FROM CLASSICAL TILTING TO TWO-TERM SILTING

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ABSTRACT. We survey some recent results generalizing classical tilting theory to a theory of two-term silting objects. In particular this includes a generalized Brenner-Butler theorem, and a homological characterization of algebras obtained by two-term silting from hereditary algebras.

INTRODUCTION

The fundamental idea of tilting theory is to relate the module categories of two algebras using so-called tilting/cotilting modules and associated tilting functors. Each of the two algebras involved can be obtained as an endomorphism algebra of a tilting or cotilting module over the other algebra.

The motivation for tilting theory stems from the introduction of reflection functors by Bernstein, Gelfand and Ponomarev [BGP]. Such functors were used to relate representations of two quivers, and in particular to prove Gabriel's theorem [G]. Auslander, Platzeck and Reiten [APR] gave a module theoretic version, and the concept was generalized by Brenner and Butler [BB] who introduced tilting functors. Happel and Ringel [HR] then defined tilted algebras and tilting modules as further generalizations of this.

Tilted algebras have particularly nice homological properties, namely: each indecomposable module has either projective or injective dimension at most one, and the global dimension is at most two. These properties do however not characterize tilted algebras, they also hold for Ringel's canonical algebras [R]. Later, tilting in abelian categories and quasi-tilted algebras were introduced by Happel, Reiten and Smalø [HRS]. The class of quasi-tilted algebras includes both tilted algebras and canonical algebras, and the above homological property actually characterizes quasi-tilted algebras. All these classical results are discussed in more detail in Section 1.

Silting complexes in the derived category were first introduced by Keller and Vossieck [KV]. We consider a particular type of silting complexes, those which are represented by a map between two finitely generated projectives, that is *two-term silting objects*. In joint work with Zhou [BZ1], we gave a version of the Brenner-Butler tilting theorem for this setting. In [BZ2], we gave a homological characterization of the *silted algebras*, which are the algebras occurring as endomorphism algebras of two-term silting objects in hereditary module categories (or more generally certain hereditary abelian categories). In a third paper [BZ3], we considered global dimensions of endomorphism algebras of two-term silting objects in more general module categories. The main results of these three papers are discussed in Section 2.

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1. Classical tilting, cotilting and quasi-tilting

Let Λ denote a basic finite dimensional k-algebra over a field k. We consider the category $\operatorname{mod}\Lambda$ of left Λ-modules. All modules are considered to be basic, when possible. For a basic module M, we let $\delta(M)$ denote the number of indecomposable direct summands of M. We let $\operatorname{add} M$ denote the full subcategory whose objects are isomorphic to direct summands in direct sums of copies of M. We always assume $\delta(\Lambda) = n$, for some positive integer n. We let $D = \operatorname{Hom}_k(-, k)$ be the ordinary duality.

1.1. Tilting and cotilting modules.

We let $\operatorname{pd} M$ and $\operatorname{id} M$ denote the projective and injective dimensions of a module M in $\operatorname{mod}\Lambda$.

Definition 1.1. *A module T in $\operatorname{mod}\Lambda$ is called a tilting (cotilting) module if*

(T1) *We have $\operatorname{pd} T \leq 1$ (we have $\operatorname{id} T \leq 1$.)*
(T2) *We have $\operatorname{Ext}^1(T, T) = 0$.*
(T3) *We have $\delta(T) = n$.*

Note that for a hereditary algebra, (T1) is void, and hence in this case a module is tilting if and only if it is cotilting.

Example 1.2. *Let $\Lambda = kQ$, where Q is the quiver $1 \xrightarrow{a} 2 \xrightarrow{b} 3$. Let S_i denote the simple corresponding to vertex i, and let P_i denote its projective cover. Then $T = P_1 \amalg P_3 \amalg S_3$ is a tilting (and cotilting) module with $\operatorname{End}_\Lambda(T)$ isomorphic to kQ/I, where I is the ideal generated by the path ba.*

1.2. Torsion pairs.

We briefly recall the notion of a torsion pair. For a full subcategory $\mathcal{X}$ of $\operatorname{mod}\Lambda$, we consider the full subcategory $\mathcal{X}^\perp = \{Y \in \operatorname{mod}\Lambda \mid \operatorname{Hom}(X, Y) = 0 \text{ for all } X \in \mathcal{X}\}$, and the similarly defined full subcategory ${}^\perp\mathcal{X}$.

