A NOTE ON WELL-POSEDNESS OF BIDIRECTIONAL WHITHAM EQUATION

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Abstract. We consider the initial-value problem for the bidirectional Whitham equation, a system which combines the full two-way dispersion relation from the incompressible Euler equations with a canonical shallow-water nonlinearity. We prove local well-posedness in classical Sobolev spaces, using a square-root type transformation to symmetrise the system.

1. Introduction and main results

We consider the bidirectional Whitham equation

$$\begin{aligned}\partial_t\eta &= -\mathcal{K}\partial_x u - \partial_x(\eta u)\\ \partial_t u &= -\partial_x\eta - u\partial_x u,\end{aligned} \tag{1.1}$$

formally derived in [1, 17] from the incompressible Euler equations to model fully dispersive shallow water waves whose propagation is allowed to be both left- and rightward. Here, η denotes the surface elevation, u is the rightward velocity at the surface, and the Fourier multiplier operator $\mathcal{K}$ is defined by

$$\widehat{\mathcal{K}v}(\xi) = \frac{\tanh(\xi)}{\xi}\,\widehat{v}(\xi), \tag{1.2}$$

for all v in the Schwartz space $\mathcal{S}(\mathbb{R})$. By duality, the operator $\mathcal{K}$ is well-defined on the space of tempered distributions, $\mathcal{S}'(\mathbb{R})$. The model (1.1) is the two-way equivalent of the Whitham equation

$$u_t + \mathcal{K}^{\frac{1}{2}}u_x + uu_x = 0, \tag{1.3}$$

a nonlocal shallow water equation that in its simple form still captures several interesting mathematical features that are present also in the full water-wave problem. The operator $\mathcal{K}^{\frac{1}{2}}$ is the square root of the operator $\mathcal{K}$ defined in (1.2), most easily defined by considering the action of these operators in Fourier space. The features of (1.3) include local well-posedness [7], travelling waves [3, 8, 10], a heighest, cusped wave [12] and wave breaking [13].

The bidirectional Whitham equation (1.1) is mathematically interesting because of its weak dispersion, and contains a logarithmically cusped wave of greatest height [9] and solitary waves [18]. Experiments and numerical results indicate surprisingly good modelling properties for this model, as well as for several other 'Whitham-like' equations and systems, see [4, 5, 19]. Still, we regard our result as a mathematical one: the system (1.1) is well-posed, but the set of initial-data for which we can control the life-span is bounded away from a zero surface deflection.[1]

It should be emphasized that (1.1)-(1.2) evolves quite delicately as η perturbs around 0: locally well-posed for η strictly positive, ill-posed if η becomes negative (see the observations in [15]), and possibly unstable for η non-negative. This indicates the significance of studying (1.1)-(1.2) mathematically besides its role as a model for water waves. In this paper we consider the well-posedness of (1.1)-(1.2) with a rigorous proof.

The weak dispersion of (1.1) clearly suggests to view it as a perturbation of a hyperbolic system. One could symmetrise the system in many ways, for example by using matrices with diagonals $(1, \eta)$ or $(1/\eta, 1)$. In this paper, we adopt the transformation $\eta \mapsto \sqrt{\eta}$, sometimes used in physical settings as a sound speed transformation and in the blow-up analysis in fluid mechanics (cf. [6]), to transfer the system (1.1) into a canonical form which may be divided two parts: the usual hyperbolic part (can be treated as [16]) and a new nonlocal part (will be mainly focused on). We also refer to [11] for the full details[2]. Recently, surface tension was taken into account in (1.1) in [14] so that $\widehat{\mathcal{K}v}(\xi) = \frac{(1+\beta\xi^2)\tanh(\xi)}{\xi}\,\widehat{v}(\xi), \beta > 0$, and local well-posedness was proved by the modified energy method; however, this method does not apply to our case ($\beta = 0$) as the authors pointed out in [14, Remark 1.2]. Finally,

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[1] It is an interesting question how this aligns with the experimental data in [4], apparently not displaying this shortcoming. One possibility is that classical Sobolev spaces are too large for (1.1). We hope to make the reader aware of these facts.

[2] This manuscript [11] on arXiv aims to provide readers with more details and will not be published elsewhere.

we view our result within a broader framework, a program to investigate the interplay between dispersive and nonlinear effects in nonlocal equations, and aim to continue to investigate what solutions and properties similar equations allow for.

Our main result is then as follows.

