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A Geometric Interpretation of Categories of Type $\tilde{A}$ and of Morphisms in the Infinite Radical

Karin Baur[1] (iD) · **Hermund André Torkildsen**[2]

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Abstract

We give a geometric realization of the module category of a hereditary algebra of type $\tilde{A}$. We work with oriented arcs to define a translation quiver isomorphic to the Auslander-Reiten quiver of the module category of type $\tilde{A}$. To get a description of the module category, we introduce *long moves* between arcs. These allow us to include the infinite radical in the geometric description. Finally, our results can also be used to describe the corresponding cluster categories by taking unoriented arcs instead.

Keywords Tame module category · Cluster category · Infinite radical · AR quiver

1 Introduction

In this article, we want to describe the module categories and cluster categories of type $\tilde{A}$ combinatorially. In particular, we will provide a geometric combinatorial interpretation of their infinite radicals.

Let k be an algebraically closed field. In this article we will first concentrate on mod $\tilde{A} = $ mod $k Q_{g,h}$ for some quiver $Q_{g,h}$ with fixed orientation, as described in Section 2.1, and on $\mathrm{rad}^{\infty}(\mathrm{mod}\,\tilde{A}) = \cap_i \mathrm{rad}^i(\mathrm{mod}\,\tilde{A})$.

Recall that the Auslander Reiten quiver $\mathrm{AR}(\mathrm{mod}\,k Q_{g,h})$ describes the isomorphism classes of indecomposable modules and the irreducible maps between them. It does not describe the elements of the infinite radical $\mathrm{rad}^{\infty}(\mathrm{mod}\,\tilde{A})$. The shape of the AR-quiver of mod $\tilde{A}$ is well-known, it consists of several connected components (cf. [9]).

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✉ Karin Baur
baurk@uni-graz.at

Hermund André Torkildsen
hermund.a.torkildsen@ntnu.no

[1] Institut für Mathematik und Wissenschaftliches Rechnen, Universität Graz, NAWI Graz, Heinrichstrasse 36, A-8010, Graz, Austria

[2] NTNU, Department of Teacher Education, Norwegian University of Science and Technology, 7004, Trondheim, Norway

We will write $\mathcal{P}, \mathcal{I}$ to denote the subcategories of $\operatorname{mod} kQ_{g,h}$ of the preprojective, preinjective modules of $\operatorname{mod} kQ_{g,h}$, $\mathcal{T}_g$, $\mathcal{T}_h$ and $\mathcal{T}^\lambda$ for $\lambda \in k \setminus \{0\}$ for the subcategories of the modules whose vertices belong to the tubes of rank g, h and to the homogeneous tubes respectively.

We describe indecomposable modules in $\mathcal{P}, \mathcal{I}, \mathcal{T}_g$ or $\mathcal{T}_h$ via oriented arcs in the annulus. This geometric model has been used in various articles on (cluster) tubes, module/cluster categories of type $\tilde{A}$, cf. [1, 2, 5, 8, 10, 11]. Note that the geometric interpretation of elements in the homogenous tubes can be achieved via closed loops in the annulus, with labels. It seems less natural to us and we thus restrict our attention to the other components.

However, we will also focus on the infinite radical of the module and the cluster categories of type $\tilde{A}$. We will introduce long moves as a new tool, and show how we can interpret them as elements of the infinite radical.

In [4], Brüstle introduces a quiver Q_m, for every $m \geq 1$, by truncating the different components in $\operatorname{AR}(\operatorname{mod} kQ_{g,h})$ at appropriate levels and then inserting additional arrows linking the different components. The author then defines full subcategories $\mathcal{J}_m Q_{g,h}$ of the module category $\operatorname{mod} kQ_{g,h}$ consisting of modules in the union of $\mathcal{P}_m$, $\mathcal{I}_m$, $(\mathcal{T}_g)_{2m(n+1)}$, $(\mathcal{T}_h)_{2m(n+1)}$, $\mathcal{T}^\lambda_{2m(n+1)}$ ($\lambda \in k \setminus \{0\}$) (see Section 3). He finally proves that the associated *path category* $\mathcal{C}(kQ_m)$ *modulo certain relations* (see Theorem 3.4) is isomorphic to $\mathcal{J}_m Q_{g,h}$.

Note that for every module $M \in \operatorname{mod} kQ_{g,h}$, there exists $m \geq 1$, such that M is isomorphic to a direct sum of objects of $\mathcal{J}_m Q_{g,h}$ and hence the categories $\mathcal{J}_m Q_{g,h}$ can be used to describe $\operatorname{mod} kQ_{g,h}$.

In Sections 2 and 3, we will fix the notation, recall the truncated quiver Q_m from [4].

In Section 4, we use oriented arcs in the annulus to define the translation quiver Γ which is isomorphic to $\operatorname{AR}(\operatorname{mod} kQ_{g,h})$.

Geometric models for categories have been studied by various authors in recent years. A geometric model for cluster categories of type $\tilde{A}$ has been described in in [5] (treating marked surfacds) and for m-cluster categories of type $\tilde{A}$ in [10] in independent work. It is based on arcs in an annulus with marked points on the boundary. Annuli have also been used to describe string modules [11] and (cluster) tubes [1, 8].

Our goal is to give a geometric model for the infinite radical in terms of moves between such arcs. To do this, we define long moves between arcs of the annulus in Section 5 and define a quiver $\overline{\Gamma}$, whose objects are the same as the objects of Γ. The arrows of Γ are kept, and we add new arrows for the long moves. The long moves are the ones corresponding to elements of the infinite radical.

We then show in Section 6 that up to the homogenous components, we obtain an isomorphism between certain subquivers $\overline{\Gamma}_m$ of $\overline{\Gamma}$ and the truncated quiver Q_m from [4]. We thus give a full description of $\mathcal{J}_m(Q_{g,h})$ for every $m \geq 1$ via the combinatorial geometric model. Finally, in Section 8 we show how we can use the above results to obtain the description of appropriate subcategories of the cluster category of type $\tilde{A}$. In the Appendix, we include the geometric description of the relations imposed in Theorem 3.4.

2 The Module Category of Type $\tilde{A}$

2.1 The AR-quiver of the Module Category

As quiver of type $\tilde{A}$ we choose a quiver $Q_{g,h}$ on $n + 1 = g + h$ vertices, for $g \geq h > 0$. Let $0, 1, 2, \ldots, g - 1, g, g + 1, \ldots, n$ be arranged clockwise on a circle. Then $Q_{g,h}$ has g

successive clockwise arrows $\beta_i : i \to i + 1$ for $i = 0 \ldots, g - 1$ and h successive anti-clockwise arrows, $\alpha_i : i + 1 \to i$ for $i = g, g + 1, \ldots, n$ (with $\alpha_n = 0 \to n$) with a source at 0 and a sink at g.
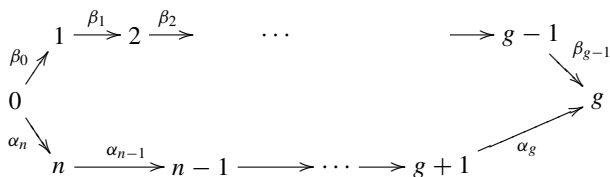
Let k be an algebraically closed field. We consider the module category mod $\tilde{A}$ of left $kQ_{g,h}$-modules. Its AR-quiver describes the isomorphism classes of indecomposable modules and the irreducible maps between them. It does not describe the elements of the infinite radical $\operatorname{rad}^\infty(\operatorname{mod} \tilde{A}) = \cap_i \operatorname{rad}^i(\operatorname{mod} \tilde{A})$.

In this article, we want to describe the whole module category mod $\tilde{A}$ by giving a geometric combinatorial interpretation of its infinite radical. We will recall the set-up necessary for our purpose. The focus of this article is the infinite radical of the module and the cluster categories of type $\tilde{A}$, restricting to the preprojective/preinjective components and the regular tubes $\mathcal{T}_g$, $\mathcal{T}_h$. We will introduce long moves as a new tool. Long moves correspond to elements in the infinite radical.

As a running example we take $n = 4$ with $g = 3$ and $h = 2$. The quiver $Q_{3,2}$ looks as follows:
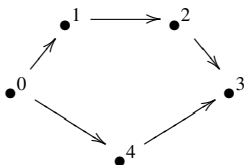
The preprojective and preinjective components $\mathcal{P}$ and $\mathcal{I}$ are viewed as infinite horizontal tubes, with $\mathcal{P}$ bounded from the left by the slice containing the projective indecomposables and $\mathcal{I}$ bounded from the right by the slice containing the injective indecomposables. We are adopting the convention of [4] to draw the tube $\mathcal{T}_h$ upside down. This will be convenient when describing relations involving the infinite radical.

In addition, there are infinitely many tubes: two regular tubes $\mathcal{T}_g$ and $\mathcal{T}_h$ of rank g and h respectively and the homogeneous tubes of rank 1, T_λ, $\lambda \in k \setminus \{0\}$. See Fig. 1 for the AR-quiver of mod($kQ_{3,2}$). Note that in the figure, the component $\mathcal{T}_2 = \mathcal{T}_h$ is drawn upside down, as mentioned above. We write M_i for the modules sitting at the mouth of $\mathcal{T}_g$ and $\overline{M}_i$ for the modules at the mouth of $\mathcal{T}_h$. These are the quasi-simple modules. In order to keep the picture readable, we have omitted the dashed lines for the AR translation τ. The effect of τ is to send a vertex horizontally to its neightbour to the left. For the homogeneous tubes $\mathcal{T}^\lambda$, this means that τ sends a vertex to itself.

3 A Truncated Translation Quiver

In [4], Brüstle recalls the description of the Auslander Reiten quiver of the module category mod $kQ_{g,h}$ (the quiver $Q_{g,h}$ is called K in [4]). Brüstle then defines subquivers Q^* of the AR-quiver formed by the vertices of the preprojective indecomposables (quiver Q^P), for the
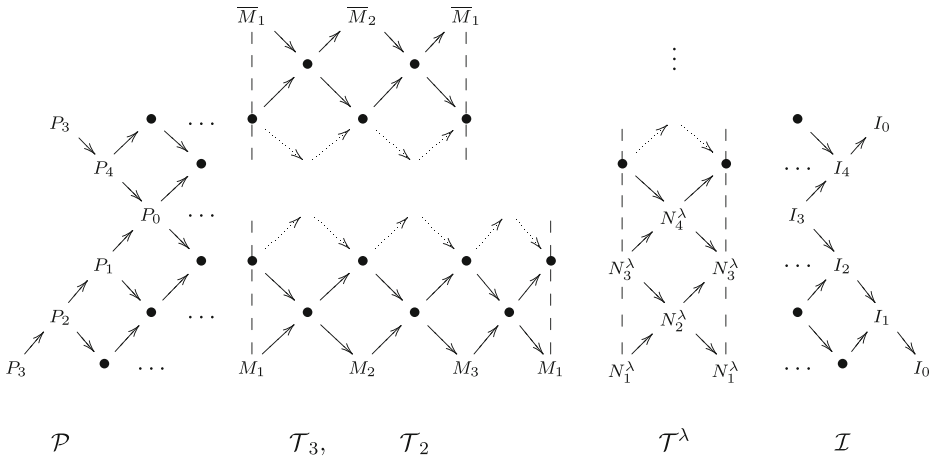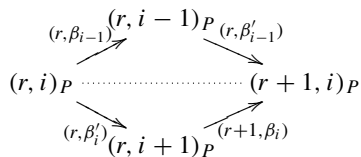
Fig. 1 AR-quiver of $\mod k Q_{3,2}$: $\mathcal{P}$ and $\mathcal{I}$ and tubes

preinjective indecomposables (quiver Q^I) and for the vertices in the tubes (quivers Q^λ) for $*$ appropriate (preprojective, preinjective, tubes) (Sections 2-4 of [4]). We will now recall his notation for the vertices of the AR-quiver. The vertices all lie in lattices of cylindrical shape. These cylinders are cut open to draw the lattices in the plane, the components Q^P and Q^I are cylinders lying on their sides, bounded on the left by the projectives or on the right by the injectives. The tubes are oriented vertically, with Q^∞ pointing downwards, all components are viewed in the plane.