Definition 1.3. *Let $(\mathcal{T}, \mathcal{F})$ be a pair of full subcategories of $\operatorname{mod}\Lambda$. Then $(\mathcal{T}, \mathcal{F})$ is called a torsion pair if $\mathcal{T} = {}^\perp\mathcal{F}$ and $\mathcal{F} = \mathcal{T}^\perp$.*

For a module M in $\operatorname{mod}\Lambda$, there is an exact sequence

$$0 \to tM \to M \to fM \to 0$$

with $tM \in \mathcal{T}$ and $fM \in \mathcal{F}$. The sequence is unique (up to isomorphism) and it is called the *canonical sequence* of M with respect to $(\mathcal{T}, \mathcal{F})$.

1.3. The Brenner-Butler tilting theorem.

Let T be a tilting module in $\operatorname{mod}\Lambda$. We let $\operatorname{Gen} T$ be the full subcategory of $\operatorname{mod}\Lambda$ whose objects are the factors of modules in $\operatorname{add} T$, and we let $\operatorname{Sub} T$ denote the full subcategory of $\operatorname{mod}\Lambda$ whose objects are the submodules of modules in $\operatorname{add} T$. Tilting and cotilting modules give rise to torsion pairs.

Proposition 1.4. (a) *For a tilting module T in $\operatorname{mod}\Lambda$, the pair*

$$(\mathcal{T}, \mathcal{F}) = (\operatorname{Gen} T, (\operatorname{Gen} T)^\perp)$$

is a torsion pair. Moreover, we have $\operatorname{Gen} T = \ker \operatorname{Ext}^1(T, -)$ and $(\operatorname{Gen} T)^\perp = \ker \operatorname{Hom}(T, -)$.

(b) *For a cotilting module U in* $\operatorname{mod}\Lambda$, *the pair*

$$(\mathcal{T}, \mathcal{F}) = ({}^{\perp}(\operatorname{Sub} U), \operatorname{Sub} U)$$

is a torsion pair.

We now recall the Brenner-Butler tilting theorem.

Theorem 1.5. [BB, HR] *Let T be a tilting module in* $\operatorname{mod}\Lambda$, *and let* $\Gamma = \operatorname{End}_{\Lambda}(T)$.

(a) *The module $D(T)$ is a cotilting Γ-module and* $\Lambda \simeq \operatorname{End}_{\Gamma}(D(T))$.
(b) *The functor* $\operatorname{Hom}(T, -)$ *restricts to an equivalence* $\operatorname{Gen} T \to \operatorname{Sub} D(T)$.
(c) *The functor* $\operatorname{Ext}^1(T, -)$ *restricts to an equivalence* $(\operatorname{Gen} T)^{\perp} \to {}^{\perp}(\operatorname{Sub} D(T))$.

1.4. **Tilted algebras.** The algebras which occur as endomorphism algebras of tilting modules over hereditary algebras are of particular interests, and their study was initiated in [HR].

Definition 1.6. *Let H be a hereditary finite dimensional algebra, and T a tilting module in* $\operatorname{mod} H$. *Then* $\Gamma = \operatorname{End}_H(T)$ *is called a* tilted algebra.

A torsion pair $(\mathcal{T}, \mathcal{F})$ is called *split* is for each indecomposable module M we have either $M \in \mathcal{T}$ or $M \in \mathcal{F}$.

