSS framework. In [6], Bona, Souganidis, and Strauss overcame this difficulty by supplementing the basic approach of GSS with a consideration of the mass. On the other hand, the general theory we develop in the present paper can be directly applied to this family of equations, meaning we are able to give a new proof of the Bona–Souganidis–Strauss theorem as a straightforward application. In fact, this also furnishes new instability results for fractional KdV; see Theorem 3.42

2 General setting and main results

Formulation and hypotheses

We will work with a scale of spaces

$$\mathbb{W} \hookrightarrow \mathbb{V} \hookrightarrow \mathbb{X},$$

where $\mathbb{X}$ is a Hilbert space, while $\mathbb{V}$ and $\mathbb{W}$ are reflexive Banach spaces. The inner product on $\mathbb{X}$ will be denoted by $(\cdot,\cdot)_{\mathbb{X}}$, and the corresponding norm by $\|\cdot\|_{\mathbb{X}}$. Likewise, let $\|\cdot\|_{\mathbb{V}}$ and $\|\cdot\|_{\mathbb{W}}$ be the norms for $\mathbb{W}$ and $\mathbb{V}$, respectively. We write $\mathbb{X}^*$ for the (continuous) dual of $\mathbb{X}$, which is naturally isomorphic to $\mathbb{X}$ via the mapping $I\colon \mathbb{X} \to \mathbb{X}^*$ taking $u \in \mathbb{X}$ to $(u,\cdot)_{\mathbb{X}} \in \mathbb{X}^*$. We will not make this identification here, but rather use I explicitly. On the other hand, we will simply identify $\mathbb{X}^{**}$ with $\mathbb{X}$, and likewise for $\mathbb{W}$ and $\mathbb{V}$. The pairing of $\mathbb{X}$ and $\mathbb{X}^*$ we denote by $\langle\cdot,\cdot\rangle_{\mathbb{X}^*\times\mathbb{X}}$, while $\langle\cdot,\cdot\rangle_{\mathbb{W}^*\times\mathbb{W}}$ is the pairing between $\mathbb{W}^*$ and $\mathbb{W}$; when there is no risk of confusion, we will omit the subscript.

Intuitively, $\mathbb{X}$ is the energy space for the system under consideration. This is where the Hamiltonian structure will be formulated and the natural setting for analyzing the spectrum. On the other hand, $\mathbb{V}$ is a space where the conserved quantities are smooth. Finally, we think of $\mathbb{W}$ as a "well-posedness space", with the norm coming from higher-order energy estimates used to prove that the Cauchy problem is at least locally well-posed in time. The norm on $\mathbb{W}$ also plays the secondary role of allowing us to get control over $\mathbb{V}$ via interpolation. More precisely, we require the following:

Assumption 3.2 (Spaces). Let $\mathbb{X}$, $\mathbb{V}$, and $\mathbb{W}$ be given as above. Assume that there exist constants $\theta \in (0,1]$ and $C > 0$ such that

$$\|u\|_{\mathbb{V}}^3 \leq C\|u\|_{\mathbb{X}}^{2+\theta}\|u\|_{\mathbb{W}}^{1-\theta} \tag{3.4}$$

for all $u \in \mathbb{W}$.

Remark 3.3. A useful consequence of (3.4) is that, if $F \in C^3(\mathbb{V};\mathbb{R})$, and $B \subset \mathbb{W}$ is a bounded set, then

$$F(x+h) - F(x) = \langle DF(x), h\rangle + \frac{1}{2}\langle D^2F(x)h, h\rangle + O(\|h\|_{\mathbb{X}}^{2+\theta})$$

for $x \in \mathbb{V}$ and $h \in B$.

It is often necessary to restrict attention to some smaller subset of these spaces in order to ensure that the problem is well-defined. For example, in the case of the traveling waves with a point vortex, there must be a positive separation between the vortex center and the air–sea interface. Abstractly, we will handle these types of situations by introducing an open set $\mathcal{O} \subset \mathbb{X}$.

Suppose that $\hat{J} \colon D(J) \subset \mathbb{X}^* \to \mathbb{X}$ is a closed linear operator, and that we for each $u \in \mathcal{O} \cap \mathbb{V}$ have a bounded linear operator $B(u) \in \operatorname{Lin}(\mathbb{X})$. We endow $\mathbb{X}$ with symplectic structure in the form of the state-dependent Poisson map

$$J(u) \coloneqq B(u)\hat{J}, \tag{3.5}$$

which is required to satisfy a number of hypotheses.

Assumption 3.4 (Poisson map).

(i) The domain $\mathcal{D}(\hat{J})$ is dense in $\mathbb{X}^*$.

(ii) $\hat{J}$ is injective.

(iii) For each $u \in \mathcal{O} \cap \mathbb{V}$, the operator $B(u)$ is bijective.

(iv) The map $u \mapsto B(u)$ is of class $C^1(\mathcal{O} \cap \mathbb{V}; \operatorname{Lin}(\mathbb{X})) \cap C^1(\mathcal{O} \cap \mathbb{W}; \operatorname{Lin}(\mathbb{W}))$.

(v) For each $u \in \mathcal{O} \cap \mathbb{V}$, $J(u)$ is skew-adjoint in the sense that

$$\langle J(u)v, w\rangle = -\langle v, J(u)w\rangle$$

for all $v, w \in \mathcal{D}(\hat{J})$.

Remark 3.5. Note that this does not assume that $J(u)$ is surjective, which is a significant departure from Grillakis, Shatah, and Strauss. Below, we will require something slightly stronger than that the range of $J(u)$ is dense in $\mathbb{X}$.

The main object of interest for this work is the abstract Hamiltonian system

$$\frac{du}{dt} = J(u)DE(u), \qquad u|_{t=0} = u_0. \tag{3.6}$$

Here $E \in C^3(\mathcal{O} \cap \mathbb{V}; \mathbb{R})$ is the *energy functional*. In addition to the energy, we suppose that there is a second conserved quantity $P \in C^3(\mathcal{O} \cap \mathbb{V}; \mathbb{R})$, which we call the *momentum*. In order to state what it means to be a solution of (3.6), and to work with it in a meaningful way, we need to be able to view $DE(u)$ and $DP(u)$ as elements of $\mathbb{X}^*$.

Assumption 3.6 (Derivative extension). There exist mappings $\nabla E, \nabla P \in C^0(\mathcal{O} \cap \mathbb{V}; \mathbb{X}^*)$ such that $\nabla E(u)$ and $\nabla P(u)$ are extensions of $DE(u)$ and $DP(u)$, respectively, for every $u \in \mathcal{O} \cap \mathbb{V}$.

We say that $u \in C^0([0,t_0);\mathcal{O}\cap\mathbb{W})$ is a solution of (3.6) on the interval $[0,t_0)$ if

$$\frac{d}{dt}\langle u(t),\, w\rangle = -\langle \nabla E(u(t)),\, J(u(t))w\rangle \qquad \text{for all } w \in \mathcal{D}(\hat{J}), \tag{3.7}$$

is satisfied in the distributional sense on $(0,t_0)$, the initial condition $u(0)=u_0$ is satisfied, and both E and P are conserved.

Of particular importance is the situation where the system (3.6) is invariant with respect to a symmetry group. Specifically, we assume that there exists a one-parameter family of affine maps $T(s)\colon \mathbb{X}\to\mathbb{X}$, with linear part $dT(s)u \coloneqq T(s)u - T(s)0$, having the properties described below. We refer to [19] for a background on affine groups on Banach spaces.

Assumption 3.7 (Symmetry group). The symmetry group $T(\cdot)$ satisfies the following.

(i) *(Invariance)* The neighborhood $\mathcal{O}$, and the subspaces $\mathbb{V}$ and $\mathbb{W}$, are all invariant under the symmetry group. Moreover, $I^{-1}\mathcal{D}(\hat{J})$ is invariant under the *linear* symmetry group.

(ii) *(Flow property)* We have $T(0) = dT(0) = \mathrm{Id}_{\mathbb{X}}$, and for all $s, r \in \mathbb{R}$,

$$T(s+r) = T(s)T(r), \qquad \text{and hence} \qquad dT(s+r) = dT(s)dT(r).$$

(iii) *(Unitarity)* The linear part $dT(s)$ is a unitary operator on $\mathbb{X}$, and an isometry on $\mathbb{V}$ and $\mathbb{W}$, for each $s \in \mathbb{R}$.

(iv) *(Strong continuity)* The symmetry group is strongly continuous on $\mathbb{X}$, $\mathbb{V}$, and $\mathbb{W}$.

(v) *(Affine part)* The function $T(\cdot)0$ belongs to $C^3(\mathbb{R};\mathbb{W})$ and there exists an increasing function $\omega\colon [0,\infty)\to[0,\infty)$ such that

$$\|T(s)0\|_{\mathbb{W}} \le \omega(\|T(s)0\|_{\mathbb{X}}), \quad \text{for all } s \in \mathbb{R}.$$

(vi) *(Commutativity with J)* For all $s \in \mathbb{R}$,

$$\begin{aligned} \hat{J} I dT(s) &= dT(s)\hat{J} I, \\ dT(s)B(u) &= B(T(s)u)dT(s), \quad \text{for all } u \in \mathcal{O}\cap\mathbb{V}. \end{aligned} \tag{3.8}$$

(vii) *(Infinitesimal generator)* The infinitesimal generator of T is the affine mapping

$$T'(0)u = \lim_{s\to 0}\Big(s^{-1}(T(s)u - u)\Big) = dT'(0)u + T'(0)0,$$

with dense domain $\mathcal{D}(T'(0)) \subset \mathbb{X}$ consisting of all $u \in \mathbb{X}$ such that the limit exists in $\mathbb{X}$ (note that $\mathcal{D}(T'(0)) = \mathcal{D}(dT'(0))$ by the first part of (v)). Similarly, we may speak of the dense subspaces $\mathcal{D}(T'(0)|_{\mathbb{V}}) \subset \mathbb{V}$ and $\mathcal{D}(T'(0)|_{\mathbb{W}}) \subset \mathbb{W}$ on which the limit exists in $\mathbb{V}$ and $\mathbb{W}$, respectively. We assume that $\nabla P(u) \in \mathcal{D}(\hat{J})$ for every $u \in \mathcal{D}(T'(0)|_{\mathbb{V}}) \cap \mathcal{O}$, and that

$$T'(0)u = J(u)\nabla P(u) \tag{3.9}$$

for all such u. Moreover, we assume that

$$\hat{J}IdT'(0) = dT'(0)\hat{J}I. \tag{3.10}$$

(viii) *(Density)* The subspace

$$\mathcal{D}(T'(0)|_{\mathbb{W}}) \cap \operatorname{Rng}\hat{J}$$

is dense in $\mathbb{X}$.

(ix) *(Conservation)* For all $u \in \mathcal{O} \cap \mathbb{V}$, the energy is conserved by flow of the symmetry group:

$$E(u) = E(T(s)u), \qquad \text{for all } s \in \mathbb{R}. \tag{3.11}$$

Remark 3.8. There are some immediate consequences of the above assumptions. We can combine parts (ii) and (vi) to deduce that

$$dT(s)J(u)I = J(T(s)u)IdT(s), \qquad \text{for all } s \in \mathbb{R},\ u \in \mathcal{O} \cap \mathbb{V},$$

and as a consequence of the unitarity of $dT(s)$, the operator $dT'(0)$ is skew-adjoint on $\mathbb{X}$. Moreover, if $u \in \mathcal{D}(T'(0)|_{\mathbb{V}}) \cap \mathcal{O}$, then $s \mapsto P(T(s)u)$ has derivative

$$\langle \nabla P(T(s)u), T'(0)T(s)u\rangle = \langle \nabla P(T(s)u), J(T(s)u)\nabla P(T(s)u)\rangle = 0$$

by (3.9) and the skew-adjointness of $J(T(s)u)$. Thus, by density of $\mathcal{D}(T'(0)|_{\mathbb{V}})$ in $\mathbb{V}$, the flow of the symmetry group also conserves the momentum for all $u \in \mathcal{O} \cap \mathbb{V}$:

$$P(u) = P(T(s)u), \qquad \text{for all } s \in \mathbb{R}. \tag{3.12}$$

We say that $u \in C^1(\mathbb{R}; \mathcal{O} \cap \mathbb{W})$ is a *bound state* of the Hamiltonian system (3.6) provided that it is a solution of the form

$$u(t) = T(ct)U_c,$$

for some $c \in \mathbb{R}$ and $U_c \in \mathcal{O} \cap \mathbb{W}$. We will also call U_c itself a bound state. If T represents translation, then bound states correspond to the familiar notion of traveling waves, such as the ones we will study later. For the general setting, we take it as given that an analogous family is available:

Assumption 3.9 (Bound states). There exists a one-parameter family of bound state solutions $\{U_c : c \in \mathcal{I}\}$ to the Hamiltonian system (3.6).

(i) The mapping $c \in \mathcal{I} \mapsto U_c \in \mathcal{O} \cap \mathbb{W}$ is C^1.

(ii) The non-degeneracy condition $T'(0)U_c \neq 0$ holds for every $c \in \mathcal{I}$. Equivalently, U_c is never a critical point of the momentum.

(iii) For all $c \in \mathcal{I}$,

$$U_c \in \mathcal{D}(T'''(0)) \cap \mathcal{D}(\hat{J}IT'(0)), \tag{3.13}$$

and

$$U_c, \hat{J}IT'(0)U_c \in \mathcal{D}(T'(0)|_{\mathbb{W}}). \tag{3.14}$$

(iv) Either $s \mapsto T(s)U_c$ is periodic, or $\liminf_{|s|\to\infty} \|T(s)U_c - U_c\|_{\mathbb{X}} > 0$.

Observe that, due to (3.11) and (3.12), the energy and momentum of $T(s)U_c$ is independent of s. For a fixed parameter c, the corresponding *augmented Hamiltonian* is the functional $E_c \in C^3(\mathbb{V} \cap \mathcal{O}; \mathbb{R})$ defined by

$$E_c(u) := E(u) - cP(u).$$

Assumption 3.7 ensures that $U_c \in \mathcal{D}(T'(0))$, and so it follows from (3.7), (3.9), and Assumption 3.4 that U_c is a critical point E_c:

$$DE_c(U_c) = DE(U_c) - cDP(U_c) = 0. \tag{3.15}$$

In that way, we can think of each of the bound state U_c as being a critical point of the energy with the constraint of a fixed momentum, and the wave speed c arises naturally as a Lagrange multiplier. Also, differentiating (3.15) with respect to c reveals that

$$\left\langle D^2E_c(U_c)\frac{dU_c}{dc}, \cdot \right\rangle = \langle DP(U_c), \cdot \rangle \,. \tag{3.16}$$

Commonly in applications, the bound states sit at a saddle point of the energy. That is, the second variation of the augmented Hamiltonian at U_c has a single simple negative (real) eigenvalue, a 0 eigenvalue generated by the symmetry group, and the rest of the spectrum lies along the positive real axis bounded uniformly away from the origin. This is the basic setting of the problem considered in Grillakis, Shatah, and Strauss [20], and it is precisely what we will encounter in our study of water waves later. We therefore make the following hypotheses about the configuration of the spectrum for the general theory.

Assumption 3.10 (Spectrum). The operator $D^2E_c(U_c) \in \operatorname{Lin}(\mathbb{V}, \mathbb{V}^*)$ extends uniquely to a bounded linear operator $H_c \colon \mathbb{X} \to \mathbb{X}^*$ such that:

(i) $I^{-1}H_c$ is self-adjoint on $\mathbb{X}$.

(ii) The spectrum of $I^{-1}H_c$ satisfies

$$\operatorname{spec}(I^{-1}H_c) = \{-\mu_c^2\} \cup \{0\} \cup \Sigma_c, \tag{3.17}$$

where $-\mu_c^2 < 0$ is a simple eigenvalue corresponding to a unit eigenvector χ_c, 0 is a simple eigenvalue generated by T, and $\Sigma \subset (0, \infty)$ is bounded away from 0.

Statement of the main results on stability and instability

The central question we wish to address is whether these bound states are stable or unstable. As there is an underlying invariance with respect to the group T, it is most natural to understand stability and instability in the orbital sense. For any $U \in \mathbb{X}$, we call the set $\{T(s)U : s \in \mathbb{R}\}$ the *U-orbit generated by T*. Formally speaking, U_c is *orbitally stable* provided that any solution to the Cauchy problem that is initially close enough to the U_c-orbit generated by T (in the $\mathbb{X}$ norm) remains near the orbit for all time. Conversely, *orbital instability* describes the situation where there exists initial data arbitrarily close to the U_c-orbit that nevertheless leaves some neighborhood of the orbit in finite time.

Making these concepts rigorous for the problem at hand is complicated both by the lack of a global well-posedness theory for the Cauchy problem (3.6), and especially the mismatch of the energy and well-posedness spaces. For that reason, all of our results will necessarily be *conditional* in that they will hold only so long as we know the solution exists and that its growth in $\mathbb{W}$ is controllable.

The *moment of instability*, which we call d, is the scalar-valued function that results from evaluating the augmented Hamiltonian along the family of bound states:

$$d(c) := E_c(U_c) = E(U_c) - cP(U_c). \tag{3.18}$$

Note that because each bound state U_c is a critical point of the augmented Hamiltonian, differentiating d gives the identity

$$d'(c) = \left\langle DE_c(U_c), \frac{dU_c}{dc} \right\rangle - P(U_c) = -P(U_c), \tag{3.19}$$

and differentiating once more yields

$$d''(c) = -\left\langle DP(U_c), \frac{dU_c}{dc} \right\rangle = -\left\langle D^2E_c(U_c)\frac{dU_c}{dc}, \frac{dU_c}{dc} \right\rangle, \tag{3.20}$$

where the last equality follows from (3.16).

For each $\rho > 0$, let

$$\mathcal{U}_\rho^{\mathbb{X}} := \left\{ u \in \mathcal{O} : \inf_{s\in\mathbb{R}} \|u - T(s)U_c\|_{\mathbb{X}} < \rho \right\}$$

be the tubular neighborhood of radius ρ in $\mathbb{X}$ for the U_c-orbit generated by T. We also define

$$\mathcal{B}_R^{\mathbb{W}} = \left\{ u \in \mathcal{O} \cap \mathbb{W} : \inf_{s\in\mathbb{R}} \|T(s)u\|_{\mathbb{W}} < R \right\}$$

for $R > 0$, which collapses to a ball if the symmetry group has no affine part.

Our first result states that if $d''(c) > 0$ at a certain wave speed $c \in \mathcal{I}$, then U_c is conditionally orbitally *stable*.

Theorem 3.11 (Stability). *Suppose that the above assumptions hold. If $d''(c) > 0$, then the bound state U_c is conditionally orbitally stable in the following sense. For any $R > 0$ and $\rho > 0$, there exists $\rho_0 > 0$ such that, if $u\colon [0, t_0) \to \mathcal{B}_R^{\mathbb{W}}$ is a solution of (3.6), with initial data $u_0 \in \mathcal{U}_{\rho_0}^{\mathbb{X}}$, then $u(t) \in \mathcal{U}_\rho^{\mathbb{X}}$ for all $t \in [0, t_0)$.*

Remark 3.12. As will become clear in the next section, the stability theorem holds under weaker hypotheses. Most notably, we can drop the intersection with $\mathcal{D}(T'(0)|_{\mathbb{W}})$ in Assumption 3.7 (viii).

In order to prove an instability result, we need to know that (3.6) can be solved at least locally around the U_c-orbit. If we introduce

$$\mathcal{U}_\nu^{\mathbb{W}} := \left\{ u \in \mathcal{O} \cap \mathbb{W} : \inf_{s\in\mathbb{R}} \|u - T(s)U_c\|_{\mathbb{W}} < \nu \right\}$$

for $\nu > 0$, we mean the following.

Assumption 3.13 (Local existence). There exists $\nu_0 > 0$ and $t_0 > 0$ such that for all initial data $u_0 \in \mathcal{U}_{\nu_0}^{\mathbb{W}}$, there exists a unique solution to (3.6) on the interval $[0, t_0)$.

With the above hypothesis, we can conclude that if $d''(c) < 0$, then U_c is conditionally orbitally *unstable.*

Theorem 3.14 (Instability). *If $d''(c) < 0$ and Assumption 3.13 is satisfied, then the bound state U_c is orbitally unstable: There exists a $\nu_0 > 0$ such for every $0 < \nu < \nu_0$ there exists initial data in $\mathcal{U}_{\nu}^{\mathbb{W}}$ whose corresponding solution exits $\mathcal{U}_{\nu_0}^{\mathbb{W}}$ in finite time.*

If $\mathbb{X} = \mathbb{W}$, we also obtain a more conventional stability result as a corollary of Theorem 3.11.

Corollary 3.15 (Stability when $\mathbb{X} = \mathbb{W}$). *If $d''(c) > 0$, Assumption 3.13 holds, and $\mathbb{X} = \mathbb{W}$, then the bound state U_c is orbitally stable: For any $\nu > 0$, there exists $\nu_0 > 0$ such that the solution for any initial data $u_0 \in \mathcal{U}_{\nu_0}^{\mathbb{W}}$ exists globally and stays in $\mathcal{U}_{\nu}^{\mathbb{W}}$.*

Together, Theorem 3.14 and Corollary 3.15 essentially recover the classical GSS theory in the special case that $\mathbb{X} = \mathbb{W}$, J is a state-independent isomorphism, $T(s)$ is linear, and the Hamiltonian system (3.6) is globally well-posed. The only exception is that, in the interest of brevity, we have not addressed the situation where $d''(c) = 0$.

Lastly, let us comment on how the above results relate to the recent monumental paper of Lin and Zeng [29], which studies the dynamics of linear Hamiltonian systems under weaker assumptions on the Poisson map than ours (for instance, they allow an infinite-dimensional kernel). While this theory concerns the linear case, under some conditions it can be applied to construct invariant manifolds for nonlinear systems as well; see the work of Jin, Lin, and Zeng [25, 26]. When this can be accomplished, it gives considerably more information than the conditional orbital stability/instability we obtain from Theorem 3.11 or Theorem 3.14. However, the methodology has difficulty attacking equations for which the solution map incurs a loss of derivatives, such as quasilinear problems. To overcome this, one needs the linear evolution to display sufficiently strong smoothing properties, which limits somewhat the applicability. By contrast, the framework we present here is adapted to the quasilinear setting by design, and does not rely on linear estimates.

3 Stability in the general setting

The purpose of this section is to prove Theorem 3.11 on the conditional orbital stability of the bound state U_c under the assumption that $d''(c) > 0$. Our basic approach follows the ideas of Grillakis, Shatah, and Strauss, but many adaptations are required due to the more complicated functional analytic setting. Interestingly, the state dependence of J is not a major issue for this argument.

We begin with a technical lemma which states that, in a sufficiently small tubular neighborhood $\mathcal{U}_\rho^{\mathbb{X}}$ of U_c, one can find a parameter value s (depending on u) such that the distance between $T(s)u$ and U_c in the energy norm is minimized.

Lemma 3.16. *If $s \mapsto T(s)U_c$ is not periodic, then then exists a $\rho > 0$ and a function $\tilde{s} \in C^2(\mathcal{U}_\rho^{\mathbb{X}}; \mathbb{R})$ such that, for all $u \in \mathcal{U}_\rho^{\mathbb{X}}$ the following holds.*

(a) $\|T(\tilde{s}(u))u - U_c\|_{\mathbb{X}} \leq \|T(r)u - U_c\|_{\mathbb{X}}$, *for all* $r \in \mathbb{R}$.

(b) $(T(\tilde{s}(u))u - U_c, T'(0)U_c)_{\mathbb{X}} = 0$.

(c) $\tilde{s}(T(r)u) = \tilde{s}(u) - r$ *for all* $r \in \mathbb{R}$.

(d) For all $u \in \mathcal{U}_\rho^{\mathbb{X}}$ *and* $v \in \mathbb{X}$,

$$\begin{aligned}
\langle D\tilde{s}(u), v\rangle &= -\frac{\langle \sigma_1(u), v\rangle}{r_1(u)}, \\
\langle D^2\tilde{s}(u)v, v\rangle &= -\frac{r_2(u)\langle \sigma_1(u), v\rangle^2}{r_1(u)^3} - 2\frac{\langle \sigma_1(u), v\rangle\langle \sigma_2(u), v\rangle}{r_1(u)^2},
\end{aligned}$$

where

$$\begin{aligned}
\sigma_k(u) &:= IT^{(k)}(-\tilde{s}(u))U_c, \\
r_1(u) &:= \|T'(0)U_c\|_{\mathbb{X}}^2 - (T(\tilde{s}(u))u - U_c, T''(0)U_c)_{\mathbb{X}}, \\
r_2(u) &:= (T(\tilde{s}(u))u - U_c, T'''(0)U_c).
\end{aligned}$$

(e) We have $D\tilde{s}(u) \in \mathcal{D}(\hat{J})$ *for every* $u \in \mathcal{U}_\rho^{\mathbb{X}}$, *and the map* $g \colon \mathcal{U}_\rho^{\mathbb{X}} \cap \mathbb{W} \to \mathbb{W}$ *defined by* $g(u) := J(u)D\tilde{s}(u)$ *is of class* $C^1(\mathcal{U}_\rho^{\mathbb{X}} \cap \mathbb{W}; \mathbb{W})$.

If instead $s \mapsto T(s)U_c$ has minimal period L, then the same result is true except $\tilde{s} \in C^2(\mathcal{U}_\rho^{\mathbb{X}}; \mathbb{R}/L\mathbb{R})$ and the equality in part (c) holds modulo L.

Proof. For $s \in \mathbb{R}$ and $u \in \mathbb{X}$, set

$$h(u,s) := \frac{1}{2}\|T(s)u - U_c\|_{\mathbb{X}}^2 = \frac{1}{2}\|u - T(-s)U_c\|_{\mathbb{X}}^2.$$

Then

$$\begin{aligned}
\partial_s h(u,s) &= (T(s)u - U_c, T'(0)U_c)_{\mathbb{X}},\\
\partial_s^2 h(u,s) &= \|T'(0)U_c\|_{\mathbb{X}}^2 - (T(s)u - U_c, T''(0)U_c)_{\mathbb{X}}.
\end{aligned}$$

Clearly $\partial_s h(U_c, 0) = 0$ and $\partial_s^2 h(U_c, 0) = \|T'(0)U_c\|_{\mathbb{X}}^2 > 0$. The implicit function theorem then ensures the existence of a ball $B_\delta \subset \mathbb{X}$ centered at U_c, an interval $(-s_0, s_0)$, and a C^2 map $\tilde{s}\colon B_\delta \to (-s_0, s_0)$ such that the equation $\partial_s h(u,s) = 0$ has a unique solution $s = \tilde{s}(u) \in (-s_0, s_0)$ for all $u \in B_\delta$. Thus $s = \tilde{s}(u)$ uniquely minimizes $h(u,\cdot)$ on $(-s_0, s_0)$ for fixed $u \in B_\delta$.

We will only present the argument for the non-periodic orbits as the proof for the periodic case requires only a simple modification. Assumption 3.9 (iv) then guarantees that there exists an $\eta > 0$ such that

$$\inf_{s \geq s_0} \|T(s)U_c - U_c\|_{\mathbb{X}} \geq \eta.$$

Let $\rho := \min(\eta/3, \delta)$. Then, if $u \in B_\rho$ and $r \in \mathbb{R}$ are such that $\|T(r)u - U_c\|_{\mathbb{X}} \leq \|T(\tilde{s}(u))u - U_c\|$, we have

$$\begin{aligned}
\|T(r)U_c - U_c\|_{\mathbb{X}} &= \|dT(r)(U_c - u) + T(r)u - U_c\|_{\mathbb{X}}\\
&\leq \|U_c - u\|_{\mathbb{X}} + \|T(\tilde{s}(u))u - U_c\|_{\mathbb{X}}\\
&\leq 2\|u - U_c\|_{\mathbb{X}} < \eta,
\end{aligned}$$

which implies that $r \in (-s_0, s_0)$ and hence $r = \tilde{s}(u)$ by uniqueness. This completes the proof of parts (a) and (b) for $u \in B_\rho$.

For part (c), note that if both u and $T(r)u$ lie in B_ρ, then

$$\|T(\tilde{s}(u) - r)T(r)u - U_c\|_{\mathbb{X}} = \|T(\tilde{s}(u))u - U_c\|_{\mathbb{X}} \leq \|T(t)u - U_c\|_{\mathbb{X}}$$

for all $t \in \mathbb{R}$. In particular, if we choose $t = \tilde{s}(T(r)u) + r$, we obtain part (c) on B_ρ by uniqueness. Moreover, as a consequence, we can proceed to extend $\tilde{s}$ to all of $\mathcal{U}_\rho^{\mathbb{X}}$ through

$$\tilde{s}(u) = \tilde{s}(T(r)u) + r,$$

where r is such that $T(r)u \in B_\rho$. This is well defined, since if both $T(r)u$ and $T(s)u$ lie in B_ρ, then

$$\tilde{s}(T(s)u) = \tilde{s}(T(s-r)T(r)u) = \tilde{s}(T(r)) - (s - r)$$

by part (c) on B_ρ.

The identities in part (d) follow by straightforward calculations.

