arXiv:1805.11943v3 [math.GM] 20 Mar 2019

ON THE PHILOSOPHY OF HIGHER STRUCTURES

NILS A. BAAS

1. Introduction

Higher structures occur and play an important role in all sciences and their applications. In a series of papers [2–15] we have developed a framework called Hyperstructures for describing and working with higher structures. The purpose of this paper is to describe and elaborate the philosophical ideas behind hyperstructures and structure formation in general and emphasize the key ideas of the Hyperstructure Program.

In Baas [4] we formulated a very general principal for forming higher structures that we now call Hyperstructures and abbreviate to $\mathscr{H}$-structures. There are six basic stages in this principle of forming $\mathscr{H}$-structures.

2. The $\mathscr{H}$-Principle

(I) *Observation and Detection.*

Given a collection of objects that we want to study and give a structure. First we observe the objects and detect their properties, states, etc. This is the *semantic* part of the process. Finally we may also select special objects.

(II) *Binding.*

A procedure to produce new objects from collections of old objects by "binding" them in some way. This is the *syntactic* part of the process.

(III) *Levels.*

Iterating the described process in the following way: forming bonds of bonds and — important! — using the detected and observed properties at one level in forming the next level. This is iteration in a new context and not a recursive procedure. It combines syntax and semantics in forming a new level. Connections between levels are given by specifying how to dissolve a

Date: March 11, 2018.

bond into lower level objects. When bonds have been formed to constitute a new level, observation and detection is like finding "emergent properties" of the process.

These three steps are the most important ones, but we include three more in the general principle.

(IV) *Local to global.*

Describing a procedure of how to move from the bottom (local) level through the intermediate levels to the top (global) level with respect to general properties and states. The importance of the level structure lies in the possibility of manipulating the systems levelwise in order to achieve a desired global goal or state. This can be done using "globalizers" — an extension of sections in sheaves on Grothendieck sites (see Baas [9]).

(V) *Composition.*

A way to produce new bonds from old ones. This means that we can compose and produce new bonds on a given level, by "gluing" (suitably interpreted) at lower levels. The rules may vary and be flexible due to the relevant context.

(VI) *Installation.*

Putting a level structure satisfying I–V on a set or collection of objects in order to perform an analysis, synthesis or construction in order to achieve a given goal. The objects to be studied may be introduced as bonds (top or bottom) in a level structure.

Synthesis: The given collection is embedded in the bottom level.

Analysis: The given collection is embedded in the top level.

Synthesis facilitates local to global processes and dually, analysis facilitates global to local processes by defining localizers dual to globalizers, see [10].

The steps I–VI are the basic ingredients of what we call the *Hyperstructure Principle* or in short the *$\mathscr{H}$-principle*. (Corresponding to "The General Principle" in Baas [4].) In our opinion it reflects the basic way in which we make or construct things. This applies to mathematics, engineering, societies and organizations. In many ways it reflects "The Structure of Everything!"

As a generic example we may think of a collection of individuals.

(a) We observe them and detect their qualities, properties or states.

(b) We use their "properties" to put them into groups interacting to achieve specific goals.

(c) We observe the groups, and bind them to groups of groups for specific reasons.

(d) We introduce mechanisms (elections, communcations,...) making it possible to pass from local states and properties to global ones through the levels. For example in democracies.

(e) Given a collection of individuals, installation is the process of organizing them in an $\mathscr{H}$-structure including (f).

(f) Composition means that at any level overlapping groups may be turned into new groups.

Evolution is a fundamental $\mathscr{H}$-structured process. The way we think also follows the $\mathscr{H}$-principle which really represents "food for thought." We think that extracting and formalizing the essential parts of these important structures and processes is very useful. How do we formalize the $\mathscr{H}$-principle?

3. Hyperstructures

We will here give the idea of how to formalize the $\mathscr{H}$-principle into a mathematical framework. Details will be given in a separate paper. Given a collection of objects and consider them as elements of a set X. Let $\mathscr{P}(X)$ be the collection of all subsets of X. We may also consider sets with structures like spaces, groups, orderings, etc.