Theorem 1.1. *Let (η_0, u_0) be initial data such that $\inf \eta_0 > 0$ and*

$$(\sqrt{\eta_0} - \sqrt{\bar{\eta}}, u_0) \in H^N(\mathbb{R}) \tag{1.4}$$

for some positive constant $\bar{\eta}$. Then the equation (1.1) is locally well-posed. There exist a positive time $T > 0$ and a classical solution $(\eta, u)^{tr}$ of (1.1) with $(\eta, u)|_{t=0} = (\eta_0, u_0)$ that is unique among solutions satisfying

$$(\sqrt{\eta} - \sqrt{\bar{\eta}}, u) \in C\big([0, T]; H^N(\mathbb{R})\big) \cap C^1\big([0, T]; H^{N-1}(\mathbb{R})\big).$$

The solution depends continuously on (η_0, u_0) with respect to the same metric.

It should be noted that in the statement of Theorem 1.1 the constant $\bar{\eta}$ is fixed, whence the metric is fixed, too. The proof of Theorem 1.1 is presented throughout Sections 2–3. Section 2 contains the statement and reformulation of the problem, as well as necessary preliminaries. In Subsection 3.1 we obtain a short-time existence result for the linearised and regularised problem. Subsection 3.2 is devoted to the study of solvability of the linearised problem. We finally give the proof of the main theorem in Subsection 3.3.

2. Preliminaries and setup of the problem

Let $L^p(\mathbb{R})$, $p \in [1, \infty]$, be the standard Lebesgue spaces. Similarly, let $H^s(\mathbb{R}) = (1 - \partial_x^2)^{-s/2} L^2(\mathbb{R})$ be the Bessel-potential spaces with norm

$$\|\cdot\|_{H^s(\mathbb{R})} = \|\big(1 - \partial_x^2\big)^{s/2} \cdot \|_{L^2(\mathbb{R})}, \qquad s \in \mathbb{R},$$

and we denote by $(\cdot, \cdot)_2$ the usual product for L^2 spaces. For any Banach space $\mathbb{Y}$, let $C^k\big([0, T]; \mathbb{Y}\big)$ be the space of functions $u \colon [0, T] \to \mathbb{Y}$ with bounded and continuous derivatives up to kth order, normed by

$$\|f\|_{C^k\left([0,T];\mathbb{Y}\right)} = \sum_{j=0}^{k} \sup_{t \in [0,T]} \big\|\partial_t^j f(t, \cdot)\big\|_{\mathbb{Y}}.$$

We write $f \lesssim g$ when $f \leq cg$ for some constant $c > 0$, and $f \eqsim g$ when $f \lesssim g \lesssim f$. Finally, for a given positive constant $\bar{\eta}$ and any function η, let

$$\bar{\lambda} = \lambda(\bar{\eta}) \quad \text{and} \quad \zeta = 2(\lambda(\eta) - \bar{\lambda}),$$

where $\lambda = \sqrt{\cdot}$ is a shorthand to ease notation. Then (1.1) may be expressed as

$$\partial_t \zeta + u \partial_x \zeta + \frac{\zeta + 2\bar{\lambda}}{2} \partial_x u + \frac{2}{\zeta + 2\bar{\lambda}} \mathcal{K} \partial_x u = 0,$$
$$\partial_t u + u \partial_x u + \frac{\zeta + 2\bar{\lambda}}{2} \partial_x \zeta = 0,$$

or, with

$$U = \begin{pmatrix} \zeta \\ u \end{pmatrix}, \quad A(U) = \begin{pmatrix} u & \frac{\zeta + 2\bar{\lambda}}{2} \\ \frac{\zeta + 2\bar{\lambda}}{2} & u \end{pmatrix} \quad \text{and} \quad B(U) = \begin{pmatrix} 0 & \dfrac{2}{\zeta + 2\bar{\lambda}} \\ 0 & 0 \end{pmatrix},$$

as

$$\partial_t U + A(U) \partial_x U + B(U) \mathcal{K} \partial_x U = 0. \tag{2.1}$$