Finally, for any $u \in \mathcal{U}_\rho^{\mathbb{X}}$, we have $\sigma_1(u) \in \mathcal{D}(\hat{J})$ by Assumption 3.9 (iii) and (3.8), and since moreover

$$J(u)\sigma_1(u) = B(u)dT(-\tilde{s}(u))\hat{J}IT'(0)U_c$$

by (3.8), part (e) follows from (3.14). □

By Assumption 3.10 (ii), we know that for $c \in \mathcal{I}$, the spectrum of $I^{-1}H_c$ consists of the simple eigenvalues $-\mu_c^2$ and 0, and a subset Σ_c of the positive real axis, bounded away from 0. It follows that $\mathbb{X}$ admits the spectral decomposition

$$\mathbb{X} = \mathbb{X}_- \oplus \mathbb{X}_0 \oplus \mathbb{X}_+,$$

where $\mathbb{X}_- := \operatorname{span}\{\chi_c\}$, $\mathbb{X}_0 = \operatorname{span}\{T'(0)U_c\}$, and $\mathbb{X}_+$ is the positive subspace of $I^{-1}H_c$. Here we are using the fact that $T'(0)U_c$ is a generator for the kernel of $I^{-1}H_c$. Observe that the restriction of $I^{-1}H_c$ to $\mathbb{X}_+$ is a positive operator, in the sense that there exists an $\alpha = \alpha(c) > 0$ such that

$$\langle H_c v, v\rangle \geq \alpha \|v\|_{\mathbb{X}}^2 \qquad \text{for all } v \in \mathbb{X}_+. \tag{3.21}$$

The following lemma describes a version of this inequality which holds also outside $\mathbb{X}_+$.

Lemma 3.17. *Suppose that $y \in \mathbb{X}$ is such that $\langle H_c y, y\rangle < 0$. Then there exists a constant $\tilde{\alpha} > 0$ such that*

$$\langle H_c v, v\rangle \geq \tilde{\alpha}\|v\|_{\mathbb{X}}^2 \tag{3.22}$$

for every $v \in \mathbb{X}$ satisfying

$$\langle H_c y, v\rangle = 0 \qquad \textit{and} \qquad (T'(0)U_c, v)_{\mathbb{X}} = 0. \tag{3.23}$$

Proof. We decompose y as

$$y = a_0\chi_c + b_0 T'(0)U_c + p_0, \qquad \text{for some } a_0, b_0 \in \mathbb{R},\ p_0 \in \mathbb{X}_+,$$

from which we compute that

$$\langle H_c y, y\rangle = -a_0^2\mu_c^2 + \langle H_c p_0, p_0\rangle,$$

or

$$a_0^2\mu_c^2 = \langle H_c p_0, p_0\rangle + |\langle H_c y, y\rangle|, \tag{3.24}$$

which in particular implies that $a_0^2 > 0$.

Now, let v be as in the statement of the lemma. Using the spectral decomposition of $\mathbb{X}$, we may likewise write

$$v = a\chi_c + p, \qquad \text{for some } a \in \mathbb{R},\ p \in \mathbb{X}_+,$$

as v has no component in $\mathbb{X}_0$, by assumption. Moreover, we have

$$0 = \langle H_c y, v\rangle = -a_0 a \mu_c^2 + \langle H_c p_0, p\rangle,$$

and therefore

$$a = \frac{\langle H_c p_0, p\rangle}{a_0 \mu_c^2}.$$

It follows that

$$\begin{aligned}
\langle H_c v, v\rangle &= -a^2\mu_c^2 + \langle H_c p, p\rangle = -\frac{\langle H_c p_0, p\rangle^2}{a_0^2\mu_c^2} + \langle H_c p, p\rangle \\
&\geq \left(1 - \frac{\langle H_c p_0, p_0\rangle}{a_0^2\mu_c^2}\right)\langle H_c p, p\rangle = \frac{|\langle H_c y, y\rangle|}{a_0^2\mu_c^2}\langle H_c p, p\rangle \\
&\geq \alpha\frac{|\langle H_c y, y\rangle|}{a_0^2\mu_c^2}\|p\|_{\mathbb{X}}^2
\end{aligned}$$

by the Cauchy–Schwarz inequality applied to $H_c|_{\mathbb{X}_+}$, together with (3.24) and (3.21). Finally, the result now follows by combining this inequality with

$$\|v\|_{\mathbb{X}}^2 = a^2 + \|p\|_{\mathbb{X}}^2 \leq \left(\frac{\|H_c p_0\|_{\mathbb{X}^*}^2}{a_0^2\mu_c^4} + 1\right)\|p\|_{\mathbb{X}}^2. \qquad \square$$

We obtain the following as a corollary.

Corollary 3.18. *Suppose that $d''(c) > 0$. Then there exists a constant $\tilde{\alpha} > 0$ such that* (3.22) *holds for every $v \in \mathbb{X}$ satisfying*

$$\langle \nabla P(U_c), v\rangle = 0 \qquad \textit{and} \qquad (T'(0)U_c, v)_{\mathbb{X}} = 0.$$

Proof. If $d''(c) > 0$, we may apply Lemma 3.17 with $y = \frac{dU_c}{dc}$, by (3.20). Furthermore, we have $H_c\frac{dU_c}{dc} = \nabla P(U_c)$ due to (3.16). $\square$

Note that in the setting of Lemma 3.16(a),

$$\|T(\tilde{s}(u)) - U_c\|_{\mathbb{X}} = \inf_{r\in\mathbb{R}} \|T(r)u - U_c\|_{\mathbb{X}} < \rho, \qquad \text{for all } u \in \mathcal{U}_\rho^{\mathbb{X}}.$$

It therefore makes sense to define the map

$$M : \mathcal{U}_\rho^{\mathbb{X}} \ni u \mapsto T(\tilde{s}(u))u \in \mathcal{U}_\rho^{\mathbb{X}}$$

whenever $\rho > 0$ is small enough for the lemma to apply. Note that M is also invariant under the action of T, as

$$M(T(s)u) = T(\tilde{s}(T(s)u))T(s)u = T(\tilde{s}(u) - s)T(s)u = T(\tilde{s}(u))u = M(u), \tag{3.25}$$

where the second equality comes from Lemma 3.16(c). Moreover, we are able to bound $M(u)$ in the smoother norm.

Lemma 3.19. *Let $R > 0$, and suppose that $\rho > 0$ is like in Lemma 3.16. Then*

$$\|M(u)\|_{\mathbb{W}} \le R + \omega(\rho + \|\iota_{\mathbb{W}\hookrightarrow\mathbb{X}}\|R + \|U_c\|_{\mathbb{X}}) \qquad \text{for all } u \in \mathcal{U}_\rho^{\mathbb{X}} \cap \mathcal{B}_R^{\mathbb{W}}.$$

Proof. If $u \in \mathcal{U}_\rho^{\mathbb{X}} \cap \mathcal{B}_R^{\mathbb{W}}$, then in particular there exists an $r \in \mathbb{R}$ such that $\|T(r)u\|_{\mathbb{W}} < R$. Set $v = T(r)u$, and observe that

$$\|M(u)\|_{\mathbb{W}} = \|M(v)\|_{\mathbb{W}} = \|dT(\tilde{s}(v))v + T(\tilde{s}(v))0\|_{\mathbb{W}} \le R + \omega(\|T(\tilde{s}(v))0\|_{\mathbb{X}})$$

by (3.25) and Assumption 3.7 (v). The result now follows by combining this inequality with

$$\begin{aligned}\|T(\tilde{s}(v))0\|_{\mathbb{X}} &= \|M(v) - U_c + U_c - dT(\tilde{s}(v))v\|_{\mathbb{X}} \\ &\le \rho + \|U_c\|_{\mathbb{X}} + \|\iota_{\mathbb{W}\hookrightarrow\mathbb{X}}\|R,\end{aligned}$$

where we have used that $v \in \mathcal{U}_\rho^{\mathbb{X}}$, since $\mathcal{U}_\rho^{\mathbb{X}}$ is invariant under T, and that $\mathbb{W}$ embeds continuously into $\mathbb{X}$. ☐

We will now use Lemmas 3.17 and 3.19 to obtain the key inequality needed to prove stability. It is convenient to introduce the notation

$$\mathcal{M}_c := \{u \in \mathcal{O} \cap \mathbb{V} : P(u) = P(U_c)\} \tag{3.26}$$

for the level set of the momentum of U_c.

Lemma 3.20. *Suppose that $d''(c) > 0$. Then, for any $R > 0$, there exist $\rho > 0$ and $\beta > 0$ such that*

$$E(u) - E(U_c) \ge \beta\|M(u) - U_c\|_{\mathbb{X}}^2 \qquad \text{for all } u \in \mathcal{U}_\rho^{\mathbb{X}} \cap \mathcal{M}_c \cap \mathcal{B}_R^{\mathbb{W}}. \tag{3.27}$$

Moreover, the assumption that $d''(c) > 0$ can be removed under the additional restriction that $\langle H_c y, M(u) - U_c\rangle = 0$ for a fixed $y \in \mathbb{X}$ such that $\langle H_c y, y\rangle < 0$.

Proof. Let u be as in the statement of the lemma, and set $v := M(u) - U_c$. Expanding E_c in a neighborhood of U_c in $\mathbb{V}$, recalling that U_c is a critical point and that both the energy and momentum are conserved by the group, yields

$$E_c(u) = E_c(U_c + v) = E_c(U_c) + \frac{1}{2}\langle H_c v, v\rangle + O(\|v\|_{\mathbb{V}}^3). \tag{3.28}$$

Note that $(v, T'(0)U_c)_{\mathbb{X}} = 0$ by Lemma 3.16 (b), so if in addition $\langle H_c y, v\rangle = 0$, then Lemma 3.17 ensures the existence of an $\tilde{\alpha} > 0$, independent of v, such that

$$\langle H_c v, v\rangle \geq \tilde{\alpha}\|v\|_{\mathbb{X}}^2.$$

If, on the other hand, $d''(c) > 0$, we decompose v as

$$v = \lambda N + w, \qquad N := I^{-1}\nabla P(U_c), \tag{3.29}$$

with $(N, w)_{\mathbb{X}} = 0$. Taking the inner product of both sides of (3.29) with N, and using that $P(U_c + v) = P(U_c)$, we find

$$\lambda\|N\|_{\mathbb{X}}^2 = (v, N)_{\mathbb{X}} = \langle DP(U_c), v\rangle = O(\|v\|_{\mathbb{V}}^2),$$

whence $\lambda = O(\|v\|_{\mathbb{V}}^2)$. It follows that

$$\langle H_c v, v\rangle = \langle H_c w, w\rangle + O(\|v\|_{\mathbb{V}}^3).$$

We wish to apply Corollary 3.18 to obtain a lower bound for $\langle H_c w, w\rangle$. In that connection, observe that $\langle \nabla P(U_c), w\rangle = 0$, as w is orthogonal to N by construction. Moreover,

$$(w, T'(0)U_c)_{\mathbb{X}} = (v, T'(0)U_c) - \lambda\langle \nabla P(U_c), T'(0)U_c\rangle = 0$$

in view of Lemma 3.16 and (3.9). Thus

$$\langle H_c v, v\rangle \geq \tilde{\alpha}\|w\|_{\mathbb{X}}^2 + O(\|v\|_{\mathbb{V}}^3),$$

where we can eliminate w in favor of v by observing that

$$\|w\|_{\mathbb{X}}^2 \geq (\|v\|_{\mathbb{X}} - |\lambda|\|N\|_{\mathbb{X}})^2 \geq \|v\|_{\mathbb{X}}^2 - O(\|v\|_{\mathbb{V}}^3).$$

In either case, the desired lower bound (3.27) follows if we can control the cubic $O(\|v\|_{\mathbb{V}}^3)$-remainder in (3.28) by using the quadratic $\|v\|_{\mathbb{X}}^2$. This is precisely the motivation behind Assumption 3.2. Indeed, (3.4) and Lemma 3.19 imply that

$$\begin{aligned} \|v\|_{\mathbb{V}}^3 &\leq C\|v\|_{\mathbb{X}}^{2+\theta}\|v\|_{\mathbb{W}}^{1-\theta} \\ &\leq C\rho^\theta[R + \omega(\rho + \|\iota_{\mathbb{W}\hookrightarrow\mathbb{X}}\|R + \|U_c\|_{\mathbb{X}}) + \|U_c\|_{\mathbb{W}}]^{1-\theta}\|v\|_{\mathbb{X}}^2, \end{aligned}$$

which enables us to absorb the remainder into the quadratic term by taking sufficiently small ρ. Note that we can replace E_c by E due to the assumption that $u \in \mathcal{M}_c$. $\square$

We are now prepared to prove the main theorem of the section on the conditional orbital stability of the bound state U_c.

Proof of Theorem 3.11. Seeking a contradiction, suppose there exist $R > 0$, $\rho > 0$, and a sequence of solutions $u_n \colon [0, t_0^n) \to \mathcal{B}_R^{\mathbb{W}}$, with initial data u_0^n, such that $\|M(u_0^n) - U_c\|_{\mathbb{X}} \to 0$, but for which

$$\|M(u_n(\tau_n)) - U_c\|_{\mathbb{X}} = \rho$$

for some $\tau_n \in (0, t_0^n)$. Without loss of generality, we may take τ_n to be the *first* time that u_n exits $\mathcal{U}_\rho^{\mathbb{X}}$. Moreover, we can shrink $\mathcal{U}_\rho^{\mathbb{X}}$ such that Lemma 3.20 applies. Together with the conservation of energy and momentum, we deduce the existence of a $\beta > 0$ such that

$$E(u_0^n) - E(U_c) \geq \beta \|M(u_n(\tau_n)) - U_c\|_{\mathbb{X}}^2 = \beta \rho^2$$

for every n. On the other hand, $E(u_0^n) = E(M(u_0^n))$, and $\|M(u_0^n) - U_c\|_{\mathbb{X}} \to 0$. Combined with the fact that $\sup_n \|M(u_0^n) - U_c\|_{\mathbb{W}} \lesssim_R 1$ by Lemma 3.19, we can use Assumption 3.2 to deduce that $M(u_0^n) \to U_c$ in $\mathbb{V}$, and therefore that $E(u_0^n) \to E(U_c)$. But this contradicts the strictly positive lower bound on $E(u_0^n) - E(U_c)$ derived above, and hence we have arrived at a contradiction. $\square$

4 Instability in the general setting

This section is devoted to proving Theorem 3.14 on the conditional orbital instability of U_c under the assumption that the moment of instability satisfies $d''(c) < 0$. In contrast to Section 3, the state-dependence of the Poisson map J presents a serious technical challenge to the analysis here.

Identification of a negative direction

Because we do not assume that $J(u)$ is surjective, and because χ_c is not necessarily in $\mathbb{W}$, we must make a further modification of the GSS program. The next lemma shows that it is possible to find a negative direction $z \in \mathbb{W}$ that is not only tangent to $\mathcal{M}_c$, but also lies in the range of a restriction of $J(U_c)$. This follows from a surprisingly simple density argument.

Lemma 3.21. *Suppose that $d''(c) < 0$. Then there exists $z \in \mathcal{D}(T'(0)|_{\mathbb{W}})$, of the form $z = J(U_c)IZ$ for some $Z \in \mathcal{D}(T'(0))$, such that*

$$\left\langle D^2 E_c(U_c)z,\, z \right\rangle < 0 \qquad \textit{and} \qquad \langle DP(U_c),\, z\rangle = 0. \tag{3.30}$$

Proof. For ease of notation, we once again set $N \coloneqq I^{-1}\nabla P(U_c)$. Defining the quadratic form $Q \in C^0(\mathbb{X}; \mathbb{R})$ by

$$Q(u) \coloneqq \langle H_c u,\, u\rangle\,,$$

we see that (3.30) can be rephrased as

$$Q(z) < 0 \qquad \text{and} \qquad (N,\, z)_{\mathbb{X}} = 0.$$

The vector

$$y \coloneqq \frac{\langle \nabla P(U_c), \chi_c\rangle}{d''(c)}\frac{dU_c}{dc} + \chi_c \in \mathbb{X}$$

satisfies both of these properties because

$$Q(y) = \frac{\langle \nabla P(U_c), \chi_c\rangle^2}{d''(c)} - \mu_c^2 < 0, \qquad (N,\, y)_{\mathbb{X}} = 0$$

by (3.16) and (3.20). However, y does not necessarily lie in $\mathcal{D}(T'(0)|_{\mathbb{W}})$, nor must it be in the range of $J(U_c)$.

Note that $B(U_c)$ restricts to an isomorphism on $\mathcal{D}(T'(0)|_{\mathbb{W}})$ by (3.8). Thus, if $J(U_c)IZ \in \mathcal{D}(T'(0)|_{\mathbb{W}})$, then $\hat{J}IZ \in \mathcal{D}(T'(0)|_{\mathbb{W}})$, and consequently $Z \in \mathcal{D}(T'(0))$ by (3.10). To complete the proof, it suffices to show that

$$\mathbb{U} \coloneqq \mathcal{D}(T'(0)|_{\mathbb{W}}) \cap \operatorname{Rng} J(U_c)$$

is dense in $N^\perp$, where $N^\perp \coloneqq \{u \in \mathbb{X} : (N, u)_{\mathbb{X}} = 0\}$. Recall that $\mathbb{U}$ is dense in $\mathbb{X}$ due to Assumption 3.7 (viii).

First we claim that there exists $v \in \mathbb{U}$ such that $(N, v)_{\mathbb{X}} \neq 0$. Were this not the case, we would have $\mathbb{U} \subset N^\perp$, which would contradict density in $\mathbb{X}$. Without loss of generality, we may choose v such that $(N, v)_{\mathbb{X}} = 1$.

Now, let $u \in N^\perp$ be given. By density, there exists an approximating sequence $\{u_n\} \subset \mathbb{U}$ with $u_n \to u$ in $\mathbb{X}$. Putting

$$w_n \coloneqq u_n - (N, u_n)_{\mathbb{X}}\, v,$$

we see that the sequence $\{w_n\} \subset N^\perp \cap \mathbb{U}$, and that

$$w_n \to u - (N, u)_{\mathbb{X}}\, v = u \text{ in } \mathbb{X}.$$

Thus $\mathbb{U}$ is indeed dense in $N^\perp$.

By the argument above, there is a sequence $\{z_n\} \subset \mathbb{U} \cap N^\perp$ such that $z_n \to y$ in $\mathbb{X}$. For n sufficiently large, $Q(z_n) < 0$ by continuity, and so the lemma is proved. □

Lyapunov function

In the previous subsection, we constructed a vector z in the negative cone of H_c that is tangent to the fixed momentum manifold $\mathcal{M}_c$ at U_c. The strategy at this point is to use z to build a Lyapunov function for the abstract Hamiltonian system (3.6) and thereby prove instability.

In the next lemma, we follow Grillakis, Shatah, and Strauss by introducing a functional A designed so that the corresponding Hamiltonian vector field (i) points in the direction z at U_c, and (ii) is in the kernel of DP in a tubular neighborhood of U_c.

Lemma 3.22. *There exists a $\rho > 0$ and a functional $A \in C^1(\mathcal{U}_\rho^{\mathbb{X}}; \mathbb{R})$ having the following properties:*

(a) $A(T(s)u) = A(u)$, for all $u \in \mathcal{U}_\rho^{\mathbb{X}}$ and $s \in \mathbb{R}$.

(b) $DA(u) \in \mathcal{D}(\hat{J})$, for all $u \in \mathcal{U}_\rho^{\mathbb{X}}$.

(c) $J(U_c)DA(U_c) = -z$, where z is like in Lemma 3.21.

(d) $\langle DP(u), J(u)DA(u)\rangle = 0$ for all $u \in \mathcal{U}_\nu^{\mathbb{W}}$, where $\nu > 0$ is such that $u \in \mathcal{U}_\nu^{\mathbb{W}} \subset \mathcal{U}_\rho^{\mathbb{X}}$; and

(e) The mapping $u \mapsto J(u)DA(u)$ is of class $C^1(\mathcal{U}_\nu^{\mathbb{W}}; \mathbb{W})$.

Proof. Let z and Z be given as in Lemma 3.21, and choose $\rho > 0$ so that Lemma 3.16 applies. Put

$$A(u) := -(Z, M(u) - U_c)_{\mathbb{X}} \qquad \text{for all } u \in \mathcal{U}_\rho^{\mathbb{X}}. \tag{3.31}$$

Part (a) follows immediately from the corresponding property of M established in (3.25). The regularity of $\tilde{s}$, and the properties of Z, also show that A is C^1 with

$$DA(u) = ((dT'(-\tilde{s}(u))Z, u)_{\mathbb{X}} - (Z, T'(\tilde{s}(u))0)_{\mathbb{X}})D\tilde{s}(u) - IdT(-\tilde{s}(u))Z.$$

Since $D\tilde{s}(u)$ lies in $\mathcal{D}(\hat{J})$ by Lemma 3.16, while $IdT(-\tilde{s}(u))Z$ is in $\mathcal{D}(\hat{J})$ by Assumption 3.7 (i), this proves part (b).

Now, choose $\nu > 0$ such that $\mathcal{U}_\nu^{\mathbb{W}} \subset \mathcal{U}_\rho^{\mathbb{X}}$. When $u \in \mathcal{U}_\nu^{\mathbb{W}} \cap \mathcal{D}(T'(0)|_{\mathbb{W}})$, the formula for the derivative simplifies to

$$DA(u) = \langle DP(u), h(u)\rangle D\tilde{s}(u) - IdT(-\tilde{s}(u))Z, \tag{3.32}$$

with

$$h(u) := J(u)IdT(-\tilde{s}(u))Z = B(u)dT(-\tilde{s}(u))B(U_c)^{-1}z. \tag{3.33}$$

Here we have used (3.9), Assumption 3.7 (vi), and the skew-adjointness of $J(u)$. By density of $D(T'(0)|_{\mathbb{W}})$ in $\mathbb{W}$, the formula in (3.32) is, in fact, valid for every $u \in \mathcal{U}_\nu^{\mathbb{W}}$.

Moreover, applying $J(u)$ to (3.32) leads to the expression

$$J(u)DA(u) = \langle DP(u), h(u)\rangle g(u) - h(u), \tag{3.34}$$

where g is the function defined in Lemma 3.16. We have already confirmed that g has the required properties for part (e), and in light of (3.33) and the fact that $z \in \mathcal{D}(T'(0)|_{\mathbb{W}})$ and (3.8), so does $u \mapsto J(u)DA(u)$. From Lemma 3.16 we see that $\tilde{s}(U_c) = 0$, and therefore $h(U_c) = z$. Evaluating (3.34) at $u = U_c$ then yields

$$J(U_c)DA(U_c) = \langle DP(U_c), z\rangle g(U_c) - z = -z, \tag{3.35}$$

by (3.30), which is part (c).

Finally, since the map $s \mapsto A(T(s)u)$ has derivative

$$0 = \langle DA(u), T'(0)u\rangle = \langle DA(u), J(u)\nabla P(u)\rangle = -\langle DP(u), J(u)DA(u)\rangle$$

at $s = 0$ for every $u \in \mathcal{U}_\nu^{\mathbb{W}} \cap \mathcal{D}(T'(0)|_{\mathbb{W}})$ by part (a), part (d) follows by density. Here we have once again made use of the identity (3.9). □

With the functional A in hand, we next consider the ordinary differential equation

$$\begin{cases} \dfrac{du}{d\lambda} = -J(u(\lambda))DA(u(\lambda)) \\ u(0) = v, \end{cases} \tag{3.36}$$

posed in $\mathcal{U}_\nu^{\mathbb{W}}$, with $\nu > 0$ taken small enough for Lemma 3.22 to apply. Part (e) of the lemma guarantees the existence of a unique solution, $\Phi = \Phi(\lambda, v) \in C^1(\mathcal{N}; \mathcal{U}_\nu^{\mathbb{W}})$, to (3.36), where

$$\mathcal{N} = \{(\lambda, v) \in \mathbb{R} \times \mathcal{U}_{\nu_0}^{\mathbb{W}} \ : |\lambda| < \lambda_0\}, \tag{3.37}$$

with $0 < \nu_0 < \nu$ and $\lambda_0 = \lambda_0(\nu_0) > 0$. By appealing to the commutation identities in (3.8) and Lemma 3.22 (a), we find

$$T(s)\Phi(\lambda, v) = \Phi(\lambda, T(s)v) \tag{3.38}$$

whenever both sides of the equation make sense, which in particular justifies that λ_0 can be taken to be a constant.

Observe that

$$\partial_\lambda \Phi(0, U_c) = z \tag{3.39}$$

as a result of Lemma 3.22(c). Furthermore, since

$$\frac{\partial}{\partial\lambda} P(\Phi(\lambda, v)) = -\left\langle DP(\Phi(\lambda, v)),\, J(\Phi(\lambda, v)) DA(\Phi(\lambda, v)) \right\rangle = 0$$

by Lemma 3.22(d), we have

$$P(\Phi(\lambda, v)) = P(v), \qquad \text{for all } (\lambda, v) \in \mathcal{N}. \tag{3.40}$$

That is, the flow of (3.36) preserves the momentum.

Lemma 3.23 (Lyapunov function)**.** *There exists a $\nu > 0$ and a functional $\Lambda \in C^1(\mathcal{U}_\nu^{\mathbb{W}}; \mathbb{R})$, vanishing on the U_c-orbit, such that*

$$E(\Phi(\Lambda(v), v)) \geq E(U_c) \qquad \textit{for all } v \in \mathcal{U}_\nu^{\mathbb{W}} \cap \mathcal{M}_c.$$

One can interpret this lemma as follows. Because of (3.40), the flow of (3.36) leaves the momentum invariant but it may change the energy in either direction near U_c. By avoiding the problematic negative direction in a suitable way, and using Lemma 3.16 (b) to deal with $\mathbb{X}_0$ and the orbit under T, we can make sure that the energy increases along a curve.

Proof of Lemma 3.23. We wish to apply Lemma 3.20. To that end, define the function $f \colon \mathcal{N} \to \mathbb{R}$ by

$$\begin{aligned} f(\lambda, v) &\coloneqq \langle H_c z, M(\Phi(\lambda, v)) - U_c \rangle \\ &= \langle H_c z, dT(\tilde{s}(\Phi))(\Phi - U_c) \rangle + \langle H_c z, T(\tilde{s}(\Phi)) U_c - U_c \rangle, \end{aligned}$$

which satisfies $f(0, U_c) = 0$. It is not obvious that this function is differentiable, but by differentiating the identity $E_c(u) = E_c(T(s)u)$, one finds that

$$\langle D^2 E_c(T(-s)u) dT(-s) v, w \rangle = \langle D^2 E_c(u) v, dT(s) w \rangle$$

for all $s \in \mathbb{R}, u \in \mathcal{O} \cap \mathbb{V}$, and $v, w \in \mathbb{V}$, and thus in particular that

$$f(\lambda, v) = \langle D^2 E_c(T(-\tilde{s}(\Phi)) U_c) dT(-\tilde{s}(\Phi)) z, \Phi - U_c \rangle + \langle H_c z, T(\tilde{s}(\Phi)) U_c - U_c \rangle,$$

holds for all $(\lambda, v) \in \mathcal{N}$. This expression shows that $f \in C^1(\mathcal{N}; \mathbb{R})$, as $E_c \in C^3(\mathcal{O} \cap \mathbb{V}; \mathbb{R})$ and both U_c and z are in $\mathcal{D}(T'(0)|_{\mathbb{W}})$. Moreover,

$$\partial_\lambda f(0, U_c) = \langle H_c z, z \rangle + \langle D\tilde{s}(U_c), z \rangle \langle H_c z, T'(0) U_c \rangle = \langle H_c z, z \rangle < 0,$$

since $\langle H_c z, T'(0)U_c\rangle = \langle z, H_c T'(0)U_c\rangle = 0$.

An application of the implicit function theorems tells us that there exists a neighborhood $\mathcal{V}$ of U_c in $\mathcal{U}_{\nu_0}^{\mathbb{W}}$, and a C^1-mapping $\Lambda\colon \mathcal{V} \to (-\lambda_0, \lambda_0)$ satisfying

$$f(\Lambda(v), v) = \langle H_c z, M(\Phi(\Lambda(v), v)) - U_c\rangle = 0 \qquad \text{for all } v \in \mathcal{V}. \tag{3.41}$$

In view of (3.38) and (3.25), we have $f(\lambda, T(s)v) = f(\lambda, v)$ for all $s \in \mathbb{R}$ and $(\lambda, v) \in \mathcal{N}$, so Λ can be extended to a tubular neighborhood $\mathcal{U}_\nu^{\mathbb{W}}$ of the U_c-orbit.

We may now use Lemma 3.20 to conclude that, possibly upon shrinking $\mathcal{U}_\nu^{\mathbb{W}}$, there exists a $\beta > 0$ such that

$$E(\Phi(\Lambda(v), v)) - E(U_c) \geq \beta \|M(\Phi(\Lambda(v), v)) - U_c\|_{\mathbb{X}}^2$$

for every $v \in \mathcal{U}_\nu^{\mathbb{W}} \cap \mathcal{M}_c$. In particular, the result follows. $\square$

Lemma 3.24. *There exists a* $\nu > 0$ *such that*

$$E(U_c) \leq E(v) + \Lambda(v)\mathcal{S}(v) \qquad \textit{for all } v \in \mathcal{U}_\nu^{\mathbb{W}} \cap \mathcal{M}_c.$$

Here

$$\mathcal{S}(v) := -\langle DE(v), J(v)DA(v)\rangle . \tag{3.42}$$

Proof. Define the $C^1(\mathcal{N}; \mathbb{R})$ function

$$g(\lambda, v) := E(\Phi(\lambda, v)) = E_c(\Phi(\lambda, v)) + cP(v).$$

Here we have exploited (3.40) in evaluating P at v. Then, suppressing the dependence of Φ on (λ, v), we find

$$\partial_\lambda g(\lambda, v) = \langle DE_c(\Phi), \partial_\lambda \Phi\rangle = -\langle DE_c(\Phi), J(\Phi)DA(\Phi)\rangle,$$

so $\partial_\lambda g \in C^1(\mathcal{N}; \mathbb{R})$ as well by Lemma 3.22(e). It therefore makes sense to compute

$$\partial_\lambda^2 g(0, U_c) = \langle D^2 E_c(U_c)\partial_\lambda \Phi(0, U_c), \partial_\lambda \Phi(0, U_c)\rangle = \langle H_c z, z\rangle < 0,$$

where we have used that U_c is a critical point of E_c, and the last inequality is Lemma 3.21. We also see that

$$\partial_\lambda g(0, v) = -\langle DE_c(v), J(v)DA(v)\rangle = -\langle DE(v), J(v)DA(v)\rangle = \mathcal{S}(v)$$

for every $v \in \mathcal{U}_{\nu_0}^{\mathbb{W}}$. It follows that

$$g(\lambda, v) \leq g(0, v) + \partial_\lambda g(0, v)\lambda$$

for small enough λ, and a possibly smaller neighborhood of U_c. This neighborhood can be made tubular, by the same reasoning as in the proof of Lemma 3.23. The desired upper bound now follows by setting $\lambda = \Lambda(v)$ and using Lemma 3.23. □

The final lemma we need in order to prove the instability theorem is the following.

