# WEIGHTED APPROXIMATION IN $\ensuremath{\mathbb{C}}$

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ABSTRACT. We prove that if  $\{\varphi_j\}_j$  is a sequence of subharmonic functions which are increasing to some subharmonic function  $\varphi$  in  $\mathbb{C}$ , then the union of all the weighted Hilbert spaces  $H(\varphi_j)$  is dense in the weighted Hilbert space  $H(\varphi)$ .

Mathematics Subject Classification (2010):30D20, 30E10, 30H50, 31A05  $\,$ 

KEYWORDS: Holomorphic approximation; subharmonic function; Bergman kernel;  $L^2\mbox{-}{\rm estimate}$ 

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### 1. INTRODUCTION

Let  $\varphi$  be a measurable function, locally bounded above on a domain  $\Omega \subset \mathbb{C}^n$ . Set

$$H(\Omega,\varphi) := \{ f \in \mathcal{O}(\Omega) : \int_{\Omega} |f|^2 e^{-\varphi} d\lambda < +\infty \}$$

where  $\mathcal{O}(\Omega)$  stands for the space of holomorphic functions on  $\Omega$  and  $d\lambda$  is the Lebesgue measure. If  $\Omega = \mathbb{C}$ , let  $H(\varphi)$  be the space of entire functions f with  $L^2(\mathbb{C}^n, \varphi)$  norm, i.e.,  $\|f\|_{L^2(\mathbb{C}^n, \varphi)}^2 = \int_{\mathbb{C}^n} |f|^2 e^{-\varphi} d\lambda < +\infty$ .

Since  $f \in H(\Omega, \varphi)$ , the function  $|f|^2$  is plurisubharmonic (psh) and  $\varphi$  is locally bounded above, then

(1.1) 
$$|f(w)| \le \frac{C_n}{r^n} ||f||_{L^2(B(w,r),0)} \le \frac{C'_n}{r^n} ||f||_{H(\Omega,\varphi)},$$

if the ball  $B(w,r) \subset \Omega$  and for  $K \subset \Omega$ 

$$\sup_{K} |f| \le C \|f\|_{H(\Omega,\varphi)}$$

where C depends only on K and  $\Omega$ . So  $H(\Omega, \varphi)$  is a closed subspace of  $L^2(\Omega, \varphi)$  and thus a Hilbert space. Let  $K_{\Omega,\varphi}(z, w)$  denote the weighted Bergman kernel corresponding to the Hilbert space  $H(\Omega, \varphi)$ . If  $\varphi = 0$ , then  $K_{\Omega}(z, w) := K_{\Omega,0}(z, w)$  is the classical kernel introduced by Stefan Bergman.

In 1971, B. A. Taylor [5] investigated weighted approximation results for entire functions on  $\mathbb{C}^n$ . He proved:

**Theorem 1.1.** Let  $\varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \cdots$  be psh functions on  $\mathbb{C}^n$ , assume  $\varphi = \lim_{j \to \infty} \varphi_j$  is psh, and suppose that  $\int_K e^{-\varphi_1} d\lambda < \infty$  for every compact set K. Then the closure of  $\bigcup_{j=1}^{\infty} H(\varphi_j + \log(1 + ||z||^2))$  in the Hilbert space  $L^2(\varphi + \log(1 + ||z||^2))$  contains  $H(\varphi)$ .

In [3] we improved Taylor's result as follows

**Theorem 1.2.** Let  $\varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \cdots$  be psh functions on  $\mathbb{C}^n$  with psh limit. For any  $\epsilon > 0$ , let  $\tilde{\varphi}_j = \varphi_j + \epsilon \log(1 + ||z||^2)$  and  $\tilde{\varphi} = \lim_{j \to +\infty} \tilde{\varphi}_j$ . Then  $\bigcup_{j=1}^{\infty} H(\tilde{\varphi}_j)$  is dense in  $H(\tilde{\varphi})$ .

It is an important question whether this theorem is true or false when  $\epsilon = 0$ . Here by using some potential theoretic properties of subharmonic functions we show that it holds in one dimension.

**Theorem 1.3.** Let  $\varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \cdots$  be subharmonic functions on  $\mathbb{C}$ . Suppose that  $\varphi = \lim_k \varphi_k$  and  $\varphi$  is locally bounded above. Then  $\bigcup_{k=1}^{\infty} H(\varphi_k)$  is dense in  $H(\varphi)$ .

**Remark 1.4.** Let  $\varphi = \lim_{k} \varphi_k$ . We define  $\phi = \varphi^* := \limsup_{\zeta \to z} \varphi(\zeta), z \in \mathbb{C}$ , which is the upper regularization of  $\varphi$ . Then  $\phi$  is subharmonic,  $\varphi \leq \phi$ ,  $\varphi = \phi$  almost everywhere on  $\mathbb{C}$  and we have  $H(\varphi) = H(\phi)$  (See Theorem 3.4.2 in [10]).

We need the strong openness theorem as follows, see [6], [7], [8], [9].

**Theorem 1.5** (strong openness theorem). Let  $V \subset \subset U \subset \mathbb{C}^n$  be two open sets. Let  $\varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \cdots$  be non-positive psh functions on U such that  $\varphi = \lim_k \varphi_k$  and  $\varphi$  is locally bounded above. If  $f \in \mathcal{O}(U)$  is such that

$$\int_U |f|^2 e^{-\varphi} d\lambda < \infty$$

then there exists  $j_0$  so that when  $j \ge j_0$ 

$$\int_V |f|^2 e^{-\varphi_j} d\lambda < \infty.$$

For convenience, we will use  $K_{\varphi_j}(z, w)$  (resp.  $K_{\varphi}(z, w)$ ) to denote the weighted Bergman kernel corresponding to the Hilbert space  $H(\varphi_j) := H(\mathbb{C}, \varphi_j)$  (resp.  $H(\varphi) := H(\mathbb{C}, \varphi)$ ). As an application of the approximation theorem 1.3, we will prove