- The vertices of Q^P are the $(r, i)_P$ with $r \geq 0$, $0 \leq i \leq n$. The $(0, i)_P$ form the slice of projectives, in Q^P, the entry r increases to the right. The one-dimensional projective indecomposable module P_g gets the notation $(0, g)_P$, the largest projective indecomposable P_0 is $(0, 0)_P$.
  There are arrows $(r, \beta_i) : (r, i + 1)_P \to (r, i)_P$ and $(r, \beta_i') : (r, i) \to (r + 1, i + 1)$ for $i = 0, \ldots, g - 1$, as well as arrows $(r, \alpha_i) : (r, i) \to (r, i + 1)$ and $(r, \alpha_i') : (r, i + 1) \to (r + 1, i)$ for $g \leq i \leq n$. For $0 < i < g - 1$ this looks as follows

$$
\begin{array}{ccc}
& (r, i - 1)_P & \\
(r, \beta_{i-1}) \nearrow & & \searrow (r, \beta_{i-1}') \\
(r, i)_P \cdots\cdots\cdots\cdots\cdots & & (r + 1, i)_P \\
(r, \beta_i') \searrow & & \nearrow (r+1, \beta_i) \\
& (r, i + 1)_P &
\end{array}
$$

where the second entry is taken modulo $n + 1$. The dashed line indicates the AR translation.

- The underlying graphs of Q^P and Q^I can be mapped one into the other by a mirror symmetry along a vertical line (for the mirror axis S see Fig. 2). The vertices of Q^I are the $(r, i)_I$ with $r \geq 0$, $0 \leq i \leq n$ with r increasing to the left. The $(0, i)_I$ form the slice of injective indecomposables. By the symmetry, for every arrow $\gamma_P : (r_1, s_1)_P \to$
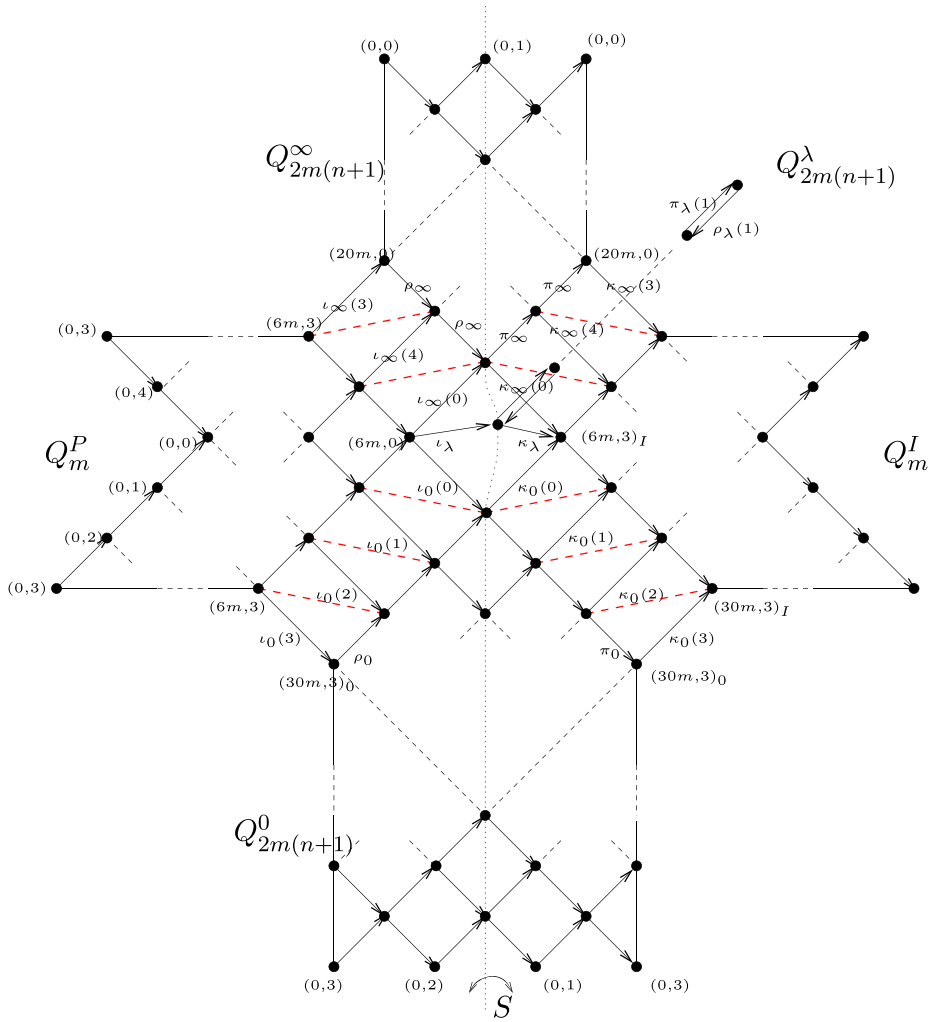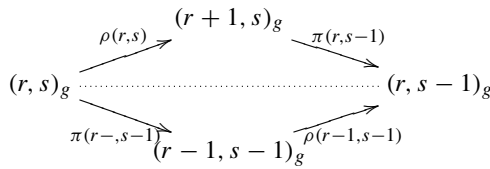
Fig. 2 Q_m for $g = 3, n = 4$

$(r_2, s_2)_P$ in Q^P, there is an arrow $\gamma_I : (r_2, s_2)_I \to (r_1, s_1)_I$ in Q^I. For $0 < i < g - 1$, we get subquivers

$$
\begin{array}{ccc}
 & (r, \beta'_{i-1}) \quad (r, i-1)_{I} (r, \beta_{i-1}) & \\
(r+1, i)_I & \cdots\cdots\cdots\cdots\cdots\cdots & (r, i)_I \\
 & (r, \beta_i) \quad (r, i+1)_{I} (r, \beta'_i) &
\end{array}
$$

with $k \geq 1$ and where the second entry is taken modulo $n + 1$.

- The vertices of Q^0 are the $(r, s)_g$ with $r \geq 0$, $1 \leq s \leq g$. The vertices at the mouth of $\mathcal{T}_g$ are $(0, s)_g$. The arrows in Q^0 are $\pi(r, s) = \pi(r, s)_0 : (r+1, s+1)_g \to (r, s)_g$ and $\rho(r, s) = \rho_0(r, s) : (r, s)_g \to (r+1, s)_g$ for $r \geq 0$. For $r > 0$, this gives
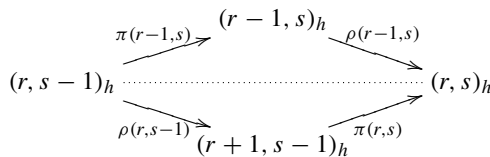
$$
\begin{array}{ccc}
 & (r+1, s)_g & \\
\nearrow {\scriptstyle \rho(r,s)} & & \searrow {\scriptstyle \pi(r,s-1)} \\
(r, s)_g \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & (r, s-1)_g \\
\searrow {\scriptstyle \pi(r-,s-1)} & & \nearrow {\scriptstyle \rho(r-1,s-1)} \\
 & (r-1, s-1)_g &
\end{array}
$$

with the second entry reduced modulo g. For $r = 0$, this becomes

$$
\begin{array}{ccc}
 & (1, s+1)_g & \\
\nearrow & & \searrow \\
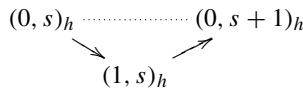(0, s+1)_g \cdots\cdots\cdots\cdots & & (0, s)_g
\end{array}
$$
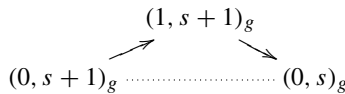
This is chosen such that the vertex $(0, g)_g$ is the $h + 1$-dimensional quasi-simple indecomposable with a non-zero morphism from $P_g = (0, g)_P$ and with a non-zero morphism to $I_0 = (0, h)_I$.

- The vertices of Q^∞ are the $(r, s)_h$, $r \geq 0$, $0 \leq s \leq h - 1$. The vertices at the mouth of $\mathcal{T}_h$ are the $(0, s)_h$. The arrows in Q^∞ are $\pi(r, s) = \pi_\infty(r, s) : (r+1, s-1)_h \to (r, s)_h$ and $\rho(r, s) = \rho_\infty(r, s) : (r, s)_h \to (r+1, s)_h$ for $r \geq 0$. For $r > 0$, we have
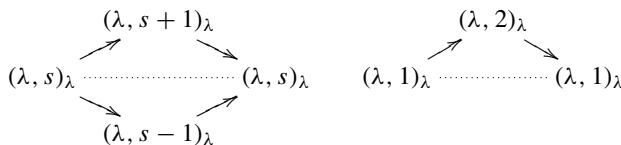
$$
\begin{array}{ccc}
 & (r-1, s)_h & \\
\nearrow {\scriptstyle \pi(r-1,s)} & & \searrow {\scriptstyle \rho(r-1,s)} \\
(r, s-1)_h \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & (r, s)_h \\
\searrow {\scriptstyle \rho(r,s-1)} & & \nearrow {\scriptstyle \pi(r,s)} \\
 & (r+1, s-1)_h &
\end{array}
$$

with $s + 1 = 0$ if $s = h - 1$. For $r = 0$, this becomes

$$
\begin{array}{ccc}
(0, s)_h \cdots\cdots\cdots\cdots & & (0, s+1)_h \\
\searrow & & \nearrow \\
 & (1, s)_h &
\end{array}
$$

This is chosen such that the vertex $(0, 0)_h$ is the $g + 1$-dimensional quasi-simple indecomposable with a non-zero morphism from $P_g = (0, g)_P$ and with a non-zero morphism to $I_0 = (0, h)_I$.

- The vertices of Q^λ ($\lambda \in k \setminus \{0\}$) are the $(\lambda, i)_\lambda$, $i \geq 1$. The vertex at the mouth of Q^λ is $(\lambda, 1)_\lambda$. The arrows in Q^λ are as follows (for $s \geq 2$):

$$
\begin{array}{ccccccc}
 & (\lambda, s+1)_\lambda & & & & (\lambda, 2)_\lambda & \\
\nearrow & & \searrow & & \nearrow & & \searrow \\
(\lambda, s)_\lambda \cdots\cdots\cdots & & (\lambda, s)_\lambda & (\lambda, 1)_\lambda \cdots\cdots & & & (\lambda, 1)_\lambda \\
\searrow & & \nearrow & & & & \\
 & (\lambda, s-1)_\lambda & & & & &
\end{array}
$$

The union of all these vertices gives all the isomorphism classes of indecomposable $kQ_{g,h}$-modules, cf. [4, Proposition §1] or [7].

In what follows, we will write $\mathcal{P}$ to denote the subcategory of $\mathrm{mod}\,kQ_{g,h}$ whose objects are the preprojective indecomposable modules, $\mathcal{I}$ for the subcategory of the preinjective indecomposable modules, $\mathcal{T}^0 = \mathcal{T}_g$ and $\mathcal{T}^\infty = \mathcal{T}_h$ (the former versions are used in Brüstle, the latter are more convenient for our geometric model) for the subcategories formed by the objects in the tubes of rank g and h respectively and $\mathcal{T}^\lambda$ for the objects forming the homogenous tubes ($\lambda \in k \setminus \{0\}$).

On the AR-quiver as described above, one imposes the mesh relations for any diamond (quadrilateral formed by four neighbours) in it and for all the triangles appearing at mouths of tubes, as indicated by the dotted lines.

Figure 2 below illustrates this AR-quiver.

