Initial participation in a reasoning-and-proving discourse in elementary school teacher education

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Based on a commognitive framework, we analyse the reasoning-and-proving processes of two teachers, and we identify the actions and routines that are visible when working on a given task. The data consist of video recordings of the teachers' attempts to validate a stated hypothesis involving multiplicative reasoning. Six categories of what characterises the teachers' initiation into a reasoning-and-proving discourse are identified. The findings reveal that some actions related to substantiation routines seem to be applicable for novice teachers. Examples of this include questioning validity and the use of words related to deductive reasoning. However, the teachers' participation in the discourse is characterised by ritualised actions, as in their use of visual mediators. Furthermore, the analysis discloses the teachers' tendency to use construction-related actions in what was designed to be a validating activity.

***Keywords**: mathematical discourse, reasoning and proving, elementary school teacher education*

Introduction

Reasoning and proving are central aspects of mathematics as a discipline, and many researchers have argued that they should be a central part of school mathematics at all grades and in all topics. In this paper, we use a broad definition of the word proof (Reid, 2010), to denote mathematical reasoning involved in the process of making sense of and establishing mathematical knowledge. Hence, we follow Stylianides (2008), who used the term 'reasoning-and-proving' to denote the activities involved in this process: identifying patterns, making conjectures and providing arguments.

The reasoning-and-proving process is difficult to learn and difficult to teach. In exploring ways to teach proof, a number of studies have shown the crucial role that a teacher plays in helping students identifying the structure of a proof, presenting arguments and distinguishing between correct and incorrect arguments (see e.g. Stylianides, 2007). Researchers have found that elementary school teachers tend to rely on external authorities, such as textbooks, college instructors or more capable peers, as the basis of their conviction. They also believe it is possible to affirm the validity of a mathematical generalisation using a few examples (see Martin & Harel, 1989). Similarly, Stylianides, Stylianides, and Philippou (2007) revealed that pre-service teachers had two main types of difficulties with proof: the lack of understanding of the logic mathematical underpinnings of different modes of argumentation and the inability to use different modes of representations appropriately.

As exemplified above, research in mathematics education has shed light on different aspects of pre-service teachers' work on reasoning-and-proving, such as their beliefs related to proofs and proving, the challenges they face when deducing proofs and their use of modes of reasoning. However, more knowledge is needed about how pre-service teachers learn to teach reasoning-and-proving, as well as how teacher education can support their learning. A vital part of teachers' learning how to teach reasoning-and-proving is learning how to reason and prove in school-relevant mathematical areas (e.g. multiplicative reasoning). That learning is the topic of our study.

We examine two elementary school teachers' work on a reasoning-and-proving task during a professional development course. Like Remillard (2014), we consider mathematics to be a specific type of discourse where reasoning-and-proving is essential. Thus, mathematics learning is seen as participation in the discourse (Sfard, 2008). The two teachers whose work we analyse have limited experience of reasoning-and-proving in mathematics, and we are interested in their initiation into that discourse. Our research question is: What characterises two in-service teachers' initial participation in a mathematical discourse on reasoning-and-proving in elementary school teacher education?

Theoretical framework

Within a commognitive framework[1], Sfard (2008) take the position that learning mathematics is learning to participate in a specific discourse. Here, discourse is a special type of communication within a specific community that is made mathematical by that community's use of words, visual mediators, narratives and routines. The use of *words* in mathematics includes the use of ordinary words that have a special meaning in mathematics, like function and proof, and mathematical words, like fraction and axiom. Furthermore, people participating in mathematical communication use *visual mediators* to identify the object about which they are talking. These visual mediators are often symbolic, but they also include graphs, illustrations and physical artefacts. Within a discourse, any sequence of utterances, spoken or written, that describes the properties of objects or the relationships between objects is called a *narrative*. Mathematical narratives can be numerical, e.g. "½ is equivalent to 2/4", or more general, e.g. "addition is commutative". Narratives are subject to endorsement or rejection, that is, being labelled as true or false, based on specific rules defined by the community. Endorsement of narratives is the main goal of the mathematics discourse; this includes the processes of constructing new endorsable narratives, substantiating them and recalling them in new situations.

