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Spin and vorticity with vanishing rigid-body rotation during shear in continuum mechanics

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ABSTRACT

A long-standing challenge in continuum mechanics has been how to separate shear deformation and corresponding shape changes from the rotation of a continuum. This can be obtained by a new decomposition of the spin tensor, i.e. of the skew part of the velocity gradient, into two parts, where one of them vanishes during shear flows. The same decomposition applies to the vorticity vector. In both cases, the two spin components are interpreted as generating plastic shear deformation and rigid body rotation. In continuum plasticity theories, the suggested rotational part of the spin tensor can be applied to avoid spurious behavior of the objective Lie derivatives of second order tensors, e.g. of the stress tensor. It provides a history-independent spin corresponding to a time-averaged angular velocity of the rotating line segments in a small homogeneous volume. In fluid mechanics, the new spin component can be used to quantify vortexes in shear flows and turbulent structures, and it provides a sound interpretation and generalization of the Δ and swirling-strength criteria for visualization of vortices.

1. Introduction

The spin tensor is the velocity gradient required to generate rigid body rotations. It is a skew symmetric tensor. In continuum theory descriptions, it is desirable to estimate a material rotation also during deformation, providing a coordinate system for the constitutive description of the material. However, when the material is changing shape, it is not possible to separate uniquely the rigid body rotation from the shape change. As a minimum requirement, the spin or spin component that generates the rotations must degenerate correctly for cases of rigid body rotations, for which only one solution is correct. A variety of spin models can do that, but all models so far, generate rotations during simple shear deformation or shear flows.

In continuum elasticity theory, a rotation will redistribute the stress between the components of the stress tensor. The calibration and validity of the constitutive description, at large strains, even the simplest case of linear isotropic elasticity, will depend on the chosen spin. To some extent the elastic constants can be re-calibrated to match different choices for the spin tensor applied. However, all previously known spin tensors introduce rotations during cases of simple shear that are not desired, unless the material generically reveals this type of behavior.

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In continuum plasticity theories most of the real materials respond anisotropically. The constitutive description of the variation of the strength in various directions must somehow co-rotate with the continuum. This rotation can be prescribed through a spin tensor that will generate the rotations. At large strains the choice of the spin tensor heavily influences the predictions of the model, e.g. with a back-stress description as by e.g. Colak (2004).

In hypo elastic-plastic material descriptions, the rate of the stress tensor must be given by an objective Lie derivative that does 'not depend on the frame of reference and can be based on a chosen spin. Two types of spin descriptions exist. In both cases the spin degenerates to the same spin for pure rigid body rotations, and the mathematical description must be invariant under rotations of the coordinate system.

In first type is a spin based on the current velocity gradient $\boldsymbol{L} = \nabla \boldsymbol{v}$, where $\boldsymbol{v}$ is the velocity. A quite general expression for the objective rate $\boldsymbol{\sigma}^{\Delta}$ of the stress tensor $\boldsymbol{\sigma}$, including the most common spin tenors of this type, can be expressed similar as suggested by (Hill, 1968).

$$\boldsymbol{\sigma}^{\Delta} = \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma} \cdot \boldsymbol{W} - \boldsymbol{W} \cdot \boldsymbol{\sigma} + \kappa \ \mathrm{tr}(\boldsymbol{D})\boldsymbol{\sigma} - m(\boldsymbol{\sigma} \cdot \boldsymbol{D} + \boldsymbol{D} \cdot \boldsymbol{\sigma}) \tag{1}$$

Here $\boldsymbol{D} = \frac{1}{2}(\boldsymbol{L} + \boldsymbol{L}^T)$ is the symmetric part of the velocity gradient, i.e. the strainrate tensor, while $\boldsymbol{W} = \frac{1}{2}(\boldsymbol{L} - \boldsymbol{L}^T)$ is the skew part, commonly denoted the spin tensor, which will be the case also here, while other spin tensors will be regarded as components of it. The most common choice is the Zaremba–Jaumann derivative (Jaumann, 1911; Zaremba, 1903), which is obtained with $\kappa = m = 0$, based on the spin description by the skew part of the velocity gradient. The Zaremba–Jaumann derivative is known to cause strong spurious oscillations at large strains (Dienes, 1979). Many other objective stress rates have been suggested, but here only the Truesdell rate (Truesdell, 1955) rate will be included as an example of a non-corotational objective rate, i.e. an objective rate that cannot equivalently be expressed by a co-rotational description in a co-ordinate system that rotates along with the material, as dictated by the spin tensor. The Truesdell rate is obtained by $\kappa = 1$ and $m = 0$, applying also the symmetric part of the velocity gradient tensor to estimate the rate. A comparison of objective stress rates are given by Szabo and Balla (1989).

The second type requires an evolution equation for the spin, which is based on the previous deformation history with appropriate initial conditions. The most common history tensor is the deformation gradient tensor $\boldsymbol{F}$, which describes the relation between a line segment $\boldsymbol{x}_0$ in the initial configuration and the line segment $\boldsymbol{x} = \boldsymbol{F} \cdot \boldsymbol{x}_0$ in the current configuration after homogeneous deformation. The evolution is given by

$$\dot{\boldsymbol{F}} = \boldsymbol{L} \cdot \boldsymbol{F} \tag{2}$$

The special case of a rigid body rotation is described by the spin $\dot{\boldsymbol{R}} \cdot \ \boldsymbol{R}^T$, where $\boldsymbol{R}$ is the rigid body rotation matrix.