Brüstle then truncates $\mathrm{AR}(\mathrm{mod}\,kQ_{g,h})$ to define quivers Q_m, for $m \geq 1$. This is obtained by cutting the preprojective component Q^P at the vertices (ghm, i), so that the resulting full subquiver Q_m^P of Q^P has $ghm + 1$ slices, starting from the vertices of the projective indecomposables, (see Section 2 of [4]). Similarly, Q^I is cut in order to get a full quiver Q_m^I of Q^I, containing $ghm + 1$ slices, ending at the injectives, (see Section 3 of [4]). The regular tubes are cut to form quivers that have cylindrical shapes with a cone on the top: Q_m^0 denotes the full subquiver with mouth given by the $(0, i)_g$, $0 \leq i \leq g$, and top row the single vertex $(gm + g, g)_g$. Similarly, Q_m^∞ is the full subquiver with mouth containing the $(0, i)_h$, $0 \leq i \leq h$, and top vertex $(hm + h, h)$. The components Q^λ are cut to get the full subquivers Q_m^λ on the $m + 1$ vertices $(\lambda, 1), \ldots, (\lambda, m + 2)$. For the tubes see Section 4 of [4].

Furthermore, let $\mathcal{P}_m$ be the full subcategory of $\mathcal{P}$ on the objects corresponding to the vertices of Q_m^P define the subcategories $\mathcal{I}_m$, $\mathcal{T}_m^0$, $\mathcal{T}_m^\infty$ and T_m^λ similarly. Brüstle proves the following statement in [4, Sections 2,3,4]. Here, $\mathcal{C}(kQ_m^*)$ is used to denoted the k-category associated to Q_m^*, modulo the relations of this quiver (as in [7, §2.1, Example 6]).

Proposition 3.1 *There are isomorphisms between the k-categories $\mathcal{C}(kQ_m^*)$, $* \in \{P, I, 0, \infty\}$ or $* = \lambda \in k \setminus \{0\}$, defined by Q_m^* and the corresponding categories $\mathcal{P}_m$, $\mathcal{I}_m$, $\mathcal{T}_m^0$, $\mathcal{T}_m^\infty$, $\mathcal{T}_m^\lambda$.*

To describe the AR-quiver of a truncated version of $\mathrm{mod}\,\tilde{A}$, we then have to define Q_m from the truncated pieces of correct size (cf. Section 5 in [4]):

Definition 3.2 For $m \geq 1$ let Q_m be the quiver whose vertices are the vertices of Q_m^P, of Q_m^I and of $Q_{2m(n+1)}^\lambda$ for $\lambda \in k \cup \{\infty\}$, the arrows are all the arrows of these quivers with additional "connecting" arrows $\iota_0(x)$, $\kappa_0(x)$ for $x = 0, \ldots, g$, $\iota_\infty(y)$, $\kappa_\infty(y)$ for $y = 0, g, g + 1, \ldots, n$ and ι_λ, κ_λ for $\lambda \in k \setminus \{0\}$.

For illustration we include a picture of Q_m for the running example ($g = 3$, $h = 1$), see Fig. 2. It is taken from [4].

Definition 3.3 Let $m \geq 1$. By $\mathcal{J}_m Q_{g,h}$ we denote the full subcategory of $\mathrm{mod}\,\tilde{A}$ whose indecomposable objects are the union of the objects of $\mathcal{P}_m$, $\mathcal{I}_m$, $\mathcal{T}_{2m(n+1)}^\sigma$, $\sigma \in k \cup \{\infty\}$.

Note that for every object M of $\mathrm{mod}\,\tilde{A}$ there exists $m \geq 1$ such that M is isomorphic to a direct sum of modules from $\mathcal{J}_m Q_{g,h}$ and hence the categories $\mathcal{J}_m Q_{g,h}$ can be used to describe $\mathrm{mod}\,kQ_{g,h}$.

We need a few abbreviations to be able to write the relations in Theorem 3.4:
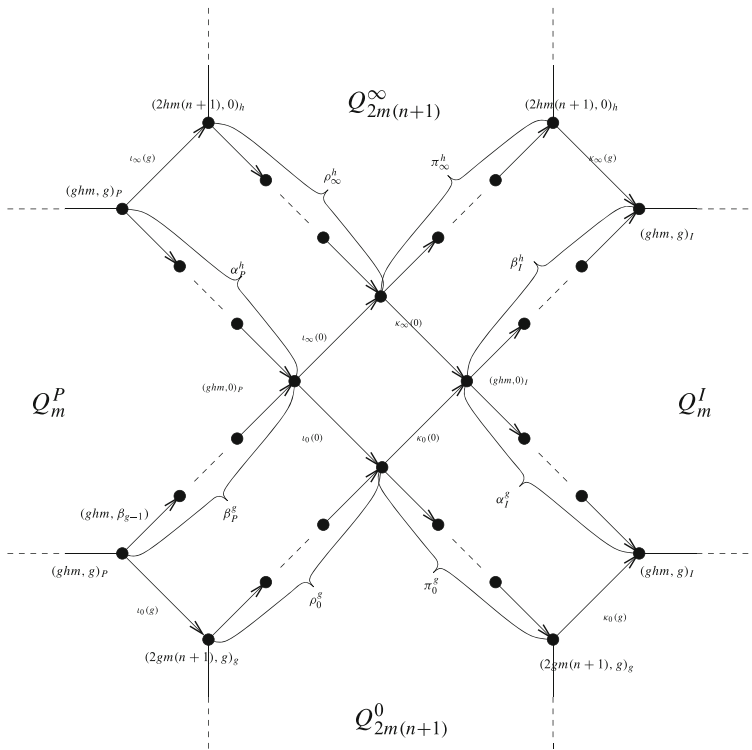
β_P^g denotes the path of length g from $(ghm, g)_P$ to $(ghm, 0)_P$ composed by the arrows $(ghm, \beta_i)_P$, $i = g - 1, \ldots, 1, 0$.

α_P^h denotes the path of length h from $(ghm, g)_P$ to $(ghm, 0)_P$ composed by the arrows $(ghm, \alpha_i)_P$, $i = g, g + 1, \ldots, n$.

The definitions of α_I^g, β_I^g, π_0^g, ρ_0^g, π_∞^h and ρ_∞^h can be understood from the picture above. Finally, we write

$\varepsilon_\lambda = \rho_\lambda(2m(n + 1) + 1)\pi_\lambda(2m(n + 1) + 1)$ for $\lambda \in k \setminus \{0\}$.

Theorem 3.4 (Hauptsatz) *Let $\mathcal{C}(kQ_m)$ be the k-category of Q_m subject to the following relations*

(a) $\gamma'\gamma = \delta'\delta$ for all arrows γ, γ', δ, δ' of Q_m in a diamond from X to Z with $X \neq (ghm, 0)_P$.

(b) $\pi_\lambda\rho_\lambda = 0$ for all $\lambda \in k \cup \{\infty\}$ and all arrows π_λ, ρ_λ of $Q_{2m(n+1)}^\lambda$ of the form $X^* \xrightarrow{\rho_\lambda} Y \xrightarrow{\pi_\lambda} Z^*$ for X^* and Z^* vertices at the mouth of a tube (possibly, $X^* = Z^*$).

(c1) $\iota_0(g) = \pi_0^g\iota_0(0)\alpha_P^h$ $\iota_\infty(g) = \pi_\infty^h\iota_\infty(0)\beta_P^g$.

(c2) $\kappa_0(g) = \beta_I^h\kappa_0(0)\rho_0^g$ $\kappa_\infty(g) = \alpha_I^g\kappa_\infty(0)\rho_\infty^h$.

(d) $\iota_\lambda\alpha_P^h = \lambda\iota_\lambda\beta_P^g + \varepsilon_\lambda\iota_\lambda\beta_P^g$ $\alpha_I^h\kappa_\lambda = \lambda\beta_I^g\kappa_\lambda + \beta_I^g\kappa_\lambda\varepsilon_\lambda$.

(e) $\kappa_0(0)\iota_0(0)(ghm, \alpha_n)_P = 0$ $(ghm, \alpha_n)_I\kappa_0(0)\iota_0(0) = 0$.

(f) $\kappa_\infty(0)(\rho_\infty^h\pi_\infty^h)^j\iota_\infty(0) = \kappa_0(0)(\rho_0^g\pi_0^g)^{2m(n+1)+1-j}\iota_0(0)$.

(g) $\kappa_\lambda\varepsilon_\lambda^j\iota_\lambda = \sum_{i=0}^{j}\binom{2m(n+1)+1-j+i}{2m(n+1)+1-j}\lambda^i\kappa_0(0)(\rho_0^g\pi_0^g)^{j-i}\iota_0(0)$

with $\lambda \in k \setminus \{0\}$ and $j = 0, \ldots, 2m(n+1) + 1$.
Then there is an isomorphism

$$\mathcal{C}(kQ_m) \to \mathcal{J}_m Q_{g,h}$$

Remark 3.5 We observe that the relations also imply

$$(e') \; \kappa_\infty(0)\iota_\infty(0)(ghm, \beta_0)_P = 0 \quad (ghm, \beta_0)_I \kappa_\infty(0)\iota_\infty(0) = 0$$

To see this, one uses (e) and (f) (with $j = 1$) and the diamond relations from (a) to push the path all the way down to the mouth of the tube, where it will include a triangle as in (b), hence the zero relation.

4 A Translation Quiver on Arcs in the Annulus

We now describe the set-up of the geometric model we use to describe mod $\tilde{A}$ and later for the cluster category $\mathcal{C}_{\tilde{A}}$. It is similar to the one appearing in [11] and in [1]. Our main focus is on the interpretation of the infinite radical of the module category resp. the cluster category in type $\tilde{A}$, in terms of the geometry.

Let $P_{g,h}$ be an annulus with g marked points on the outer boundary ∂ and h marked points on the inner boundary ∂', $gh \neq 0$, let $g \geq h$. .

We assume that the marked points are distributed in equidistance on the two boundaries. We will identify $P_{g,h}$ with a cylinder $\mathrm{Cyl}_{g,h}$ of height 1 with g marked points on the lower boundary and h marked points on the upper boundary. We can view this cylinder as a rectangle of height 1 and width gh in $\mathbb{R}^2$, identifying its two vertical sides. We will always label the marked points of $\mathrm{Cyl}_{g,h}$ from left to right on the lower boundary and from right to left on the upper boundary.

On the lower boundary (the outer boundary of the annulus), we choose the points $(0, 0)$, $(h, 0)$, $\ldots$, $(gh - h, 0)$ and $(gh, 0)$ as marked points. On the upper boundary (the inner boundary of the annulus), we choose the points $(0, 1)$, $(g, 1)$, $\ldots$, $(gh - g, 1)$ and $(gh, 1)$ to be marked points, as in Fig. 3.

With the annulus in mind, we denote the points on the lower boundary by i_∂, for $0 \leq i \leq g - 1$:

$$0_\partial := (0, 0) = (gh, 0), \; 1_\partial = (h, 1), \ldots, (g - 1)_\partial := (hg - h, 0)$$
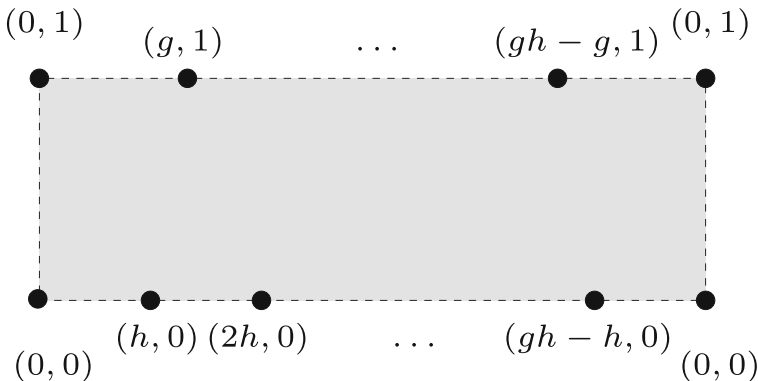
Fig. 3 Annulus via rectangle $\mathrm{Cyl}_{g,h}$

and the points on the upper boundary by $j_{\partial'}$, for $0 \leq j \leq h-1$:

$$0_{\partial'} := (gh, 1) = (0, 1), 1_{\partial'} := (g, 1), \ldots, (h-1)_{\partial'} := (g(h-1), 1),$$

We thus label marked points on both boundaries from the left to the right, by $\mathbb{Z}_b$ with $b \in \{\partial, \partial'\}$.