The system (2.1) is hyperbolic with a nonlocal dispersive perturbation and we shall look for solutions in Sobolev spaces embedded into $L^\infty(\mathbb{R})$. One notes that the initial data $\zeta_0 = 2(\lambda(\eta_0) - \bar{\lambda})$ satisfies $\zeta_0 + 2\bar{\lambda} \geq 2\sqrt{\inf \eta_0} > 0$ and may thus pick a positive constant μ such that $\bar{\lambda} \leq \mu^{-1}$ and

$$2\mu \leq \zeta_0 + 2\bar{\lambda} \leq (2\mu)^{-1}, \tag{2.2}$$

that we will use below. The initial data $U(0, x)$ for our problem shall be denoted by

$$U_0 = (\zeta_0, u_0)^{tr}, \tag{2.3}$$

where tr denotes the transpose of a matrix. Finally, let $U^{(k)} = (\partial_x^k \zeta, \partial_x^k u)^{tr}$, and define the partial and total energy functionals as

$$E^{(k)}(t,U) = \left\|U^{(k)}(t,\cdot)\right\|_{L^2}^2 = \left\|\zeta^{(k)}(t,\cdot)\right\|_{L^2}^2 + \left\|u^{(k)}(t,\cdot)\right\|_{L^2}^2$$
$$E_N(t,U) = \sum_{k=0}^{N} E^{(k)}(t,U),$$

respectively. We will assume the integer $N \geq 2$ and sometimes write simply $E_N(t)$, and use $E_N(U_0)$ for $E_N(0,U_0)$.

3. Proof of the main theorem

3.1. **The regularised and linearised problem.** For $0 < \varepsilon \ll 1$, let $\mathcal{J}_\varepsilon$ be a standard mollifier based on some smooth and compactly supported function ϱ on $\mathbb{R}$. Denote by $\mathbb{N}_0$ the set of non-negative integers. We consider first the regularised problem

$$\partial_t U_\varepsilon + \mathcal{J}_\varepsilon\big[\mathcal{J}_\varepsilon(A(V))\partial_x(\mathcal{J}_\varepsilon U_\varepsilon)\big] + \mathcal{J}_\varepsilon\big[\mathcal{J}_\varepsilon(B(V))\mathcal{K}\partial_x(\mathcal{J}_\varepsilon U_\varepsilon)\big] = 0, \tag{3.1}$$

with initial data $U_\varepsilon(0,x) = U_0(x)$. Here, for any positive number T_1, it is assumed that

$$V = (\varphi, v)^{tr} \in C\big([0,T_1]; H^N(\mathbb{R})\big) \cap C^1\big([0,T_1]; H^{N-1}(\mathbb{R})\big)$$

satisfies

$$\begin{aligned} E_N(t,V) &\leq 2E_N(U_0), \\ \mu \leq \varphi &+ 2\bar{\lambda} \leq \mu^{-1}, \end{aligned} \tag{3.2}$$

for all $(t,x) \in [0,T_1] \times \mathbb{R}$. We will make repeated use of the following standard estimates [16].

Lemma 3.1. *Mollification is continuous $L^\infty \to BUC$, and for $k,l \in \mathbb{N}_0$,*

$$\|\mathcal{J}_\varepsilon f\|_{H^{k+l}} \lesssim \varepsilon^{-l}\|f\|_{H^k},$$
$$\|(\mathcal{J}_\varepsilon - \mathcal{J}_{\varepsilon'})f\|_{H^k} \lesssim |\varepsilon - \varepsilon'|\|\partial_x f\|_{H^k}.$$

Proposition 3.2. *For any $0 < \varepsilon \ll 1$, $N \geq 2$ and $T_1 > 0$ as in (3.2) the regularised problem (3.1) has a unique solution $U_\varepsilon \in C^1\big([0,T_1]; H^N(\mathbb{R})\big)$.*

Proof. We express (3.1) as an ODE in the Hilbert space $H^N(\mathbb{R})$:

$$\partial_t U_\varepsilon = F(U_\varepsilon), \qquad U_\varepsilon(0,x) = U_0(x), \tag{3.3}$$

with

$$F(U_\varepsilon) = -\mathcal{J}_\varepsilon\big[\mathcal{J}_\varepsilon(A(V))\partial_x(\mathcal{J}_\varepsilon U_\varepsilon)\big] - \mathcal{J}_\varepsilon\big[\mathcal{J}_\varepsilon(B(V))\mathcal{K}\partial_x(\mathcal{J}_\varepsilon U_\varepsilon)\big] =: F_1(U_\varepsilon) + F_2(U_\varepsilon).$$

To use Picard's theorem to prove the existence of a positive time T_ε and a unique solution $U_\varepsilon \in C^1\big([0,T_\varepsilon]; H^N(\mathbb{R})\big)$ of the regularised problem (3.3), one needs to verify:
(i) the map F is bounded from $H^N(\mathbb{R})$ to $H^N(\mathbb{R})$;
(ii) F is locally Lipschitz continuous on any open set in $H^N(\mathbb{R})$.