Routines are well-defined practices that a given community regularly employs in a discourse. Sfard (2008) describe routines as patterns that are guided by two sets of rules: those telling the participants *how* to act, and those indicating *when* to do the given action. In contrast to rules on the object-level, which describe regularities on actions on and relations of objects, routines describe the participants' patterns of actions in a given discourse, and they can be considered to be rules on a meta-level. Lavie, Steiner, and Sfard (2018) emphasise the role of routines when participating in a specific discourse, and they suggest that learning routines can be seen as the routinisation of actions in a given discourse. Thus, on their way to new routines, learners must pass, if only briefly, through the stage of ritualised performance or imitation (Sfard, 2008). Here, *rituals* are understood to be socially oriented; they are acts of solidarity with co-performers. At this transitory stage, learners may become very familiar with the *how* of the new routine, but they will be much less aware of *when* it is used.

Our research question focuses on participation in a mathematical discourse on reasoning-and-proving, involving the processes of identifying patterns, making conjectures and providing arguments for whether or not conjectures are true. Hence, from a discursive stance, we are primarily interested in the routines associated with the construction and substantiation of narratives. New narratives are constructed mainly through operations on previously endorsed narratives. To substantiate a constructed mathematical narrative, one produces a proof—a sequence of endorsed narratives, each

[1] The term "commognition" is a combination of the words "communication" and "cognitive", and it stresses that thinking is a way of communicating with oneself and others.

of which is deductively inferred from previous ones, the last of which is the narrative that is being endorsed. Thus, learning to reason and prove in mathematics is about individualising both the *when* and the *how* of the construction and substantiation routines. In this paper, we focus on substantiation routines as framed by a task in which a hypothesis is already given.

To obtain insight into how individuals learn about reasoning-and-proving, it is useful to delineate the possible patterns of the processes and actions involved in constructing and substantiating narratives. Applying a commognitive perspective, Jeannotte and Kieran (2017) have developed a conceptual model of mathematical reasoning based on exhaustive analyses of mathematics education research. They propose the following definition of mathematical reasoning through commognition lenses: "Mathematical reasoning processes are commognitive processes that are meta-discursive, that is, that derive narratives about objects or relations by exploring the relations between objects" (Jeannotte & Kieran 2017, p. 9). Because their notion of mathematical reasoning involves proving, it coincides with our use of reasoning-and-proving. However, we use the latter to indicate that we look at a special kind of reasoning that is used when validating mathematical hypotheses, as mathematical reasoning *per se* does not necessarily include proving. Furthermore, Jeannotte and Kieran (2017) distinguishbetween processes related to the search for similarities and differences and processes related to validating. Searching for similarities and differences includes generalising, conjecturing, identifying a pattern, comparing and classifying. All these processes infer narratives about mathematical objects or relations (although on a partly different basis); thus, they are related to the routine of constructing narratives (Sfard, 2008). The processes related to validating include validating, justifying, proving and formal proving (defined inclusively, with an increasing degree of deductive structure and stringency). These processes aim to change the epistemic value (e.g. true, false) of a given narrative; therefore, they are related to substantiation routines (Sfard, 2008).

This study investigated two teachers' initial participation based on their utterances and actions in a mathematical discourse on reasoning-and-proving in elementary school teacher education. We aim to illustrate how key concepts from the commognitive framework proposed by Sfard (2008) can provide insight into how mathematics teachers learn about the process of reasoning-and-proving.

Method

The two elementary school teachers, Sandra and Nora (pseudonyms), who participated in the research study, were part of a professional developmental course in mathematics for teachers in grades 1–7 in Norway that was held by two of the authors of this paper. Both teachers are 45-50-year-old females, and both completed general teacher education with less than 15 ECST credits in mathematics education. Sandra and Nora represent typical teachers attending the course, due to their age, educational background, gender and having more than 10 years of experience as general teachers.

The course contained materials on mathematics and mathematics education, and it was organised as six, three-day seminars distributed over one year, in addition to the teachers' individual work on literature and assignments. The topics were sense-making in mathematics, pattern seeking and exploration, use of different representations (e.g. the array model for multiplication) and reasoning-and-proving (in particular, representation-based proofs). The participants noted that the coursework invited them to use new ways of thinking about and working with mathematics. This paper presents an analysis of the data collected through video recordings (in total 24 minutes) of Sandra and Nora working on a task (Teddy's hypothesis, see Figure 1) on the second day of the fourth seminar of the course. The day before data collection, the topic was multiplication: different properties, strategies, models and reasoning-and-proving.