For our purposes, it will be most convenient to work in the universal cover $\mathbb{U} = (\mathbb{U}, \pi_{gh})$ of $\text{Cyl}_{g,h}$ with $\mathbb{U} = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1\}$ an infinite strip in the plane. It inherits the orientation from its embedding in $\mathbb{R}^2$. The covering map $\pi_{gh} : \mathbb{U} \to \text{Cyl}_{g,h}$ is induced from wrapping $\mathbb{U}$ around $\text{Cyl}_{g,h}$, it takes the first entry of $(x, y) \in \mathbb{U}$ modulo gh:

$$\pi := \pi_{gh} : \mathbb{U} \to \text{Cyl}_{g,h}, \qquad (x, y) \mapsto (\overline{x}, y)$$

where $\overline{x}$ denotes the class of x modulo gh. As we can identify the annulus $P_{g,h}$ with $\text{Cyl}_{g,h}$, π is also a covering map of $P_{g,h}$.

We take as marked points on the lower boundary the points $\{(hx, 0) \mid x \in \mathbb{Z}\}$ and as marked points on the upper boundary of $\mathbb{U}$ the points $\{(gx, 1) \mid x \in \mathbb{Z}\}$, see Fig. 4.

When working in the universal cover, it is most convenient to use integers to denote marked points on the lower and upper boundary. We will write subscripts to indicate the boundary on which the points sit. So for $i, j \in \mathbb{Z}$, i_∂ is a marked point on the lower boundary and $j_{\partial'}$ a marked point on the upper boundary. We do this in such a way that the copy of $\text{Cyl}_{g,h}$ with vertices $(0, 0)$ and $(gh, 0)$ obtains the labels $0_\partial, 1_\partial, \ldots, g_\partial$ on the lower boundary and the labels $0_{\partial'}, 1_{\partial'}, \ldots, h_{\partial'}$ on the upper boundary, as shown for the case $g = 3, h = 2$ in Fig. 4. More generally, for fixed g, h, with $g \geq h \geq 1$, we endow $\mathbb{U}$ with the following set of marked points

$$\{i_\partial = (ih, 0) \mid i \in \mathbb{Z}\}$$
$$\{j_{\partial'} = (jg, 1) \mid j \in \mathbb{Z}\}$$

placing $0_{\partial'}$ above 0_∂, $h_{\partial'}$ above g_∂ etc.

4.1 Arcs in $P_{g,h}$ and in the Universal Cover

We want to consider certain oriented arcs between marked points of the annulus $P_{g,h}$ to define a stable translation quiver Γ. We recall that our goal is to show that Γ is isomorphic to the AR-quiver of the module category of type $\tilde{A}$, with underlying quiver $Q_{g,h}$ and to describe the infinite radical in terms of moves between arcs. Passing to unoriented arcs, we will then obtain a quiver which is isomorphic to the AR-quiver of the cluster category of type $\tilde{A}$ of the same underlying quiver $Q_{g,h}$.
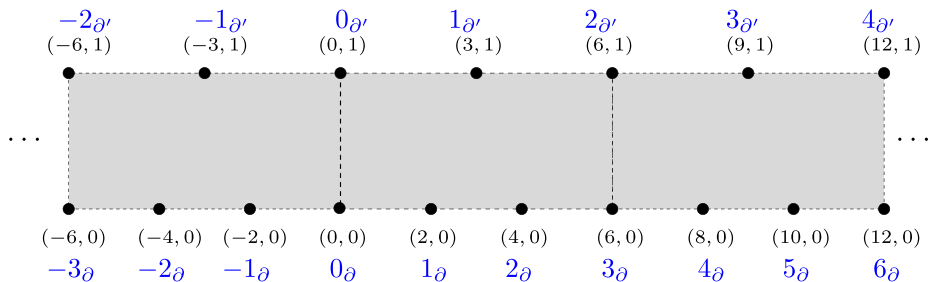

Fig. 4 Universal cover for $g = 3, h = 2$

We will call the arrows in this quiver *moves*. There are two types of moves, namely *elementary moves* and *long moves*. We describe these moves below. In the oriented case, elementary moves between arcs of an annulus have appeared in [11] and [1] independently to describe (components of) module categories of type $\tilde{A}$. In the unoriented case, elementary moves appeared in [5] for the cluster category of a marked surface, in [8] for tubes and in [10] for the m-cluster category of an annulus. To our knowledge, long moves of (oriented/unoriented) arcs have not been considered so far.

To define an oriented arc of $\mathbb{U}$, we need to give its start and end point as well as its winding number around the inner boundary. A convenient way to define such arcs is to work in the universal cover instead.

We need to recall a few facts before we can do this, cf. [1, §2.2] in the case with marked points on one boundary. Let $\sigma : \mathbb{U} \to \mathbb{U}$ be the translation $(x, y) \mapsto (x + gh, y)$, with inverse $\sigma^{-1}(x, y) = (x - gh, y)$. The group $G = \langle \sigma \rangle$ acts naturally on $\mathbb{U}$. Note that in terms of z_∂ and $z_{\partial'}$, σ acts on vertices of the lower boundary as $z_\partial \mapsto (z + g)_\partial$ and on the upper boundary as $z_{\partial'} \mapsto (z + h)_{\partial'}$.

An *oriented arc* in $\mathbb{U}$ is an isotopy class of arcs joining marked points of the boundary of $P_{g,h}$. If an arc starts at a marked point x_{b_1} and ends at a marked point y_{b_2} with $x \in \mathbb{Z}$, $b_i \in \{\partial, \partial'\}$, we write $[x_{b_1}, y_{b_2}]$ for this arc (defined only up to isotopy fixing endpoints).

We will distinguish two main types of arcs in $\mathbb{U}$ and in $P_{g,h}$.

Definition 4.1 Let $\alpha = [x_{b_1}, y_{b_2}]$ be an arc in $\mathbb{U}$. If $b_1 \neq b_2$, then we say that α is a *bridging arc*.
If $b_1 = b_2$ and $x \leq y - 2$, we say that α is a *peripheral arc* and that α is *based at b_1*.

We do not consider arcs in the remaining cases.

Definition 4.2 A *bridging* arc in $P_{g,h}$ is an arc $\pi_{g,h}(\alpha)$ where α is a bridging arc of $\mathbb{U}$. Similarly, a *peripheral* arc of $P_{g,h}$ is an arc $\pi_{g,h}(\alpha)$ for α a peripheral arc of $\mathbb{U}$.

Remark 4.3 If α is an arc of $P_{g,h}$ starting at ∂ (at ∂') it has a unique lift $\tilde{\alpha}$ in $\mathbb{U}$ with starting point in $\{0_\partial, 1_\partial, \ldots, (g-1)_\partial\}$ (with starting point $\{0_{\partial'}, 1_{\partial'}, \ldots, (h-1)_{\partial'}\}$).
All other lifts of α (of β) can be obtained by applying σ or σ^{-1} repeatedly to $\tilde{\alpha}$, hence $\pi_{g,h}^{-1}(\alpha) = G\tilde{\alpha}$.

Bridging arcs will serve as models for the preprojective and preinjective components, peripheral arcs correspond to objects of the tubes $\mathcal{T}_g, \mathcal{T}_h$, as we will explain in Sections 4.3 and 4.4.

4.2 Elementary Moves

The geometric idea is that a move rotates an arc clockwise around one of its endpoints. For elementary moves, this is a "shortest" clockwise rotation. We keep the idea of rotating in mind for the long moves.

Let $g \geq h > 0$ be fixed, let $\pi = \pi_{g,h} : \mathbb{U} \to P_{g,h}$ be the covering map. Let α be an arc of $P_{g,h}$ and let $[x_{b_1}, y_{b_2}]$ be a lift of α, with $b_1, b_2 \in \{\partial, \partial'\}$. Since an elementary move gives rise to an irreducible map in the module category, it should be a minimal rotation: it fixes one endpoint of an arc and moves one endpoint by ± 1 (depending on the boundary this sits on).

So there are at most two elementary moves from α, namely to the arcs $\pi([x_{b_1}, (y\pm 1)_{b_2}])$ and $\pi([(x+1)_{b_1}, y_{b_2}])$, where ± 1 is -1 on the boundary ∂ and $+1$ on ∂'.

Note that when drawing vertices and moves between them, we will always use the convention that arrows (moves) go to the right. Furthermore, we will draw moves fixing the starting point, i.e. the first entry of an arc in $\mathbb{U}$ as pointing down and moves fixing the endpoint, i.e. the second entry of the arc in $\mathbb{U}$ will be pointing up.

We will sometimes also need to work with moves between arcs in $\mathbb{U}$ itself, they are defined analogously.

Observe that if $b_1 = b_2$, one of the images might be isotopic to a segment of the boundary. In this case, there is only one elementary move from α.

On the other hand, if an arc corresponds to an injective indecomposable, it should only have an elementary move to another arc for an injective indecomposable or no elementary move, if α is the terminal object, i.e. if α corresponds to the 1-dimensional injective module.

We use lifts to describe the elementary moves case-by-case.

- Bridging arcs: Let α be a bridging arc of $P_{g,h}$. Then we define two elementary moves starting at α.

  (i) If the lift of α in $\mathbb{U}$ is $[i_\partial, y_{\partial'}]$ for some $i, y \in \mathbb{Z}$, they are of the form

$$\alpha = \pi([i_\partial, y_{\partial'}]) \quad \longrightarrow \quad \begin{array}{l} \pi([(i-1)_\partial, y_{\partial'}]) \\ \\ \pi([i_\partial, (y+1)_{\partial'}]) \end{array}$$
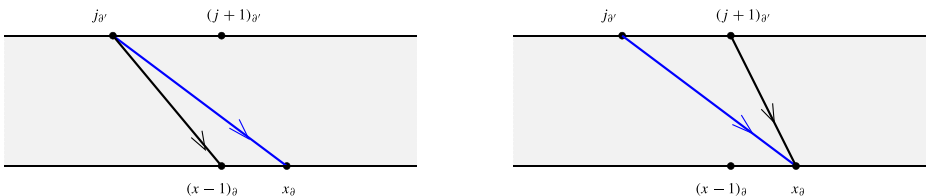
The corresponding picture in $\mathbb{U}$ is:
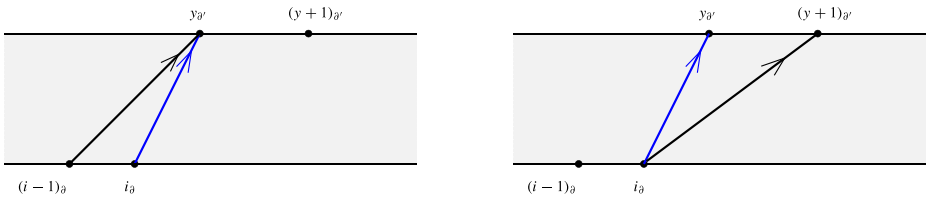
  (ii) If the lift of α in $\mathbb{U}$ is $[j_{\partial'}, x_\partial]$ with $j, x \in \mathbb{Z}$, they are of the form

$$\alpha = \pi([j_{\partial'}, x_\partial]) \quad \longrightarrow \quad \begin{array}{l} \pi([(j+1)_{\partial'}, x_\partial]) \\ \\ \pi([j_{\partial'}, (x-1)_\partial]) \end{array}$$

In pictures in $\mathbb{U}$:

- Peripheral arcs: Let α be a peripheral arc of $P_{g,h}$.

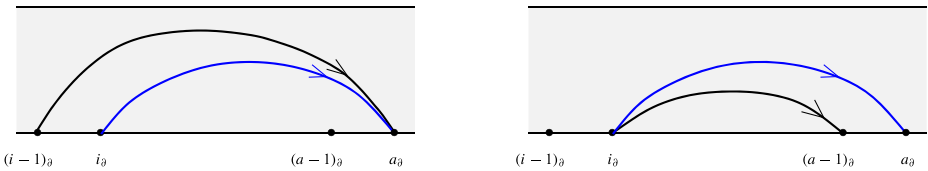(i) Assume that the lift of α is $[i_\partial, a_\partial]$ with $i, a \in \mathbb{Z}$.