Proposition 1.7. [HR] *Let T be a tilting module over an hereditary algebra H. Then the torsion pair* $({}^{\perp}(\operatorname{Sub} D(T), \operatorname{Sub} D(T))$ *in* $\operatorname{mod}\Gamma$ *is a split torsion pair.*

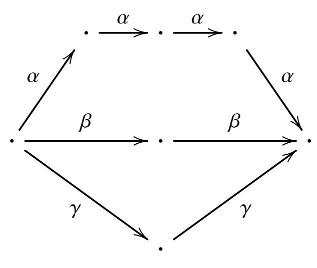
Proposition 1.7 is an important ingredient for proving the following.

Theorem 1.8. [HR] *Let* $\Gamma = \operatorname{End}_H(T)$ *be a tilted algebra. Then the following hold.*

(S1) *For each indecomposable M in* $\operatorname{mod}\Gamma$, *we have that either* $\operatorname{id} M \leq 1$ *or* $\operatorname{pd} M \leq 1$.
(S2) $\operatorname{gl.dim}\Gamma \leq 2$.

However, (S1) and (S2) do not characterize tilted algebras. Ringel's canonical algebras also satisfies these properties and are in general not tilted.

Example 1.9. *Consider the quiver Q*



and let I be the ideal generated by $\alpha + \beta - \gamma$. Then kQ/I is an example of a canonical algebra. By varying the length (≥ 2) of the paths and number of paths, and properly defining relations, one obtains all canonical algebras.

1.5. **Hereditary abelian categories and quasi-tilted algebras.** Geigle and Lenzing [GL] introduced certain curves called *weighted projective lines*, relative to a sequence of integers (the weight sequence) $(p_1, \ldots, p_t)$ with $p_i \geq 2$. For a weighted projective line $\mathbb{X}$ they showed that the category of coherent sheaves $\operatorname{coh}(\mathbb{X})$ is an abelian hereditary category which admits tilting objects in the following sense.

Definition 1.10. *An object T in a hereditary abelian category $\mathcal{H}$ is called a* tilting object *if*

(a) *We have* $\mathrm{Ext}^1(T, T) = 0$.
(b) *If* $\mathrm{Hom}(T, X) = 0 = \mathrm{Ext}^1(T, X)$ *for an object X in $\mathcal{H}$, then $X = 0$.*

The canonical algebras were shown to occur as endomorphism algebras of tilting objects in the category of coherent sheaves over a weighted projective line, where the weight sequence determines the length of the paths.

Happel, Reiten and Smalø [HRS] introduced the following class of algebras.

Definition 1.11. *If $\mathcal{H}$ is a hereditary abelian category with a tilting object T, then* $\mathrm{End}_{\mathcal{H}}(T)$ *is called a quasi-tilted algebra.*

So, by the above, both tilted and canonical algebras are quasi-tilted. Happel [H3] later proved all quasi-tilted algebras are either tilted or derived equivalent to canonical algebras.

1.6. **Algebras of small homological dimension.** Inspired by the homological properties of tilted and quasi-tilted algebras, the following two classes of algebras were introduced.

Definition 1.12. *Let Λ be a finite dimensional algebra.*

(a) Λ *is said to be of* small homological dimension *(shod), if for each indecomposable module X, we have either* $\mathrm{pd}\, X \leq 1$ *or* $\mathrm{id}\, X \leq 1$.
(b) *A shod algebra Λ is said to be* almost hereditary *if in addition* $\mathrm{gl.dim}\, \Lambda \leq 2$.

Almost hereditary algebras were first introduced in [HRS], and later Coelho and Lanzilotta [CL] introduced the shod algebras.

Happel, Reiten and Smalø proved the following characterization.

Theorem 1.13. [HRS] *Λ is almost hereditary if and only if it is quasi-tilted.*

They also proved the following.

Proposition 1.14. [HRS] *A shod algebra has global dimension at most three.*

A shod algebra Λ is called *strictly shod* [CL], if $\mathrm{gl.dim}\, \Lambda = 3$. In other words, a shod algebra is either strictly shod or it is quasi-tilted (or equivalently almost hereditary).