Lemma 3.25. *Suppose that $d''(c) < 0$. Then there exists a C^2-curve $\psi\colon (-1, 1) \to \mathbb{W}$ such that*

(i) $\psi(0) = U_c$ *and* $\psi'(0) = z$*;*

(ii) $\psi(s) \in \mathcal{M}_c$ *for all* $s \in (-1, 1)$*;*

(iii) $E \circ \psi$ *has a strict local maximum at* 0.

Proof. Define $\psi\colon (-\lambda_0, \lambda_0) \to \mathbb{W}$ by $\psi(s) := \Phi(s, U_c)$. Then $\psi(0) = U_c$ by definition of Φ, while $\psi'(0) = z$ is (3.39). We also know that the flow of (3.36) conserves momentum, whence $\psi(s) \in \mathcal{M}_c$ for all $s \in (-\lambda_0, \lambda_0)$. Finally, the proof of Lemma 3.24 shows that ψ is C^2, and that $E \circ \psi$ has a strict local maximum at 0. The result is now obtained by a possible reparameterization. □

Remark 3.26. The properties of the curve in Lemma 3.25 show that U_c is *not* a local minimizer of the constrained minimization problem

$$\min \{E(u) : u \in \mathcal{M}_c\}.$$

Proof of the instability theorem

Proof. Assume, to the contrary, that we do not have instability. Then for every $\nu_0 > 0$, small enough for local existence from Assumption 3.13, there exists a $0 < \nu < \nu_0$ such that solutions corresponding to initial data in $\mathcal{U}_{\nu}^{\mathbb{W}}$ exist globally in time and stay inside $\mathcal{U}_{\nu_0}^{\mathbb{W}}$. Fix such a ν_0, which we also require to satisfy the hypotheses of the lemmas in this section.

By the above reasoning, there exists a unique global in time solution $u^s \in C^0([0, \infty), \mathcal{U}_{\nu_0}^{\mathbb{W}})$ to (3.6), with initial data $u^s(0) = \psi(s)$, for all $|s| \ll 1$.

Here ψ is the curve from Lemma 3.25. Since $\psi(s) \in \mathcal{M}_c$, these solutions all live on $\mathcal{M}_c$ by conservation of momentum.

From Lemma 3.24, and conservation of energy, we obtain the inequality

$$E(U_c) - E(\psi(s)) \le \Lambda(u^s(t))\mathcal{S}(u^s(t))$$

for all $|s| \ll 1$ and $t \in [0,\infty)$. By choosing $\lambda_0 \le 1$ in (3.37), we can assume that $|\Lambda(u)| \le 1$ for all $u \in \mathcal{U}_{\nu_0}^{\mathbb{W}}$. Thus

$$|\mathcal{S}(u^s(t))| \ge E(U_c) - E(\psi(s)) > 0 \tag{3.43}$$

for all $0 < |s| \ll 1$ and $t \in [0,\infty)$, where the strict inequality stems from Lemma 3.25(iii). Moreover, by continuity, this implies that $\mathcal{S} \circ u^s$ does not change sign.

Since $\hat{J}\colon \mathcal{D}(\hat{J}) \subset \mathbb{X}^* \to \mathbb{X}$ is a closed operator, we may view $\mathbb{D} := \mathcal{D}(\hat{J})$ as a Banach space with the graph norm

$$\|v\|_{\mathbb{D}} := \|v\|_{\mathbb{X}^*} + \|\hat{J}v\|_{\mathbb{X}},$$

and in this norm the map $u \mapsto J(u)$ is of class $C^0(\mathcal{O} \cap \mathbb{W}; \operatorname{Lin}(\mathbb{D},\mathbb{X}))$ by Assumption 3.4. It follows that the map $u \mapsto J(u)^*$ is in $C^0(\mathcal{O} \cap \mathbb{W}; \operatorname{Lin}(\mathbb{X}^*,\mathbb{D}^*))$. From (3.7) we obtain

$$\frac{d}{dt}\langle u^s(t), v\rangle = -\langle \nabla E(u^s(t)), J(u^s(t))v\rangle = -\langle J(u^s(t))^*\nabla E(u^s(t)), v\rangle$$

for every $v \in \mathbb{D}$. Now, since the embedding $\mathbb{D} \hookrightarrow \mathbb{X}^*$ is dense, we have $\mathbb{X} \hookrightarrow \mathbb{D}^*$. We can therefore view u^s as a member of $C^0([0,\infty),\mathcal{U}_{\nu_0}^{\mathbb{W}}) \cap C^1((0,\infty),\mathbb{D}^*)$, with

$$(u^s)'(t) = -J(u^s(t))^*\nabla E(u^s(t))$$

for all $t \in (0,\infty)$. Furthermore, the functional A in Lemma 3.22 can be viewed as a member of $C^1(\mathcal{U}_{\nu_0}^{\mathbb{W}};\mathbb{R})$, with the derivative DA in $C^0(\mathcal{U}_{\nu_0}^{\mathbb{W}};\mathbb{D})$. As the embedding $\mathbb{W} \hookrightarrow \mathbb{X}$ is dense and $\mathbb{W}$ is reflexive, the embeddings $\mathbb{X}^* \hookrightarrow \mathbb{W}^*$ and $\mathbb{D} \hookrightarrow \mathbb{W}^*$ are likewise dense.

We may now apply [20, Lemma 4.6] to conclude that the composition $A \circ u^s$ is a member of $C^1([0,\infty),\mathbb{R})$, and that

$$\begin{aligned}(A \circ u^s)'(t) &= -\langle J(u^s(t))^*\nabla E(u^s(t)), DA(u^s(t))\rangle \\ &= -\langle DE(u^s(t)), J(u^s(t))DA(u^s(t))\rangle = \mathcal{S}(u^s(t)),\end{aligned}$$

whence

$$|A(u^s(t)) - A(\psi(s))| \ge t(E(U_c) - E(\psi(s)))$$

for all $0 < |s| \ll 1$ and $t \in [0,\infty)$ by (3.43). This shows that $A \circ u^s$ is unbounded, but we also have

$$|A(u)| \leq \|Z\|_{\mathbb{X}} \|M(u) - U_c\|_{\mathbb{X}} \leq \|Z\|_{\mathbb{X}} \|\iota_{\mathbb{W}\hookrightarrow\mathbb{X}}\| \nu_0$$

for every $u \in \mathcal{U}^{\mathbb{W}}_{\nu_0}$ by the definition of A. We have arrived at a contradiction, and must conclude that the U_c-orbit is unstable. □

5 Hamiltonian structure for the water wave problem with a point vortex

With our general machinery in place, we turn to the question of stability of solitary capillary-gravity waves with a submerged point vortex. The next subsection recalls how this system was formulated by Shatah, Walsh, and Zeng in [43]. In Section 5, we show that the problem can be rewritten once more as an abstract Hamiltonian system of the general form (3.6), and verify that the corresponding energy, momentum, Poisson map, symmetry group, and bound states meet the many requirements of Section 2.

Nonlocal formulation

Consider the capillary-gravity water wave problem with a point vortex described in (3.1). In any simply connected subset of $\Omega_t \setminus \{\bar{x}\}$, the velocity v can be decomposed as

$$v = \nabla\Phi + \epsilon\nabla\Theta, \tag{3.44}$$

where Φ is harmonic on Ω_t and Θ is harmonic on the subset. The latter represents the vortical contribution of the point vortex. Since the surface is a graph, we will use $\Theta = \Theta_1 - \Theta_2$, where

$$\begin{aligned}
\Theta_1(x) &= -\frac{1}{\pi}\arctan\left(\frac{x_1 - \bar{x}_1}{|x-\bar{x}| + x_2 - \bar{x}_2}\right),\\
\Theta_2(x) &= \frac{1}{\pi}\arctan\left(\frac{x_1 - \bar{x}_1}{|x-\bar{x}'| - x_2 - \bar{x}_2}\right).
\end{aligned}$$

Then Θ is harmonic on the open set $\{x \in \mathbb{R}^2 : (x_1 \neq \bar{x}_1) \vee (|x_2| < -\bar{x}_2)\}$, and $\nabla\Theta$ extends to a smooth velocity field on $\Omega_t \setminus \{\bar{x}\}$. The purpose of Θ_2, which corresponds to a mirror vortex at $\bar{x}' := (\bar{x}_1, -\bar{x}_2)$, is to make $\nabla\Theta$ decay faster as $|x| \to \infty$. Indeed, $\nabla\Theta$ is L^2 on the complement of any neighborhood of $\bar{x}$ in Ω_t.

For later use, we also introduce notation for the harmonic conjugate of Θ, $\Gamma = \Gamma_1 - \Gamma_2$, which takes the form

$$\Gamma_1(x) = \frac{1}{2\pi} \log |x - \bar{x}|, \quad \Gamma_2(x) = \frac{1}{2\pi} \log |x - \bar{x}'|. \tag{3.45}$$

Note that the convention here is that $\nabla\Theta = \nabla^\perp \Gamma$, where $\nabla^\perp := (-\partial_{x_2}, \partial_{x_1})$.

The rationale behind splitting v according to (3.44) is that it nearly decouples the tasks of determining the rotational and irrotational parts of the velocity. Indeed, Θ is entirely explicit given $\bar{x}$, which solves the differential equation (3.1b). The main analytical challenge is determining Φ and η. But for this we can proceed as in the classical Zakharov–Craig–Sulem formulation of the *irrotational* water wave problem: Because Φ is harmonic, it is enough to know η and the trace

$$\varphi = \varphi(x_1) := \Phi(x_1, \eta(x_1))$$

of Φ on the surface. Notice that η and φ then have the fixed spatial domain $\mathbb{R}$. The problem can then be reduced to the boundary, where the rotational part $\nabla\Theta|_{S_t}$ can be viewed as a forcing term.

On S_t, we must ensure that the kinematic condition and Bernoulli condition are satisfied. Naturally, these will now involve tangential and normal derivatives of Φ and Θ. Here and in the sequel, we will therefore make use of the shorthand

$$\nabla_\perp := (-\eta' \partial_{x_1} + \partial_{x_2})\,|_{S_t}, \qquad \nabla_\top := (\partial_{x_1} + \eta' \partial_{x_2})\,|_{S_t},$$

which arise naturally when parameterizing the free surface using η. Note also that we are using the convention that spatial derivatives of quantities restricted to the boundary are denoted by a prime, while ∂_{x_1} is reserved for functions of two or more spatial variables. Exceptions will be made for certain differential operators when this does not cause ambiguities.

The tangential derivative $\nabla_\top \Phi$ is simply φ', but to express a normal derivative $\nabla_\perp \Phi$ requires using the nonlocal Dirichlet–Neumann operator $G(\eta)\colon \dot{H}^k(\mathbb{R}) \to \dot{H}^{k-1}(\mathbb{R})$, which is defined by

$$G(\eta)\phi := \nabla_\perp(\mathcal{H}(\eta)\phi), \tag{3.46}$$

where $\mathcal{H}(\eta)\phi \in \dot{H}^1(\Omega_t)$ is uniquely determined as the harmonic extension of $\phi \in \dot{H}^k(\mathbb{R})$ to Ω_t.

It is well known that, for any $k_0 > 1$, $k \in [1/2 - k_0, 1/2 + k_0]$, and $\eta \in H^{k_0+1/2}(\mathbb{R})$, the operator $G(\eta)$ is an isomorphism. Moreover, the mapping $\eta \mapsto G(\eta)$ is analytic, $G(\eta)\colon \dot{H}^{1/2}(\mathbb{R}) \to \dot{H}^{-1/2}(\mathbb{R})$ is self-adjoint, and $G(0) = |\partial_{x_1}|$, the Calderón operator. We refer the reader to [43, Appendix A], [44, Section 6], or [28, Chapter 3 and Appendix A] for more details.

Finally, using the Dirichlet–Neumann operator, we can rewrite the water wave problem with a point vortex in terms of $(\eta, \varphi, \bar{x})$ as

$$\begin{cases} \partial_t \eta = G(\eta)\varphi + \epsilon \nabla_\perp \Theta, \\ \partial_t \varphi = \; - \dfrac{(\varphi')^2 - 2\eta'\varphi' G(\eta)\varphi - (G(\eta)\varphi)^2}{2\langle \eta' \rangle^2} - g\eta + b\Big(\dfrac{\eta'}{\langle \eta' \rangle}\Big)' \\ \qquad\qquad\qquad - \epsilon \varphi' \Theta_{x_1}|_S - \dfrac{\epsilon^2}{2}\Big(|\nabla \Theta|^2\Big)|_S + \epsilon \xi|_S \cdot \partial_t \bar{x}, \\ \partial_t \bar{x} = \nabla \Phi(\bar{x}) - \epsilon \partial_{x_1} \Theta_2(\bar{x}) e_1, \end{cases} \tag{3.47}$$

where, to simplify the notation, we have introduced $\Xi := \Theta_1 + \Theta_2$ and $\xi := (\Theta_{x_1}, \Xi_{x_2})$. The motivation for this being that $\nabla_{\bar{x}} \Theta = -\xi$.

The first equation in (3.47) is simply the kinematic condition in (3.1c), while the second follows from evaluating Bernoulli's law along the free surface using the dynamic condition to replace the trace of the pressure with the (signed) curvature. Finally, the third equation is just (3.1b) in view of the splitting (3.44). Observe that $\partial_t \bar{x}$ can easily be eliminated from the equation for $\partial_t \varphi$, but we opt not to do so.

Hamiltonian formulation

We now endeavor to rewrite (3.47) as a Hamiltonian system for the state variable $u = (\eta, \varphi, \bar{x})$. The first step is to fix a functional analytic framework. For that, we introduce the continuous scale of spaces

$$\mathbb{X}^k = \mathbb{X}_1^k \times \mathbb{X}_2^k \times \mathbb{X}_3 := H^{k+1/2}(\mathbb{R}) \times \Big(\dot{H}^k(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R})\Big) \times \mathbb{R}^2, \qquad k \geq 1/2 \tag{3.48}$$

For each $k \geq 1/2$, $\mathbb{X}^k$ is a Hilbert space, and the embedding $\mathbb{X}^k \hookrightarrow \mathbb{X}^{k'}$ is dense for all $1/2 \leq k' \leq k$.

For the energy space, we take

$$\mathbb{X} := \mathbb{X}^{1/2} = H^1(\mathbb{R}) \times \dot{H}^{1/2}(\mathbb{R}) \times \mathbb{R}^2, \tag{3.49}$$

which has the space

$$\mathbb{X}^* = H^{-1}(\mathbb{R}) \times \dot{H}^{-1/2}(\mathbb{R}) \times \mathbb{R}^2$$

as its dual. The isomorphism $I\colon \mathbb{X} \to \mathbb{X}^*$ takes the explicit form

$$I = \left(1 - \partial_{x_1}^2,\, |\partial_{x_1}|,\, \mathrm{id}_{\mathbb{R}^2}\right).$$

This choice for $\mathbb{X}$ ensures that $\nabla\Phi \in L^2(\Omega_t)$, and therefore that the kinetic energy corresponding to the irrotational part of the velocity is finite.

On the other hand, anticipating the Dirichlet–Neumann operator, we expect to need $k > 1$ to ensure that the energy is smooth. With that in mind, set

$$\mathbb{V} := \mathbb{X}^{1+} = H^{3/2+}(\mathbb{R}) \times (\dot{H}^{1+}(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R})) \times \mathbb{R}^2, \tag{3.50}$$

where by $\mathbb{X}^{1+}$ we mean $\mathbb{X}^{1+s}$ for a fixed $0 < s \ll 1$. For the well-posedness space we use

$$\mathbb{W} := \mathbb{X}^{5/2+} = H^{3+}(\mathbb{R}) \times (\dot{H}^{5/2+}(\mathbb{R}) \cap \dot{H}^{\frac{1}{2}}(\mathbb{R})) \times \mathbb{R}^2. \tag{3.51}$$

A local well-posedness result at this level of regularity was obtained for irrotational capillary-gravity water waves by Alazard, Burq, and Zuily [1]. While the Cauchy problem for (3.47) has not yet been studied, it is reasonable to suppose that local well-posedness will hold in the same space. In our setting, this is the minimal regularity required to have the traces of the velocity be Lipschitz on the surface. Note that our results hold with any smoother choice of $\mathbb{W}$ as well. The Gagliardo–Nirenberg interpolation inequality yields the following.

Lemma 3.27 (Function spaces)**.** *Let $\mathbb{X}$, $\mathbb{V}$, and $\mathbb{W}$ be defined by* (3.49)*,* (3.50)*, and* (3.51)*, respectively. Then there exists a constant $C > 0$ and $\theta \in (0, 1/4)$ such that Assumption 3.2 is satisfied.*

Lastly, recall that for the problem to be well-defined, the surface must lie between the point vortex at $\bar{x} \in \Omega_t$, and its mirror at $\bar{x}'$. We therefore let

$$\mathcal{O} := \{u \in \mathbb{X} : \bar{x}_2 < \eta(\bar{x}_1) < -\bar{x}_2\},$$

and seek solutions taking values in $\mathcal{O} \cap \mathbb{W}$ at each time.

We endow $\mathbb{X}$ with symplectic structure by prescribing a Poisson map. First, consider the linear operator $\hat{J}\colon \mathcal{D}(\hat{J}) \subset \mathbb{X}^* \to \mathbb{X}$ defined by

$$\hat{J} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon^{-1} \\ 0 & 0 & -\epsilon^{-1} & 0 \end{pmatrix}, \tag{3.52}$$

with the natural domain

$$\mathcal{D}(\hat{J}) := (H^{-1}(\mathbb{R}) \cap \dot{H}^{1/2}(\mathbb{R})) \times (H^{1}(\mathbb{R}) \cap \dot{H}^{-1/2}(\mathbb{R})) \times \mathbb{R}^2.$$

One can understand $\hat{J}$ as encoding the Hamiltonian structure for the point vortex and water wave in isolation. To get the full system, we must incorporate wave-vortex interaction terms. For each $u \in \mathcal{O} \cap \mathbb{V}$, define

$$\operatorname{Lin}(\mathbb{X}) \ni B(u) := \mathrm{id}_{\mathbb{X}} + \mathcal{K}(u), \tag{3.53}$$

where $\mathcal{K}(u) \in \operatorname{Lin}(\mathbb{X})$ is the finite-rank operator given by

$$\mathcal{K}(u)\dot{w} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\epsilon \Xi_{x_2}|_S & \epsilon \Theta_{x_1}|_S & \epsilon \Theta_{x_1}|_S & \epsilon \Xi_{x_2}|_S \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} \langle \Theta_{x_1}|_S, \dot{\eta} \rangle \\ \langle \Xi_{x_2}|_S, \dot{\eta} \rangle \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

for all $\dot{w} \in \mathbb{X}$. The full Poisson map is formed, like in (3.5), by composing $\hat{J}$ with $B(u)$:

Lemma 3.28 (Properties of J). *For each $u \in \mathcal{O} \cap \mathbb{V}$, the operator*

$$J(u) \colon \mathcal{D}(\hat{J}) \subset \mathbb{X}^* \to \mathbb{X}$$

is given by

$$J(u) := B(u)\hat{J} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & J_{22} & J_{23} & J_{24} \\ 0 & J_{32} & 0 & \epsilon^{-1} \\ 0 & J_{42} & -\epsilon^{-1} & 0 \end{pmatrix}, \tag{3.54}$$

where

$$\begin{aligned} J_{22} &= -\epsilon \Xi_{x_2}|_S \langle \cdot, \Theta_{x_1} \rangle + \epsilon \Theta_{x_1} \langle \cdot, \Xi_{x_2}|_S \rangle, \\ J_{23} &= -\Xi_{x_2}|_S, \\ J_{24} &= \Theta_{x_1}|_S, \\ J_{32} &= \langle \cdot, \Xi_{x_2}|_S \rangle, \\ J_{42} &= -\langle \cdot, \Theta_{x_1}|_S \rangle, \end{aligned}$$

and Assumption 3.4 is satisfied.

Proof. It is clear from its definition in (3.52) that $\hat{J}$ is injective and closed, and its domain $\mathcal{D}(\hat{J})$ is dense in $\mathbb{X}^*$ by Lemma 3.43. Thus parts (i) and (ii) of Assumption 3.4 hold. Now, fix $u \in \mathcal{O} \cap \mathbb{V}$ and consider the operator

$B(u)$ given by (3.53). The map $\mathcal{K}(u)$ has finite rank, so $B(u)$ is a compact perturbation of identity. In particular, $B(u)$ is Fredholm index 0. On the other hand, $B(u)$ is clearly injective, and thus it must be an isomorphism on $\mathbb{X}$. This proves part (iii). The properties of the mapping $u \mapsto B(u)$ asked for in part (iv) are obvious from the definition (3.53). Finally, the skew-adjointness of $J(u)$ is apparent from the formula (3.54). $\square$

Next, we must determine the energy associated to a water wave with a point vortex. Classically, the kinetic energy is given by $\frac{1}{2}\int |v(t)|^2\,dx$. To adapt this to the point vortex case, we use the splitting (3.44) and formally integrate by parts. This produces traces on S_t, plus terms at the vortex center. We neglect the singular one, corresponding to Γ_1, which is equivalent to removing the self-advection of the point vortex as in the Helmholtz–Kirchhoff model. Ultimately, this leads us to define the energy functional $E = E(u)$ to be

$$E(u) \coloneqq K(u) + V(u), \tag{3.55}$$

where

$$\begin{aligned} K(u) &\coloneqq K_0(u) + \epsilon K_1(u) + \epsilon^2 K_2(u) \\ &\coloneqq \frac{1}{2}\int_{\mathbb{R}} \varphi G(\eta)\varphi\,dx_1 + \epsilon\int_{\mathbb{R}} \varphi \nabla_\perp \Theta\,dx_1 \\ &\qquad + \frac{1}{2}\epsilon^2\Big(\int_{\mathbb{R}} \Theta|_S \nabla_\perp\Theta\,dx_1 + \Gamma_2(\bar{x})\Big) \end{aligned} \tag{3.56}$$

is the kinetic energy, and

$$V(u) \coloneqq \int_{\mathbb{R}} \Big(\frac{1}{2} g\eta^2 + b(\langle \eta' \rangle - 1)\Big)\,dx_1 \tag{3.57}$$

is the potential energy. Notice that V depends solely on the surface profile. A similar procedure also shows that

$$P = P(u) \coloneqq \epsilon \bar{x}_2 - \int_{\mathbb{R}} \eta'(\varphi + \epsilon\Theta|_S)\,dx_1. \tag{3.58}$$

is the momentum carried by a water wave with a submerged point vortex.

It is easy to see that $E, P \in C^\infty(\mathcal{O}\cap\mathbb{V};\mathbb{R})$. For the convenience of the reader, the first and second Fréchet derivatives of E and P are recorded in Appendix C. By inspection, we see that DE and DP admit the explicit extensions

$$\nabla E(u) \coloneqq (E'_\eta(u), E'_\varphi(u), \nabla_{\bar{x}} E(u)), \tag{3.59}$$

$$\nabla P(u) \coloneqq (P'_\eta(u), P'_\varphi(u), \nabla_{\bar{x}} P(u)), \tag{3.60}$$

in $C^\infty(\mathcal{O} \cap \mathbb{V}; \mathbb{X}^*)$, with

$$\begin{aligned}
E'_\eta(u) &:= \frac{(\varphi')^2 - 2\eta'\varphi' G(\eta)\varphi - (G(\eta)\varphi)^2}{2\langle \eta' \rangle^2} + g\eta - b\Big(\frac{\eta'}{\langle \eta' \rangle}\Big)' \\
&\qquad + \epsilon\varphi'\Theta_{x_1}|_{S_t} + \frac{\epsilon^2}{2}\Big(|\nabla\Theta|^2\Big)|_{S_t}, \\
E'_\varphi(u) &:= G(\eta)\varphi + \epsilon\nabla_\perp\Theta, \\
\nabla_{\bar{x}}E(u) &:= -\frac{1}{2}\epsilon^2\int_{\mathbb{R}} \nabla_\perp(\Theta\xi)\,dx - \epsilon\int_{\mathbb{R}} \varphi\nabla_\perp\xi\,dx_1 - \epsilon^2\partial_{x_1}\Theta_2(\bar{x})e_2,
\end{aligned} \tag{3.61}$$

and

$$\begin{aligned}
P'_\eta(u) &:= \varphi' + \epsilon\Theta_{x_1}|_{S_t}, \\
P'_\varphi(u) &:= -\eta', \\
\nabla_{\bar{x}}P(u) &:= \epsilon e_2 + \epsilon\int_{\mathbb{R}} \eta'\xi|_{S_t}\,dx_1.
\end{aligned} \tag{3.62}$$

Thus Assumption 3.6 is indeed satisfied. The next lemma confirms that the Hamiltonian system for this choice of the energy and Poisson map correspond to the water wave with a point vortex problem.

Theorem 3.29 (Hamiltonian formulation). *$u := (\eta, \varphi, \bar{x}) \in C^1([0,t_0); \mathbb{W} \cap \mathcal{O})$ is a solution of the capillary-gravity water wave problem with a point vortex* (3.47) *if and only if it is a solution to the abstract Hamiltonian system*

$$\frac{du}{dt} = J(u)\nabla E(u), \tag{3.63}$$

where $J = J(u)$ is the Poisson map (3.54) *and ∇E is the energy functional defined in* (3.55).

Proof. Written out more explicitly using (3.54), the Hamiltonian system (3.63) is

$$\begin{cases}
\partial_t\eta = E'_\varphi(u) \\
\partial_t\varphi = -E'_\eta(u) + \epsilon\xi|_{S_t} \cdot \Big(\langle E'_\varphi(u), \Xi_{x_2}\rangle + \epsilon^{-1}\partial_{\bar{x}_2}E(u), \\
\qquad\qquad\qquad\qquad - \langle E'_\varphi(u), \Theta_{x_1}\rangle - \epsilon^{-1}\partial_{\bar{x}_1}E(u)\Big) \\
\partial_t\bar{x} = \Big(\langle E'_\varphi(u), \Xi_{x_2}\rangle + \epsilon^{-1}\partial_{\bar{x}_2}E(u), -\langle E'_\varphi(u), \Theta_{x_1}\rangle - \epsilon^{-1}\partial_{\bar{x}_1}E(u)\Big)
\end{cases} \tag{3.64}$$

Using (3.61), we see that the first of Hamilton's equations is

$$\partial_t \eta = G(\eta)\varphi + \epsilon \nabla_\perp \Theta,$$

which is equivalent to the kinematic boundary condition in (3.47). Moreover, the equation for $\partial_t \varphi$ above agrees with the corresponding one in (3.47), as the final term is simply $\epsilon \xi|_{S_t} \cdot \partial_t \bar{x}$ by the third equation in (3.64).

Only the equation for the motion of the point vortex remains. Written out explicitly, we find that

$$\begin{aligned}
\partial_t \bar{x}_2 &= \int_{\mathbb{R}} (\varphi \nabla_\perp \Theta_{x_1} - \Theta_{x_1}|_{S_t} G(\eta)\varphi)\, dx_1 \\
&\qquad + \frac{\epsilon}{2} \int_{\mathbb{R}} (\Theta|_{S_t} \nabla_\perp \Theta_{x_1} - \Theta_{x_1}|_{S_t} \nabla_\perp \Theta)\, dx_1 \\
&= \int_{S_t} N \cdot (\Gamma_{x_1} \nabla \Psi - \Psi \nabla \Gamma_{x_1})\, dS + \frac{\epsilon}{2} \int_{S_t} N \cdot (\Gamma_{x_1} \nabla \Gamma - \Gamma \nabla \Gamma_{x_1})\, dS,
\end{aligned}$$

where Ψ is the harmonic conjugate to Φ in Ω_t and N is the outward-pointing unit normal. Now, owing to the fact that Γ and Γ_{x_1} are harmonic on $\mathbb{R}^2 \setminus \{\bar{x}, \bar{x}'\}$, the final integral is equal to

$$\int_{x_2=0} (\Gamma_{x_1} \Gamma_{x_2} - \Gamma \Gamma_{x_1 x_2})\, dx_1 = 0$$

by path independence. Here we have used that $\Gamma = \Gamma_{x_1} = 0$ on $\{x_2 = 0\}$.