Since the term $F_1(U_\varepsilon)$ comes from the usual hyperbolic part, we only focus on $F_2(U_\varepsilon)$ involving the nonlocal operator $\mathcal{K}$. First notice that since $\tanh(|\xi|) \leq 1$, it holds that

$$\|\mathcal{K}\partial_x f\|_{H^s}^2 = \int_{\mathbb{R}} \frac{\xi^2 \tanh^2(\xi)}{\xi^2}(1+\xi^2)^s|\hat{f}(\xi)|^2\,\mathrm{d}\xi \leq \|f\|_{H^s}^2. \tag{3.4}$$

To annihilate the constant term appearing in $\varphi + 2\bar{\lambda}$ in $B(V)$ for estimates in Sobolev spaces, we then shall use the following homogeneous estimates (cf. [16])

$$\|\partial_x^k(fg)\|_{L^2} \lesssim \|f\|_{L^\infty}\|\partial_x^k g\|_{L^2} + \|g\|_{L^\infty}\|\partial_x^k f\|_{L^2},\ k \in \mathbb{N}_0, \tag{3.5}$$

and assumption (3.2) and Lemma 3.1 to obtain

$$\|\partial_x^{N+1}\big(\mathcal{J}_\varepsilon(B(V))\mathcal{K}\partial_x(\mathcal{J}_\varepsilon U_\varepsilon)\big)\|_{L^2} \lesssim \|B(V)\|_{L^\infty}\|\partial_x^{N+1}\mathcal{K}\partial_x(\mathcal{J}_\varepsilon U_\varepsilon)\|_{L^2} + \|\mathcal{K}\partial_x U_\varepsilon\|_{L^\infty}\|\partial_x^{N+1}\mathcal{J}_\varepsilon(B(V))\|_{L^2}. \tag{3.6}$$

On the other hand, one has

$$\|\mathcal{J}_\varepsilon(B(V))\mathcal{K}\partial_x(\mathcal{J}_\varepsilon U_\varepsilon)\|_{L^2} \lesssim \|B(V)\|_{L^\infty}\|\mathcal{K}\partial_x(\mathcal{J}_\varepsilon U_\varepsilon)\|_{L^2}. \tag{3.7}$$

Thus, by Gagliardo–Nirenberg interpolation, we use (3.2), (3.4) and (3.6)-(3.7) to estimate

$$\begin{aligned}\|F_2(U_\varepsilon)\|_{H^N} &\lesssim \sum_{i=0}^{N} \|\mathcal{J}_\varepsilon(B(V))\mathcal{K}\partial_x(\mathcal{J}_\varepsilon U_\varepsilon)\|_{L^2}^{1-\frac{i}{N+1}} \|\partial_x^{N+1}\big(\mathcal{J}_\varepsilon(B(V))\mathcal{K}\partial_x(\mathcal{J}_\varepsilon U_\varepsilon)\big)\|_{L^2}^{\frac{i}{N+1}} \\ &\lesssim \|B(V)\|_{L^\infty}\|\mathcal{K}\partial_x(\mathcal{J}_\varepsilon U_\varepsilon)\|_{H^{N+1}} + \|\mathcal{K}\partial_x U_\varepsilon\|_{L^\infty}\|\partial_x^{N+1}\mathcal{J}_\varepsilon(B(V))\|_{L^2} \\ &\lesssim (\mu\varepsilon)^{-1}\|U_\varepsilon\|_{H^N} + (\mu\varepsilon)^{-N-1}\|U_\varepsilon\|_{H^1}\sum_{i=0}^{N} E_N(t,V)^{\frac{i}{2}} \\ &\lesssim (\mu\varepsilon)^{-N-1}\|U_\varepsilon\|_{H^N}.\end{aligned} \tag{3.8}$$

The local Lipschitz continuity of F on any open set of $H^N(\mathbb{R})$ results from its linear dependence in U and similar estimates as above:

$$\|F(U_\varepsilon^1) - F(U_\varepsilon^2)\|_{H^N} \lesssim (\mu\varepsilon)^{-N-1}\|U_\varepsilon^1 - U_\varepsilon^2\|_{H^N}.$$

□

3.2. **Solvability of the linearised problem.** In this subsection we develop a priori estimates enabling us to take a limit in the regularised equation (3.1), thereby solving the linearised problem

$$\partial_t U + A(V)\partial_x U + B(V)\mathcal{K}\partial_x U = 0, \tag{3.9}$$

with $U(0,x) = U_0(x)$. The main estimates appear in the proof of the following result.