**Theorem 1.6.** Let  $\varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \cdots$  be subharmonic functions on  $\mathbb{C}$ . Suppose that  $\varphi = \lim_k \varphi_k$  and  $\varphi$  is locally bounded above. Then

$$\lim_{j} K_{\varphi_j}(z, z) = K_{\varphi}(z, z), \quad \forall z \in \mathbb{C}.$$

Ligocka showed that the classical Bergman kernel of certain Hartogs domains can be expressed as the sum of a series of weighted Bergman kernels defined on another domain of lower dimension. We set

$$\Omega_j = \{ (z, w) \in \mathbb{C} \times \mathbb{C} : |w| < e^{-\varphi_j(z)} \}$$

and

$$\Omega = \{ (z, w) \in \mathbb{C} \times \mathbb{C} : |w| < e^{-\phi(z)} \}.$$

We note that a domain  $\{(z, w) \in \mathbb{C} \times \mathbb{C} : |w| < e^{-\ell(z)}\}$  is open exactly when  $\ell(z)$  is upper semicontinuous. Hence we use  $\phi$  in the definition of  $\Omega$ and not  $\varphi$ .

We denote by  $K_{\Omega_j}[(z,t),(w,s)]$  (resp.  $K_{\Omega}[(z,t),(w,s)]$ ) the Classical Bergman kernel of the Hilbert space  $H(\Omega_j, 0)$  (resp.  $H(\Omega, 0)$ ), where  $z, t, w, s \in \mathbb{C}$ . Ligocka's formula [4] implies that

$$K_{\Omega_j}[(z,t),(w,s)] = \sum_{k=0}^{\infty} 2(k+1)K_{2(k+1)\varphi_j}(z,w)\langle t,s\rangle^k, \quad z,t,w,s \in \mathbb{C}$$

where  $K_{2(k+1)\varphi_j}(z, w)$  is the weighted Bergman kernel of the Hilbert space  $H(\mathbb{C}, 2(k+1)\varphi_j)$ .

According to the result of Theorem 1.6 we obtain that

**Theorem 1.7.** Let  $\{\varphi_j\}$  and  $\varphi$  be as in theorem 1.6, then the sequence  $K_{\Omega_j}[(z,t),(w,s)]$  converges to  $K_{\Omega}[(z,t),(w,s)]$  locally uniformly in  $\Omega \times \Omega$ .

The set-up of the paper is as follows: We prove in Section 2 an integral estimate for subharmonic weights. In Section 3 we discuss some potential theoretic properties of subharmonic functions which we will need. In Section 4, we prove the Main Theorem, Theorem 1.3. In Section 5, we prove the convergence of the Bergman Kernel which can be seen as an application of the main Theorem 1.3.

### 2. Subharmonic functions in $\mathbb C$

**Lemma 2.1.** Let  $\alpha_i > 0, z_i \in \mathbb{C}, i = 1, ..., N$  and let  $\alpha = \sum_i \alpha_i$ . For any z which is not one of the  $z_i$  we have the inequality

$$\Pi_{i=1}^{N} \left( \frac{1}{|z-z_i|} \right)^{\alpha_i} \le \sum_{i=1}^{N} \frac{\alpha_i}{\alpha} \left( \frac{1}{|z-z_i|} \right)^{\alpha}.$$

Proof.

$$\Pi_{i=1}^{N} \left( \frac{1}{|z-z_{i}|} \right)^{\alpha_{i}} = e^{\sum_{i=1}^{N} \frac{\alpha_{i}}{\alpha} \log\left( \left( \frac{1}{|z-z_{i}|} \right)^{\alpha} \right)}$$
  
Since exp is convex,

$$\leq \sum_{i=1}^{N} \frac{\alpha_i}{\alpha} e^{\log\left(\left(\frac{1}{|z-z_i|}\right)^{\alpha}\right)}$$
$$= \sum_{i=1}^{N} \frac{\alpha_i}{\alpha} \left(\frac{1}{|z-z_i|}\right)^{\alpha}.$$

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**Lemma 2.2.** Let  $|z_0| < R$  and suppose  $0 < \alpha < 2$ . Then

$$\int_{|z|< R} \left(\frac{1}{|z-z_0|}\right)^{\alpha} d\lambda(z) \le \frac{28R^2}{2-\alpha}$$

Proof.

$$\begin{split} \int_{|z|$$

As a direct consequence, we obtain

**Theorem 2.3.** Let  $\alpha_i > 0, |z_i| < R, i = 1, ..., N$  and let  $\alpha = \sum_i \alpha_i$ . Suppose that  $\alpha < 2$ . Then

$$\int_{|z|< R} \prod_{i=1}^N \left(\frac{1}{|z-z_i|}\right)^{\alpha_i} d\lambda(z) \le \frac{28R^2}{2-\alpha}.$$

**Lemma 2.4.** Let K be a compact set in  $\mathbb{R}^n$ . Suppose  $f(x, y) : K \times K \to \mathbb{R}$ is continuous. Let  $\mu$  be a positive measure on K with finite total mass  $\alpha$ . Let  $\epsilon > 0$ . Then there are  $p_i \in K$  and  $r_i > 0$  for  $i = 1, 2, \ldots, N$ , such that the measure  $\sigma = \sum_{i=1}^N r_i \delta_{p_i}$  has total mass  $\alpha$ , and if  $\phi(x) = \int_{y \in K} f(x, y) d\mu(y)$ and  $\psi(x) = \int_{y \in K} f(x, y) d\sigma(y)$  then  $\psi(x) < \phi(x) + \epsilon$ . *Proof.* Divide K into finitely many small sets  $K_i$  and pick  $p_i \in K_i$ . By uniform continuity of f we may assume that for any  $y \in K_i$  we have that  $|f(x,y) - f(x,p_i)| < \epsilon/\alpha$ . Let  $r_i = \mu(K_i)$ . Define  $\sigma = \sum_{i=1}^N r_i \delta_{p_i}$ . Let  $x \in K$ . Define

$$\begin{split} \psi(x) &= \int_{y \in K} f(x, y) d\sigma(y) \\ &= \sum_{i} f(x, p_{i}) r_{i} \\ &= \sum_{i} f(x, p_{i}) \int_{K_{i}} d\mu(y) \\ &= \sum_{i} \int_{K_{i}} f(x, p_{i}) d\mu(y) \\ &\leq \sum_{i} \int_{K_{i}} (f(x, y) + \epsilon/\alpha) d\mu \\ &= \phi(x) + (\epsilon/\alpha) \int_{K} d\mu \\ &= \phi(x) + \epsilon. \end{split}$$