  – If $a = i + 2$, there is only one elementary move from α:

$$\alpha = \pi([i_\partial, (i+2)_\partial]) \qquad \nearrow \quad \pi([(i-1)_\partial, (i+2)_\partial])$$

  – If $a > i + 2$, there are two elementary moves from α:

$$\alpha = \pi([i_\partial, a_\partial]) \quad \begin{array}{l} \nearrow \pi([(i-1)_\partial, a_\partial]) \\ \searrow \pi([i_\partial, (a-1)_\partial]) \end{array}$$

In $\mathbb{U}$, the case $a > i + 2$ looks as follows:

$(i-1)_\partial \qquad i_\partial \qquad\qquad (a-1)_\partial \quad a_\partial \qquad\qquad (i-1)_\partial \qquad i_\partial \qquad\qquad (a-1)_\partial \quad a_\partial$

(ii) Let $[j_{\partial'}, b_{\partial'}]$ be a lift of α, with $b, j \in \mathbb{Z}$.

  – If $b = j + 2$, there is one elementary move from α:

$$\alpha = \pi([j_\partial, (j+2)_\partial]) \qquad \searrow \quad \pi([j_\partial, (j+3)_\partial])$$

  – If $b > j + 2$, there are two elementary moves from α:

$$\beta = \pi([j_{\partial'}, b_{\partial'}]) \quad \begin{array}{l} \nearrow \pi([(j+1)_{\partial'}, b_{\partial'}]) \\ \searrow \pi([j_{\partial'}, (b+1)_{\partial'}]) \end{array}$$

For $b > j + 2$, we have the following picture in $\mathbb{U}$:

$b_{\partial'} \quad (b+1)_{\partial'} \qquad\quad j_{\partial'} \quad (j+1)_{\partial'} \qquad\qquad b_{\partial'} \quad (b+1)_{\partial'} \qquad\quad j_{\partial'} \quad (j+1)_{\partial'}$
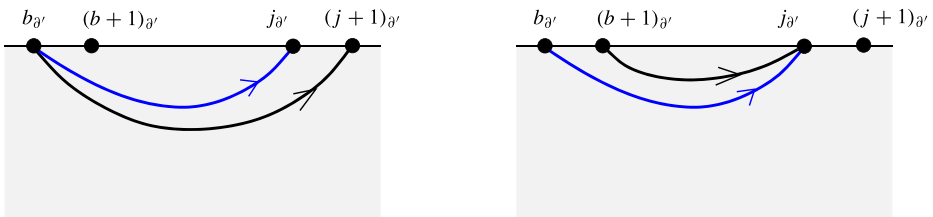
4.3 Preprojective Arcs

Recall that the quiver $Q_{g,h}$ has its source at vertex 0, its sink at vertex g (cf. Section 2.1). Hence, among the projective indecomposable modules, P_g is one-dimensional, with irreducible morphisms to P_{g-1} and to P_{g+1}. Also, P_0 is the largest projective indecomposable

module, with irreducible morphisms from P_1 and from P_n. The preprojective component of $\mod k Q_{g,h}$ starts with the projective indecomposable modules (cf. Figure 1 for $g = 3$, $h = 2$). All other indecomposable modules in $\mathcal{P}$ are reached through compositions of irreducible maps from the P_i.

We now describe a subset of the bridging arcs from ∂ to ∂' that will play the role of projective indecomposable modules.

Let

$$\beta_i := \begin{cases} \pi([(i-g)_\partial, 0_{\partial'}]) & i = 0, 1, \ldots, g-1, g \\ \pi([0_\partial, (i-g)_{\partial'}]) & i = g+1, g+2, \ldots, n \end{cases}$$

Definition 4.4 The arcs $\beta_0, \ldots, \beta_n$ of $P_{g,h}$ are called *projective arcs*. Figure 5 displays the β_i. We call an arc $\alpha = [x_\partial, y_{\partial'}]$ of $P_{g,h}$ *preprojective* if it can be reached from β_g through a finite (possibly empty) sequence of elementary moves. In particular, the β_i $(i = 0, \ldots, n)$ are preprojective arcs.

Roughly speaking, the preprojective arcs of $P_{g,h}$ are winding positively from the outer to the inner boundary, with initial object β_g.

4.4 Preinjective Arcs

Among the injective indecomposable modules of $\mod k Q_{g,h}$, the module I_0 is the one-dimensional one, with irreducible morphisms from I_n and from I_1, whereas I_g is the largest one, with irreducible morphisms to I_{g+1} and to I_{g-1} (cf. Figure 1 for $g = 3$, $h = 2$). All other indecomposable modules in $\mathcal{I}$ map to the injectives through compositions of irreducible maps.

We now describe a subset of the bridging arcs from ∂' to ∂ for the injective indecomposable modules. Let

$$\gamma_i := \begin{cases} \pi([-2_{\partial'}, (i-g+2)_\partial]) & i = 0, 1, \ldots, g \\ \pi([(i-g-2)_{\partial'}, 2_\partial]) & i = g+1, g+2, \ldots, n \end{cases}$$

Definition 4.5 The arcs $\gamma_0, \ldots, \gamma_n$ of $P_{g,h}$ are called *injective arcs*. Figure 6 displays the γ_i. We call an arc $\alpha = [x_{\partial'}, y_\partial]$ of $P_{g,h}$ *preinjective* if γ_0 is reached from α through a finite sequence of elementary moves. In particular, the γ_i $(i = 0, \ldots, n)$ are preinjective arcs.

Up to rotating the inner boundary by $2\frac{2\pi}{h}$, the preinjective arcs are winding positively from the inner to the outer boundary towards the terminal object γ_0.

Definition 4.6 An oriented arc of $P_{g,h}$ is called *admissible*, if it is either preprojective, peripheral or preinjective.

4.5 A Translation Quiver

With the definitions from the last sections, we are ready to define a translation quiver structure on arcs in $P_{g,h}$. The vertices of $\Gamma = \Gamma(P_{g,h})$ are the admissible arcs of $P_{g,h}$. The arrows of Γ are given by the elementary moves. Furthermore, let τ be induced by the map $i_\partial \mapsto (i+1)_\partial$ and $j_{\partial'} \mapsto (i-1)_{\partial'}$.
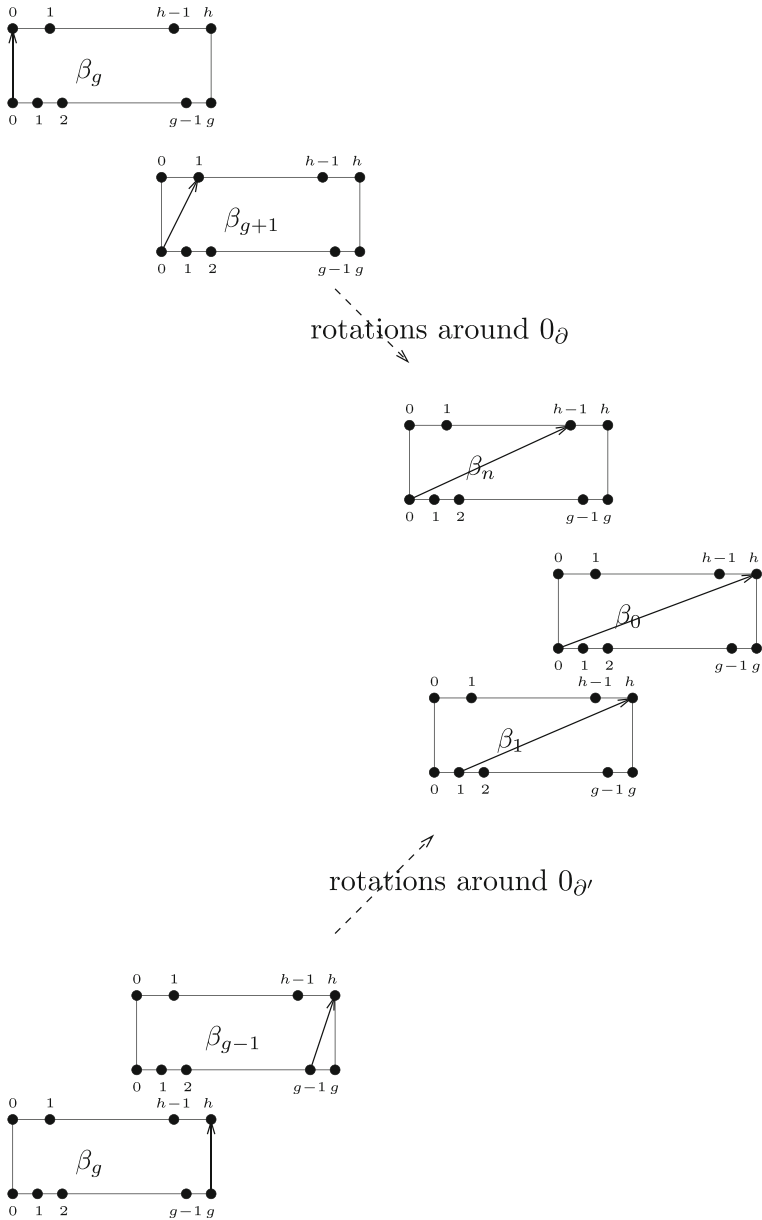
Fig. 5 The projective arcs

In terms of $\mathbb{U}$, τ shifts endpoints on ∂ to the right and endpoints on ∂' to the left. Peripheral arcs are thus shifted to the right/to the left, whereas the endpoints of bridging arcs of Γ move in different directions under τ.

The translation τ on Γ makes (Γ, τ) a translation quiver. This means that whenever we have two vertices $\tau(\alpha)$, $\alpha \in \Gamma$, there is a mesh relation involving the subgraph on these two
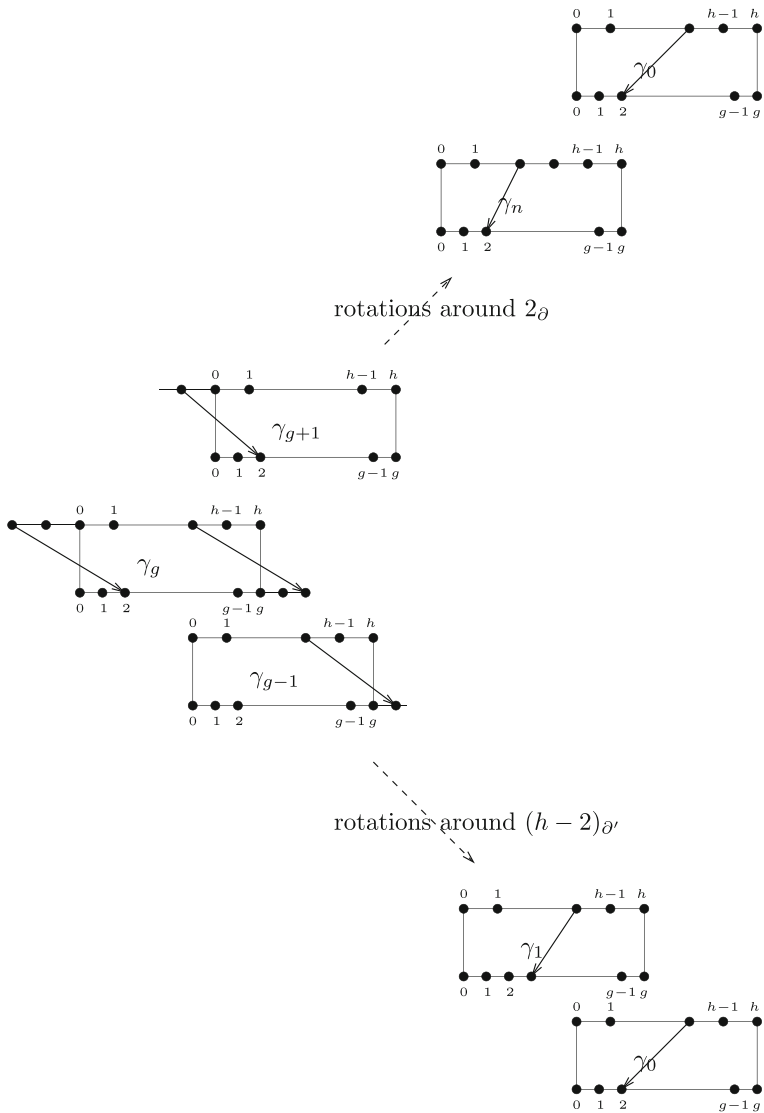
Fig. 6 The injective arcs

vertices and their common neighbours. In our case, these meshes have either four or three vertices:
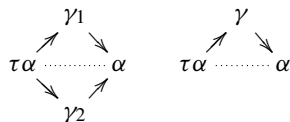
The meshes on three vertices only occur at the mouth of tubes formed by peripheral arcs. As relations on the quiver (Γ, τ), we impose on meshes with four vertices that the two paths between $\tau(\alpha)$ and α are equal and on meshes with three vertices, that the composition of the two arrows is zero.

