ESTIMATE OF THE SQUEEZING FUNCTION FOR A CLASS OF BOUNDED DOMAINS

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Abstract. We construct a class of bounded domains, on which the squeezing function is not uniformly bounded from below near a smooth and pseudoconvex boundary point.

1. Introduction

In [14, 15], the authors introduced the notion of *holomorphic homogeneous regular*. Then in [16], the equivalent notion of *uniformly squeezing* was introduced. Motivated by these studies, in [3], the authors introduced the *squeezing function* as follows.

Denote by $\mathbb{B}(r)$ the ball of radius $r > 0$ centered at the origin 0. Let Ω be a bounded domain in $\mathbf{C}^n$, and $p \in \Omega$. For any holomorphic embedding $f : \Omega \to \mathbb{B}(1)$, with $f(p) = 0$, set

$$s_{\Omega,f}(p) := \sup\{r > 0 : \ \mathbb{B}(r) \subset f(\Omega)\}.$$

Then, the squeezing function of Ω at p is defined as

$$s_\Omega(p) := \sup_f \{s_{\Omega,f}(p)\}.$$

Many properties and applications of the squeezing function have been explored by various authors, see e.g. [3, 4, 6, 8, 10, 11, 12].

It is clear that squeezing functions are invariant under biholomorphisms, and they are positive and bounded above by 1. It is a natural and interesting problem to study the uniform lower and upper bounds of the squeezing function.

It was shown recently in [12] that the squeezing function is uniformly bounded below for bounded convex domains. On the other hand, in [3], the authors showed that the squeezing function is not uniformly bounded below on certain domains with non-smooth boundaries, such as punctured balls. In [5], the authors constructed a smooth pseudoconvex domain in $\mathbf{C}^3$ on which the quotient of the Bergman metric and the Kobayashi metric is not bounded above near an infinite type point. By [4, Theorem 3.3], the squeezing function is not uniformly bounded below on this domain.

These studies raise the question: Is the squeezing function always uniformly bounded below near a smooth finite type point? In this paper, we answer the question negatively. More precisely, we have the following

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Theorem 1. *Let Ω be a bounded domain in $\mathbf{C}^3$, and $q \in \partial\Omega$. Assume that Ω is smooth and pseudoconvex in a neighborhood of q and the Bloom-Graham type of Ω at q is $d < \infty$. Moreover, assume that the regular order of contact at q is greater than $2d$ along two smooth complex curves not tangent to each other. Then the squeezing function $s_\Omega(p)$ has no uniform lower bound near q.*

Remark 1. The proof gives the estimate $s_\Omega(p) \leq C\delta^{\frac{1}{2d(2d+1)}}$ for some points approaching the boundary.

In section 2, we recall some preliminary notions and results. In section 3, we prove Theorem 1.

2. Preliminaries

Let Ω be a bounded domain in $\mathbf{C}^n$, $n \geq 2$, and $q \in \partial\Omega$. Assume that Ω is smooth and pseudoconvex in a neighborhood of q. The *Bloom-Graham type* of Ω at q is the maximal order of contact of complex manifolds of dimension $n-1$ tangent to $\partial\Omega$ at q (see e.g. [1]). Choose local coordinates $(z,t) \in \mathbf{C}^{n-1} \times \mathbf{C}$ such that the complex manifold of dimension $n-1$ with the maximal order of contact is given by $\{t = 0\}$. Then Ω is locally given by $\rho(z,t) < 0$, where $\rho(z,t) = \mathsf{Re}t + P(z) + Q(z,t)$ with $Q(z,0) \equiv 0$ and $\deg P(z) = d$. (We say that the degree of P is d if the Taylor expansion of P has no nonzero term of degree less than d.) Since Ω is pseudoconvex, we actually have $d = 2k$ (see e.g. [2]).

For $1 \leq k \leq n-1$, let $\varphi : \mathbf{C}^k \to \mathbf{C}^n$ be analytic with $\varphi(0) = q$ and $\mathrm{rank} d\varphi(0) = k$. Then the *regular order of contact* at q along the k-dimensional complex manifold defined by φ is defined as $\deg \rho \circ \varphi$ (see e.g. [2]).