Proposition 3.3. *For any $N \geq 2$ and any μ as in (2.2) and (3.2) there exist a positive number T_2 and a unique solution $U \in C\big([0,T_2]; H^N(\mathbb{R})\big) \cap C^1\big([0,T_2]; H^{N-1}(\mathbb{R})\big)$ of (3.9) that satisfies*

$$\max_{0\le t\le T_2} E_N(t,U) \le 2E_N(U_0), \tag{3.10}$$

where the above norms of U for a fixed N depend only on μ and $E_N(U_0)$.

Proof. **Uniform bound.** We apply ∂_x^k, $0 \le k \le N$ to (3.1) and integrate by parts to get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E^{(k)}(t,U_\varepsilon) = &-\sum_{l=0}^{k} C_k^l\big(\mathcal{J}_\varepsilon(A(V^{(l)}))\partial_x(\mathcal{J}_\varepsilon U_\varepsilon^{(k-l)}), \mathcal{J}_\varepsilon U_\varepsilon^{(k)}\big)_2 \\ &-\sum_{l=0}^{k} C_k^l\big(\mathcal{J}_\varepsilon(B^{(l)}(V))\mathcal{K}\partial_x(\mathcal{J}_\varepsilon U_\varepsilon^{(k-l)}), \mathcal{J}_\varepsilon U_\varepsilon^{(k)}\big)_2.\end{aligned} \tag{3.11}$$

For the same reason as before, we only focus on the term involving the nonlocal operator $\mathcal{K}$. Observe first that

$$\big(\mathcal{J}_\varepsilon(B^{(l)}(V))\mathcal{K}\partial_x(\mathcal{J}_\varepsilon U_\varepsilon^{(k-l)}), \mathcal{J}_\varepsilon U_\varepsilon^{(k)}\big)_2 = 2\int_{\mathbb{R}} \mathcal{J}_\varepsilon\left(\frac{1}{\varphi+2\bar{\lambda}}\right)^{(l)} \mathcal{K}\partial_x(\mathcal{J}_\varepsilon u_\varepsilon^{(k-l)})\mathcal{J}_\varepsilon\zeta_\varepsilon^{(k)}\,\mathrm{d}x.$$

The case $l = 0$ is straightforward, as

$$\int_{\mathbb{R}} \mathcal{J}_\varepsilon\left(\frac{1}{\varphi+2\bar{\lambda}}\right)\mathcal{K}\partial_x(\mathcal{J}_\varepsilon u_\varepsilon^{(k)})\mathcal{J}_\varepsilon\zeta_\varepsilon^{(k)}\,\mathrm{d}x \lesssim \mu^{-1}\|\mathcal{K}\partial_x u_\varepsilon^{(k)}\|_{L^2}\|\zeta_\varepsilon^{(k)}\|_{L^2} \lesssim \mu^{-1}\|u_\varepsilon^{(k)}\|_{L^2(\mathbb{R})}\|\zeta_\varepsilon^{(k)}\|_{L^2}.$$

On the other hand, when $1 \le l \le k$, Leibniz's rule and the assumptions (3.2) on V yield that

$$\Big\|\mathcal{J}_\varepsilon\big((\frac{1}{\varphi+2\bar{\lambda}})^{(l)}\big)\Big\|_{L^2} \lesssim \mu^{-(l+1)}\big(\|\varphi^{(l)}\|_{L^2} + \cdots + \|\partial_x\varphi\|_{L^2}\|\partial_x\varphi\|_{L^\infty}^{l-1}\big) \lesssim \mu^{-(l+1)}\sum_{i=1}^{N}\big(2E_N(U_0)\big)^{\frac{i}{2}}.$$