**Theorem 2.5.** Let  $d\mu$  be a positive measure on the disc of radius R with total mass  $\alpha < 2$ . Set  $\phi(z) = \int_{|\zeta| < R} \log |z - \zeta| d\mu(\zeta)$ . Then

$$\int_{|z| < R} e^{-\phi(z)} d\lambda(z) \le \frac{28R^2}{2 - \alpha}.$$

*Proof.* Define  $\psi_n(z,\zeta) = \max\{\log |z-\zeta|, -n\}$ . Define

$$\phi_n(z) = \int_{|\zeta| < R} \psi_n(z,\zeta) d\mu(\zeta) \quad \text{for} \quad z \in \Delta(R).$$

Then  $\phi_n : \Delta(R) \to \mathbb{R}$  is continuous and  $\phi_n \searrow \phi$  pointwise for  $z \in \Delta(R)$ . Hence  $e^{-\phi_n(z)} \nearrow e^{-\phi}$  on  $\Delta(R)$ . Therefore, in order to show that

$$\int_{|z| < R} e^{-\phi(z)} d\lambda(z) \le \frac{28R^2}{2 - \alpha},$$

it suffices to show that

$$\int_{|z| < R} e^{-\phi_n(z)} d\lambda(z) \le \frac{28R^2}{2 - \alpha} + \frac{1}{n} \qquad \forall n.$$

We fix n. Let  $\delta > 0$ . Since  $\psi_n$  is continuous, according to Lemma 2.4 we can find a finite positive measure  $\mu_n = \sum_{i=1}^N \alpha_i \delta_{z_i}$  with total mass  $\alpha$  so that

$$\tilde{\phi}_n := \int_{|\zeta| < R} \psi_n(z,\zeta) d\mu_n(\zeta) \le \phi_n(z) + \delta.$$

By Theorem 2.3,

$$\int_{|z|< R} \prod_{i=1}^N \left(\frac{1}{|z-z_i|}\right)^{\alpha_i} d\lambda(z) \le \frac{28R^2}{2-\alpha}.$$

Hence

$$\int_{|z| < R} e^{-\sum_i \alpha_i \log |z - z_i|} d\lambda \le \frac{28R^2}{2 - \alpha}.$$

 $\operatorname{So}$ 

$$\int_{|z|< R} e^{-\int_{|\zeta|< R} \log |z-\zeta| d\mu_n(\zeta)} d\lambda(z) \le \frac{28R^2}{2-\alpha}.$$

Since  $\max\{\log |z-\zeta|, -n\} \ge \log |z-\zeta|$ , it follows that

$$-\max\{\log|z-\zeta|,-n\}\leq -\log|z-\zeta|.$$

Hence  $\int_{|z| < R} e^{-\tilde{\phi}_n} d\lambda \leq \frac{28R^2}{2-\alpha}$ . Choosing  $\delta$  small enough we get that

$$\int_{|z| < R} e^{-\phi_n} d\lambda \le \frac{28R^2}{2 - \alpha} + \frac{1}{n}$$

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# 3. Comparison of weights

**Lemma 3.1.** Suppose that  $|\zeta| < R$  and  $z \in \mathbb{C}$ . Then

$$\log|z-\zeta| \le \frac{1}{2}\log(1+|z|^2) + \log 2 + \frac{1}{2}\log(1+R^2).$$

Proof.

$$\begin{aligned} \log |z - \zeta| &\leq \log(|z| + |\zeta|) \\ &\leq \log 2 + \max\{\log |z|, \log |\zeta|\} \\ &\leq \log 2 + \frac{1}{2}\log(1 + |z|^2) + \frac{1}{2}\log(1 + |\zeta|^2). \end{aligned}$$

**Proposition 3.2.** Let  $\mu$  be a nonnegative measure on the disc  $|\zeta| < R$  with mass M. Let  $\phi_1(z) = \int_{|\zeta| < R} \log |z - \zeta| d\mu(\zeta)$  and  $\phi_2(z) = M/2 \log(1 + |z|^2)$ . Suppose that  $\phi = \phi_1 + \sigma$  and  $\psi = \phi_2 + \sigma$ . Then there exists constant C so that  $\|f\|_{\psi}^2 \leq C \|f\|_{\phi}^2$ . In particular,  $H(\phi) \subset H(\psi)$ .

*Proof.* By the previous lemma,

$$\phi_1 \le \phi_2 + M\left(\log 2 + \frac{1}{2}\log(1+R^2)\right),$$

hence

$$\phi \leq \psi + M\left(\log 2 + \frac{1}{2}\log(1+R^2)\right).$$

It follows that  $e^{-\psi} \leq C e^{-\phi}$ . Thus we have  $H(\phi) \subset H(\psi)$ .

For the other direction we need some extra hypothesis.

**Proposition 3.3.** Let  $\mu$  be a nonnegative measure on the disc  $|\zeta| < (R+\epsilon)$ ( $\epsilon > 0$  some constant) with mass  $\beta \in (0,2)$ . Let  $\alpha$  be the  $\mu$  mass of  $\Delta(R)$ . Suppose that  $\phi$  is subharmonic on  $\mathbb{C}$  and that  $\frac{1}{2\pi}\Delta\phi = \mu$  on  $\Delta(R+\epsilon)$ . Let  $\phi_1(z) = \int_{|\zeta| < R} \log |z - \zeta| d\mu(\zeta)$  and  $\phi_2(z) = \alpha/2 \log(1+|z|^2)$ . Write  $\phi = \phi_1 + \sigma$ and  $\psi = \phi_2 + \sigma$ . Then  $H(\psi) \subset H(\phi)$ .