On the other hand, we have the identity

$$\int_{S_t} N \cdot (\Gamma_{x_1} \nabla \Psi - \Psi \nabla \Gamma_{x_1})\, dS = \int_{|x-\bar{x}|=r} N \cdot (\Gamma_{x_1} \nabla \Psi - \Psi \nabla \Gamma_{x_1})\, dS$$

for all $0 < r \ll 1$. Notice that Γ_2 is harmonic in Ω_t, so only Γ_1 contributes in the limit $r \to 0$. Setting $x - \bar{x} = re^{i\theta}$, under the natural identification, we have

$$\begin{aligned}
\int_{|x-\bar{x}|=r} \Gamma_{1,x_1} N \cdot \nabla \Psi\, dS &= \int_0^{2\pi} \frac{\cos(\theta)}{2\pi r} (\cos(\theta), \sin(\theta)) \cdot \nabla \Psi(\bar{x} + re^{i\theta}) r\, d\theta \\
&= \frac{1}{4\pi} \int_0^{2\pi} \left(1 + \cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right)\right) \cdot \nabla \Psi(\bar{x} + re^{i\theta})\, d\theta
\end{aligned}$$

and

$$\begin{aligned}
-\int_{|x-\bar{x}|=r} \Psi N \cdot \nabla \Gamma_{1,x_1}\, dS &= \frac{1}{2\pi} \int_0^{2\pi} \cos(\theta) \frac{\Psi(\bar{x} + re^{i\theta})}{r}\, d\theta \\
&= \frac{1}{4\pi} \int_0^{2\pi} \int_0^1 \left(1 + \cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right)\right) \cdot \nabla \Psi(\bar{x} + tre^{i\theta})\, dt\, d\theta.
\end{aligned}$$

In total, then,

$$\partial_t \bar{x}_2 = \lim_{r\to 0} \int_{|x-\bar{x}|=r} N \cdot (\Gamma_{x_1} \nabla \Psi - \Psi \nabla \Gamma_{x_1})\, dS = \Psi_{x_1}(\bar{x}),$$

and an essentially identical argument shows that

$$\partial_t \bar{x}_1 = -\partial_{x_2}\Psi(\bar{x}) - \epsilon \partial_{x_1}\Theta_2(\bar{x}).$$

Recalling that Ψ and Φ are harmonic conjugates, these two equations are equivalent to the vortex dynamics equation in (3.47).

Finally, conservation of the energy E is immediate from the fact that u is a C^1 solution of (3.63). The conservation of momentum P is simply a consequence of (3.9), which we verify below in Lemma 3.30. □

Symmetry

Let $T = T(s)\colon \mathbb{X} \to \mathbb{X}$ be the one-parameter family of affine mappings given by

$$T(s)u := (\eta(\cdot - s), \varphi(\cdot - s), \bar{x} + se_1), \qquad s \in \mathbb{R}, \tag{3.65}$$

representing the invariance of the underlying system with respect to horizontal translations. The linear part of the family is

$$dT(s)u = (\eta(\cdot - s), \varphi(\cdot - s), \bar{x}), \ \ s \in \mathbb{R}, \tag{3.66}$$

and the infinitesimal generator of T is the affine operator

$$T'(0) = dT'(0) + T'(0)0 = -(\partial_{x_1}, \partial_{x_1}, 0) + (0, 0, e_1), \tag{3.67}$$

with domain $\mathcal{D}(T'(0)) = \mathbb{X}^{3/2}$.

Lemma 3.30 (Properties of T)**.** *The group $T(\cdot)$ satisfies Assumption 3.7.*

Proof. Parts (i) to (iii) are obvious from the definition of T. The strong continuity of the group in the respective spaces is likewise straightforward. Observe also that $T(t)0 = t(0, 0, e_1)$, which has norm $|t|$ in both $\mathbb{X}$ and $\mathbb{W}$. Thus part (v) holds with $\omega(t) = t$.

For part (vi), note that $dT(s)$ is invariant on $I^{-1}\mathcal{D}(\hat{J})$, which is therefore the common domain of definition for both sides at the top of (3.8). Verifying that we have equality in the two equations for all $s \in \mathbb{R}$ is then just a matter of inserting the definitions. For part (vii), observe that $\mathcal{D}(T'(0)|_{\mathbb{V}}) = \mathbb{X}^{2+}$. That $\nabla P(u) \in \mathcal{D}(\hat{J})$ for any $u \in \mathcal{O} \cap \mathcal{D}(T'(0)|_{\mathbb{V}})$ follows from its formula in (3.60) and (3.62). Moreover, (3.9) and (3.10) can be obtained by direct computation.

To verify part (viii), note that

$$\operatorname{Rng} \hat{J} = (H^1(\mathbb{R}) \cap \dot{H}^{-1/2}) \times (H^{-1}(\mathbb{R}) \cap \dot{H}^{1/2}) \times \mathbb{R}^2, \qquad \mathcal{D}(T'(0)|_{\mathbb{W}}) = \mathbb{X}^{7/2},$$

so

$$\mathcal{D}(T'(0)|_{\mathbb{W}}) \cap \operatorname{Rng} \hat{J} = (H^4(\mathbb{R}) \cap \dot{H}^{-1/2}) \times (H^{-1}(\mathbb{R}) \cap \dot{H}^{7/2}(\mathbb{R}) \cap \dot{H}^{1/2}) \times \mathbb{R}^2,$$

which is certainly dense in $\mathbb{X}$ (cf. Lemma 3.43). Finally, the conservation of energy under the group (ix) is immediate given the translation invariant nature of E in (3.55), (3.56), and (3.57). □

Traveling waves

In Theorem 3.44, we prove the existence of a surface of small-amplitude traveling wave solutions of the point vortex problem, parameterized by the vortex strength ϵ and the depth of the point vortex a. For the stability analysis, however, it is important to fix ϵ, as it appears as part of the equation. We will therefore consider the families

$$\mathscr{C}_{\mathcal{I}}^{\epsilon} := \{U_{c(\epsilon,a)} := (\eta(\epsilon,a), \varphi(\epsilon,a), -ae_2) : a \in \mathcal{I}\} \subset \mathcal{O} \cap \mathbb{W} \tag{3.68}$$

of traveling water waves with a point vortex of strength ϵ at $-ae_2$, traveling at speed $c(\epsilon, a)$, for nontrivial compact intervals $\mathcal{I} \subset (0,\infty)$ and $0 < \epsilon \ll 1$. From (3.93) we see that $a \mapsto c(\epsilon, a)$ is a diffeomorphism onto its image when $\epsilon \neq 0$ is sufficiently small, which justifies viewing $\mathscr{C}_{\mathcal{I}}^{\epsilon}$ as being parameterized by the wave speed c.

We emphasize that the family $\mathscr{C}_{\mathcal{I}}^{\epsilon}$ comprises *all* traveling wave solutions of (3.47) with (η, φ, c) and ϵ in a neighborhood of 0 in a certain function space setting; see Appendix B. Thus our stability result applies to any waves that is sufficiently small-amplitude, slow moving, and has small enough vortex strength.

Lemma 3.31. *For each nontrivial compact intervals $\mathcal{I} \subset (0,\infty)$ and $0 < \epsilon \ll 1$, the family $\mathscr{C}_{\mathcal{I}}^{\epsilon}$ satisfies Assumption 3.9.*

Proof. From the construction of $\mathscr{C}_{\mathcal{I}}^{\epsilon}$ in Theorem 3.44, we know that the mapping $c \mapsto U_c$ is of class C^1. Since the existence theory can be carried out for any $k > 3/2$, we can ensure that U_c and $\frac{dU_c}{dc}$ satisfy Assumption 3.9 (iii).

Also, the non-degeneracy condition (ii) holds for small enough ϵ in view of (3.93). Finally,

$$\|T(s)U_c - U_c\|_{\mathbb{X}} \geq |(se_1 - ae_2) - ae_2| = |s|,$$

so the second option in (iv) holds. $\square$

Formally, the traveling waves on $\mathscr{C}_{\mathcal{I}}^{\epsilon}$ are stable if we can show that the moment of instability defined in (3.18) has positive second derivative. This can be shown to be the case when ϵ is small.

Lemma 3.32. *Fix a nontrivial compact interval $\mathcal{I} \subset (0, \infty)$. Then*

$$d''(c(\epsilon, a)) > 0, \qquad \textit{for all } a \in \mathcal{I},$$

when $0 < |\epsilon| \ll 1$.

Proof. By (3.20),

$$\partial_a c(\epsilon, a) d''(c(\epsilon, a)) = -\langle DP(U_{c(\epsilon,a)}), \partial_a U_{c(\epsilon,a)}\rangle, \tag{3.69}$$

and from Appendix B

$$\begin{aligned} c(\epsilon, a_0) &= -\frac{1}{4\pi a}\epsilon + O(\epsilon^3) \\ U_c(\epsilon, a_0) &= (0, 0, -ae_2) + (\eta_2(a), 0, 0)\epsilon^2 + O(\epsilon^3) \end{aligned}$$

in $C^1(\mathcal{I}; \mathbb{W})$. From the latter expression, and (3.62), we find that

$$\nabla P(U_{c(\epsilon,a)}) = (\Theta_{x_1}(\cdot, 0), 0, e_2)\epsilon - (0, \eta_2'(a), 0)\epsilon^2 + O(\epsilon^3)$$

in $C^0(\mathcal{I}, \mathbb{X}^*)$, and we can finally deduce from (3.69) that

$$\left(\frac{1}{4\pi a^2}\epsilon + O(\epsilon^3)\right) d''(c(\epsilon, a)) = \epsilon + O(\epsilon^3)$$

or

$$d''(c(\epsilon, a)) = 4\pi a^2 + O(\epsilon^2)$$

in $C^0(\mathcal{I}, \mathbb{R})$. The right hand side is positive on $\mathcal{I}$ for sufficiently small $\epsilon \neq 0$. $\square$

6 Stability of solitary waves with a point vortex

In the previous section, we confirmed that the capillary-gravity water wave problem with a point vortex has a Hamiltonian formulation (3.63) that is invariant under the translation group $T(\cdot)$ defined in (3.65), and we introduced the corresponding trio of Banach spaces $\mathbb{W} \hookrightarrow \mathbb{V} \hookrightarrow \mathbb{X}$ in (3.49)–(3.51). We are now prepared to state and prove the main theorem:

Theorem 3.33 (Main theorem)**.** *Fix a nontrivial compact interval $\mathcal{I} \subset (0,\infty)$ and $0 < |\epsilon| \ll 1$. Then the family $\mathscr{C}_{\mathcal{I}}^{\epsilon}$ of solitary capillary-gravity water waves with a submerged point vortex are conditionally orbitally stable in the sense of Theorem 3.11.*

Again, it is important to note that $\mathscr{C}_{\mathcal{I}}^{\epsilon}$ contains all even traveling waves solutions to (3.47) in a certain neighborhood of 0, and hence the stability furnished by Theorem 3.33 is not specific to this family but extends to all sufficiently small-amplitude, slow moving, and small vortex strength waves.

We have already addressed a number of the hypotheses of the general theory. Moreover, Lemma 3.32 shows that the family $\mathscr{C}_{\mathcal{I}}^{\epsilon}$ is *formally* orbitally stable for $0 < |\epsilon| \ll 1$. The only remaining task — which is by far the most difficult — is to verify that the waves in $\mathscr{C}_{\mathcal{I}}^{\epsilon}$ lie at a saddle point of the energy with a one-dimensional negative subspace, as required by Assumption 3.10. Our basic approach follows along the lines of Mielke's study of the irrotational case [34], with many modifications necessitated by the presence of the point vortex.

Recall that the family of traveling waves $\{U_c\}$ are critical points of the augmented Hamiltonian $E_c := E - cP$. Because φ occurs quadratically in E, and

$$\langle D_\varphi E_c, \dot{\varphi} \rangle = \int_{\mathbb{R}} \dot{\varphi}(G(\eta)\varphi + \epsilon \nabla_\perp \Theta + c\eta')\, dx_1,$$

we can eliminate φ by introducing

$$\begin{aligned} \varphi_*(v) &:= -G(\eta)^{-1}(c\eta' + \epsilon \nabla_\perp \Theta), \\ u_*(v) &:= (\eta, \varphi_*(v), \bar{x}) \in \mathbb{V}, \end{aligned} \tag{3.70}$$

and the *augmented potential*

$$\mathcal{V}_c^{\mathrm{aug}}(v) := \min_{\varphi \in \mathbb{V}_2} E_c(\eta, \varphi, \bar{x}) = E_c(u_*(v)). \tag{3.71}$$

for $v = (\eta, \bar{x}) \in \mathbb{V}_{1,3} \cap \mathcal{O}_{1,3}$. Here

$$\mathbb{V}_{1,3} := \mathbb{V}_1 \times \mathbb{V}_3, \qquad \mathcal{O}_{1,3} := \{(\eta, \bar{x}) \in \mathbb{X}_1 \times \mathbb{X}_3 : \bar{x}_2 < \eta(\bar{x}_1) < -\bar{x}_2\}.$$

Note that $\varphi_* \in C^\infty(\mathbb{V}_{1,3} \cap \mathcal{O}_{1,3}; \mathbb{X}_2^{3/2+})$ and $u_* \in C^\infty(\mathbb{V}_{1,3} \cap \mathcal{O}_{1,3}; \mathbb{V})$, whence in particular $\mathcal{V}_c^{\mathrm{aug}} \in C^\infty(\mathbb{V}_{1,3}; \mathbb{R})$.

For later use, we also define

$$\mathfrak{a} = \mathfrak{a}(v) := (\nabla(\mathcal{H}\varphi_*))|_S, \qquad \mathfrak{b} = \mathfrak{b}(v) := \mathfrak{a} + \epsilon\nabla\Theta|_S - ce_1.$$

Thus $\mathfrak{a}$ is the irrotational part of the velocity field, and $\mathfrak{b}$ is the relative velocity field, both restricted to the surface. Observe that $\mathfrak{b}_2 = \eta'\mathfrak{b}_1$ by (3.70). Because we are working with the steady problem, in what follows we simply write S rather than S_t.

Lemma 3.34. *For all $v \in \mathbb{V}_{1,3} \cap \mathcal{O}_{1,3}$ and $\dot{v} = (\dot{\eta}, \dot{\bar{x}}) \in \mathbb{V}_{1,3}$, we have*

$$\begin{aligned}\langle D^2\mathcal{V}_c^{\mathrm{aug}}(v)\dot{v}, \dot{v}\rangle_{\mathbb{V}_{1,3}^* \times \mathbb{V}_{1,3}} &= \langle D_v^2 E_c(u_*(v))\dot{v}, \dot{v}\rangle_{\mathbb{V}_{1,3}^* \times \mathbb{V}_{1,3}} \\ &\qquad - \langle \mathcal{L}(v)\dot{v}, G(\eta)^{-1}\mathcal{L}(v)\dot{v}\rangle_{\mathbb{X}_2^* \times \mathbb{X}_2},\end{aligned} \tag{3.72}$$

where

$$\mathcal{L}(v)\dot{v} := G(\eta)(\mathfrak{a}_2\dot{\eta}) + (\mathfrak{b}_1\dot{\eta})' + \epsilon\nabla_\perp\xi \cdot \dot{\bar{x}} \tag{3.73}$$

defines a bounded linear operator $\mathcal{L}(v) \in \operatorname{Lin}(\mathbb{X}_{1,3}; \mathbb{X}_2^*)$.

Proof. By the definitions of φ_* in (3.70) and $\mathcal{V}_c^{\mathrm{aug}}$ in (3.71), it follows that

$$\langle D\mathcal{V}_c^{\mathrm{aug}}(v), \dot{v}\rangle = \langle D_\varphi E_c(u_*(v)), \dot{v}\rangle + \langle D_v E_c(u_*(v)), \dot{v}\rangle = \langle D_v E_c(u_*(v)), \dot{v}\rangle,$$

and

$$\begin{aligned}\langle D^2\mathcal{V}_c^{\mathrm{aug}}(v)\dot{v}, \dot{v}\rangle &= \langle D_v D_\varphi E_c(u_*(v))\langle D\varphi_*(v), \dot{v}\rangle, \dot{v}\rangle + \langle D_v^2 E_c(u_*(v))\dot{v}, \dot{v}\rangle \\ &= -\langle D_\varphi^2 E_c(u_*(v))\langle D\varphi_*(v), \dot{v}\rangle, \langle D\varphi_*(v), \dot{v}\rangle\rangle \\ &\qquad + \langle D_v^2 E_c(u_*(v))\dot{v}, \dot{v}\rangle,\end{aligned}$$

which yields the claimed formula after computing that

$$\begin{aligned}G(\eta)\langle D\varphi_*(v), \dot{v}\rangle &= -\langle D_\eta G(\eta)\dot{\eta}, \varphi_*(v)\rangle + ([\epsilon\Theta_{x_1}|_S - c]\dot{\eta})' + \epsilon\nabla_\perp\xi \cdot \dot{\bar{x}} \\ &= G(\eta)(\mathfrak{a}_2\dot{\eta}) + (\mathfrak{b}_1\dot{\eta})' + \epsilon\nabla_\perp\xi \cdot \dot{\bar{x}}.\end{aligned}$$ ☐

The next lemma further unpacks the expression (3.72) to obtain a quadratic form representation on the energy space, in preparation for the verification of Assumption 3.10.

Lemma 3.35 (Extension of $D^2\mathcal{V}_c^{\mathrm{aug}}$)**.** *For all $v \in \mathbb{V}_{1,3} \cap \mathcal{O}_{1,3}$, there is a self-adjoint linear operator $A(v) \in \operatorname{Lin}(\mathbb{X}_{1,3}; \mathbb{X}_{1,3}^*)$ such that*

$$\langle D^2\mathcal{V}_c^{\mathrm{aug}}(v)\dot{v}, \dot{w}\rangle_{\mathbb{V}_{1,3}^* \times \mathbb{V}_{1,3}} = \langle A(v)\dot{v}, \dot{w}\rangle_{\mathbb{X}_{1,3}^* \times \mathbb{X}_{1,3}}$$

for all $\dot{v}, \dot{w} \in \mathbb{V}_{1,3}$. Explicitly,

$$A = \begin{pmatrix} A_{11} & A_{13} \\ A_{13}^* & A_{33} \end{pmatrix}, \tag{3.74}$$

with entries given by

$$\begin{aligned} A_{11}\dot{\eta} &:= (g + \mathfrak{b}_2'\mathfrak{b}_1)\dot{\eta} - \left(\frac{b}{\langle \eta' \rangle^3}\dot{\eta}'\right)' - \mathcal{M}\dot{\eta}, \\ A_{13}\dot{\bar{x}} &:= \epsilon\mathfrak{b}_1\nabla_\top(G(\eta)^{-1}\nabla_\perp\xi - \xi) \cdot \dot{\bar{x}} \\ A_{13}^*\dot{\eta} &:= \epsilon\int_{\mathbb{R}} \dot{\eta}\mathfrak{b}_1\nabla_\top(G(\eta)^{-1}\nabla_\perp\xi - \xi)\, dx_1, \\ A_{33} &:= D_{\bar{x}}^2 E_c(u_*) + \epsilon^2\int_{\mathbb{R}} \nabla_\perp\xi \odot G(\eta)^{-1}\nabla_\perp\xi\, dx_1. \end{aligned}$$

Here $\mathcal{M}\dot{\eta} := -\mathfrak{b}_1(G(\eta)^{-1}(\mathfrak{b}_1\dot{\eta})')'$, $x \odot y = (x \otimes y + y \otimes x)/2$ is the symmetric outer product, and an explicit expression for $D_{\bar{x}}^2 E_c(u_*)$ is given in (3.75)*.*

Proof. Due to symmetry, it is sufficient to consider the diagonal. A series of rather lengthy, but direct, computations show that

$$\begin{aligned} \int_{\mathbb{R}} (\mathcal{L}(v)\dot{v})G(\eta)^{-1}\mathcal{L}(v)\dot{v}\, dx_1 &= \int_{\mathbb{R}} \mathfrak{a}_2\dot{\eta}G(\eta)(\mathfrak{a}_2\dot{\eta})\, dx_1 + \int_{\mathbb{R}} \dot{\eta}\mathcal{M}\dot{\eta}\, dx_1 \\ &+ \int_{\mathbb{R}} (\mathfrak{a}_2\mathfrak{b}_1' - \mathfrak{a}_2'\mathfrak{b}_1)\dot{\eta}^2\, dx_1 + 2\epsilon\dot{\bar{x}} \cdot \int_{\mathbb{R}} (\mathfrak{a}_2\nabla_\perp\xi - \mathfrak{b}_1(G(\eta)^{-1}\nabla_\perp\xi)')\dot{\eta}\, dx_1 \\ &+ \epsilon^2\dot{\bar{x}}^T\left(\int_{\mathbb{R}} \nabla_\perp\xi \odot G(\eta)^{-1}\nabla_\perp\xi\, dx_1\right)\dot{\bar{x}}, \end{aligned}$$

while

$$\begin{aligned} \langle D_\eta^2 E_c(u_*)\dot{\eta}, \dot{\eta}\rangle &= \int_{\mathbb{R}} \mathfrak{a}_2\dot{\eta}G(\eta)(\mathfrak{a}_2\dot{\eta})\, dx_1 \\ &+ \int_{\mathbb{R}} (g + \epsilon\mathfrak{b}_1\nabla_\top\Theta_{x_2} + \mathfrak{a}_2\mathfrak{b}_1')\dot{\eta}^2\, dx_1 \\ &+ \int_{\mathbb{R}} \frac{b}{\langle \eta' \rangle^3}(\dot{\eta}')^2\, dx_1, \end{aligned}$$

$$\nabla_{\bar{x}}\langle D_\eta E_c(u_*), \dot\eta\rangle = \epsilon \int_{\mathbb{R}} (\mathfrak{a}_2 \nabla_\perp \xi - \mathfrak{b}_1 \nabla_\top \xi)\dot\eta \, dx_1,$$

and

$$\begin{aligned} D_{\bar{x}}^2 E_c(u_*) = 2\epsilon^2 D_x^2 \Gamma_2(\bar{x}) - \epsilon \int_{\mathbb{R}} (G(\eta)\varphi_* D_{\bar{x}}^2 \Theta + \varphi_*' D_{\bar{x}}^2 \Gamma)|_S \, dx_1 \\ + \epsilon^2 \int_{\mathbb{R}} \nabla_\perp \xi \odot \xi \, dx_1 \\ - \frac{\epsilon^2}{2} \int_{\mathbb{R}} (\nabla_\perp \Theta D_{\bar{x}}^2 \Theta + \nabla_\top \Theta D_{\bar{x}}^2 \Gamma)|_S \, dx_1. \end{aligned} \tag{3.75}$$

Thus, using Lemma 3.34, we find

$$\begin{aligned} \langle D^2 \mathcal{V}_c^{\mathrm{aug}}(v)\dot v, \dot v\rangle = \int_{\mathbb{R}} (g + \mathfrak{b}_2' \mathfrak{b}_1)\dot\eta^2 \, dx_1 - \int_{\mathbb{R}} \left(\frac{b}{\langle \eta' \rangle^3} \dot\eta'\right)' \dot\eta \, dx_1 - \int_{\mathbb{R}} \dot\eta \mathcal{M} \dot\eta \, dx_1 \\ + 2\epsilon \dot{\bar{x}} \cdot \int_{\mathbb{R}} \dot\eta \mathfrak{b}_1 \nabla_\top (G(\eta)^{-1} \nabla_\perp \xi - \xi) \, dx_1 \\ + \dot{\bar{x}}^T \left(D_{\bar{x}}^2 E_c(u_*) - \epsilon^2 \int_{\mathbb{R}} \nabla_\perp \xi \odot G(\eta)^{-1} \nabla_\perp \xi \, dx_1 \right) \dot{\bar{x}}, \end{aligned}$$

which yields the claimed operator A. □

Remark 3.36. Under natural symmetry assumptions on v, the expression for A_{33} can be simplified further. Specifically, if η is even and $\bar{x}_1 = 0$, then

$$\begin{aligned} A_{33} = 2\epsilon^2 D_x^2 \Gamma_2(\bar{x}) - \epsilon \int_{\mathbb{R}} (G(\eta)\varphi_* D_{\bar{x}}^2 \Theta + \varphi_*' D_{\bar{x}}^2 \Gamma)|_S \, dx_1 \\ + \epsilon^2 \int_{\mathbb{R}} \nabla_\perp \xi \odot (\xi - G(\eta)^{-1} \nabla_\perp \xi) \, dx_1, \end{aligned}$$

and all three terms are diagonal matrices.

We can now confirm that the augmented Hamiltonian admits an extension to the energy space.

Lemma 3.37 (Extension of $D^2 E_c$)**.** *For all $v \in \mathbb{V}_{1,3} \cap \mathcal{O}_{1,3}$, there is a self-adjoint operator $H_c(v) \in \mathrm{Lin}(\mathbb{X}, \mathbb{X}^*)$ such that*

$$\langle D^2 E_c(u_*(v))\dot u, \dot w\rangle_{\mathbb{V}^* \times \mathbb{V}} = \langle H_c(v)\dot u, \dot w\rangle_{\mathbb{X}^* \times \mathbb{X}} \tag{3.76}$$

for all $\dot u, \dot w \in \mathbb{V}$. The operator is given by

$$H_c(v)\dot u = \begin{pmatrix} \mathrm{id}_{\mathbb{X}_1^*} & 0 & 0 \\ 0 & 0 & \mathrm{id}_{\mathbb{X}_2^*} \\ 0 & \mathrm{id}_{\mathbb{R}^2} & 0 \end{pmatrix} \begin{pmatrix} A(v) + \mathcal{L}(v)^* G(\eta)^{-1} \mathcal{L}(v) & -\mathcal{L}(v)^* \\ -\mathcal{L}(v) & G(\eta) \end{pmatrix} \begin{bmatrix} \dot v \\ \dot\varphi \end{bmatrix},$$

where $\mathcal{L}(v)$ and $A(v)$ are as defined in Lemmas 3.34 and 3.35, respectively. The adjoint $\mathcal{L}(v)^* \in \mathrm{Lin}(\mathbb{X}_2; \mathbb{X}_{1,3}^*)$ is given by

$$\mathcal{L}(v)^* \dot{\varphi} = \left(\mathfrak{a}_2 G(\eta)\dot{\varphi} - \mathfrak{b}_1 \dot{\varphi}', \epsilon\langle \nabla_\perp \xi, \dot{\varphi}\rangle\right),$$

and we have

$$\begin{aligned}\langle H_c \dot{u}, \dot{u}\rangle_{\mathbb{X}^* \times \mathbb{X}} &= \langle A(v)\dot{v}, \dot{v}\rangle_{\mathbb{X}_{1,3}^* \times \mathbb{X}_{1,3}} \\ &\quad + \langle G(\eta)(\dot{\varphi} - G(\eta)^{-1}\mathcal{L}\dot{v}), (\dot{\varphi} - G(\eta)^{-1}\mathcal{L}\dot{v})\rangle_{\mathbb{X}_2^* \times \mathbb{X}_2}\end{aligned} \tag{3.77}$$

for all $\dot{u} \in \mathbb{X}$.

Proof. Again, we need only consider the diagonal. By Lemmas 3.34 and 3.35 one has

$$\begin{aligned}\langle D^2 E_c(u_*(v))\dot{u}, \dot{u}\rangle_{\mathbb{V}^* \times \mathbb{V}} &= \langle A\dot{v} + \mathcal{L}(v)^* G(\eta)^{-1}\mathcal{L}(v)\dot{v}, \dot{v}\rangle_{\mathbb{X}_{1,3}^* \times \mathbb{X}_{1,3}} \\ &\quad + \langle D_\varphi^2 E_c(u_*(v))\dot{\varphi} + 2D_\varphi D_v E_c(u_*(v))\dot{v}, \dot{\varphi}\rangle_{\mathbb{V}_2^* \times \mathbb{V}},\end{aligned}$$

for all $\dot{u} \in \mathbb{V}$, and it is simple to verify that

$$\langle D_\varphi D_v E_c(u_*(v))\dot{v}, \dot{\varphi}\rangle_{\mathbb{V}_2^* \times \mathbb{V}_2} = -\langle \mathcal{L}(v)\dot{v}, \dot{\varphi}\rangle_{\mathbb{X}_2^* \times \mathbb{X}_2}$$

for all $\dot{v} \in \mathbb{V}_{1,3}$ and $\dot{\varphi} \in \mathbb{V}_2$. $\square$

Using the representation for $D^2\mathcal{V}_c^{\mathrm{aug}}$ furnished by Lemma 3.37 in conjunction with the asymptotics derived in Appendix B, we are at last able to prove that Assumption 3.10 is satisfied.

Theorem 3.38. *Let $\mathcal{I} \subset (0,\infty)$ be a nontrivial compact interval, and consider the family of bound states $\mathscr{C}_{\mathcal{I}}^\epsilon$ defined in* (3.68)*, furnished by Theorem 3.44. Fix $0 < |\epsilon| \ll 1$. Then the spectrum of $H_c = H_{c(\epsilon,a)}(v(\epsilon,a))$ has the form*

$$\mathrm{spec}(I^{-1}H_c) = \{-\mu_c^2\} \cup \{0\} \cup \Sigma_c$$

for all $a \in \mathcal{I}$, with $-\mu_c^2 < 0$ and 0 being simple eigenvalues, and $\Sigma_c \subset (0,\infty)$ bounded away from 0.

Proof. Under this hypothesis, we may view H_c as a small perturbation of the block diagonal operator

$$\begin{pmatrix} g - b\partial_{x_1}^2 & 0 & 0 \\ 0 & |\partial_{x_1}| & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathrm{Lin}(\mathbb{X}, \mathbb{X}^*),$$

whose spectrum clearly consists of a part $\tilde{\Sigma} \subset (0,\infty)$ bounded away from 0, plus the eigenvalue 0 with multiplicity two. Thus the spectrum of H_c will have a part $\Sigma_c \subset (0,\infty)$ bounded away from 0, plus two eigenvalues near the origin. We know that one of these is exactly 0, with corresponding eigenvector $T'(0)U_c$. Finally, from Lemma 3.32 and (3.20) we see that $\frac{dU}{dc}$ is a negative direction for H_c. Thus the other eigenvalue has to be negative. $\square$

At this stage, we have completely verified that the myriad hypotheses of the abstract stability theory are satisfied for the solutions constructed in Appendix B. Theorem 3.33 therefore follows immediately from Theorem 3.11.

7 Stability for a class of dispersive PDEs modeling water waves

As a second illustration of the abstract theory, we devote this section to studying the stability properties of solitary wave solutions to the nonlinear dispersive PDE

$$\partial_t u = \partial_x(\Lambda^\alpha u - u^p), \tag{3.78}$$

where $u = u(t,x)\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is the unknown, $\Lambda := |\partial_x|$, $\alpha \in (1/3, 2]$, and

$$p \in \mathbb{N} \cap \begin{cases} (1, (1+\alpha)(1-\alpha)^{-1}) & \alpha \in (1/3, 1), \\ (1,\infty) & \alpha \in [1,2]. \end{cases} \tag{3.79}$$

Heuristically, α describes the strength of the dispersion, while p describes the strength of the nonlinearity.