Before concluding our summary of classical results concerning tilting and cotilting, we should also point out that a different characterization of shod algebras, in terms of certain so-called double sections of the AR-quiver, was given by Reiten and Skowronski in [RS].

2. Two-term silting objects

We now turn our attention from tilting and cotilting objects in module categories to silting objects in derived categories. Our aim in this section is to summarize results from [BZ1, BZ2, BZ3], including giving the necessary background for these results.

Let $\mathcal{P}(\Lambda)$ denote the full subcategory of $\mathrm{mod}\, \Lambda$ of finitely generated projective Λ-modules. Let $K^b(\mathcal{P}(\Lambda))$ be the bounded homotopy category of complexes of projectives, which we regard as a full subcategory of the bounded derived category $D^b(\Lambda)$ which is equivalent to $K^{-,b}(\mathcal{P}(\Lambda))$, the category of complexes of projectives, bounded to the right, and bounded in homology to the left.

Definition 2.1. *A complex $\mathbb{P}$ in $K^b(\mathcal{P}(\Lambda))$ is called a two-term silting complex if*

(a) *It is of the form*

$$\cdots 0 \to 0 \to P^{-1} \to P^0 \to 0 \to 0 \cdots$$

(b) $\mathrm{Hom}(\mathbb{P}, \mathbb{P}[1]) = 0$

(c) *It generates $K^b(\mathcal{P}(\Lambda))$ in the sense that it is contained in no proper triangulated subcategory of $K^b(\mathcal{P}(\Lambda))$.*

Tilting modules give rise to two-term silting complexes in the following way: Let T be a tilting module in $\mathrm{mod}\,\Lambda$, and let

$$P^{-1} \xrightarrow{u} P^0 \to T \to 0$$

be a projective presentation. Then

$$\mathbb{P} = \cdots \to 0 \to 0 \to P^{-1} \xrightarrow{u} P^0 \to 0 \to 0 \cdots$$

is a two-term silting object. On the other hand, for a two-term silting complex $\mathbb{P}$ in $K^b(\mathcal{P}(\Lambda))$ we have that $T = H^0(\mathbb{P})$ is a tilting module over $\Lambda/\operatorname{ann} T$, while $U = H^{-1}(\nu\mathbb{P})$ is a cotilting module over $\Lambda/\operatorname{ann} U$, where ν denotes the Nakayama functor.

A more general notion of silting objects first appeared in [KV] in the context of t-structures in bounded derived categories of Dynkin algebras. This definition allows bounded complexes $\mathbb{P}$ of any size, generating $K^b(\mathcal{P}(\Lambda))$, and with $\mathrm{Hom}(\mathbb{P}, \mathbb{P}[i]) = 0$ for all $i > 0$. More recently the concept appeared in work of many authors, see e.g [AI] or [KY].

Two-term silting has been of particular interest due to the link to τ-tilting theory, as introduced and explored by Adachi, Iyama and Reiten [AIR], and later generalized beyond finite dimensional algebras by Angeleri Hügel, Marks and Vitoria [AMV]. See furthermore [IJY] and [BY].

2.1. **The silting theorem.** Hoshino, Kato and Miyachi considered torsion pairs induced from two-term silting objects already in [HKM]. They mainly worked with abelian categories with arbitrary coproducts, but many of their results easily adapt to our setting. In particular they proved the following.

Theorem 2.2. [HKM] *Let $\mathbb{P}$ be a 2-term silting complex in $K^b(\mathcal{P}(\Lambda))$, and let*

$$\mathcal{T}(\mathbb{P}) = \{X \in \mathrm{mod}\,\Lambda \mid \mathrm{Hom}(\mathbb{P}, X[1]) = 0\}$$

and

$$\mathcal{F}(\mathbb{P}) = \{X \in \mathrm{mod}\,\Lambda \mid \mathrm{Hom}(\mathbb{P}, X) = 0\}$$

Then $(\mathcal{T}(\mathbb{P}), \mathcal{F}(\mathbb{P}))$ is a torsion pair in $\mathrm{mod}\,\Lambda$.