More complex evolution equations for the spin depend on the detailed microstructure of the considered material and its evolution. An example is the crystal plasticity theory, in which plastic deformation occurs by simple shear on a limited number of slip systems. From the calculated slip activity, a contribution to the spin (the skew part of the velocity gradient) is obtained. The remaining counterpart of the spin can further be decomposed into spin components generating elastic and lattice rigid-body rotations. In metals, however, the elastic deformation spin component is very small, hence the lattice spin is approximately the difference between the total spin and this plastic spin component. The crystal plasticity theory inspired the idea that the plastic part of the spin can be modeled phenomenologically by continuum theories (Kratochvil, 1973; Mandel, 1973). This spin component is referred to as the plastic spin (Dafalias, 1984), whereas its counterpart, generating rigid body rotations of the substructure, is denoted the constitutive spin (Dafalias, 1998) or the Mandel spin (Dafalias, 1985; Peeters et al., 2001). Phenomenological models have been suggested based on tensor representation theory, e.g. with application for the behavior of soil, (Voyiadjis and Song, 2005), where a phenomenological correlation between the plastic spin and the back-stress tensor is applied, as proposed by Dafalias (1983).

In fluid mechanics a long-standing challenge has been to mathematically identify and quantify vortexes in the flow. In turbulence, vortex-like structures carry most of the energy and play an essential role. It is critically important to understand kinetic energy production and dissipation through the length and time scales, and a clear identification and quantification of these important vortex structures is required.

Progress was made in the 1980–1990s by several important contributions. The work by Chong et al. (1990), defines a vortex core to be the connected region in which the velocity gradient has complex eigenvalues. Similar approaches were taken by Hunt et al. (1988) by the Q method and Jeong and Hussain (1995) by the λ_2 method using another invariant of the velocity gradient tensor to identify the connected vortex regions.

Zhou et al. (1999) suggested the swirling strength method as a generalization of the Δ method, where the magnitude of the imaginary part of the complex conjugate pair of eigenvalues of the velocity gradient tensor was recognized as the strength of the vortex identified by the Δ method.

Ideally one would use the vorticity vector, which provides both a direction and a magnitude for the vortex description, but unfortunately it cannot distinguish shear flow from rotations. Dong et al. (2016) suggested to use a hybrid approach of the λ_2 iso-surface and vortex filaments to visualize the structures with directional information.

The latest progress in the identification of vortexes and the local rigid rotation of fluids is the eigenvector-based Liutex method (Liu et al., 2019, 2018), which involves a decomposition of the vorticity vector into rigid body rotational and shear components. Similar as for the swirling strength method and the Δ method, the rotation is recognized as occurring when two of the eigenvalues of the velocity gradient are complex conjugated, whereas there is no rotation if all three eigenvalues are real-valued. The direction of the vortex vector component of interest is assumed to be the same as the eigenvector of the real valued third eigenvalue. Its magnitude is estimated as the minimum angular rotation velocity obtained by any rotating line segment around this eigenvector axis. It has been proven that the Liutex vector is Galilean invariant (Wang et al., 2018). The physical meaning of the Liutex vector is discussed in Wang et al. (2019). They concluded that the absolute value of the imaginary part of the complex conjugate eigenvalue pair (if existing) of the velocity gradient tensor is a pseudo-time average angular velocity, while the Liutex vorticity is based on an estimate for the minimum angular velocity. A recent review of vortex identification methods, including the Liutex method, is given by Liu et al. (2019).

Obviously, the Liutex component of the vorticity vector also defines a corresponding spin tensor that can be applied in continuum elasticity and plasticity theories, and that does not generate rotations during simple shear deformation. However, since the Liutex method has been recently proposed within the fluid dynamics society, this important fact has so far not been realized by other authors. In the following, an alternative way of decomposing the vorticity vector will be derived, providing an alternative to the Liutex vector.

2. Spin, angular velocity and vorticity

In continuum mechanics, the deformation, i.e. rotation and shape change, of a small homogeneous volume element is described by the velocity gradient tensor $\boldsymbol{L}$.

This rigid body rotation can also be described by the angular velocity vector $\boldsymbol{\omega}$. During pure rigid body rotations, $\boldsymbol{L} = \boldsymbol{W}$ will be corresponding skew velocity gradient tensor, which is related to the angular velocity vector by that $\boldsymbol{W} \cdot \boldsymbol{x} = \boldsymbol{\omega} \times \boldsymbol{x}$ for any position vector $\boldsymbol{x}$.

$$\boldsymbol{W} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \tag{3}$$

The skew velocity gradient tensor $\boldsymbol{W}$ is denoted the spin tensor. Along with the corresponding angular velocity vector $\boldsymbol{\omega}$, it is defined also for cases that are not pure rigid body rotation, by the skew part $\boldsymbol{W} = \frac{1}{2}(\boldsymbol{L} - \boldsymbol{L}^{\boldsymbol{T}})$ of any velocity gradient tensor $\boldsymbol{L}$. The spin $\boldsymbol{W}$ then defines a rigid body rotation $\boldsymbol{R}$, for which $\dot{\boldsymbol{R}} \cdot \boldsymbol{R}^T = \boldsymbol{W}$. It follows that the angular velocity vector $\boldsymbol{\omega}$ (corresponding to $\boldsymbol{W}$) equals half of the vorticity vector, $\hat{\boldsymbol{\omega}}$.

$$\boldsymbol{\omega} = \frac{1}{2}\hat{\boldsymbol{\omega}} = \frac{1}{2}\nabla \times \boldsymbol{v} \tag{4}$$

where $\boldsymbol{v}$ is the velocity vector. The spin $\boldsymbol{W}$ may be regarded as the velocity gradient required to generate the rotation of an infinitesimal line segment in a homogeneous continuum, but for non-skew velocity gradient tensors, different line segments with different orientations will experience different angular velocities. In such cases, the angular velocity $\boldsymbol{\omega}$ and the spin tensor $\boldsymbol{W}$ estimate a rotation of the line segments contained by the considered small volume in the considered continuum.