Denote by Δ the unit disc in $\mathbf{C}$. Let $p \in \Omega$ and $\zeta \in \mathbf{C}^n$. The *Kobayashi metric* is defined as

$$K_\Omega(p,\zeta) := \inf\{\alpha : \ \alpha > 0, \ \exists\ \phi : \Delta \to \Omega, \ \phi(0) = p, \ \alpha\phi'(0) = \zeta\}.$$

Then the *Kobayashi indicatrix* is defined as (see e.g. [13])

$$D_\Omega(p) := \{\zeta \in \mathbf{C}^n : \ K_\Omega(p,\zeta) < 1\}.$$

For each unit vector $e \in \mathbf{C}^n$, set $D_\Omega(p,e) := \max\{|\eta| : \ \eta \in \mathbf{C}, \ \eta e \in D_\Omega(p)\}$. By the definition of Kobayashi indicatrix, the following three lemmas are clear.

Lemma 1. $D_{\mathbb{B}(r)}(0) = \mathbb{B}(r)$.

Lemma 2. *Let Ω_1 and Ω_2 be two domains in $\mathbf{C}^n$ with $\Omega_1 \subset \Omega_2$. Then for each $p \in \Omega_1$, $D_{\Omega_1}(p) \subset D_{\Omega_2}(p)$.*

Lemma 3. *Let Ω be a domain in $\mathbf{C}^n$ and $f : \Omega \to \mathbf{C}^n$ a biholomorphic map. Then for each $p \in \Omega$, $D_{f(\Omega)}(f(p)) = f'(p)D_\Omega(p)$.*

We also need the following localization lemma (see e.g. [9, Lemma 3]).

Lemma 4. *Let Ω be a domain in $\mathbf{C}^n$, $q \in \partial\Omega$ and U a neighborhood of q. If $V \subset\subset U$ and $q \in V$, then*

$$K_\Omega(p,\zeta) \simeq K_{\Omega\cap U}(p,\zeta), \quad \forall\ p \in V, \ \zeta \in \mathbf{C}^n.$$

By the above lemma, when we consider the size of the Kobayashi indicatrix in the next section, we will work in $\Omega \cap U$.

3. Estimate of the squeezing function

We first choose local coordinates adapted to our purpose. We will use $\gtrsim$ (resp. $\lesssim$, $\simeq$) to mean $\geq$ (resp. $\leq$, $=$) up to a positive constant.

Lemma 5. *Let Ω be a bounded domain in $\mathbf{C}^{n+1}$, $n \geq 1$, and $q \in \partial\Omega$. Assume that Ω is smooth and pseudoconvex in a neighborhood of q and the Bloom-Graham type of Ω at q is $2k$, $k \geq 1$. Then there exist local coordinates $(z,t) = (z_1, \cdots, z_n, u+iv)$ such that $q = (0,0)$ and Ω is locally given by $\rho(z,t) < 0$ with*

$$\rho(z,t) = u + P(z) + Q(z) + vR(z) + v^2 + o(u^2, uv, v^2, uz), \tag{1}$$

where $P(z)$ is plurisubharmonic, homogeneous of degree $2k$, but not pluriharmonic, $\deg Q(z) \geq 2k+1$ and $\deg R(z) \geq k+1$.

Proof. By assumption, we have a local defining function of the form

$$\rho(z,t) = u + P(z) + Q(z) + au^2 + buv + cv^2 + uA(z) + vB(z) + o(|t|^2),$$

where $P(z)$ is plurisubharmonic, homogeneous of degree $2k$, but not pluriharmonic, and $\deg Q(z) \geq 2k+1$. By changing t to $t + dt^2$ and multiplying with $1 + eu$ or $1 + ev$, we can freely change the quadratic terms in u, v. Thus, we can assume that

$$\rho(z,t) = u + P(z) + Q(z) + u^2 + v^2 + uA(z) + vB(z) + o(|t|^2).$$

Multiplying with $1 - uA(z)$, we can further assume that

$$\rho(z,t) = u + P(z) + Q(z) + u^2 + v^2 + vB(z) + o(|t|^2).$$

Write $B(z) = B_s(z) + B'(z)$, where $B_s(z)$ is the lowest order homogeneous term of degree $s \geq 1$. Assume that $B_s(z)$ is pluriharmonic. Then there exists a holomorphic function $F(z) = A(z) - iB_s(z)$. Change again ρ to add the term $uA(z)$ with this new $A(z)$. Then ρ takes the form

$$\begin{aligned}\rho(z,t) &= u + P(z) + Q(z) + u^2 + v^2 + uA(z) + vB_s(z) + vB'(z) + o(|t|^2)\\ &= u + P(z) + Q(z) + u^2 + v^2 + \mathsf{Re}(tF(z)) + vB'(z) + o(|t|^2).\end{aligned}$$

By absorbing $\mathsf{Re}(tF(z))$ into u, we get

$$\rho(z,t) = u + P(z) + Q(z) + u^2 + v^2 + vB'(z) + o(|t|^2).$$

Continuing this process, we can assume that ρ takes the form

$$\rho(z,t) = u + P(z) + Q(z) + u^2 + v^2 + vB_l(z) + vB'(z) + o(|t|^2),$$

where $B_l(z)$ is not pluriharmonic. Suppose that $l \leq k$. We will arrive at a contradiction to pseudoconvexity.