Note that for a two-term silting object $\mathbb{P}$, obtained from a projective presentation of a classical tilting module T, we have that

$$(\mathcal{T}(\mathbb{P}), \mathcal{F}(\mathbb{P})) = (\ker \mathrm{Ext}^1(T, -), \ker \mathrm{Hom}(T, -)),$$

so this is a natural generalization of Proposition 1.4 (a).

Now consider a two-term silting complex $\mathbb{P}$ and let $\Gamma = \mathrm{End}_{D^b(\Lambda)}(\mathbb{P})$. In order to construct a two-term silting complex over Γ, the following result of Wei [W] is crucial.

Proposition 2.3. [W] *Let $\mathbb{P}$ be a two-term silting complex in $K^b(\mathcal{P}(\Lambda))$. Then there is a triangle*

$$\Lambda \to \mathbb{P}' \xrightarrow{p} \mathbb{P}'' \to$$

with $\mathbb{P}', \mathbb{P}''$ in $\operatorname{add}\mathbb{P}$.

Now consider the two-term complex $\mathbb{Q}$ in $K^b(\mathcal{P}(\Gamma))$ induced by the map

$$\operatorname{Hom}(\mathbb{P}, p)\colon \operatorname{Hom}(\mathbb{P}, \mathbb{P}') \to \operatorname{Hom}(\mathbb{P}, \mathbb{P}'').$$

The following generalization of the Brenner-Butler tilting theorem was proved in [BZ1].

Theorem 2.4. *Let $\mathbb{P}$ be a two-term silting complex in $K^b(\mathcal{P}(\Lambda))$, and let $\Gamma = \operatorname{End}_{D^b(\Lambda)}(\mathbb{P})$. With notation as above, the following hold.*

(a) *The complex $\mathbb{Q}$ is a two-term silting complex in $K^b(\mathcal{P}(\Gamma))$.*
(b) *There is an algebra epimorphism $\Phi_{\mathbb{P}}\colon \Lambda \to \overline{\Lambda} = \operatorname{End}_{D^b(\Gamma)}(\mathbb{Q})$.*
(c) *$\Phi_{\mathbb{P}}$ is an isomorphism if and only if $\mathbb{P}$ is tilting.*
Let $\Phi_*\colon \operatorname{mod}\overline{\Lambda} \hookrightarrow \operatorname{mod}\Lambda$ be the induced inclusion functor.
(d) *The functors $\operatorname{Hom}_{D^b(\Lambda)}(\mathbb{P}, -)$ and $\Phi_* \operatorname{Hom}_{D^b(\Gamma)}(\mathbb{Q}, -[1])$ restrict to inverse equivalences between $\mathcal{T}(\mathbb{P})$ and $\mathcal{F}(\mathbb{Q})$.*
(e) *The functors $\operatorname{Hom}_{D^b(\Lambda)}(\mathbb{P}, -[1])$ and $\Phi_* \operatorname{Hom}_{D^b(\Gamma)}(\mathbb{Q}, -)$ restrict to inverse equivalences between $\mathcal{F}(\mathbb{P})$ and $\mathcal{T}(\mathbb{Q})$.*

We remark that (a) and (b) could also have been deduced directly from [BY, Propositions A.3 and A.5], going via the differential graded endomorphism algebra of $\mathbb{P}$.

2.2. **Silted algebras.** We consider two-term silting complexes over hereditary algebras, and more generally hereditary abelian categories.

If $\mathbb{P}$ is a two-term silting complex in $K^b(\mathcal{P}(H))$ for a hereditary algebra H, then $\Gamma = \operatorname{End}_{D^b(H)}(\mathbb{P})$ is called a *silted algebra*. Proposition 1.7 now generalizes as follows.

