The Narimanov-Moiseev multimodal analysis of nonlinear sloshing in circular conical tanks

A.V. Solodun[1*] and A.N. Timokha[1,2†]

[1]Institute of Mathematics, National Academy of Sciences of Ukraine, Tereschenkivska 3 str., Kiev, 01004, Ukraine

[2]Centre for Autonomous Marine Operations and Systems, Department of Marine Technology, Norwegian University of Science and Technology, NO-7491, Trondheim, Norway

Abstract

The chapter reports mathematical aspects of the Narimanov-Moiseev multimodal modelling for the liquid sloshing in rigid circular conical tanks, which perform small-magnitude oscillatory motions with the forcing frequency close to the lowest natural sloshing frequency. To derive the corresponding nonlinear modal system (of ordinary differential equations), we introduce an infinite set of the sloshing-related generalised coordinates governing the free-surface elevation but the velocity potential is posed as a Fourier series by the natural sloshing modes where the time-depending coefficients are treated as the generalised velocities. The employed approximate natural sloshing modes exactly satisfy both the Laplace equation and the zero-Neumann boundary condition on the wetted tank walls. The Lukovsky non-conformal mapping technique transforms the inner (conical) tank (physical) domain to an artificial upright circular cylinder, for which the single-valued representation of the free surface is possible. Occurrence of secondary resonances for the V-shaped truncated conical tanks is evaluated. The Narimanov-Moiseev modal equations allow for deriving an analytical steady-state (periodic) solution, whose stability is studied. The latter procedure is illustrated for the case of longitudinal harmonic excitations. Standing (planar) waves and swirling as well as irregular sloshing (chaos) are established in certain frequency ranges. The corresponding amplitude response curves are drawn and extensively discussed.

*solodun@imath.kiev.ua
†alexander.timokha@ntnu.no

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Contents

1 Introduction

Practical interest to sloshing in truncated circular conical tanks is, mainly, associated with water towers (figure 1 a). Exposed to earthquake and wind loads, the towers may become most severe resonantly excited when the forcing frequency is close to the lowest natural sloshing frequency. Large water tonnage generates resonant hydrodynamic loads on tank wall and bottom, which are of serious hazard. To predict these loads, compute associated resulting (integral) force and moment, one must solve, analytically or numerically, a rather complicated free-boundary (sloshing) problem.

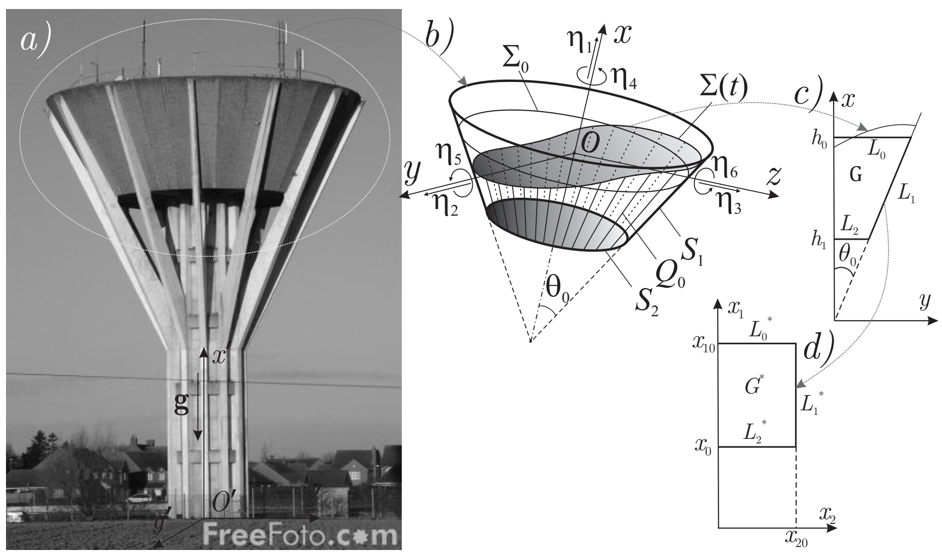

Figure 1: Pictures and drawings, which illustrate appropriate engineering applications, geometric notations of the original free-boundary (sloshing) problem, and ideas of the Lukovsky non-conformal mapping technique, respectively. Panel (a) shows a mega-liter water tower container of the circular conical shape. Panel (b) presents the geometric and physical nomenclature for the original problem (section 2); here, the tank motion is described by the six small-magnitude generalised coordinates $\eta_i(t)$. Panels (c) and (d) specify the original (physical) and transformed meridional tank cross-sections of the conical tank as they follow from the Lukovsky non-conformal mapping technique [22, 30].

Proposed in the famous paper [11], the multimodal method became a popular analytically approximate approach to examine the liquid sloshing dynamics. The method reduces, in a rigorous mathematical way, the original free-boundary problem to a system of nonlinear ordinary differential equations (multidimensional modal equations) governing the sloshing-related generalised coordinates, which describe amplifications (perturbations) of the natural sloshing modes. Employing the nonlinear multimodal equations facilitates both direct numerical simulations and analytical studies of the nonlinear liquid sloshing, provides a rather accurate description of the free-surface elevation (wave patterns) and hydrodynamic loads (resulting forces and moments). Newbies and interested readers are referred to the recent books [2, 25] and the papers [17, 19, 20, 26, 37, 32], in which history, abilities and open problems of the multimodal method are discussed. These works review all previously-derived nonlinear modal equations, which are mostly obtained and studied for upright cylindrical tanks of the rectangular and circular (annular) cross-sections when a single-valued (natural) representation of the free surface is possible as well as exact analytical natural sloshing modes exist.

Combining the nonlinear multimodal method with the non-conformal mapping technique by Lukovsky [22, 30], or its modifications [14, 21, 24, 25], theoretically enables generalising the method for containers with non-vertical walls. However, the nonlinear modal systems for containers with non-vertical walls remain a rare exception in the literature. The latter fact could be partly clarified by a sensitivity of the multimodal method to an error in satisfying the volume (mass) conservation condition. The error is zero for upright tanks when the aforementioned exact analytical natural sloshing modes (solutions of the corresponding spectral boundary problem) exist and, therefore, both the continuity (Laplace) equation and the boundary conditions on the wetted tank surface are exactly and analytically fulfilled. Because the spectral sloshing problem has no analytical solutions for tank shapes with non-vertical walls, to guarantee the mass conservation, one should construct analytically approximate natural sloshing modes, which are obligated to exactly satisfy the Laplace equation and the zero Neumann condition on the wetted tank wall. This is a rather complicated mathematical task. It is solved, to date with, only for non-truncated circular conical tanks [14], two-dimensional circular and spherical tanks [3, 4], as well as, recently, for truncated circular conical tanks [15]. By employing the latter approximate natural sloshing modes from [15], we will report applied mathematical procedures, derivations and keystone formulas, which are attributed to the so-called Narimanov-Moiseev (weakly-nonlinear modal) theory, by starting with the original differential/variational statement of the nonlinear free-boundary (sloshing) problem. The Narimanov-Moiseev modal theory effectively describes sloshing in tanks, which move almost periodically with the forcing frequency close to the lowest natural sloshing frequency, when there are no secondary resonances. A difficulty is that the nonlinear Narimanov-Moiseev modal systems should, for axisymmetric tanks, have an infinite number of degrees of freedom for the second- and third-order generalised coordinates [31]. That is why, the mathematically-complete (i.e., infinite-dimensional) Narimanov-Moiseev modal systems are rare exceptions in the literature. Up to date with, those modal systems only exit for upright annular [6, 36] and spherical [5] containers. All other existing Narimanov-Moiseev's modal systems include a few second- and third-order sloshing-related generalised coordinates.

The primary goal of the present chapter is to describe, in some technical detail, mathematical aspects of the Narimanov-Moiseev asymptotic multimodal method for the free-boundary problem of the liquid sloshing dynamics in rigid circular (truncated) conical tanks, which perform small-magnitude oscillatory motions with the forcing frequency close to the lowest natural sloshing frequency. Being strictly limited in the journal length, the traditionally-formatted research papers are, normally, not able to present all derivation nuances and report specific but important formulas, especially, when dealing with weakly-nonlinear (approximate) mathematical models, which are the best represented by the Narimanov-Moiseev multimodal theory. The book chapter format makes it possible to fill up the gaps. We start with the needed mathematical background and some fundamentals whose keystone is the Bateman-Luke variational formulation of the original free-boundary problem and, thereafter, derive a generalisation of the Miles-Lukovsky nonlinear modal system, which is fully equivalent to the original mathematical problem. The latter system (of ordinary differential equations) is well known for sloshing in rigid upright tanks. To account for non-vertical walls and derive the corresponding generalised Miles-Lukovsky system, one should postulate that instant (unknown) free-surface shapes can be implicitly defined by introducing an infinite set of the sloshing-related generalised coordinates while the velocity potential is, as usually, posed as a Fourier-type solution by natural sloshing modes where the time-depending coefficients play the role of the generalised velocities.

Because the multimodal method requires similar Fourier-type solution for the free surface, but the non-vertical tank walls do not allow for the single-valued (normal) representation of the free surface (which is necessary condition), we utilise the so-called Lukovsky non-conformal mapping technique. The non-conformal mapping transforms the inner (conical) tank (physical) domain to an artificial upright circular cylinder, for which the single-valued representation of the free surface becomes possible. The transformation is applied, in parallel way, to the Bateman-Luke variational formulation, the Miles-Lukosvky modal system, and, finally, to the spectral boundary problem whose eigensolution corresponds to the natural sloshing modes.

Owing to requirements in the volume (mass) conservation, the multimodal method effectively describes nonlinear sloshing, if and only if, the spectral boundary problem has analytically-approximate solutions, which exactly satisfy both the Laplace equation and the zero-Neumann boundary condition on the wetted tank walls, including in the 'ullage' domain over the mean free surface; in other words, the eigenfunctions should be analytically continuable through the free surface. This kind of approximate natural sloshing modes was already constructed for the truncated conical tank shapes. We shortly outline how to get these modes and, furthermore, adopt them in derivations of the generalised Miles-Lukovsky modal equations and their simplified forms. By mentioning the simplified forms, we mean weakly-nonlinear modal systems, which may facilitate analytical studies of the resonant (nonlinear) sloshing. The weakly-nonlinear modal systems normally possess either adaptive (account for the so-called secondary resonance in the hydrodynamic system) or Narimanov-Moiseev-type (no secondary resonances) form.

Occurrence of the secondary resonances for sloshing in the V-shaped truncated conical tanks is estimated. Further, we derive a generic third-order infinite-dimensional system of nonlinear ordinary differential equations, in which the unknowns, sloshing-related generalised coordinates hold equal asymptotic order so that all cubic polynomial quantities in the weakly-nonlinear modal system are asymptotically similar to the nondimensional tank magnitude. On the next stage, the generic modal system reduces to a more convenient (for mathematical studies) analytical form by using assumptions of the Narimanov-Moiseev asymptotic theory.

The Narimanov-Moiseev (modal) system of ordinary differential equations also has infinite number of degrees of freedom but only for the second- and third-order generalised coordinates. The two lowest-order generalised coordinates are associated with the primary excited natural sloshing modes. Due to this very special analytical structure, the Narimanov-Moiseev modal equations allow for implementing diverse analytical approaches and, thereby, getting analytical solutions whose analysis establishes important features of transient and steady-state resonant waves. Ideas of those appropriate approaches are illustrated in the present work for the case of the longitudinal harmonic tank excitation with the forcing frequency close to the lowest natural sloshing frequency. Primary focus is on on the steady-state sloshing regimes.

In section 2, we write down both differential and variational formulations of the free-boundary problem whose physical details can be found in the books [2, 24, 25]. The problem requires either initial or periodicity conditions. Adopting different initial scenarios (conditions) implies modelling the corresponding transient surface waves. The periodicity condition is used for modelling the steady-state (periodic) sloshing regimes, which are expected when the tank moves periodically.

Generally speaking, the nonlinear free-surface sloshing problem has no unique periodic (steady-state) solution [2]. This yields the so-called *classification* problem, a twofold task, which consists of identifying all possible steady-state (periodic) solutions and studying their stability as well as describing the corresponding amplitude (force, moment, etc.) response curves. Because traditional CFD methods solve, normally, the Cauchy (initial) problem, they may fail for solving the classification problem. The multimodal method reduces the original free-boundary problem to system(s) of nonlinear ordinary equations. There exists a variety of analytical methods and approaches, which can effectively solve the two-point (periodic) problem for these differential equations, analyse the obtained solutions and, thereby, classify the steady-state wave regimes.

Employing the Bateman-Luke variational formulation of the original sloshing problem, we further derive a generalisation of the Miles-Lukovsky nonlinear modal (ordinary differential) equations [24, 25], which couple the sloshing-related generalised coordinates $\{\beta_K(t)\}$ (which describe the free-surface shape) and the generalised velocities $\{F_N(t)\}$ (represent the velocity potential). The Miles-Lukovsky modal system is fully equivalent to the original free-surface problem. Getting the modal system in its canonic form, normally, requires the single-valued (normal) representation of the free surface, $x = f(y, z, t)$ (x is the vertical coordinate). The single-valued representation is impossible for tanks with non-vertical walls. That is why, we assume the implicitly-defined free surface, $\zeta(y, z, \{\beta_K(t)\}) = 0$. The generalised velocities $\{F_N(t)\}$ appear as time-dependent coefficients in the Fourier representation of the velocity potential. The Miles-Lukovsky modal system consists of kinematic and dynamic sub-systems.

Section 3 reports analytical and technical details of a non-conformal mapping technique, which was proposed by Lukovsky [22]. The technique transforms the non-cylindrical physical (inner tank) domain to an auxiliary cylindrical domain by using the curvilinear coordinates $Ox_1x_2x_3$. The goal consists of replacing the implicit free-surface representation $\zeta(y, z, \{\beta_K(t)\}) = 0$ in the physical space to the single-valued Fourier-type representation $\zeta = x_1 - \beta_0(t) + \sum \beta_N(t) f_N(x_2, x_3)$ in the transformed space ($\{f_N\}$ is the Fourier basis, normally, the transformed natural sloshing modes). The non-conformal mapping should be simultaneously applied to both the spectral boundary problem on the natural sloshing modes and the Miles-Lukovsky modal equations. Following [15], we construct the analytically-approximate natural sloshing modes (eigenfunctions of the transformed spectral boundary problem) for the case of the circular truncated conical tank. Furthermore, by adopting the single-valued representation of the free surface in the transformed space, we rewrite the generalised Miles-Lukovsky equations in a more convenient analytical form.

In section 4, we use the Miles-Lukovsky modal equations from section 3 for derivation of a generic weakly-nonlinear modal system, which exclusively couples the sloshing-related generalised coordinates. The generalised velocities are found, in an explicit form, by resolving the kinematic subsystem of the Miles-Lukovsky system; the result is substituted into the dynamic subsystem. The derivation utilises ideas of the so-called third-order adaptive multimodal modelling [7, 10], which suggests that the forcing magnitude has the third asymptotic order in terms of the lowest-order sloshing-related generalised coordinates. The generic weakly-nonlinear modal equations keep only the cubic polynomial terms with respect to the generalised coordinates.

Details of the Narimanov-Moiseev multimodal asymptotic theory, as these appear for axisymmetric tanks [31], are reported in section 5. The Narimanov-Moiseev modal equations are derived for the circular conical tank shape. The asymptotic theory assumes that there are no secondary resonances and the forcing frequency is close to the lowest natural sloshing frequency. The secondary resonance phenomenon for sloshing in conical tanks was investigated in [28]. These results are shortly outlined in the present chapter to detect the critical geometric pairs, the semi-apex angle and the liquid depth (for truncated conical tanks), when the second- or third-order generalised coordinates can be resonantly amplified to a lower asymptotic order due to the secondary resonance phenomenon.

In section 5, we demonstrate how to construct an analytic asymptotic periodic solution of the Narimanov-Moiseev system from the previous section and study its stability. These periodic solutions implies the steady-state resonant sloshing regimes. Finding all these regimes and drawing the corresponding response curves (*versus* the forcing frequency) implies the so-called *classification* problem [9]. The wave-amplitude response curves are illustrated for the case of the lateral (horizontal) harmonic tank forcing that is one of the classical benchmark sloshing problems.

2 Statement

We consider a rigid truncated conical tank of the semi-apex angle θ_0, which performs a small-magnitude oscillatory motion with six degrees of freedom as shown in figure 1 (b). These degrees of freedom are associated with translatory tank motions (generalised coordinates η_1, η_2, and η_3; $\boldsymbol{v}_O = (\dot{\eta}_1, \dot{\eta}_2, \dot{\eta}_3)$) and angular tank motions, which are defined by the instant angular velocity $\boldsymbol{\omega}(t) = (\dot{\eta}_4, \dot{\eta}_5, \dot{\eta}_6)$. The circular conical tank is partially filled by an ideal incompressible liquid with irrotational flows.

The absolute fluid velocity field is considered in the tank-fixed coordinate system $Oxyz$ whose origin O is superposed with the artificial cone vertex so that the Ox-axis coincides with the symmetry axis (figure 1,a). Whereas the tank does not move, the gravity acceleration vector $\boldsymbol{g}$ has opposite direction to Ox.

2.1 Free-boundary problem

After introducing the absolute velocity potential $\Phi(x, y, z, t)$ and function $\zeta(x, y, z, t)$ implicitly determining the free surface $\Sigma(t) : \zeta(x, y, z, t) = 0$, the free-boundary problem on the liquid sloshing dynamics in a movable rigid tank can be written down in the form (see, the physical derivation details in [2, 24])

$$\nabla^2 \Phi = 0, \quad \boldsymbol{r} \in Q(t), \tag{1a}$$

$$\frac{\partial \Phi}{\partial \nu} = \boldsymbol{v}_O \cdot \boldsymbol{\nu} + \boldsymbol{\omega} \cdot (\boldsymbol{r} \times \boldsymbol{\nu}), \quad \boldsymbol{r} \in S(t), \tag{1b}$$

$$\frac{\partial \Phi}{\partial \nu} = \boldsymbol{v}_O \cdot \boldsymbol{\nu} + \boldsymbol{\omega} \cdot (\boldsymbol{r} \times \boldsymbol{\nu}) - \frac{\partial \zeta / \partial t}{|\nabla \zeta|}, \quad \boldsymbol{r} \in \Sigma(t), \tag{1c}$$

$$\frac{\partial \Phi}{\partial t} + \tfrac{1}{2} |\nabla \Phi|^2 - \nabla \Phi \cdot (\boldsymbol{v}_O + \boldsymbol{\omega} \times \boldsymbol{r}) + U = 0, \quad \boldsymbol{r} \in \Sigma(t), \tag{1d}$$

$$\int_{Q(t)} dQ = V_l = \text{const}, \tag{1e}$$

where $\boldsymbol{v}_O(t)$ is the velocity of the origin O, $\boldsymbol{\omega}(t)$ is the instant angular velocity vector of the $Oxyz$ coordinate system, $\boldsymbol{\nu}$ is the outer normal vector, $S(t) = S_1(t) \cup S_2$ is the wetted tank surface, $\boldsymbol{r} = (x, y, z)$ is the radius vector, $U = \boldsymbol{r} \cdot \boldsymbol{g}$ is the gravity potential ($\boldsymbol{g}$ is the gravity acceleration vector) defined in the $Oxyz$-coordinate system. These notations are illustrated in figure 1 (b). Equation (1e) implies the liquid volume (mass) conservation, which can be treated as a necessary solvability condition of the Neumann boundary problem (1a)-(1c).

The pressure field $p(x, y, z, t)$ can be determined by using the Bernoulli equation rewritten in the non-inertial coordinate system $Oxyz$,

$$\frac{\partial \Phi}{\partial t} + \tfrac{1}{2} |\nabla \Phi|^2 - \nabla \Phi \cdot (\boldsymbol{v}_O + \boldsymbol{\omega} \times \boldsymbol{r}) + U = -\frac{p - p_0}{\rho} \tag{2}$$

where p_0 is the ullage pressure and ρ is the liquid density.

2.2 Initial and periodicity conditions

The free-boundary problem (1) requires either initial conditions

$$\zeta(x, y, z, t_0) = \zeta_0(x, y, z), \quad \left.\frac{\partial \Phi}{\partial \nu}\right|_{\Sigma(t_0)} = \Phi_0(x, y, z)|_{\Sigma(t_0)}, \tag{3}$$

which define the initial free-surface pattern $\Sigma(t_0)$ and the normal velocity on $\Sigma(t_0)$ ($\zeta_0(x, y, z)$ and $\Phi_0(x, y, z)|_{\Sigma(t_0)}$ are the two given functions), or, alternatively, the periodicity conditions

$$\zeta(x, y, z, t + T) = \zeta(x, y, z, t), \quad \Phi(x, y, z, t + T) = \Phi(x, y, z, t), \tag{4}$$

which could be used when the tank moves periodically with the forcing period T.

Solutions of the time-periodic problem (1) + (4) imply the steady-state surface waves. The latter problem has a non-unique solution for each fixed T (see, details in chapters 8 and 9 of [2]). Full description of all theoretically possible steady-state solutions and identification of their stability are often called the *classification*.

2.3 Bateman-Luke variational formulation

Instead of dealing with the free-boundary problem (1), whose steady-state resonant solutions are difficult to classify when using the Computational Fluid Dynamics, we will employ the multimodal method, which reduces the free-boundary problem (1) to a system of nonlinear ordinary differential equations.

The derivation procedure utilises the Bateman–Luke variational formulation whose equivalence to (1) is, for instance, proven in [2] (Sect. 2.5.3.2) and Chapt. 2 by [29]. According to this variational formulation, *the solution (the pair of independent unknowns Φ and ζ) of the sloshing problem (1) coincides with extrema points of the action*

$$
\begin{aligned}
A(\zeta,\Phi) &= \int_{t_1}^{t_2}\left(\int_{Q(t)}(p-p_0)\,dx\,dy\,dz\right)dt\\
&= -\int_{t_1}^{t_2}\left[\int_{Q(t)}\left(\frac{\partial\Phi}{\partial t}+\tfrac{1}{2}\left|\nabla\Phi\right|^2-\nabla\Phi\cdot(\boldsymbol{v}_O+\boldsymbol{\omega}\times\boldsymbol{r})+U\right)dx\,dy\,dz\right]dt
\end{aligned}
\tag{5}
$$

for arbitrary fixed t_1 *and* t_2 *(*$t_1 < t_2$*) subject to variations satisfying*

$$
\delta\Phi|_{t_1,t_2}=0,\quad \delta\zeta|_{t_1,t_2}=0. \tag{6}
$$

Here, $(p-p_0)$ *is the formal mathematical expression taken from the Bernoulli equation (2).*

2.4 Miles-Lukovsky modal equations

The Bateman-Luke variational formulation (5), (6) was used by many authors to derive the so-called Miles-Lukovsky system of nonlinear ordinary differential equations with respect to the sloshing-related generalised coordinates $\{\beta_N(t)\}$ and velocities $\{F_N(t)\}$. The system is fully equivalent to the the original free-boundary problem (1) but its derivation requires *a priori* satisfying a series of special conditions, which are listed in chapter 7 of [2].