Equations of the general form (3.78) include a number of extremely important hydrodynamical models. In particular, when $p = 2$, the cases $\alpha = 1$ and $\alpha = 2$ are known as the Benjamin–Ono equation (BO) and Korteweg–de Vries equation (KdV), respectively. KdV, among many other things, governs surface waves in shallow water. Benjamin–Ono models the motion of waves along the interface between two infinitely deep fluid regions in a certain long wave regime [4, 38].

In [6], Bona, Souganidis, and Strauss investigated the orbital stability and instability of solitary wave solutions to (3.78) for $\alpha \in [1,2]$. Their strategy relied on many of the ideas underlying the GSS method. However, as we will see below, the corresponding Poisson map J was not surjective, and hence a number of adaptations were necessary. Specifically, the authors made use of another conserved quantity — the mass $\int u\,dx$ — requiring them to obtain estimates on the spatial decay rates of solitary waves in order to ensure the persistence of integrability.

Our purpose in this section is to offer a new proof of the Bona, Souganidis, and Strauss theorem that follows directly from the stability machinery presented in Section 3 and Section 4. Because we do not appeal to the mass, no asymptotic estimates are then required. Notice also that we treat "fractional" dispersive model equations for which $\alpha \in (1/3, 1)$. Orbital stability results for $\alpha \in (1/2, 1)$ have been obtained by Linares, Pilod, and Saut [30], and Pava [39]; we discuss the connections between these works and the present paper further below. Theorem 3.42 below gives conditional orbital instability for fKdV ($p = 2$) when $\alpha \in (1/3, 1/2)$, and this appears to be new. Indeed, Linares, Pilod, and Saut observe that the Bona, Souganidis, and Strauss approach *almost* works in this regime, except that the tail estimates fail to hold.

While we do not pursue it here, one can also consider more general nonlinearities at the expense of some sharpness. Another interesting possible extension is to study dispersive PDEs like the Whitham equation, where Λ^α in (3.78) is replaced by a Fourier multiplier with an inhomogeneous symbol.

It is also important to note that, by specializing to specific choices of α and p, one can say much more. As one example, for $\alpha = 2$ (gKdV), Pego and Weinstein [40], Mizumachi [35], Martel and Merle [33], and Germain, Pusateri, and Rousset [17] obtain asymptotic stability results (in different topologies) for various subcritical cases $p < 5$. For supercritical waves $p > 5$, Jin, Lin, and Zeng [25] were able to completely classify the H^1 dynamics near the family of solitary waves using invariant manifold techniques. The main appeal of our approach is its relative simplicity, and the fact that it simultaneously addresses the range of dispersion strengths $\alpha \in (1/3, 2]$ and nonlinearities (3.79).

Reformulation as a Hamiltonian system

Formally, the expression inside the parentheses on the right-hand side of (3.78) is the derivative of the energy

$$E(u) := \frac{1}{2}\int_{\mathbb{R}} (\Lambda^{\frac{\alpha}{2}} u)^2 \, dx - \frac{1}{p+1}\int_{\mathbb{R}} u^{p+1} \, dx. \tag{3.80}$$

This suggests that the natural energy space is $\mathbb{X} := H^{\frac{\alpha}{2}}(\mathbb{R})$, with the dual space $\mathbb{X}^* = H^{-\frac{\alpha}{2}}(\mathbb{R})$, and the isomorphism $I : \mathbb{X} \to \mathbb{X}^*$ given by $\langle \Lambda \rangle^\alpha$. The condition $\alpha > 1/3$ ensures the existence of admissible p, those satisfying (3.79), which in particular implies that $\mathbb{X} \hookrightarrow L^{p+1}(\mathbb{R})$. Observe that E defined according to (3.80) then lies in $C^\infty(\mathbb{X}; \mathbb{R})$, and that indeed

$$DE(u) = \Lambda^\alpha u - u^p$$

for all $u \in \mathbb{X}$. We may therefore take

$$\mathbb{V} \coloneqq \mathbb{X}. \tag{3.81}$$

The local and global well-posedness of the Cauchy problem for (3.78) is still an active subject of research, and what is currently known depends considerably on α and p. To state things concisely, we suppose that (3.78) is known to be locally well-posed in H^s for $s > s_0 = s_0(\alpha, p)$, and set

$$\mathbb{W} \coloneqq \begin{cases} \mathbb{X} & \text{if } \frac{\alpha}{2} > s_0(\alpha, p), \\ H^{s_0+}(\mathbb{R}) & \text{if } \frac{\alpha}{2} \leq s_0(\alpha, p). \end{cases} \tag{3.82}$$

At present, the best known result when $p = 2$ is $s_0(\alpha, 2) = 3/2 - 5\alpha/4$, and hence (3.78) is *globally* well-posed in $\mathbb{X}$ when $\alpha > 6/7$ and $p = 2$; see [36, 37]. This is conjectured to hold for all $\alpha > 1/2$, which corresponds to the L^2 subcritical case.

However, for fKdV with $\alpha \in (1/3, 6/7)$, the functional analytic setup in (3.82) will lead to a conditional stability or instability result. This is essentially what is done by Pava in [39, Theorem 1.1], as well as Linares, Pilod, and Saut in [30, Theorem 2.14], who treat the range $\alpha \in (1/2, 1)$. We caution, however, that in both of these papers the definition of "conditional stability" is less conditional than ours: we require the solution to remain in the ball $\mathcal{B}_R^{\mathbb{W}}$, while they only ask for it to exist.

Next, define the Poisson map $J \colon \mathcal{D}(J) \subset \mathbb{X}^* \to \mathbb{X}$ by

$$J \coloneqq \partial_x, \tag{3.83}$$

with domain $\mathcal{D}(J) \coloneqq H^{1+\frac{\alpha}{2}}(\mathbb{R})$. As J is independent of state, it can be identified with $\hat{J}$ in Assumption 3.4. Moreover, J is clearly injective, and skew-adjoint. The Cauchy problem for (3.78) can now be restated rigorously as the abstract Hamiltonian system

$$\frac{d}{dt}\langle u(t), w\rangle = \langle u^p - \Lambda^\alpha u, \partial_x w\rangle \quad \text{for all } w \in H^{1+\frac{\alpha}{2}}(\mathbb{R}), \quad u(0) = u_0, \tag{3.84}$$

by specializing the general system in (3.7).

The equation (3.78) possesses a number of symmetries, but the one of most interest to us is spatial translation invariance. For each $s \in \mathbb{R}$, we define $T(s) \in \operatorname{Lin}(\mathbb{X})$ by

$$T(s)u \coloneqq u(\cdot - s), \tag{3.85}$$

and this forms a group of unitary operators on $\mathbb{X}$. Its infinitesimal generator is $T'(0) = -\partial_x$, with domain $\mathcal{D}(T'(0)) = H^{1+\frac{\alpha}{2}}(\mathbb{R})$. Moreover, since $T'(0) = J(-\iota_{\mathbb{X}\to\mathbb{X}^*})$, the group generates the momentum

$$P(u) := \frac{1}{2}\langle -u, u\rangle = -\frac{1}{2}\int_{\mathbb{R}} u^2\, dx, \tag{3.86}$$

which also is of class $C^\infty(\mathbb{X};\mathbb{R})$.

The next lemma collects and expands upon these observations to confirm that the Hamiltonian formulation meets the requirements of the general theory.

Lemma 3.39. *The Hamiltonian formulation* (3.84) *of the dispersive model equation* (3.78) *satisfies Assumptions 3.2 and 3.7.*

Proof. Because $\mathbb{V} = \mathbb{X}$, both Assumption 3.2 and Assumption 3.6 hold trivially. Likewise, for J defined as in (3.83), we have already verified that the relevant requirements of Assumption 3.4 are met. To show that the symmetry group satisfies Assumption 3.7 requires chasing the definitions. Both invariance and the commutativity are readily checked, as differentiation commutes with translation. Finally, the only remaining property that requires elaboration is (viii). We see that

$$\begin{aligned}\mathcal{D}(T'(0)|_{\mathbb{W}}) \cap \operatorname{Rng} J &= \begin{cases} H^{1+\frac{\alpha}{2}}(\mathbb{R}) \cap \partial_x H^{1+\frac{\alpha}{2}}(\mathbb{R}) & \text{if } \frac{\alpha}{2} > s_0(\alpha,p),\\ H^{(1+s_0)+}(\mathbb{R}) \cap \partial_x H^{1+\frac{\alpha}{2}}(\mathbb{R}) & \text{if } \frac{\alpha}{2} \le s_0(\alpha,p),\end{cases}\\ &\supset \partial_x H^{2+\frac{\alpha}{2}}(\mathbb{R}),\end{aligned}$$

which is dense in $\mathbb{X}$ by the same kind of argument as in Lemma 3.43. □

Solitary waves and spectral properties

It is well known that the dispersive models captured by (3.78) support solitary waves $u(t) = T(ct)U_c = U_c(\cdot - ct)$ for all $c > 0$. Recall that such $U_c \in \mathbb{X}$ must satisfy

$$DE_c(U_c) = \Lambda^\alpha U_c - U_c^p + cU_c = 0, \qquad (\text{in } \mathbb{X}^*) \tag{3.87}$$

and by introducing the scaling

$$U_c = c^{\frac{1}{p-1}} Q(c^{\frac{1}{\alpha}}\cdot) \tag{3.88}$$

we see that all such waves are just scaled versions of solutions of the equation

$$Q + \Lambda^\alpha Q = Q^p. \tag{3.89}$$

Lemma 3.40. *If $Q \in \mathbb{X}$ is a nontrivial solution of* (3.89)*, then the family $\{U_c \colon c \in (0,\infty)\}$ defined through* (3.88) *satisfies Assumption 3.9.*

Proof. By a standard bootstrapping argument, we have that any such solution Q lies in $H^r(\mathbb{R})$ for every $r \geq 0$. Since U_c is defined by (3.88), parts (i) to (iii) are therefore immediate. Finally,

$$\liminf_{|s|\to\infty} \|T(s)U_c - U_c\|_{\mathbb{X}} = 2\|U_c\|_{\mathbb{X}} > 0,$$

so the second option of part (iv) holds. ☐

It is also easily seen that if $Q \neq 0$ solves (3.89), then Q is a critical point of the Weinstein functional $\mathcal{J} \in C^2(\mathbb{X} \setminus \{0\}; (0,\infty))$ defined by

$$\mathcal{J}(u) := \frac{\|u\|_{\dot{H}^{\frac{\alpha}{2}}(\mathbb{R})}^{\frac{p-1}{\alpha}} \|u\|_{L^2(\mathbb{R})}^{p+1-\frac{p-1}{\alpha}}}{\|u\|_{L^{p+1}(\mathbb{R})}^{p+1}}.$$

We say that a solution Q of (3.89) is a ground state if Q is not just a critical point of $\mathcal{J}$, but also an even, positive minimizer. Since $\mathcal{J}$ is invariant under scaling, the same is then true of each U_c, solving (3.87), defined through (3.88). Note that the corresponding operator $H_c \in \operatorname{Lin}(\mathbb{X}, \mathbb{X}^*)$ is given by

$$H_c u := D^2 E_c(U_c)u = \Lambda^\alpha u - pU_c^{p-1}u + cu,$$

which is clearly self-adjoint. We have the following result, due to Frank and Lenzmann [15], vastly generalizing earlier results for KdV [48] and BO [2, 3].

Lemma 3.41. *There exists a unique ground state solution $Q \in \mathbb{X}$ of* (3.89)*. Moreover, for each $c \in (0,\infty)$, the spectrum of the operator H_c corresponding to the bound state solution U_c defined by* (3.88) *satisfies*

$$\operatorname{spec} I^{-1}H_c = \{-\mu_c^2\} \cup \{0\} \cup \Sigma_c,$$

where $-\mu_c^2 < 0$ and 0 are simple eigenvalues, and $\Sigma_c \subset (0,\infty)$ is bounded away from zero. That is, Assumption 3.10 is satisfied.

Of course, Q cannot be written down explicitly for most choices of α and p. Famously, for KdV

$$Q_{\mathrm{KdV}}(x) = \frac{3}{2}\operatorname{sech}^2\left(\frac{x}{2}\right),$$

while Benjamin [4] exhibited the ground state

$$Q_{\mathrm{BO}}(x) = \frac{2}{1+x^2}$$

for BO in his original paper on the topic.

Stability and instability

The analysis of the previous subsection confirms that the family $\{U_c : c \in (0,\infty)\}$ corresponding to the unique ground state Q of (3.89) furnished by Lemma 3.41 falls into the scope of the general stability theory developed in Section 3 and Section 4. We therefore obtain the following extended version of the classical result of Bona, Souganidis, and Strauss [6]:

Theorem 3.42. *If $p < 2\alpha + 1$, then each solitary wave in the family $\{U_c : c \in (0,\infty)\}$ is conditionally orbitally stable in the sense of Theorem 3.11 when $\frac{\alpha}{2} \leq s_0(\alpha, p)$, and orbitally stable in the sense of Corollary 3.15 when $\frac{\alpha}{2} > s_0(\alpha, p)$. When $p > 2\alpha + 1$, the solitary waves are orbitally unstable in the sense of Theorem 3.14.*

Proof. Whether U_c is stable or not reduces to the sign of $d''(c)$, where we recall that $d(c) \coloneqq E_c(U_c)$ is the moment of instability. Exploiting the scaling (3.88) and the identity (3.19), we find

$$d'(c) = -P(U_c) = \frac{1}{2}\int_{\mathbb{R}} U_c^2\, dx = \frac{1}{2} c^{\frac{2}{p-1} - \frac{1}{\alpha}} \|Q\|_{L^2(\mathbb{R})}^2,$$

whence

$$\operatorname{sgn} d''(c) = \operatorname{sgn}\left(\frac{2}{p-1} - \frac{1}{\alpha}\right) \begin{cases} > 0 & \text{if } p < 2\alpha + 1, \\ < 0 & \text{if } p > 2\alpha + 1, \end{cases}$$

which gives the statement in the theorem. □

A Function spaces

Define the Schwartz class $\mathcal{S}(\mathbb{R})$ to be the set of all $f \in C^\infty(\mathbb{R})$ such that $x^n f^{(m)}(x) \in L^\infty(\mathbb{R})$ for all $n, m \in \mathbb{N}_0$, and also the subspace $\mathcal{S}_0(\mathbb{R})$ of those $f \in \mathcal{S}(\mathbb{R})$ for which $\hat{f}^{(n)}(0) = 0$ for all $n \in \mathbb{N}_0$. For every $s \in \mathbb{R}$, we define the inhomogeneous Sobolev space $H^s(\mathbb{R})$ to be the completion of $\mathcal{S}(\mathbb{R})$ with respect to

$$\|f\|_{H^s(\mathbb{R})} \coloneqq \|\langle\cdot\rangle^s \hat{f}\|_{L^2(\mathbb{R})}, \tag{3.90}$$

and can be realized as the space of all $f \in \mathcal{S}'(\mathbb{R})$ for which $\hat{f} \in L^1_{\text{loc}}(\mathbb{R})$ and $\|f\|_{H^s(\mathbb{R})} < \infty$.

The homogeneous Sobolev space $\dot{H}^s(\mathbb{R})$, on the other hand, is defined to be the completion of $\mathcal{S}_0(\mathbb{R})$ with respect to

$$\|f\|_{\dot{H}^s(\mathbb{R})} \coloneqq \||\cdot|^s \hat{f}\|_{L^2(\mathbb{R})} < \infty, \tag{3.91}$$

and for $s < 1/2$ it can be realized like before as the space of all $f \in \mathcal{S}'(\mathbb{R})$ for which $\hat{f} \in L^1_{\mathrm{loc}}(\mathbb{R})$ and $\|f\|_{\dot{H}^s(\mathbb{R})} < \infty$. If $s \geq 1/2$, set $n := \lfloor s + 1/2 \rfloor$, so that $s = n + \alpha$ with $\alpha \in [-1/2, 1/2)$. Then $\dot{H}^s(\mathbb{R})$ can be realized as the space of all $f \in \mathcal{S}'(\mathbb{R})$ such that $f^{(n)} \in \dot{H}^\alpha(\mathbb{R})$, *modulo* polynomials of degree at most $n-1$, with the norm (3.91) interpreted as $\|f^{(n)}\|_{\dot{H}^\alpha}$.

On domains $\Omega \subset \mathbb{R}^2$, we shall only have use for $\dot{H}^1(\Omega)$, defined as the space of $f \in L^1_{\mathrm{loc}}(\Omega)/\mathbb{R}$ for which $\nabla f \in L^2(\Omega)$.

Lemma 3.43 (Density)**.** *For all $s, r \in \mathbb{R}$, the space $H^s(\mathbb{R}) \cap \dot{H}^r(\mathbb{R})$ is dense in both $H^s(\mathbb{R})$ and $\dot{H}^r(\mathbb{R})$.*

Proof. Define $\chi_n := \chi_{1/n < |\xi| < n}$ for all $n \in \mathbb{N}$. Suppose first that $f \in H^s(\mathbb{R})$, and set $f_n := \mathscr{F}^{-1}(\chi_n \hat{f})$ for $n \in \mathbb{N}$. Then $f_n \in H^s(\mathbb{R}) \cap \dot{H}^r(\mathbb{R})$ and

$$\|f - f_n\|_{H^s(\mathbb{R})} = \|\langle\cdot\rangle^s (1 - \chi_n)\hat{f}\|_{L^2(\mathbb{R})},$$

so $f_n \to f$ in $H^s(\mathbb{R})$. Next, suppose that $f \in \dot{H}^r$ and choose $k \in \mathbb{N}$ large enough so that $r - k < 1/2$. Then the sequence

$$f_n := \mathscr{F}^{-1}\Big(\chi_n \widehat{f^{(k)}}/(i\xi)^k\Big) \in H^s(\mathbb{R}) \cap \dot{H}^r(\mathbb{R})$$

converges to f in $\dot{H}^r(\mathbb{R})$. □

B Existence theory

In this appendix, we present a slightly modified version of the existence theory for capillary-gravity waves with a point vortex due to Shatah, Walsh, and Zeng [43]. The original paper fixes the location of the vortex, which is ill-suited for us. Since ϵ appears in the Poisson map, it must be held fixed on any family of waves to which we wish to apply the general stability theory. We can obtain such families by also allowing the location of the point vortex to vary.

Specifically, we suppose that the point vortex is situated at $\bar{x} = -ae_2$, where $a > 0$. A symmetric traveling wave solution to (3.47) having wave speed c must then satisfy the abstract operator equation

$$\mathscr{F}(\eta, \varphi, c; \epsilon, a) = 0, \tag{3.92}$$

with $\mathscr{F} = (\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3) \colon O \subset (X \times \mathbb{R} \times (0,\infty)) \to Y$ defined by

$$\begin{aligned}
\mathscr{F}_1(\eta, \varphi, c; \epsilon, a) &:= \frac{(\varphi')^2 - 2\eta'\varphi' G(\eta)\varphi - (G(\eta)\varphi)^2}{2\langle \eta' \rangle^2} - c\varphi' + g\eta - b\left(\frac{\eta'}{\langle \eta' \rangle}\right)' \\
&\qquad + \epsilon\varphi'\Theta_{x_1}|_S + \frac{\epsilon^2}{2}\left(|\nabla\Theta|^2\right)|_S - \epsilon c\Theta_{x_1}|_S \\
\mathscr{F}_2(\eta, \varphi, c; \epsilon, a) &:= c\eta' + G(\eta)\varphi + \epsilon\nabla_\perp\Theta, \\
\mathscr{F}_3(\eta, \varphi, c; \epsilon, a) &:= c - (\mathcal{H}(\eta)\varphi)_{x_1}(0, -a) + \frac{\epsilon}{4\pi a},
\end{aligned}$$

where $\mathcal{H}(\eta)$ denotes the harmonic extension operator. We will use the spaces

$$\begin{aligned}
X &:= H_e^k(\mathbb{R}) \times (\dot{H}_o^k(\mathbb{R}) \cap \dot{H}_o^{1/2}(\mathbb{R})) \times \mathbb{R}, \\
Y &:= H_e^{k-2}(\mathbb{R}) \times (\dot{H}_o^{k-1}(\mathbb{R}) \cap \dot{H}_o^{-1/2}) \times \mathbb{R}
\end{aligned}$$

for any $k > 3/2$ fixed, with the subscripts indicating odd and even, and the open set

$$O := \{(\eta, \varphi, c; \epsilon, a) \in X \times \mathbb{R} \times (0,\infty) : |\eta(0)| < a\}.$$

The map $\mathscr{F}$ is then C^∞ (even analytic), and we have the following existence theorem.

Theorem 3.44. *There exists a C^∞-surface*

$$\{(\eta(\epsilon, a), \varphi(\epsilon, a); \epsilon, a) : (\epsilon, a) \in U\} \subset O \times \mathbb{R} \times (0, \infty)$$

of solutions to (3.92)*, with U an open neighborhood of $\{0\} \times (0,\infty)$. Asymptotically, the solutions are of the form*

$$\begin{aligned}
\eta(\epsilon, a) &= \epsilon^2\eta_2(a) + O(\epsilon^4), \\
\varphi(\epsilon, a) &= O(\epsilon^3), \\
c(\epsilon, a) &= \epsilon c_1(a) + O(\epsilon^3),
\end{aligned} \tag{3.93}$$

in $C^1_{loc}((0,\infty); X)$, with

$$c_1(a) := -\frac{1}{4\pi a}, \qquad \eta_2(a) := \frac{1}{4\pi^2}(g - b\partial_{x_1}^2)^{-1}\left(\frac{x_1^2 - a^2}{(x_1^2 + a^2)^2}\right). \tag{3.94}$$

Proof. As in [43], this result follows from the implicit function theorem applied to $\mathscr{F}$ at the trivial solutions $(0; 0, a)$ for $a \in (0, \infty)$. We compute

$$D_X\mathscr{F}(0; 0, a) = \begin{pmatrix} g - \partial_{x_1}^2 & 0 & 0 \\ 0 & |\partial_{x_1}| & 0 \\ 0 & -(\mathcal{H}(0)\cdot)_{x_1}(0, -a) & 1 \end{pmatrix},$$

which is clearly an isomorphism $X \to Y$, as it is a lower diagonal matrix with isomorphisms on the diagonal. Finally, the asymptotic expansions listed in (3.93) can be found by implicit differentiation. $\square$

Interestingly, it is possible to write the leading order surface term η_2 in terms of the so-called exponential integral E_1.

Theorem 3.45. *Define the holomorphic function $f\colon \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ by*

$$f(z) := -e^z E_1(z) = (\gamma + \log(z))e^z - \sum_{k=1}^{\infty} \frac{H_k}{k!} z^k,$$

where γ is the Euler–Mascheroni constant and H_k is the k-th harmonic number. If we write $w = x + i\alpha = \sqrt{g/b}(x_1 + ia)$, then

$$\eta_2(x_1) = \frac{1}{4\pi^2 b}\tilde{\eta}_2(x), \qquad \tilde{\eta}_2(x) := \operatorname{Re}\left(\frac{f(w) + f(-w)}{2}\right).$$

More explicitly,

$$\begin{aligned}\eta_2(x) = {} & \left[\left(\gamma + \log\left(\sqrt{x^2 + \alpha^2}\right)\right)\cos(\alpha) - \frac{\pi}{2}\sin(\alpha)\right]\cosh(x) \\ & + \sin(\alpha)\arctan(x/\alpha)\sinh(x) - \sum_{k=1}^{\infty} \frac{H_{2k}}{(2k)!}(x^2 + \alpha^2)^k T_{2k}\left(\frac{x}{\sqrt{x^2 + \alpha^2}}\right)\end{aligned}$$

for all $x \in \mathbb{R}$, with T_k being the k-th Chebyshev polynomial.

Proof. By using (3.94) and the scaling, we see that $\tilde{\eta}_2$ solves the differential equation

$$\tilde{\eta}_2(x) - \tilde{\eta}_2''(x) = \frac{x^2 - \alpha^2}{(x^2 + \alpha^2)^2} = \operatorname{Re} w^{-2},$$

and one may directly verify that $g(z) := (f(z) + f(-z))/2$ satisfies $g(z) - g''(z) = z^{-2}$ in $\mathbb{C} \setminus \mathbb{R}$. Moreover, this is the unique solution that vanishes at infinity, as it can be shown using a well-known asymptotic series for E_1 that $g(z) = 1/z^2 + O(z^{-4})$ as $|z| \to \infty$. $\square$

C Derivatives of the energy and momentum

We record here the derivatives of E and P up to order two. Fix $u = (\eta, \varphi, \bar{x}) \in \mathbb{V} \cap \mathcal{O}$, and let $\dot{u} = (\dot{\eta}, \dot{\varphi}, \dot{\bar{x}}) \in \mathbb{V}$ represent a variation. Some of the integrals must be understood in the dual-pairing sense. To simplify the notation, we also introduce

$$\mathfrak{a} := (\nabla\varphi_{\mathcal{H}})|_S, \qquad \xi := (\Theta_{x_1}, \Xi_{x_2}) = -\nabla_{\bar{x}}\Theta,$$

and note that

$$D_{\bar{x}}^2\Theta = \begin{pmatrix} \Theta_{x_1x_1} & \Xi_{x_1x_2} \\ \Xi_{x_1x_2} & \Theta_{x_2x_2} \end{pmatrix}.$$

Variations of K_0

From (3.56) we compute

$$\begin{aligned} \langle D_\varphi K_0(u), \dot\varphi\rangle &= \int_{\mathbb{R}} \dot\varphi G(\eta)\varphi\, dx_1, \\ \langle D_\eta K_0(u), \dot\eta\rangle &= \frac{1}{2}\int_{\mathbb{R}} \varphi\langle D_\eta G(\eta)\dot\eta, \varphi\rangle\, dx_1 = \int_{\mathbb{R}} \dot\eta\left(\frac{1}{2}|\mathfrak{a}|^2 - \mathfrak{a}_2 G(\eta)\varphi\right) dx_1, \end{aligned}$$

with second variations

$$\begin{aligned} \langle D_\varphi^2 K_0(u)\dot\varphi, \dot\varphi\rangle &= \int_{\mathbb{R}} \dot\varphi G(\eta)\dot\varphi\, dx_1, \\ \langle D_\varphi D_\eta K_0(u)\dot\varphi, \dot\eta\rangle &= \int_{\mathbb{R}} \dot\varphi\langle D_\eta G(\eta)\dot\eta, \varphi\rangle\, dx_1 = \int_{\mathbb{R}} \dot\eta(\mathfrak{a}_1\dot\varphi' - \mathfrak{a}_2 G(\eta)\dot\varphi)\, dx_1, \\ \langle D_\eta^2 K_0(u)\dot\eta, \dot\eta\rangle &= \frac{1}{2}\int_{\mathbb{R}} \varphi\langle\langle D_\eta^2 G(\eta)\dot\eta, \dot\eta\rangle, \varphi\rangle\, dx_1 \\ &= \int_{\mathbb{R}} (\mathfrak{a}_1'\mathfrak{a}_2\dot\eta^2 + \mathfrak{a}_2\dot\eta G(\eta)(\mathfrak{a}_2\dot\eta))\, dx_1, \end{aligned}$$

where the explicit expressions for the shape-derivatives only hold when $\varphi \in \mathbb{X}_2^{3/2}$.

Variations of K_1

From (3.56) we find quickly that

$$\langle D_\eta K_1(u), \dot\eta\rangle = \int_{\mathbb{R}} \dot\eta\varphi'\Theta_{x_1}|_S\, dx_1, \quad \langle D_\varphi K_1(u), \dot\varphi\rangle = \int_{\mathbb{R}} \dot\varphi\nabla_\perp\Theta\, dx_1,$$

and

$$\nabla_{\bar{x}} K_1(u) = -\int_{\mathbb{R}} \varphi\nabla_\perp\xi\, dx_1.$$

The second variations are thus

$$\begin{aligned}
\langle D_\eta^2 K_1(u)\dot\eta, \dot\eta\rangle &= \int_{\mathbb{R}} \dot\eta^2 \varphi' \Theta_{x_1 x_2}|_S \, dx_1, \\
\langle D_\eta D_\varphi K_1(u)\dot\eta, \dot\varphi\rangle &= \int_{\mathbb{R}} \dot\eta \dot\varphi' \Theta_{x_1}|_S \, dx_1, \\
\nabla_{\bar x}\langle D_\eta K_1(u), \dot\eta\rangle &= -\int_{\mathbb{R}} \dot\eta \varphi' \xi_{x_1}|_S \, dx_1, \\
\nabla_{\bar x}\langle D_\varphi K_1(u), \dot\varphi\rangle &= -\int_{\mathbb{R}} \dot\varphi \nabla_\perp \xi \, dx_1,
\end{aligned}$$

and

$$D_{\bar x}^2 K_1(u) = \int_{\mathbb{R}} \varphi \nabla_\perp D_{\bar x}^2 \Theta \, dx_1.$$

Note that in the above computations we have made repeated use of the fact that Θ is harmonic in a neighborhood of the surface S. In particular, this implies that

$$\nabla_\perp \Theta_{x_1} = \nabla_\top \Theta_{x_2} = (\Theta_{x_2}|_S)', \quad \nabla_\perp \Theta_{x_2} = -\nabla_\top \Theta_{x_1} = -(\Theta_{x_1}|_S)', \tag{3.95}$$

with similar identities holding for Ξ as well.