Proposition 2.5. *Let H be a hereditary algebra and $\mathbb{P}$ a two-term silting complex, with $\Gamma = \operatorname{End}_{D^b(H)}(\mathbb{P})$. Let $\mathbb{Q}$ be the corresponding silting complex in $K^b(\mathcal{P}(\Gamma))$ as defined in Section 2.1. Then $(\mathcal{T}(\mathbb{Q}), \mathcal{F}(\mathbb{Q}))$ is a split torsion pair.*

Recall that a subcategory $\mathcal{X}$ of $\operatorname{mod}\Lambda$ is called *functorially finite* if each module M has both a left and a right $\mathcal{X}$-approximation. Here a right $\mathcal{X}$-approximation is a morphism $u\colon X' \to M$, with X' in $\mathcal{X}$, and such that $\operatorname{Hom}(X, u)$ is an epimorphism for each X in $\mathcal{X}$, and left $\mathcal{X}$-approximations are defined dually. A torsion pair $(\mathcal{X}, \mathcal{Y})$ is called functorially finite, if both $\mathcal{X}$ and $\mathcal{Y}$ are functorially finite. These are exactly the torsion pairs associated to two-term silting objects.

Proposition 2.6. [AIR] *A torsion pair $(\mathcal{X}, \mathcal{Y})$ in $\operatorname{mod}\Lambda$ is functorially finite if and only if there is a two-term silting object $\mathbb{P}$ in $K^b(\mathcal{P}(\Lambda))$ with $(\mathcal{X}, \mathcal{Y}) = (\mathcal{T}(\mathbb{P}), \mathcal{F}(\mathbb{P}))$.*

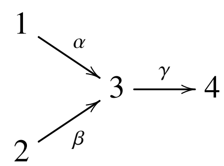
This fact is important for the proof of the following, which is the main result of [BZ2].

Theorem 2.7. *Let Λ be a connected finite dimensional algebra over an algebraically closed field k. Then the following are equivalent:*

(a) *Λ is a silted algebra;*
(b) *there is a split functorially finite torsion pair $(\mathcal{T}, \mathcal{F})$ in $\operatorname{mod}\Lambda$ such that $\operatorname{id}_\Lambda X \leq 1$ for any $X \in \mathcal{T}$ and $\operatorname{pd}_\Lambda Y \leq 1$ for any $Y \in \mathcal{F}$;*
(c) *Λ is a tilted algebra or a strictly shod algebra.*

2.3. **Example.** The smallest example of an algebra which is silted, but not tilted, is given as follows. Let Q be the quiver $1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{c} 4$, and consider the algebra $\Gamma = kQ/I$ where I is generated by $\{ba, cb\}$. It is easy to see that this is a shod algebra, since the only indecomposable modules which are not projective or injective are S_2 and S_3 and we have that $\operatorname{pd} S_2 = \operatorname{id} S_3 = 2$ and $\operatorname{pd} S_3 = \operatorname{id} S_2 = 1$. Here S_i denotes the simple module associated to vertex i.

It turns out that there is a two-term silting complex $\mathbb{P}$ over the path algebra H of the quiver



such that $\Gamma = \operatorname{End}_{D^b(H)}(\mathbb{P})$. Let P_i denote the projective H-module corresponding to vertex i and consider the complex given by $\mathbb{P} = P_2[1] \amalg P_1 \amalg P_4 \amalg \mathbb{P}'$, with $\mathbb{P}' = (P_3 \to P_1)$. Then, it is easy to verify that this is a two-term silting complex and that $\operatorname{End}_{D^b(H)}(\mathbb{P}) \cong \Gamma$.

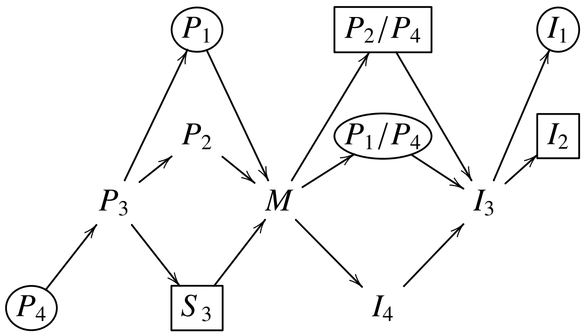
Let us also use this example to illustrate Theorem 2.4.