We prove first a lemma:

**Lemma 3.4.** There exists a constant C so that if  $|z| \ge R + \frac{\epsilon}{2}$  and  $|\zeta| < R$ , then

$$\log|z - \zeta| \ge \frac{1}{2}\log(1 + |z|^2) - C.$$

*Proof.* We get

$$\begin{split} \log |z - \zeta| &= \log |z| \left| 1 - \frac{\zeta}{z} \right| \\ &\geq \log |z| + \log \left( 1 - \frac{2R}{2R + \epsilon} \right) \\ &\geq \frac{1}{2} \log \left( |z|^2 \left( \frac{1}{R^2} + 1 \right) \right) - \frac{1}{2} \log \left( \frac{1}{R^2} + 1 \right) \\ &\quad + \log \left( 1 - \frac{2R}{2R + \epsilon} \right) \\ &\geq \frac{1}{2} \log (1 + |z|^2) - \frac{1}{2} \log \left( \frac{1}{R^2} + 1 \right) + \log \left( 1 - \frac{2R}{2R + \epsilon} \right). \end{split}$$

The proof gives that we can choose  $C = \frac{1}{2} \log \left(\frac{1}{R^2} + 1\right) - \log(1 - \frac{2R}{2R + \epsilon})$ .  $\Box$ 

In order to prove the above Proposition 3.3 we also need the following well known Riesz Decomposition Theorem (see Ransford [10] Theorem 3.7.9).

**Theorem 3.5** (Riesz Decomposition Theorem). Let u be a subharmonic function on a domain D in  $\mathbb{C}$ , with  $u \not\equiv -\infty$ . Then, given a relatively compact open subset U of D, we can decompose u as

$$u = \int_{\zeta \in U} \log |z - \zeta| d\mu(\zeta) + h$$

on U, where  $\mu = \frac{1}{2\pi} \Delta u|_U$  and h is harmonic on U.

We prove Proposition 3.3:

*Proof.* Let F be an entire function so that  $\int_{\mathbb{C}} |F|^2 e^{-\psi} d\lambda < \infty$ . If  $|z| > R + \frac{\epsilon}{2}$ , by the previous Lemma,

$$\phi_1(z) = \int_{|\zeta| < R} \log |z - \zeta| d\mu(\zeta)$$
  

$$\geq \frac{\alpha}{2} \log(1 + |z|^2) - \alpha C$$
  

$$= \phi_2(z) - \alpha C.$$

Here C is the explicit constant from Lemma 3.4. It follows that

$$\int_{|z|\ge R+\epsilon/2} |F|^2 e^{-\phi} d\lambda < e^{\alpha C} \int_{|z|\ge R+\epsilon/2} |F|^2 e^{-\psi} d\lambda < +\infty.$$

On the disc of radius  $R+\epsilon$ , according to Riesz decomposition theorem we can write  $\phi(z) = \int_{|\zeta| < R+\epsilon} \log |z-\zeta| d\mu(\zeta) + \tau := \Phi + \tau$  where  $\tau$  is a subharmonic function which is harmonic of radius  $R + \epsilon$ . In particular,  $\tau$  is bounded on the disc of radius  $R + \frac{2}{3}\epsilon$ . By Theorem 2.5 we have

$$\int_{|z|< R+\frac{1}{2}\epsilon} e^{-\Phi} d\lambda < \int_{|z|< R+\epsilon} e^{-\Phi} d\lambda < \frac{2\pi}{2-\beta} (2(R+\epsilon))^2 < +\infty$$

and hence the same is true for the integral  $|F|^2 e^{-\phi}$  on the disc of radius  $R + \frac{1}{2}\epsilon$ . That means we have  $\int_{\mathbb{C}} |F|^2 e^{-\phi} d\lambda < +\infty$ . Thus

$$H(\psi) \subset H(\phi).$$

## 4. Proof of Theorem 1.3

In this section we prove the Main Theorem, Theorem 1.3. We prove first the case when the upper regularization  $\phi$  is a harmonic function. We can suppose that  $\varphi_1$  is not identically  $-\infty$ .

**Lemma 4.1.** If  $\phi$  is harmonic, then there are constants  $c_1 \leq c_2 \leq \cdots, c_j \rightarrow 0$  so that  $\varphi_j = \varphi + c_j = \phi + c_j$ .

To prove the lemma, observe that  $\varphi_j - \phi \leq \varphi_j - \varphi \leq 0$ . Then  $\varphi_j - \phi$  must be constant. Thus we have  $\varphi = \phi$ . The Lemma follows.

Then the theorem follows in the case when  $\phi$  is harmonic.

We can generalize this to the following case:

Condition (A): The upper regularization  $\phi$  of  $\varphi$  is a subharmonic function with the following property:  $\Delta \phi = \sum_i a_i \delta_{z_i}$  where  $z_i$  is a sequence in  $\mathbb{C}$  and  $a_i > 0$ .

**Lemma 4.2.** In the case of (A), there exist non-positive constants  $c_j$  so that  $\varphi_j = \varphi + c_j = \phi + c_j$ .

*Proof.* Fix N. We can write

$$\phi = \sum_{i=1}^{N} a_i \log |z - z_i| + \psi_N$$

where  $\psi_N$  is subharmonic and  $\Delta \psi_N = \sum_{i>N} a_i \delta_{z_i}$ . We get for any j, N

$$\varphi_j - \sum_{i=1}^N a_i \log |z - z_i| \le \psi_N$$

for  $z \neq z_i$ . Then  $\varphi_j - \sum_{i=1}^N a_i \log |z - z_i|$  extends across  $z_i$  as a subharmonic function  $\psi_j^N$ . That is  $\varphi_j = \sum_{i=1}^N a_i \log |z - z_i| + \psi_j^N$  for any N on  $\mathbb{C}$ . Thus  $\varphi_j = -\infty$  at  $z_i$  and  $\Delta \varphi_j \geq \sum_{i=1}^N a_i \delta_{z_i}$ . It follows that  $\Delta \varphi_j \geq \Delta \phi$  on  $\mathbb{C}$  for all j. So we can find some subharmonic function  $\lambda_j$  such that  $\varphi_j = \phi + \lambda_j$ . But  $\lambda_j \leq 0$  thus it must be constant. The Lemma follows and hence the theorem also follows in this case.