For the same range of l, we thus deduce that

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{J}_\varepsilon\left(\frac{1}{\varphi+2\bar{\lambda}}\right)^{(l)} \mathcal{K}\partial_x(\mathcal{J}_\varepsilon u_\varepsilon^{(k-l)})\mathcal{J}_\varepsilon\zeta_\varepsilon^{(k)}\,\mathrm{d}x &\lesssim \sum_{i=1}^{N}\big(2E_N(U_0)\big)^{\frac{i}{2}}\|\mathcal{K}\partial_x u_\varepsilon^{(k-l)}\|_{L^\infty}\|\zeta_\varepsilon^{(k)}\|_{L^2} \\ &\lesssim \sum_{i=1}^{N}\big(2E_N(U_0)\big)^{\frac{i}{2}}\|u_\varepsilon^{(k-l)}\|_{H^1}\|\zeta_\varepsilon^{(k)}\|_{L^2}.\end{aligned}$$

We then insert the above three estimates into (3.11) and conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}E_N(t,U_\varepsilon)\lesssim\left(1+\sum_{i=1}^{N}\big(2E_N(U_0)\big)^{\frac{i}{2}}\right)E_N(t,U_\varepsilon),$$

and Grönwall's inequality now guarantees the existence of

$$T_2\eqsim\min\left(T_1,\frac{\ln 2}{1+\sum_{i=1}^{N}\big(2E_N(U_0)\big)^{\frac{i}{2}}}\right)$$

such that

$$\max_{0\le t\le T_2}E_N(t,U_\varepsilon)\le 2E_N(U_0). \tag{3.12}$$

The family $\{U_\varepsilon\}_\varepsilon$ is therefore uniformly bounded in $C\big([0,T_2];H^N(\mathbb{R})\big)$.

Convergence. We shall now prove that a subsequence of the family $\{U_\varepsilon\}_\varepsilon$ defines a Cauchy sequence in $C\big([0,T_2];L^2(\mathbb{R})\big)$. By (3.1), the difference $U_\varepsilon-U_{\varepsilon'}$ of two solutions of the regularised problem satisfies

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U_\varepsilon-U_{\varepsilon'}\|_{L^2}^2&=-\big(\mathcal{J}_\varepsilon[\mathcal{J}_\varepsilon(A(V))\partial_x(\mathcal{J}_\varepsilon U_\varepsilon)]-\mathcal{J}_{\varepsilon'}[\mathcal{J}_{\varepsilon'}(A(V))\partial_x(\mathcal{J}_{\varepsilon'}U_{\varepsilon'})],U_\varepsilon-U_{\varepsilon'}\big)_2\\&\quad-\big(\mathcal{J}_\varepsilon[\mathcal{J}_\varepsilon(B(V))\mathcal{K}\partial_x(\mathcal{J}_\varepsilon U_\varepsilon)]-\mathcal{J}_{\varepsilon'}[\mathcal{J}_{\varepsilon'}(B(V))\mathcal{K}\partial_x(\mathcal{J}_{\varepsilon'}U_{\varepsilon'})],U_\varepsilon-U_{\varepsilon'}\big)_2\\&=:I+J.\end{aligned}$$

Again we only estimate J, for this, we split it as follows:

$$\begin{aligned}J&=-\big(\mathcal{J}_\varepsilon[\mathcal{J}_\varepsilon(B(V))\mathcal{K}\partial_x(\mathcal{J}_\varepsilon(U_\varepsilon-U_{\varepsilon'}))],U_\varepsilon-U_{\varepsilon'})_2-(\mathcal{J}_\varepsilon[\mathcal{J}_\varepsilon(B(V))\mathcal{K}\partial_x((\mathcal{J}_\varepsilon-\mathcal{J}_{\varepsilon'})U_{\varepsilon'})],U_\varepsilon-U_{\varepsilon'}\big)_2\\&\quad-\big(\mathcal{J}_\varepsilon[(\mathcal{J}_\varepsilon-\mathcal{J}_{\varepsilon'})(B(V))\mathcal{K}\partial_x(\mathcal{J}_{\varepsilon'}U_{\varepsilon'})],U_\varepsilon-U_{\varepsilon'})_2-((\mathcal{J}_\varepsilon-\mathcal{J}_{\varepsilon'})[\mathcal{J}_{\varepsilon'}(B(V))\mathcal{K}\partial_x(\mathcal{J}_{\varepsilon'}U_{\varepsilon'})],U_\varepsilon-U_{\varepsilon'}\big)_2\\&\quad=:J_1+J_2+J_3+J_4.\end{aligned}$$