In particular, the derivation normally assumes the single-valued (normal) representation of the free surface $\Sigma(t)$: $\zeta = x - f(y,z,t) = 0$, in which a Fourier series for $f(y,z,t)$ is employed with the time-dependent coefficients (generalised coordinates) $\{\beta_N(t)\}$. For the non-vertical tank walls, the single-valued representation is impossible. However, one can implicitly introduce the generalised coordinates by postulating

$$
\zeta = \zeta(x,y,z;\{\beta_N(t)\}) \tag{7}
$$

subject to the volume conservation condition (1e), which is considered as a holonomic constraint.

In parallel way, the multimodal method needs the Fourier-type representation of the velocity potential

$$
\Phi(x,y,z,t) = \boldsymbol{v}_O\cdot\boldsymbol{r}+\boldsymbol{\omega}\cdot\boldsymbol{\Omega}+\sum_{N=1}^{\infty}F_N(t)\varphi_N(x,y,z), \tag{8}
$$

where $\boldsymbol{\Omega}(x,y,z;\{\beta_N(t)\}) = (\Omega_1,\Omega_2,\Omega_3)$ are the Stokes-Joukowski potentials, which parametrically depend on $\{\beta_N(t)\}$ as they are found from the

Neumann boundary value problem in the time-varied liquid domain $Q(t)$,

$$\begin{aligned}
&\nabla^2\Omega_i = 0 \ \text{ in } \ Q(t),\\
&\frac{\partial\Omega_1}{\partial\nu} = y\nu_z - z\nu_y;\ \frac{\partial\Omega_2}{\partial\nu} = z\nu_x - x\nu_z;\ \frac{\partial\Omega_3}{\partial\nu} = x\nu_y - y\nu_x \text{ on } \Sigma(t)\cup S(t).
\end{aligned} \tag{9}$$

Here, ν_* are the projections of the outer normal vector on the corresponding coordinate axes.

The Fourier basis $\{\varphi_N\}$ in (8) is normally associated with the natural sloshing modes, eigenfunctions of the spectral boundary problem,

$$\nabla^2\varphi = 0,\ \boldsymbol{r}\in Q_0,\ \frac{\partial\varphi}{\partial\nu} = 0,\ \ \boldsymbol{r}\in S_0,\ \frac{\partial\varphi}{\partial\nu} = \bar{\kappa}\,\varphi,\ \ \boldsymbol{r}\in\Sigma_0,\ \int_{\Sigma_0}\frac{\partial\varphi}{\partial\nu}dS = 0, \tag{10}$$

defined in the hydrostatic (mean) liquid domain Q_0, which is bounded by the mean free surface Σ_0 and the mean wetted tank surface S_0.

According to the spectral theorems [12], the functional set $\{\varphi_N\}$ constitutes a harmonic (functions $\{\varphi_N\}$ exactly satisfy the Laplace equation) functional basis in Q_0. The multimodal method requires that $\{\varphi_N\}$ is defined in any admissible instant liquid domain $Q(t)$. In other words, the eigensolution of (10) should be analytically continuable over the mean free surface Σ_0. Furthermore, the method says that the Fourier solution (8) must exactly satisfy the volume (mass) conservation condition. The latter means that the base functions $\{\varphi_N\}$ exactly satisfy the zero-Neumann boundary condition on the wetted tank surface for any instant time t.

Because ζ and Φ are independent variables in the Bateman-Luke formulation, the generalised coordinates $\{\beta_N(t)\}$ and velocities $\{F_N(t)\}$ are also independent time-depending functions and, due to (6), these must satisfy the condition

$$\delta F_N|_{t=t_1,t_2} = \delta\beta_N|_{t=t_1,t_2} = 0.$$

Substituting (8) into (5) and varying $\{F_N(t)\}$ leads to the *kinematic* modal equations

$$\frac{dA_N}{dt} \equiv \sum_K \frac{\partial A_N}{\partial\beta_K}\dot{\beta}_K = \sum_K A_{NK}F_K \quad \text{for all} \quad N, \tag{11}$$

which are mathematically equivalent to the Neumann boundary value problem (1a)-(1c). Derivation of (11) is algebraically similar to those reported in chapter 7 of [2] and we refer interested readers to this book for analytical details.

Tedious derivations in [2] (pages 301-303) explain how varying the generalised coordinates $\{\beta_N(t)\}$ in the Bateman-Luke formulation leads to the *dynamic* modal equations

$$\sum_K \frac{\partial A_K}{\partial \beta_N}\dot{F}_K + \frac{1}{2}\sum_{K,L}\frac{\partial A_{KL}}{\partial \beta_N}F_K F_L + (\boldsymbol{\omega}\times\boldsymbol{v}_O - \mathbf{g})\cdot\frac{\partial \boldsymbol{l}}{\partial \beta_i} - \frac{1}{2}\boldsymbol{\omega}\cdot\frac{\partial \boldsymbol{J}^1}{\partial \beta_i}\cdot\boldsymbol{\omega}$$
$$+\dot{\boldsymbol{\omega}}\cdot\left(\frac{\partial \boldsymbol{l}_\omega}{\partial \beta_i} - \frac{\partial \boldsymbol{l}_{\omega t}}{\partial \dot{\beta}_i}\right) + \boldsymbol{\omega}\left(\frac{\partial \boldsymbol{l}_{\omega t}}{\partial \beta_i} - \frac{d}{dt}\frac{\partial \boldsymbol{l}_{\omega t}}{\partial \dot{\beta}_i}\right) = 0 \quad \text{for all} \quad N, \quad (12)$$

which are mathematically equivalent to the dynamic boundary condition (1d).

The modal equations (11), (12) govern to the generalised coordinates and velocities so that

$$\begin{aligned}
&A_N = \int_{Q(t)} \varphi_N dQ, \quad A_{NK} = \int_{Q(t)} (\nabla\varphi_N \cdot \nabla\varphi_K) dQ,\\
&l_1 = \int_{Q(t)} x dQ, \quad l_2 = \int_{Q(t)} y dQ, \quad l_3 = \int_{Q(t)} z dQ,\\
&l_{k\omega} = \rho\int_{Q(t)} \Omega_k dQ, \quad l_{k\omega t} = \rho\int_{Q(t)} \frac{\partial \Omega_k}{\partial t} dQ,\\
&J^1_{ij} = \rho\int_{S(t)+\Sigma(t)} \Omega_i \frac{\partial \Omega_j}{\partial t} dQ; \quad k = 1,2,3, \quad J^1_{ij} = J^1_{ji},
\end{aligned} \quad (13)$$

are, in fact, the implicitly-defined nonlinear functions of $\{\beta_N(t)\}$ ($Q(t)$ is determined by (7)).

3 Non-conformal mapping technique

To have the single-valued (normal) representation of the free surface, which is impossible within the framework of the Cartesian parametrisation, we follow the Lukovsky non-conformal mapping technique [14, 22, 30] and utilise the curvilinear coordinate system $Ox_1x_2x_3$,

$$x = x_1, \quad y = x_1 x_2 \cos x_3, \quad z = x_1 x_2 \sin x_3, \quad (14)$$

where $x_3 = \eta$ is, in fact, the angular coordinate.

The coordinate transformation (14) should be applied to both the spectral boundary problem (10) and the Miles-Lukovsky modal system (11), (12).

3.1 Natural sloshing modes

The natural sloshing modes (eigenfunctions of (10)) are normally defined only in the unperturbed domain Q_0. However, to make integrals (13) correctly defined, these eigenfunctions (natural modes), exact or approximate, must be analytically continuable over the mean free surface Σ_0 from the liquid into ullage domain. Another requirement is that $\{\varphi_N\}$ should exactly satisfy the Laplace equation and the zero-Neumann condition on the wetted tank surface.

The curvilinear coordinate system $Ox_1x_2x_3$ by (14) transforms the original conical (physical) domain to an artificial circular cylindrical shape. Figure 1 (c,d) demonstrates the meridional cross-section of the original (mean) liquid domain in the physical G and transformed G^* planes. Considering the eigensolution of (10) in the curvilinear coordinate system

$$\varphi(x_1,x_2,x_3)=\psi_m(x_1,x_2)\begin{matrix}\sin m\,x_3\\ \cos m\,x_3\end{matrix},\quad m=0,1,2,\dots \tag{15}$$

makes it possible to separate the spatial variables (x_1,x_2) and x_3 so that it yields the following m-family of spectral boundary problems

$$p\frac{\partial^2\psi_m}{\partial x_1^2}+2q\frac{\partial^2\psi_m}{\partial x_1\partial x_2}+s\frac{\partial^2\psi_m}{\partial x_2^2}+d\frac{\partial\psi_m}{\partial x_2}-m^2c\psi_m=0 \qquad \text{in} \quad G^*, \tag{16a}$$

$$s\frac{\partial\psi_m}{\partial\, x_2}+q\frac{\partial\psi_m}{\partial\, x_1}=0 \qquad \text{on} \quad L_1^*, \tag{16b}$$

$$p\frac{\partial\psi_m}{\partial x_1}+q\frac{\partial\psi_m}{\partial x_2}=\bar{\kappa}_m p\psi_m \qquad \text{on} \quad L_0^*, \tag{16c}$$

$$p\frac{\partial\psi_m}{\partial\, x_1}+q\frac{\partial\psi_m}{\partial\, x_2}=0 \qquad \text{on} \quad L_2^*, \tag{16d}$$

$$|\psi_m(x_1,0)|<\infty,\quad m=0,1,2,\dots, \tag{16e}$$

$$\int_0^{x_{20}}\psi_0x_2dx_2=0, \tag{16f}$$

where $G^*=\{(x_1,x_2):x_0\le x_1\le x_{10},\ 0\le x_2\le x_{20}\}$, $p=x_1^2x_2$, $q=-x_1x_2^2$, $s=x_2(x_2^2+1)$, $d=1+2x_2^2$, $c=1/x_2$, and L_0^*, L_1^* and L_2^* are defined in figure 1 (c,d).

The natural sloshing frequencies are

$$\sigma_{mn}=\sqrt{g\bar{\kappa}_{mn}}=\sqrt{\frac{g\kappa_{mn}}{\mathtt{r}_0}}, \tag{17}$$

where $\kappa_{mn} = \mathtt{r_0}\bar{\kappa}_{mn}$ are the nondimensional eigenvalues.

By using the Trefftz method, [15] constructed an analytically approximate Treftz solution of (16), which exactly satisfies (16a), (16b), and (16d). This solution takes the form

$$\psi_m = \psi_{mn}(x_1, x_2) = \sum_{k=1}^{q_1} a_{n,k}^{(m)} w_k^{(m)} + \sum_{l=1}^{q_2} \bar{a}_{n,l}^{(m)} \bar{w}_l^{(m)}, \tag{18}$$

where functions $w_k^{(m)}(x_1, x_2)$ and $\bar{w}_k^{(m)}(x_1, x_2)$ are

$$\begin{aligned} w_k^{(m)}(x_1, x_2) &= N_k^{(m)}\, x_1^{\nu_{mk}}\, T_{\nu_{mk}}^{(m)}(x_2), \\ \bar{w}_k^{(m)}(x_1, x_2) &= \bar{N}_k^{(m)}\, x_1^{-1-\nu_{mk}}\, \bar{T}_{\nu_{mk}}^{(m)}(x_2) \end{aligned} \tag{19}$$

with $T_{\nu_{mk}}^{(m)}(x_2)$ and $\bar{T}_{\nu_{mk}}^{(m)}(x_2)$ expressed via the associate Legendre polynomials of the first kind, $P_\nu^{(m)}(\mu)$ (see [23]), as follows,

$$\begin{aligned} T_{\nu_{mk}}^{(m)}(x_2) &= (1 + x_2^2)^{\frac{\nu_{mk}}{2}} P_{\nu_{mk}}^{(m)}\left(\frac{1}{\sqrt{1 + x_2^2}}\right), \\ \bar{T}_{\nu_{mk}}^{(m)}(x_2) &= (1 + x_2^2)^{\frac{-1-\nu_{mk}}{2}} P_{\nu_{mk}}^{(m)}\left(\frac{1}{\sqrt{1 + x_2^2}}\right). \end{aligned}$$

The numbers ν_{mk} are roots of the equation $\partial P_\nu^{(m)}(\cos\theta)/\partial\theta\Big|_{\theta=\theta_0} = 0$ and $N_k^{(m)}$ and $\bar{N}_k^{(m)}$ are the normalizing multipliers introduced to satisfy the condition $\left\| w_k^{(m)} \right\|^2_{L_2^* \cup L_0^*} = \left\| \bar{w}_k^{(m)} \right\|^2_{L_2^* \cup L_0^*} = 1$, where $\|\cdot\|$ implies the mean square-root norm on $L_2^* \cup L_0^*$. The paper [15] reports the Trefftz variational scheme, which makes it possible to find the coefficients $a_{n,k}^{(m)}$ and $\bar{a}_{n,k}^{(m)}$ in (18).

3.2 Alternative form of the Miles-Lukovsky modal equations

We start with the implicitly-given free-surface representation (7) rewritten in the $x_1x_2x_3$-coordinates and, furthermore, assume, because the tank walls become vertical in these coordinates (figure 1 c,d), that (7) may be written down in the form

$$\zeta = x_1 - f(x_2, x_3, \{\beta_{mi}\}) = 0,$$

where

$$f = f(x_2, x_3, \{p_{mi}\}, \{r_{mi}\}) = x_{10} + \beta_0(t) + \sum_{m=1}^{\infty}\sum_{i=1}^{\infty} r_{mi}(t)\sin(mx_3)f_{mi}(x_2)$$
$$+ \sum_{M=0}^{\infty}\sum_{i=1}^{\infty} p_{Mi}(t)\cos(Mx_3)f_{Mi}(x_2), \quad (20)$$

and

$$f_{Mi}(x_2) = \frac{\sigma_{Mi}}{g}\psi_{Mi}(x_{10}, x_2) \quad (21)$$

defines the radial profiles of the natural sloshing modes but σ_{Mi} are the natural sloshing frequencies introduced in (17).

Specifically, the free-surface representation (20) contains the non-zero generalised coordinate $\beta_0(t)$, which is yielded by the volume conservation condition (1e) playing the role of the holonomic constraint. Resolving this constraint makes the generalised coordinate $\beta_0(t)$ by a function of other generalised coordinates, namely, one can write down

$$\beta_0(t) = \beta_0(\{p_{Mi}(t)\}, \{r_{mi}(t)\}). \quad (22)$$

The latter function is derived in an explicit analytical form in Appendix A.1.

Along with the multimodal representation of the free surface (20), the multimodal method also requires the Fourier–type (multimodal) representation of the velocity potential (8)

$$\Phi(x_1, x_2, x_3, t) = \boldsymbol{v}_O \cdot \boldsymbol{r} + \boldsymbol{\omega} \cdot \boldsymbol{\Omega} + \sum_{M=0}^{\infty}\sum_{i=1}^{\infty} P_{Mj}(t)\cos(Mx_3)\psi_{mj}(x_2, x_3)$$
$$+ \sum_{m=1}^{\infty}\sum_{i=1}^{\infty} R_{mj}(t)\sin(mx_3)\psi_{mj}(x_2, x_3). \quad (23)$$

The multimodal representations (20) and (23) are employed, instead of (7) and (8), in the Miles-Lukosky modal equations (11), (12), where integrals (13) are fully determined by the generalised coordinates $p_{Mi}(t)$ and $r_{mi}(t)$, in which capital indices should be replaced by the complex indices $(Mi, \cos)$ and $(mi, \sin)$ so that, for instance, when $N = (Mi, \cos)$,

$$A_N = A_{(Mi,\cos)}$$
$$= \int_{-\pi}^{\pi}\int_{0}^{r_0}\int_{x_0}^{f(x_2,x_3,\{p_{Mi}\},\{r_{mi}\})} x_1^2 x_2 \psi_{Mi}(x_1, x_2)\cos(Mx_3)dx_1dx_2dx_3. \quad (24)$$

The capital letter M implies changing index from zero to infinity ($M = 0, 1, 2, \ldots$), and small m means $m = 1, 2, \ldots$.

According to (20) and (23), the Miles-Lukovsky multimodal equations (11), (12) can be rewritten in a more suitable form. The *kinematic* modal equations (11) take then the form

$$\begin{aligned}
&\sum_{Mn} \frac{\partial A^p_{Ab}}{\partial p_{Mn}} \dot{p}_{Mn} + \sum_{mn} \frac{\partial A^p_{Ab}}{\partial r_{mn}} \dot{r}_{mn} = \sum_{Mn} A^{pp}_{Ab,Mn} P_{Mn} + \sum_{mn} A^{pr}_{Ab,mn} R_{mn} = 0,\\
&\sum_{Mn} \frac{\partial A^r_{ab}}{\partial p_{Mn}} \dot{p}_{Mn} + \sum_{mn} \frac{\partial A^r_{ab}}{\partial r_{mn}} \dot{r}_{mn} = \sum_{Mn} A^{pr}_{Mn,ab} P_{Mn} + \sum_{mn} A^{rr}_{Ab,mn} R_{mn} = 0,
\end{aligned} \tag{25}$$

and the *dynamic* modal equations (12) are

$$\begin{aligned}
&\sum_{Mn} \frac{\partial A^p_{Mn}}{\partial p_{Ab}} \dot{P}_{Mn} + \sum_{mn} \frac{\partial A^r_{mn}}{\partial p_{Ab}} \dot{R}_{mn} + \frac{1}{2} \sum_{MnLk} \frac{\partial A^{pp}_{Mn,Lk}}{\partial p_{Ab}} P_{Mn} P_{Lk} + \\
&+ \frac{1}{2} \sum_{mnlk} \frac{\partial A^{rr}_{mn,lk}}{\partial p_{Ab}} R_{mn} R_{lk} + \sum_{Mnlk} \frac{\partial A^{pr}_{Mn,lk}}{\partial p_{Ab}} P_{Mn} R_{lk} + g\Lambda_{AA} p_{Ab} + \\
&\qquad + (\ddot{\eta}_2 - g\eta_6 - S_b \ddot{\eta}_6)\Lambda_{A1} e_b = 0,\\
&\sum_{Mn} \frac{\partial A^p_{Mn}}{\partial r_{ab}} \dot{P}_{Mn} + \sum_{mn} \frac{\partial A^r_{mn}}{\partial r_{ab}} \dot{R}_{mn} + \frac{1}{2} \sum_{MnLk} \frac{\partial A^{pp}_{Mn,Lk}}{\partial r_{ab}} P_{Mn} P_{Lk} + \\
&+ \frac{1}{2} \sum_{mnlk} \frac{\partial A^{rr}_{mn,lk}}{\partial r_{ab}} R_{mn} R_{lk} + \sum_{Mnlk} \frac{\partial A^{pr}_{Mn,lk}}{\partial r_{ab}} P_{Mn} R_{lk} + g\Lambda_{aa} r_{ab} + \\
&\qquad + (\ddot{\eta}_3 - g\eta_5 - S_b \ddot{\eta}_5)\Lambda_{a1} e_b = 0,
\end{aligned} \tag{26}$$

where $e_b = \hat{\lambda}_{1b}$ from (79)

$$\Lambda_{IJ} = \begin{cases} 2\pi, & I = J = 0,\\ \pi\delta_{IJ}, & \text{otherwise}, \end{cases} \qquad \delta_{IJ} = \begin{cases} 1, & I = J,\\ 0, & I \neq J. \end{cases} \tag{27}$$

By using the free-surface representation (20) and accounting for (22), one can derive explicit analytical expressions for (13). Components of the vector $A_N = \big\{\{A^p_{Ab}\}, \{A^r_{ab}\}\big\}$ come from

$$\begin{aligned}
A^p_{Ab} &= \rho \int_0^{2\pi} \int_0^{x_{20}} \cos Ax_3 \Theta^0_{Ab}(x_1, x_2, p_{Ij}, r_{ij}) dx_2 dx_3,\\
A^r_{ab} &= \rho \int_0^{2\pi} \int_0^{x_{20}} \sin ax_3 \Theta^0_{ab}(x_1, x_2, p_{Ij}, r_{ij}) dx_2 dx_3,
\end{aligned} \tag{28}$$

where $A=0,1,\dots$ and $a,b=1,2,\dots$, but components of the matrix $A_{NK}=\{\{A^{pp}_{Ab,Cd},A^{pr}_{Ab,cd}\},\{A^{pr}_{Ab,cd},A^{rr}_{ab,cd}\}\}$ are defined by

$$
\begin{aligned}
A^{pp}_{Ab,Cd} &= \rho\int_0^{2\pi}\int_0^{x_{20}}\big(\cos Ax_3\cos Cx_3\Theta^1_{AbCd}(x_1,x_2,p_{Ij},r_{ij})+\\
&\qquad +\sin Ax_3\sin Cx_3\Theta^2_{AbCd}(x_1,x_2,p_{Ij},r_{ij})\big)\,dx_2dx_3,\\
A^{rr}_{ab,cd} &= \rho\int_0^{2\pi}\int_0^{x_{20}}\big(\sin ax_3\sin cx_3\Theta^1_{abcd}(x_1,x_2,p_{Ij},r_{ij})+\\
&\qquad +\cos ax_3\cos cx_3\Theta^2_{abcd}(x_1,x_2,p_{Ij},r_{ij})\big)\,dx_2dx_3,\\
A^{pr}_{Ab,cd} &= \rho\int_0^{2\pi}\int_0^{x_{20}}\big(\cos Ax_3\sin cx_3\Theta^1_{Abcd}(x_1,x_2,p_{Ij},r_{ij})-\\
&\qquad -\sin Ax_3\cos cx_3\Theta^2_{Abcd}(x_1,x_2,p_{Ij},r_{ij})\big)\,dx_2dx_3,
\end{aligned}\tag{29}
$$

where

$$
\begin{aligned}
\Theta^0_N(x_1,x_2,p_{Ij},r_{ij}) &= \int_0^{f^*+x_{10}} x_1^2\psi_N\,dx_1,\\
\Theta^1_{NK}(x_1,x_2,p_{Ij},r_{ij}) &= \int_0^{f^*+x_{10}}\Bigg(x_1^2x_2\frac{\partial\psi_N}{\partial x_1}\frac{\partial\psi_K}{\partial x_1}+x_2\left(1+x_2^2\right)\frac{\partial\psi_N}{\partial x_2}\frac{\partial\psi_K}{\partial x_2}\\
&\qquad -x_1x_2^2\left(\frac{\partial\psi_N}{\partial x_1}\frac{\partial\psi_K}{\partial x_2}+\frac{\partial\psi_N}{\partial x_2}\frac{\partial\psi_K}{\partial x_1}\right)\Bigg)dx_1,\\
\Theta^2_{NK}(x_1,x_2,p_{Ij},r_{ij}) &= \int_0^{f^*+x_{10}}\frac{1}{x_2}\frac{\partial\psi_N}{\partial x_3}\frac{\partial\psi_K}{\partial x_3}dx_1.
\end{aligned}\tag{30}
$$

4 Generic weakly-nonlinear modal equations

The derived fully-nonlinear modal equations (25)–(30) are difficult to use in analytical studies; these are also not efficient in numerical simulations. Moreover, they involve the generalised velocities that is not typical for dynamic equations for oscillatory mechanical systems, which normally appear as the second-order differential equations with respect to the generalised coordinates.