Condition (B): Let  $\phi$  be a subharmonic function on  $\mathbb{C}$ . Let  $\mu$  denote the Laplacian of  $\frac{1}{2\pi}\phi$ . We say that  $\phi$  satisfies condition (B) if there exist some constant R > 0 and c > 0 such that on the disc  $|\zeta| < R + c$ , the mass of  $\mu$  is equal to  $\beta$ , with  $0 < \beta < 2$  and the mass of  $\mu$  on the disc  $|\zeta| < R$  is  $\alpha > 0$ . According to Proposition 3.2 and Proposition 3.3 with the same notation as there we have the following Corollary:

**Corollary 4.3.** If  $\phi$  satisfies the above condition (B), then the spaces  $L^2(\mathbb{C}, \phi)$ and  $L^2(\mathbb{C}, \psi)$  are the same. Moreover the norms are equivalent.

**Lemma 4.4.** Let  $\varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \cdots$  be subharmonic functions on  $\mathbb{C}$  with  $\varphi = \lim_k \varphi_k$ . Suppose the upper regularization  $\phi$  of  $\varphi$  satisfies the above

Condition (B). Then  $\bigcup_{k=1}^{\infty} H(\varphi_k)$  is dense in  $H(\varphi)$ .

To prove Lemma 4.4 we need the following  $L^2$ -estimate by Berndtsson (see [1]).

**Lemma 4.5** ([1]). Let  $\Omega \subset \mathbb{C}^n$  be a pseudoconvex domain and  $\varphi \in psh(\Omega)$ . Suppose  $\psi$  is a  $C^2$  real function satisfying

$$ri\partial\overline{\partial}(\varphi+\psi) \ge i\partial\psi \wedge \overline{\partial}\psi$$

for some 0 < r < 1. Then for each  $\overline{\partial}$ -closed (0,1)-form v, there is a solution u to  $\overline{\partial}u = v$  satisfying

(4.1) 
$$\int_{\Omega} |u|^2 e^{\psi - \varphi} d\lambda \le \frac{6}{(1-r)^2} \int_{\Omega} |v|^2_{i\partial\overline{\partial}(\varphi + \psi)} e^{\psi - \varphi} d\lambda.$$

Proof of Lemma 4.4. Put  $\frac{1}{2\pi}\Delta\phi = \mu$ ,  $\frac{1}{2\pi}\Delta\varphi_j = \mu_j$  for each j. The mass of  $\mu$  (resp.  $\mu_j$ ) on the disc  $\Delta(R)$  will be denoted by  $\alpha$  (resp.  $\alpha_j$ ). Since  $\varphi_j$ is increasing to  $\varphi$  and  $\varphi = \phi$  a.e., we have that  $\Delta\varphi_j$  converges to  $\Delta\phi$  in the sense of distributions. That means we can find some 0 < c' < c with the mass of  $\mu_j$  on the disc  $\Delta(R + c')$  belongs to (0, 2) when j is sufficient large. Moreover the mass of  $\mu_j$  on the disc  $\Delta(R)$  is  $\alpha_j > \alpha/2$  for all sufficiently large j. By the Riesz decomposition theorem we can write

$$\phi = \widetilde{\phi} + \int_{|\zeta| < R} \log |z - \zeta| d\mu(\zeta), \quad \varphi_j = \widetilde{\varphi}_j + \int_{|\zeta| < R} \log |z - \zeta| d\mu_j(\zeta), \quad \forall j.$$

Here  $\widetilde{\phi}$  and  $\widetilde{\varphi}_j$  are subharmonic functions on  $\mathbb{C}$ . Put

$$\varphi'_j = \widetilde{\varphi}_j + \frac{\alpha_j}{2}\log(1+|z|^2), \quad \forall j$$

and

$$\phi' = \widetilde{\phi} + \frac{\alpha}{2}\log(1+|z|^2).$$

By Corollary 4.3 we have that  $H(\varphi'_j) = H(\varphi_j)$  for each j and  $H(\phi') = H(\phi) = H(\varphi)$ . The following proof is similar to [3]. Let  $\chi : \mathbb{R} \to [0, 1]$  be a smooth function satisfying  $\chi|_{(-\infty, \log \frac{1}{2})} = 1$ ,  $\chi|_{(0, +\infty)} = 0$  and  $|\chi'| \leq 3$ . Set

$$\psi = -\log\left(\log(1+|z|^2)\right).$$

Then we have

$$i\partial\overline{\partial}\psi = -i\frac{\partial\overline{\partial}\log(1+|z|^2)}{\log(1+|z|^2)} + i\frac{\partial\log(1+|z|^2)\wedge\overline{\partial}\log(1+|z|^2)}{(\log(1+|z|^2))^2}$$

and

$$i\partial\psi\wedge\overline{\partial}\psi = i\frac{\partial\log(1+|z|^2)\wedge\overline{\partial}\log(1+|z|^2)}{(\log(1+|z|^2))^2}.$$

For each j, put  $\Phi_j := \varphi'_j + \frac{\alpha_j}{4}\psi = \widetilde{\varphi}_j + \frac{\alpha_j}{2}\log(1+|z|^2) + \frac{\alpha_j}{4}\psi$  and  $\Psi_j = \frac{\alpha_j}{4}\psi$ . By calculation, if |z| > 2

$$\begin{split} i\partial\overline{\partial}(\Phi_j + \Psi_j) &\geq \frac{\alpha_j}{2}i\partial\overline{\partial}\log(1+|z|^2) - \frac{\alpha_j}{2}i\frac{\partial\partial\log(1+|z|^2)}{\log(1+|z|^2)} \\ &+ \frac{\alpha_j}{2}i\frac{\partial\log(1+|z|^2)\wedge\overline{\partial}\log(1+|z|^2)}{(\log(1+|z|^2))^2} \\ &\geq \frac{\alpha_j}{2}\partial\psi\wedge\overline{\partial}\psi \\ &= \frac{8}{\alpha_j}\partial\Psi_j\wedge\overline{\partial}\Psi_j, \end{split}$$

while  $\frac{\alpha}{16} < \frac{\alpha_j}{8} < \frac{1}{4}$ . Let  $f \in H(\varphi) = H(\phi')$ . We fix  $0 < \epsilon < \frac{1}{2}$ . Put  $v_{\epsilon} := f \cdot \overline{\partial} \chi \left( \log(-\psi) + \log \epsilon \right)$ .