First, we *observe* the subsets in terms of an assignment

$$\Omega\colon \mathscr{P}(X) \to \text{Sets}$$

(mathematically like a functor). This observation mechanism detects properties and (or) states $\Omega(S)$ of a subset $S \subseteq X$. This is *detection*.

Then we consider subsets with properties, i.e. pairs $(S, \omega), \omega \in \Omega(S)$. To each pair we want to assign another set $B(S, \omega)$ — the set of *bonds* — meaning "mechanisms" that can bind S into some kind of unity. This means that we consider the collection of all these:

$$\Gamma = \{(S, \omega) \mid S \subseteq X, \omega \in \Omega(S)\}$$

We will elsewhere discuss how these assignments for S and S' with $S' \subseteq S$, $S' \cap S = \emptyset$ or $S' \cap S \neq \emptyset$ are related.

Form bonds of these pairs in terms of an assignment:

$$B\colon \Gamma \to \text{Sets}$$

— in mathematical terms also possibly like a functor.

In step III we want to create a new level, proceeding as follows: Giving everything considered so far an index 0 we form a new set or collection of objects:

$$X_1 = \{b_0 \mid b_0 \in B(S_0, \omega_0), S_0 \in \mathscr{P}(X_0), \omega_0 \in \Omega_0(S_0)\}$$

Based on X_1 we proceed with a new Ω_1, Γ_1 and B_1 to form X_2 in the same way. B_1 will then represent bonds of bonds. In this way we continue the process, but notice there are new choices of assignments at each level. Hence we end up with a hyperstructure of order $N - \mathscr{H}$, specified by:

$$\begin{aligned}
\mathcal{X} &= \{X_0, \ldots, X_N\} \\
\Omega &= \{\Omega_0, \ldots, \Omega_N\} \\
\mathcal{B} &= \{B_0, \ldots, B_N\} \\
\partial &= \{\partial_0, \ldots, \partial_N\}
\end{aligned}$$

where ∂ consists of level connections

$$\partial_i\colon X_{i+1} \to \mathscr{P}(X_i),$$

such that $\partial_i b_i = S_i$.

Definition. The system $\mathscr{H} = (\mathcal{X}, \Omega, \mathcal{B}, \partial)$ is called a hyperstructure of order N.

This is a semi-formal definition. In order to make it into a formal mathematical definition more technical conditions shall be added. This will be done in a separate paper, [11].

Hyperstructures extend the ideas involved in the mathematical theory of higher categories, see [19, 20].

4. Discussion

Once a hyperstructure has been constructed the essence lies in the bonds: $B_0, B_1, \ldots, B_n$. To each level of bonds we may assign new properties or states (dynamic or static). In order to "glue" or put together properties or states in a compatible way we use a Grothendieck topology and a generalized site on $\mathscr{H}$ (see Baas [9]) denoted by J. Then we consider the pair:

$$(\mathscr{H}, J) \qquad (\mathscr{H} \text{ of level } n).$$

We consider a new type of property (or state) assignment

$$(\mathscr{H}, J) \xrightarrow{\Lambda} \mathscr{S}$$

where $\mathscr{S}$ is a kind of n-level structure for example another hyperstructure where possibly the levels themselves can be hyperstructures:

$$\begin{aligned} B_n &\xrightarrow{\Lambda_n} \mathscr{S}_0 \\ B_{n-1} &\longrightarrow \mathscr{S}_1 \\ \vdots \quad & \quad \vdots \\ B_0 &\xrightarrow{\Lambda_0} \mathscr{S}_n. \end{aligned}$$

As discussed in Baas [9] this gives a mechanism to pass from level B_0 (Bottom level) to B_n (Top level) of properties or states — the local to global process IV in the $\mathscr{H}$-principle. For example there may be a preferred state s_0 in $\mathscr{S}_0$ that we want the system to achieve by suitable dynamical actions at the lower levels. Complex state structures (hyperstructures) may be useful in manipulating a system in giving a wide variety of possibilities.