It follows from Lemma 3.1 that

$$\begin{aligned}J_1&\lesssim\|B(V)\|_{L^\infty}\|U_\varepsilon-U_{\varepsilon'}\|_{L^2}^2\lesssim\mu^{-1}\|U_\varepsilon-U_{\varepsilon'}\|_{L^2}^2,\\J_2&\lesssim|\varepsilon-\varepsilon'|\|B(V)\|_{L^\infty}\|U_{\varepsilon'}\|_{H^1}\|U_\varepsilon-U_{\varepsilon'}\|_{L^2}\lesssim|\varepsilon-\varepsilon'|\mu^{-1}E_N(U_0)^{\frac{1}{2}}\|U_\varepsilon-U_{\varepsilon'}\|_{L^2}.\end{aligned}$$

Similarly, by the assumption (3.2) on V, we obtain

$$\|(\mathcal{J}_\varepsilon-\mathcal{J}_{\varepsilon'})(B(V))\|_{L^\infty}\lesssim\mu^{-2}\|(\mathcal{J}_\varepsilon-\mathcal{J}_{\varepsilon'})\varphi\|_{L^\infty},$$

which via Lemma 3.1 leads to

$$\|(\mathcal{J}_\varepsilon-\mathcal{J}_{\varepsilon'})(B(V))\|_{L^\infty}\lesssim|\varepsilon-\varepsilon'|\mu^{-2}E_N(U_0)^{\frac{1}{2}}.$$

One thus obtains

$$J_3\lesssim\|(\mathcal{J}_\varepsilon-\mathcal{J}_{\varepsilon'})(B(V))\|_{L^\infty}\|U_{\varepsilon'}\|_{L^2}\|U_\varepsilon-U_{\varepsilon'}\|_{L^2}\lesssim|\varepsilon-\varepsilon'|\mu^{-2}E_N(U_0)\|U_\varepsilon-U_{\varepsilon'}\|_{L^2}.$$

and

$$\begin{aligned}J_4&\lesssim\|(\mathcal{J}_\varepsilon-\mathcal{J}_{\varepsilon'})[\mathcal{J}_{\varepsilon'}(B(V))\mathcal{K}\partial_x(\mathcal{J}_{\varepsilon'}U_{\varepsilon'})]\|_{L^2}\|U_\varepsilon-U_{\varepsilon'}\|_{L^2}\\&\lesssim|\varepsilon-\varepsilon'|\big(\|B(V)\|_{L^\infty}\|\mathcal{K}\partial_x^2U_{\varepsilon'}\|_{L^2}+\|\mathcal{K}\partial_xU_\varepsilon\|_{L^\infty}\|\partial_xB(V)\|_2\big)\|U_\varepsilon-U_{\varepsilon'}\|_{L^2}\\&\lesssim|\varepsilon-\varepsilon'|(E_N(U_0)^{\frac{1}{2}}+\mu^{-1})\mu^{-1}E_N(U_0)^{\frac{1}{2}}\|U_\varepsilon-U_{\varepsilon'}\|_{L^2}.\end{aligned}$$

We conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|U_\varepsilon-U_{\varepsilon'}\|_{L^2}\lesssim_{\mu,E_N(U_0)}\|U_\varepsilon-U_{\varepsilon'}\|_{L^2}+|\varepsilon-\varepsilon'|,$$

which by Grönwall's inequality gives that

$$\max_{0\le t\le T_2}\|U_\varepsilon-U_{\varepsilon'}\|_{L^2}\lesssim|\varepsilon-\varepsilon'|. \tag{3.13}$$

The remaining part is a standard procedure based on (3.12) and (3.13) to complete the proof of Proposition 3.3. □

3.3. **Proof of Theorem 1.1.** In this subsection we give the proof of the main result. We note first the following lemma, which is immediate from the uniform bound (3.10) in Proposition 3.3.

Lemma 3.4. *There exists $T_3 \in (0, T_2]$, depending only on N and μ, such that if the initial data U_0 satisfies (2.2), then the assumption (3.2) holds with V replaced by U on $[0, T_3] \times \mathbb{R}$, where U solves the linearised equation (3.9).*

We have now come to the proof of the main result.