Apply Lemma 4.5 for  $\Omega = \mathbb{C}$  with  $\varphi$  and  $\psi$  replaced by  $\Phi_j$  and  $\Psi_j$  respectively, we then obtain a solution  $u_{j,\epsilon}$  of  $\overline{\partial}u = v_{\epsilon}$  satisfying

$$\int_{\mathbb{C}} |u_{j,\epsilon}|^2 e^{-\varphi'_j} d\lambda \leq \frac{6}{(1-\frac{\alpha_j}{8})^2} \int_{\mathbb{C}} |v_{\epsilon}|^2_{\partial\overline{\partial}(\Phi_j+\Psi_j)} e^{\Psi_j - \Phi_j} d\lambda$$
$$\leq \frac{C}{\alpha_j} \epsilon^2 \int_{\frac{1}{2\epsilon} \leq -\psi \leq \frac{1}{\epsilon}} |f|^2 e^{-\varphi'_j} d\lambda.$$

Put  $K := \{z : z \in \mathbb{C}, -\psi \leq \frac{1}{\epsilon}\}$ . Since  $f \in H(\varphi)$ , according to the strong openness theorem there exists  $j_0$  so that when  $j \geq j_0$  we have  $\int_K |f|^2 e^{-\varphi_j} d\lambda < \infty$ .

Next by using Lemma 3.1 we obtain that for some constant C, independent of j and  $\epsilon$ 

$$\begin{split} \int_{K} |f|^{2} e^{-\varphi_{j}'} d\lambda &= \int_{K} |f|^{2} e^{-\frac{\alpha_{j}}{2} \log(1+|z|^{2}) - \widetilde{\varphi}_{j}} d\lambda \\ &\leq C \int_{K} |f|^{2} e^{-\int_{|\zeta| < R} \log|z-\zeta| d\mu_{j}(\zeta) - \widetilde{\varphi}_{j}} d\lambda \\ &= C \int_{K} |f|^{2} e^{-\varphi_{j}} d\lambda < \infty. \end{split}$$

 $\operatorname{Set}$ 

$$F_{j,\epsilon} = f \cdot \chi \left( \log(-\psi) + \log \epsilon \right) - u_{j,\epsilon}.$$

Then  $F_{j,\epsilon}$  is an entire function for each  $j \ge j_0 \gg 1$  with

$$\|F_{j,\epsilon}\|_{L^2(\mathbb{C},\varphi'_j)} \le (1 + \frac{C}{\sqrt{\alpha_j}}\epsilon) \|f\|_{L^2(K,\varphi'_j)} < +\infty.$$

That is  $F_{j,\epsilon} \in \bigcup_{j=1}^{\infty} H(\varphi'_j) = \bigcup_{j=1}^{\infty} H(\varphi_j)$ . We also obtain

$$\begin{split} \|F_{j,\epsilon} - f\|_{L^{2}(\mathbb{C},\varphi)}^{2} &\leq 2\int_{-\psi \geq \frac{1}{2\epsilon}} |f|^{2} e^{-\varphi} d\lambda + C \int_{\mathbb{C}} |u_{j,\epsilon}|^{2} e^{-\varphi'_{j}} d\lambda \\ &\leq 2\int_{-\psi \geq \frac{1}{2\epsilon}} |f|^{2} e^{-\varphi} d\lambda + C' \epsilon^{2} \int_{K} |f|^{2} e^{-\varphi_{j}} d\lambda. \end{split}$$

Still keeping  $\epsilon$  fixed, but letting  $j \to \infty$ , we get

$$\limsup_{j \to \infty} \|F_{j,\epsilon} - f\|_{L^2(\mathbb{C},\varphi)}^2 \le 2 \int_{-\psi \ge \frac{1}{2\epsilon}} |f|^2 e^{-\varphi} d\lambda + C' \epsilon^2 \int_K |f|^2 e^{-\varphi} d\lambda.$$

Finally we let  $\epsilon \to 0$ . Then  $\bigcup_{j=1}^{\infty} H(\varphi_j)$  is dense in  $H(\varphi)$ , which completes the proof.

For any subharmonic  $\phi$  on  $\mathbb{C}$ , we let  $\mu = \frac{1}{2\pi}\Delta\phi$  which is a locally finite positive measure on  $\mathbb{C}$ . Then  $\mu$  decomposes into a sum  $\mu = \mu_1 + \mu_2$  where  $\mu_2 = \sum_i a_i \delta_{z_i}$  is an at most countable sum of Dirac masses and where  $\mu_1$  has no point mass. Thus Theorem 1.3 follows as above, using Condition (B) because of the following lemma.

**Lemma 4.6.** Suppose  $\mu_1$  is not identically zero. Then there exist a point  $z_0 \in \mathbb{C}$  and 0 < r < s so that  $\mu(\Delta(z_0, r)) > 0, \mu(\Delta(z_0, s)) < 2$ .

*Proof.* The support of the measure  $\mu_1$  is uncountable. Hence we can choose a point  $z_0$  in the support of  $\mu_1$  which is not one of the  $z_i$ . Then  $\mu(\Delta(z_0, s)) \to 0$  as  $s \to 0$  while  $\mu(\Delta(z_0, r)) > 0$  for all r > 0.