3. The Liutex minimum spin tensor

In vector notation (Wang et al., 2019), the Liutex vorticity component $\hat{\omega}_{Liu}$ can be obtained as

$$\hat{\boldsymbol{\omega}}_{\mathrm{Liu}} = \mathrm{sign}\left(\hat{\boldsymbol{\omega}} \cdot \boldsymbol{x}_{\mathrm{r}}\right)\left(\left|\hat{\boldsymbol{\omega}} \cdot \boldsymbol{x}_{\mathrm{r}}\right| - \sqrt{\left(\hat{\boldsymbol{\omega}} \cdot \boldsymbol{x}_{\mathrm{r}}\right)^2 - 4\mathrm{Im}(\lambda_2)^2}\right)\boldsymbol{x}_{\mathrm{r}} \tag{5}$$

Here $\boldsymbol{x}_{\mathrm{r}}$ is the unit length normalized eigenvector (arbitrarily chosen in positive or negative direction) of the velocity gradient tensor $\boldsymbol{L}$, belonging to the real-valued eigenvalue. For cases when a complex conjugate eigenvalue pair with imaginary parts $\pm\,\mathrm{Im}(\lambda_2)$ does 'not exists, $\hat{\boldsymbol{\omega}}_{\mathrm{Liu}}=\boldsymbol{0}$.

The vorticity can be decomposed as

$$\hat{\boldsymbol{\omega}} = \hat{\boldsymbol{\omega}}_{\mathrm{Liu}} + \hat{\boldsymbol{\omega}}_{\mathrm{Liu,s}} \tag{6}$$

$$\hat{\boldsymbol{\omega}}_{\mathrm{Liu,s}} = \nabla \times \boldsymbol{v} - \hat{\boldsymbol{\omega}}_{\mathrm{Liu}} \tag{7}$$

Here $\hat{\boldsymbol{\omega}}_{\mathrm{Liu,s}}$ is the shear component, and $\hat{\boldsymbol{\omega}}_{\mathrm{Liu}}$ is the asymptotic rigid body rotation component. The corresponding angular velocity vector $\boldsymbol{\omega}_{\mathrm{Liu}} = \frac{1}{2}\hat{\boldsymbol{\omega}}_{\mathrm{Liu}}$, and the Liu spin tensor $\boldsymbol{\Omega}_{\mathrm{Liu}}$ can be expressed

$$\boldsymbol{\Omega}_{\mathrm{Liu}} = \frac{1}{2}\begin{bmatrix} 0 & -\hat{\omega}_{\mathrm{Liu}_3} & \hat{\omega}_{\mathrm{Liu}_2} \\ \hat{\omega}_{\mathrm{Liu}_3} & 0 & -\hat{\omega}_{\mathrm{Liu}_1} \\ -\hat{\omega}_{\mathrm{Liu}_2} & \hat{\omega}_{\mathrm{Liu}_1} & 0 \end{bmatrix} \tag{8}$$

When all three eigenvalues are real-valued, the Liu spin vanishes, e.g. in the simple shear deformation mode. Hence, the Liu spin is recognized as the firstly proposed proper spin tensor that does 'not generate rotation during pure shear deformation.

The velocity gradient can now formally be split into three parts

$$\boldsymbol{L} = \boldsymbol{\Omega}_{\mathrm{Liu}} + \boldsymbol{\Omega}_{\mathrm{Liu,s}} + \boldsymbol{D} \tag{9}$$

$$\mathbf{\Omega}_{\mathrm{Liu,s}} = \boldsymbol{W} - \mathbf{\Omega}_{\mathrm{Liu}} \tag{10}$$

Here $\mathbf{\Omega}_{\mathrm{Liu,\,s}}$ is an estimate of the shear part of the spin, i.e. the plastic spin. The other spin component $\mathbf{\Omega}_{\mathrm{Liu}}$ is the rotational spin and provides an estimate for the minimum contribution from rigid rotation with the real-valued eigenvector as the spin axis (for cases when the two other eigenvalues are complex valued).

4. A time averaged spin tensor

The main idea is to split the spin $\boldsymbol{W}$ into two components, where one part vanishes during pure shear flows. It is noted that during simple shear deformation, one of the line segments corresponding to one particular direction in the shear plane, will not rotate. The other line segments in this plane will rotate towards this orientation but never pass it. Hence, by maintaining the velocity gradient corresponding to simple shear, the asymptotic rotation of any line segment will eventually vanish. The estimate of the rotational spin component will be constructed based on the average angular velocity of a rotating line segment during time. In the case of simple shear, all segments will stop rotating and such a time average will vanish.