The following result is well-known, it appears in [11] for the module category and in [1] for tubes.

Proposition 4.7 (Γ, τ) *is a translation quiver,* (Γ, τ) *is isomorphic to the AR-quiver of* $\mod k Q_{g,h}$.

By the above result, (Γ, τ) describes $\mod k Q_{g,h}$ up to maps in the infinite radical.

Remark 4.8 (1) Observe that by definition, $\tau(\beta_i), 0 \le i \le n$ is not admissible and neither are the $\tau^{-1}(\gamma_i)$.

(2) However, if we forget the orientations of arcs, there is a way to go from β_i to γ_i via τ^2 or back via τ^{-2} as follows: Let $0 \le i \le n$. Then γ_i can be obtained from β_i by applying τ^2 to the arc (the result is not admissible, but bridging) and reversing the orientation of the image. I.e. if β_i has lift $[a_\partial, b_{\partial'}]$ in $\mathbb{U}$, then the arc $[\tau^2(b_{\partial'}), \tau^2(a_\partial)]$ is a lift of γ_i.

5 The Infinite Radical

In this section, we describe the elements of the infinite radical of $\mod k Q_{g,h}$. We will do this using the geometric interpretation of objects in terms of admissible arcs. With this, we will obtain a quiver $\overline{\Gamma} = \overline{\Gamma}(P_{g,h})$ from $\Gamma = \Gamma(P_{g,h})$. The vertices of $\overline{\Gamma}$ are the vertices of Γ, the arrows of $\overline{\Gamma}$ are the arrows of Γ together with the arrows arising from elements of the infinite radical (corresponding to the "long moves", see below).

Elementary moves are combinatorial models for irreducible morphisms, i.e. *correspond to nonzero* elements of $\mathrm{rad}(X, Y)/\mathrm{rad}^2(X, Y)$ for X, Y the two indecomposable objects corresponding to the starting and end point of α. We have seen that an elementary move sends an arc α to another arc α' with a common endpoint by rotating the arc α clockwise around the common endpoint. The other endpoint is moved by ± 1 along the corresponding boundary.

Long moves are models for maps in the infinite radical of $\mod \tilde{A}$, i.e. for elements of $\mathrm{rad}^\infty(\mod \tilde{A}) = \cap_i \mathrm{rad}^i(\mod \tilde{A})$. In terms of the AR-quiver, maps in the infinite radical arise from infinitely many compositions of irreducible morphisms, keeping the direction fixed, i.e. moving along a ray or coray in the quiver $\Gamma_{g,h}$.
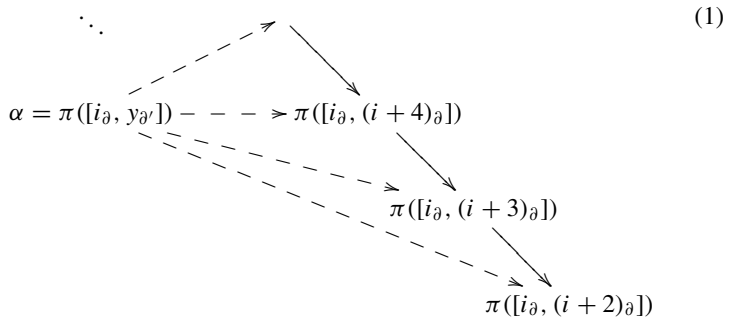
5.1 Long Moves

Long moves *correspond to nonzero* elements of *of the vector space* $\mathrm{rad}^\infty(X, Y)/(\mathrm{rad}^\infty (X, Y))^2$ (for X, Y indecomposable). In our model, we view them as limits of sequences of elementary moves, fixing one common vertex for all the elementary moves involved. This way, long moves follow a line in the quiver $\Gamma(P_{g,h})$.

Geometrically, this is achieved by rotating an arc α clockwise to an arc α' which has one endpoint in common with α and where the other endpoint has changed boundary components. The endpoint switching boundary components is allowed to go to any marked point

on the new boundary, as long as the image is an admissible arc (preprojective, peripheral or preinjective).

In particular, there are infinitely many long moves from α. This is now described case-by-case. We will indicate the long moves by drawing dashed arrows. Let $g \geq h > 0$ be fixed, let $\pi = \pi_{g,h} : \mathbb{U} \to P_{g,h}$ be the covering map.

- Bridging arcs: Let α be a preprojective arc, let $[i_\partial, y_{\partial'}]$ be a lift of α.

    - There are long moves from α to every vertex of the coray of $\pi([i_\partial, i + 2_\partial])$ in the tube $\mathcal{T}_g$, i.e. to every arc $\pi([i_\partial, (i + k)_\partial])$ for $k \geq 2$.

$$\begin{array}{c} \tag{1} \end{array}$$

Geometrically this means that from every preprojective arc starting at i_∂, there are long moves to every peripheral arc starting at i_∂.

    - There are long moves from α to every vertex of the coray of $\pi([(y - 2)_{\partial'}, y_{\partial'}]$ in the tube $\mathcal{T}_h$, i.e. to every arc $[(y - k)_{\partial'}, y_{\partial'}]$ with $k \geq 2$. (Recall that the quiver of $\mathcal{T}_h$ is drawn upside down, the vertex $\pi([(y - 2)_{\partial'}, y_{\partial'}])$ sits at the mouth of this component).

$$\begin{array}{c} \tag{2} \end{array}$$
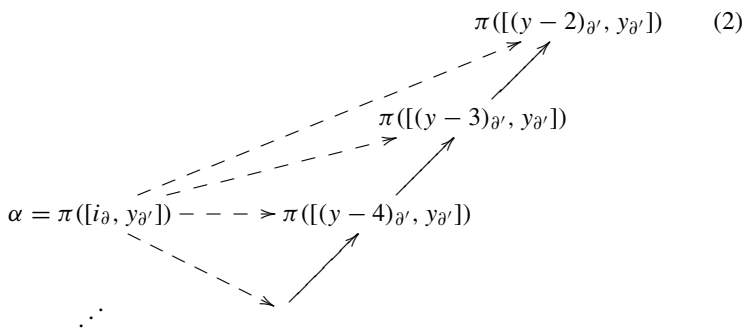
Geometrically this means that from every preprojective arc ending at $y_{\partial'}$, there are long moves to every peripheral arc ending at y_∂.

- Peripheral arcs:

– Let α be peripheral and based at ∂, let $[a_\partial, y_\partial]$ be a lift of α. Then we define a long move to every preinjective arc $\pi([x_{\partial'}, y_\partial])$ of $P_{g,h}$.

(3)
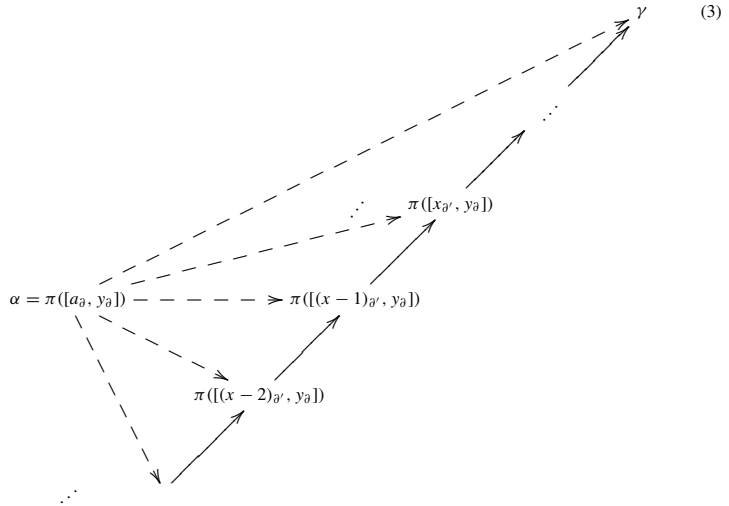
Note that the terminal object of this sequence is one of the injective arcs $\gamma_0, \dots, \gamma_{g-1}$. More precisely, if we assume that the lift of α satisfies $-2 \leq y < g-2$, then the terminal object is $\pi[2_{\partial'}, y_\partial] = \gamma_0$ if $y = -2$ and $\gamma = \gamma_{y+2}$ otherwise. Geometrically the description above means that from every peripheral arc ending at y_∂, there are long moves to every admissible bridging arc ending at y_∂.

– Let α be a peripheral arc based at ∂' and let $[x_{\partial'}, b_{\partial'}]$ a lift of α. Then we define a long move from α to every arc $\pi([x_{\partial'}, z_\partial])$.

(4)
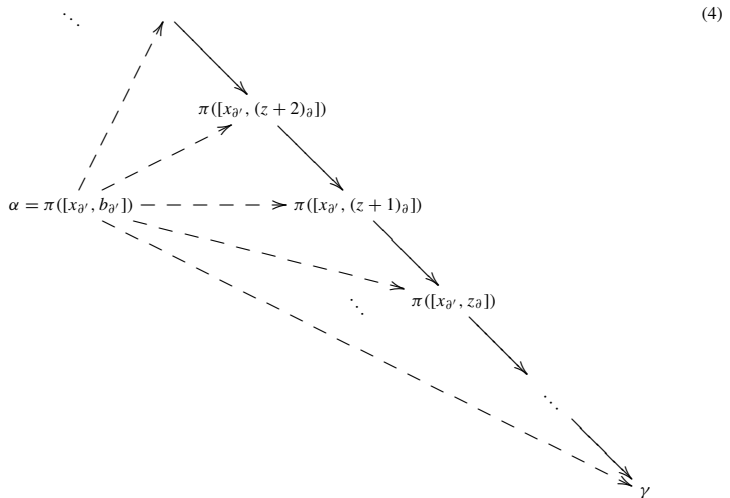
Again, the terminal object of this sequence is an injective arc, it is γ_0 or one of the arcs $\gamma_{g+1}, \dots, \gamma_n$. If we assume that the lift of α satisfies $0 \leq x < -h-1$, then this injective arc is $\pi[x_{\partial'}, -2] = \gamma_0$ if $x = 2$ and $\gamma = \gamma_{x+g-2}$ otherwise.

Geometrically the above description means that from every peripheral arc starting at $x_{\partial'}$, there are long moves to every admissible bridging arc starting at $x_{\partial'}$.

*Remark 5.1* Note that unlike Γ, the quiver $\overline{\Gamma}$ is connected.

5.2 Preimages Under Long Moves

Now we make a few observations which follow from the definitions of the long moves above. We will need this when we define relations in the Section 7.