For this we need a "globalizer" which is a sequence of assignments

$$\mathscr{S}_n \rightsquigarrow \mathscr{S}_{n-1} \rightsquigarrow \cdots \rightsquigarrow \mathscr{S}_0$$

compatible with the ∂_i's and the site structure J (see Baas [9]). Often it may be easier to perturb (take dynamic actions) at lower levels. We may here perturb at the lowest level — states in $\mathscr{S}_n$ — and via level changes at the $\mathscr{S}_i$'s let the changes propagate through the structure to achieve the desired state s_0.

This is a formal description of what happens often in social systems and organizations. A formal framework may help applying this to many more situations.

Often one may want to use a collection of objects or systems (think of a group of individuals) to achieve a goal. In order to do so one may have to structure (organize) the collection X by putting a hyperstructure on it — $\mathscr{H}(X)$ where X is embedded in e.g. the top or bottom level. Then one can use a globalizer or its dual to act on the collection X to achieve the desired goal.

This is *Installation* — we will install a hyperstructure on X. Once we have formed a collection of various hyperstructures we may apply the $\mathscr{H}$-principle to form $\mathscr{H}$-structures of such collections again and this goes on to any order. Our point is to show that there is a lot of room in these new higher order universes for forming new and interesting objects both in the abstract and physical sense.

Installation may apply to physical, biological and abstract systems. It may also be useful in Quantum Systems making local effects global, and controlling global states by local manipulations.

Evolutionary processes are examples of $\mathscr{H}$-structure. Nature uses object properties in forming new objects, whose properties again are used to form the next level of objects.

Let us illustrate this by an example.

An $\mathscr{H}$-structure as defined in Section 3 is given by:

Level 0:

- Basic objects (X_0): cells
- Properties (Ω_0): receptors (selected properties)
- Bonds (B_0): aggregates of cells formed by the selected receptors.

Level 1:

- Objects (X_1): Organs formed by aggregates.
- Properties (Ω_1): Products made by organs (selected).
- Bonds (B_1): Aggregates of organs using the product properties as bonds — forming new units.

Level 2:

- Objects (X_2): Individuals formed by aggregates of organs.
- Properties (Ω_2): Various types of skills for selection.
- Bonds (B_2): Combination of skills of individuals forming a unity — a population.

This structure may be refined and extended both to higher and lower levels. The objects of each level may be subject to selection, See Buss [16] for a thorough discussion of how selection of units leads to the formation of higher order (hierarchical) organizations. Hyperstructures capture this in a general sense showing the potential of a plethora of applications where mathematical structures will be needed.

Another illustrating example is as follows. Given a finite set X of agents and the goal is to maximize interactions in X. Ideally everybody would interact with everybody realizing the complete graph on X, but normally there will be constraints giving a subinteraction graph.

What next?

Then subsets of X may interact and we lift the interactions to the power set level $\mathscr{P}(X)$. Lots of new interactions may occur and when all possibilites are exhausted within the given context, we may proceed to the next level of subsets in $\mathscr{P}^2(X)$ using the newly created properties to form new bonds (subsets), etc. The potential is enormous, and $\mathscr{H}$-structures lead to a lot of structural novelty.

Note. From now on we will let the assignments of properties, phases and states just be called states with the understanding of this broad interpretation. The new thing here is that we have levels of observables, states, properties, etc. — not just local and global.

5. Elaborations

5.1. **$\mathscr{H}$-formation.** Once we have constructed a hyperstructure — basically satisfying I–III — it may be used as a target for state (and property) assignments in the formation and construction of new $\mathscr{H}$-structures, etc. This is a very important idea. It is similar to and extends categorification and enrichment in forming higher categories in mathematics.

First we use the general $\mathscr{H}$-principle with state and bond assignment in Sets or some other known mathematical structure, e.g. categories like described in Baas [4]. We call these first level basic $\mathscr{H}$-structures and denote them by

$$\mathscr{H}\text{yp}_0 = \{\mathscr{H}(0)\}.$$

In this case the Ω_i^0's and B_i^0's take values in structures $\mathscr{S}_1^0, \ldots, \mathscr{S}_n^0$, and we put $\mathscr{S}(0) = \{\mathscr{S}_i^0\}$.