Let Q_i denote the indecomposable projective Γ-module associated to vertex i. Consider the two-term complex $\mathbb{Q}$ in $K^b(\mathcal{P}(\Gamma))$, given by

$$\mathbb{Q} = Q_1 \amalg Q_3[1] \amalg Q_4[1] \amalg \mathbb{Q}'$$

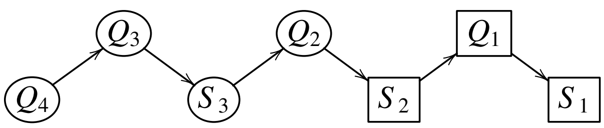
where $\mathbb{Q}' = (Q_3 \to Q_2)$. Then it is easily verified that $\mathbb{Q}$ is a two-term silting object with $\operatorname{End}_{D^b(\Gamma)}(\mathbb{Q}) \cong H/I$, where I is the ideal generated by the path $\gamma\alpha$.

The AR-quiver of $\operatorname{mod} H$ is



where the objects in $\mathcal{T}(\mathbb{P})$ are encircled and the objects in $\mathcal{F}(\mathbb{P})$ are boxed.

The AR-quiver of $\operatorname{mod} \Gamma$ is



where the objects in $\mathcal{T}(\mathbb{Q})$ are boxed and the objects in $\mathcal{F}(\mathbb{Q})$ are encircled. Now, the equivalences of Theorem 2.4 are easily verified.

2.4. **Hereditary abelian categories and quasi-silted algebras.** We can also define two-term silting complexes in the setting of hereditary abelian categories. Let $\mathcal{A}$ be an Ext-finite abelian category. That is: for all X, Y in $\mathcal{A}$ and all $i \geq 0$, we have that $\mathrm{Ext}^i(X, Y)$ is finite dimensional. Then in particular $\mathcal{A}$ is Hom-finite and Krull-Schmidt.

Definition 2.8. *A complex $\mathbb{P}$ in $D^b(\mathcal{A})$ is called two-term silting if*

- *We have* $\mathrm{Hom}(\mathbb{P}, M[i]) = 0$ *for any M in $\mathcal{A}$ and $i \notin \{0, 1\}$.*
- *We have* $\mathrm{Hom}(\mathbb{P}, \mathbb{P}[1]) = 0$.
- *If* $\mathrm{Hom}(\mathbb{P}, M[i]) = 0$ *for all i, then $M = 0$.*

This definition is compatible with the definition for module categories, more precisely we have:

Proposition 2.9. [BZ2] *Let Λ be a finite dimensional algebra, and let $\mathcal{A} = \mathrm{mod}\,\Lambda$. Then an object T in $K^b(\mathcal{A})$ satisfies Definition 2.1 if and only if it satisfies Definition 2.8.*

Happel proved in [H3], that for an Ext-finite hereditary abelian category $\mathcal{H}$, we either have that $\mathcal{H}$ has enough projectives or that $\mathcal{H}$ have no projective objects. In the former case, $\mathcal{H}$ is equivalent to $\mathrm{mod}\,H$ for a hereditary finite dimensional algebra. In the latter case, we have the following.

Proposition 2.10. *If $\mathcal{H}$ is an* Ext*-finite hereditary abelian category with no projectives, then for any two-term silting object $\mathbb{P}$, we have that $\mathbb{P} \simeq H^0(\mathbb{P})$ and that $H^0(\mathbb{P})$ is a tilting object in $\mathcal{H}$.*

Now, let an algebra Γ be called *quasi-silted* if $\Gamma = \mathrm{End}_{\mathcal{D}^b(\mathcal{H})}(\mathbb{P})$ for a two-term silting object $\mathbb{P}$ for a hereditary Ext-finite abelian category $\mathcal{H}$. We then have the following consequence of the above.