Simplifying (25)–(30) to a weakly-nonlinear, adaptive form [7, 10] implies postulating the asymptotic relations

$$p_{Mi}\sim P_{Mi}\sim r_{mi}\sim R_{mi}=O(\epsilon),\tag{31}$$

provided by

$$\eta_i(t)=O(\epsilon^3)\tag{32}$$

as well as neglecting all quantities in the modal equations, which have the asymptotic order $O(\epsilon^4)$. Furthermore, one should resolve the kinematic equations (25) with respect to the generalised velocities and substitute the result into the dynamic equations (26) where, again, the asymptotic terms $O(\epsilon^4)$ must be omitted. The derivation of the generic weakly-nonlinear equations is a rather complicated and tedious analytical procedure. Its details are reported in Appendix A.

The procedure consists of several stages. At the first stage, we derive a weakly-nonlinear form of (28) for both symmetric A^p_{Ab} and antisymmetric A^r_{ab} components up to the third polynomial order with respect to the sloshing-related generalised coordinates (Appendix A.2), and, in parallel way, we derive analogous weakly-nonlinear expressions for A^{pp}_{Ab}, A^{pr}_{Ab}, A^{rr}_{ab} keeping the second-order polynomial terms (Appendix A.3).

At the second stage, we asymptotically resolve (25) with respect to the generalised velocities, whose weakly-nonlinear structure possesses the form

$$\begin{aligned} P_{Cd} = \mathbb{Z}^p_{Cd}\dot{p}_{Cd} + \sum_{Mnlijk} \mathbb{Z}^{prr,Cd}_{Mi,nj,lk} p_{Mi} r_{nj} \dot{r}_{lk} + \sum_{MNLijk} \mathbb{Z}^{ppp,Cd}_{Mi,Nj,Lk} p_{Mi} p_{Nj} \dot{p}_{Lk} \\ + \sum_{MNij} \mathbb{Z}^{pp,Cd}_{Mi,Nj} p_{Mi} \dot{p}_{Nj} + \sum_{mnij} \mathbb{Z}^{rr,Cd}_{mi,nj} r_{mi} \dot{r}_{nj} + \sum_{mnLijk} \mathbb{Z}^{rrp,Cd}_{mi,nj,Lk} r_{mi} r_{nj} \dot{p}_{Lk}, \end{aligned} \tag{33a}$$

$$\begin{aligned} R_{cd} = \mathbb{Z}^r_{cd}\dot{r}_{cd} + \sum_{MnLijk} \mathbb{Z}^{prp,cd}_{Mi,nj,Lk} p_{Mi} r_{nj} \dot{p}_{Lk} + \sum_{mnlijk} \mathbb{Z}^{rrr,cd}_{mi,nj,lk} r_{mi} r_{nj} \dot{r}_{lk} \\ + \sum_{Mnij} \mathbb{Z}^{pr,cd}_{Mi,nj} p_{Mi} \dot{r}_{nj} + \sum_{mNij} \mathbb{Z}^{rp,cd}_{mi,Nj} r_{mi} \dot{p}_{Nj} + \sum_{MNlijk} \mathbb{Z}^{ppr,cd}_{Mi,Nj,lk} p_{Mi} p_{Nj} \dot{r}_{lk}. \end{aligned} \tag{33b}$$

Explicit expressions for the $\mathbb{Z}$-coefficients are given in Appendix A.4.

Elements of the vector $\boldsymbol{l}$ by (13) are presented in the curvilinear coordinate system and expressed as follows

$$
\begin{aligned}
l_1 &= \sum_{MNLijk} \mathbf{l}^{xppp}_{Mi,Nj,Lk} p_{Mi} p_{Nj} p_{Lk} + \sum_{Mnlijk} \mathbf{l}^{xprr}_{Mi,nj,lk} p_{Mi} r_{nj} r_{lk} \\
&\quad + \sum_{MNij} \mathbf{l}^{xpp}_{Mi,Nj} p_{Mi} p_{Nj} + \sum_{mnij} \mathbf{l}^{xrr}_{mi,nj} r_{mi} r_{nj} + \mathbf{l}^{x}, \\
l_2 &= \sum_{Mi} \hat{\mathbf{l}}^{yp}_{Mi} p_{Mi} + \sum_{MNij} \hat{\mathbf{l}}^{ypp}_{Mi,Nj} p_{Mi} p_{Nj} + \sum_{mnij} \hat{\mathbf{l}}^{yrr}_{mi,nj} r_{mi} r_{nj} \\
&\quad + \sum_{MNLijk} \hat{\mathbf{l}}^{yppp}_{Mi,Nj,Lk} p_{Mi} p_{Nj} p_{Lk} + \sum_{Mnlijk} \hat{\mathbf{l}}^{yprr}_{Mi,nj,lk} p_{Mi} r_{nj} r_{lk}, \\
l_3 &= \sum_{mi} \hat{\mathbf{l}}^{zp}_{mi} r_{mi} + \sum_{Mnij} \hat{\mathbf{l}}^{zpr}_{Mi,nj} p_{Mi} r_{nj} + \\
&\quad + \sum_{MNlijk} \hat{\mathbf{l}}^{zppr}_{Mi,Nj,lk} p_{Mi} p_{Nj} r_{lk} + \sum_{mnlijk} \hat{\mathbf{l}}^{zrrr}_{mi,nj,lk} r_{mi} r_{nj} r_{lk},
\end{aligned}
\tag{34}
$$

where the coefficients $\hat{\mathbf{l}}^{\mathbf{r}\beta}_{Mi}$, $\hat{\mathbf{l}}^{\mathbf{r}\beta\beta}_{Mi,Nj}$, $\hat{\mathbf{l}}^{\mathbf{r}\beta\beta\beta}_{Mi,Nj,Lk}$ are defined in Appendix A.5.

Finally, at the final stage, we derive the following infinite-dimensional modal equations

$$
\begin{aligned}
L_{p_{Eh}} &= \sum_{Mi} \delta_{ME} \delta_{ih} \mathbf{d}^{p,Eh}_{Mi} \ddot{p}_{Mi} + \sum_{Mi} \delta_{ME} \delta_{ih} \mathbf{g}^{p,Eh}_{Mi} p_{Mi} + \sum_{mnij} \mathbf{g}^{rr,Eh}_{mi,nj} r_{mi} r_{nj} \\
&+ \sum_{MNij} \mathbf{g}^{pp,Eh}_{Mi,Nj} p_{Mi} p_{Nj} + \sum_{MNij} \mathbf{t}^{pp,Eh}_{Mi,Nj} \dot{p}_{Mi} \dot{p}_{Nj} + \sum_{MNLijk} \mathbf{g}^{ppp,Eh}_{Mi,Nj,Lk} p_{Mi} p_{Nj} p_{Lk} \\
&+ \sum_{Mnlijk} \mathbf{g}^{prr,Eh}_{Mi,nj,lk} p_{Mi} r_{nj} r_{lk} + \sum_{MNij} \mathbf{d}^{pp,Eh}_{Mi,Nj} p_{Mi} \ddot{p}_{Nj} + \sum_{Mnlijk} \mathbf{d}^{prr,Eh}_{Mi,nj,lk} p_{Mi} r_{nj} \ddot{r}_{lk} \\
&+ \sum_{mnij} \mathbf{t}^{rr,Eh}_{mi,nj} \dot{r}_{mi} \dot{r}_{nj} + \sum_{mnij} \mathbf{d}^{rr,Eh}_{mi,nj} r_{mi} \ddot{r}_{nj} + \sum_{MNLijk} \mathbf{t}^{ppp,Eh}_{Mi,Nj,Lk} p_{Mi} \dot{p}_{Nj} \dot{p}_{Lk} \\
&+ \sum_{MNLijk} \mathbf{d}^{ppp,Eh}_{Mi,Nj,Lk} p_{Mi} p_{Nj} \ddot{p}_{Lk} + \sum_{Mnlijk} \mathbf{t}^{prr,Eh}_{Mi,nj,lk} p_{Mi} \dot{r}_{nj} \dot{r}_{lk} \\
&+ \sum_{mNlijk} \mathbf{t}^{rpr,Eh}_{mi,Nj,lk} r_{mi} \dot{p}_{Nj} \dot{r}_{lk} + \sum_{mnLijk} \mathbf{d}^{rrp,Eh}_{mi,nj,Lk} r_{mi} r_{nj} \ddot{p}_{Lk} \\
&= -(\ddot{\eta}_2 - g\eta_6 - S_h \ddot{\eta}_6) \Lambda_{E1} e_h,
\end{aligned}
\tag{35a}
$$

$$
\begin{aligned}
L_{r_{eh}} &= \sum_{mi} \delta_{me} \delta_{ih} \mathbf{d}^{r,eh}_{mi} \ddot{r}_{mi} + \sum_{mi} \delta_{me} \delta_{ih} \mathbf{g}^{r,eh}_{mi} r_{mi} + \sum_{Mnij} \mathbf{g}^{pr,eh}_{Mi,nj} p_{Mi} r_{nj} \\
&+ \sum_{MNlijk} \mathbf{g}^{ppr,eh}_{Mi,Nj,lk} p_{Mi} p_{Nj} r_{lk} + \sum_{mnlijk} \mathbf{g}^{rrr,eh}_{mi,nj,lk} r_{mi} r_{nj} r_{lk} + \sum_{Mnij} \mathbf{t}^{pr,eh}_{Mi,nj} \dot{p}_{Mi} \dot{r}_{nj}
\end{aligned}
$$

$$+\sum_{Mnij}\mathbf{d}^{pr,eh}_{Mi,nj}p_{Mi}\ddot{r}_{nj}+\sum_{mNLijk}\mathbf{t}^{rpp,eh}_{mi,Nj,Lk}r_{mi}\dot{p}_{Nj}\dot{p}_{Lk}+\sum_{MnLijk}\mathbf{d}^{prp,eh}_{Mi,nj,Lk}p_{Mi}r_{nj}\ddot{p}_{Lk}$$
$$+\sum_{mNij}\mathbf{d}^{rp,eh}_{mi,Nj}r_{mi}\ddot{p}_{Nj}+\sum_{MNlijk}\mathbf{t}^{ppr,eh}_{Mi,Nj,lk}p_{Mi}\dot{p}_{Nj}\dot{r}_{lk}+\sum_{MNlijk}\mathbf{d}^{ppr,eh}_{Mi,Nj,lk}p_{Mi}p_{Nj}\ddot{r}_{lk}$$
$$+\sum_{mnlijk}\mathbf{t}^{rrr,eh}_{mi,nj,lk}r_{mi}\dot{r}_{nj}\dot{r}_{lk}+\sum_{mnlijk}\mathbf{d}^{rrr,eh}_{mi,nj,lk}r_{mi}r_{nj}\ddot{r}_{lk}$$
$$=-(\ddot{\eta}_3-g\eta_5-S_h\ddot{\eta}_5)\Lambda_{e1}e_h. \quad (35b)$$

Computational formulas for the hydrodynamic coefficients **d**, **g**, and **t** are presented in Appendix A.6. These are much more complicated than those for upright rectangular [8, 11] and circular [13, 26] containers. Many of these coefficients are zero or equal to each other (see, examples in Appendix A.7). This fact was analytically established in [6, 26] for the vertical annular cylindrical tank, in [14] for the V-shape tank, as well as in [5] for the spherical tank.

5 Narimanov-Moiseev multimodal theory

5.1 Modal equations

As we remarked in Introduction, one can simplify the generic weakly-nonlinear modal equations (35) by postulating specific asymptotic relationships between the generalised coordinates $p_{Mi}(t)$ and $r_{mi}(t)$, specifying among them the first-, second- and third-order coordinates in terms of ϵ. For finite liquid depths, the most popular relationship follows from the Moiseev-Narimanov theory [31, 34, 35], which effectively handles the resonant sloshing in tanks exposed to the non-parametric harmonic excitations, i.e., when

$$\eta_1(t)\equiv 0 \quad (36)$$

with the forcing frequency close to the lowest natural sloshing frequency and the secondary resonance in the hydromechanical system can be neglected [11, 14, 16, 19, 31, 37].

For axisymmetric containers, in general, and circular conical tanks, in particular, the Narimanov-Moiseev asymptotic relationships suggest that the $\mathtt{r}_0$-scaled forcing magnitude is small, of the order $\epsilon^3 \ll 1$, but only the two primary excited lowest natural sloshing modes, differing only by the $\pi/2$-azimuthal drift, and associated with the $\mathtt{r}_0$-scaled generalised coordinates p_{11} and r_{11} possess dominant character and have the asymptotic order $O(\epsilon)$.

The trigonometric algebra by the angular coordinate leads to the following asymptotic relations for the $\mathtt{r}_0$-scaled generalised coordinates and velocities [31, 26]

$$\begin{aligned}&P_{11} \sim R_{11} \sim p_{11} \sim r_{11} = O(\epsilon),\\ &P_{2n} \sim R_{2n} \sim P_{0n} \sim p_{2n} \sim r_{2n} \sim p_{0n} = O(\epsilon^2),\\ &P_{3n} \sim R_{3n} \sim P_{1(n+1)} \sim R_{1(n+1)} \sim p_{3n} \sim r_{3n}\\ &\qquad\qquad\qquad \sim p_{1(n+1)} \sim r_{1(n+1)} = O(\epsilon^3), \quad n \geq 1,\end{aligned} \tag{37}$$

but all other generalised coordinates and velocities are of the order $o(\epsilon^3)$ and can be neglected within the framework of the Narimanov-Moiseev theory.

Applying the asymptotic rules (37) to the generic modal equations (35) and going through tedious and time-consuming derivations lead to the following infinite-dimensional Narimanov-Moiseev nonlinear modal equations

$$\begin{aligned}L_{p_{0h}} = \mu_{0h}\left(\ddot{p}_{0h} + \sigma_{0h}^2 p_{0h}\right) + d_{8,h}\left(\dot{p}_{11}^2 + \dot{r}_{11}^2\right)\\ + d_{10,h}\left(p_{11}\ddot{p}_{11} + r_{11}\ddot{r}_{11}\right) + \mathcal{G}_{0h}\left(p_{11}^2 + r_{11}^2\right) = 0,\end{aligned} \tag{38a}$$

$$\begin{aligned}L_{p_{2h}} = \mu_{2h}\left(\ddot{p}_{2h} + \sigma_{2h}^2 p_{2h}\right) + d_{7,h}\left(\dot{p}_{11}^2 - \dot{r}_{11}^2\right)\\ + d_{9,h}\left(p_{11}\ddot{p}_{11} - r_{11}\ddot{r}_{11}\right) + \mathcal{G}_{4,h}\left(p_{11}^2 - r_{11}^2\right) = 0,\end{aligned} \tag{38b}$$

$$\begin{aligned}L_{r_{2h}} = \mu_{2h}\left(\ddot{r}_{2h} + \sigma_{2h}^2 r_{2h}\right) + 2d_{7,h}\left(\dot{p}_{11}\dot{r}_{11}\right)\\ + d_{9,h}\left(p_{11}\ddot{r}_{11} + r_{11}\ddot{p}_{11}\right) + 2\mathcal{G}_{4,h} p_{11} r_{11} = 0,\end{aligned} \tag{38c}$$

$$\begin{aligned}L_{p_{11}} &= \mu_{11}\left(\ddot{p}_{11} + \sigma_{11}^2 p_{11}\right) + d_1\left(p_{11}^2\ddot{p}_{11} + p_{11}r_{11}\ddot{r}_{11} + p_{11}\dot{p}_{11}^2 + p_{11}\dot{r}_{11}^2\right)\\ &+ d_2\left(r_{11}^2\ddot{p}_{11} - p_{11}r_{11}\ddot{r}_{11} + 2r_{11}\dot{p}_{11}\dot{r}_{11} - 2p_{11}\dot{r}_{11}^2\right) + \mathcal{G}_1\left(p_{11}^3 + p_{11}r_{11}^2\right)\\ &+ \sum_{j=1}\Big(d_3^j\left(\ddot{p}_{11}p_{2j} + \ddot{r}_{11}r_{2j} + \dot{p}_{11}\dot{p}_{2j} + \dot{r}_{11}\dot{r}_{2j}\right) + d_4^j\left(p_{11}\ddot{p}_{2j} + r_{11}\ddot{r}_{2j}\right)\\ &\qquad + d_5^j\left(p_{0j}\ddot{p}_{11} + \dot{p}_{0j}\dot{p}_{11}\right) + d_6^j\left(\ddot{p}_{0j}p_{11}\right) + \mathcal{G}_2^j\left(p_{0j}p_{11}\right)\\ &\qquad + \mathcal{G}_3^j\left(p_{11}p_{2j} + r_{11}r_{2j}\right)\Big) = -(\ddot{\eta}_2 - g\eta_6 - S_1\ddot{\eta}_6)\kappa_{11}e_1,\end{aligned} \tag{38d}$$

$$L_{r_{11}} = \mu_{11}\left(\ddot{r}_{11} + \sigma_{11}^2 r_{11}\right) + d_1\left(p_{11}r_{11}\ddot{p}_{11} + r_{11}^2\ddot{r}_{11} + r_{11}\dot{p}_{11}^2 + r_{11}\dot{r}_{11}^2\right)$$

$$
\begin{aligned}
&+ d_2\left(p_{11}^2\ddot{r}_{11} - p_{11}r_{11}\ddot{p}_{11} + 2p_{11}\dot{p}_{11}\dot{r}_{11} - 2r_{11}\dot{p}_{11}^2\right) + \mathcal{G}_1\left(p_{11}^2 r_{11} + r_{11}^3\right) \\
&+ \sum_{j=1}\Big(d_3^j\left(\ddot{p}_{11}r_{2i} - \ddot{r}_{11}p_{2j} + \dot{p}_{11}\dot{r}_{2j} - \dot{r}_{11}\dot{p}_{2j}\right) + d_4^j\left(p_{11}\ddot{r}_{2j} - r_{11}\ddot{p}_{2j}\right) \\
&\qquad + d_5^j\left(p_{0j}\ddot{r}_{11} + \dot{p}_{0j}\dot{r}_{11}\right) + d_6^j\left(\ddot{p}_{0j}r_{11}\right) + \mathcal{G}_2^j\left(p_{0j}r_{11}\right) \\
&\qquad\quad + \mathcal{G}_3^j\left(p_{11}r_{2j} - r_{11}p_{2j}\right)\Big) = -(\ddot{\eta}_3 - g\eta_5 - S_1\ddot{\eta}_5)\kappa_{11}e_1, \quad \text{(38e)}
\end{aligned}
$$

$$
\begin{aligned}
L_{p_{3h}} &= \mu_{3h}\left(\ddot{p}_{3h} + \sigma_{3h}^2 p_{3h}\right) + d_{11,h}\left(p_{11}^2\ddot{p}_{11} - r_{11}^2\ddot{p}_{11} - 2p_{11}r_{11}\ddot{r}_{11}\right) \\
&\quad d_{12,h}\left(p_{11}\dot{p}_{11}^2 - p_{11}\dot{r}_{11}^2 - 2r_{11}\dot{p}_{11}\dot{r}_{11}\right) + \mathcal{G}_{6,h}\left(p_{11}^3 - 3p_{11}r_{11}^2\right) \\
&\quad + \sum_{j=1}\Big(d_{13,h}^j\left(\ddot{p}_{11}p_{2j} - \ddot{r}_{11}r_{2j}\right) + d_{14,h}^j\left(p_{11}\ddot{p}_{2j} - r_{11}\ddot{r}_{2j}\right) \\
&\qquad + d_{15,h}^j\left(\dot{p}_{11}\dot{p}_{2j} - \dot{r}_{11}\dot{r}_{2j}\right) + \mathcal{G}_{5,h}^j\left(p_{11}p_{2j} - r_{11}r_{2j}\right)\Big) = 0, \quad \text{(38f)}
\end{aligned}
$$

$$
\begin{aligned}
L_{r_{3h}} &= \mu_{3h}\left(\ddot{r}_{3h} + \sigma_{3h}^2 r_{3h}\right) + d_{11,h}\left(p_{11}^2\ddot{r}_{11} - r_{11}^2\ddot{r}_{11} + 2p_{11}r_{11}\ddot{p}_{11}\right) \\
&\quad d_{12,h}\left(r_{11}\dot{p}_{11}^2 - r_{11}\dot{r}_{11}^2 + 2p_{11}\dot{p}_{11}\dot{r}_{11}\right) + \mathcal{G}_{6,h}\left(3p_{11}^2 r_{11} - r_{11}^3\right) \\
&\quad + \sum_{j=1}\Big(d_{13,h}^j\left(\ddot{p}_{11}r_{2j} + \ddot{r}_{11}p_{2j}\right) + d_{14,h}^j\left(p_{11}\ddot{r}_{2j} + r_{11}\ddot{p}_{2j}\right) \\
&\qquad + d_{15,h}^j\left(\dot{p}_{11}\dot{r}_{2j} + \dot{r}_{11}\dot{p}_{2j}\right) + \mathcal{G}_{5,h}^j\left(p_{11}r_{2j} + r_{11}p_{2j}\right)\Big) = 0, \quad \text{(38g)}
\end{aligned}
$$

$$
\begin{aligned}
L_{p_{1k}} &= \mu_{1k}\left(\ddot{p}_{1k} + \sigma_{1k}^2 p_{1k}\right) + d_{16,k}\left(p_{11}^2\ddot{p}_{11} + p_{11}r_{11}\ddot{r}_{11}\right) \\
&\quad + d_{18,k}\left(p_{11}\dot{p}_{11}^2 + p_{11}\dot{r}_{11}^2\right) + d_{17,k}\left(r_{11}^2\ddot{p}_{11} - p_{11}r_{11}\ddot{r}_{11}\right) \\
&\quad + d_{19,k}\left(r_{11}\dot{p}_{11}\dot{r}_{11} - p_{11}\dot{r}_{11}^2\right) + \mathcal{G}_{1k}\left(p_{11}^3 + p_{11}r_{11}^2\right) \\
&+ \sum_{j=1}\Big(d_{20,k}^j\left(\ddot{p}_{11}p_{2j} + \ddot{r}_{11}r_{2j}\right) + d_{22,k}^j\left(\dot{p}_{11}\dot{p}_{2i} + \dot{r}_{11}\dot{r}_{2j}\right) + d_{23,k}^j p_{0j}\ddot{p}_{11} \\
&\qquad + d_{21,k}^j\left(p_{11}\ddot{p}_{2j} + r_{11}\ddot{r}_{2j}\right) + d_{25,k}^j\dot{p}_{0j}\dot{p}_{11} + d_{24,k}^j\ddot{p}_{0j}p_{11} \\
&+ \mathcal{G}_{3,k}^j p_{0j}p_{11} + \mathcal{G}_{2,k}^j\left(p_{11}p_{2j} + r_{11}r_{2j}\right)\Big) = -(\ddot{\eta}_2 - g\eta_6 - S_k\ddot{\eta}_6)\kappa_{1k}e_k, \quad \text{(38h)}
\end{aligned}
$$

$$
\begin{aligned}
L_{r_{1k}} &= \mu_{1k}\left(\ddot{r}_{1k} + \sigma_{1k}^2 r_{1k}\right) + d_{16,k}\left(p_{11}r_{11}\ddot{p}_{11} + r_{11}^2\ddot{r}_{11}\right) \\
&\quad + d_{18,k}\left(r_{11}\dot{p}_{11}^2 + r_{11}\dot{r}_{11}^2\right) + d_{17,k}\left(p_{11}^2\ddot{r}_{11} - p_{11}r_{11}\ddot{p}_{11}\right) \\
&\quad + d_{19,k}\left(p_{11}\dot{p}_{11}\dot{r}_{11} - r_{11}\dot{p}_{11}^2\right) + \mathcal{G}_{1k}\left(p_{11}^2 r_{11} + r_{11}^3\right)
\end{aligned}
$$

$$
+\sum_{j=1}\Big(d^j_{20,k}\left(\ddot{p}_{11}r_{2j}-\ddot{r}_{11}p_{2j}\right)+d^j_{22,k}\left(\dot{p}_{11}\dot{r}_{2j}-\dot{r}_{11}\dot{p}_{2j}\right)
$$

$$
+\,d^j_{21,k}\left(r_{11}\ddot{p}_{2j}-p_{11}\ddot{r}_{2j}\right)+d^j_{25,k}\dot{p}_{0j}\dot{r}_{11}+d^j_{23,k}p_{0j}\ddot{r}_{11}+d^j_{24,k}\ddot{p}_{0j}r_{11}
$$

$$
+\mathcal{G}^j_{3,k}p_{0j}r_{11}+\mathcal{G}^j_{2,k}\left(p_{11}r_{2j}-r_{11}p_{2j}\right)\Big)=-(\ddot{\eta}_3-g\eta_5-S_k\ddot{\eta}_5)\kappa_{1k}e_k, \quad \text{(38i)}
$$

where all the hydrodynamic coefficients are functions of the mean conical liquid shape and they can be computed by using formulas in Appendix A.7. If we keep only first seven harmonics ($m=0,1,2,3,\ i,j,h=1$) in (38), the system becomes identical to the seven-dimensional nonlinear modal system in [27].