For a constant velocity gradient $\boldsymbol{L}$, an arbitrary line segment $\boldsymbol{x}$ will evolve according to

$$\dot{\boldsymbol{x}} = \boldsymbol{L}\cdot\boldsymbol{x} \tag{11}$$

Keeping $\boldsymbol{L}$ constant the general solution will be

$$\boldsymbol{x} = \sum_{i=1}^{3} \alpha_i \boldsymbol{x}_i \exp\left(\lambda_i t\right) \tag{12}$$

Here $\boldsymbol{x}_i$ are the complex-valued eigenvectors, λ_i the corresponding eigenvalues of $\boldsymbol{L}$ and α_i are complex valued constants. In rigid body rotation, i.e. when $\boldsymbol{L}$ is skew, two eigenvalues are complex conjugate and the third is zero. In the general case, either all three eigenvalues are real, corresponding to no rotation, or two of them are complex conjugate and one is real-valued. The general solution can be written in terms of real valued numbers. Two possible cases are then distinguished.

The first case corresponds to three real-valued eigenvalues λ_i and corresponding real-valued eigenvectors $\boldsymbol{x}_i$. In this case, the time averaged angular velocity vector of any rotating line segment, denoted $\boldsymbol{\omega}_{\mathrm{r}}$, will vanish, since the solution is not periodic.

The second case involves one real-valued eigenvalue λ_1 with corresponding real-valued eigenvector $\boldsymbol{x}_1$. Furthermore, the two other eigenvalues are complex conjugate, $\lambda_2 = \bar{\lambda}_3$, where the overline denotes the complex conjugate. The corresponding eigenvectors are also complex conjugate, i.e. $\boldsymbol{x}_2 = \bar{\boldsymbol{x}}_3$. The real-valued solution $\boldsymbol{x}$ can be written as:

$$\boldsymbol{x}\exp(-\mathrm{Re}(\lambda_2)t) = \boldsymbol{x}_1\hat{\alpha}_1\exp\left((\lambda_1 - \mathrm{Re}(\lambda_2))t\right) + \hat{\alpha}_2\left(\mathrm{Re}(\boldsymbol{x}_2)\cos\left(\mathrm{Im}(\lambda_2)t + \psi\right) + \mathrm{Im}(\boldsymbol{x}_2)\sin\left(\mathrm{Im}(\lambda_2)t + \psi\right)\right) \tag{13}$$

Here the phase ψ and $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are real valued constants that can be chosen to fulfill arbitrary initial conditions. The solution clearly has a periodic nature with a time period equal to $2\pi/|\mathrm{Im}(\lambda_2)|$, hence $\mathrm{Im}(\lambda_2)$ corresponds to the magnitude of the average angular velocity vector. The rotation plane is spanned by the two complex conjugate eigenvectors and the corresponding angular velocity vector $\boldsymbol{\omega}_{\mathrm{r}}$ can be derived as

$$\boldsymbol{\omega}_{\mathrm{r}} = \frac{\mathrm{Re}(\boldsymbol{x}_2)\times\ \mathrm{Im}(\boldsymbol{x}_2)}{|\mathrm{Re}(\boldsymbol{x}_2)\times\ \mathrm{Im}(\boldsymbol{x}_2)|}\ \mathrm{Im}(\lambda_2) \tag{14}$$

The corresponding spin tensor $\mathbf{\Omega}_{\mathrm{r}}$ for the rigid body rotation follows as

$$\mathbf{\Omega}_{\mathrm{r}} = \begin{bmatrix} 0 & -\omega_{\mathrm{r}_3} & \omega_{\mathrm{r}2} \\ \omega_{\mathrm{r}3} & 0 & -\omega_{\mathrm{r}1} \\ -\omega_{\mathrm{r}2} & \omega_{\mathrm{r}1} & 0 \end{bmatrix} \tag{15}$$

The velocity gradient can now formally be split into three parts

$$\boldsymbol{L} = \mathbf{\Omega}_{\mathrm{r}} + \mathbf{\Omega}_{\mathrm{s}} + \boldsymbol{D} \tag{16}$$

$$\mathbf{\Omega}_{\mathrm{s}} = \boldsymbol{W} - \mathbf{\Omega}_{\mathrm{r}} \tag{17}$$

Here the shear spin component, $\mathbf{\Omega}_{\mathrm{s}}$, is the counterpart of the rotational spin component $\mathbf{\Omega}_{\mathrm{r}}$. Following the discussion by Dafalias (1998), $\mathbf{\Omega}_{\mathrm{r}}$ may be regarded as the constitutive spin, since it determines the rotation of the coordinate system that describes the anisotropy of the stress tensor. Accordingly, $\mathbf{\Omega}_{\mathrm{s}}$, as its counterpart, is the plastic spin, with contributions from the shape change occurring without constitutive rotations. Note that the plastic spin in this interpretation is determined solely from the kinematical quantities, independently of the material properties. Alternatively, a separate constitutive model for the plastic spin is required, for which the constitutive spin follows as its counterpart and an explicit model for $\mathbf{\Omega}_{\mathrm{r}}$ isn't needed. This is the case in crystal plasticity theories, where the plastic spin is given by the slip activites, and the constitutive spin follows as the counterpart and can be applied to update the crystal orientation.