Then we proceed with a new set of objects — possibly from $\mathscr{H}\text{yp}_0$ — and then let the Ω's and B's take values in $\mathscr{H}$-structures of type $\mathscr{H}\text{yp}_0$: $\mathscr{S}_1^1, \ldots, \mathscr{S}_n^1$ such that even $\mathscr{S}(1) = \{\mathscr{S}_1^1, \ldots, \mathscr{S}_n^1\}$ may be an $\mathscr{H}(0)$-structure. The results are $\mathscr{H}(1)$ structures.

This gives us $\mathscr{H}\text{yp}_1 = \{\mathscr{H}(1)\}$ and in this way we proceed to form

$$\begin{gathered}\mathscr{H}\text{yp}_2 = \{\mathscr{H}(2)\}\\ \vdots \\ \mathscr{H}\text{yp}_n = \{\mathscr{H}(n)\}.\end{gathered}$$

Similarly if we are given an $\mathscr{H}$-structure $\mathscr{H}$ of some type, say, $\mathscr{H}\text{yp}_k$, we may want to make new state (property) assignments and study local to global relations.

As already described and discussed in Baas [9], we form a generalized site $(\mathscr{H}, J)$ and give assignments

$$\begin{aligned} B_n &\rightsquigarrow \mathscr{S}_0 \\ B_{n-1} &\rightsquigarrow \mathscr{S}_1 \\ &\vdots \\ B_0 &\rightsquigarrow \mathscr{S}_n \end{aligned}$$

where the $\mathscr{S}_i$'s are $\mathscr{H}$-structures at some level, even such that $\mathscr{S} = \{\mathscr{S}_i\}$ is an $\mathscr{H}$-structure itself, for example of type $\mathscr{H}\text{yp}_k$. In this context one may then consider globalizers (as in Baas [9]) relating local and global states. A globalizer will permit levelwise dynamics or actions in order to get to a desired global state from given local ones, as illustrated in Figure 1.

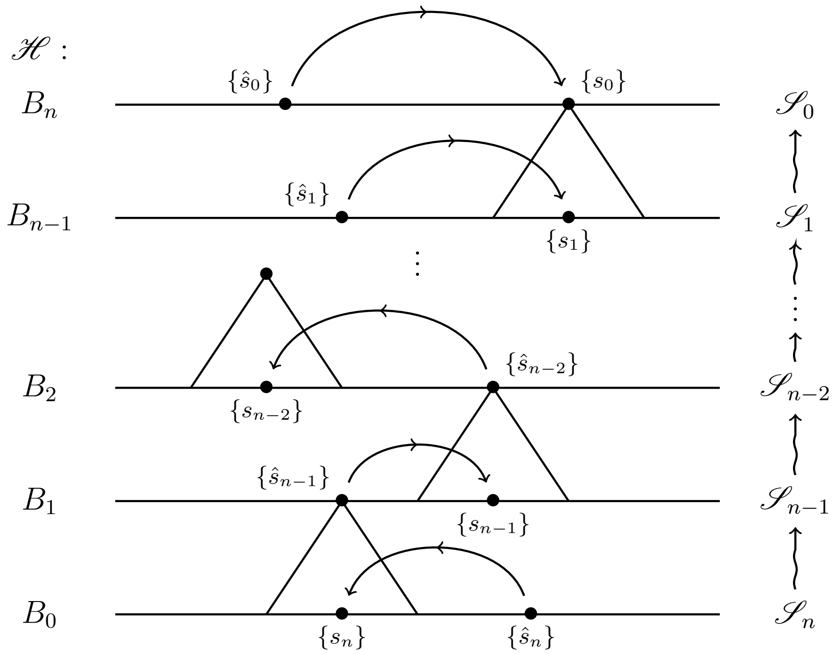


FIGURE 1.

5.2. **$\mathscr{H}$-processes.** In many types of systems: atomic and molecular, biological, sociological and organizational one wants to change global states by performing local actions. $\mathscr{H}$-structures are useful in organizing such actions.