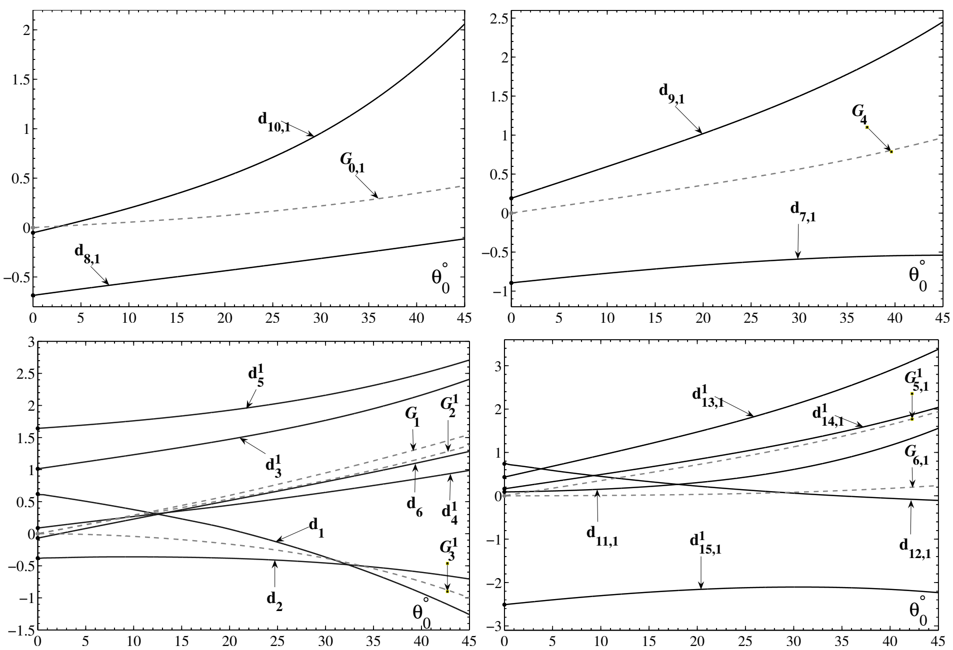

Figure 2: The nondimensional (scaled by the radius $\mathtt{r_0}$) hydrodynamic coefficients $d^j_{i,h}$, $\mathcal{G}^j_{i,k}$ of the Narimanov-Moiseev modal system (38) as functions of θ_0. The non-truncated V-shaped conical container.

Derivation and computation of the hydrodynamic coefficients require a quality control including a comparison with the limiting cases. Such a limiting case could be, for example, the vertical circular cylinder ($\theta_0 \to 0$), and the case $\mathtt{r_1} \to 0$, which corresponds to the non-truncated cone. For the last limiting case, the hydrodynamic coefficients of (38) can be compared with analogous coefficients in the five-dimensional modal system from

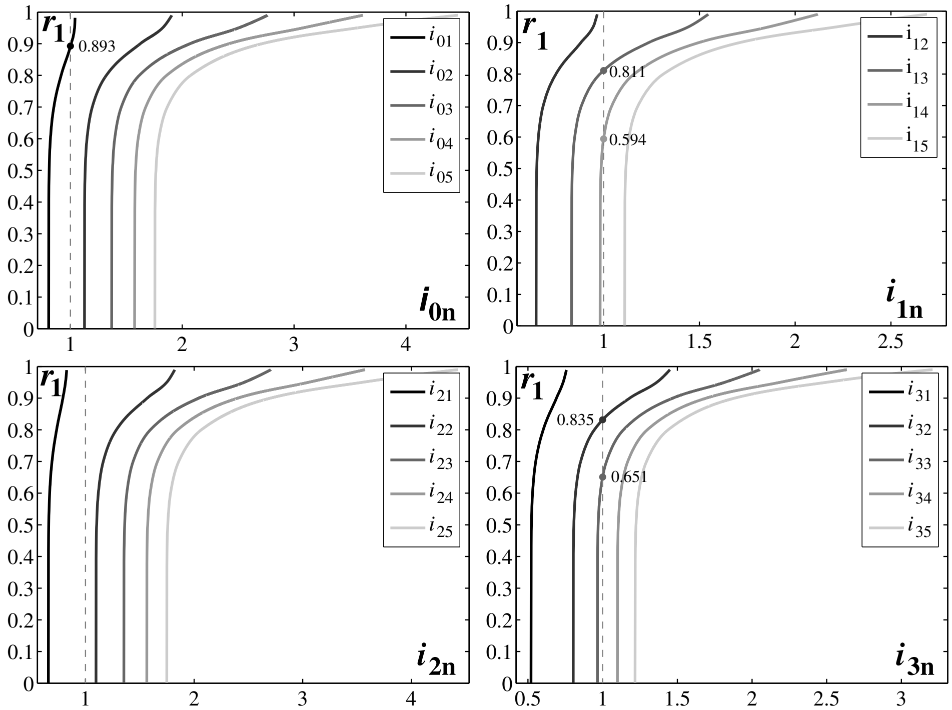

Figure 3: The graphs of $i_{mn}(\theta_0, r_1)$, which illustrate occurrence of the secondary resonance phenomena within the framework of the Narimanov-Moiseev modal theory. The calculations are done for the semi-apex cone angle $\theta_0 = 30°$; r_1 is the r_0-normalised radius of the tank bottom (truncated conical tank).

[14]. Calculations show that the hydrodynamic coefficients coincide with the tabulated coefficients from the latter paper.

Figure 2 depicts normalised (nondimesnionalised) coefficients $d^j_{i,h}$, $\mathcal{G}^j_{i,k}$ *versus* the semi-apex angle θ_0 for the V-shaped (non-truncated) tanks. The limiting case $\theta_0 = 0$ corresponds to the circular cylindrical tank with an infinite liquid depth. We compared the computed values with those for the circular tank in [27]; the limiting case is well fitted by our computations. Note that there are the $\mathcal{G}$-type coefficients in (38), which are an attribute of non-vertical walls. The graphs in figure 2 show that the limiting numerical values $\mathcal{G}$ are zeros when the semi-apex angle tends to zero.

5.2 Secondary resonances

Applying the Narimanov-Moiseev multimodal theory implicitly assumes that there are no secondary resonances in the hydromechanic system when the forcing frequency σ is close to the lowest natural sloshing frequency σ_{11}, i.e.

$$\sigma \approx \sigma_{11}.$$

The secondary resonance concept for sloshing in a circular conical tank was described in [28]. The resonance may happen when 2σ is close to one from the natural sloshing frequencies σ_{0i} and σ_{2i}, $i > 1$, or, alternatively, when 3σ tends to one from the natural sloshing frequencies σ_{3i}, $i > 1$ and σ_{1i}, $i > 2$. Necessary condition of the secondary resonance takes the form

$$\sigma_{0i} \approx \sigma, \quad \sigma_{2i} \approx \sigma, \quad \sigma_{3i} \approx \sigma, \quad \sigma_{1(i+1)} \approx \sigma, \quad i > 1, \tag{39}$$

in a neighborhood of the primary resonance zone, i.e., provided by $\sigma \approx \sigma_{11}$.

To analyze the secondary resonance with the strict equalities in (39), [28] studied $i_{mn}\ (\theta_0, r_1)$ as functions of the non-dimensional parameter r_1 (r_1 is the ratio of the bottom and free surface radii) with a fixed value of the semi-apex angle

$$i_{0n}(\theta_0, r_1) = \frac{\sigma_{0n}}{2\sigma_{11}} = \frac{1}{2}\sqrt{\frac{\kappa_{0n}}{\kappa_{11}}}, \quad i_{2n}(\theta_0, r_1) = \frac{\sigma_{2n}}{2\sigma_{11}} = \frac{1}{2}\sqrt{\frac{\kappa_{2n}}{\kappa_{11}}}, \tag{40}$$

$$\begin{aligned} i_{3n}(\theta_0, r_1) &= \frac{\sigma_{3n}}{3\sigma_{11}} = \frac{1}{2}\sqrt{\frac{\kappa_{3n}}{\kappa_{11}}}, \\ i_{1(n+1)}(\theta_0, r_1) &= \frac{\sigma_{1(n+1)}}{3\sigma_{11}} = \frac{1}{3}\sqrt{\frac{\kappa_{3(n+1)}}{\kappa_{11}}}, \quad n \geq 1. \end{aligned} \tag{41}$$

The functions $i_{mn} = i_{mn}(\theta_0, r_1)$ do not depend on the forcing frequency σ and one can see that condition $i_{mn} = 1$, for certain indices m and n, is equivalent to a strict equality in the corresponding m, n-equation of (39), which should be simultaneously fulfilled. The case $r_1 = 0$ corresponds to the V-shaped conical tank but the limit $r_1 \to 1$ implies the shallow water condition.

The calculations were done for the three semi-apex angles $\theta_0 = 30°$, $45°$ and $60°$. The strict equality $i_{01} = 1$ occurs for $r_1 = 0.8926$ implying that the first axisymmetric mode is subject to the secondary resonance for larger r_1; the double harmonics 2σ can then be resonantly amplified. As for the triple harmonics 3σ, the secondary resonance can occur for the modes $(1, 3)$, $(1, 4)$, $(3, 2)$ and $(3, 3)$. So, for $r_1 = 0.651$, the modes $(3, 3)$ are subject to the secondary resonance but the modes $(3,2)$ are resonantly excited at $r_1 = 0.835$. Finally, the modes $(1,3)$ are exposed to the secondary resonance at $r_1 = 0.8116$ and the modes $(1,4)$ – at $r_1 = 0.5939$. The secondary resonances for the semi-apex angle $\theta_0 = 30°$ are not possible for the non-dimensional radius $r_1 \lesssim 0.5$.

5.3 Steady-state (periodic) solutions and their stability

We consider the forced steady-state resonant liquid sloshing caused by the lateral horizontal harmonic tank excitation

$$\eta_2(t) = \eta_{02}\cos(\sigma t); \quad \eta_i(t) = 0, \quad i \neq 2. \tag{42}$$

The task consists of finding all periodic solutions of the Narimanov-Moiseev modal equations and analysing their stability. To find these solutions, we pose $r_{11}(t)$ and $p_{11}(t)$ as the Fourier series with unknown coefficients

$$\begin{aligned} p_{Mi}(t) =& \sum_{k=1}^{\infty} \left(B_{M(2k-1)} \cos k\sigma t + B_{M(2k)} \sin k\sigma t \right), \\ r_{mi}(t) =& \sum_{k=1}^{\infty} \left(A_{m(2k-1)} \cos k\sigma t + A_{m(2k)} \sin k\sigma t \right), \end{aligned} \tag{43}$$

where, according to the Narimanov-Moiseev asymptotics, the lowest-order asymptotic terms are

$$\begin{aligned} p_{11}(t) &= B_c \cos\sigma t + B_s \sin\sigma t + o(\epsilon), \\ r_{11}(t) &= A_c \cos\sigma t + A_s \sin\sigma t + o(\epsilon). \end{aligned} \tag{44}$$

Substituting (44) into the modal equations (38a)-(38c) and gathering the second-harmonic quantities lead to the following solutions

$$\begin{aligned} p_{0h}(t) = \left(A_c^2 + A_s^2 + B_c^2 + B_s^2\right) \mathbf{o}_{0h0} + 2\left(A_c A_s + B_c B_s\right) \mathbf{o}_{0h2} \sin 2\sigma t \\ + \left(A_c^2 - A_s^2 + B_c^2 - B_s^2\right) \mathbf{o}_{0h2} \cos 2\sigma t, \end{aligned} \tag{45a}$$

$$\begin{aligned} p_{2h}(t) = \left(-A_c^2 - A_s^2 + B_c^2 + B_s^2\right) \mathbf{o}_{2h0} + 2\left(B_c B_s - A_c A_s\right) \mathbf{o}_{2h2} \sin 2\sigma t \\ + \left(-A_c^2 + A_s^2 + B_c^2 - B_s^2\right) \mathbf{o}_{2h2} \cos 2\sigma t, \end{aligned} \tag{45b}$$

$$r_{2h}(t) = 2\left(A_c B_c + A_s B_s\right) \mathbf{o}_{2h0} + 2\left(A_s B_c + A_c B_s\right) \mathbf{o}_{2h2} \sin 2\sigma t$$

$$+ 2\left(A_cB_c - A_sB_s\right)\mathbf{o}_{2h2}\cos 2\sigma t \quad (45c)$$

for the second-order generalised coordinates, but inserting (45) and (44) into (38f)-(38i) produces

$$\begin{aligned} p_{3h}(t) = &\left(\left(3A_c^2 + A_s^2 - B_c^2 - B_s^2\right)B_c + 2A_cA_sB_s\right)\mathbf{o}_{3h1}\cos\sigma t \\ &+ \left(\left(A_c^2 + 3A_s^2 - B_c^2 - B_s^2\right)B_s + 2A_cA_sB_c\right)\mathbf{o}_{3h1}\sin\sigma t \\ &+ \left(\left(3A_c^2 - 3A_s^2 - 3B_c^2 + B_s^2\right)B_s + 6A_cA_sB_c\right)\mathbf{o}_{3h3}\sin 3\sigma t \\ &+ \left(\left(3A_c^2 - 3A_s^2 - B_c^2 + 3B_s^2\right)B_c - 6A_cA_sB_s\right)\mathbf{o}_{3h3}\cos 3\sigma t, \end{aligned} \quad (46a)$$

$$\begin{aligned} r_{3h}(t) = &\left(\left(A_c^2 + A_s^2 - 3B_c^2 - B_s^2\right)A_c - 2A_sB_cB_s\right)\mathbf{o}_{3h1}\cos\sigma t \\ &+ \left(\left(A_c^2 + A_s^2 - B_c^2 - 3B_s^2\right)A_s - 2A_cB_cB_s\right)\mathbf{o}_{3h1}\sin\sigma t \\ &+ \left(\left(A_c^2 - 3A_s^2 - 3B_c^2 + 3B_s^2\right)A_c + 6A_sB_cB_s\right)\mathbf{o}_{3h3}\cos 3\sigma t \\ &+ \left(\left(3A_c^2 - A_s^2 - 3B_c^2 + 3B_s^2\right)A_s - 6A_cB_cB_s\right)\mathbf{o}_{3h3}\sin 3\sigma t, \end{aligned} \quad (46b)$$

$$\begin{aligned} p_{1k}(t) = &\Big(\left(\left(-A_c^2 - B_c^2 - B_s^2\right)\mathbf{o}_{1k11} - A_s^2\mathbf{o}_{1k12}\right)B_c \\ &+ A_cA_sB_s\mathbf{o}_{1k13}\Big)\cos\sigma t + \Big(\left(\left(-A_s^2 - B_c^2 - B_s^2\right)\mathbf{o}_{1k11} \\ &-A_c^2\mathbf{o}_{1k12}\right)B_s + A_cA_sB_c\mathbf{o}_{1k13}\Big)\sin\sigma t + \Big(2A_cA_sB_s \\ &+ \left(-A_c^2 + A_s^2 - B_c^2 + 3B_s^2\right)B_c\Big)\mathbf{o}_{1k3}\cos 3\sigma t \\ &+ \Big(-2A_cA_sB_c + \left(-A_c^2 + A_s^2 + B_s^2 - 3B_c^2\right)B_s\Big)\mathbf{o}_{1k3}\sin 3\sigma t, \end{aligned} \quad (46c)$$

$$\begin{aligned} r_{1k}(t) = &\Big(\left(\left(-A_c^2 - A_s^2 - B_c^2\right)\mathbf{o}_{1k11} - B_s^2\mathbf{o}_{1k12}\right)A_c \\ &+ A_sB_cB_s\mathbf{o}_{1k13}\Big)\cos\sigma t + \Big(\left(\left(-A_c^2 - A_s^2 - B_s^2\right)\mathbf{o}_{1k11} \\ &- B_c^2\mathbf{o}_{1k12}\right)A_s + A_cB_cB_s\mathbf{o}_{1k13}\Big)\sin\sigma t + \Big(2A_sB_cB_s \\ &\left(-A_c^2 + 3A_s^2 - B_c^2 + B_s^2\right)A_c\Big)\mathbf{o}_{1k3}\cos 3\sigma t \\ &+ \Big(-2A_sB_cB_s + \left(-3A_c^2 + A_s^2 - B_c^2 + B_s^2\right)A_s\Big)\mathbf{o}_{1k3}\sin 3\sigma t. \end{aligned} \quad (46d)$$

Here the coefficients $\mathbf{o}_{mhk}$ are computed by the following formulas

$$\begin{aligned} \mathbf{o}_{0h0} &= \frac{d_{10,h} - d_{8,h}}{2\bar{\sigma}_{0h}^2} - \frac{\mathcal{G}_{0,h}}{2}, \quad \mathbf{o}_{0h2} = \frac{d_{10,h} + d_{8,h} - \mathcal{G}_{0,h}\bar{\sigma}_{0h}^2}{2\left(\bar{\sigma}_{0h}^2 - 4\right)}, \\ \mathbf{o}_{2h0} &= \frac{d_{9,h} - d_{7,h}}{2\bar{\sigma}_{2h}^2} - \frac{\mathcal{G}_{4,h}}{2}, \quad \mathbf{o}_{2h2} = \frac{d_{9,h} + d_{7,h} - \mathcal{G}_{4,h}\bar{\sigma}_{2h}^2}{2\left(\bar{\sigma}_{2h}^2 - 4\right)}, \end{aligned} \quad (47a)$$

$$\mathbf{o}_{3h1} = \frac{1}{4\left(\bar{\sigma}_{3h}^2 - 1\right)}\Big(\bar{\sigma}_{3h}^2\left(3\mathcal{G}_{6,h} + 4S_0^{\mathcal{G}_{5,h}} + 2S_2^{\mathcal{G}_{5,h}}\right) - 4S_0^{d_{13,h}} - 2S_2^{d_{13,h}}$$
$$- 8S_2^{d_{14,h}} + 4S_2^{d_{15,h}} - 3d_{11,h} + d_{12,h}\Big),$$
$$\mathbf{o}_{3h3} = \frac{1}{4\left(\sigma_{3h}^2 - 9\right)}\Big(- d_{11,h} - d_{12,h} - 2S_2^{d_{13,h}} - 8S_2^{d_{14,h}} - 4S_2^{d_{15,h}}$$
$$+ \bar{\sigma}_{3h}^2\left(\mathcal{G}_{6,h} + 2S_2^{\mathcal{G}_{5,h}}\right)\Big) \quad (47b)$$

and the coefficients $\mathbf{o}_{mk1i}$ ($\mathbf{o}_{1k13} = \mathbf{o}_{1k12} - \mathbf{o}_{1k11}$) are determined by

$$\mathbf{o}_{1k11} = \frac{1}{4(\bar{\sigma}_{1k}^2 - 1)}\Big(- 4C_0^{d_{23,k}} - 2C_2^{d_{23,k}} - 8C_2^{d_{24,k}} + 4C_2^{d_{25,k}}$$
$$- 4S_0^{d_{20,k}} - 2S_2^{d_{20,k}} - 8S_2^{d_{21,k}} + 4S_2^{d_{22,k}} - 4d_{19,k} + 3d_{18,k} - d_{16,k}$$
$$+ \bar{\sigma}_{1,k}^2\left[3\mathcal{G}_{1,k} + 4C_0^{\mathcal{G}_{3,k}} + 2C_2^{\mathcal{G}_{3,k}} + 4S_0^{\mathcal{G}_{2,k}} + 2S_2^{\mathcal{G}_{2,k}}\right]\Big),$$
$$\mathbf{o}_{1k12} = \frac{1}{4(\bar{\sigma}_{1k}^2 - 1)}\Big(- 4C_0^{d_{23,k}}2C_2^{d_{23,k}} + 8C_2^{d_{24,k}} - 4C_2^{d_{25,k}}$$
$$+ \bar{\sigma}_{1k}^2\left[\mathcal{G}_{1k} + 4C_0^{\mathcal{G}_{3,k}} - 2C_2^{\mathcal{G}_{3,k}} - 4S_0^{\mathcal{G}_{2,k}} + 6S_2^{\mathcal{G}_{2,k}}\right] + 4S_0^{d_{20,k}}$$
$$- 6S_2^{d_{20,k}} - 24S_2^{d_{21,k}} + 12S_2^{d_{22,k}} - 4d_{19,k} + 3d_{18,k} - d_{16,k}\Big),$$
$$\mathbf{o}_{1k3} = \frac{1}{4(\bar{\sigma}_{1k}^2 - 9)}\Big(- d_{16,k} - d_{18,k} - 2C_2^{d_{23,k}} - 8C_2^{d_{24,k}} - 4C_2^{d_{25,k}}$$
$$- 2S_2^{d_{20,k}} - 8S_2^{d_{21,k}} - 4S_2^{d_{22,k}} + \bar{\sigma}_{1k}^2\left[\mathcal{G}_{1k} + 2C_2^{\mathcal{G}_{3,k}} + 2S_2^{\mathcal{G}_{2,k}}\right]\Big) \quad (47c)$$

so that

$$\bar{\sigma}_{mi}^2 = \frac{\sigma_{mi}^2}{\sigma^2}, \quad C_i^{d_{k,h}} = \sum_j d_{k,h}^j \mathbf{o}_{0ji}, \quad S_i^{d_{k,h}} = \sum_j d_{k,h}^j \mathbf{o}_{2ji},$$
$$S_i^{\mathcal{G}_{k,h}} = \sum_j \mathcal{G}_{k,h}^j \mathbf{o}_{2ji}, \quad C_i^{\mathcal{G}_{k,h}} = \sum_j \mathcal{G}_{k,h}^j \mathbf{o}_{0ji}. \quad (48)$$