Variations of K_2

From (3.56) we find

$$\begin{aligned}
\langle D_\eta K_2(u), \dot\eta\rangle &= \frac{1}{2}\int_{\mathbb{R}} \dot\eta (|\nabla\Theta|^2)|_S \, dx_1, \\
\nabla_{\bar x} K_2(u) &= \nabla\Gamma_2(\bar x) - \frac{1}{2}\int_{\mathbb{R}} \nabla_\perp(\Theta\xi) \, dx_1.
\end{aligned}$$

The second variations are thus

$$\begin{aligned}
\langle D_\eta^2 K_2(u)\dot\eta, \dot\eta\rangle &= \int_{\mathbb{R}} (\nabla\Theta \cdot \nabla\Theta_{x_2})|_S \dot\eta^2 \, dx_1, \\
\nabla_{\bar x}\langle D_\eta K_2(u), \dot\eta\rangle &= -\int_{\mathbb{R}} \dot\eta ((D_x\xi)\nabla\Theta)|_S \, dx_1, \\
D_{\bar x}^2 K_2(u) &= 2D_x^2\Gamma_2(\bar x) + \frac{1}{2}\int_{\mathbb{R}} \nabla_\perp(\Theta D_{\bar x}^2\Theta + \xi\xi^T) \, dx_1.
\end{aligned}$$

Variations of V

From (3.57) we have

$$\begin{aligned}
\langle D_\eta V(u), \dot\eta\rangle &= \int_{\mathbb{R}} \left(g\eta - b\left(\frac{\eta'}{\langle \eta'\rangle}\right)'\right)\dot\eta\, dx_1,\\
\langle D_\eta^2 V(u)\dot\eta, \dot\eta\rangle &= \int_{\mathbb{R}} \left(g\dot\eta^2 + b\frac{1}{\langle \eta'\rangle^3}(\dot\eta')^2\right) dx_1.
\end{aligned}$$

Variations of P

Lastly we consider the momentum. The first variations are given by

$$\langle D_\eta P(u), \dot\eta\rangle = \int_{\mathbb{R}} \dot\eta(\varphi' + \epsilon\Theta_{x_1}|_S)\, dx_1, \qquad \langle D_\varphi P(u), \dot\varphi\rangle = -\int_{\mathbb{R}} \eta'\dot\varphi\, dx_1,$$

and

$$\nabla_{\bar x} P(u) = \epsilon e_2 + \epsilon \int_{\mathbb{R}} \eta'\xi|_S\, dx_1.$$

Likewise, we find that the second variations are

$$\begin{aligned}
\langle D_\eta^2 P(u)\dot\eta, \dot\eta\rangle &= \epsilon \int_{\mathbb{R}} \dot\eta^2 \Theta_{x_1x_2}|_S\, dx_1,\\
\langle D_\eta D_\varphi P(u)\dot\eta, \dot\varphi\rangle &= -\int_{\mathbb{R}} \dot\eta'\dot\varphi\, dx_1,\\
\nabla_{\bar x}\langle D_\eta P(u), \dot\eta\rangle &= -\epsilon \int_{\mathbb{R}} \dot\eta\xi_{x_1}|_S\, dx_1,\\
D_{\bar x}^2 P(u) &= -\epsilon \int_{\mathbb{R}} \eta'(D_{\bar x}^2\Theta)|_S\, dx_1.
\end{aligned}$$

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PAPER 4

GLOBAL BIFURCATION OF WAVES WITH MULTIPLE CRITICAL LAYERS

In preparation for submission

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Abstract. Analytic global bifurcation theory is used to construct a large variety of families of steady periodic two-dimensional gravity water waves with real-analytic vorticity distributions, propagating in an incompressible fluid. The waves that are constructed can possess an arbitrary number of interior stagnation points in the fluid, and corresponding critical layers consisting of closed streamlines. This is made possible by the use of the so-called naive flattening transform, which has previously only been used for local bifurcation.

1 Introduction

In this paper, our concern shall be two-dimensional traveling water waves, propagating in an inviscid and incompressible fluid of finite depth atop a flat bed. The waves will be purely gravitational — that is, we neglect the influence of surface tension. Moreover, we make the assumption that the waves have not overturned, meaning that the free surface can be described as the graph of a function, which we shall call $\eta\colon \mathbb{R} \to \mathbb{R}$ in the sequel.

The steady-frame fluid domain, stationary with respect to the wave, will be denoted by

$$\Omega_\eta \coloneqq \Big\{(x, y) \in \mathbb{R}^2 : 0 < y < d + \eta(x)\Big\},$$

where $d > 0$ represents the unperturbed fluid depth; with x marking the horizontal direction, and y the vertical direction. Furthermore, we will write

$$S_\eta \coloneqq \{(x, d + \eta(x)) : x \in \mathbb{R}\}$$

for the free surface, and

$$B \coloneqq \{(x, 0) : x \in \mathbb{R}\}$$

to signify the flat bed. These components of $\partial\Omega_\eta$ are assumed to be positively separated, with S_η above B.

The waves are required to satisfy the steady incompressible Euler equations

$$
\begin{aligned}
\nabla \cdot [(u - ce_x) \otimes (u - ce_x)] + \nabla(p + gy) = 0,\\
\nabla \cdot u = 0,
\end{aligned}
\tag{4.1a}
$$

in Ω_η, where $u\colon \Omega_\eta \to \mathbb{R}^2$ is the velocity field, and $p\colon \Omega_\eta \to \mathbb{R}$ is the pressure. The constant $c > 0$ is the wave speed, and $g > 0$ is known as the acceleration due to gravity, while $e_x := (1, 0)$ is the horizontal unit vector. We may interpret the individual equations in (4.1a) as representing conservation of momentum and mass, respectively.

To finish the description of the governing equations for steady water waves, we also require boundary conditions: First, we have the *kinematic* boundary conditions, which read

$$
\eta^\perp \cdot (u - ce_x) = 0 \qquad \text{on } S_\eta, \tag{4.1b}
$$

and

$$
e_y \cdot u = 0 \qquad \text{on } B, \tag{4.1c}
$$

naturally "attaching" the boundary of the fluid domain to the velocity field. Here, $\eta^\perp := (-\partial_x \eta, 1)$ yields the non-normalized normal vector on S_η in terms of the surface profile η.

The final boundary condition is the *dynamic* boundary condition

$$
p = 0 \qquad \text{on } S_\eta, \tag{4.1d}
$$

ensuring that the pressure is continuous across the interface. This is where surface tension would have entered, had we not neglected it. Collectively, (4.1a)–(4.1d) is known as the *steady water-wave problem.*

Of particular interest to us are *rotational* steady waves, for which the scalar circulation density

$$
\omega := \nabla^\perp \cdot u, \qquad \text{where } \nabla^\perp := (-\partial_y, \partial_x),
$$

does not vanish identically. This quantity is known as the *vorticity* of the fluid. Conveniently, if a *stream function* $\psi\colon \Omega_\eta \to \mathbb{R}$ is introduced through $\nabla^\perp \psi := u - ce_x$, then the identity

$$
\omega = \nabla^\perp \cdot \nabla^\perp \psi = \Delta \psi
$$

holds throughout Ω_η.

We can go further than this still: If we have a smooth solution of the steady water-wave problem (4.1), and $c - u \cdot e_x = \partial_y \psi > 0$ in Ω_η, then there exists a *vorticity distribution* $\gamma\colon \mathbb{R} \to \mathbb{R}$ such that

$$\Delta\psi + \gamma(\psi) = 0 \qquad \text{in } \Omega_\eta, \tag{4.2a}$$

see e.g. [4, Section 2]. Of course, this merely constitutes a sufficient condition for γ to exist. There is nothing preventing us from postulating the existence of a vorticity distribution, even when the hypothesis above fails in the presence of interior *stagnation points* where $u - ce_x = \nabla^\perp\psi = 0$. Having a vorticity distribution is highly convenient mathematically.

By employing elementary vector-calculus identities, we find that

$$\nabla \cdot \Big(\nabla^\perp\psi \otimes \nabla^\perp\psi\Big) = \nabla\Big(\frac{1}{2}|\nabla\psi|^2 + \Gamma(\psi)\Big), \qquad \Gamma(t) := \int_0^t \gamma(s)\,ds,$$

for solutions of (4.2a). Combining this identity with (4.1a) and (4.1d), we are led to the free-surface *Bernoulli equation*

$$\frac{1}{2}|\nabla\psi|^2 + g\eta = Q \qquad \text{on } S_\eta, \tag{4.2b}$$

for some constant $Q > -gd$. Together with (4.2a) and the demand that

$$\psi = \mu \qquad \text{on } S_\eta, \tag{4.2c}$$

$$\psi = \Upsilon \qquad \text{on } B, \tag{4.2d}$$

for two constants $\mu, \Upsilon \in \mathbb{R}$, which is the form that the kinematic boundary conditions (4.1b) and (4.1c) take for stream functions, these equations give rise to solutions of the steady water-wave problem in (4.1).

If we appeal to the integral

$$\Upsilon - \mu = \int_0^{d+\eta(x)} (u(x,y)\cdot e_x - c)\,dy,$$

which is independent of the choice of $x \in \mathbb{R}$, we see that the difference between the constants in (4.2c) and (4.2d) may be interpreted as a relative mass flux.

Previous work

The mathematical study of solutions to variants of the steady water-wave problem (4.1) has a rich and extensive history — going back hundreds of years. Most of the earlier literature concerned irrotational waves, where the reader may find surveys such as [16, 27] of interest. Comparatively, the study of *rotational* steady waves specifically is much more recent, especially when they are allowed to stagnate. Rotation is of course ubiquitous in nature, essentially being induced whenever there are non-conservative forces at play, but also introduces new non-trivial mathematical challenges.

One could argue that the first rotational water-wave result was the explicit infinite-depth Gerstner wave [14]; see also [3] for a modern treatment, including recent developments in the field of nonlinear water waves more generally. However, a much more compelling case can be made for the thesis [9]. There, Dubreil-Jacotin introduced the semi-hodograph transform for (4.2), treating the stream function as the vertical variable. They subsequently used the transform in an existence theorem for small-amplitude periodic solutions, and it has since seen wide use and become a staple tool in the field. In particular, we must single out its use in the seminal paper [4], which was the first large-amplitude result in the same setting.

One significant downside of the semi-hodograph transform is that it precludes the presence of interior stagnation points, or their corresponding critical layers of closed streamlines, in the fluid. The reason for this is that ψ_y must necessarily have a definite sign in order to enable the use of ψ as a vertical variable. Therefore, if stagnation is a desired feature, a different way of dealing with the free boundary must be utilized.

An early existence result for stagnant waves was [12], furnishing *linear* stagnant waves with constant vorticity. This paper would lead to the first nonlinear existence result in [31], where Wahlén constructed small-amplitude waves with one critical layer for constant γ. Instead of the semi-hodograph transform, they used the same naive flattening transform that we shall soon introduce (see (4.5)). In the decade following this paper, there has been a flurry of activity concerning waves with stagnation points: In [6] a different approach than [31] was used, restating the problem as a pseudodifferential equation using conformal mapping. While highly specialized for constant vorticity, the framework is elegant and potentially allows for overhanging waves. The authors of [6] would later go on to further develop this framework with Strauss in [5], establishing the existence of *large* amplitude waves. We should also mention here that there is another global result in the presence of capillary effects [23].

At the same time, there has been a parallel endeavor of considering more “interesting” vorticity distributions, expanding on the use of the flattening transform from [31]. Even taking the step up to affine vorticity distributions [1, 10, 11, 13] admits waves with an arbitrary number of critical layers. These works also provide results for bimodal [1, 11], or even trimodal [13] waves. Small-amplitude solutions for very general vorticity distributions γ were examined in [19]. See also [20] for a recent result involving n-modal waves for "almost"-affine vorticity distributions, giving a partial answer to a question posed in [13].

In this paper, our goal is to, in a sense, unite these two efforts: We show that the framework of [11, 31] can be extended in such a way that it can be used for large-amplitude waves as well.

There are, of course, several papers on stagnant waves that do not fit neatly into our categorizations above. For instance, a recent paper [21] constructs small-amplitude *non-symmetric* solitary waves with critical layers using a spatial-dynamics approach, as opposed to the bifurcation-theoretic nature of the above results. There is also a preprint [18] on solitary waves with constant vorticity and a critical layer connecting with the bed, again using spatial dynamics. Finally, there are several papers on solitary capillary-gravity waves with compactly supported vorticity [25, 28], including two preprints dealing with the stability of such waves [22, 29]. These waves are stagnant, but we remark that the waves with immersed point vortices do not possess vorticity distributions in the traditional sense.

Plan for this article

In Section 2, we formulate the problem, and describe its linearization. Of noteworthy importance is the key generalization of the so-called $\mathcal{T}$-isomorphism from [11] to non-trivial solutions. Section 3 is used to study the kernel of the linearization, in particular resulting in Theorem 4.9. This section is also used to state our local bifurcation result, Theorem 4.17. Finally, Section 4 extends the local curves to *global* ones, which is the central event of this paper. Our main result here is Theorem 4.18, which concerns the global solution curves obtained by applying analytic global bifurcation theory. An interesting feature of the proof is using an alternative near-surface flattening to prove the necessary compactness.

2 Formulation

After a convenient choice of scaling, we may set the unperturbed depth and gravitational acceleration to $d = g \equiv 1$, switching out (4.2b) for

$$\frac{1}{2}|\nabla\psi|^2 + \eta = Q \qquad \text{on } S_\eta, \tag{4.2b$\star$}$$

instead. At this point, we make a regularity assumption, the first half of which is necessary for us to be able to apply *analytic* global bifurcation theory later.

Assumption 4.1 (Regularity of γ)*.* The vorticity distribution $\gamma\colon \mathbb{R} \to \mathbb{R}$ is real analytic, with bounded derivative.

The trivial solutions of (4.2) are those corresponding to parallel flows beneath a flat surface, and in particular those for which $\eta \equiv 0$. We *define* the trivial stream function $\overline{\psi} = \overline{\psi}(\Lambda)$ to be the unique solution — the existence of which is ensured by Assumption 4.1 — of the initial value problem

$$\begin{aligned} \overline{\psi}'' + \gamma(\overline{\psi}) &= 0, \quad \text{for } y \in (0,1) \\ \overline{\psi}(1) = \mu, &\qquad \overline{\psi}'(1) = \lambda, \end{aligned} \tag{4.3}$$

for $\Lambda := (\mu, \lambda)$ in the set

$$\mathcal{U} := \{\underbrace{(\mu,\lambda)}_{\Lambda} \in \mathbb{R}^2 : \lambda \neq 0\}$$

of permissible parameters. Here, the restriction on λ ensures that there is no technically problematic surface stagnation present at the trivial solution. The corresponding values of Q and Υ are determined from (4.2b$\star$) and (4.2d), namely

$$\begin{aligned} Q(\Lambda) &= \frac{1}{2}\lambda^2, \\ \Upsilon(\Lambda) &= \overline{\psi}(0;\Lambda). \end{aligned} \tag{4.4}$$

We will often leave out dependence on Λ from notation for readability, especially for $\overline{\psi}$.

Remark 4.2*.* We mention that the solutions of Equation (4.3) can be written down explicitly only for very special choices of γ, such as when the vorticity distribution is either constant or affine.

By flattening the fluid domain through what we shall call the *naive flattening transform* $\Pi\colon \Omega_\eta \to \Omega_0$, defined by

$$\Pi(x,y) = \left(x, \frac{y}{1+\eta(x)}\right), \tag{4.5}$$

the water wave problem in (4.2) becomes

$$\begin{aligned}
\left(\partial_x - \frac{s\eta_x}{1+\eta}\partial_s\right)^2 \hat{\psi} + \frac{1}{(1+\eta)^2}\hat{\psi}_{ss} + \gamma(\hat{\psi}) &= 0 \quad \text{in } \Omega_0,\\
\frac{1+\eta_x^2}{2(1+\eta)^2}\hat{\psi}_s^2 + \eta &= Q \quad \text{on } S_0,\\
\hat{\psi} &= \mu \quad \text{on } S_0,\\
\hat{\psi} &= \Upsilon \quad \text{on } B,
\end{aligned} \tag{4.6}$$

where s is used to distinguish the vertical variable in the flattened domain. For notational simplicity, we will henceforth use $\Omega \coloneqq \Omega_0$ and $S \coloneqq S_0$.

Write now

$$\hat{\psi} = \hat{\psi}(\hat{\varphi}, \Lambda) \coloneqq \overline{\psi}(\Lambda) + \hat{\varphi}, \tag{4.7}$$

where $\hat{\varphi}$ is a disturbance from $\overline{\psi}$ that vanishes at both the bottom and surface. The trivial solution in this definition takes care of the Dirichlet boundary conditions in (4.6). We will typically use the notation $w = (\eta, \hat{\varphi})$, for the pairs living in the space

$$X = X_1 \times \hat{X}_2 \coloneqq C^{2,\beta}_{\kappa,\mathrm{e}}(\mathbb{R}) \times \{\hat{\varphi} \in C^{2,\beta}_{\kappa,\mathrm{e}}(\overline{\Omega}) : \hat{\varphi}|_S = \hat{\varphi}|_B = 0\}, \tag{4.8}$$

for some fixed Hölder exponent $\beta \in (0,1)$. Here, the subscripts denote $2\pi/\kappa$-periodicity and evenness in the horizontal direction, respectively.

If we further define the open subset

$$\mathcal{O} \coloneqq \{(w,\Lambda) \in X \times \mathcal{U} : 1+\eta > 0, \operatorname{sgn}(\lambda)\hat{\psi}_s|_S > 0\} \tag{4.9}$$

of $X \times \mathbb{R}^2$, we may define the *analytic* map $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2) \colon \mathcal{O} \to Y$ by

$$\begin{aligned}
\mathcal{F}_1(w,\Lambda) &\coloneqq \frac{1+\eta_x^2}{2(1+\eta)^2}(\hat{\psi}_s|_S)^2 + \eta - \frac{1}{2}\lambda^2,\\
\mathcal{F}_2(w,\Lambda) &\coloneqq \left(\partial_x - \frac{s\eta_x}{1+\eta}\partial_s\right)^2 \hat{\psi} + \frac{1}{(1+\eta)^2}\hat{\psi}_{ss} + \gamma(\hat{\psi}),
\end{aligned} \tag{4.10}$$

where the codomain of $\mathcal{F}$ is the space

$$Y = Y_1 \times Y_2 \coloneqq C^{1,\beta}_{\kappa,\mathrm{e}}(\mathbb{R}) \times C^{\beta}_{\kappa,\mathrm{e}}(\overline{\Omega}).$$

Note in particular that $\Lambda \mapsto \overline{\psi}(\Lambda)$ defines an analytic map from $\mathbb{R}^2$ into the space $C^{2,\beta}([0,1]) =: V$, by a simple argument involving the implicit function theorem applied to the map $F \colon V \times \mathbb{R}^2 \to V$ defined through

$$F(\zeta,\Lambda)(s) \coloneqq \zeta(s) - \mu - \lambda(s-1) + \int_1^s \int_1^t \gamma(\zeta(r))\,dr\,dt.$$

The idea of the definition in (4.10) is that it combines (4.6) with (4.7), and the value of of Q from (4.4). Thus $(0, \Lambda)$ is a solution of the equation

$$\mathcal{F}(w, \Lambda) = 0 \tag{4.11}$$

in $\mathcal{O}$ for every $\Lambda \in \mathcal{U}$, by construction. Moreover, these are the only solutions with the flat surface $\eta \equiv 0$.

Remark 4.3. The last condition for membership in $\mathcal{O}$ ensures that there is no stagnation on the surface, and that

$$\operatorname{sgn}(\hat{\psi}_s|_S) = \operatorname{sgn}(\lambda),$$

whereupon the line segment between $(0, \Lambda)$ and (w, Λ) is always contained in $\mathcal{O}$ for any $(w, \Lambda) \in \mathcal{O}$. A further implication is that the slice

$$\mathcal{O}_\lambda := \{(w, \mu) \in X \times \mathbb{R} : (w, \Lambda) \in \mathcal{O}\}$$

is connected for any fixed $\lambda \neq 0$.

Our objective from here on is to further investigate the solution set of (4.11) in $\mathcal{O}$. It turns out that solutions of (4.11) are more regular than generic elements of $\mathcal{O}$, as a consequence of the following theorem:

Theorem 4.4 (Analyticity of solutions)**.** *Suppose that* $(w, \Lambda) \in \mathcal{O}$ *is a solution of* (4.11) *under Assumption 4.1. Then:*

(i) The surface profile η *is analytic.*

(ii) The stream function $\hat{\psi}$ *extends to an analytic function on an open set containing* $\overline{\Omega}$.

Proof. The proof is essentially the same as the one for [1, Theorem 2.5], but using *nonlinear* elliptic regularity theory instead of linear theory; see for instance [24]. □

Linearization around a solution

In preparation for both local and global bifurcation, we require the linearization of (4.11) around its solutions. It is straightforward to compute the partial derivatives

$$D_\eta \mathcal{F}_1(w, \Lambda) H = \left(1 - \frac{1 + \eta_x^2}{(1+\eta)^3} \hat{\psi}_s^2\right) H + \frac{\eta_x \hat{\psi}_s^2}{(1+\eta)^2} H_x$$

$$D_{\hat{\varphi}} \mathcal{F}_1(w, \Lambda) \hat{\Phi} = \frac{1 + \eta_x^2}{(1+\eta)^2} \hat{\psi}_s \hat{\Phi}_s$$

for $\mathcal{F}_1$, where restrictions to S are implied, and

$$
\begin{aligned}
D_\eta \mathcal{F}_2(w,\Lambda)H &= \left(\frac{s\eta_{xx}\hat{\psi}_s}{(1+\eta)^2} - \frac{4s\eta_x^2\hat{\psi}_s}{(1+\eta)^3} + \frac{2s\eta_x\hat{\psi}_{xs}}{(1+\eta)^2} - 2\frac{1+s^2\eta_x^2}{(1+\eta)^3}\hat{\psi}_{ss} \right) H \\
&\quad + \left(\frac{4s\eta_x\hat{\psi}_s}{(1+\eta)^2} - \frac{2s\hat{\psi}_{xs}}{1+\eta} + \frac{2s^2\eta_x\hat{\psi}_{ss}}{(1+\eta)^2} \right) H_x - \frac{s\hat{\psi}_s}{1+\eta} H_{xx} \\
D_{\hat{\varphi}} \mathcal{F}_2(w,\Lambda)\hat{\Phi} &= \left(\partial_x - \frac{s\eta_x}{1+\eta}\partial_s \right)^2 \hat{\Phi} + \frac{1}{(1+\eta)^2}\hat{\Phi}_{ss} + \gamma'(\hat{\psi})\hat{\Phi}
\end{aligned}
$$

for $\mathcal{F}_2$. These expressions are of course valid for any $(w,\Lambda) \in \mathcal{O}$, regardless of whether this pair is a solution of (4.11), but appear quite formidable. It turns out that *if* (w,Λ) is a solution of (4.11), then $D_w\mathcal{F}(w,\Lambda)$ can be transformed into an operator that is easier to study.

To that end, let us introduce the space

$$
X_2 \coloneqq \{\Phi \in C^{2,\beta}_{\kappa,\mathrm{e}}(\overline{\Omega}) : \Phi|_B = 0\},
$$

which differs from $\hat{X}_2$, see (4.8), only through the relaxation of the Dirichlet condition on S. The purpose of introducing this space is to "encode" both H (i.e. capital η) and $\hat{\Phi}$ in a single variable.

If $(w,\Lambda) \in \mathcal{O}$ is such that $\eta \in C^{3,\beta}_{\kappa,\mathrm{e}}(\mathbb{R})$ and $\hat{\varphi} \in C^{3,\beta}_{\kappa,\mathrm{e}}(\overline{\Omega})$, we may define a — soon to be motivated — bounded linear operator $\mathcal{L}(w,\Lambda) \in \mathrm{Lin}(X_2, Y)$ by

$$
\begin{aligned}
\mathcal{L}_1(w,\Lambda)\Phi &\coloneqq \frac{1+\eta_x^2}{(1+\eta)^2}\hat{\psi}_s\Phi_s + \left(\gamma(\mu) - \frac{1+\eta}{\hat{\psi}_s}\right)\Phi - \left(\frac{\eta_x\hat{\psi}_s}{1+\eta}\Phi\right)_x \\
\mathcal{L}_2(w,\Lambda)\Phi &\coloneqq D_{\hat{\varphi}}\mathcal{F}_2(w,\Lambda)\Phi,
\end{aligned} \tag{4.12}
$$

with the functions in the definition of $\mathcal{L}_1(w,\Lambda)$ evaluated on S. Furthermore, for $\mathcal{L}_2(w,\Lambda)$, we interpret $D_{\hat{\varphi}}\mathcal{F}_2(w,\Lambda)$ as extended to $X_2 \supset \hat{X}_2$ in the natural way. For convenience, we note that the operator defined in (4.12) simplifies to

$$
\begin{aligned}
\mathcal{L}_1(\Lambda) &\coloneqq \mathcal{L}_1(0,\Lambda)\Phi = \lambda\Phi_s + \left(\gamma(\mu) - \frac{1}{\lambda}\right)\Phi \\
\mathcal{L}_2(\Lambda) &\coloneqq \mathcal{L}_2(0,\Lambda)\Phi = (\Delta + \gamma'(\overline{\psi}))\Phi
\end{aligned} \tag{4.13}
$$

at the trivial solutions.

In particular, we have the required increased regularity for $\mathcal{L}(w,\Lambda)$ to be well defined when (w,Λ) is a solution of (4.11), due to Theorem 4.4. Moreover, in this case we can relate $D_w\mathcal{F}(w,\Lambda)$ to the operator $\mathcal{L}(w,\Lambda)$, which is the reason for its introduction. This is done by utilizing a suitable generalization of the so-called $\mathcal{T}$-isomorphism from [11].

Theorem 4.5 ($\mathcal{T}$-isomorphism)**.** *Suppose that* $(w,\Lambda)\in\mathcal{O}$ *satisfies* (4.11)*. Then*

$$\mathcal{T}(w,\Lambda)\Phi := \left(-\frac{1+\eta}{\hat{\psi}_s|_S}\Phi|_S, \Phi - \frac{s\hat{\psi}_s}{\hat{\psi}_s|_S}\Phi|_S\right). \tag{4.14}$$

defines an isomorphism $\mathcal{T}(w,\Lambda)\colon X_2\to X$*, and*

$$\mathcal{L}(w,\Lambda) = D_w\mathcal{F}(w,\Lambda)\mathcal{T}(w,\Lambda). \tag{4.15}$$

Proof. Inspired by the procedure that was presumably used to arrive at the $\mathcal{T}$-isomorphism for trivial solutions in [11], we suppose the existence of some element $\hat{f}\in X_2$ which enjoys the property

$$D_{\hat{\varphi}}\mathcal{F}_2(w,\Lambda)(\hat{f}H) = -D_\eta\mathcal{F}_2(w,\Lambda)H \tag{4.16}$$

for all $H\in C^{2,\beta}_{\kappa,\mathrm{e}}(\mathbb{R})$, and which furthermore does not vanish at any point on the surface. Again, we view the derivative $D_{\hat{\varphi}}\mathcal{F}_2(w,\Lambda)$ as naturally extended to an operator on $X_2\supset\hat{X}_2$. Let $g:=\hat{f}|_S$, which by supposition has a definite sign. Then the operator $\mathcal{T}(w,\Lambda)\colon X_2\to X$ defined by

$$\mathcal{T}(w,\Lambda)\Phi := \left(-\frac{\Phi|_S}{g}, \Phi - \hat{f}\frac{\Phi|_S}{g}\right) \tag{4.17}$$

is easily seen to be an isomorphism, and yields

$$\begin{aligned} D_w\mathcal{F}_2(w,\Lambda)\mathcal{T}(w,\Lambda)\Phi &= -D_\eta\mathcal{F}_2(w,\Lambda)\left(\frac{\Phi|_S}{g}\right) + D_{\hat{\varphi}}\mathcal{F}_2(w,\Lambda)\left(\Phi - \hat{f}\frac{\Phi|_S}{g}\right)\\ &= D_{\hat{\varphi}}\mathcal{F}_2(w,\Lambda)\left(\hat{f}\frac{\Phi|_S}{g}\right) + D_{\hat{\varphi}}\mathcal{F}_2(w,\Lambda)\left(\Phi - \hat{f}\frac{\Phi|_S}{g}\right)\\ &= D_{\hat{\varphi}}\mathcal{F}_2(w,\Lambda)\Phi, \end{aligned}$$

whence the second component of (4.15) is satisfied.

We have shown that the property in (4.16) is key to establishing the theorem, and this equation turns out to be simpler to consider on the unflattened Ω_η instead. Define therefore the pullbacks $\psi=\hat{\psi}\circ\Pi$ and $f=\hat{f}\circ\Pi$, where we recall that Π is the flattening transform from (4.5). Then the left-hand side of (4.16) becomes

$$(\Delta+\gamma'(\psi))(fH) = (\Delta+\gamma'(\psi))fH + 2f_xH_x + fH_{xx}, \tag{4.18}$$

while the right-hand side turns into

$$\begin{aligned} &\left(\frac{2\eta_x^2 y\psi_y}{(1+\eta)^3} + \frac{2\psi_{yy}}{1+\eta} - \frac{y\eta_{xx}\psi_y}{(1+\eta)^2} - \frac{2\eta_x y\psi_{xy}}{(1+\eta)^2}\right)H\\ &\qquad + 2\left(\frac{y\psi_y}{1+\eta}\right)_x H_x + \frac{y\psi_y}{1+\eta}H_{xx}, \end{aligned} \tag{4.19}$$

after a lengthy computation. By comparing (4.18) and (4.19), we see that the only possible solution candidate of (4.16) is

$$f = \frac{y\psi_y}{1+\eta},$$

and that we need only verify that the coefficients in front of H in (4.18) and (4.19) are equal. Note that f has the correct regularity for $\mathcal{T}$ to be well-defined, because ψ and η are analytic by Theorem 4.4. Additionally, ψ_y does not vanish on the surface, because of the postulation that $(w,\Lambda) \in \mathcal{O}$.