The task was chosen for the purpose of reasoning-and-proving, starting with a hypothesis proposed by Teddy, an imaginary student (Figure 1). The first step of the task (part a) involves validation of the hypothesis; the second step of the task (part b) entails both stating and validating the new hypotheses. In our analysis, we study validation processes, as exemplified by Sandra and Nora's work on the first step of the task (part a).

Teddy is a grade 5 student. He and his classmates are working on square and cubic numbers. After completing some tasks, Teddy says to the teacher: "Look here, if you multiply … take two numbers and multiply… and both numbers end with 5… then the result also will end with 5".

a) Give a proof that shows that Teddy's observation is correct for all such numbers.
b) The situation can be used to propose and solve other problems, for instance:
    1. Is it only when both numbers end with 5 that the result ends with 5?
    2. Does the result hold only for 5, or when two numbers ending with the same digit are multiplied, does the product also end with that same digit?
    3. Which digits can square numbers end with?

Figure 1. Teddy's hypothesis task (adapted from Skott, Jess, & Hansen, 2008, pp. 223–224)

Teddy's hypothesis can be proved by using a generic example and array model of multiplication. Given any two numbers, both ending with 5, say 125 and 35, one can use the array model for multiplication to represent the multiplication 125x35, as shown in Figure 2.

	100	20	5
30	100×30	20×30	5×30
5	100×5	20×5	5×5

Figure 2. The array model for multiplication used in a representation-based proof of the hypothesis

Every cell in this array, except the cell with 5x5, is a multiple of 10. Thus, these cells do not contribute to the ones in the product. Only the cell with 5x5 does, and since 5x5 equals 25, we find that the product ends with 5. There is nothing special about the two numbers (35 and 125) in the example. The product of any two numbers both ending with 5 will have the same structure; thus, the number resulting from such a multiplication will end with 5.

Data analysis

The video recordings of Sandra and Nora's work on the task were transcribed, and then coded. The coding was guided by the research question. The aim was to describe the actions and utterances in the teachers' work, rather than to evaluate the mathematical and logical correctness of their arguments. Four researchers made a descriptive coding of the collected data, individually (Miles & Huberman, 1994). Next, the researchers compared and contrasted their coding and grouped the codes into six categories describing the two teachers' reasoning-and-proving efforts. The following categories were agreed upon: *confirming; proposing hypotheses*; *questioning validity*; *warranting; searching for patterns;* and *making drawings*.

To illustrate the categories and our findings, we present the teachers' work from part a of the hypothesis task. However, the above-mentioned categories apply to the teachers' discussions on all the given tasks. The teachers' utterances are sometimes imprecise and difficult to interpret, and we tried to preserve this in the translation. In the analysis presented below, we use italics to emphasise the categories.

Sandra and Nora start their work by reading the hypothesis proposed by Teddy.

1	N:	It is true, what he says.
2	S:	Yeah.
3	N:	So, the argument is correct.
4	S:	Yes, it is, eh; but it's more. It works for all odd numbers; so, the answer is 5.
5	N:	Yes, exactly.
6	S:	As long as one is a 5, one of …
7	N:	Yes, in the 5 times table, no matter what you multiply with something with a 5 in, then you'll get a 5 at the end of the answer.
8	S:	Yes, ehm …
9	N:	But, that's also because 5 is an odd number.
10	S:	Yeah, but do you have an argument that shows that Teddy is correct? Yes, it is, but is it ... is it enough? Now, we have actually, sort of gone further.