By substituting the expressions (44) and (45) into (38d) and (38e) and using the Fredholm alternative

$$\int_0^{\frac{2\pi}{\sigma}} L_{\{p_{11},r_{11}\}} \cos\sigma t\,dt = 0, \quad \int_0^{\frac{2\pi}{\sigma}} L_{\{p_{11},r_{11}\}} \sin\sigma t\,dt = 0, \quad (49)$$

we arrive at the following four nonlinear algebraic equations with respect to the amplitude parameters A_s, A_c, B_s, B_c

$$\begin{cases} A_c \left((\bar\sigma_{11}^2 - 1) + \left(A_c^2 + A_s^2 + B_c^2 \right) m_1 + B_s^2 m_2 \right) + A_s B_c B_s m_3 = e_1 \eta_{2a}, \\ A_s \left((\bar\sigma_{11}^2 - 1) + \left(A_c^2 + A_s^2 + B_s^2 \right) m_1 + B_c^2 m_2 \right) + A_c B_c B_s m_3 = 0, \\ B_c \left((\bar\sigma_{11}^2 - 1) + \left(A_c^2 + B_c^2 + B_s^2 \right) m_1 + A_s^2 m_2 \right) + A_c A_s B_s m_3 = 0, \\ B_s \left((\bar\sigma_{11}^2 - 1) + \left(A_s^2 + B_c^2 + B_s^2 \right) m_1 + A_c^2 m_2 \right) + A_c A_s B_c m_3 = 0, \end{cases} \tag{50}$$

where coefficient m_i $m_1, m_2, m_3 = (m_1 - m_2)$ are computed by the formulas

$$m_1 = \bar\sigma_{11}^2 \left[\tfrac{3}{4}\mathcal{G}_1 + C_0^{\mathcal{G}_2} + \tfrac{1}{2} C_2^{\mathcal{G}_2} + S_0^{\mathcal{G}_3} + \tfrac{1}{2} S_2^{\mathcal{G}_3} \right] - C_0^{d_5} + \tfrac{1}{2} C_2^{d_5} - 2C_2^{d_6} - S_0^{d_3} + \tfrac{1}{2} S_2^{d_3} + 2S_2^{d_4} - \tfrac{1}{2} d_1, \tag{51a}$$

$$m_2 = \tfrac{1}{2} d_1 - 2 d_2 - C_0^{d_5} - \tfrac{1}{2} C_2^{d_5} + 2C_2^{d_6} + S_0^{d_3} + \tfrac{3}{2} S_2^{d_3} - 6S_2^{d_4} + \bar\sigma_{11}^2 \left[\tfrac{1}{4}\mathcal{G}_1 + C_0^{\mathcal{G}_2} - \tfrac{1}{2} C_2^{\mathcal{G}_2} - S_0^{\mathcal{G}_3} + \tfrac{3}{2} S_2^{\mathcal{G}_3} \right]. \tag{51b}$$

An analysis of the (secular) system (50) in [6] *proved* that $A_s = B_c = 0$ and, therefore, (50) reduces to the system of two algebraic equations

$$\begin{cases} A_c \left((\bar\sigma_{11}^2 - 1) + A_c^2 m_1 + B_s^2 m_2 \right) = e_1 \eta_{2a}, \\ B_s \left((\bar\sigma_{11}^2 - 1) + B_s^2 m_1 + A_c^2 m_2 \right) = 0, \end{cases} \tag{52}$$

whose solutions depend on the coefficients m_i, which are, in turn, functions of $\bar{\mathtt{r}}_1$, $\bar\sigma_1(\bar{\mathtt{r}}_1)$ and θ_0 ($m_i = m_i(\bar\sigma_1, \bar{\mathtt{r}}_1, \theta_0)$).

The secular system (52) has two types of analytical solutions. The first type implies $B_s = 0$ and corresponds to the so-called planar steady-state sloshing, but the second solution means $B_s \neq 0$; it determines swirling (angularly propagating wave). The *planar* waves ($A_c \neq 0$, $A_s = B_c = B_s = 0$) correspond to the solution

$$\begin{aligned} r_{11}(t) &= A_c \cos\sigma t, \\ r_{1k}(t) &= -A_c^3\, \mathbf{o}_{1k11} \cos\sigma t - A_c^3\, \mathbf{o}_{1k3} \cos 3\sigma t, \\ p_{0h}(t) &= A_c^2\, \mathbf{o}_{0h0} + A_c^2\, \mathbf{o}_{0h2} \cos 2\sigma t, \\ p_{2h}(t) &= -A_c^2\, \mathbf{o}_{2h0} - A_c^2\, \mathbf{o}_{2h2} \cos 2\sigma t, \\ r_{3h}(t) &= A_c^3\, \mathbf{o}_{3h1} \cos\sigma t + A_c^3\, \mathbf{o}_{3h3} \cos 3\sigma t, \\ p_{11}(t) &= p_{1k}(t) = p_{3h}(t) = r_{2h}(t) = 0, \end{aligned} \tag{53}$$

where A_c comes from the cubic equation

$$m_1 A_c^3 + (\bar{\sigma}_{11}^2 - 1)A_c - e_1\eta_{2a} = 0. \tag{54}$$

The *swirling* ($A_c \neq 0$, $B_s \neq 0$, $B_c = A_s = 0$) corresponds to

$$\begin{aligned}
&r_{11}(t) = A_c\cos\sigma t, \quad p_{11}(t) = B_s\sin\sigma t, \quad r_{2h}(t) = 2A_cB_s\,\mathbf{o}_{2h2}\sin 2\sigma t,\\
&p_{2h}(t) = -\left(A_c^2 - B_s^2\right)\mathbf{o}_{2h0} - \left(A_c^2 + B_s^2\right)\mathbf{o}_{2h2}\cos 2\sigma t,\\
&p_{0h}(t) = \left(A_c^2 + B_s^2\right)\mathbf{o}_{0h0} + \left(A_c^2 - B_s^2\right)\mathbf{o}_{0h2}\cos 2\sigma t,\\
&r_{3h}(t) = \left(\left(A_c^2 - B_s^2\right)A_c\right)\mathbf{o}_{3h1}\cos\sigma t + \left(\left(A_c^2 + 3B_s^2\right)A_c\right)\mathbf{o}_{3h3}\cos 3\sigma t,\\
&p_{3h}(t) = \left(\left(A_c^2 - B_s^2\right)B_s\right)\mathbf{o}_{3h1}\sin\sigma t + \left(\left(3A_c^2 + B_s^2\right)B_s\right)\mathbf{o}_{3h3}\sin 3\sigma t,\\
&r_{1k}(t) = -\left(A_c^2\,\mathbf{o}_{1k11} + B_s^2\,\mathbf{o}_{1k12}\right)A_c\cos\sigma t - \left(A_c^2 - B_s^2\right)A_c\mathbf{o}_{1k3}\cos 3\sigma t,\\
&p_{1k}(t) = -\left(A_c^2\,\mathbf{o}_{1k12} + B_s^2\,\mathbf{o}_{1k11}\right)B_s\sin\sigma t - \left(A_c^2 - B_s^2\right)B_s\mathbf{o}_{1k3}\sin 3\sigma t,
\end{aligned} \tag{55}$$

where A_c and B_s are roots of

$$A_c\left(\bar{\sigma}_{11}^2 - 1 + A_c^2 m_1 + B_s^2 m_3\right) = e_1\eta_{2a}, \quad B_s^2 m_1 + A_c^2 m_3 = 1 - \bar{\sigma}_{11}^2. \tag{56}$$

To study the hydrodynamic stability of the constructed asymptotic periodic solutions, we use the multi-timing technique combined with the linear Lyapunov method. Limitations of this approach was extensively discussed in [2] (chapters 8 and 9). The stability analysis implies introducing the slow time $\tau(t) = \epsilon^2\sigma t/2$ and considering the small (linear) perturbations of the lowest-order generalised coordinates (44)

$$\begin{aligned}
p_{11}(t) &= (B_c + \tilde{\beta}(\tau))\cos\sigma t + (B_s + \beta(\tau))\sin\sigma t + O(\epsilon),\\
r_{11}(t) &= (A_c + \alpha(\tau))\cos\sigma t + (A_s + \tilde{\alpha}(\tau))\sin\sigma t + O(\epsilon),
\end{aligned} \tag{57}$$

where A_c, B_s are known and come from the secular equations (56) but the unknowns $\alpha, \tilde{\alpha}, \beta, \tilde{\beta}$ imply linear perturbations; they are functions of τ.

Inserting (57) into the Narimanov-Moiseev modal equations and linearising relative to $\alpha, \tilde{\alpha}, \beta, \tilde{\beta}$ leads to the linear system of ordinary differential equations $d\boldsymbol{c}/d\tau + \boldsymbol{C}\boldsymbol{c} = 0$, where $\boldsymbol{c} = (\alpha, \tilde{\alpha}, \beta, \tilde{\beta})^T$ and the matrix $\boldsymbol{C}$ consists of the elements

$$
\begin{aligned}
&c_{11} = -c_{22}, \quad c_{13} = -c_{32} = -c_{42},\\
&c_{11} = -2m_1 A_c A_s - m_3 B_c B_s, \quad c_{13} = -2m_1 A_s B_s - m_3 A_c B_c,\\
&c_{14} = -2m_1 A_s B_c - m_3 A_c B_s, \quad c_{23} = 2m_2 A_c B_s + m_3 A_s B_c,\\
&c_{24} = 2m_1 A_c B_c + m_3 A_s B_s, \quad c_{33} = 2m_1 B_c B_s + m_3 A_c A_s,\\
&c_{12} = -(\bar{\sigma}_{11}^2 - 1) - m_1(A_c^2 + B_s^2 + 3A_s^2) - m_2 B_c^2,\\
&c_{21} = (\bar{\sigma}_{11}^2 - 1) + m_1(A_s^2 + B_c^2 + 3A_c^2) + m_2 B_s^2, \ c_{23} = -c_{41},\\
&c_{34} = (\bar{\sigma}_{11}^2 - 1) + m_1(A_c^2 + B_s^2 + 3B_c^2) + m_2 A_s^2, \ c_{24} = -c_{31},\\
&c_{43} = -(\bar{\sigma}_{11}^2 - 1) - m_1(A_s^2 + B_c^2 + 3B_s^2) - m_2 A_c^2, \ c_{33} = -c_{44},
\end{aligned}
\tag{58}
$$

The instability occurs when at least one eigenvalue of the 4x4 matrix $\boldsymbol{C}$ has a nonzero positive real part. Computations give the following characteristic polynomials

$$\lambda^4 + c_1\lambda^2 + c_0 = 0, \tag{59}$$

where c_0 is the determinant of matrix $\boldsymbol{C}$, and c_1 is a complicated function of the elements of $\boldsymbol{C}$ As [5] shows, the stability requires

$$c_0 > 0, \quad c_1 > 0, \quad c_1^2 - 4c_0 > 0. \tag{60}$$

5.4 Illustrative response curves

The amplitude response curves of the steady-state resonance sloshing regimes can be best interpreted in terms of the two lowest-order wave amplitude parameters A_c and B_s (scaled by $\mathtt{r}_0$) *versus* the normalised forcing frequency σ/σ_{11}. Figure 4 exemplifies the amplitude response curves by using computations done with the fixed mean liquid domain, which is defined by the semi-apex angle $\theta_0 = 30°$ and the ratio $\mathtt{r}_1/\mathtt{r}_0 = 0.7427$. The nondimensional forcing amplitude is $\eta_{2a} = 0.00125$.

The solid lines correspond to the stable steady-state sloshing but the dashed ones imply the hydrodynamic instability. Panel (c) demonstrates the three-dimensional response curves in the $(\sigma/\sigma_{11}, |A_c|, |B_s|)$-space but other panels (a) and (b) show projections of the branching on the $(\sigma/\sigma_{11}, |A_c|)$ and $(\sigma/\sigma_{11}, |B_s|)$ planes. The planar steady-state waves of (53) are easily distinguished in (c) as belonging to the $(\sigma/\sigma_{11}, |A_c|, |B_s|)$ plane. All the three-dimensional curves ($B_s \neq 0$) correspond to swirling.

The branching contains three bifurcation points U, H and P whose positions determine the effective frequency ranges where stable planar, swirling or irregular waves are theoretically expected. This fact is illustrated in (a).

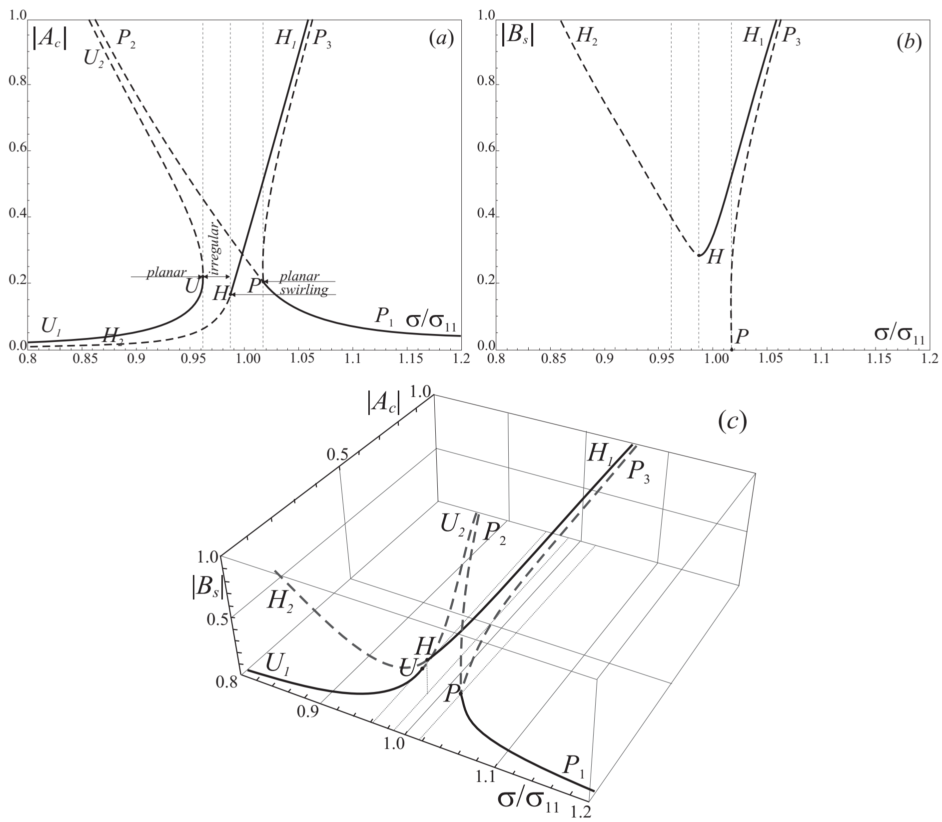

Figure 4: The amplitude response curves $(\sigma/\sigma_{11}, |A_c|, |B_s|)$ for the lateral harmonic excitation of a circular truncated conical tank with the semi-apex angle $\theta_0 = 30°$ and the bottom radius $\mathtt{r}_1/\mathtt{r}_0 = 0.7427$. The nondimensional forcing amplitude is $\eta_{2a} = 0.00125$. The three-dimensional view in the panel (c) and its projection on the $(\sigma/\sigma_{11}, |A_c|)$ (panel a) and $(\sigma/\sigma_{11}, |B_s|)$ (panel c) planes. Planar (standing) waves ($B_s = 0$) and swirling are detected. The solid lines imply the stability. All steady-state wave regimes are not stable in the frequency range determined by the turning point U and the Hopf bifurcation point H.

The forcing frequencies to the left of U lead to the planar steady-state wave. In the frequency range between U and H, both planar and swirling waves are unstable and one should expect irregular, chaotic wave patterns where switches between planar and swirling occur on a long time scale (the range is marked as irregular). In the frequency range between H and P, only stable swirling exists, but the forcing frequencies on the right of P may lead to either planar or swirling steady-state waves depending on the initial transients.

Specifically, the planar wave response demonstrates the soft-spring behaviour but the response curves associated with swirling have the hard-spring behaviour. This is similar to sloshing in a circular base tank with a fairly deep liquid depth [25]. This kind of branching may change with varying the geometric parameters θ_0 and $\mathtt{r}_1/\mathtt{r}_0$ as it happened for the annular base containers [6], where two geometric parameters were the liquid depth and the inner radius. A dedicated parameter study is required to identify what kind of branching occurs for different values of θ_0 and $\mathtt{r}_1/\mathtt{r}_0$. One should remember that some values of these two parameters can lead to the secondary resonance phenomenon when the Narimanov-Moiseev asymptotic theory is not applicable and an adaptive multimodal theory is required [10].

6 Concluding remarks

The authors took an opportunity for reporting specific details of the Narimanov-Moiseev analysis of the nonlinear sloshing in containers with nonvertical walls exemplifying the related formulas and derivation procedures for the case of circular conical tanks. The Narimanov-Moiseev multimodal theory is, perhaps, the only analytical approach to resonant and strongly nonlinear sloshing in rigid tanks, which makes it possible to both conduct analytical studies and perform simulations. Getting the Narimanov-Moiseev modal equations is a complicated task consisting of several stages. Tedious derivations with huge expressions are normally hidden from readers, these simply cannot be fully presented by the regular journal format. The present chapter is, most probably, the first publication where the interested readers can find and investigate them.

The Narimanov-Moiseev multimodal theory is limited to the case of no secondary resonances in the hydrodynamic system. As we showed for the circular conical tanks, the resonances may happen for certain values of the semi-apex angle θ_0 and the lower-to-upper radius ratio $\mathtt{r}_1/\mathtt{r}_0$. Handling these critical values needs an adaptive multimodal analysis.

Another problem is a lack of experimental studies devoted to the nonlinear resonant sloshing in truncated conical tanks. Being interested in these experiments to validate our theoretical results, we paid an attention to [1] where appropriate experiments were mentioned in the context of the tuned liquid dampers equipped with conical tanks. However, these experiments as well as the PhD thesis [33] basically deal with either linear sloshing or the input geometric parameters imply the secondary resonance phenomenon.

A Details of derivation

A.1 Generalised coordinate $\beta_0(t)$

The generalised coordinate $\beta_0(t)$ follows from the volume conservation condition appearing in the sloshing problem as the geometric constraint

$$V_0 = \int_0^{2\pi}\int_0^{x_{20}} x_2\left(x_{10}^2 f + x_{10}f^2 + \frac{1}{3}f^3\right)dx_2dx_3 = 0. \tag{61}$$

Resolving this constraint makes this generalised coordinate $\beta_0(t)$ an explicitly-given function of other generalised coordinates, $p_{Mi}(t)$ and $r_{mi}(t)$. The function can be found in an asymptotic sense keeping up to the $O(\epsilon^3)$-order terms (here, all generalised coordinates have the first order of smallness)

$$\beta_0 = \sum_{Mi}\beta^{pp}_{Mi,Mi}p^2_{Mi} + \sum_{mi}\beta^{rr}_{mi,mi}r^2_{mi} + \sum_{MNLijk}\beta^{ppp}_{Mi,Nj,Lk}p_{Mi}p_{Nj}p_{Lk} + \sum_{Mnlijk}\beta^{prr}_{Mi,nj,lk}p_{Mi}r_{nj}r_{lk}, \tag{62}$$

The β-coefficients in (62) are as follows

$$\beta^{pp}_{Mi,Mi} = -\frac{\Lambda^{cc}_{MM}\lambda_{Mi,Mi}}{\pi x_{10}x_{20}^2}, \quad \beta^{rr}_{mi,mi} = -\frac{\Lambda^{ss}_{mm}\lambda_{mi,mi}}{\pi x_{10}x_{20}^2},$$
$$\beta^{ppp}_{Mi,Nj,Lk} = -\frac{\Lambda^{ccc}_{MNL}\lambda_{Mi,Nj,Lk}}{3\pi x_{10}^2x_{20}^2}, \quad \beta^{prr}_{Mi,nj,lk} = -\frac{\Lambda^{css}_{Mnl}\lambda_{Mi,nj,lk}}{\pi x_{10}^2x_{20}^2}, \tag{63}$$

where we introduced the tensor-type coefficient

$$\Lambda^{\overbrace{c...c}^{N_1}\overbrace{s...s}^{N_2}}_{\underbrace{i..j}_{N_1}\underbrace{k...l}_{N_2}} = \int_{-\pi}^{\pi}\underbrace{\cos(ix_3)\cdot\ldots\cdot\cos(jx_3)}_{N_1}\cdot\underbrace{\sin(kx_3)\cdot\ldots\cdot\sin(lx_3)}_{N_2}dx_3 \tag{64}$$

for the angular coordinate and the tensor-type coefficients are responsible for the radial direction

$$\lambda_{\underbrace{Mi,\ldots,Nj}_{N_3}} = \int_0^{x_{20}} x_2\underbrace{f_{Mi}(x_2)\cdot\ldots\cdot f_{Nj}(x_2)}_{N_3}dx_2. \tag{65}$$

A.2 Integrals A^p_{Mi} and A^r_{mi} defined by (28)

Expanding A^p_{Mi} and A^r_{mi} up to the third polynomial order in p_{Mi} and r_{mi} gives

$$\begin{aligned}A^p_{Ab} &= \mathbb{A}^p_{Ab} + \mathbb{A}^{p.p}_{Ab,Ab}\,p_{Ab} + \sum_{MNij}\mathbb{A}^{p.pp}_{Ab,Mi,Nj}p_{Mi}p_{Nj} + \sum_{mnij}\mathbb{A}^{p.rr}_{Ab,mi,nj}r_{mi}r_{nj} \\ &+ \sum_{MNLijk}\mathbb{A}^{p.ppp}_{Ab,Mi,Nj,Lk}p_{Mi}p_{Nj}p_{Lk} + \sum_{Mnlijk}\mathbb{A}^{p.prr}_{Ab,Mi,nj,lk}p_{Mi}r_{nj}r_{lk},\end{aligned} \tag{66a}$$

$$\begin{aligned}A^r_{ab} &= \mathbb{A}^{r.r}_{ab,ab}\,r_{ab} + \sum_{Mnij}\mathbb{A}^{r.pr}_{ab,Mi,nj}p_{Mi}r_{nj} + \sum_{MNlijk}\mathbb{A}^{r.ppr}_{ab,Mi,Nj,lk}p_{Mi}p_{Nj}r_{lk} \\ &+ \sum_{mnlijk}\mathbb{A}^{r.rrr}_{ab,mi,nj,lk}r_{mi}r_{nj}r_{lk}.\end{aligned} \tag{66b}$$