Corollary 2.11. *Any quasi-silted algebra is shod.*

Summarizing we obtain the following.

Corollary 2.12. *An algebra is quasi-silted if and only if it is shod.*

2.5. **Endomorphism rings of two-term silting objects.** Having in mind that a silted algebra, that is $\mathrm{End}_{K^b(H)}(\mathbb{P})$ for a two-term silting complex over a hereditary algebra H, in particular has global dimension at most 3, it is natural to ask if there is a more general statement for two-term silting complexes over arbitrary finite dimensional algebras. This problem was studied in [BZ3].

In case of classical tilting, there is the following bound.

Theorem 2.13. [H1, III, Section 3.4] *Let T be a tilting module in* $\mathrm{mod}\,\Lambda$*, and let* $\Gamma = \mathrm{End}_\Lambda(T)$*. Then* $\mathrm{gl.dim}\,\Gamma \leq \mathrm{gl.dim}\,\Lambda + 1$.

It turns out that the silting case is less well behaved. For global dimension at most two, however, we get the following bounds.

Theorem 2.14. *Let $\mathbb{P}$ be a two-term silting complex in $K^b(\mathcal{P}(\Lambda))$ for a finite dimensional algebra Λ and let $\Gamma = \mathrm{End}_{D^b(\Lambda)}(\mathbb{P})$. Then the following hold.*

(a) *If* $\mathrm{gl.dim}\,\Lambda = 1$*, then* $\mathrm{gl.dim}\,\Gamma \leq 3$.
(b) *If* $\mathrm{gl.dim}\,\Lambda = 2$*, then* $\mathrm{gl.dim}\,\Gamma \leq 7$.

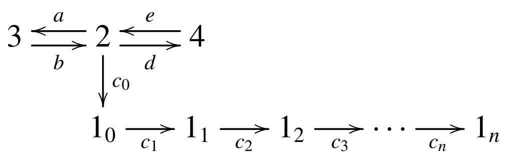
But beyond global dimension two, there is generally no bound.

Theorem 2.15. *For any $n > 2$, there is an algebra Λ, with* $\mathrm{gl.dim}\,\Lambda = n$*, such that* $D^b(\Lambda)$ *admits a two-term silting complex* $\mathbb{P}$ *with* $\mathrm{gl.dim}\,\mathrm{End}_{D^b(\Lambda)}(\mathbb{P}) = \infty$.

Putting further restrictions on $\mathbb{P}$ we still do obtain a bound.

Theorem 2.16. *Let $\mathbb{P}$ be a two-term silting complex in $K^b(\mathcal{P}(\Lambda))$ for a finite dimensional algebra Λ and let $\Gamma = \mathrm{End}_{D^b(\Lambda)}(\mathbb{P})$. Assuming in addition that* $\mathrm{pd}\,H^0(\mathbb{P}) \leq 1$*, we have* $\mathrm{gl.dim}\,\Gamma \leq 2(\mathrm{gl.dim}\,\Lambda) + 2$.

Let us define the algebras needed for Theorem 2.15. For any n, consider the quiver Q_n given by



the ideal $I_n = \langle ba, bd, abc_0, de, c_0c_1, c_1c_2, \ldots, c_{n-1}c_n\rangle$, and the algebra $\Lambda_n = kQ_n/I_n$. Then it is straightforward to check that $\mathrm{gl.dim}\,\Lambda_n = n + 3$. However, let $\mathbb{Q}$ be the complex

$$\cdots \to 0 \to P_{1_0} \amalg P_3 \amalg P_4 \to P_2 \to 0 \to \cdots$$

concentrated in degree -1 and 0 and let $\mathbb{P}_n = \mathbb{Q} \amalg \coprod_{i=1}^{n} P_{1_i}[1] \amalg P_3[1] \amalg P_4[1]$. Then for any n, we have that $\mathbb{P}_n$ is a two-term silting complex with $\mathrm{gl.dim}\,\mathrm{End}_{D^b(\Lambda_n)}(\mathbb{P}) = \infty$.

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