Note that $P(z)$ is plurisubharmonic but not pluriharmonic. This implies that there exists a complex line through the origin on which the restriction of P is subharmonic, but not harmonic. Pick a tangent vector $\xi = (\xi_1, \dots \xi_n)$ so that the Levi form of P calculated at a point $\eta\xi$, $\eta = |\eta|e^{i\theta}$, in the direction of ξ is $|\eta|^{2k-2}G(\theta)\|\xi\|^2$. Here G is a smooth nonnegative function which at most vanishes at finitely many angles. Choose λ such that $\sigma = (\xi, \lambda)$ is a complex tangent vector to $\partial\Omega$, i.e.

$$\sum_{j=1}^{n} \frac{\partial\rho}{\partial z_j}\xi_j + \frac{\partial\rho}{\partial t}\lambda = 0.$$

Then we have $|\lambda| = O(|\eta|^{2k-1} + v|\eta|^{l-1} + |t|^2)\|\xi\|$.

The Levi form of ρ at a boundary point $(\eta\xi, t)$, along the tangent vector σ is

$$\begin{aligned}\mathcal{L}(r,\sigma) =&|z|^{2k-2}G(\theta)\|\xi\|^2 + |\lambda|^2 + \mathcal{L}(vB_l(z),\sigma) + \cdots \\ =&|z|^{2k-2}G(\theta)\|\xi\|^2 + |\lambda|^2 + \mathsf{Re}(\sum_{j=1}^{n}\frac{\partial B_l}{\partial z_j} i\xi_j\overline{\lambda}) + v\sum_{k,m}\frac{\partial^2 B_l}{\partial z_k \overline{z}_m}\xi_k\overline{\xi}_m + \cdots .\end{aligned}$$

Since B_l is not pluriharmonic, we can assume after changing ξ slightly that $\frac{\partial^2 B_l}{\partial z_k \partial \overline{z}_m}\xi_k\overline{\xi}_m \neq 0$. Next choose $v = \pm C|\eta|^k$ with $C > \max_\theta\{G(\theta)\}$. The second term is $o(|\eta|^{2k-2}\|\xi\|^2)$ and the third terms is $o(|\eta|^{k+l-2}\|\xi\|^2)$. The last term is $O(|\eta|^{k+l-2}\|\xi\|^2)$ and, since $l \leq k$, at least $O(|\eta|^{2k-2}\|\xi\|^2)$. Thus we have $\mathcal{L}(r,\sigma) < 0$. This is a contraction. □

By Lemma 5, we can choose local coordinates $(z, w, t) = (z, w, u + iv)$ near q such that $q = (0, 0, 0)$ and Ω is locally given by $\rho(z, w, t) < 0$, where

$$\rho(z,w,t) = u + P(z,w) + Q(z,w) + vR(z,w) + v^2 + o(u^2, uv, v^2, uz, uw). \tag{2}$$

Here $P(z, w)$ is homogeneous of degree $2k$ with $P(z, 0) = P(0, w) = 0$, $\deg Q(z, w) \geq 2k+1$ with $\deg Q(z, 0) \geq 4k+1$ and $\deg Q(0, w) \geq 4k+1$, and $\deg R(z, w) \geq k+1$. Set $p = (-\delta, 0, 0)$ with $0 < \delta \ll 1$.

Lemma 6. *Let $\zeta_1 = (1, 0, 0)$ and $\zeta_2 = (0, 1, 0)$. Then $K_\Omega(p, \zeta_1), K_\Omega(p, \zeta_2) \lesssim \delta^{-\frac{1}{4k+1}}$.*

Proof. Consider the linear map $\phi : \Delta \to \mathbf{C}^3$ with $\phi(\tau) = (\beta\tau, 0, -\delta)$ for $\tau \in \Delta$, and $|\beta| = \epsilon\delta^{\frac{1}{4k+1}}$ for $0 < \epsilon \ll 1$. Then

$$\rho \circ \phi(\tau) \leq -\delta + C|\beta\tau|^{4k+1} + o(\delta) < -\delta + \epsilon\delta + o(\delta) < 0.$$

Therefore, $K_\Omega(p, u) \lesssim \delta^{-\frac{1}{4k+1}}$. The argument in the direction v is similar. □

Let $(a, b, 0)$ be a point so that $P(a\tau, b\tau)$ is a subharmonic homogeneous polynomial of degree $2k$ which is not harmonic. Then both a and b must be nonzero. By scaling in each variable, we can assume that $a = b = 1/\sqrt{2}$.