All generalised coordinates have the first order of smallness ($p_{Mi} \sim r_{mi} \sim \epsilon$). The $\mathbb{A}$-coefficients take the following form

$$\begin{aligned}&\mathbb{A}^p_{Ab} = \Lambda^c_A\hat{\mathcal{E}}^{Ab,0}, \quad \mathbb{A}^{p.pp}_{Ab,Mi,Nj} = \Lambda^{ccc}_{AMN}\hat{\mathcal{E}}^{Ab,2}_{Mi,Nj} + \delta_{MN}\delta_{ij}\Lambda^c_A\hat{\mathcal{E}}^{Ab,1}\beta^{pp}_{Mi,Nj}, \\ &\mathbb{A}^{p.p}_{Ab,Ab} = \Lambda^{cc}_{AA}\hat{\mathcal{E}}^{Ab,1}_{Ab}, \quad \mathbb{A}^{p.rr}_{Ab,mi,nj} = \Lambda^{css}_{Amn}\hat{\mathcal{E}}^{Ab,2}_{mi,nj} + \delta_{mn}\delta_{ij}\Lambda^c_A\hat{\mathcal{E}}^{Ab,1}\beta^{rr}_{mi,nj},\end{aligned}$$

$$\begin{aligned}\mathbb{A}^{p.ppp}_{Ab,Mi,Nj,Lk} &= \Lambda^c_A\hat{\mathcal{E}}^{Ab,1}\beta^{ppp}_{Mi,Nj,Lk} + \Lambda^{cccc}_{AMNL}\hat{\mathcal{E}}^{Ab,3}_{Mi,Nj,Lk} \\ &+ 2\delta_{MA}\delta_{ib}\Lambda^{cc}_{AM}\hat{\mathcal{E}}^{Ab,2}_{Mi}\delta_{NL}\delta_{jk}\beta^{pp}_{Nj,Lk}, \\ \mathbb{A}^{p.prr}_{Ab,Mi,nj,lk} &= \Lambda^c_A\hat{\mathcal{E}}^{Ab,1}\beta^{prr}_{Mi,nj,lk} + 3\Lambda^{ccss}_{AMnl}\hat{\mathcal{E}}^{Ab,3}_{Mi,nj,lk} \\ &+ 2\delta_{MA}\delta_{ib}\Lambda^{cc}_{AM}\hat{\mathcal{E}}^{Ab,2}_{Mi}\delta_{nl}\delta_{jk}\beta^{rr}_{nj,lk},\end{aligned} \tag{67a}$$

$$\begin{aligned}&\mathbb{A}^{r.r}_{ab,ab} = \Lambda^{ss}_{aa}\hat{\mathcal{E}}^{ab,1}_{ab}, \quad \mathbb{A}^{r.pr}_{ab,Mi,nj} = 2\Lambda^{css}_{Mna}\hat{\mathcal{E}}^{ab,2}_{Mi,nj}, \\ &\mathbb{A}^{r.ppr}_{ab,Mi,Nj,lk} = 3\Lambda^{ccss}_{MNla}\hat{\mathcal{E}}^{ab,3}_{Mi,Nj,lk} + 2\delta_{al}\delta_{bk}\Lambda^{ss}_{al}\hat{\mathcal{E}}^{ab,2}_{lk}\delta_{MN}\delta_{ij}\beta^{pp}_{Mi,Nj}, \\ &\mathbb{A}^{r.rrr}_{ab,mi,nj,lk} = \Lambda^{ssss}_{mnla}\hat{\mathcal{E}}^{ab,3}_{mi,nj,lk} + 2\delta_{al}\delta_{bk}\Lambda^{ss}_{al}\hat{\mathcal{E}}^{ab,2}_{lk}\delta_{mn}\delta_{ij}\beta^{rr}_{mi,nj},\end{aligned} \tag{67b}$$

where

$$\hat{\mathcal{E}}^{Ab,e}_{\underbrace{Mi,\ldots,Nj}_{N_3}} = \int_0^{x_{20}} x_2 B^{Ab}_e(x_2)\,\underbrace{f_{Mi}(x_2)\cdot\ldots\cdot f_{Nj}(x_2)}_{N_3}\,dx_2. \tag{68}$$

Partial derivatives of A^p_{Ab} and A^r_{ab} and A_N ($A_N = \big\{\{A^p_{Ab}\}, \{A^r_{ab}\}\big\}$) by the generalised coordinates p_{Mi} and r_{mi} take the following form

$$\begin{aligned}
\frac{\partial A^p_{Ab}}{\partial p_{Eh}} &= \mathbb{V}^p_{Ab,Eh} + \sum_{Mi} \mathbb{V}^{p.p}_{Ab,Eh,Mi} p_{Mi} \\
&\quad + \sum_{MNij} \mathbb{V}^{p.pp}_{Ab,Eh,Mi,Nj} p_{Mi} p_{Nj} + \sum_{mnij} \mathbb{V}^{p.rr}_{Ab,Eh,mi,nj} r_{mi} r_{nj}, \\
\frac{\partial A^p_{Ab}}{\partial r_{eh}} &= \sum_{mi} \mathbb{V}^{p.r}_{Ab,mi,eh} r_{mi} + \sum_{Mnij} \mathbb{V}^{p.pr}_{Ab,Mi,nj,eh} p_{Mi} r_{nj}, \\
\frac{\partial A^r_{ab}}{\partial p_{Eh}} &= \sum_{mi} \mathbb{V}^{r.r}_{ab,Eh,mi} r_{mi} + \sum_{Mnij} \mathbb{V}^{r.pr}_{ab,Eh,Mi,nj} p_{Mi} r_{nj}, \\
\frac{\partial A^r_{ab}}{\partial r_{eh}} &= \mathbb{V}^r_{ab,eh} + \sum_{Mi} \mathbb{V}^{r.p}_{ab,Mi,eh} p_{Mi} \\
&\quad + \sum_{MNij} \mathbb{V}^{r.pp}_{ab,Mi,Nj,eh} p_{Mi} p_{Nj} + \sum_{mnij} \mathbb{V}^{r.rr}_{ab,mi,nj,eh} r_{mi} r_{nj},
\end{aligned} \tag{69}$$

where $\mathbb{V}$-coefficients are expressed in terms of (67) as follows

$$\begin{aligned}
&\mathbb{V}^p_{Ab,Eh} = \mathbb{A}^{p.p}_{Ab,Eh}, \quad \mathbb{V}^{p.p}_{Ab,Eh,Mi} = 2\mathbb{A}^{p.pp}_{Ab,Eh,Mi} p_{Mi}, \\
&\mathbb{V}^r_{ab,eh} = \mathbb{A}^{r.r}_{ab,eh}, \quad \mathbb{V}^{p.pp}_{Ab,Eh,Mi,Nj} = \mathbb{A}^{p.ppp}_{Ab,Eh,Mi,Nj} + 2\mathbb{A}^{p.ppp}_{Ab,Mi,Eh,Nj}, \\
&\mathbb{V}^{p.rr}_{Ab,Eh,mi,nj} = \mathbb{A}^{p.prr}_{Ab,Eh,mi,nj}, \quad \mathbb{V}^{p.pr}_{Ab,Mi,nj,eh} = 2\mathbb{A}^{p.prr}_{Ab,Mi,nj,eh} p_{Mi} r_{nj}, \\
&\mathbb{V}^{p.r}_{Ab,mi,eh} = 2\mathbb{A}^{p.rr}_{Ab,mi,eh}, \quad \mathbb{V}^{r.pr}_{ab,Eh,Mi,Nj} = 2\mathbb{A}^{r.ppr}_{ab,Eh,Mi,nj} p_{Mi} r_{nj}, \\
&\mathbb{V}^{r.p}_{ab,Mi,eh} = \mathbb{A}^{r.pr}_{ab,Mi,eh}, \quad \mathbb{V}^{r.rr}_{ab,mi,nj,eh} = 2\mathbb{A}^{r.rrr}_{ab,eh,mi,nj} + \mathbb{A}^{r.rrr}_{ab,mi,nj,eh}, \\
&\mathbb{V}^{r.r}_{ab,Eh,mi} = \mathbb{A}^{r.pr}_{ab,Eh,mi}, \quad \mathbb{V}^{r.pp}_{ab,Mi,Nj,eh} = \mathbb{A}^{r.ppr}_{ab,Mi,Nj,eh}.
\end{aligned} \tag{70}$$

A.3 Integrals A_{NK} defined by (29)

By expanding elements of (29) ($A_{NK} = \big\{\{A^{pp}_{NK}, A^{pr}_{NK}\},\ \{A^{pr}_{NK}, A^{rr}_{NK}\}\big\}$) to the second polynomial order by the generalised coordinates p_{Mi} and r_{mi}, we get the following expressions

$$
\begin{aligned}
A^{pp}_{Ab,Cd} &= \mathbb{B}^{pp.0}_{Ab,Cd} + \sum_{Mi} \mathbb{B}^{pp.p}_{Ab,Cd,Mi} p_{Mi} \\
&\quad + \sum_{MNij} \mathbb{B}^{pp.pp}_{Ab,Cd,Mi,Nj} p_{Mi} p_{Nj} + \sum_{mnij} \mathbb{B}^{pp.rr}_{Ab,Cd,mi,nj} r_{mi} r_{nj}, \\
A^{rr}_{ab,cd} &= \mathbb{B}^{rr.0}_{ab,cd} + \sum_{Mi} \mathbb{B}^{rr.p}_{ab,cd,Mi} p_{Mi} \\
&\quad + \sum_{MNij} \mathbb{B}^{rr.pp}_{ab,cd,Mi,Nj} p_{Mi} p_{Nj} + \sum_{mnij} \mathbb{B}^{rr.rr}_{ab,cd,mi,nj} r_{mi} r_{nj}, \\
A^{pr}_{Ab,cd} &= \sum_{mi} \mathbb{B}^{pr.r}_{Ab,cd,mi} r_{mi} + \sum_{Mnij} \mathbb{B}^{pr.pr}_{Ab,cd,Mi,nj} p_{Mi} r_{nj}.
\end{aligned}
\tag{71}
$$

The $\mathbb{B}$-coefficients are as follows

$$
\begin{aligned}
\mathbb{B}^{pp.0}_{Ab,Cd} &= \Lambda^{cc}_{AC} \tilde{\mathcal{E}}^{Ab,Cd,0} + \Lambda^{ss}_{AC} \bar{\mathcal{E}}^{Ab,Cd,0}, \\
\mathbb{B}^{pp.p}_{Ab,Cd,Mi} &= \Lambda^{ccc}_{ACM} \tilde{\mathcal{E}}^{Ab,Cd,1}_{Mi} + \Lambda^{css}_{MAC} \bar{\mathcal{E}}^{Ab,Cd,1}_{Mi}, \\
\mathbb{B}^{pp.pp}_{Ab,Cd,Mi,Nj} &= \Lambda^{cccc}_{ACMN} \tilde{\mathcal{E}}^{Ab,Cd,2}_{Mi,Nj} + \Lambda^{ccss}_{MNAC} \bar{\mathcal{E}}^{Ab,Cd,2}_{Mi,Nj} \\
&\quad + \left(\Lambda^{cc}_{AC} \tilde{\mathcal{E}}^{Ab,Cd,1} + \Lambda^{ss}_{AC} \bar{\mathcal{E}}^{Ab,Cd,1} \right) \delta_{MN} \delta_{ij} \beta^{pp}_{Mi,Nj}, \\
\mathbb{B}^{pp.rr}_{Ab,Cd,mi,nj} &= \Lambda^{ccss}_{A,C,m,n} \tilde{\mathcal{E}}^{Ab,Cd,2}_{mi,nj} + \Lambda^{ssss}_{A,C,m,n} \bar{\mathcal{E}}^{Ab,Cd,2}_{mi,nj} \\
&\quad + \left(\Lambda^{cc}_{AC} \tilde{\mathcal{E}}^{Ab,Cd,1} + \Lambda^{ss}_{AC} \bar{\mathcal{E}}^{Ab,Cd,1} \right) \delta_{mn} \delta_{ij} \beta^{rr}_{mi,nj},
\end{aligned}
\tag{72a}
$$

$$
\begin{aligned}
\mathbb{B}^{rr.0}_{ab,cd} &= \delta_{ac} \Lambda^{ss}_{ac} \tilde{\mathcal{E}}^{ab,cd,0} + \delta_{ac} \Lambda^{cc}_{ac} \bar{\mathcal{E}}^{ab,cd,0}, \\
\mathbb{B}^{rr.p}_{ab,cd,Mi} &= \Lambda^{css}_{Mac} \tilde{\mathcal{E}}^{ab,cd,1}_{Mi} + \Lambda^{ccc}_{acM} \bar{\mathcal{E}}^{ab,cd,1}_{Mi}, \\
\mathbb{B}^{rr.pp}_{ab,cd,Mi,Nj} &= \Lambda^{ccss}_{MNac} \tilde{\mathcal{E}}^{ab,cd,2}_{Mi,Nj} + \Lambda^{cccc}_{acMN} \bar{\mathcal{E}}^{ab,cd,2}_{Mi,Nj} \\
&\quad + \left(\Lambda^{ss}_{ac} \tilde{\mathcal{E}}^{ab,cd,1} + \Lambda^{cc}_{ac} \bar{\mathcal{E}}^{ab,cd,1} \right) \delta_{m1} \delta_{i1} \delta_{MN} \delta_{ij} \beta^{pp}_{Mi,Nj}, \\
\mathbb{B}^{rr.rr}_{ab,cd,mi,nj} &= \Lambda^{ssss}_{mnac} \tilde{\mathcal{E}}^{ab,cd,2}_{mi,nj} + \Lambda^{ccss}_{acmn} \bar{\mathcal{E}}^{ab,cd,2}_{mi,nj} \\
&\quad + \left(\Lambda^{ss}_{ac} \tilde{\mathcal{E}}^{ab,cd,1} + \Lambda^{cc}_{ac} \bar{\mathcal{E}}^{ab,cd,1} \right) \delta_{m1} \delta_{i1} \delta_{mn} \delta_{ij} \beta^{rr}_{mi,nj},
\end{aligned}
\tag{72b}
$$

$$
\begin{aligned}
\mathbb{B}^{pr.r}_{Ab,cd,mi} &= \Lambda^{css}_{Acm} \tilde{\mathcal{E}}^{Ab,cd,1}_{mi} - \Lambda^{css}_{cAm} \bar{\mathcal{E}}^{Ab,cd,1}_{mi}, \\
\mathbb{B}^{pr.pr}_{Ab,cd,Mi,nj} &= 2 \left(\Lambda^{ccss}_{AMcn} \tilde{\mathcal{E}}^{Ab,cd,2}_{Mi,nj} - \Lambda^{ccss}_{cMAn} \bar{\mathcal{E}}^{Ab,cd,2}_{Mi,nj} \right),
\end{aligned}
\tag{72c}
$$

where

$$
\tilde{\mathcal{E}}^{Ab,Cd,e}_{\underbrace{Mi,\ldots,Nj}_{N_3}} = \int_0^{x_{20}} F^{AbCd}_e (x_2) \underbrace{f_{Mi}(x_2) \cdot \ldots \cdot f_{Nj}(x_2)}_{N_3} dx_2, \tag{73a}
$$

$$\underbrace{\bar{\mathcal{E}}^{Ab,Cd,e}_{Mi,\dots,Nj}}_{N_3} = AC \int_0^{x_{20}} \frac{1}{x_2} B_e^{AbCd}(x_2) \underbrace{f_{Mi}(x_2) \cdot \ldots \cdot f_{Nj}(x_2)}_{N_3} dx_2. \tag{73b}$$

The partial derivatives of A^{pp}_{MiNj}, A^{rr}_{minj} and A^{pr}_{Minj} by p_{Mi}, r_{mi} are

$$\begin{aligned}
\frac{\partial A^{pp}_{Ab,Cd}}{\partial p_{Eh}} &= \mathbb{W}^{pp.p}_{Ab,Cd,Eh} + \sum_{Mi} \mathbb{W}^{pp.pp}_{Ab,Cd,Eh,Mi} p_{Mi}, \\
\frac{\partial A^{pp}_{Ab,Cd}}{\partial r_{eh}} &= \sum_{mi} \mathbb{W}^{pp.rr}_{Ab,Cd,mi,eh} r_{mi}, \\
\frac{\partial A^{rr}_{ab,cd}}{\partial p_{Eh}} &= \mathbb{W}^{rr.p}_{ab,cd,Eh} + \sum_{Mi} \mathbb{W}^{rr.pp}_{ab,cd,Eh,Mi} p_{Mi}, \\
\frac{\partial A^{rr}_{ab,cd}}{\partial r_{eh}} &= \sum_{m,i} \mathbb{W}^{rr.rr}_{ab,cd,mi,eh} r_{mi}, \\
\frac{\partial A^{pr}_{Ab,cd}}{\partial p_{Eh}} &= \sum_{mi} \mathbb{W}^{pr.pr}_{Ab,cd,Eh,mi} r_{mi}, \\
\frac{\partial A^{pr}_{Ab,cd}}{\partial r_{eh}} &= \mathbb{W}^{pr.r}_{eh} + \sum_{Mi} \mathbb{W}^{pr.pr}_{Ab,cd,Mi,eh} p_{Mi},
\end{aligned} \tag{74}$$

where the $\mathbb{W}$-coefficients are expressed in terms of the matrix A_{NK} (72)

$$\begin{aligned}
&\mathbb{W}^{pp.p}_{Ab,Cd,Eh} = \mathbb{B}^{pp.p}_{Ab,Cd,Eh}, \\
&\mathbb{W}^{pp.pp}_{Ab,Cd,Eh,Mi} = 2\mathbb{B}^{pp.pp}_{Ab,Cd,Eh,Mi} = 2\mathbb{B}^{pp.pp}_{Ab,Cd,Mi,Eh}, \\
&\mathbb{W}^{pp.rr}_{Ab,Cd,mi,eh} = 2\mathbb{B}^{pp.rr}_{Ab,Cd,eh,mi} = 2\mathbb{B}^{pp.rr}_{Ab,Cd,mi,eh}, \\
&\mathbb{W}^{rr.pp}_{ab,cd,Eh,Mi} = 2\mathbb{B}^{rr.pp}_{ab,cd,Eh,Mi} = 2\mathbb{B}^{rr.pp}_{ab,cd,Mi,Eh}, \\
&\mathbb{W}^{rr.rr}_{ab,cd,mi,eh} = 2\mathbb{B}^{rr.rr}_{ab,cd,eh,mi} = 2\mathbb{B}^{rr.rr}_{ab,cd,mi,eh}, \\
&\mathbb{W}^{rr.p}_{ab,cd,Eh} = \mathbb{B}^{rr.p}_{ab,cd,Eh}, \quad \mathbb{W}^{pr.pr}_{Ab,cd,Eh,mi} = \mathbb{B}^{pr.pr}_{Ab,cd,Eh,mi}, \\
&\mathbb{W}^{pr.r}_{Ab,cd,eh} = \mathbb{B}^{pr.r}_{Ab,cd,eh}, \quad \mathbb{W}^{pr.pr}_{Ab,cd,Mi,eh} = \mathbb{B}^{pr.pr}_{Ab,cd,Mi,eh}.
\end{aligned} \tag{75}$$

A.4 Generalised velocities P_{Cd} and R_{cd}

After substituting expressions for the generalised velocities (33) into the kinematic equation (25), accounting for the derivatives (69) and (74) and collecting similar terms, we derive the $\mathbb{Z}$-coefficients as follows

$$\mathbb{Z}^{p}_{Ab} = \frac{\mathbb{V}^{p}_{Ab,Ab}}{\mathbb{B}^{pp.0}_{Ab,Ab}}, \quad \mathbb{Z}^{pp,Ab}_{Mi,Nj} = \frac{\mathbb{V}^{p.p}_{Ab,Nj,Mi} - \mathbb{B}^{pp.p}_{Ab,Nj,Mi}\mathbb{Z}^{p}_{Nj}}{\mathbb{B}^{pp.0}_{Ab,Ab}},$$

$$\mathbb{Z}^{ppp,Ab}_{Mi,Nj,Lk} = \frac{\mathbb{V}^{p.pp}_{Ab,Lk,Mi,Nj} - \mathbb{B}^{pp.pp}_{Ab,Lk,Mi,Nj}\mathbb{Z}^{p}_{Lk} - \sum_{Cd}\mathbb{B}^{pp.p}_{Ab,Cd,Mi}\mathbb{Z}^{pp,Cd}_{Nj,Lk}}{\mathbb{B}^{pp.0}_{Ab,Ab}},$$

$$\mathbb{Z}^{rr,Ab}_{mi,nj} = \frac{\mathbb{V}^{p.r}_{Ab,mi,nj} - \mathbb{B}^{pr.r}_{Ab,nj,mi}\mathbb{Z}^{r}_{nj}}{\mathbb{B}^{pp.0}_{Ab,Ab}}, \quad \mathbb{Z}^{prr,Ab}_{Mi,nj,lk} =$$

$$\frac{\mathbb{V}^{p.pr}_{Ab,Mi,nj,lk} - \mathbb{B}^{pr.pr}_{Ab,lk,Mi,nj}\mathbb{Z}^{r}_{lk} - \mathbb{B}^{pr.r}_{Ab,cd,nj}\mathbb{Z}^{pr,cd}_{Mi,lk} - \sum_{Cd}\mathbb{B}^{pp.p}_{Ab,Cd,Mi}\mathbb{Z}^{rr,Cd}_{nj,lk}}{\mathbb{B}^{pp.0}_{Ab,Ab}},$$

$$\mathbb{Z}^{rrp,Ab}_{mi,nj,Lk} = \frac{\mathbb{V}^{p.rr}_{Ab,Lk,mi,nj} - \mathbb{B}^{pp.rr}_{Ab,Lk,mi,nj}\mathbb{Z}^{p}_{Lk} - \mathbb{B}^{pr.r}_{Ab,cd,nj}\mathbb{Z}^{rp,cd}_{mi,Lk}}{\mathbb{B}^{pp.0}_{Ab,Ab}}, \quad (76a)$$

$$\mathbb{Z}^{r}_{ab} = \frac{\mathbb{V}^{r}_{ab,ab}}{\mathbb{B}^{rr.0}_{ab,ab}}, \quad \mathbb{Z}^{pr,ab}_{Mi,nj} = \frac{\mathbb{V}^{r.p}_{ab,Mi,nj} - \mathbb{B}^{rr.p}_{ab,nj,Mi}\mathbb{Z}^{r}_{nj}}{\mathbb{B}^{rr.0}_{ab,ab}},$$

$$\mathbb{Z}^{rp,ab}_{mi,Nj} = \frac{\mathbb{V}^{r.r}_{ab,Nj,mi} - \mathbb{B}^{pr.r}_{Nj,ab,mi}\mathbb{Z}^{p}_{Nj}}{\mathbb{B}^{rr.0}_{ab,ab}},$$

$$\mathbb{Z}^{ppr,ab}_{Mi,Nj,lk} = \frac{\mathbb{V}^{r.pp}_{ab,Mi,Nj,lk} - \mathbb{B}^{rr.pp}_{ab,lk,Mi,Nj}\mathbb{Z}^{r}_{lk} - \sum_{cd}\mathbb{B}^{rr.p}_{ab,cd,Mi}\mathbb{Z}^{pr,cd}_{Nj,lk}}{\mathbb{B}^{rr.0}_{ab,ab}},$$

$$\mathbb{Z}^{rrr,ab}_{mi,nj,lk} = \frac{\mathbb{V}^{r.rr}_{ab,mi,nj,lk} - \mathbb{B}^{rr.rr}_{ab,lk,mi,nj}\mathbb{Z}^{r}_{lk} - \sum_{Cd}\mathbb{B}^{pr.r}_{Cd,ab,nj}\mathbb{Z}^{rr,Cd}_{mi,lk}}{\mathbb{B}^{rr.0}_{ab,ab}},$$

$$\mathbb{Z}^{prp,ab}_{Mi,nj,Lk} = \Bigg(\mathbb{V}^{r.pr}_{ab,Lk,Mi,nj} - \mathbb{B}^{pr.pr}_{Lk,ab,Mi,nj}\mathbb{Z}^{p}_{Lk} - \sum_{Cd}\mathbb{B}^{pr.r}_{Cd,ab,nj}\mathbb{Z}^{pp,Cd}_{Mi,Lk}$$
$$- \sum_{cd}\mathbb{B}^{rr.p}_{ab,cd,Mi}\mathbb{Z}^{rp,cd}_{nj,Lk}\Bigg)\Bigg/\mathbb{B}^{rr.0}_{ab,ab}. \quad (76b)$$