### 5. Proof of Theorem 1.6 and Theorem 1.7

Proof of theorem 1.6. We assume that the upper regularization  $\phi$  of  $\varphi$  satisfies Condition (B). For other  $\phi$ , we may use the same method as in the proof of Theorem 1.3, we skip the details. Fix  $r < +\infty$ . We prove that

$$\lim_{j \to \infty} K_{\varphi_j}(w, w) = K_{\varphi}(w, w)$$

for all |w| < r. Set  $B_r = \{|z| < r\}$ . Let  $\epsilon \ll 1$ . Let  $\chi, \psi, \Psi_j, \Phi_j$  and K as before in the proof of Lemma 4.4. The proof is similar to [2]. Set

$$\lambda_{\epsilon} = \chi(\log(-\psi) + \log \epsilon).$$

Let  $w \in B_R := \{|z| < R\}$ . Applying Lemma 4.5 with  $\varphi$  and  $\psi$  replaced by  $\Phi_j$  and  $\Psi_j$  respectively, we get a solution  $u_{j,\epsilon}$  of

$$\overline{\partial}u = K_{\varphi}(\cdot, w)\overline{\partial}\lambda_{\epsilon}$$

such that

$$\int_{\mathbb{C}} |u_{j,\epsilon}|^2 e^{-\varphi'_j} d\lambda \le \frac{C}{\alpha_j} \epsilon^2 \int_{\frac{1}{2\epsilon} \le -\psi \le \frac{1}{\epsilon}} |K_{\varphi}(\cdot, w)|^2 e^{-\varphi_j} d\lambda \le \frac{C'}{\alpha_j} \epsilon^2 K_{\varphi}(w, w)$$

for sufficiently large j. The last inequality holds because of the following argument.

By the strong openness theorem  $|K_{\varphi}(\cdot, w)|^2 e^{-\varphi_j}$  is integrable on K for some j, hence by the monotone convergence theorem we have

$$\int_{K} |K_{\varphi}(\cdot, w)|^{2} e^{-\varphi_{j}} d\lambda \longrightarrow \int_{K} |K_{\varphi}(\cdot, w)|^{2} e^{-\varphi} d\lambda \leq K_{\varphi}(w, w)$$

for sufficiently large j. Note that if  $K_{\varphi}(w, w) = 0$ , then  $K_{\varphi_j}(z, w) \equiv 0$  for all  $z \in \mathbb{C}$ .

Since  $B_{R+1} \subset \{-\psi \leq \frac{1}{2\epsilon}\}$ . We claim that  $\varphi'_j$  is uniformly bounded above on  $\partial B_{R+1}$ . The reason is that  $|\varphi_j - \varphi'_j|$  is uniformly bounded on  $\partial B_{R+1}$  independent of j and  $\varphi_j \leq \varphi$  is uniformly bounded.  $u_{j,\epsilon}$  is holomorphic on  $\{-\psi \leq \frac{1}{2\epsilon}\}$ , the mean value inequality yields

$$\begin{aligned} |u_{j,\epsilon}(w)|^2 &\leq C_n \int_{B_{R+1}} |u_{j,\epsilon}|^2 d\lambda \\ &\leq C'_{n,R} \int_{B_{R+1}} |u_{j,\epsilon}|^2 e^{-\varphi'_j} d\lambda \\ &\leq \frac{C''_{n,R}}{\alpha_j} \epsilon^2 K_{\varphi}(w,w). \end{aligned}$$

It follows that

$$f_{j,\epsilon} := \lambda_{\epsilon} K_{\varphi}(\cdot, w) - u_{j,\epsilon}$$

is an entire function satisfying

$$|f_{j,\epsilon}(w)| \ge K_{\varphi}(w,w) - \frac{C_{n,R}}{\sqrt{\alpha_j}}\epsilon$$

and

$$\begin{split} \|f_{j,\epsilon}\|_{H(\varphi_j)} &\leq \|K_{\varphi}(\cdot,w)\|_{L^2(K,\varphi_j)} + C\|u_{j,\epsilon}\|_{L^2(\mathbb{C},\varphi_j)} \\ &\leq (1 + \frac{C}{\sqrt{\alpha_j}}\epsilon)\|K_{\varphi}(\cdot,w)\|_{L^2(K,\varphi_j)} \\ &\leq (1 + \frac{C}{\sqrt{\alpha_j}}\epsilon)\sqrt{K_{\varphi}(w,w)}. \end{split}$$

Thus we have

$$\frac{|f_{j,\epsilon}(w)|}{\|f_{j,\epsilon}\|_{H(\varphi_j)}} \ge \frac{K_{\varphi}(w,w) - C_{n,R,\alpha_j}\epsilon}{(1 + C_{\alpha_j}\epsilon)\sqrt{K_{\varphi}(w,w)}},$$

that is

$$\liminf_{j \to +\infty} K_{\varphi_j}(w, w) \ge K_{\varphi}(w, w).$$

Since  $\varphi_j \leq \varphi$  we know that  $K_{\varphi_j}(w, w) \leq K_{\varphi}(w, w)$  for each  $j \geq 1$ , thus we obtain

$$\lim_{j \to +\infty} K_{\varphi_j}(w, w) = K_{\varphi}(w, w), \quad \forall w \in \mathbb{C}.$$

This completes the proof.

Proof of Theorem 1.7. For each compact  $F \subset \mathbb{C}$ , each fixed  $w \in F$  and  $z \in F$ , according to the mean value inequality we know that

$$|K_{\varphi_{j}}(z,w) - K_{\varphi}(z,w)|^{2}$$

$$\leq C' ||K_{\varphi_{j}}(\cdot,w) - K_{\varphi}(\cdot,w)||^{2}_{H(U,0)}$$

$$\leq C ||K_{\varphi_{j}}(\cdot,w) - K_{\varphi}(\cdot,w)||^{2}_{H(U,\varphi)}$$

$$\leq C ||K_{\varphi_{j}}(\cdot,w) - K_{\varphi}(\cdot,w)|^{2}_{H(\mathbb{C},\varphi)}$$

$$= C \left( \int_{\mathbb{C}} |K_{\varphi_{j}}(\cdot,w)|^{2} e^{-\varphi} d\lambda + \int_{\mathbb{C}} |K_{\varphi}(\cdot,w)|^{2} e^{-\varphi} d\lambda - 2K_{\varphi_{j}}(w,w) \right)$$

$$\leq C \left( \int_{\mathbb{C}} |K_{\varphi_{j}}(\cdot,w)|^{2} e^{-\varphi_{j}} d\lambda + K_{\varphi}(w,w) - 2K_{\varphi_{j}}(w,w) \right)$$

$$(5.1) = C(K_{\varphi}(w,w) - K_{\varphi_{j}}(w,w))$$

where U is some neighborhood of the compact set F. By Theorem 1.6,  $K_{\varphi_j}(z, w)$  pointwise converges to  $K_{\varphi}(z, w)$  in  $\mathbb{C} \times \mathbb{C}$ . Similarly