Consider a peripheral arc $\pi([i_\partial, (i + k)_\partial])$ with $k \geq 2$. Then for every $y_{\partial'}$ such that $\pi([i_\partial, y_{\partial'}])$ is a preprojective arc there exists a long move $\pi([i_\partial, y_{\partial'}]) \to \pi([i_\partial, (i + k)_\partial])$. Combining this with elementary moves between preprojective arcs starting at i_∂, we have the following picture:

(5)
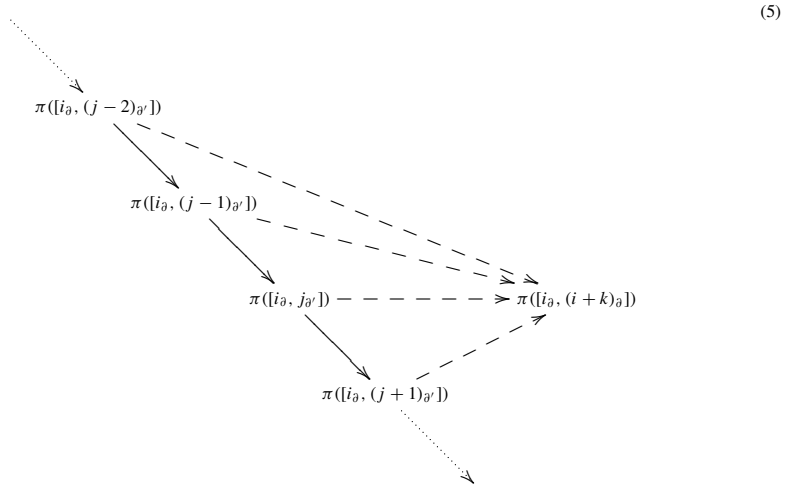
Similarly, if we consider long moves from $\mathcal{P}$ to the tube of peripheral arcs at the inner boundary: Let $\pi([(y - k)_{\partial'}, y_{\partial'}])$ be a peripheral arc, $k \geq 2$. Then there is a long move $\pi([j_\partial, y_{\partial'}]) \to \pi((y - k)_{\partial'}, y_{\partial'}])$ for every preprojective arc of the form $\pi([j_\partial, y_{\partial'}])$. Combining this with elementary moves between preprojective arcs ending at $y_{\partial'}$, we get
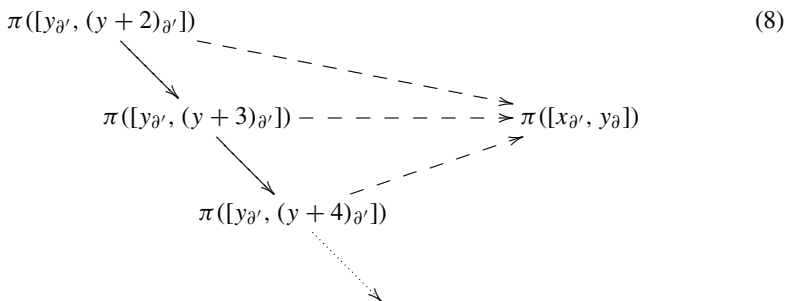
(6)
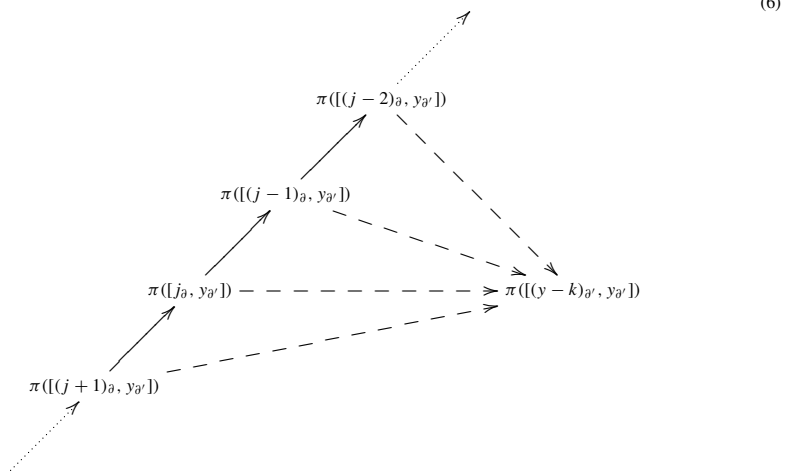
Now we consider long moves from the tubes to $\mathcal{I}$. Let $\pi([x_{\partial'}, y_\partial])$ be an admissible bridging arc in $\mathcal{I}$. Then there is a long move $\pi([(j - k)_\partial, y_\partial]) \to \pi([x_{\partial'}, y_\partial])$ for every $k \geq 2$.

In other words, there are long moves from every arc in the ray of $\pi([(j-2)_\partial, y_\partial])$ to $\pi([x_{\partial'}, y_\partial])$.

(7)
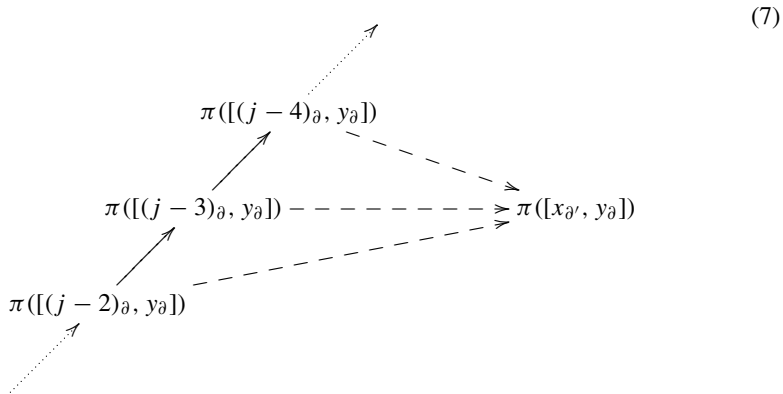
Similarly we get long moves from every arc in the ray of $\pi([y_{\partial'}, (y+2)_{\partial'}])$ to $\pi([x_{\partial'}, y_\partial)]$.

(8)

Remark 5.2 The connected quiver $\overline{\Gamma}$ is **not** locally finite.

5.3 Mesh Relations in $\overline{\Gamma}$

In what follows, when we refer to connected components, we will usually refer to the connected components $\mathcal{P}, \mathcal{I}, \mathcal{T}_g, \mathcal{T}_h$ of Γ.

For $\overline{\Gamma}$, we keep the mesh relations involving the arrows of Γ. Within $\overline{\Gamma}$, there are new diamonds of arrows arising, formed by moves of different kinds. If such a diamond involves vertices from two connected components of Γ, we define a new relation from it. We do not impose any relations on diamonds involving four long moves.
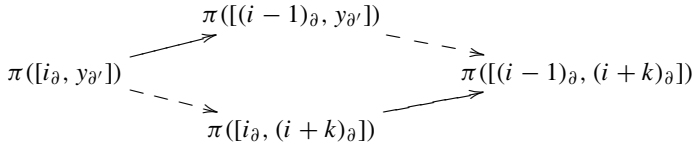
The diagrams giving rise to new relations are of the form
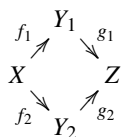
where one pair of opposite arrows come from elementary moves (i.e. f_1 and g_2 or g_1 and f_2) and the two other arrows from long moves.
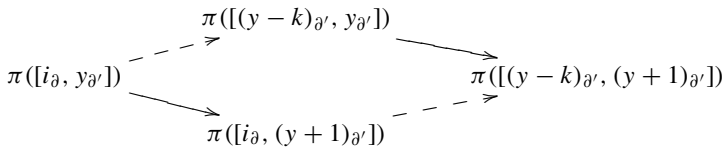
5.4 Diamonds from Preprojective Arcs to Peripheral Arcs

Recall that elementary moves in $\mathcal{P}$ are either of the form $\pi([i_\partial, y_{\partial'}]) \to \pi([(i-1)_\partial, y_{\partial'}])$ or $\pi([i_\partial, y_{\partial'}]) \to \pi([i_\partial, (y+1)_{\partial'}])$. As before, we write dashed arrows to indicate long moves. Combining long moves with elementary moves, there exist diamonds

$$
\begin{array}{ccc}
 & \pi([(i-1)_\partial, y_{\partial'}]) & \\
\pi([i_\partial, y_{\partial'}]) & & \pi([(i-1)_\partial, (i+k)_\partial]) \\
 & \pi([i_\partial, (i+k)_\partial]) &
\end{array}
$$

for all $k \geq 2$. The vertices to the left and above belong to $\mathcal{P}$, the other two vertices are peripheral arcs based at the outer boundary.
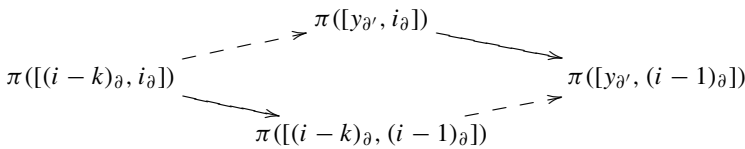
And diamonds

$$
\begin{array}{ccc}
 & \pi([(y-k)_{\partial'}, y_{\partial'}]) & \\
\pi([i_\partial, y_{\partial'}]) & & \pi([(y-k)_{\partial'}, (y+1)_{\partial'}]) \\
 & \pi([i_\partial, (y+1)_{\partial'}]) &
\end{array}
$$

for all $k \geq 2$. Here the left vertex and the one below are preprojective, the other two peripheral and based at the inner boundary.

In $\overline{\Gamma}$ we require that all such diamonds commute. The geometric descriptions of these relations is in Cases A) and B) in Section 7.2.
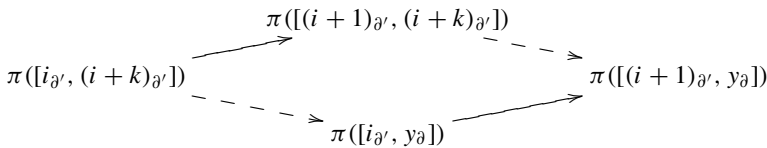
5.5 Diamonds from Peripheral Arcs to Preinjective Arcs

We also have diamonds involving peripheral and preinjective arcs.

$$
\begin{array}{ccc}
 & \pi([y_{\partial'}, i_\partial]) & \\
\pi([(i-k)_\partial, i_\partial]) & & \pi([y_{\partial'}, (i-1)_\partial]) \\
 & \pi([(i-k)_\partial, (i-1)_\partial]) &
\end{array}
$$

for all $k \geq 3$. The first vertex and the one below are peripheral arcs of ∂, two are preinjective arcs.

Also, we have diamonds

$$
\begin{array}{ccc}
 & \pi([(i+1)_{\partial'}, (i+k)_{\partial'}]) & \\
\pi([i_{\partial'}, (i+k)_{\partial'}]) & & \pi([(i+1)_{\partial'}, y_\partial]) \\
 & \pi([i_{\partial'}, y_\partial]) &
\end{array}
$$

for $k \geq 3$. The first vertex and the one on top are peripheral arcs based at ∂', the other two are preinjective arcs.

In $\overline{\Gamma}$ we require that all such diamonds commute. The geometric descriptions of these relations is in Cases C) and D) in Section 7.2.

5.6 Relations in $\overline{\Gamma}$

Recall that τ is defined as moving endpoints on ∂ by $+1$ and endpoints on ∂' by -1, inducing meshes and thus relations on Γ. There are additional relations we need to impose on $\overline{\Gamma}$. In Section 5.1 we defined long moves (cf. (1)-(4)) and and then in Section 5.2 we described diagrams coming from the long moves (see (5)-(8)). In these diagrams we have that all triangles commute.

From these relations we get the following facts.

- From (1) and (2) we get that any map (long move) from $\mathcal{P}$ to a tube factors through infinitely many arcs in that tube. In particular, from (1), any map of the form $\pi([i_\partial, y_{\partial'}]) \to \pi([i_\partial, (i+k)_\partial])$, with $k \geq 2$, factors through all arcs $\pi([i_\partial, j_\partial])$ where $j > i + k$. Similarly, from (2), maps from $\mathcal{P}$ to $\mathcal{T}_h$ factor through infinitely many arcs in $\mathcal{T}_h$.

- From (3) and (4) we get that any map from a tube to $\mathcal{I}$ factors through infinitely many preinjective arcs. In particular, from (3), any map of the form $\pi([i_\partial, y_\partial]) \to \pi([x_{\partial'}, y_\partial])$ factors through all arcs $\pi([(x-k)_{\partial'}, y_\partial])$ where $k > 1$. Similarly, from (4), with maps from $\mathcal{T}_h$ to $\mathcal{I}$.