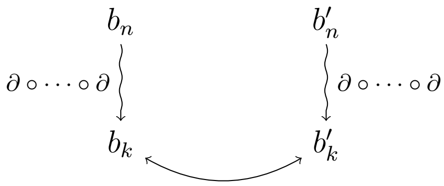
Suppose we have a system X of objects and an associated system or structure of states $\mathscr{S}(X)$. Often a general situation is that one wants to change the state of the system by finding a suitable action: Given $s_0, \hat{s}_0 \in \mathscr{S}(X)$ and construct an action $A\colon \mathscr{S}(X) \to \mathscr{S}(X)$ changing the state. A possible way to do this is by installing $\mathscr{H}$-structures: $\mathscr{H}(X)$ and $\mathscr{H}(\mathscr{S}(X))$ and construct a hyperstructured action $A(\mathscr{H})$ such that $A(\mathscr{H})(\hat{s}_0) = s_0$ where $A(\mathscr{H}) = \{A_k(\mathscr{H})\}$ and in the notation of Figure 1:

$$A_k(\mathscr{H})(\hat{s}_{n-k}) = s_{n-k}.$$

Such a change of state could be a change of production, changing a material property, fusion or splitting of systems, change of political attitudes, etc. The point is that the lower level, local actions may be easier to perform than the global ones. A higher order state structure allows for more varied actions and propagation through the levels. Local to global processes in $\mathscr{H}$-structures are sometimes like structured and controlled "butterfly effects" as seen in non-linear chaotic systems — amplifying small effects. See Nicolis and Nicolis [21].

5.3. **$\mathscr{H}$-algebras.** In an $\mathscr{H}$-structure with bonds $\{B_0, B_1, \ldots, B_n\}$ we may define operations or products of bonds by "gluing." If b_n and b'_n

are bonds in B_n that are "gluable" at level k, then we "glue" them to a new bond $b_n \,\square_k^n\, b'_n$:



"gluable" (having similar parts to be identified).

$(\mathscr{H}, \{\square_k^n\})$ gives new forms of higher algebraic structures. We have *level operations* $\{\square_k^k\}$ and *interlevel operations* $\{\square_k^n\}$.

For geometric objects X and Y one may define a "fusion" product

$$X \,\square_{\mathscr{H}}\, Y$$

by using installed $\mathscr{H}$-structures on $\mathscr{H}(X)$, $\mathscr{H}(Y)$ and $\mathscr{H}(X \sqcup Y)$, see Baas [9].

If in an $\mathscr{H}$-structure we are given a bond b_k binding $\{b_{k-1}^i\}$ the state assignments will give a levelwise assignment via a globalizer

$$\Lambda_{k-1}(\{b_{k-1}^i\}) \rightsquigarrow \Lambda_k(b_k).$$

The globalizers act as generalized pairings connecting levels. In some cases factorization algebras connect local to global observables. The global observables may be obtained from the local ones up to isomorphism in perturbative field theories, see [1, 17, 18], but not in general.

Often a tensor product $\otimes_k$ may be provided in $\mathscr{S}_k$ we may have

$$\Lambda_{k-1}(\{b_{k-1}^i\}) = \bigotimes_i \Lambda_{k-1}(b_{k-1}^i)$$

and sometimes when it makes sense, $\mathscr{S}_{k-1} = \mathscr{S}_k$, like in topological quantum field theory:

$$\Lambda_k(b_k) \in \bigotimes_i \Lambda_{k-1}(b_{k-1}^i).$$

See also Baas [8].

An *$\mathscr{H}$-algebra* will be an $\mathscr{H}$-structure $\mathscr{H}$ with "fusion" operations $\square = \{\square_k^n\}$. One may also add a "globalizer" (see Baas [9]) and tensor-type products as just described. The combination of a tensor product and a globalizer is a kind of extension of a "multilevel operad."

5.4. **$\mathscr{H}$-scaffolds.** In many situations we have systems with resources that we want to release. This could be energy, products, human resources, etc. Often it takes resources to get resources released. In such situations it may be advantageous to put a suitable hyperstructure $\mathscr{H}(X)$ on the system X. Often release of resources at the lowest level may require small inputs, the outputs being then inputs of the next level as in the following scheme.