(5.2) 
$$\lim_{j \to \infty} K_{2(k+1)\varphi_j}(z, w) = K_{2(k+1)\varphi}(z, w) \quad \forall z, w \in \mathbb{C}.$$

On the other hand, from Ligocka's formula

$$K_{\Omega_j}[(z,t),(w,s)] = \sum_{k=0}^{\infty} 2(k+1)K_{2(k+1)\varphi_j}(z,w)\langle t,s\rangle^k, \quad z,t,w,s \in \mathbb{C},$$

we can easily obtain that

$$2(k+1)K_{2(k+1)\varphi_j}(z,w) = \frac{\partial^{2k}}{\partial t^k \partial \overline{s}^k} K_{\Omega_j}[(z,t),(w,s)]|_{t=s=0}.$$

For each  $(z_0, t_0) \in \Omega \subset \mathbb{C}^2$ , there exist  $r_1, r_2 > 0$  so that

$$(z_0, t_0) \in P := \Delta(z_0, r_1) \times \Delta(0, r_2) \subset \subset \Omega \subset \Omega_j, \ j \ge 1.$$

Since  $\varphi_j$  is increasing to  $\varphi$ , for each  $j \ge 1$ 

(5.3) 
$$|K_{\Omega_j}[(z,t),(w,s)]| \leq K_{\Omega_j}[(z,t),(z,t)]^{\frac{1}{2}}K_{\Omega_j}[(w,s),(w,s)]^{\frac{1}{2}}$$
  
 $\leq K_{\Omega}[(z,t),(z,t)]^{\frac{1}{2}}K_{\Omega}[(w,s),(w,s)]^{\frac{1}{2}}.$ 

Put  $M := \sup_{j} \sup_{\overline{P} \times \overline{P}} |K_{\Omega_j}[(z,t),(w,s)]|$ , we have  $M < +\infty$ . By Cauchy estimates we obtain

$$\left|\frac{\partial^{2k}}{\partial t^k \partial \overline{s}^k} K_{\Omega_j}[(z,t),(w,s)]\right|_{t=s=0} \le C_k \frac{M}{r_2^{2k}}, \quad \forall z, w \in \Delta(z_0,r_1).$$

Let  $0 < r'_1 < r_1$ ,  $0 < r'_2 < r_2$ , then for each  $\epsilon > 0$ , there exists  $k_{\epsilon} \gg 1$ , so that

$$\sum_{k \ge k_{\epsilon}} 2(k+1) \left| K_{2(k+1)\varphi_j}(z,w)(t\overline{s})^k \right| < \epsilon, \ \forall z, w \in \Delta(z_0, r_1'), \ \forall t, s \in \Delta(0, r_2'),$$

$$\sum_{k \ge k_{\epsilon}} 2(k+1) |K_{2(k+1)\varphi}(z,w)(t\overline{s})^k| < \epsilon, \quad \forall z, w \in \Delta(z_0, r_1'), \ \forall t, s \in \Delta(0, r_2').$$

Since

and

$$\sum_{k=0}^{k_{\epsilon}} 2(k+1)K_{2(k+1)\varphi_j}(z,w)(t\overline{s})^k \longrightarrow \sum_{k=0}^{k_{\epsilon}} 2(k+1)K_{2(k+1)\varphi}(z,w)(t\overline{s})^k$$

by (5.2), it follows that  $K_{\Omega_j}[(z,t), (w,s)]$  pointwise converges to  $K_{\Omega}[(z,t), (w,s)]$ in  $\Delta(z_0, r'_1) \times \Delta(0, r'_2) \subset \Omega \times \Omega$ . By (5.3), the functions  $K_{\Omega_j}[(z,t), (w,s)]$ form a normal family in  $\Omega \times \Omega$ . It follows from the normality and the pointwise convergence just proved that the convergence is uniform on compact subsets of  $\Omega \times \Omega$ . This completes the proof.  $\Box$ 

Acknowledgements The authors were supported in part by the Norwegian Research Council grant number 240569, the first author was also supported by NSFC grant 11601120. The first author also thanks Professor Bo-Yong Chen for his valuable comments.

### References

- [1] Berndtsson, B.: Weighted estimates for the  $\overline{\partial}$ -equation. Complex Analysis and Geometry. Columbus, 1999. Ohio State University Mathematical Research Institute, **9**, 43–57 (2001)
- [2] Chen, B.Y.: Parameter dependence of the Bergman kernels. Adv. Math. **299**, 108–138 (2016)
- [3] Fornæss, J. E. and Wu, J.: A global approximation result by Bert Alan Taylor and the strong openness conjecture in  $\mathbb{C}^n$ . J. Geom. Anal. **28**, 1-12(2018)

[4] Ligocka E.: On the Forelli-Rudin construction and weighted Bergman projections. Studia Math, **94**(3), 257–272(1989)

 [5] Taylor, B.A.: On weighted polynomial approximation of entire functions. Pacific. J. Math. 36, 523–539 (1971)

- [6] Guan, Q.A. and Zhou, X.Y.: Strong openness conjecture and related problems for plurisubharmonic functions. arxiv:1401.7158.
- [7] Guan, Q.A. and Zhou, X.Y.: A proof of Demailly's strong openness conjecture. Ann. Math. 182(2), 605–616 (2015)
- [8] Guan, Q.A. and Zhou, X.Y.: Effectiveness of Demailly's strong openness conjecture and related problems. Invent. math. **22**, 635–676 (2015)

[9] Lempert, L.: Modules of square integrable holomorphic germs. Analysis meets geometry, 311-333, Trends Math., Birkhuser/Springer, Cham, 2017.

[10] Ransford, T.: Potential theory in the complex plane (Vol. 28). Cambridge university press (1995)

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