Lemma 7. *Let $\zeta = \frac{1}{\sqrt{2}}(1, 1, 0)$. Then $K_\Omega(p, \zeta) \gtrsim \delta^{-\frac{1}{4k}}$.*

Proof. For z, w small, we have

$$v^2+vR(z,w) \geq v^2-2Cv\|z,w\|^{k+1}+C^2\|z,w\|^{2k+2}-C^2\|z,w\|^{2k+2} \geq -C^2\|z,w\|^{2k+2}.$$

Therefore,

$$\begin{aligned}\rho &\geq u + P(z,w) + Q(z,w) - C^2\|z,w\|^{2k+2} + o(u^2, uv, v^2, uz, uw) \\ &= u + P(z,w) + \tilde{Q}(z,w) + o(u^2, uv, v^2, uz, uw) =: \tilde{\rho}.\end{aligned}$$

Consider an analytic map $\phi : \Delta \to \Omega$ with

$$\phi(\tau) = (\beta\tau + f(\tau), \beta\tau + g(\tau), -\delta + h(\tau)), \quad f(\tau), g(\tau), h(\tau) = O(\tau^2).$$

Then $\tilde{\rho} \circ \phi(\tau) \leq \rho \circ \phi(\tau) < 0$. For terms containing u, the dominant term of $\tilde{\rho} \circ \phi(\tau)$ is $-\delta$. Thus, we have

$$\varphi(\tau) := \mathsf{Re}h(\tau) + P(\beta\tau + f(\tau), \beta\tau + g(\tau)) + \tilde{Q}(\beta\tau + f(\tau), \beta\tau + g(\tau)) + o(v^2) \lesssim \delta,$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi(|\tau|e^{i\theta})d\theta \lesssim \delta. \tag{3}$$

On the left-hand side of (3), only the average of $|\cdot|^2$ terms remain. For any analytic function $a(\tau)=\sum_{n\geq 0} a_n\tau^n$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} |a(|\tau|e^{i\theta})|^2 d\theta = \frac{1}{2\pi}\sum_{n\geq 0}|a_n|^2|\tau|^{2n}.$$

Thus by the homogeneous expansion of $P(z,w)$, we have for $|\tau|$ small

$$|\beta\tau|^{2k} - \sum_{i=0}^{2k-1}|\beta|^i|\tau|^{4k-i} \lesssim \delta. \tag{4}$$

Choose $|\tau| = \frac{1}{2}|\beta|$. Then (4) gives

$$|\frac{\beta}{2}|^{4k} \lesssim \delta.$$

Hence, $K_\Omega(p,u) \gtrsim \delta^{-\frac{1}{4k}}$. □

Lemma 8. *Let D be a bounded domain in $\mathbf{C}^n$, $n\geq 2$, containing the origin. Assume that there exist two linearly independent nonzero vectors $\zeta_1,\zeta_2\in D$ and $\epsilon>0$ such that $\epsilon(\zeta_1+\zeta_2)\notin D$. Then there does not exist a linear map $L: D\to\mathbf{C}^n$, with $L(0)=0$, such that $\mathbb{B}(3\epsilon)\subset L(D)\subset\mathbb{B}(1)$.*

Proof. Let $L: D\to\mathbf{C}^n$ be a linear map with $L(0)=0$ and suppose $\mathbb{B}(3\epsilon)\subset L(D)$. Since $\epsilon(\zeta_1+\zeta_2)\notin D$ and L is linear, we have $\epsilon(L(\zeta_1)+L(\zeta_2))\notin L(D)$. This implies that $\epsilon(L(\zeta_1)+L(\zeta_2))\notin\mathbb{B}(3\epsilon)$ and thus $\|L(\zeta_1)+L(\zeta_2)\|\geq 3$. However, $\|L(\zeta_1)+L(\zeta_2)\|\leq\|L(\zeta_1)\|+\|L(\zeta_2)\|\leq 1+1=2$. This completes the proof. □

Proof of Theorem 1. Choose local coordinates (z,w,t) such that $q=(0,0,0)$ and let $p=(-\delta,0,0)$ for $\delta>0$ small. Let $\zeta_1=(1,0,0)$ and $\zeta_2=(0,1,0)$. By Lemma 6, $K_\Omega(p,\zeta_1), K_\Omega(p,\zeta_2)\lesssim\delta^{-\frac{1}{4k+1}}$. By Lemma 7, $K_\Omega(p,\frac{1}{\sqrt{2}}(\zeta_1+\zeta_2))\gtrsim\delta^{-\frac{1}{4k}}$.