$\mathscr{H}(X)$	
Bonds	Resources
B_n	R_n
$\vdots$	$\vdots$
B_1	R_1
B_0	R_0

	$\mathscr{S}(X)$	
States	Actions	Mechanisms
$\mathscr{S}_0$	$s_0 \xrightarrow{A_0} t_0$	M_0
$\vdots$	$\vdots$	$\vdots$
$\mathscr{S}_{n-1}$	$s_{n-1} \xrightarrow{A_{n-1}} t_{n-1}$	M_{n-1}
$\mathscr{S}_n$	$s_n \xrightarrow{A_n} t_n$	M_n

Then we may imagine that a small (in a precise sense) input of resources r_0 will cause an action A_n by the help of a mechanism M_n to change the state from s_n to t_n, which again will cause a larger release of resources r_1. This will be used as an input and proceed through the levels upwards until a release r_n is obtained.

In favourable situations with a well designed $\mathscr{H}$-structure we may have:

$$r_0 \ll r_1 \ll \cdots \ll r_n.$$

In this way we may think of $\mathscr{H}(X)$ as a *"scaffold"* on X. The actions will take place as in Figure 1. The mechanisms providing the actions are designed levelwise. We may think of this in a metaphorical way: We have a dam and want to release energy. Start with a mechanism M_n drilling small holes at the top. The released water is organized ("by bonds") to act on another mechanism M_{n-1} drilling bigger holes, releasing more water, etc. In this picture the $\mathscr{H}$-structure acts like a scaffold.

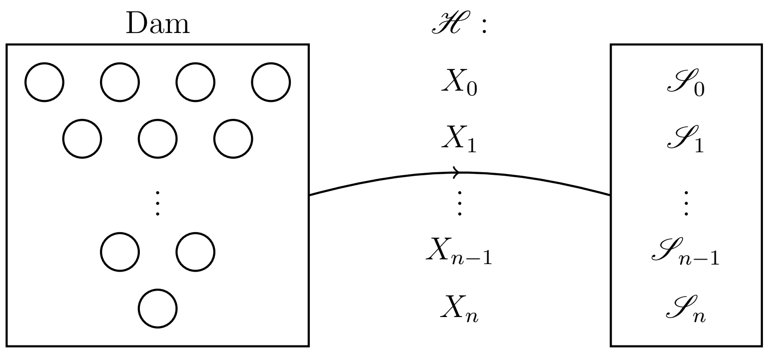


FIGURE 2.

This is meant to illustrate how "small" local actions may cause "large" global effects on a system suitably organized into an $\mathscr{H}$-structure. We see this in biological systems, societies and organizations as well as in systems releasing various types of energy: chemical and nuclear (fission and fusion). Hyperstructures may improve such processes.

5.5. **$\mathscr{H}$-states.** States of an $\mathscr{H}$-structure that are levelwise connected by a globalizer we call $\mathscr{H}$-states. Desired states of bonds b_0 at the lowest level may be thought of as being "protected" by the higher ones from external "noise." On the other hand we may think of desired states b_n at the top level as being supported or created by the lower level ones, and may also be protected by an underlying geometric $\mathscr{H}$-structure of the system.

Consider $\mathscr{H} = \{B_0, B_1, \ldots, B_n\}$. Start with (S_0, ω_0), $\omega_0 \in \Omega_0(S_0)$, $b_0 \in B_0(S_0, \omega_0)$. Proceed to $b_1(\{(b_0^{j_0}, \omega_0^{j_0})\}, \omega_1)$, etc.

$$\begin{gathered} \omega_0^{j_0} \text{ state is "protected" by } b_1 \\ \omega_1^{j_1} \text{ state is "protected" by } b_2 \\ \vdots \\ \omega_{n-1}^{j_{n-1}} \text{ state is "protected" by } b_n. \end{gathered}$$