The other, rotational spin component $\mathbf{\Omega}_{\mathrm{r}}$ is given by the time averaged angular velocity vector, hence it can be regarded as the time averaged rotational spin. It provides an estimate for the time averaged contribution from rigid rotation to the

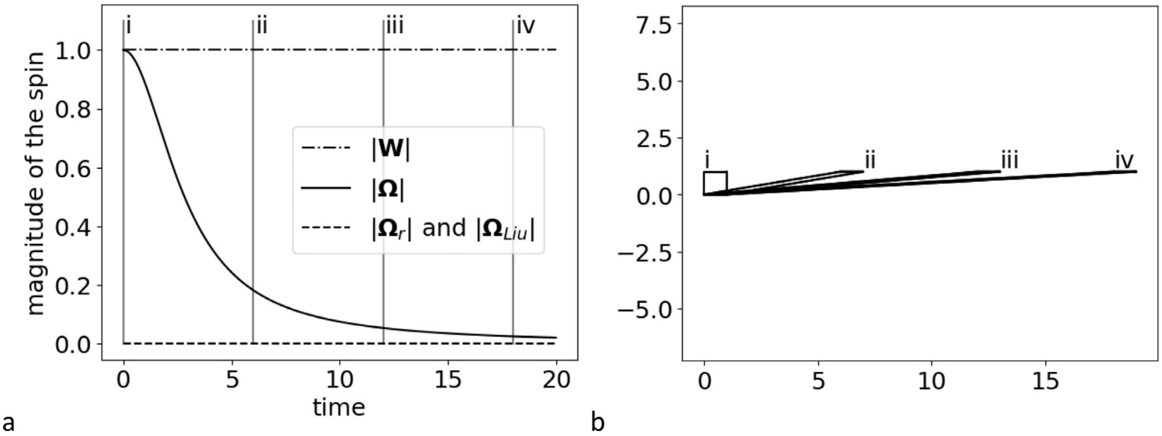

Fig. 1. Simple shear ($\beta = r = 0$). In (a), the L_2-norm of the spin, estimated by the skew part of the deformation gradient $\boldsymbol{W}$, the polar decomposition of the deformation gradient $\boldsymbol{\Omega}$, the new rigid rotation $\boldsymbol{\Omega}_{\mathrm{r}}$ and the Liu spin $\boldsymbol{\Omega}_{\mathrm{Liu}}$, are compared. In (b), the deformation of an initial square element in the x_1-x_2 plane is shown at the times indicated by vertical lines in a.

total spin. This estimate is consistent with the periodic nature of the deformation, occurring when the velocity gradient is held constant and has complex valued eigenvalues.

In fluid mechanics, the vorticity vector is commonly applied. It is a pseudovector field measuring both shear and rigid body rotation contributions to the spinning motion of a continuum. The vorticity can now be decomposed as

$$\hat{\boldsymbol{\omega}} = \hat{\boldsymbol{\omega}}_{\mathrm{r}} + \hat{\boldsymbol{\omega}}_{\mathrm{s}} \tag{18}$$

$$\hat{\boldsymbol{\omega}}_{\mathrm{r}} = 2\boldsymbol{\omega}_{\mathrm{r}}, \;\; \hat{\boldsymbol{\omega}}_{s} = \nabla \times \boldsymbol{v} - 2\boldsymbol{\omega}_{\mathrm{r}} \tag{19}$$

Here $\hat{\boldsymbol{\omega}}_{s}$ is the shear component, and $\hat{\boldsymbol{\omega}}_{\mathrm{r}}$ is the asymptotic rigid body rotation component of the vorticity vector.

5. Discussion

The following velocity gradient will be applied for the discussion.

$$\boldsymbol{L} = \begin{bmatrix} -2\beta & 1 & 0 \\ -r & 0 & 0 \\ 0 & 0 & 2\beta \end{bmatrix} \tag{20}$$

When $\beta = r = 0$, this spin tensor corresponds to simple shear, whereas with $r = 1$ and $\beta = 0$, it is the spin of a rigid-body rotation around the third axis. The eigenvalues of $\boldsymbol{L}$ are

$$\lambda_1 = 2\beta, \;\; \lambda_{2,3} = -\beta \pm i\sqrt{r - \beta^2} \tag{21}$$

For the velocity gradient tensor in Eq. (20), the rotation axis, i.e. the direction of the angular velocity vector, will be in the x_3 direction for all considered spin estimates, for all choices of the parameters β and r. For this velocity gradient the magnitude of the angular velocity equal $|\boldsymbol{\omega}_{\mathrm{Liu}}| = \frac{1}{2}(1 + r - \sqrt{(1-r)^2 + \beta^2})$ for the Liutex decomposition. Clearly, $|\boldsymbol{\omega}_{\mathrm{Liu}}| = r$ for cases when $\beta = 0$. The new decomposition has $|\boldsymbol{\omega}_{\mathrm{r}}| = \sqrt{r - \beta^2}$. Both angular velocities vanish for the case of pure shear ($r = 0$) and both are pointing in the x_3-direction for all selections of r and β.

The first case to consider is the simple shear flow, for which $\beta = r = 0$ in Eq. (20). The norm of the spins and the change of an initial square element in the $x_1 - x_2$ plane with increasing time are shown in Fig. 1. The spin $\boldsymbol{W}$ corresponds to the angular velocity experienced by a diagonal line segment in the initial square element in Fig. 1(b). The angular velocity equals $\omega_3 = |\boldsymbol{\omega}| = \sqrt{2}|\boldsymbol{W}|/2$. The corresponding vorticity equals $\hat{\omega}_3 = \sqrt{2}|\boldsymbol{W}|$. The spin $\boldsymbol{\Omega}$ based on the polar decomposition, corresponds to updated angular velocity following the rotation of the initially diagonal line segment. In the new decomposition of the spin, the entire spin is shear induced, i.e. $\boldsymbol{\Omega}_{\mathrm{s}} = \boldsymbol{W}$. The rigid rotation component left is $\boldsymbol{\Omega}_{\mathrm{r}} = 0$. This spin equals the time average of any line segment and also the time average of the spin $\boldsymbol{\Omega}$, if the time-average is made over a sufficiently long period of time. The Liu spin corresponds to the non-rotating horizontal line segment, hence $\boldsymbol{\Omega}_{\mathrm{Liu}} = 0$ for simple shear.