A.5 Integrals l_i

Expressions for $\boldsymbol{l}$ (see, (13)) appearing in the dynamic equations (26) take the form

$$\begin{aligned} l_1 &= \rho \int_0^{2\pi} \int_0^{x_{20}} \int_0^{f^*(x_2,x_3,t)+x_{10}} x_1^3 x_2 dx_1 dx_2 dx_3, \\ l_2 &= \rho \int_0^{2\pi} \int_0^{x_{20}} \int_0^{f^*(x_2,x_3,t)+x_{10}} x_1^3 x_2^2 \cos(x_3)\, dx_1 dx_2 dx_3, \\ l_3 &= \rho \int_0^{2\pi} \int_0^{x_{20}} \int_0^{f^*(x_2,x_3,t)+x_{10}} x_1^3 x_2^2 \sin(x_3)\, dx_1 dx_2 dx_3. \end{aligned} \quad (77)$$

Coefficients $\hat{\mathbf{l}}^{\mathbf{r}\beta}_{Mi}$, $\hat{\mathbf{l}}^{\mathbf{r}\beta\beta}_{Mi,Nj}$, $\hat{\mathbf{l}}^{\mathbf{r}\beta\beta\beta}_{Mi,Nj,Lk}$ in (34) are determined by the following expressions (h_t and h_b are distances from the cone vertex to the unperturbed free surface and the bottom, respectively; the $\beta^{pp}_{Mi,Nj}$ coefficients appear in expression for β_0 (20), and δ_{ij} is the Kronecker delta):

$$\begin{aligned} &\mathbf{l}^x = \frac{\pi}{4}\left(h_t^4 - h_b^4\right) x_{20}^2, \quad \mathbf{l}^{xpp}_{Mi,Nj} = \frac{h_t^2}{2}\delta_{MN}\delta_{ij}\Lambda^{cc}_{MN}\lambda_{Mi,Nj}, \\ &\mathbf{l}^{xrr}_{mi,nj} = \frac{h_t^2}{2}\delta_{mn}\delta_{ij}\Lambda^{ss}_{mn}\lambda_{mi,nj}, \quad \mathbf{l}^{xppp}_{Mi,Nj,Lk} = \frac{2}{3}h_t\Lambda^{ccc}_{MNL}\lambda_{Mi,Nj,Lk}, \\ &\mathbf{l}^{xprr}_{Mi,nj,lk} = 2h_t\Lambda^{css}_{Mnl}\lambda_{Mi,nj,lk}, \quad \hat{\mathbf{l}}^{yp}_{Mi} = h_t^3\delta_{1,M}\Lambda^{cc}_{1M}\hat{\lambda}_{Mi}, \\ &\hat{\mathbf{l}}^{yppp}_{Mi,Nj,Lk} = h_t\Lambda^{cccc}_{1MNL}\hat{\lambda}_{Mi,Nj,Lk} + 3h_t^2\delta_{1M}\Lambda^{cc}_{1M}\hat{\lambda}_{Mi}\delta_{NL}\delta_{jk}\beta^{pp}_{Nj,Lk}, \\ &\hat{\mathbf{l}}^{ypp}_{Mi,Nj} = \frac{3}{2}h_t^2\Lambda^{ccc}_{1MN}\hat{\lambda}_{Mi,Nj}, \quad \hat{\mathbf{l}}^{yrr}_{mi,nj} = \frac{3}{2}h_t^2\Lambda^{css}_{1mn}\hat{\lambda}_{mi,nj}, \\ &\hat{\mathbf{l}}^{yprr}_{Mi,nj,lk} = 3h_t\Lambda^{ccss}_{1Mnl}\hat{\lambda}_{Mi,nj,lk} + 3h_t^2\delta_{1M}\Lambda^{cc}_{1M}\hat{\lambda}_{Mi}\delta_{nl}\delta_{jk}\beta^{rr}_{nj,lk}, \\ &\hat{\mathbf{l}}^{zp}_{mi} = h_t^3\delta_{1m}\Lambda^{ss}_{m1}\hat{\lambda}_{mi}, \quad \hat{\mathbf{l}}^{zpr}_{Mi,nj} = 3h_t^2\Lambda^{css}_{Mn1}\hat{\lambda}_{Mi,nj}, \\ &\hat{\mathbf{l}}^{zppr}_{Mi,Nj,lk} = 3h_t\Lambda^{ccss}_{MNl1}\hat{\lambda}_{Mi,Nj,lk} + 3h_t^2\delta_{1l}\Lambda^{ss}_{l1}\hat{\lambda}_{lk}\delta_{MN}\delta_{ij}\beta^{pp}_{Mi,Nj}, \\ &\hat{\mathbf{l}}^{zrrr}_{mi,nj,lk} = h_t\Lambda^{ssss}_{mnl1}\hat{\lambda}_{mi,nj,lk} + 3h_t^2\delta_{1l}\Lambda^{ss}_{l1}\hat{\lambda}_{lk}\delta_{mn}\delta_{ij}\beta^{rr}_{mi,nj}. \end{aligned} \quad (78)$$

The following notation is adopted

$$\hat{\lambda}\underbrace{Mi,\ldots,Nj}_{N_3} = \int_0^{x_{20}} x_2^2 \underbrace{f_{Mi}(x_2)\cdot\ldots\cdot f_{Nj}(x_2)}_{N_3} dx_2, \quad (79)$$

in addition to (64) and (65).

When using the Moiseev-Narimanov asymptotics (37) in (34), we deduce that only the following components should be kept

$$
\begin{aligned}
l_1 &= \mathbf{l}^x + \mathbf{l}^{xpp}_{11,11} p_{11}^2 + \mathbf{l}^{xrr}_{11,11} r_{11}^2 + \mathbf{l}^{xprr}_{11,11,11} p_{11} r_{11}^2 + \mathbf{l}^{xppp}_{11,11,11} p_{11}^3,\\
l_2 &= \hat{\mathbf{l}}^{ypp}_{11,11} p_{11}^2 + \hat{\mathbf{l}}^{yrr}_{11,11} r_{11}^2 + \hat{\mathbf{l}}^{yprr}_{11,11,11} p_{11} r_{11}^2 + \hat{\mathbf{l}}^{yppp}_{11,11,11} p_{11}^3\\
&\quad + \sum_i \hat{\mathbf{l}}^{yp}_{1i} p_{1i} + \sum_i \left(\hat{\mathbf{l}}^{ypp}_{0i,11} + \hat{\mathbf{l}}^{ypp}_{11,0i}\right) p_{11} p_{0i}\\
&\quad + \sum_i \left(\hat{\mathbf{l}}^{ypp}_{2i,11} + \hat{\mathbf{l}}^{ypp}_{11,2i}\right) p_{11} p_{2i} + \sum_i \left(\hat{\mathbf{l}}^{yrr}_{2i,11} + \hat{\mathbf{l}}^{yrr}_{11,2i}\right) r_{11} r_{2i},\\
l_3 &= \hat{\mathbf{l}}^{zpr}_{11,11} p_{11} r_{11} + \hat{\mathbf{l}}^{zrrr}_{11,11,11} r_{11}^3 + \hat{\mathbf{l}}^{zppr}_{11,11,11} p_{11}^2 r_{11}\\
&\quad + \sum_i \hat{\mathbf{l}}^{zp}_{1i} r_{1i} + \sum_i \hat{\mathbf{l}}^{zpr}_{0i,11} r_{11} p_{0i} + \sum_i \hat{\mathbf{l}}^{zpr}_{11,2i} p_{11} r_{2i} + \sum_i \hat{\mathbf{l}}^{zpr}_{2i,11} r_{11} p_{2i}.
\end{aligned}
\tag{80}
$$

The derivatives $\partial l_1/\partial \beta_N$ by p_{Mi} and r_{mi} take the following form

$$
\begin{aligned}
\frac{\partial l_1}{\partial p_{Eh}} &= \bar{\mathbf{l}}^{xpp}_{Eh,Eh} p_{Eh} + \sum_{MNij} \bar{\mathbf{l}}^{xppp}_{Eh,Mi,Nj} p_{Mi} p_{Nj} + \sum_{mnij} \bar{\mathbf{l}}^{xprr}_{Eh,mi,nj} r_{mi} r_{nj}\\
&\quad + \sum_{MNLijk} \bar{\mathbf{l}}^{xpppp}_{Eh,Mi,Nj,Lk} p_{Mi} p_{Nj} p_{Lk} + \sum_{Mnlijk} \bar{\mathbf{l}}^{xpprr}_{Eh,Mi,nj,lk} p_{Mi} r_{nj} r_{lk},
\end{aligned}
\tag{81a}
$$

$$
\begin{aligned}
\frac{\partial l_1}{\partial r_{eh}} &= \bar{\mathbf{l}}^{xrr}_{eh,eh} r_{eh} + \sum_{Mnij} \bar{\mathbf{l}}^{xprr}_{Mi,nj,eh} p_{Mi} r_{nj}\\
&\quad + \sum_{MNlijk} \bar{\mathbf{l}}^{xpprr}_{Mi,Nj,lk,eh} p_{Mi} p_{Nj} r_{lk} + \sum_{mnlijk} \bar{\mathbf{l}}^{xrrrr}_{mi,nj,lk,eh} r_{mi} r_{nj} r_{lk},
\end{aligned}
\tag{81b}
$$

where the derived $\bar{\mathbf{l}}$-coefficients are expressed in terms of l_1 as follows

$$
\begin{aligned}
&\bar{\mathbf{l}}^{xpp}_{Eh,Eh} = 2\mathbf{l}^{xpp}_{Eh,Eh}, \quad \bar{\mathbf{l}}^{xppp}_{Eh,Mi,Nj} = 3\mathbf{l}^{xppp}_{Eh,Mi,Nj}, \quad \bar{\mathbf{l}}^{xprr}_{Eh,mi,nj} = \mathbf{l}^{xprr}_{Eh,mi,nj},\\
&\bar{\mathbf{l}}^{xpppp}_{Eh,Mi,Nj,Lk} = 4\mathbf{l}^{xpppp}_{Eh,Mi,Nj,Lk}, \quad \bar{\mathbf{l}}^{xpprr}_{Eh,Mi,nj,lk} = 2\mathbf{l}^{xpprr}_{Eh,Mi,nj,lk},\\
&\bar{\mathbf{l}}^{xprr}_{Mi,nj,eh} = 2\mathbf{l}^{xprr}_{Mi,nj,eh}, \quad \bar{\mathbf{l}}^{xpprr}_{Mi,Nj,lk,eh} = 2\mathbf{l}^{xpprr}_{Mi,Nj,lk,eh},\\
&\bar{\mathbf{l}}^{xrr}_{eh,eh} = 2\mathbf{l}^{xrr}_{eh,eh}, \quad \bar{\mathbf{l}}^{xrrrr}_{mi,nj,lk,eh} = 4\mathbf{l}^{xrrrr}_{mi,nj,lk,eh}.
\end{aligned}
\tag{82}
$$

For the steady-state sloshing regimes (53), (55), using the Moiseev-Narimanov asymptotics derives the second time derivative for horizontal components of the vector $\boldsymbol{l}$ as

$$
\begin{aligned}
\ddot{l}_2 &= B_s \left(\lambda^s_{y1} + A_c^2 \lambda^{ccs}_{y1} + B_s^2 \lambda^{sss}_{y1}\right) \sigma^2 \sin \sigma t + B_s \left(A_c^2 - B_s^2\right) \lambda^{sss}_{y3} \sigma^2 \sin 3\sigma t,\\
\ddot{l}_3 &= A_c \left(\lambda^c_{z1} + A_c^2 \lambda^{ccc}_{z1} + B_s^2 \lambda^{css}_{z1}\right) \sigma^2 \cos \sigma t + A_c \left(A_c^2 - B_s^2\right) \lambda^{ccc}_{z3} \sigma^2 \cos 3\sigma t,
\end{aligned}
\tag{83}
$$

where coefficients λ_{ijk} are

$$
\begin{aligned}
&\lambda^s_{y1} = \lambda^c_{z1} = -\pi h_t^3 \hat{\lambda}_{11}, \quad \hat{\lambda}_{111} = \frac{x_{20}^2 \hat{\lambda}_{11,11,11} - 4\hat{\lambda}_{11}\lambda_{11,11}}{4h_t x_{20}^2}, \\
&\lambda^{sss}_{y1} = \lambda^{sss}_{yo1} + \lambda^{sss}_{yn1}, \ \lambda^{ccs}_{y1} = \lambda^{ccs}_{yo1} + \lambda^{ccs}_{yn1}, \ \lambda^{sss}_{y3} = \lambda^{sss}_{yo3} + \lambda^{sss}_{yn3}, \\
&\lambda^{ccc}_{z1} = \lambda^{ccc}_{zo1} + \lambda^{ccc}_{zn1}, \ \lambda^{css}_{z1} = \lambda^{css}_{zo1} + \lambda^{css}_{zn1}, \ \lambda^{ccc}_{z3} = \lambda^{ccc}_{zo3} + \lambda^{ccc}_{zn3}, \\
&\lambda^{sss}_{yo1} = \lambda^{ccc}_{zo1} = -\tfrac{3}{4}\pi h_t^2 \Big(3\hat{\lambda}_{111} + 2\left(2\mathbf{o}_{010} + \mathbf{o}_{012}\right)\hat{\lambda}_{01,11} \\
&\qquad +2\left(\mathbf{o}_{210} + \mathbf{o}_{212}\right)\hat{\lambda}_{21,11}\Big), \\
&\lambda^{ccs}_{yo1} = \lambda^{css}_{zo1} = -\tfrac{3}{4}\pi h_t^2 \Big(\hat{\lambda}_{111} + 2\left(2\mathbf{o}_{010} - \mathbf{o}_{012}\right)\hat{\lambda}_{01,11} \\
&\qquad -\left(2\mathbf{o}_{210} - 3\mathbf{o}_{212}\right)\hat{\lambda}_{21,11}\Big), \\
&\lambda^{sss}_{yo3} = \lambda^{ccc}_{z3} = -\tfrac{27}{4}\pi h_t^2 \left(\hat{\lambda}_{111} + 2\mathbf{o}_{012}\hat{\lambda}_{01,11} + \mathbf{o}_{212}\hat{\lambda}_{21,11}\right), \\
&\lambda^{ccc}_{yn1} = \tfrac{1}{2}\pi h_t^2 \left(2h_t G^{\hat{\lambda}_1}_{11} - 3\left(2C^{\hat{\lambda}_{01}}_0 + C^{\hat{\lambda}_{01}}_2 + S^{\hat{\lambda}_{21}}_0 + \tfrac{1}{2}S^{\hat{\lambda}_{21}}_2\right)\right), \\
&\lambda^{css}_{yn1} = \tfrac{1}{2}\pi h_t^2 \left(2h_t G^{\hat{\lambda}_1}_{12} - 3\left(2C^{\hat{\lambda}_{01}}_0 + C^{\hat{\lambda}_{01}}_2 + S^{\hat{\lambda}_{21}}_0 - \tfrac{3}{2}S^{\hat{\lambda}_{21}}_2\right)\right), \\
&\lambda^{ccc}_{yn3} = \tfrac{9}{2}\pi h_t^2 \left(2h_t G^{\hat{\lambda}_1}_3 - 3C^{\hat{\lambda}_{01}}_2 - \tfrac{3}{2}S^{\hat{\lambda}_{21}}_2\right),
\end{aligned}
\tag{84}
$$

and

$$
\begin{aligned}
&C^{\hat{\lambda}_{k1}}_j = \sum_{i=2}^{\infty} \hat{\lambda}_{ki11}\mathbf{o}_{0ij}, \quad S^{\hat{\lambda}_{k1}}_j = \sum_{i=2}^{\infty} \hat{\lambda}_{ki11}\mathbf{o}_{2ij}, \\
&G^{\hat{\lambda}_1}_3 = \sum_{i=2}^{\infty} \hat{\lambda}_{1i}\mathbf{o}_{1i3}, \quad G^{\hat{\lambda}_1}_{jk} = \sum_{i=2}^{\infty} \hat{\lambda}_{1i}\mathbf{o}_{1ijk}.
\end{aligned}
\tag{85}
$$

A.6 The d-, g-, t-coefficients in (35)

The **d**-, **g**-, **t**-coefficients of the infinite-dimensional nonlinear modal equation (35) are computed by the formulas

$$
\begin{aligned}
&\mathbf{d}^{p,Eh}_{Mi} = \delta_{M,E}\delta_{i,h}\mathbb{V}^p_{Mi,Eh}\mathbb{Z}^p_{Mi}, \quad \mathbf{g}^{p,Eh}_{Mi} = \delta_{M,E}\delta_{i,h}\bar{\mathbf{l}}^{opp}_{Eh,Mi}, \\
&\mathbf{g}^{pp,Eh}_{Mi,Nj} = \bar{\mathbf{l}}^{oppp}_{Eh,Mi,Nj}, \quad \mathbf{g}^{prr,Eh}_{Mi,nj,lk} = \bar{\mathbf{l}}^{opprr}_{Eh,Mi,nj,lk}, \\
&\mathbf{d}^{pp,Eh}_{Mi,Nj} = \mathbb{V}^{p.p}_{Nj,Eh,Mi}\mathbb{Z}^p_{Nj} + \sum_{Ab}\delta_{A,E}\delta_{b,h}\mathbb{V}^p_{Ab,Eh}\mathbb{Z}^{pp,Ab}_{Mi,Nj}, \\
&\mathbf{d}^{rr,Eh}_{mi,nj} = \mathbb{V}^{r.r}_{nj,Eh,mi}\mathbb{Z}^r_{nj} + \sum_{Ab}\delta_{A,E}\delta_{b,h}\mathbb{V}^p_{Ab,Eh}\mathbb{Z}^{rr,Ab}_{mi,nj}, \\
&\mathbf{t}^{pp,Eh}_{Mi,Nj} = \frac{1}{2}\mathbb{W}^{pp.p}_{Mi,Nj,Eh}\mathbb{Z}^p_{Mi}\mathbb{Z}^p_{Nj} + \sum_{Ab}\delta_{A,E}\delta_{b,h}\mathbb{V}^p_{Ab,Eh}\mathbb{Z}^{pp,Ab}_{Mi,Nj}, \\
&\mathbf{t}^{rr,Eh}_{mi,nj} = \frac{1}{2}\mathbb{W}^{rr.p}_{mi,nj,Eh}\mathbb{Z}^r_{mi}\mathbb{Z}^r_{nj} + \sum_{Ab}\delta_{AE}\delta_{bh}\mathbb{V}^p_{Ab,Eh}\mathbb{Z}^{rr,Ab}_{mi,nj},
\end{aligned}
$$