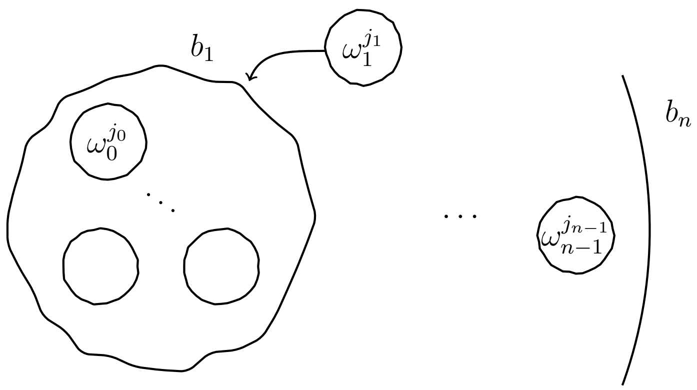


FIGURE 3.

The local states are protected by the higher bonds which may be realized in various ways as fields, subspaces, etc.

If we have a globalizer of states

$$\mathscr{S}_n \rightsquigarrow \mathscr{S}_{n-1} \rightsquigarrow \cdots \rightsquigarrow \mathscr{S}_0$$

we may think of levels of states

$$\{s_n\} \rightsquigarrow \{s_{n-1}\} \rightsquigarrow \cdots \rightsquigarrow \{s_0\}$$

as states of condensation, like a multilevel Bose–Einstein condensation (s_0). Another interpretation is as an $\mathscr{H}$ organized form of entanglement (see Baas [5]).

Sometimes when coherence of bonds and states fail ("equations do not hold") higher bonds may be introduced and used in order to solve the "frustration" of lack of equations and introduce a new form of unity (with proper equations). This may go on for several levels.

6. The structure of everything

Suitably formulated hyperstructures cover most types of structures in nature and science like e.g. trees, graphs, hypergraphs, networks, spaces, categories, etc. often considered as structures with just one level. But in nature — in biology — we see lots of structures built up with several natural levels — the effects of evolution.

The way we think and evolution have shown us the importance of observing "things" and detect properties that we use in binding old "things" into new "things." This is the key idea in forming higher structures, and hyperstructures give us a framework in which we can design and perform these constructions.

Our main point is that using higher structures in the sense of hyperstructures represents an enormous unused potential for *new* kinds of design and construction. There is a lot of room in the "space" of structures and the $\mathscr{H}$-principle is a way to create new structures by constructing $\mathscr{H}$-structures on sets or collection of objects. This gives a plethora of new structures described as hyperstructures which is useful both for construction and handling of these new universes.

Hyperstructures represent a universal and unifying mechanism to study both existing objects, making new objects and studying them as well. For example, this should lead to higher structures as follows:

$\mathscr{H}$-states
$\mathscr{H}$-materials
$\mathscr{H}$-brain states
$\mathscr{H}$-gene structure
$\mathscr{H}$-organization, economy
$\mathscr{H}$-language
⋮

In science symmetries are important. So are limits of all kinds ("More is different," see Baas [10]). We would like to add "Higher is different" — meaning that lots of new phenomena occur in higher structures (hyperstructures) with several levels.

As pointed out in [2–10, 12–15] there are many important areas of science where we think that $\mathscr{H}$-structures may turn out to be very useful and important. Let us sum up by mentioning some:

(1) Evolution, Genome structure, Cancer.
(2) The brain and AI systems.
(3) New materials in chemistry and physics (e.g. molecular $\mathscr{H}$-links, high temperature superconductors).
(4) Fusing systems (including nuclear fusion) and energy production in general.
(5) Organizations, Societies, Economics, Production systems.
(6) Engineering and Architecture design.
(7) New higher QM-states — extending Efimov states and GHZ-states. $\mathscr{H}$-type condensates — extending Bose–Einstein condensates.

(8) New universes of Abstract Matter for design and construction.
(9) Mathematics in many ways. Extending for example higher categories.

What more? Time will tell! Only our imagination limits the list.

Note. $\mathscr{H}$-structures are structures that I imagined and "saw" as a child, but it took a lifetime to describe what I saw!

Acknowledgements. I would like to thank M. Thaule for his kind technical assistance in preparing the manuscript.