Choose $\lambda>0$ with $\lambda\gtrsim\delta^{\frac{1}{4k+1}}$ such that $\lambda\zeta_1,\lambda\zeta_2\in D_\Omega(p)$. Then for $\epsilon\simeq\delta^{\frac{1}{4k(4k+1)}}$, we have $\epsilon(\lambda\zeta_1+\lambda\zeta_2)\notin D_\Omega(p)$. Thus, by Lemma 8, there does not exist a linear map $L: D_\Omega(p)\to\mathbf{C}^3$ such that $\mathbb{B}(3\epsilon)\subset L(D_\Omega(p))\subset\mathbb{B}(1)$.

Let f be a biholomorphism of Ω into $\mathbb{B}(1)$ such that $f(p)=0$ and $\mathbb{B}(c)\subset f(\Omega)$ for some $c>0$. Set $L=f'(p)$. Then, by Lemmas 1, 2 and 3, $\mathbb{B}(c)\subset L(D_\Omega(p))\subset\mathbb{B}(1)$. Therefore, we have $c\lesssim\delta^{\frac{1}{4k(4k+1)}}$. Since f is arbitrary, we get $s_\Omega(p)\lesssim\delta^{\frac{1}{4k(4k+1)}}$. Since δ can be arbitrarily small, this completes the proof. □

Remark 2. Theorem 1 does not hold if only assuming that the regular order of contact at q is greater than $2d$ along one smooth complex curve. For instance, consider Ω given by

$$\{(z,w,t)\in\mathbf{C}^3: \ |t|^2+|z|^2+|w|^6<1\}.$$

Then at $q=(0,0,1)$, the Bloom-Graham type is 2 and the regular order of contact along $(0,1,0)$ is $6>4$. But Ω is a bounded convex domain and thus the squeezing function has a uniform lower bound by [12].

Remark 3. Using similar arguments, one can extend Theorem 1 to higher dimensions as follows.

Theorem 2. *Let Ω be a bounded domain in $\mathbf{C}^n$, $n \geq 4$, and $q \in \partial\Omega$. Assume that Ω is smooth and pseudoconvex in a neighborhood of q and the Bloom-Graham type of Ω at q is d. Moreover, assume that the regular order of contact at q is d along a two-dimensional complex surface Σ and the regular order of contact at q is greater than $2d$ along two smooth complex curves not tangent to each other contained in Σ. Then the squeezing function $s_\Omega(p)$ has no uniform lower bound near q.*

Remark 4. After the completion of this work, it was brought to our attention by Gregor Herbort that a similar comparison result to [5] was obtained for the following domain in [7]:

$$\Omega := \{(z,w,t) \in \mathbf{C}^3 : \ \mathsf{Re}t + |z|^{12} + |w|^{12} + |z|^2|w|^4 + |z|^6|w|^2 < 0\}.$$

Therefore, by our remark in the introduction, the squeezing function does not have a uniform lower bound on this domain. More generally, we have the following

Theorem 3. *Let Ω be a bounded domain in $\mathbf{C}^3$, and $q \in \partial\Omega$. Assume that Ω is smooth and pseudoconvex in a neighborhood of q and the Bloom-Graham type of Ω at q is $d < \infty$. Let ρ be a defining function of Ω near q in the normal form* (1) *and assume that the leading homogeneous term $P(z)$ only contains positive terms. Moreover, assume that the regular order of contact at q is greater than d along two smooth complex curves not tangent to each other. Then the squeezing function $s_\Omega(p)$ has no uniform lower bound near q.*

Sketch of proof. In Lemma 6, we get $K_\Omega(p,u), K_\Omega(p,v) \lesssim \delta^{-\frac{1}{2k+1}}$, by the same argument. In Lemma 7, we get $K_\Omega(p,u) \gtrsim \delta^{-\frac{1}{2k}}$, by noticing that instead of (4) we have $|\xi\tau|^{2k} \lesssim \delta$ since all terms of $P(z)$ are positive. Then arguing exactly as in the proof of Theorem 1, we get $s_\Omega(p) \lesssim \delta^{\frac{1}{2k(2k+1)}}$. □

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