Light can be shed on the limiting case of simple shear, by inspecting the superposition of a small rotation to it, as illustrated with $r = 0.05$ in Fig. 2. The element then rotates as illustrated in Fig. 2(b), with a period equal to $2\pi/|\mathrm{Im}(\lambda_2)|$, i.e. a time period of 28.1 in this example. The spin by $\boldsymbol{W}$ corresponds to the largest angular velocity amongst the spins shown in Fig. 2, but it is still lower than the angular velocity of the fastest rotating vertical line segment. The polar spin $\boldsymbol{\Omega}$ is periodic, but with a strong asymmetry between the lower and upper part of its oscillations. The upper part is a steep peek

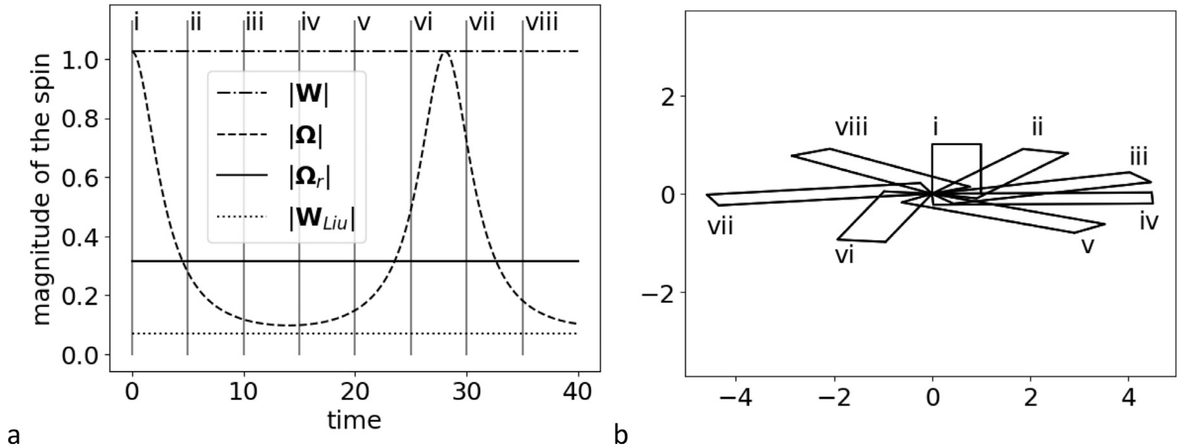

Fig. 2. Simple shear with slight rotation ($\beta = 0, r = 0.05$). In (a), the L_2-norm of the spin, estimated by the skew part of the deformation gradient $\boldsymbol{W}$, the polar decomposition of the deformation gradient $\boldsymbol{\Omega}$, the new rigid rotation $\boldsymbol{\Omega}_{\mathrm{r}}$ and the Liu spin $\boldsymbol{\Omega}_{\mathrm{Liu}}$, are compared. In (b), the deformation of an initial square element in the $x_1 - x_2$ plane is shown at the times indicated by vertical lines in (a).

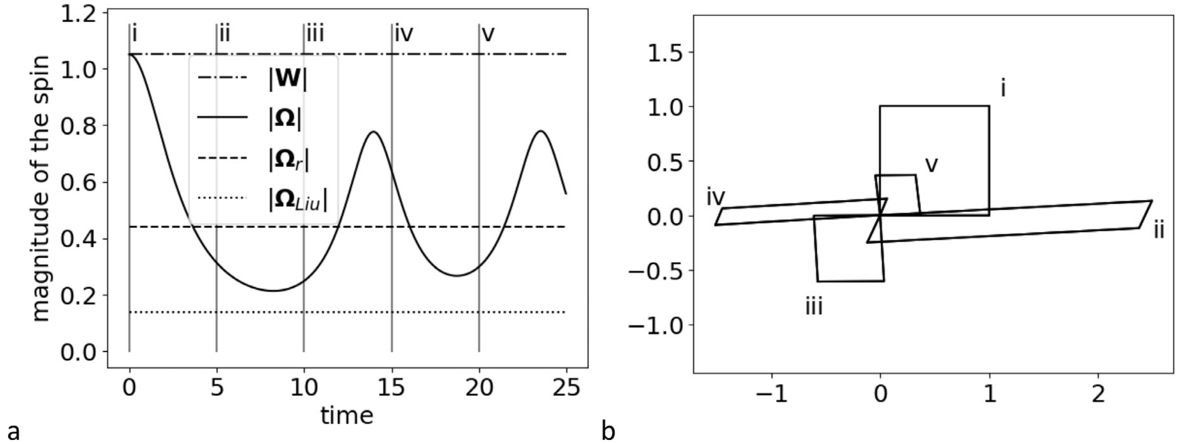

Fig. 3. Simple shear with rotation and transverse tension ($\beta = 0.05, = r = 0.1$). In (a), the L_2-norm of the spin, estimated by the skew part of the deformation gradient $\boldsymbol{W}$, the polar decomposition of the deformation gradient $\boldsymbol{\Omega}$, the new rigid rotation $\boldsymbol{\Omega}_{\mathrm{r}}$ and the Liu spin $\boldsymbol{\Omega}_{\mathrm{Liu}}$, are compared. In (b), the deformation of an initial square element in the $x_1 - x_2$ plane is shown at the times indicated by vertical lines in a.