Proof of Theorem 1.1. We consider the following series of linearised problems for $m \in \mathbb{N}_0$.

$$\begin{aligned} \partial_t U_{m+1} + A(U_m)\partial_x U_{m+1} + B(U_m)\mathcal{K}\partial_x U_{m+1} = 0,\\ U_{m+1}(0,\cdot) = U_0. \end{aligned} \tag{3.14}$$

Note that u_0 satisfies (1.4), and that the positive constant $\mu \le \bar{\lambda}$ is chosen so that (2.2) holds. By induction on m and using Proposition 3.3 and Lemma 3.4, for each m, there exists a solution $U_m \in C\big([0, T_3]; H^N(\mathbb{R})\big) \cap C^1\big([0, T_3]; H^{N-1}(\mathbb{R})\big)$ of (3.14) satisfying the assumption (3.2) on V in (3.9). Therefore, for any $1 \le l \le N$,

$$\Big\| \Big(\frac{1}{\zeta_m(t,\cdot) + 2\bar{\lambda}} \Big)^{(l)} \Big\|_{L^2} \lesssim_\mu \sum_{i=1}^{N} E_N(t, U_m)^{\frac{i}{2}}.$$

We suppress now the dependence on μ^{-1}, since it is a fixed and bounded number. Similar to (3.2), we now have

$$\frac{\mathrm{d}}{\mathrm{d}t} E_N(t, U_{m+1}) \lesssim \left(1 + \sum_{i=1}^{N} E_N(t, U_m)^{\frac{i}{2}} \right) E_N(t, U_{m+1}),$$

where the estimate is independent of m. By induction on m, one has

$$\max_{0 \le t \le T_3} E_N(t, U_m) \le 2E_N(U_0) \quad \text{ for all } m \in \mathbb{N}_0.$$

The family $\{U_m\}$ is thus uniformly bounded in $C\big([0, T_3]; H^N(\mathbb{R})\big)$.

We shall now prove that $\{U_m\}_m$ forms a Cauchy sequence in $C([0, T_3]; L^2(\mathbb{R}))$. For each $m \ge 1$, let $W_{m+1} = U_{m+1} - U_m$. It then follows from (3.14) that

$$\begin{aligned} \tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|W_{m+1}\|_{L^2}^2 = &-\big(A(U_m)\partial_x W_{m+1}, W_{m+1}\big)_2 \quad - \big(B(U_m)\mathcal{K}\partial_x W_{m+1}, W_{m+1}\big)_2 \\ &- \big((A(U_m) - A(U_{m-1}))\partial_x U_m, W_{m+1}\big)_2 \quad - \big((B(U_m) - B(U_{m-1}))\mathcal{K}\partial_x U_m, W_{m+1}\big)_2. \end{aligned}$$

It is straightforward to estimate

$$-\big(B(U_m)\mathcal{K}\partial_x W_{m+1}, W_{m+1}\big)_2 \lesssim \|W_{m+1}\|_{L^2}^2,$$

and

$$-\big((B(U_m) - B(U_{m-1}))\partial_x U_m, W_{m+1}\big)_2 \lesssim E_N(U_0)^{\frac{1}{2}} \|W_m\|_{L^2} \|W_{m+1}\|_{L^2}.$$

We may thus conclude that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\|W_{m+1}\|_{L^2} \lesssim_{\mu, E_N(U_0)} \|W_{m+1}\|_{L^2} + \|W_m\|_{L^2}.$$

By Grönwall's inequality,

$$\max_{0 \le t \le T} \|W_{m+1}\|_{L^2} \lesssim_{\mu, E_N(U_0)} T \exp(c_{\mu, E_N(U_0)} T) \max_{0 \le t \le T} \|W_m\|_{L^2},$$

and we may choose $T \le T_3$ such that

$$\|W_{m+1}\|_{C\left([0,T]; L^2(\mathbb{R})\right)} \le \tfrac{1}{2} \|W_m\|_{C\left([0,T]; L^2(\mathbb{R})\right)}.$$

This immediately implies that $\{U_m\}_m$ is a Cauchy sequence in the same space, and there thus exists a pair (ζ, u) such that

$$\|\zeta_m - \zeta\|_{C\left([0,T]; L^2(\mathbb{R})\right)} + \|u_m - u\|_{C\left([0,T]; L^2(\mathbb{R})\right)} \to 0, \tag{3.15}$$

as $m \to \infty$. In view of (3.15), one can show that U is a unique classical solution of (2.1) in the sense of $C\big([0, T]; H^N(\mathbb{R})\big) \cap C^1\big([0, T]; H^{N-1}(\mathbb{R})\big)$. That the solution U depends continuously on the initial data U_0 follows from a Bona–Smith type argument [2]. $\square$

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