The category *confirming* in our analysis is a social act of support. Examples of this are seen in turns [2] and [5]. Furthermore, two new *hypotheses are proposed* in this excerpt of the discussion, one in turn [4] and another in [7]. Both hypotheses are related to Teddy's, but they are partly different, as [4] is more general than Teddy's hypothesis and [7] concerns properties of the 5 times table. In turn [9], the teachers *warrant* the hypothesis stated in turn [7]; in turn [10], they *question the validity* of Teddy's hypothesis. A few turns later, the discussion continues, as follows:

17	N:	Because, eh, when you, right, in the 5 times table [S: Yeah], when you multiply with an odd number, you'll always end with 5, [S: Yeah] the answer will, the sum will always end, the answer will always be 5, thus …
18	N:	And, anything that ends with 5 is an odd number, so if you multiply… 35 is an odd number, right.
19	S:	Yes, yes, because of the 5.
20	N:	Yes, so because of the 5 there it will be an odd number.
21	S:	Yeah.
22	N:	And, therefore, it will end with 5.
23	S:	Yeah.
24	S:	Yeah, but, but, if one should have made such a, representation-based proof for it, is that what they want? Or is it enough that we … it is probably not enough that we say this. [N: [laughs]] Believe me [in English].

In the utterance in turn [17] a new *hypothesis is proposed* (stating that when multiplying any odd number by 5, the product always ends with 5), which can be seen as a generalisation of the hypothesis proposed in turn [7]. At the same time, the purpose of the utterance in turn [17], and also several other utterances in this excerpt ([18, 19, 20, 22]), is *warranting,* as recognised by their use of the words "because" and "therefore". The utterance in turn [24] *questions the validity* of the argument given ("is it enough"). Following her own request for a representation-based proof, Sandra is *making drawings* of arrays on a sheet of graph paper (Figure 3).

26 S: Yeah, but if you have a, 1-2-3-4-5, [N: Yes] (draws a 1x5 array on the sheet), that's there. How do I draw this here, then? Eh … So, you have … (draws a 2x5 array)

27 N: So, each 5 you'll get… Now, there it is an even number. [S: Yeah (draws a 3x5 array)] Then, there is an odd number.

28 S: Yeah, do we get a … pattern? (draws a 4x5 array)

29 N: Yes… even number.

30 S: Yeah (draws a 5x5 array).

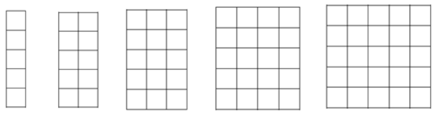

Figure 3. Sandra's drawings for part a of the hypothesis task

31 N: Odd number.

32 S: Yeah. But it's a … eh, even number (writes e below the 1x5 array). No, (corrects to the letter o below the 1x5 array) odd number. Odd number plus odd number is always … even number [N: even number, yeah] (writes o+o=e below the 2x5 array. And here it's [N: odd number] odd number plus odd number plus odd number, equals odd number (writes o+o+o=o below the 3x5 array, then o+o+o+o=e below the 4x5 array)

33 N: And five is an odd number.

34 S: Yeah, … Shall we drop this now, and try the next question?

While drawing the arrays, the teachers are *searching for patterns*. The patterns they discuss concern even and odd numbers in the 5 times table [32]. After turn [34], the teachers leave the task in step a, and proceed to step b. It is not clear if they are dissatisfied with the pattern discovered or if they are finding it difficult to identify a way to use the pattern to prove the hypothesis in turn [17] or Teddy's hypothesis, when they choose to leave the task.

Results and discussion

Our analysis shows that Sandra and Nora use several actions related to construction and substantiation routines. According to Sfard (2008), one of the distinct characteristics of discourses is the keywords that are used. In a mathematical reasoning-and-proving discourse, these keywords relate to deductive reasoning, which is "the only form of reasoning that can change the epistemic value of mathematical knowledge from likely to true" (Jeannotte & Kieran, 2017, p. 8). Sandra and Nora use words that are distinctive of a reasoning-and-proving discourse, namely their warranting of statements by their use of the word "because" followed by "then" or "therefore", as seen in excerpts [18–23]. Moreover, the teachers question the validity of the arguments they provide, and they make drawings and search for patterns. In general, the use of drawings as visual mediators is one of the main aspects of mathematical discourse, and its role in reasoning-and-proving was emphasised in the professional development course. Sandra and Nora's drawings and their search for patterns is initiated by their act of questioning the validity of the arguments, which is part of the process of convincing (oneself or another) and is fundamental to mathematical reasoning (Jeannotte & Kieran, 2017).