Notes on the contributor

Nils A. Baas was born in Arendal, Norway, 1946. He was educated at the University of Oslo where he got his final degree in 1969. Later on he studied in Aarhus and Manchester. He was a Visiting Assistant Professor at U. Va. Charlottesville, USA in 1971–1972. Member of IAS, Princeton in 1972–1975 and IHES, Paris in 1975. Associate Professor at the University of Trondheim, Norway in 1975–1977 and since 1977, Professor at the same university till date. He conducted research visits to Berkeley in 1982–1983 and 1989–1990; Los Alamos in 1996; Cambridge, UK in 1997, Aarhus in 2001 and 2004. Member IAS, Princeton 2007, 2010, 2013 and 2016. His research interests include: algebraic topology, higher categories and hyperstructures and topological data analysis.

References

[1] D. Ayala, J. Francis, and N. Rozenblyum. Factorization homology I: higher categories. Preprint arXiv:1504.04007, 2015.
[2] N.A. Baas. Hyper-structures as a tool in nanotechnology. *Nanobiology*, 3(1):49–60, 1994.
[3] N.A. Baas. Higher order cognitive structures and processes. In *Toward a Science of Consciousness*, pages 633–648. MIT Press, 1996.
[4] N.A. Baas. Hyperstructures as abstract matter. *Adv. Complex Syst.*, 9(3):157–182, 2006.
[5] N.A. Baas. New structures in complex systems. *Eur. Phys. J. Special Topics*, 178:25–44, 2009.
[6] N.A. Baas. Hyperstructures, topology and datasets. *Axiomathes*, 19(3):281–295, 2009.
[7] N.A. Baas. New states of matter suggested by new topological structures. *Int. J. Gen. Syst.*, 42(2):137–169, 2013.
[8] N.A. Baas. On structure and organization: an organizing principle. *Int. J. Gen. Syst.*, 42(2):170–196, 2013.

[9] N.A. Baas. Higher order architecture of collections of objects. *Int. J. Gen. Syst.*, 44(1):55–75, 2015.

[10] N.A. Baas. On higher structures. *Int. J. Gen. Syst.*, 45(6):747–762, 2016.

[11] N.A. Baas. On the mathematics of higher structures. Preprint, arXiv:1805.11944, 2018.

[12] N.A. Baas and N.C. Seeman. On the chemical synthesis of new topological structures. *J. Math. Chem.*, 50(1):220–232, 2012.

[13] N.A. Baas, A.C. Ehresmann, and J.-P. Vanbremeersch. Hyperstructures and memory evolutive systems. *Int. J. Gen. Syst.*, 33 (5):553–568, 2004.

[14] N.A. Baas, D.V. Fedorov, A.S. Jensen, K. Riisager, A.G. Volosniev, and N.T. Zinner. Higher-order brunnian structures and possible physical realizations. *Physics of Atomic Nuclei*, 77(3):336–343, 2014.

[15] N.A. Baas, N.C. Seeman, and A. Stacey. Synthesising topological links. *J. Math. Chem.*, 53(1):183–199, 2015.

[16] L.W. Buss. *The evolution of individuality.* Princeton University Press, Princeton, NJ, 1987.

[17] K. Costello and O. Gwilliam. *Factorization algebras in quantum field theory. Vol. 1*, volume 31 of *New Mathematical Monographs.* Cambridge University Press, Cambridge, 2017.

[18] G. Ginot. Notes on factorization algebras, factorization homology and applications. In *Mathematical aspects of quantum field theories*, Math. Phys. Stud., pages 429–552. Springer, Cham, 2015.

[19] J. Lurie. On the classification of topological field theories. In *Current developments in mathematics, 2008*, pages 129–280. Int. Press, Somerville, MA, 2009.

[20] J. Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, NJ, 2009.

[21] C.R. Nicolis and G. Nicolis. Butterfly effect. *Scholarpedia*, 4(5), 1720, 2009.

Department of Mathematical Sciences, NTNU, N-7491 Trondheim, Norway

E-mail address: nils.baas@ntnu.no