One may now check by direct calculation that

$$\begin{aligned}(\Delta + \gamma'(\psi))\left(\frac{y\psi_y}{1+\eta}\right) = \frac{2\eta_x^2 y\psi_y}{(1+\eta)^3} + \frac{2\psi_{yy}}{1+\eta} - \frac{y\eta_{xx}\psi_y}{(1+\eta)^2} - \frac{2\eta_x y\psi_{xy}}{(1+\eta)^2}\\ + \frac{y}{1+\eta}(\Delta\psi + \gamma(\psi))_y,\end{aligned}$$

where the first terms are precisely the ones in front of H in (4.19), while the last term vanishes because ψ solves (4.2a). Hence (4.16) holds for

$$\hat{f} = f \circ \Pi^{-1} = \frac{s\hat{\psi}_s}{1+\eta},$$

for which (4.17) becomes (4.14). Finally, direct computation yields

$$\begin{aligned}D_w\mathcal{F}_1(w,\Lambda)\mathcal{T}(w,\Lambda) = \frac{1+\eta_x^2}{(1+\eta)^2}\hat{\psi}_s\Phi_s - \frac{\eta_x\hat{\psi}_s}{1+\eta}\Phi_x\\ - \left(\frac{1+\eta}{\hat{\psi}_s} + \frac{1+\eta_x^2}{(1+\eta)^2}\hat{\psi}_{ss} + \frac{\eta_x^2\hat{\psi}_s}{(1+\eta)^2} - \frac{\eta_x\hat{\psi}_{xs}}{1+\eta}\right)\Phi,\end{aligned}$$

where

$$\frac{1+\eta_x^2}{(1+\eta)^2}\hat{\psi}_{ss} + \frac{\eta_x^2}{(1+\eta)^2}\hat{\psi} - \frac{\eta_x}{1+\eta}\hat{\psi}_{xs} = \left(\frac{\eta_x\hat{\psi}_s}{1+\eta}\right)_x - \gamma(\mu)$$

on S because $\mathcal{F}_2(w,\Lambda) = 0$, and so (4.15) holds. □

Remark 4.6. It is worth noting that if $\mathcal{F}_2(w,\Lambda) = 0$, and we define the pullbacks $\psi = \hat{\psi} \circ \Pi$ and $\tilde{\Phi} = \Phi \circ \Pi$, then

$$\begin{aligned}\mathcal{L}_1(w,\Lambda)\Phi = \psi_y\partial^\perp\tilde{\Phi} - \left(\partial^\perp\psi_y + \frac{1}{\psi_y}\right)\tilde{\Phi},\\ (\mathcal{L}_2(w,\Lambda)\Phi) \circ \Pi = (\Delta + \gamma'(\psi))\tilde{\Phi},\end{aligned}$$

where the functions in the expression for $\mathcal{L}_1$ are evaluated on S_η. By $\partial^\perp$, we here mean the non-normalized normal derivative $\partial^\perp := \eta^\perp \cdot \nabla$ for S_η. Viewed through this lens, $\mathcal{L}(w, \Lambda)$ closely resembles the operator $\mathcal{L}(\Lambda)$ for the trivial solutions in (4.13).

We also mention that, while outside the scope of this paper, the $\mathcal{T}$-isomorphism can be employed even with the pseudo-stream function of waves in a stratified, incompressible fluid.

3 Kernel and local bifurcation

The purpose of this section is to describe the kernel of the operator $\mathcal{L}(\Lambda)$ from (4.13), and to give the corresponding local bifurcation results for one-dimensional kernels. This extends parts of [1, 11] to more general vorticity distributions, albeit with a slightly different bifurcation parameter. The paper [19] deals with the same problem, in more detail, but with a quite different approach. Since our primary concern is global bifurcation, we present the results with this goal in mind. Note that Assumption 4.1 is much stronger, especially the analyticity, than what is actually necessary for most of this section.

To simplify the description of the kernel of $\mathcal{L}(\Lambda)$, we define $u = u(s; z)$ to be the solution of the initial value problem

$$\begin{gathered} u''(s; z) + (\gamma'(\overline{\psi}(s)) - z)u(s; z) = 0, \\ u(0; z) = 0, \qquad u'(0; z) = 1, \end{gathered} \tag{4.20}$$

where z acts as a parameter, and primes indicate derivatives with respect to s. Note that u is entire in the parameter z (see for instance [26, Chapter 5]), and that u also has a suppressed dependence on Λ through $\overline{\psi}$. For real values of z we may introduce the corresponding Prüfer angle $\vartheta = \vartheta(s; z)$ as the unique continuous representative of $\arg(u' + iu)$ with $\vartheta(0; z) = \arg(1) = 0$. This representative is well defined since u and u' cannot vanish simultaneously, due to u being the solution of (4.20). The Prüfer angle satisfies the first order equation

$$\vartheta'(s; z) = \cos(\vartheta(s; z))^2 + (\gamma'(\overline{\psi}(s)) - z) \sin(\vartheta(s; z))^2,$$

at the cost of this equation being nonlinear.

Recall that the derivative of γ is bounded by Assumption 4.1. To facilitate the remainder of this section, we introduce the two quantities

$$\rho := \inf \gamma' \quad \text{and} \quad R := \sup \gamma',$$

as they are ubiquitous. The next lemma describes the behavior of the Prüfer angle $\vartheta(1;z)$ with respect to the parameter z. We will encounter this angle while describing the kernel.

Lemma 4.7 (Properties of ϑ)**.** *The Prüfer angle $\vartheta(1;\cdot)$ is strictly decreasing, and in fact $\vartheta_z(1;\cdot) < 0$. Moreover, it satisfies the bounds*

$$\sigma(z-\rho) \le \vartheta(1;z) \le \sigma(z-R), \tag{4.21}$$

for all $z \in \mathbb{R}$, where $\sigma\colon \mathbb{R} \to (0,\infty)$ is the (single-valued) function defined by

$$\sigma(z) = \arg\left(\cosh(\sqrt{z}) + i\frac{\sinh(\sqrt{z})}{\sqrt{z}}\right),$$

with $\sigma(0) = \pi/4$. In particular, $\vartheta(1,-\infty) = \infty$ and $\vartheta(1,\infty) = 0$ (in the sense of limits).

Proof. By differentiating (4.20) with respect to z, multiplying by u and integrating by parts, one arrives at the identity

$$u_z'(1;z)u(1;z) - u_z(1;z)u'(1;z) = \int_0^1 u(s;z)^2\,ds,$$

which implies that

$$\begin{aligned}\vartheta_z(1;z) &= \frac{u_z(1;z)u'(1;z) - u(1;z)u_z'(1;z)}{u(1;z)^2 + u'(1;z)^2}\\ &= -\int_0^1 \frac{u(s;z)^2\,ds}{u(1;z)^2 + u'(1;z)^2} < 0,\end{aligned}$$

proving the first part of the proposition.

Define now $\tilde{u} = \tilde{u}(s;z)$ by

$$\tilde{u}(s;z) := \frac{\sinh(s\sqrt{z})}{\sqrt{z}},$$

and $\tilde{\sigma} = \tilde{\sigma}(s;z)$ by

$$\tilde{\sigma}(s;z) := \arg(\tilde{u}'(s;z) + i\tilde{u}(s;z)),$$

choosing the representative in the same way we did for ϑ. To obtain the bounds described in (4.21), it suffices to observe that $\tilde{\sigma}(0;z) = 0$, and that the differential inequalities

$$\tilde{\sigma}'(s;z-R) \ge \cos(\tilde{\sigma}(s;z-R))^2 + (\gamma'(\overline{\psi}(s)) - z)\sin(\tilde{\sigma}(s;z-R))^2$$

and

$$\tilde{\sigma}'(s; z-\rho) \leq \cos(\tilde{\sigma}(s; z-\rho))^2 + (\gamma'(\overline{\psi}(s)) - z)\sin(\tilde{\sigma}(s; z-\rho))^2$$

hold by direct computation. Then

$$\tilde{\sigma}(s; z-\rho) \leq \vartheta(s; z) \leq \tilde{\sigma}(s; z-R)$$

for all $s \geq 0$, and in particular we have (4.21) from the special case $s = 1$. □

Since the function u introduced in (4.20) is entire in the parameter z, we may define a function l by

$$l(z, \Lambda) := \frac{u'(1; z)}{u(1; z)}, \tag{4.22}$$

which consequently is meromorphic in z. Observing that $l(z, \Lambda) = \cot(\vartheta(1; z))$ on the real axis, we immediately obtain the following from Lemma 4.7:

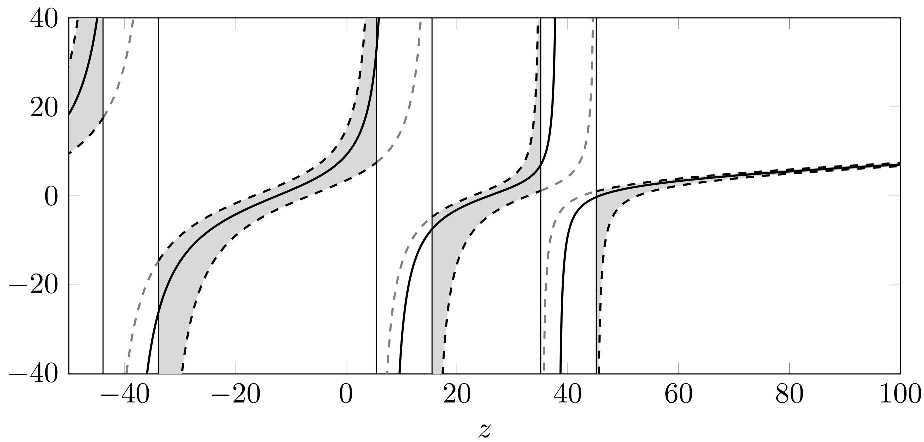

Figure 4.1: The graph of a particular instance of l, with the lower and upper bounds furnished by Proposition 4.8.

Proposition 4.8 (Properties of l). *The derivative of l is positive on the real axis, except at the poles of l, which are all simple. Moreover, l satisfies the bounds*

$$v(z-R) \leq l(z, \Lambda) \leq v(z-\rho), \tag{4.23}$$

where

$$v(z) := \cot(\sigma(z)) = \frac{\sqrt{z}}{\tanh(\sqrt{z})},$$

on the (possibly empty) intervals

$$I_j := \begin{cases} (R-(j+1)^2\pi^2, \rho - j^2\pi^2) & j \geq 1 \\ (R-\pi^2, \infty) & j = 0. \end{cases}$$

Finally, the point $z = 0$ is a pole of l if and only if $\overline{\psi}_\lambda(0) = 0$, and if not then

$$l(0;\Lambda) = -\frac{\overline{\psi}_\mu(0)}{\overline{\psi}_\lambda(0)}. \tag{4.24}$$

Proof. In order to show that (4.24) holds, it suffices to observe that

$$u(s;0) = \overline{\psi}_\mu(0)\overline{\psi}_\lambda(s) - \overline{\psi}_\lambda(0)\overline{\psi}_\mu(s)$$

for all $s \in [0,1]$. Indeed, the right-hand side satisfies (4.20) (with $z = 0$) by differentiation of (4.3), and vanishes at $s = 0$. It also has the correct derivative at $s = 0$ since

$$\overline{\psi}_\mu(0)\overline{\psi}'_\lambda(0) - \overline{\psi}_\lambda(0)\overline{\psi}'_\mu(0) = \overline{\psi}_\mu(1)\overline{\psi}'_\lambda(1) - \overline{\psi}_\lambda(1)\overline{\psi}'_\mu(1) = 1,$$

where we have used that the Wronskian of $\overline{\psi}_\mu$ and $\overline{\psi}_\lambda$ is constant. □

We are now equipped with everything we need to describe the kernel of the operator $\mathcal{L}(\Lambda)$.

Theorem 4.9 (Kernel of $\mathcal{L}(\Lambda)$)**.** *Let $\Lambda \in \mathcal{U}$. A basis for $\ker \mathcal{L}(\Lambda)$ is then given by $\{\Phi_n\}_{n\in M}$, where*

$$\Phi_n(x,s) := \cos(n\kappa x)u(s; n^2\kappa^2)$$

and $M = M(\Lambda)$ is the finite set of all $n \in \mathbb{N}_0$ satisfying the kernel equation

$$l(n^2\kappa^2, \Lambda) = r(\Lambda), \tag{4.25}$$

where

$$r(\Lambda) := \frac{1}{\lambda^2} - \frac{\gamma(\mu)}{\lambda}, \tag{4.26}$$

and l is the function defined in (4.22)*.*

Proof. Suppose that $\Phi \in X_2$, and write it as a Fourier series

$$\Phi(x,s) = \sum_{n=0}^{\infty} a_n(s)\cos(n\kappa x)$$

in the horizontal direction. By inserting the series into (4.13), we deduce that $\Phi \in \ker \mathcal{L}(\Lambda)$ if and only if each coefficient a_n solves the regular Sturm–Liouville problem

$$\begin{gathered} a_n''(s) + (\gamma'(\overline{\psi}(s)) - n^2\kappa^2)a_n(s) = 0 \\ a_n(0) = 0, \qquad \lambda a_n'(1) + \left(\gamma(\mu) - \frac{1}{\lambda}\right)a_n(1) = 0 \end{gathered}$$

for all $n \in \mathbb{N}_0$. Trivially, $a_n = 0$ is always a solution, but not always the only one: It is well known that this Sturm–Liouville problem has nonzero solutions, spanned by $u(\cdot, n^2\kappa^2)$, if and only if (4.25) is satisfied. There are only finitely many solutions of (4.25), as $n^2\kappa^2 \in I_0$ for all sufficiently large $n \in \mathbb{N}_0$, and l is strictly increasing there. □

Remark 4.10. The function l will depend non-trivially on Λ unless γ' is a constant, namely when the vorticity is either constant or affine. We also mention that we would typically like to avoid the degenerate case where $n = 0$ solves (4.25). For this reason, (4.24) can occasionally be useful.

Exactly one-dimensional kernels can be found under certain assumptions on γ, especially if we are willing to relinquish control of the wavenumber κ. One way is to, in essence, require that γ is close enough to affine to enable us to exploit the bounds in (4.23). Note that there is no loss of generality in limiting the scope to $M(\Lambda) = \{1\}$, by redefining κ, as long as the only interest is in one-dimensional kernels $M(\Lambda) \neq \{0\}$.

Proposition 4.11 (Kernel construction)**.**

(i) Suppose that $n^2\kappa^2 \in I_j$ for some $j \in \mathbb{N}_0$, and further that μ is such that

$$\gamma(\mu)^2 > -4v(n^2\kappa^2 - R),$$

with v as defined in Proposition 4.8. Then there exist $\lambda \neq 0$ such that $n \in M(\Lambda)$. More precisely, such λ can always be chosen to satisfy

$$\lambda \in \begin{cases} (0, -2/\gamma(\mu)) & \gamma(\mu) < 0, \\ (-2/\gamma(\mu), 0) & \gamma(\mu) > 0, \end{cases} \tag{4.27}$$

if $v(n^2\kappa^2 - R) \leq 0$, and

$$\lambda \in \begin{cases} (-\infty, 0) \text{ or } (0, -1/\gamma(\mu)) & \gamma(\mu) < 0 \\ (-\infty, 0) \text{ or } (0, \infty) & \gamma(\mu) = 0 \\ (-1/\gamma(\mu), 0) \text{ or } (0, \infty) & \gamma(\mu) > 0 \end{cases} \tag{4.28}$$

otherwise.

(ii) *Assume that* $0 \in I_j$ *for some* $j \in \mathbb{N}_0$*, and let* $\mu \in \mathbb{R}$*. For any* $\kappa > 0$ *such that* $\kappa^2 \in I_0$ *and*

$$v(\kappa^2 - R) > \max(v(-\rho), -\gamma(\mu)^2/4)$$

there is a $\lambda \neq 0$*, which can be chosen according to* (4.27) *or* (4.28)*, such that* $M(\Lambda) = \{1\}$*. In particular, this is the case for all sufficiently large* $\kappa > 0$*.*

Proof. We know by Proposition 4.8 that if $n^2\kappa^2 \in I_j$, then

$$v(n^2\kappa^2 - R) \leq l(n^2\kappa^2, \Lambda) \leq v(n^2\kappa^2 - \rho),$$

where it is crucial that the bounds do not depend on Λ. Observe that

$$\inf_{\lambda \neq 0} r(\mu, \lambda) = -\frac{1}{4}\gamma(\mu)^2$$

for every $\mu \in \mathbb{R}$, with the infimum attained at $\lambda = -2/\gamma(\mu)$ as long as $\gamma(\mu) \neq 0$. In the same event, we also have $r(\mu, -1/\gamma(\mu)) = 0$. Moreover, $r(\mu, \lambda) \to \infty$ as $\lambda \to 0$, and $r(\mu, \lambda) \to 0$ as $|\lambda| \to \infty$. The first part of the proposition now follows from the intermediate value theorem applied to $r(\mu, \cdot) - l(n^2\kappa^2, \mu, \cdot)$, on appropriate intervals chosen according to either (4.27) or (4.28).

For the second part of the proposition, observe that the hypothesis of the first part is satisfied with $n = 1$ and $j = 0$. Thus, there is some $\lambda \neq 0$ such that $1 \in M(\Lambda)$. Moreover, by the assumptions and Proposition 4.8 we have

$$l(0, \Lambda) \leq v(-\rho) < v(\kappa^2 - R) \leq l(\kappa^2, \Lambda),$$

whence $0 \notin M(\Lambda)$. Finally, $l(n^2\kappa^2, \Lambda) > l(\kappa^2, \Lambda)$ for all $n \geq 2$, since l is strictly increasing on I_0. Thus $M(\Lambda) = \{1\}$. $\square$

Remark 4.12. Kernels of arbitrarily large finite dimension exist when γ is affine [1], but it is unclear if, and in what sense, the existence of multi-dimensional kernels generalizes to more general vorticity distributions. We will not pursue this question here.

The Fredholm property of $\mathcal{L}(\Lambda)$

Let us introduce the notation

$$\mathcal{T}(\Lambda) = (\mathcal{T}_1, \mathcal{T}_2)(\Lambda) := \mathcal{T}(0, \Lambda), \qquad \Lambda \in \mathcal{U},$$

for the $\mathcal{T}$-isomorphism at the trivial solutions, mirroring our use of $\mathcal{L}(\Lambda)$ in (4.13). If we also equip Y with the inner product (inducing a finer topology)

$$\langle w_1, w_2\rangle_Y := \langle \eta_1, \eta_2\rangle_{L^2_\kappa(\mathbb{R})} + \langle \hat\varphi_1, \hat\varphi_2\rangle_{L^2_\kappa(\Omega)}, \qquad w_i = (\eta_i, \hat\varphi_i), \tag{4.29}$$

we can state a useful lemma. In particular, we will employ it to describe the image of $\mathcal{L}(\Lambda)$.

Lemma 4.13 (Symmetry for $\mathcal{L}(\Lambda)$)**.** *The identity*

$$\langle (\mathcal{T}_1(\Lambda)\Phi, \Phi), \mathcal{L}(\Lambda)\Psi\rangle_Y = \langle \mathcal{L}(\Lambda)\Phi, (\mathcal{T}_1(\Lambda)\Psi, \Psi)\rangle_Y$$

holds for all $\Phi, \Psi \in X_2$*.*

Proof. By one of Green's identities, we have

$$\int_\Omega (\Phi\Delta\Psi - \Psi\Delta\Phi)\,dx\,ds = \int_S (\Phi\Psi_s - \Phi_s\Psi)\,dx,$$

where the integrals are understood to be over one period. Therefore

$$\langle \Phi, \mathcal{L}_2(\Lambda)\Psi\rangle_{L^2_\kappa(\Omega)} = \langle \mathcal{L}_2(\Lambda)\Phi, \Psi\rangle_{L^2_\kappa(\Omega)} + \int_S (\Phi\Psi_s - \Phi_s\Psi)\,dx,$$

and as a consequence, we find

$$\begin{aligned}\langle (\mathcal{T}_1(\Lambda)\Phi, \Phi), \mathcal{L}(\Lambda)\Psi\rangle_Y &= \langle \mathcal{L}_2(\Lambda)\Phi, \Psi\rangle_{L^2_\kappa(\Omega)} + \int_S (\Phi\Psi_s - \Phi_s\Psi)\,dx \\ &\qquad + \int_S \Phi(r(\Lambda)\Psi - \Psi_s)\,dx \\ &= \langle \mathcal{L}_2(\Lambda)\Phi, \Psi\rangle_{L^2_\kappa(\Omega)} + \langle \mathcal{L}_1(\Lambda)\Phi, \mathcal{T}_1(\Lambda)\Psi\rangle_{L^2_\kappa(\mathbb{R})} \\ &= \langle \mathcal{L}(\Lambda)\Phi, (\mathcal{T}_1(\Lambda)\Psi, \Psi)\rangle_Y\end{aligned}$$

by direct computation. □

Since $\mathcal{L}(\Lambda)$ is a simple elliptic operator (with boundary conditions), it is a standard result that it, and by consequence $D_w\mathcal{F}(0,\Lambda)$ through Theorem 4.5, is Fredholm of index zero. Stated more precisely, we have the following:

Lemma 4.14 (Fredholm property of $\mathcal{L}(\Lambda)$)**.** *Suppose that* $\Lambda \in \mathcal{U}$*. Then* $\mathcal{L}(\Lambda)$ *is a Fredholm operator of index zero. Moreover, its image is the orthogonal complement of the subspace*

$$Z(\Lambda) := \{(\mathcal{T}_1(\Lambda)\Phi, \Phi) : \Phi \in \ker \mathcal{L}(\Lambda)\}$$

in Y *with respect to the inner product in* (4.29)*.*

We omit the proof of Lemma 4.14, opting only to motivate the result by noting that the inclusion

$$\operatorname{im} \mathcal{L}(\Lambda) \subset Z(\Lambda)^{\perp}$$

is an immediate corollary of Lemma 4.13. The opposite inclusion is less trivial. See for instance [30] for a proof in a similar setting.

Transversality and local bifurcation

An additional benefit of introducing the kernel equation (4.25) is that the transversality condition for local bifurcation, appearing in the Crandall–Rabinowitz theorem, can be expressed using a a differentiated version of this equation. To prove this, we exploit the characterization of $\operatorname{im} \mathcal{L}(\Lambda)$ given in Lemma 4.14.

Proposition 4.15 (Transversality condition)**.** *Suppose that* $\Lambda \in \mathcal{U}$*, and that* $M(\Lambda) = \{n\}$ *for some* $n \in \mathbb{N}_0$*. Then*

$$D_{w\mu}\mathcal{F}(0,\Lambda)\mathcal{T}(\Lambda)\Phi_n \notin \operatorname{im} D_w\mathcal{F}(0,\Lambda) \tag{4.30}$$

if and only if the transversality condition

$$l_\mu(n^2\kappa^2,\Lambda) \neq r_\mu(\Lambda) \tag{4.31}$$

is satisfied. Here, the functions l *and* r *are those defined in* (4.22) *and* (4.26)*, respectively, and the subscripts denote partial derivatives.*

Proof. We first observe that by the identity

$$D_{w\mu}\mathcal{F}(0,\Lambda)\mathcal{T}(\Lambda) + D_w\mathcal{F}(0,\Lambda)\mathcal{T}_\mu(\Lambda) = \mathcal{L}_\mu(\Lambda),$$

which follows from (4.15), we have that the condition

$$\mathcal{L}_\mu(\Lambda)\Phi_n \notin \operatorname{im} \mathcal{L}(\Lambda)$$

is equivalent to (4.30). Further, this condition is, in turn, equivalent to

$$\langle \mathcal{L}_\mu(\Lambda)\Phi_n, (\mathcal{T}_1(\Lambda)\Phi_n, \Phi_n)\rangle_Y \neq 0,$$

or

$$\int_0^1 \gamma''(\overline{\psi}(s))\overline{\psi}_\mu(s)u(s;n^2\kappa^2)^2\,ds \neq \frac{\gamma'(\mu)}{\lambda}u(1;n^2\kappa^2)^2, \tag{4.32}$$

by Lemma 4.14.

We immediately recognize that $\gamma'(\mu)/\lambda = -r_\mu(\Lambda)$ on the right-hand side of (4.32). The result finally follows by observing that

$$\int_0^1 \gamma''(\overline{\psi}(s))\overline{\psi}_\mu(s)u(s;n^2\kappa^2)^2\,ds = -u(1;n^2\kappa^2)^2 l_\mu(n^2\kappa^2,\Lambda),$$

which is obtained by differentiating (4.20) with respect to μ, multiplying by $u(\cdot;n^2\kappa^2)$, and integrating by parts with respect to s. Note that $u(1;n^2\kappa^2)$ is necessarily nonzero, since (4.25) is satisfied by hypothesis. □

Remark 4.16. A completely analogous transversality condition to (4.31) holds if λ is used as the bifurcation parameter instead of μ. The only change needed is to exchange the partial derivatives for ones with respect to λ.

We can now apply the Crandall–Rabinowitz theorem, see [7] (or [2] for a more modern exposition), to obtain small-amplitude waves that solve (4.11). This extends the corresponding theorem in [11] to more general vorticity distributions than affine. Note that a similar result to Theorem 4.17, for small-amplitude waves, has previously been obtained in [19].

Theorem 4.17 (Local bifurcation). *Let $\Lambda^* \in \mathcal{U}$ and suppose that $M(\Lambda^*) = \{n\}$ for some $n \in \mathbb{N}$, so that*

$$\ker D_w\mathcal{F}(0,\Lambda^*) = \operatorname{span}\{\mathcal{T}(\Lambda^*)\Phi_n\},$$

where Φ_n is as in Theorem 4.9. If the transversality condition (4.31) *holds, there exists an analytic curve $\mathcal{K}^{\mathrm{loc}}_{\Lambda^*} = \{(w(t),\mu(t)) : |t| < \varepsilon\}$ of solutions to*

$$\mathcal{F}(w,\mu,\lambda^*) = 0 \tag{4.33}$$

in $\mathcal{O}_{\lambda^*}$, with

$$w(t) = t\mathcal{T}(\Lambda^*)\Phi_n + O(t^2) \quad (\textit{in } X) \tag{4.34}$$

and

$$\mu(t) = \mu^* + O(t^2) \tag{4.35}$$

as $t \to 0$. The solutions on the curve $\mathcal{K}^{\mathrm{loc}}_{\Lambda^*}$ have wavenumber $n\kappa$, and

$$\begin{aligned} \mu(-t) &= \mu(t), \\ \eta(-t)(x) &= \eta(t)\left(x + \frac{\pi}{n\kappa}\right), \\ \hat{\varphi}(-t)(x,s) &= \hat{\varphi}(t)\left(x + \frac{\pi}{n\kappa}, s\right) \end{aligned} \tag{4.36}$$

for all $|t| < \epsilon$ *and* $(x, s) \in \overline{\Omega}$.

There is a neighborhood of $(0, \mu^*) \in \mathcal{O}_{\lambda^*}$ *in which all solutions of* (4.33) *are either trivial or on the curve.*

Proof. The only parts of the theorem that do not follow directly from the Crandall–Rabinowitz theorem are:

(i) The claim that the solutions have wavenumber $n\kappa$,

(ii) the symmetry properties in (4.36), and

(iii) the asymptotics in (4.35).

Here, part (i) follows by redefining κ such that $n = 1$ before applying the Crandall–Rabinowitz theorem, while part (ii) can be obtained by observing that (w, μ) is a solution of (4.33) if and only if

$$\left(x \mapsto \eta\left(x + \frac{\pi}{n\kappa}\right), (x, s) \mapsto \hat{\varphi}\left(x + \frac{\pi}{n\kappa}, s\right), \mu\right).$$

is a solution. Lastly, part (iii) is an immediate corollary of the just-proved symmetry of μ in (ii). $\square$

Explicit examples

We believe there is value in pausing to record the two simplest forms of vorticity distributions here, for which many aspects of the theory become significantly more explicit.

Constant vorticity

If the vorticity distribution is constant; that is, of the form

$$\gamma(t) \equiv \omega_0,$$

for some fixed $\omega_0 \in \mathbb{R}$, then the trivial solutions (solving (4.3)) are given by the quadratic polynomials

$$\overline{\psi}(s; \Lambda) = \mu + \lambda(s - 1) - \frac{1}{2}\omega_0(s - 1)^2$$

for every $\Lambda \in \mathcal{U}$. Consequently

$$\Upsilon(\Lambda) = \mu - \lambda - \frac{1}{2}\omega_0,$$

and we see that

$$u(s;z) = \tilde{u}(s;z) = \frac{\sinh(s\sqrt{z})}{\sqrt{z}},$$

with the notation taken from the proof of Lemma 4.7, solves (4.20). Hence

$$l(z,\Lambda) = v(z) = \frac{\sqrt{z}}{\tanh(\sqrt{z})}$$

in the kernel equation (4.25). This can also obtained directly from Proposition 4.8, as $\rho = R = 0$.

Bifurcation with respect to μ is never possible for constant vorticity, as (4.31) can *never* be satisfied. Therefore Theorem 4.17 does not apply, in the way it is stated here. This is not unexpected, as changing μ merely constitutes a constant shift of $\overline{\psi}$, and one may equally well set $\mu \equiv 0$. Bifurcation with respect to λ, on the other hand, can be done from either of the bifurcation points

$$\frac{1}{\lambda_{n,\pm}} = \frac{\omega_0}{2} \pm \sqrt{\left(\frac{\omega_0}{2}\right)^2 + \frac{n\kappa}{\tanh(n\kappa)}},$$

for any $n \in \mathbb{N}$. Moreover, the transversality condition with respect to λ becomes $\omega_0\lambda_{n,\pm} \neq 2$, which is always satisfied. Global bifurcation for the constant case, including stagnation, has already been studied in great detail in [5].