At the same time, Sandra and Nora's explicit reference to representation-based proving (as seen in statement [24]), within the framework of Sfard (2008) and Lavie, Steiner, and Sfard (2018), indicate that the teachers' participation is ritualised. As previously explained, rituals are understood to be socially oriented; they are acts of solidarity with co-performers or authorities. With their questioning of validity, Sandra and Nora express what they assume to be expected by the community, i.e. the teacher educators, regarding substantiation routines ("is that what they want?" [24]). The making of drawings and the search for patterns emerge in the teachers' work as a result of stating this question. This stands in contrast to questioning validity on the basis of the given hypothesis and a discussion of what narratives can be considered to already be endorsed by the community. Apart from the question referring to the teacher educators, Sandra and Nora also frequently confirm each other's contributions. Because their questioning of the validity and confirming each other's statements appears to be an attempt to gain social acceptance rather than their need to support and strengthen their substantiation of Teddy's hypothesis, their initial participation in the reasoning-and-proving discourse appears to be ritualised.

Our analysis also reveals ritualised participation in terms of *how* to act, in particular, how to use drawings. As shown in Figure 3, Sandra has made a drawing based on her own request for a representation-based proof of Teddy's hypothesis. The drawing and the following search for a pattern are related to the teachers' hypothesis (as seen in [17]), and not Teddy's hypothesis. Nevertheless, the chosen drawing does not advance the teachers 'reasoning-and-proving process.

Sandra and Nora's actions related to construction and substantiation of narratives also indicate ritualised participation in terms of *when* to do a given action. As previously discussed, meta-discursive processes of reasoning-and-proving can be divided into processes of searching for similarities and differences (constructing narratives) and validating processes (substantiation of narratives). Throughout Sandra and Nora's conversation, these processes seem to intersect: several actions that they use, e.g. proposing hypotheses and searching for patterns, are mainly related to the processes of construction of narratives, and they are not appropriate for modifying the epistemic value of a narrative from likely to true. In a substantiation routine, a sequence of endorsed narratives is used, each of which is deductively inferred from previous narratives. Sandra and Nora propose several *new* hypotheses during their work (e.g. in [17]), and they are not explicit about whether the new narratives are (or can be seen to be) endorsed by the community and how they connect to Teddy's hypothesis. Moreover, Sandra and Nora search for patterns related to even and odd numbers, and it seems that the aim of this action is proving a hypothesis given in [17]. However, their search for patterns does not help them validate the hypothesis, and they leave the task. It is worth noting that the teachers' use of actions related to the construction of narratives happens, even though the Teddy's hypothesis task was designed to direct the teachers to participate in the validating process.

Conclusions and implications

Ritualised participation and challenges in knowing how and when a given action can be used are not surprising results when studying novices' initial participation in a given discourse (Sfard, 2008). Yet, within the frames of commognition, we have highlighted that some reasoning-and-proving actions seem to be more visible and applicable for novice participants than other actions; thus, they are easier to imitate. The teachers in this study employed several actions that are not directly related to substantiation but are regularly applied in a mathematical discourse. They search for patterns, propose a hypothesis and make drawings. They also perform actions related to substantiation routines, such as warranting and questioning validity. Yet, other actions related to substantiation of narratives seem to be more hidden. For example, being critical is central to substantiation routines; however, Sandra and Nora continuously confirmed each other's contributions.

Moreover, the analysis discloses the two teachers' tendency to use construction-related actions (searching for patterns, proposing hypotheses) in what was designed to be a validating activity. Thus, the findings imply a need in teacher education to be more explicit about what actions are specific for reasoning-and-proving, and also, to be explicit about changes in actions when moving from construction to substantiation of narratives.

In this paper, we have reported on the characteristics of two in-service teachers' learning of reasoning-and-proving in a professional development context in the field of elementary education. Nevertheless, our study is limited by the number of participants, and further research from a commognitive standpoint is needed to shed more light on elementary education teachers' learning of reasoning-and-proving. For example, longitudinal studies are needed to learn more about teachers' evolving routines. Another topic for further research is the role of visual mediators in a reasoning-and-proving context, and how participants can routinise the use of visual mediators in the discourse.

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