that lasts shortly, whereas the lower part is flat and lasts long. In the limit of simple shear, the start is at the top of the short upper peak, but the second upper peak is shifted to infinite long times and is therefore never reached. The Liu spin $\boldsymbol{\Omega}_{\mathrm{Liu}}$ here corresponds to the rotation of the slowest rotating line segment in the $x_1 - x_2$ plane, i.e. a line segment in the x_1 direction. Note that in this case, the magnitude of the spin from the polar decomposition, $|\boldsymbol{\Omega}|$, stays always larger than $|\boldsymbol{\Omega}_{\mathrm{Liu}}|$, which means it provides larger angular velocities than the minimum angular velocity of the segments orthogonal to its spin axis. Neither will it ever reach the angular velocity of the fastest rotating line segment. The new rotation spin $\boldsymbol{\Omega}_{\mathrm{r}}$, reflects an average of the angular velocities of a segment rotating in the $x_1 - x_2$ plane, and its magnitude remains between $|\boldsymbol{W}|$ and $|\boldsymbol{\Omega}_{\mathrm{Liu}}|$ and equals the time average of the polar spin $|\boldsymbol{\Omega}|$.

A more general example is illustrated in Fig. 3, where an elongation in the x_3 direction, by $\beta = 0.05$, is superposed to simple shear in the $x_1 - x_2$ plane with a rotation around the x_3 axis, by $r = 0.1$. The volume is conserved during this deformation mode, but contractions in the $x_1 - x_2$ plane make the initial square element in Fig. 3(b) shrink in this plane. In addition, the shear deformation changes the shape periodically, in a similar manner as in Fig. 2. In this case, the Liu spin $\boldsymbol{\Omega}_{\mathrm{Liu}}$, and the new rigid spin $\boldsymbol{\Omega}_{\mathrm{r}}$ have the same spin axis, i.e. in the x_3 direction, but the magnitude of the Liu spin is considerable smaller than the magnitude of the polar spin $\boldsymbol{\Omega}$ at any time, while $|\boldsymbol{\Omega}_{\mathrm{r}}|$ approaches the time average of $|\boldsymbol{\Omega}|$.

In all the examples in Figs. 1–3, the new rigid spin $\boldsymbol{\Omega}_{\mathrm{r}}$ approaches the time average of the polar spin $\boldsymbol{\Omega}$. Note, that while the Liu spin $\boldsymbol{\Omega}_{\mathrm{Liu}}$, corresponds to the slowest rotating line segment around the eigenvector (the x_3-axis in the examples here) of the real-valued eigenvalue of $\boldsymbol{L}$, it is a trivial modification to formulate a similar spin of the fastest rotating line segment instead. This spin would have a magnitude larger than both $|\boldsymbol{W}|$ and the maximum of $|\boldsymbol{\Omega}|$. In the case of simple shear in Fig. 1, the maximum spin would correspond to the angular velocity of the vertical line segment, as compared to of the diagonal by $\boldsymbol{W}$.

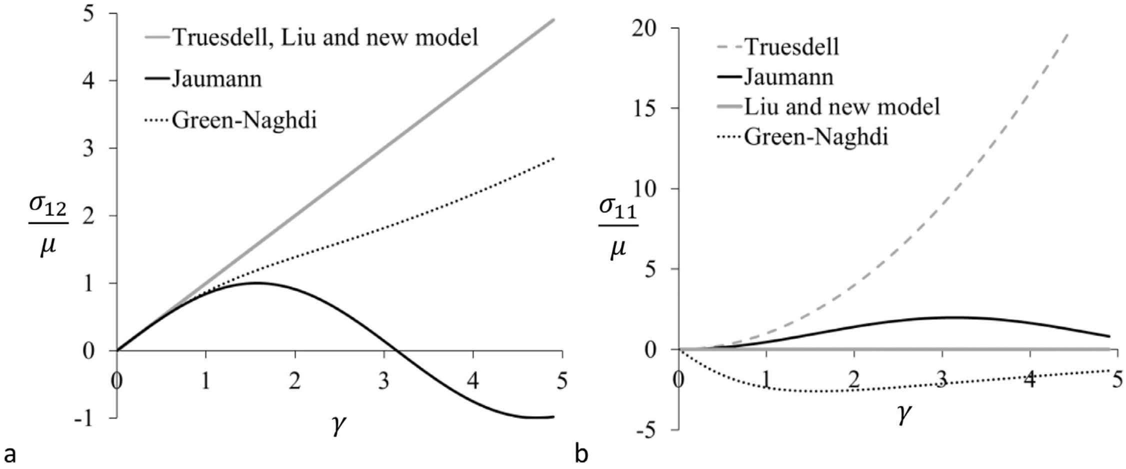

Fig. 4. Simple shear of a linear elastic isotropic material. Comparison of evolution of (a) shear stress and (b) normal stress, as functions of the engineering shear strain γ, obtained by the Zaremba–Jaumann rate, the Green–Naghdi rate, the Truesdell rate, the Liu rate and the stress rate based on the new spin $\boldsymbol{\Omega}_{\mathrm{r}}$.