Affine vorticity

As discussed in [11], it is sufficient to consider *linear* vorticity distributions of the form

$$\gamma(t) = \omega_0 t \tag{4.37}$$

for fixed $\omega_0 \neq 0$. The trivial solutions take the form

$$\overline{\psi}(s;\Lambda) = \mu\cos(\sqrt{\omega_0}(s-1)) + \lambda\frac{\sin\bigl(\sqrt{\omega_0}(s-1)\bigr)}{\sqrt{\omega_0}},$$

and are trigonometric when $\omega_0 > 0$, and hyperbolic when $\omega_0 < 0$. Accordingly,

$$\Upsilon(\Lambda) = \mu\cos(\sqrt{\omega_0}) + \lambda\frac{\sin\bigl(\sqrt{\omega_0}\bigr)}{\sqrt{\omega_0}},$$

and

$$u(s,z) = \tilde{u}(s,z-\omega_0), \qquad l(z,\Lambda) = v(z-\omega_0),$$

with $\tilde{u}$ and v as defined above.

There is a rich structure of bifurcation points, even with such simple vorticity distributions like (4.37). Furthermore, the transversality condition for one-dimensional bifurcation is trivially satisfied for μ, while the condition reduces to

$$\omega_0 \mu \lambda \neq 2$$

when λ is used as the bifurcation parameter. See the works [1, 11, 13] for local bifurcation results with slightly different choices of parameters (which are quite hard to generalize to non-affine γ), including thorough studies of the resulting kernel equation.

4 Global bifurcation

Our local bifurcation result, Theorem 4.17, establishes the existence of *local* curves of small solutions to (4.11). We will now proceed to the main event of this paper, which is to use analytic global bifurcation theory, due to Buffoni, Dancer & Toland [2, 8], to extend these local curves to *global* curves. The principal result is the following theorem:

Theorem 4.18 (Global bifurcation)**.** *Suppose that $\Lambda^* \in \mathcal{U}$ is such that $M(\Lambda^*) = \{n\}$, with $n \in \mathbb{N}$, and that the transversality condition* (4.31) *holds. Then the local curve obtained in Theorem 4.17 can be uniquely extended (up to reparametrization) to a continuous curve*

$$\mathcal{K}_{\Lambda^*} = \{(w(t), \mu(t)) : t \in \mathbb{R}\} \supset \mathcal{K}_{\Lambda^*}^{\mathrm{loc}}$$

of solutions to (4.33)*, such that the following properties hold:*

(i) The curve can be reparametrized analytically in a neighborhood of any point on the curve.

(ii) The solutions have wavenumber $n\kappa$, and satisfy the symmetry properties (4.36)*, for all $t \in \mathbb{R}$.*

(iii) One of the following alternatives occur:

(A) Either

$$\min\left\{\frac{1}{1 + \|w(t)\|_X + |\mu(t)|}, \min_{x\in\mathbb{R}} (1 + \eta(t)), \min_{S} |\hat{\psi}_s(t)|\right\} \to 0$$

as $t \to \infty$,

(B) or the curve is closed.

Remark 4.19. Alternative (A) would imply the existence of subsequences $(t_n)_{n\in\mathbb{N}}$, with $t_n \to \infty$, along which at least one of (i) the solutions are unbounded, (ii) the surface approaches the bed, or (iii) surface stagnation is approached, hold true.

Theorem 4.18 will follow directly from a slightly modified version of [2, Theorem 9.1.1], stated in [5, Theorem 6], if we can prove the required Fredholm and compactness properties. Namely, that:

(i) The derivative $D_w\mathcal{F}(w,\Lambda)$ is Fredholm of index zero not only when $(w,\mu) = (0,\mu^*)$, but on the entire solution set of (4.33).

(ii) For an appropriately chosen increasing sequence $(\mathcal{Q}_j^{\lambda^*})_{j\in\mathbb{N}}$ of closed and bounded subsets of $\mathcal{O}_{\lambda^*}$ such that

$$\mathcal{O}_{\lambda^*} = \bigcup_{j\in\mathbb{N}} \mathcal{Q}_j^{\lambda^*},$$

the intersection

$$\{(w,\mu) \in \mathcal{O}_{\lambda^*} : \mathcal{F}(w,\Lambda) = 0\} \cap \mathcal{Q}_j^{\lambda^*}$$

is compact in X for each $j \in \mathbb{N}$.

Thus, to establish Theorem 4.18, we are left to verify that these two conditions are satisfied. We have already made some of the necessary preparations for this in previous sections.

Verification of the global Fredholm property

The central tool we will use to show that the Fréchet derivative of $\mathcal{F}$ is Fredholm on the solution set of (4.11), is the generalized $\mathcal{T}$-isomorphism from Theorem 4.5. Like for the trivial solutions, this reduces the problem to one for the simpler operator $\mathcal{L}$. First, we show semi-Fredholmness whenever $\mathcal{L}$ is well defined. Note that, importantly, (w,Λ) need *not* be a solution of (4.11) in the lemma.

Lemma 4.20. *Let $(w,\Lambda) \in \mathcal{O}$ be such that $\eta \in C^{3,\beta}_{\kappa,\mathrm{e}}(\mathbb{R})$ and $\hat{\varphi} \in C^{3,\beta}_{\kappa,\mathrm{e}}(\overline{\Omega})$. Then $\mathcal{L}(w,\Lambda)$ has finite-dimensional kernel and closed range.*

Proof. Recall the definition of $\mathcal{L}(w,\Lambda)\colon X_2 \to Y$ in (4.12). Since

$$\begin{aligned} \xi_1^2 - \frac{2s\eta_x}{1+\eta}\xi_1\xi_2 + \frac{1+s^2\eta_x^2}{(1+\eta)^2}\xi_2^2 &= \xi^\intercal \begin{pmatrix} 1 & -\dfrac{s\eta_x}{1+\eta} \\ -\dfrac{s\eta_x}{1+\eta} & \dfrac{1+s^2\eta_x^2}{(1+\eta)^2} \end{pmatrix} \xi \\ &\geq \frac{1}{(1+\eta)^2+1+s^2\eta_x^2}|\xi|^2, \end{aligned} \tag{4.38}$$

for all $\xi = (\xi_1,\xi_2) \in \mathbb{R}^2$, the operator component $\mathcal{L}_2(w,\Lambda) : X_2 \to Y_1$ is strictly elliptic. The inequality in (4.38) can be deduced from the eigenvalues of the matrix.

Furthermore, the coefficient in front of ∂_s in $\mathcal{L}_1(w,\Lambda)$ is uniformly separated away from 0, by how we defined $\mathcal{O}$ in (4.9). Combining now the Schauder estimates from Theorems 6.6 and 6.30 in [15], we deduce that there is a constant $C = C(w,\Lambda) \geq 0$ such that

$$\|\Phi\|_{X_2} \leq C(\|\Phi\|_{L^\infty} + \|\mathcal{L}(w,\Lambda)\Phi\|_Y) \tag{4.39}$$

for all $\Phi \in X_2$. Standard arguments based on (4.39) are in turn used to establish the lemma:

Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence in the closed unit ball of $\ker \mathcal{L}(w,\Lambda)$. By compact embedding of Hölder spaces, this sequence has a subsequence that converges with respect to $\|\cdot\|_{L^\infty}$. It then follows by (4.39) that the subsequence is Cauchy also in X_2, and therefore converges. The closed unit ball of $\ker \mathcal{L}(w,\Lambda)$ is thus compact, whence $\ker \mathcal{L}(w,\Lambda)$ is finite dimensional.

Next, we argue that the range of $\mathcal{L}(w,\Lambda)$ is closed. Since $\ker \mathcal{L}(w,\Lambda)$ is finite dimensional, there is a closed subspace $E \subseteq X_2$ such that

$$X_2 = \ker \mathcal{L}(w,\Lambda) \oplus E.$$

Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence in E such that $(\mathcal{L}(w,\Lambda)\Phi_n)_{n\in\mathbb{N}}$ converges in Y. If $(\Phi_n)_{n\in\mathbb{N}}$ is bounded in X_2, it follows from (4.39) and a similar argument to the one above that we can extract a convergent subsequence. It only remains to rule out the possibility that $(\Phi_n)_{n\in\mathbb{N}}$ is unbounded. To that end, suppose so, and also without loss of generality that $\|\Phi_n\|_{X_2} \to \infty$. Arguing as before, we extract a limit $\Phi \in E$ of a subsequence of $(\Phi_n/\|\Phi_n\|_{X_2})_{n\in\mathbb{N}}$, satisfying $\|\Phi\|_{X_2} = 1$, but $\mathcal{L}(w,\Lambda)\Phi = 0$. This contradicts $E \cap \ker \mathcal{L}(w,\Lambda) = \varnothing$. ☐

Armed with Lemma 4.20, we can prove the desired Fredholm property, by relating $\mathcal{L}(w,\Lambda)$ and $\mathcal{L}(\Lambda)$ and then employing the stability of the Fredholm index.

Theorem 4.21 (Global Fredholm property)**.** *Suppose that* $(w, \Lambda) \in \mathcal{O}$ *is a solution of* (4.11)*. Then the operator* $D_w \mathcal{F}(w, \Lambda)$ *is Fredholm of index zero.*

Proof. Recalling Remark 4.3, we have $(tw, \Lambda) \in \mathcal{O}$ for every $t \in [0, 1]$. By Theorem 4.4 used on (w, Λ), the necessary regularity for Lemma 4.20 to apply is also present. Thus $\mathcal{L}(tw, \Lambda)$ has finite-dimensional kernel and closed range for each $t \in [0, 1]$. This is the case even if (tw, Λ) need not be a solution of (4.11) in general, except at the endpoints. In particular, the operators $\mathcal{L}(tw, \Lambda)$ are semi-Fredholm, so their Fredholm index is well defined (albeit not necessarily finite), and stable under perturbation.

We can now use continuity of

$$t \mapsto \operatorname{ind} \mathcal{L}(tw, \Lambda),$$

see [17, Theorem IV-5.17], to conclude that

$$\operatorname{ind} \mathcal{L}(w, \Lambda) = \operatorname{ind} \mathcal{L}(\Lambda) = 0$$

for every solution of (4.11). We have shown that $\mathcal{L}(w, \Lambda)$ is a Fredholm operator of index zero, and the same is then true for $D_w \mathcal{F}(w, \Lambda)$ by (4.15), completing the proof. □

Verification of the compactness property

Inspecting (4.9), it is clear that a reasonable definition of the increasing sequences $(\mathcal{Q}_j^\lambda)_{j \in \mathbb{N}}$ is to let

$$\mathcal{Q}_j^\lambda := \left\{ (w, \mu) \in \mathcal{O}_\lambda : 1 + \eta \geq \frac{1}{j}, \operatorname{sgn}(\lambda) \hat{\psi}_s|_S \geq \frac{1}{j}, \|w\|_X + |\mu| \leq j \right\}$$

for each $j \in \mathbb{N}$. These sets are certainly both closed and bounded, and it is evident that, indeed,

$$\mathcal{O}_\lambda = \bigcup_{j \in \mathbb{N}} \mathcal{Q}_j^\lambda$$

for every $\lambda \neq 0$.

We will use Schauder estimates to obtain compactness of the intersections

$$\mathcal{Q}_j^\lambda \cap \{(w, \mu) \in \mathcal{O}_\lambda : \mathcal{F}(w, \Lambda) = 0\} \tag{4.40}$$

for every $\lambda \neq 0$ and $j \in \mathbb{N}$. In order to do so, we will use a *different* way of flattening (4.2) than the naive (4.5), but only in a neighborhood of the surface. This strategy will first give us control of η, which in turn can be leveraged to control $\hat{\varphi}$.

Proposition 4.22 (Compactness)**.** *The intersection in* (4.40) *is a compact subset of* $X \times \mathbb{R}$ *for every* $\lambda \neq 0$ *and* $j \in \mathbb{N}$*.*

Proof. Without loss of generality, we will assume that $\lambda > 0$, which fixes the sign of $\hat{\psi}_s|_S$. Let (w, μ) be any point of $\mathcal{Q}_j^\lambda$ such that (w, Λ) solves (4.11). Pull back $\hat{\psi}$ to $\psi = \hat{\psi} \circ \Pi$ on Ω_η using the naive flattening transform from (4.5), and observe that

$$\psi_y(x, 1 + \eta(x)) = \frac{\hat{\psi}_s(x, 1)}{1 + \eta(x)} \geq \frac{1/j}{1 + j} \geq \frac{1}{(1 + j)^2} \tag{4.41}$$

for all $x \in \mathbb{R}$. Also,

$$|\psi_{yy}(x, y)| = \frac{|\hat{\psi}_{ss}(x, y/(1 + \eta(x)))|}{(1 + \eta(x))^2} \leq (1 + j)^3 \tag{4.42}$$

for all $(x, y) \in \Omega_\eta$. Together, (4.41) and (4.42) imply the lower bound

$$\psi_y(x, y) \geq \frac{1}{2(1 + j)^2} \tag{4.43}$$

whenever

$$y \geq 1 + \eta(x) - \frac{1}{2(1 + j)^5},$$

through the mean value theorem.

Furthermore

$$\psi\left(x, 1 + \eta(x) - \frac{1}{2(1 + j)^5}\right) - \mu \leq -\epsilon_j, \quad \epsilon_j := \frac{1}{4(1 + j)^7},$$

and so we deduce the existence of a streamline $\tilde{\eta} \in X_1$ satisfying both

$$-\frac{1}{2(1 + j)^5} \leq \tilde{\eta}(x) - \eta(x) < 0$$

and

$$\psi(x, 1 + \tilde{\eta}(x)) - \mu = -\epsilon_j$$

for all $x \in \mathbb{R}$.

If we proceed to define the strips

$$R_\epsilon := \mathbb{R} \times (-\epsilon, 0)$$

for $\epsilon > 0$, then the semi-hodograph transform $\Gamma \colon \Omega_\eta \setminus \Omega_{\tilde{\eta}} \to R_{\epsilon_j}$ defined by

$$\Gamma(x, y) = (x, \psi(x, y) - \mu)$$

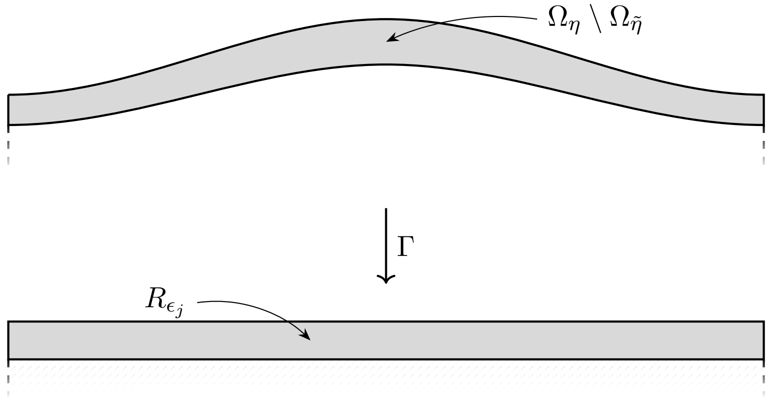

Figure 4.2: The setup in the proof of Proposition 4.22. We ignore whatever is occurring outside of $\Omega_\eta \setminus \Omega_{\tilde\eta}$.

is a diffeomorphism between the closures of the same sets. The transform has an inverse of the form

$$\Gamma^{-1}(q,p) = (q, h(q,p)), \tag{4.44}$$

where the choice of letters for the variables is a matter of convention.

It is well known that the function $h\colon R_{\epsilon_j} \to \mathbb{R}$ implicitly defined by (4.44), see e.g. [4, 9], satisfies the second order quasi-linear elliptic boundary value problem

$$\begin{aligned} \mathcal{S}(h)h &= \gamma(p+\mu)h_p^3 && \text{in } R_{\epsilon_j},\\ 1 + h_q^2 + (2h - 2 - \lambda^2)h_p^2 &= 0 && \text{on } p = 0, \end{aligned} \tag{4.45}$$

where we have introduced the differential operator

$$\mathcal{S}(h) \coloneqq h_p^2\partial_q^2 - 2h_qh_p\partial_q\partial_p + (1+h_q^2)\partial_p^2.$$

For the same reasons as in earlier results (such as [4]), Schauder estimates applied directly to (4.45) do not help us here. However, while (4.45) is not suitable, it follows by straight-forward differentiation that the partial derivative $\theta \coloneqq h_q$ (h is again actually analytic on $\overline{R}_{\epsilon_j}$ due to Theorem 4.4) satisfies a similar boundary value problem, namely

$$\begin{aligned} \mathcal{S}(h)\theta &= 2h_q(h_{qp}^2 - h_{pp}h_{qq}) + 3\gamma(p+\mu)h_p^2h_{qp} && \text{in } R_{\epsilon_j},\\ h_p^3\theta + h_ph_q\theta_q - (1+h_q^2)\theta_p &= 0 && \text{on } p = 0, \end{aligned} \tag{4.46}$$

which *can* be used.

Just like in (4.38), we have strict ellipticity in (4.46), because

$$h_p^2\xi_1^2 - 2h_ph_q\xi_1\xi_2 + (1+h_q^2)\xi_2^2 \geq \frac{h_p^2}{1+h_p^2+h_q^2}|\xi|^2$$

for all $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$. Moreover, it can be shown that (where the exact constant is unimportant)

$$\frac{h_p^2}{1+h_p^2+h_q^2} \geq \frac{1}{(j+1)^6},$$

whence the strict ellipticity is uniform in the choice of $(w, \mu) \in \mathcal{Q}_j^\lambda$. The boundary condition at $p = 0$ in (4.46) is also trivially uniformly oblique, in the sense of [15, (6.76)], because

$$1 + h_q^2 \geq 1,$$

and this is again obviously uniform in the choice of $(w, \mu) \in \mathcal{Q}_j^\lambda$.

Suppose that $(w_n, \mu_n)_{n\in\mathbb{N}}$, where as usual $w_n = (\eta_n, \hat{\varphi}_n)$, is a sequence in $\mathcal{Q}_j^\lambda$ such that

$$\mathcal{F}(w_n, \mu_n, \lambda) = 0$$

for all $n \in \mathbb{N}$. Using (4.44), we may define an associated sequence $(h_n)_{n\in\mathbb{N}}$ of functions $h_n\colon R_{\epsilon_j} \to \mathbb{R}$ solving (4.45) with $\mu = \mu_n$. Due to the bounds in the definition of $\mathcal{Q}_j^\lambda$, and the uniform lower bound on ψ_y from (4.43), we infer that this sequence is bounded in $C^{2,\beta}_{\kappa,\mathrm{e}}(\overline{R}_{\epsilon_j})$. As the terms of the corresponding sequence $(\theta_n)_{n\in\mathbb{N}} = (\partial_q h_n)_{n\in\mathbb{N}}$ satisfies (4.46) with $\mu = \mu_n$ for each $n \in \mathbb{N}$, we deduce from the Schauder estimate in [15, Theorem 6.30] that $(\theta_n)_{n\in\mathbb{N}}$ is bounded in $C^{2,\beta}_{\kappa,\mathrm{o}}(\overline{R}_{\epsilon_j/2})$. Note that, crucially, we do not need a boundary condition at $p = -\epsilon_j$, as we can use *interior* estimates on R_{ϵ_j} to procure a global estimate on the smaller rectangle.

It follows that

$$(\partial_x\eta_n)_{n\in\mathbb{N}} = (\theta_n(\cdot, 0))_{n\in\mathbb{N}}$$

is bounded in $C^{2,\beta}_{\kappa,\mathrm{o}}(\mathbb{R})$, and therefore that the sequence $(\eta_n)_{n\in\mathbb{N}}$ is bounded in $C^{3,\beta}_{\kappa,\mathrm{e}}(\mathbb{R})$. Recall next that (4.6) is strictly elliptic due to (4.38). This ellipticity is again uniform in the choice of $(w, \mu) \in \mathcal{Q}_j^\lambda$, because

$$\frac{1}{(1+\eta)^2 + 1 + s^2\eta_x^2} \geq \frac{1}{(1+j)^2 + 1 + j^2} \geq \frac{1}{2(1+j)^2}$$

on $\overline{\Omega}$ by our definition of $\mathcal{Q}_j^\lambda$.

Having gained an additional bounded derivative for the surface profile, we can now use the Schauder estimate in [15, Theorem 6.6] on (4.6) to infer that $(\hat{\psi}_n)_{n\in\mathbb{N}}$, and therefore $(\hat{\varphi}_n)_{n\in\mathbb{N}}$, is bounded in $C^{3,\beta}_{\kappa,\mathrm{e}}(\overline{\Omega})$. Finally, by boundedness of $(\mu_n)_{n\in\mathbb{N}}$ and the usual compact embedding of Hölder spaces [15, Lemma 6.36], we conclude that the sequence $(w_n, \mu_n)_{n\in\mathbb{N}}$ has a convergent subsequence. The intersection (4.40) is therefore compact, concluding the proof. □

5 Properties of the global curve

It is highly desirable to narrow down the alternatives in Theorem 4.18. In particular, like in most global bifurcation results, one would typically like to rule out alternative (B) entirely. The reason for this, of course, is that this would guarantee the existence of truly large-amplitude solutions to (4.11). That is not to say that solutions on a hypothetical closed curve are small, but they do not "blow up" in the same way that solutions do in the event that (A) occurs.

Due to the loss of global maximum principles on Ω_η, ruling out alternatives in Theorem 4.18 is significantly more difficult than for non-stagnant waves (and perhaps even impossible in general without making further assumptions on γ). Alternative (B) was substantially ruled out for the special case of waves with constant vorticity in [5], albeit in an entirely different framework, but it is not at all clear how to generalize this to more general vorticity distributions.

Various nodal properties are preserved *near* the surface on the local curve $\mathcal{K}^{\mathrm{loc}}_{\Lambda^*}$, but extending these near-surface properties to all of $\mathcal{K}_{\Lambda^*}$ is challenging. One may imagine an argument akin to the one in the proof of Proposition 4.22, where one works in a neighborhood of the surface, only this time for the nodal properties. Despite much effort, we have not been able to obtain conclusive results from this, and certainly nothing close to being able to rule out alternative (B). An additional challenge, is that there are *more* trivial solutions of (4.2) than those we have described; namely those with a flat surface $\eta \neq 0$. It could indeed be that the curve loops back to the original bifurcation point, by first passing through one of these trivial solutions.

For these reasons, and more, we will leave the matter of exploring these alternatives to future work. Still, there are certain things we can quite easily conclude, and which we find worth mentioning. For instance, from (4.2b$\star$)

we can immediately infer the upper bound

$$\eta(t) < \frac{1}{2}(\lambda^*)^2 \tag{4.47}$$

for every curve parameter $t \in \mathbb{R}$, where the inequality is strict since there are no stagnation points on the surface. Moreover, we note that *if* $\mathcal{K}_{\Lambda^*}$ were to have a subsequence ending in a wave of greatest height, with a stagnation point at the crest, then it would necessarily have surface deviation η precisely equal to the right-hand side of (4.47) at the crest.

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APPENDIX A

PERIODIC POINT VORTICES ON FINITE DEPTH

It is possible to obtain periodic analogs of the results in Paper 1, but we opted not to do so at the time because they would be less explicit. It is not at all obvious how to construct the conformal map for Theorem 1.30, but we can use a more brute force method: By summing the stream functions for the localized waves, it is evident that

$$\Phi(x, y) = \frac{1}{4\pi} \sum_{k=-\infty}^{\infty} \log\left(\frac{\cosh(\pi(x - 2kl)/h) + \cos(\pi(y/h - \theta))}{\cosh(\pi(x - 2kl)/h) + \cos(\pi(y/h + \theta))}\right) \tag{A.1}$$

is the appropriate generalization of the stream function in Proposition 1.5 for describing $2l$-periodic waves. We will show that there is a nice closed form representation of Φ in terms of so-called Weierstrass functions, which is convenient for computation.

Definition A.1 (Weierstrass functions)**.** The Weierstrass sigma function $\sigma(\cdot; \Lambda)$ corresponding to a lattice $\Lambda \subset \mathbb{C}$ is defined as the product

$$\sigma(z; \Lambda) \coloneqq z \prod_{w \in \Lambda \setminus \{0\}} \left(1 - \frac{z}{w}\right) \exp\left(\frac{z}{w} + \frac{1}{2}\left(\frac{z}{w}\right)^2\right),$$

while the Weierstrass zeta function $\zeta(\cdot; \Lambda)$ associated with the same lattice is its logarithmic derivative,

$$\zeta(z; \Lambda) \coloneqq \frac{1}{z} + \sum_{w \in \Lambda \setminus \{0\}} \left(\frac{1}{z - w} + \frac{1}{w} + \frac{z}{w^2}\right).$$

If we fix the lattice

$$\Lambda \coloneqq 2h\mathbb{Z} \times 2l\mathbb{Z} \subset \mathbb{C},$$

and introduce

$$q \coloneqq \exp(-\pi l/h)$$

then it is well known, [2, (6.5.2)], that the resulting sigma function can be expressed in the form

$$\sigma(z) = \frac{2h}{\pi} \sin\left(\frac{\pi z}{2h}\right) \exp\left(\frac{\zeta(h) z^2}{2h}\right) \prod_{j=1}^{\infty} \frac{1 - 2q^{2j} \cos(\pi z/h) + q^{4j}}{(1 - q^{2j})^2},$$

where we have suppressed the lattice from the notation. In particular,

$$\frac{\sigma(z+a)}{\sigma(z-a)} = \exp\left(\frac{2\zeta(h)az}{h}\right)\frac{\sin\left(\frac{\pi(z+a)}{2h}\right)}{\sin\left(\frac{\pi(z-a)}{2h}\right)}\prod_{j=1}^{\infty}\frac{1-2q^{2j}\cos\left(\frac{\pi(z+a)}{h}\right)+q^{4j}}{1-2q^{2j}\cos\left(\frac{\pi(z-a)}{h}\right)+q^{4j}}$$

for any fixed $a \in \mathbb{C}$.

If we now write $z = y + ix$ and let $a := (1-\theta)h$, then one may check that

$$\begin{aligned}
\left|\exp\left(\frac{2\zeta(h)az}{h}\right)\right|^2 &= \exp\left(\frac{4\zeta(h)ay}{h}\right), \\
\left|\frac{\sin(\pi(z+a)/(2h))}{\sin(\pi(z-a)/(2h))}\right|^2 &= \frac{\cosh(\pi x/h)-\cos(\pi(y+a)/h)}{\cosh(\pi x/h)-\cos(\pi(y-a)/h)}.
\end{aligned}$$

and

$$\left|\prod_{j=1}^{\infty}\frac{1-2q^{2j}\cos\left(\frac{\pi(z+a)}{h}\right)+q^{4j}}{1-2q^{2j}\cos\left(\frac{\pi(z-a)}{h}\right)+q^{4j}}\right|^2 = \prod_{j\in\mathbb{Z}\setminus\{0\}}\frac{\cosh\left(\frac{\pi(x-2jl}{h}\right)-\cos\left(\frac{\pi(y+a)}{h}\right)}{\cosh\left(\frac{\pi(x-2jl}{h}\right)-\cos\left(\frac{\pi(y-a)}{h}\right)},$$

whence

$$\left|\frac{\sigma(z+a)}{\sigma(z-a)}\right|^2 = \exp\left(\frac{4\zeta(h)ay}{h}\right)\prod_{j\in\mathbb{Z}}\frac{\cosh\left(\frac{\pi(x-2jl}{h}\right)-\cos\left(\frac{\pi(y+a)}{h}\right)}{\cosh\left(\frac{\pi(x-2jl}{h}\right)-\cos\left(\frac{\pi(y-a)}{h}\right)},$$

and therefore finally

$$\begin{aligned}
\Phi(x,y) &= \frac{1}{4\pi}\log\left(\left|\frac{\sigma(z+a)}{\sigma(z-a)}\right|^2\exp\left(-\frac{4\zeta(h)ay}{h}\right)\right) \\
&= \frac{1}{2\pi}\log\left(\left|\frac{\sigma(z+a)}{\sigma(z-a)}\right|\right) - \frac{1-\theta}{\pi}\zeta(h)y
\end{aligned}$$

by comparison with (A.1)

It follows that (identifying $\mathbb{C}$ and $\mathbb{R}^2$ for simplicity)

$$\nabla^{\perp}\Phi = -\Phi_y + i\Phi_x = \frac{1}{2\pi}(\zeta(z-a)-\zeta(z+a)) + \frac{1-\theta}{\pi}\zeta(h),$$

where we note that

$$\zeta(w) = \frac{1}{w} + O(w^3)$$

as $w \to 0$; see [1, 18.5.5]. In particular,

$$\begin{aligned}\nabla^{\perp}(\Phi - \Gamma(x, y+a))(0, -a) &= \frac{1}{2\pi}\zeta(-2a) + \frac{1-\theta}{\pi}\zeta(h) \\ &= \frac{1}{2\pi}\zeta(2\theta h) - \frac{\theta}{\pi}\zeta(h)\end{aligned}$$

is the leading order velocity for periodic waves on finite depth, where we have used the quasiperiodicity of ζ [2, (6.6.4)].

This is consistent with the expression for solitary waves, cf. (1.18), as

$$\zeta(2\theta h) = 2\theta\zeta(h) + \frac{\pi}{2h}\cot(\pi\theta) + \frac{\pi}{h}\sum_{j=1}^{\infty}\frac{\sin(2\pi\theta)}{\cosh(2\pi jl/h) - \cos(2\pi\theta)}$$

by [2, p. 183], and therefore

$$\nabla^{\perp}(\Phi - \Gamma(x, y+a))(0, -a) = \frac{1}{4h}\cot(\pi\theta) + \frac{1}{2h}\sum_{j=1}^{\infty}\frac{\sin(2\pi\theta)}{\cosh(2\pi jl/h) - \cos(2\pi\theta)}$$

where the series vanishes exponentially in the limit $l \to \infty$. Moreover, each term always has the same sign as $\cot(\pi\theta)$, so the same sign change as for solitary waves happens when $\theta = 1/2$.

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