$$\mathbf{d}^{ppp,Eh}_{Mi,Nj,Lk} = \mathbb{V}^{p.pp}_{Lk,Eh,Mi,Nj}\mathbb{Z}^{p}_{Lk} + \sum_{Ab} \mathbb{V}^{p.p}_{Ab,Eh,Mi}\mathbb{Z}^{pp,Ab}_{Nj,Lk}$$
$$+ \sum_{Ab} \delta_{AE}\delta_{bh}\mathbb{V}^{p}_{Ab,Eh}\mathbb{Z}^{ppp,Ab}_{Mi,Nj,Lk},$$
$$\mathbf{d}^{prr,Eh}_{Mi,nj,lk} = \mathbb{V}^{r.pr}_{lk,Eh,Mi,nj}\mathbb{Z}^{r}_{lk} + \sum_{ab} \mathbb{V}^{r.r}_{ab,Eh,nj}\mathbb{Z}^{pr,ab}_{Mi,lk}$$
$$+ \sum_{Ab} \mathbb{V}^{p.p}_{Ab,Eh,Mi}\mathbb{Z}^{rr,Ab}_{nj,lk} + \sum_{Ab} \delta_{AE}\delta_{bh}\mathbb{V}^{p}_{Ab,Eh}\mathbb{Z}^{prr,Ab}_{Mi,nj,lk},$$
$$\mathbf{g}^{rr,Eh}_{mi,nj} = \bar{\mathbf{l}}^{oprr}_{Eh,mi,nj}, \quad \mathbf{g}^{ppp,Eh}_{Mi,Nj,Lk} = \bar{\mathbf{l}}^{opppp}_{Eh,Mi,Nj,Lk},$$
$$\mathbf{d}^{rrp,Eh}_{mi,nj,Lk} = \mathbb{V}^{p.rr}_{Lk,Eh,mi,nj}\mathbb{Z}^{p}_{Lk} + \sum_{ab} \mathbb{V}^{r.r}_{ab,Eh,mi}\mathbb{Z}^{rp,ab}_{nj,Lk}$$
$$+ \sum_{Ab} \delta_{AE}\delta_{bh}\mathbb{V}^{p}_{Ab,Eh}\mathbb{Z}^{rrp,Ab}_{mi,nj,Lk},$$
$$\mathbf{t}^{ppp,Eh}_{Mi,Nj,Lk} = \frac{1}{2}\mathbb{W}^{pp.pp}_{Nj,Lk,Eh,Mi}\mathbb{Z}^{p}_{Nj}\mathbb{Z}^{p}_{Lk} + \sum_{Ab} \mathbb{V}^{p.p}_{Ab,Eh,Mi}\mathbb{Z}^{pp,Ab}_{Nj,Lk}$$
$$+ \sum_{Cd} \frac{1}{2}\left(\mathbb{W}^{pp.p}_{Cd,Nj,Eh} + \mathbb{W}^{pp.p}_{Nj,Cd,Eh}\right)\mathbb{Z}^{p}_{Nj}\mathbb{Z}^{pp,Cd}_{Mi,Lk}$$
$$+ \sum_{Ab} \delta_{AE}\delta_{bh}\mathbb{V}^{p}_{Ab,Eh}\left(\mathbb{Z}^{ppp,Ab}_{Mi,Nj,Lk} + \mathbb{Z}^{ppp,Ab}_{Nj,Mi,Lk}\right),$$
$$\mathbf{t}^{prr,Eh}_{Mi,nj,lk} = \sum_{Ab} \mathbb{V}^{p.p}_{Ab,Eh,Mi}\mathbb{Z}^{rr,Ab}_{nj,lk} + \sum_{Ab} \delta_{AE}\delta_{bh}\mathbb{V}^{p}_{Ab,Eh}\mathbb{Z}^{prr,Ab}_{Mi,nj,lk}$$
$$+ \frac{1}{2}\mathbb{W}^{rr.pp}_{nj,lk,Eh,Mi}\mathbb{Z}^{r}_{nj}\mathbb{Z}^{r}_{lk} + \sum_{cd} \frac{1}{2}\left(\mathbb{W}^{rr.p}_{cd,lk,Eh} + \mathbb{W}^{rr.p}_{lk,cd,Eh}\right)\mathbb{Z}^{r}_{lk}\mathbb{Z}^{pr,cd}_{Mi,nj},$$
$$\mathbf{t}^{rpr,Eh}_{mi,Nj,lk} = \mathbb{W}^{pr.pr}_{Nj,lk,Eh,mi}\mathbb{Z}^{p}_{Nj}\mathbb{Z}^{r}_{lk} + \sum_{Cd} \frac{1}{2}\left(\mathbb{W}^{pp.p}_{Cd,Nj,Eh} + \mathbb{W}^{pp.p}_{Nj,Cd,Eh}\right)$$
$$\times \mathbb{Z}^{p}_{Nj}\mathbb{Z}^{rr,Cd}_{mi,lk} + \sum_{cd} \frac{1}{2}\left(\mathbb{W}^{rr.p}_{cd,lk,Eh} + \mathbb{W}^{rr.p}_{lk,cd,Eh}\right)\mathbb{Z}^{r}_{lk}\mathbb{Z}^{rp,cd}_{mi,Nj}$$
$$+ \sum_{ab} \mathbb{V}^{r.r}_{ab,Eh,mi}\left(\mathbb{Z}^{pr,ab}_{Nj,lk} + \mathbb{Z}^{rp,ab}_{lk,Nj}\right)$$
$$+ \sum_{Ab} \delta_{AE}\delta_{bh}\mathbb{V}^{p}_{Ab,Eh}\left(\mathbb{Z}^{prr,Ab}_{Nj,mi,lk} + \mathbb{Z}^{rrp,Ab}_{mi,lk,Nj} + \mathbb{Z}^{rrp,Ab}_{lk,mi,Nj}\right),$$
$$\mathbf{d}^{r,eh}_{mi} = \delta_{m,e}\delta_{i,h}\mathbb{V}^{r}_{mi,eh}\mathbb{Z}^{r}_{mi}, \quad \mathbf{g}^{r,eh}_{mi} = \delta_{m,e}\delta_{i,h}\bar{\mathbf{l}}^{orr}_{mi,eh},$$
$$\mathbf{g}^{pr,eh}_{Mi,nj} = \bar{\mathbf{l}}^{oprr}_{Mi,nj,eh}, \quad \mathbf{g}^{ppr,eh}_{Mi,Nj,lk} = \bar{\mathbf{l}}^{opprr}_{Mi,Nj,lk,eh}, \quad \mathbf{g}^{rrr,eh}_{mi,nj,lk} = \bar{\mathbf{l}}^{orrrr}_{mi,nj,lk,eh},$$

$$
\begin{aligned}
\mathbf{t}^{pr,eh}_{Mi,nj} &= \mathbb{W}^{pr.r}_{eh}\mathbb{Z}^{p}_{Mi}\mathbb{Z}^{r}_{nj} + \sum_{ab}\delta_{ae}\delta_{bh}\mathbb{V}^{r}_{ab,eh}\left(\mathbb{Z}^{pr,ab}_{Mi,nj} + \mathbb{Z}^{rp,ab}_{nj,Mi}\right),\\
\mathbf{d}^{pr,eh}_{Mi,nj} &= \mathbb{V}^{r.p}_{nj,Mi,eh}\mathbb{Z}^{r}_{nj} + \sum_{ab}\delta_{ae}\delta_{bh}\mathbb{V}^{r}_{ab,eh}\mathbb{Z}^{pr,ab}_{Mi,nj},\\
\mathbf{d}^{rp,eh}_{mi,Nj} &= \mathbb{V}^{p.r}_{Nj,mi,eh}\mathbb{Z}^{p}_{Nj} + \sum_{ab}\delta_{ae}\delta_{bh}\mathbb{V}^{r}_{ab,eh}\mathbb{Z}^{rp,ab}_{mi,Nj},\\
\mathbf{d}^{prp,eh}_{Mi,nj,Lk} &= \mathbb{V}^{p.pr}_{Lk,Mi,nj,eh}\mathbb{Z}^{p}_{Lk} + \sum_{Ab}\mathbb{V}^{p.r}_{Ab,nj,eh}\mathbb{Z}^{pp,Ab}_{Mi,Lk}\\
&\quad + \sum_{ab}\mathbb{V}^{r.p}_{ab,Mi,eh}\mathbb{Z}^{rp,ab}_{nj,Lk} + \sum_{ab}\delta_{ae}\delta_{bh}\mathbb{V}^{r}_{ab,eh}\mathbb{Z}^{prp,ab}_{Mi,nj,Lk},\\
\mathbf{d}^{ppr,eh}_{Mi,Nj,lk} &= \mathbb{V}^{r.pp}_{lk,Mi,Nj,eh}\mathbb{Z}^{r}_{lk} + \sum_{ab}\mathbb{V}^{r.p}_{ab,Mi,eh}\mathbb{Z}^{pr,ab}_{Nj,lk}\\
&\quad + \sum_{ab}\delta_{ae}\delta_{bh}\mathbb{V}^{r}_{ab,eh}\mathbb{Z}^{ppr,ab}_{Mi,Nj,lk},\\
\mathbf{d}^{rrr,eh}_{mi,nj,lk} &= \mathbb{V}^{r.rr}_{lk,mi,nj,eh}\mathbb{Z}^{r}_{lk} + \sum_{Ab}\mathbb{V}^{p.r}_{Ab,mi,eh}\mathbb{Z}^{rr,Ab}_{nj,lk}\\
&\quad + \sum_{ab}\delta_{ae}\delta_{bh}\mathbb{V}^{r}_{ab,eh}\mathbb{Z}^{rrr,ab}_{mi,nj,lk},\\
\mathbf{t}^{rpp,eh}_{mi,Nj,Lk} &= \frac{1}{2}\mathbb{W}^{pp.rr}_{Nj,Lk,mi,eh}\mathbb{Z}^{p}_{Nj}\mathbb{Z}^{p}_{Lk} + \sum_{Ab}\mathbb{V}^{p.r}_{Ab,mi,eh}\mathbb{Z}^{pp,Ab}_{Nj,Lk}\\
&\quad + \sum_{cd}\mathbb{W}^{pr.r}_{Nj,cd,eh}\mathbb{Z}^{rp,cd}_{mi,Lk}\mathbb{Z}^{p}_{Nj} + \sum_{ab}\delta_{ae}\delta_{bh}\mathbb{V}^{r}_{ab,eh}\mathbb{Z}^{prp,ab}_{Nj,mi,Lk},\\
\mathbf{t}^{ppr,eh}_{Mi,Nj,lk} &= \mathbb{W}^{pr.pr}_{Nj,lk,Mi,eh}\mathbb{Z}^{p}_{Nj}\mathbb{Z}^{r}_{lk} + \sum_{cd}\mathbb{W}^{pr.r}_{Nj,cd,eh}\mathbb{Z}^{pr,cd}_{Mi,lk}\mathbb{Z}^{p}_{Nj}\\
&\quad + \sum_{Ab}\mathbb{W}^{pr.r}_{Ab,lk,eh}\mathbb{Z}^{pp,Ab}_{Mi,Nj}\mathbb{Z}^{r}_{lk} + \sum_{ab}\mathbb{V}^{r.p}_{ab,Mi,eh}\left(\mathbb{Z}^{pr,ab}_{Nj,lk} + \mathbb{Z}^{rp,ab}_{lk,Nj}\right)\\
&\quad + \sum_{ab}\delta_{ae}\delta_{bh}\mathbb{V}^{r}_{ab,eh}\left(\mathbb{Z}^{ppr,ab}_{Mi,Nj,lk} + \mathbb{Z}^{ppr,ab}_{Nj,Mi,lk} + \mathbb{Z}^{prp,ab}_{Mi,lk,Nj}\right),\\
\mathbf{t}^{rrr,eh}_{mi,nj,lk} &= \frac{1}{2}\mathbb{W}^{rr.rr}_{nj,lk,mi,eh}\mathbb{Z}^{r}_{nj}\mathbb{Z}^{r}_{lk} + \sum_{Ab}\mathbb{W}^{pr.r}_{Ab,lk,eh}\mathbb{Z}^{rr,Ab}_{mi,nj}\mathbb{Z}^{r}_{lk}\\
&\quad + \sum_{Ab}\mathbb{V}^{p.r}_{Ab,mi,eh}\mathbb{Z}^{rr,Ab}_{nj,lk} + \sum_{ab}\delta_{ae}\delta_{bh}\mathbb{V}^{r}_{ab,eh}\left(\mathbb{Z}^{rrr,ab}_{mi,nj,lk} + \mathbb{Z}^{rrr,ab}_{nj,mi,lk}\right).
\end{aligned}
$$

A.7 Coefficients of the modal system (38)

The nonzero hydrodynamic coefficients in (38) take the form

$$\mu_{0h}^p = \mathbf{d}_{1i}^{p,1i} = \mu_{0h}^r = \mathbf{d}_{1i}^{r,li},\ \sigma_{0h}^2 = \mathbf{g}_{1i}^{p,1i}/\mathbf{d}_{1i}^{p,1i},\ \mathcal{G}_{0h} = \mathbf{g}_{11,11}^{pp,1i} = \mathbf{g}_{11,11}^{rr,1i},$$
$$d_{8,h} = \mathbf{t}_{11,11}^{pp,1i} = \mathbf{t}_{11,11}^{rr,1i},\quad d_{10,h} = \mathbf{d}_{11,11}^{pp,1i} = \mathbf{d}_{11,11}^{rr,1i},$$
$$\mu_{2h}^p = \mathbf{d}_{2h}^{p,2h} = \mu_{1k}^r = \mathbf{d}_{2h}^{r,2h},\quad \sigma_{2h}^2 = \mathbf{g}_{2h}^{p,2h}/\mathbf{d}_{2h}^{p,2h} = \mathbf{g}_{2h}^{r,2h}/\mathbf{d}_{2h}^{r,2h},$$
$$\mathcal{G}_{4,h} = \mathbf{g}_{11,11}^{pp,2h} = -\mathbf{g}_{11,11}^{rr,2h} = \tfrac{1}{2}\mathbf{g}_{11,11}^{pr,2h},\quad d_{7,h} = \mathbf{t}_{11,11}^{pp,2h} = -\mathbf{t}_{11,11}^{rr,2h} = \tfrac{1}{2}\mathbf{t}_{11,11}^{pr,2h},$$
$$d_{9,h} = \mathbf{d}_{11,11}^{pp,2h} = -\mathbf{d}_{11,11}^{rr,2h} = \mathbf{d}_{11,11}^{pr,2h} = \mathbf{d}_{11,11}^{rp,2h},$$
$$\mu_{11}^p = \mathbf{d}_{11}^{p,11} = \mu_{1k}^r = \mathbf{d}_{11}^{r,11},\ \sigma_{11}^2 = \mathbf{g}_{11}^{p,11}/\mathbf{d}_{11}^{p,11} = \mathbf{g}_{11}^{r,11}/\mathbf{d}_{11}^{r,11},$$
$$\mathcal{G}_1 = \mathbf{g}_{11,11,11}^{ppp,11} = \mathbf{g}_{11,11,11}^{prr,11} = \mathbf{g}_{11,11,11}^{ppr,11} = \mathbf{g}_{11,11,11}^{rrr,11},\quad \mathcal{G}_2^j = \mathbf{g}_{0j,11}^{pp,11} + \mathbf{g}_{11,0j}^{pp,11} = \mathbf{g}_{0j,11}^{pr,11},$$
$$\mathcal{G}_3^j = \mathbf{g}_{11,2j}^{pp,11} + \mathbf{g}_{2j,11}^{pp,11} = \mathbf{g}_{11,2j}^{rr,11} + \mathbf{g}_{2j,11}^{rr,11} = \mathbf{g}_{11,2j}^{pr,11} = -\mathbf{g}_{2j,11}^{pr,11},$$
$$\begin{aligned} d_1 = \mathbf{d}_{11,11,11}^{ppp,11} = \mathbf{d}_{11,11,11}^{prr,11} = \mathbf{t}_{11,11,11}^{ppp,11} = \mathbf{t}_{11,11,11}^{prr,11} = \mathbf{d}_{11,11,11}^{prp,11} = \mathbf{d}_{11,11,11}^{rrr,11} \\ = \mathbf{t}_{11,11,11}^{rpp,11} = \mathbf{t}_{11,11,11}^{rrr,11}, \end{aligned}$$
$$\begin{aligned} d_2 &= \mathbf{d}_{11,11,11}^{rrp,11} = -\mathbf{d}_{11,11,11}^{prr,11} = \tfrac{1}{2}\mathbf{t}_{11,11,11}^{rpr,11} = -\tfrac{1}{2}\mathbf{t}_{11,11,11}^{prr,11} = \mathbf{d}_{11,11,11}^{ppr,11} \\ &= -\mathbf{d}_{11,11,11}^{prp,11} = \tfrac{1}{2}\mathbf{t}_{11,11,11}^{ppr,11} = -\tfrac{1}{2}\mathbf{t}_{11,11,11}^{rpp,11}, \end{aligned}$$
$$\begin{aligned} d_3^j = \mathbf{d}_{2j,11}^{pp,11} = \mathbf{d}_{2j,11}^{rr,11} = \mathbf{t}_{2j,11}^{pp,11} + \mathbf{t}_{11,2j}^{pp,11} = \mathbf{t}_{2j,11}^{rr,11} + \mathbf{t}_{11,2j}^{rr,11} = \mathbf{d}_{2j,11}^{rp,11} \\ = -\mathbf{d}_{2j,11}^{pr,11} = \mathbf{t}_{11,2j}^{pr,11} = -\mathbf{t}_{2j,11}^{pr,11}, \end{aligned}$$
$$d_4^j = \mathbf{d}_{11,2j}^{pp,11} = \mathbf{d}_{11,2j}^{rr,11} = \mathbf{d}_{11,2j}^{pr,11} = -\mathbf{d}_{11,2j}^{rp,11},$$
$$d_5^j = \mathbf{d}_{0j,11}^{pp,11} = \mathbf{t}_{0j,11}^{pp,11} + \mathbf{t}_{11,0j}^{pp,11} = \mathbf{d}_{0j,11,11}^{pr,11} = \mathbf{t}_{0j,11,11}^{pr},$$
$$d_6^j = \mathbf{d}_{11,0j}^{pp,11} = \mathbf{d}_{11,0j}^{rp,11},$$
$$\mu_{3h}^p = \mathbf{d}_{3h}^{p,3h} = \mu_{3h}^r = \mathbf{d}_{3h}^{r,3h},\ \sigma_{3h}^2 = \mathbf{g}_{3h}^{p,3h}/\mathbf{d}_{3h}^{p,3h} = \mathbf{g}_{3h}^{r,3h}/\mathbf{d}_{3h}^{r,3h},$$
$$\mathcal{G}_{6,h} = \mathbf{g}_{11,11,11}^{ppp,3h} = -\tfrac{1}{3}\mathbf{g}_{11,11,11}^{prr,3h} = \tfrac{1}{3}\mathbf{g}_{11,11,11}^{ppr,3h} = -\mathbf{g}_{11,11,11}^{rrr,3h},$$
$$\mathcal{G}_{5,h}^j = \mathbf{g}_{11,2j}^{pp,3h} + \mathbf{g}_{2j,11}^{pp,3h} = -\mathbf{g}_{11,2j}^{rr,3h} - \mathbf{g}_{2j,11}^{rr,3h} = \mathbf{g}_{11,2j}^{pr,3h} = \mathbf{g}_{2j,11}^{pr,3h},$$
$$d_{11,h} = \mathbf{d}_{11,11,11}^{ppp,3h} = -\mathbf{d}_{11,11,11}^{rrp,3h} = -\tfrac{1}{2}\mathbf{d}_{11,11,11}^{prr,3h} = \mathbf{d}_{11,11,11}^{ppr,3h} = -\mathbf{d}_{11,11,11}^{rrr,3h} = \tfrac{1}{2}\mathbf{d}_{11,11,11}^{prp,3h},$$
$$d_{12,h} = \mathbf{t}_{11,11,11}^{ppp} = -\mathbf{t}_{11,11,11}^{prr,3h} = -\tfrac{1}{2}\mathbf{t}_{11,11,11}^{rpr,3h} = \mathbf{t}_{11,11,11}^{rpp,3h} = -\mathbf{t}_{11,11,11}^{rrr,3h} = \tfrac{1}{2}\mathbf{t}_{11,11,11}^{ppr,3h},$$
$$d_{13,h}^j = \mathbf{d}_{2j,11}^{pp,3h} = -\mathbf{d}_{2j,11}^{rr,3h} = \mathbf{d}_{2j,11}^{rp,3h} = \mathbf{d}_{2j,11}^{pr,3h},$$
$$d_{14,h}^j = \mathbf{d}_{11,2j}^{pp,3h} = -\mathbf{d}_{11,2j}^{rr,3h} = \mathbf{d}_{11,2j}^{pr,3h} = \mathbf{d}_{11,2j}^{rp,3h},$$
$$d_{15,h}^j = \mathbf{t}_{2j,11}^{pp,3h} + \mathbf{t}_{11,2j}^{pp,3h} = -\mathbf{t}_{2j,11}^{rr,3h} - \mathbf{t}_{11,2j}^{rr,3h} = \mathbf{t}_{11,2j}^{pr,3h} = \mathbf{t}_{2j,11}^{pr,3h},$$
$$\mu_{1k}^p = \mathbf{d}_{1k}^{p,1k} = \mu_{1k}^r = \mathbf{d}_{1k}^{r,1k},\quad \sigma_{1k}^2 = \mathbf{g}_{1k}^{p,1k}/\mathbf{d}_{1k}^{p,1k} = \mathbf{g}_{1k}^{r,1k}/\mathbf{d}_{1k}^{r,1k},$$

$$
\begin{aligned}
&\mathcal{G}_{1k} = \mathbf{g}^{ppp,1k}_{11,11,11} = \mathbf{g}^{prr,1k}_{11,11,11} = \mathbf{g}^{ppr,1k}_{11,11,11} = \mathbf{g}^{rrr,1k}_{11,11,11},\\
&\mathcal{G}^{j}_{2,k} = \mathbf{g}^{pp,1k}_{11,2j} + \mathbf{g}^{pp,1k}_{2j,11} = \mathbf{g}^{rr,1k}_{11,2j} + \mathbf{g}^{rr,1k}_{2j,11} = \mathbf{g}^{pr,1k}_{1k,11,2j} = -\mathbf{g}^{pr,1k}_{2j,11},\\
&\mathcal{G}^{j}_{3,k} = \mathbf{g}^{pp,1k}_{0j,11} + \mathbf{g}^{pp,1k}_{11,0j} = \mathbf{g}^{pr}_{1k,0j,11},\\
&d^{j}_{16,k} = \mathbf{d}^{ppp,1k}_{11,11,11} = \mathbf{d}^{prr,1k}_{11,11,11} = \mathbf{d}^{prp,1k}_{11,11,11} = \mathbf{d}^{rrr,1k}_{11,11,11},\\
&d^{j}_{17,k} = \mathbf{d}^{rrp,1k}_{11,11,11} = -\mathbf{d}^{prr,1k}_{11,11,11} = \mathbf{d}^{ppr,1k}_{11,11,11} = -\mathbf{d}^{prp,1k}_{11,11,11},\\
&d^{j}_{18,k} = \mathbf{t}^{ppp,1k}_{11,11,11} = \mathbf{t}^{prr,1k}_{11,11,11} = \mathbf{t}^{rpp,1k}_{11,11,11} = \mathbf{t}^{rrr,1k}_{11,11,11},\\
&d^{j}_{19,k} = \mathbf{t}^{rpr,1k}_{11,11,11} = -\mathbf{t}^{prr,1k}_{11,11,11} = \mathbf{t}^{ppr,1k}_{11,11,11} = -\mathbf{t}^{rpp,1k}_{11,11,11},\\
&d^{j}_{20,k} = \mathbf{d}^{pp,1k}_{2j,11} = \mathbf{d}^{rr,1k}_{2j,11} = \mathbf{d}^{rp,1k}_{2j,11} = -\mathbf{d}^{pr,1k}_{2j,11},\\
&d^{j}_{21k} = \mathbf{d}^{pp,1k}_{11,2j} = \mathbf{d}^{rr,1k}_{11,2j} = -\mathbf{d}^{rp,1k}_{11,2j} = \mathbf{d}^{pr,1k}_{11,2j},\\
&d^{j}_{22,k} = \mathbf{t}^{pp,1k}_{2j,11} + \mathbf{t}^{pp,1k}_{11,2j} = \mathbf{t}^{rr,1k}_{2j,11} + \mathbf{t}^{rr,1k}_{11,2j} = \mathbf{t}^{pr,1k}_{11,2j} = -\mathbf{t}^{pr,1k}_{2j,11},\\
&d^{j}_{23,k} = \mathbf{d}^{pp,1k}_{0j,11} = \mathbf{d}^{pr,1k}_{0j,11}, \quad d^{j}_{24,k} = \mathbf{d}^{pp,1k}_{11,0j} = \mathbf{d}^{rp,1k}_{11,0j},\\
&d^{j}_{25,k} = \mathbf{t}^{pp,1k}_{0j,11} + \mathbf{t}^{pp,1k}_{11,0j} = \mathbf{t}^{pr,1k}_{0j,11}.
\end{aligned}
$$

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