A major advantage with $\boldsymbol{\Omega}_{\mathrm{r}}$ and $\boldsymbol{\Omega}_{\mathrm{Liu}}$ is that both vanish during simple shear. This is illustrated in Fig. 4 for the classical example of simple shear and isotropic Hooke elasticity, given by:

$$\boldsymbol{\sigma}^{\Delta} = 2\mu\boldsymbol{D} + \lambda\boldsymbol{I}\ \mathrm{tr}(\boldsymbol{D}) \tag{22}$$

Here μ and λ are the Lamé constants and $\boldsymbol{I}$ is the identity matrix. The objective rate $\boldsymbol{\sigma}^{\Delta}$ is obtained similar as in the Jaumann and Green–Naghdi stress rates but by applying $\boldsymbol{\Omega}_{\mathrm{r}}$ or $\boldsymbol{\Omega}_{\mathrm{Liu}}$ as the spin, here formulated for $\boldsymbol{\Omega}_{\mathrm{r}}$:

$$\boldsymbol{\sigma}^{\Delta} = \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\cdot\boldsymbol{\Omega}_{\mathrm{r}} - \boldsymbol{\Omega}_{\mathrm{r}}\cdot\boldsymbol{\sigma} \tag{23}$$

Note that $\boldsymbol{\Omega}_{\mathrm{r}}$ or $\boldsymbol{\Omega}_{\mathrm{Liu}}$ are spins that generate rotations $\dot{\mathbf{R}} = \boldsymbol{\Omega}_{\mathrm{r}}\cdot\mathbf{R}$ and $\dot{\mathbf{R}} = \boldsymbol{\Omega}_{\mathrm{Liu}}\cdot\mathbf{R}$, respectively, from which corresponding co-rotational formulations can be made. The elastic shear result is compared for Liu spin $\boldsymbol{\Omega}_{\mathrm{Liu}}$, $\boldsymbol{\Omega}_{\mathrm{r}}$, $\boldsymbol{W}$ (Zaremba–Jaumann rate) and $\boldsymbol{\Omega}$ (Green–Naghdi rate) in Fig. 4. In addition, the non-co-rotational Truesdell rate, based on $\boldsymbol{L}$, is shown. The Green–Naghdi rate behaves oscillatory, but less than the Jaumann rate. The Truesdell rate transfers the stress to the σ_{11} component. The Liu spin $\boldsymbol{\Omega}_{\mathrm{Liu}}$ and the new spin $\boldsymbol{\Omega}_{\mathrm{r}}$ both ensure that the linear elasticity law gives a linear evolution with the engineering shear strain γ.

The main part of this work is focusing on spin estimates for finite continuum deformations, of importance for elasticity and plasticity theories. However, in fluid mechanics, material flow usually is described by the vorticity vector, and the decomposition of the vorticity vector is of importance.

The Δ-criterion and the swirling-strength criterion state that vorticity is obtained in the flow field at locations where the imaginary part of the complex conjugate eigenvalue pair $\lambda_{2,3}$ of the velocity gradient tensor is non-zero. As a result of the theory above, the vorticity component $\hat{\boldsymbol{\omega}}_{\mathrm{r}}$ defines the missing rotation axis and provides a clear interpretation of these earlier proposed criteria.

Another recent attempt to decompose the vorticity vector, is by the Liutex method. Instead of the cross product of the complex conjugated eigenvectors, the eigenvector of the real valued eigenvalue is taken as the direction of the desired vorticity vector component. In cases where the two other eigenvalues do 'not form a complex conjugate pair, this vorticity component vanishes. Also the magnitude of Liu's vorticity component is estimated in a different way than for $\hat{\boldsymbol{\omega}}_{\mathrm{r}}$. One interesting point is that the introduction of a shear component out of the rotation plane, will alter the eigenvector with the real-valued eigenvalue. A simple example can be the following velocity gradient tensor:

$$\boldsymbol{L} = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{24}$$

In this case, both the vorticity $\hat{\boldsymbol{\omega}} = [0, 1, -2]$, the Liutex vector $\hat{\boldsymbol{\omega}}_{\mathrm{Liu}} = [0, 1, -1]$ will have directions that are not perpendicular to the x_1–x_2 plane. The direction of the Liutex vector is given by the eigenvector of the real valued eigenvalue of $\boldsymbol{L}$ in Eq. (24), while the direction of the new rigid body rotation component of the vorticity, $\hat{\boldsymbol{\omega}}_{\mathrm{r}} = [0, 0, -2]$, is given by the eigenvectors of the complex conjugate eigenvalues, with a spin axis in the x_3 direction. The Liutex $\hat{\boldsymbol{\omega}}_{\mathrm{Liu}}$ and the $\hat{\boldsymbol{\omega}}_{\mathrm{r}}$ vorticity components are related, both being Galilean invariants designed to remove the contribution from shear flow to the vorticity, but they are different, both in terms of direction and magnitude in general cases. It will be of importance to apply and test $\hat{\boldsymbol{\omega}}_{\mathrm{r}}$ to visualize fluid flows, but the scope of this work is limited to present the new theory required to do that.

6. Conclusions

A new decomposition of the spin tensor into a part that vanishes during shear and a rotational counterpart, provides objective stress rates in continuum plasticity theories without spurious oscillations or undesired behavior, e.g. during simple shear of an isotropic Hooke elastic material. It is pointed out that the recent Liutex vorticity vector decomposition also provides a comparable spin tensor decomposition with a component that vanishes during shear deformations. In comparison to the polar decomposition, the Liu spin and the new spin suggested here, are history-independent. In fluid mechanics, the corresponding new decomposition of the vorticity, provides an alternative to the recent Liutex decomposition, enabling a new way to quantify and visualize vortexes without shear-flow contributions in fluid flows, e.g. in turbulent structures.

Declaration of Competing Interest

None.

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