- From (5) and (6) any map from $\mathcal{P}$ to a tube factors through infinitely many preprojective arcs.

- From (7) and (8) we have that any map from a tube to $\mathcal{I}$ factors through infinitely many arcs in that tube.

In addition, we have relations coming from the diamonds involving two different kind of arcs, see Sections 5.4 and 5.5.

6 An Isomorphism of Translation Quivers

6.1 Definition of $\overline{\Gamma}_m$

We now want to define a subquiver $\overline{\Gamma}_m$ of $\overline{\Gamma}$ and prove that it is isomorphic to the translation quiver Q_m defined by Brüstle in [4], except from the homogeneous tubes. See Section 3.

By Proposition 4.7, the quiver (Γ, τ) is isomorphic to the AR-quiver of $\mod kQ_{g,h}$. Let $\overline{\Gamma}^P, \overline{\Gamma}^I, \overline{\Gamma}^0$ and $\overline{\Gamma}^\infty$ be the subquivers of $\overline{\Gamma}$ consisting of the preprojective arcs, preinjective arcs, the arcs in the tube of rank g and the tube of rank h respectively. Then we have that $\overline{\Gamma}^P$ is isomorphic to Q^P, $\overline{\Gamma}^I$ is isomorphic to Q^I, $\overline{\Gamma}^0$ is isomorphic to Q^0 and $\overline{\Gamma}^\infty$ is isomorphic to Q^∞.

Following [4], we define a subquiver $\overline{\Gamma}_m^P$ of $\overline{\Gamma}^P$ which is isomorphic to Q_m^P. We take the arcs in $\overline{\Gamma}_m^P$ to be all arcs that can be reached from a projective arc by applying τ^{-ghm}, for some integer $m \geq 0$. In other words, the arcs in $\overline{\Gamma}_m^P$ are all arcs of the form $\tau^{-r} P$, with P a projective arc and $0 \leq r \leq ghm$. It follows directly from the definition of Q_m^P that $\overline{\Gamma}_m^P$ is isomorphic to Q_m^P. Similarly we define $\overline{\Gamma}_m^I$ to be all arcs in $\overline{\Gamma}^I$ of the form $\tau^r I$, with I an injective arc and $0 \leq r \leq ghm$. Clearly $\overline{\Gamma}_m^I$ is isomorphic to Q_m^I.

When cut open to lie in the plane, the components $\overline{\Gamma}_m^0$ and $\overline{\Gamma}_m^\infty$ look like rectangles with an equilateral triangle on top. We start from this top vertex to describe $\overline{\Gamma}_m^0$. We take the peripheral arcs at ∂, the longest one we want in $\overline{\Gamma}_m^0$ is the arc $\pi[0_\partial, (gm + g + 2)_\partial]$. We take

$g - 1$ predecessors along the ray of this top vertex, namely $\pi[0_\partial, (gm + g + 2 - i)_\partial]$ for $i = 1, \ldots, g - 1$. Then $\overline{\Gamma}_m^0$ consists of the full subgraph on the vertices in the corays from the mouth up to these g vertices.

For $\overline{\Gamma}_m^\infty$, we start with $\pi[-2_{\partial'}, (hm + h)_{\partial'}]$ and take $h - 1$ predecessors of this vertex along its ray, $\pi[-2_{\partial'}, (hm + h - i)_{\partial'}]$ for $i = 1, \ldots, h - 1$. Then $\overline{\Gamma}_m^\infty$ consists of the full subgraph on the vertices in the corays from the mouth down to these h vertices.

Definition 6.1 We define $\overline{\Gamma}_m$ to be the full subquiver of $\overline{\Gamma}$ consisting of the objects in $\overline{\Gamma}_m^P$, $\overline{\Gamma}_m^I$, $\overline{\Gamma}_{2m(n+1)}^0$ and $\overline{\Gamma}_{2m(n+1)}^\infty$ m.

Let Q'_m be the full subquiver of Q_m on all vertices apart from the homogeneous tubes. It is clear that the quivers $\overline{\Gamma}_m$ and Q'_m have the same vertices. Within the components, they also have the same arrows. Since we have arrows for long moves in $\overline{\Gamma}_m$, the quiver $\overline{\Gamma}_m$ has many more arrows than Q'_m, the latter only has very few arrows linking the components, namely the $\iota_0(i)$ and $\kappa_0(i)$ with $0 \leq i \leq g$, the $\iota_\infty(i)$ and the $\kappa_\infty(i)$ for $i = g, g + 1, \ldots, n$. The translation maps on both quivers work the same, so we will denote both by τ.

To be able to go between $\overline{\Gamma}_m$ and Q'_m, we need to know exactly which vertex of $\overline{\Gamma}_m$ corresponds to which vertex of Q'_m. We do this now by giving a function $F : \overline{\Gamma}_m \to Q'_m$

Recall from Section 5 the bridging arcs that play the role of projective and injective indecomposable modules. This gives the function from the projective arcs to the projective objects in Q_m, i.e. $F(\beta_i) = F(\pi[(i - g)_\partial, 0_{\partial'}]) = (0, i)_P$ for $i = 0, 1, \ldots, g - 1, g$ and $F(\beta_i) = F(\pi[0_\partial, (i - g)_{\partial'}]) = (0, i)_P$ for $i = g + 1, g + 2, \ldots, n$. This describes how F works on $\overline{\Gamma}_m^P$ since every preprojective arc is in the τ-orbit of a projective arc, as every vertex of Q_m^P belongs to the τ-orbit of some projective $(0, j)_P$. E.g. if $0 \leq i \leq g$, we have

$$F(\tau^j \beta_i) = F(\tau^j \pi[(i - g)_\partial, 0_{\partial'}]) = F(\pi[(i - g + j)_\partial, (-j)_{\partial'}])$$
$$= \tau^j (0, i)_P = (-j, i)_P \tag{9}$$

for $-ghm \leq j \leq 0$

Now we consider the preinjective component.

We define $F(\gamma_i) = F(\pi[-2_{\partial'}, (i - g + 2)_\partial]) = (0, i)_I$ for $i = 0, 1, \ldots, g$ and $F(\gamma_i) = F(\pi[(i - g - 2)_{\partial'}, 2_\partial]) = (0, i)_I$ for $i = g + 1, g + 2, \ldots, n$.

As above, we can apply τ to get the other objects in the preinjective component.

For the tube $\overline{\Gamma}_{2m(n+1)}^0$ we only describe the function on the arcs at the top in the tube, because we only need those for the proof. Note that we can easily describe the function on all objects by considering the rays of the arcs at the top. The arc at the top is $\pi([0_\partial, (2m(n + 1) + g + 2)_\partial])$, so we set $F(\pi([0_\partial, (2gm(n+1) + g + 2)_\partial])) = (2gm(n+1) + g, g)_g$. Then $F(\pi([k_\partial, (2gm(n + 1) + g + 2)_\partial])) = (2gm(n + 1) + g - k, g)_g$ for $0 \leq k \leq g$. Also, we have $F(\pi([0_\partial, (2gm(n + 1) + g + 2 - k)_\partial])) = (2gm(n + 1) + g - k, g - k)_g, 0 \leq k \leq g$.

The arc at the mouth of $\overline{\Gamma}_{2m(n+1)}^\infty$ is $\pi[-2_{\partial'}, (2m(n + 1) + h)_{\partial'}]$ and from this, one can describe the map F similarly as for $\overline{\Gamma}_{2m(n+1)}^0$.

Remark 6.2 Observe that in the quiver $\overline{\Gamma}_m$, the relations (c1),(c2), (e), (f) of Theorem 3.4 hold.

To see that relations (c1) and (c2) hold, we use similar arguments as in the proof of Theorem 6.3, using the relations from Section 5.6.

To see that (f) holds in $\overline{\Gamma}_m$, one looks at the corresponding compositions into elementary moves, one long move and elementary moves and observes that both paths have the same effect on the arc corresponding to $(ghm, 0)_P$. The proof of (e) then uses (f) with $j = 0$ or $2m(n+1)+1$ and the fact, that these compositions will pass through two successive vertices at the boundary of the corresponding tube, hence the zero relation. A detailed geometric interpretation of these four relations is given in the Appendix.

6.2 Isomorphism

Recall that Q'_m is the full subquiver of Q_m on the vertices not belonging to the homogeneous tubes. Let R denote the relations (c1),(c2),(e) and (f) on Q'_m, write Q'_m/R for the bound quiver (Q'_m, R). Then we can formulate our result.

Theorem 6.3 *Let Q'_m be the full subquiver of Q_m consisting of all vertices not in a homogeneous tube. Then $\overline{\Gamma}_m$ and Q'_m/R are isomorphic as quivers with relations.*

Proof We already have that the components $\overline{\Gamma}^*_m$ are isomorphic to Q^*_m ($* \in \{P, I, 0, \infty\}$). We need to show that every long move in $\overline{\Gamma}_m$ factors through a long move corresponding to one of the appropriate $\iota_0, \iota_\infty, \kappa_0$ or κ_∞.

Without loss of generality, assume we have a long move $f : \alpha \to \beta$ with $\alpha \in \overline{\Gamma}^P_m$ and $\beta \in \overline{\Gamma}^0_m$ (all other cases follow completely analogously). Since there exists a long move between α and β, their starting points in $P_{g,h}$ must be the same.

We first describe the long moves corresponding to the $\iota_0(i)$ for $i = g, g - 1, \ldots, 1, 0$. These maps start from $g + 1$ different vertices of the last slice in $\overline{\Gamma}^P_m$, namely from the $\tau^{-ghm}\beta_i$ with $i = g, g - 1, \ldots, 0$. By (9), they are arcs of the form $\pi[(-ghm + i - g)_\partial, ghm_{\partial'}]$ for $i = g, \ldots, 0$. We can choose lifts for them as follows:

$$[i_\partial, (hm(n + 1) + h)_{\partial'}] \text{ for } i = g, \ldots, 0.$$

The $\iota_0(i)$ map these vertices to the $g + 1$ vertices sitting on the ray at the top level of $\overline{\Gamma}^0_m$, namely the vertices $\pi[i_\partial, (2gm(n + 1) + g + 2)_\partial]$ for $i = g, \ldots, 0$. We also use $\iota_0(i)$ to denote the corresponding long move,

$$\iota_0(i) : [i_\partial, (hm(n + 1) + h)_{\partial'}] \to \pi[i_\partial, (2gm(n + 1) + g + 2)_\partial]$$

Observe that the rays in $\overline{\Gamma}^P_m$ passing through the $g + 1$ vertices $\pi[i_\partial, (hm(n + 1) + h)_{\partial'}]$ cover the whole component. Each of these rays consists of a (finite) sequence of elementary moves. In terms of arcs, every such ray starts at β_i, $i = g, \ldots, 0$ and consists of all vertices obtained through a sequence of rotations around the common starting point of the arcs, the vertex i of $P_{g,h}$

Similarly, $\overline{\Gamma}^0_m$ is formed by the vertices of all corays between the top vertices $\pi[i_\partial, (2gm(n + 1) + g + 2)_\partial]$ and the mouth. The corays in $\overline{\Gamma}^0_m$ are formed by all arcs obtained from the top one by sequence of rotations around the common starting point i_∂.

Denote the sequence of elementary moves (fixing starting points) from α to $\pi[i_\partial, (hm(n + 1) + h)_{\partial'}]$ by $(\mu_r)_r$ and the sequence of elementary moves (fixing starting points) from $\pi[i_\partial, (2gm(n + 1) + g + 2)_\partial]$ to β by $(\nu_t)_t$.

Then the composition $(\nu_t)_t \circ \iota_0(i) \circ (\mu_r)_r$ is a long move from α to β. By using both commutativity relations for triangles between $\overline{\Gamma}^P_m$ and $\overline{\Gamma}^0_m$ (Section 5.6, for a geometric

interpretation cf. Section 7.3.1), it is then straightforward to see that f is equal to this composition. □

Recall the equivalence of categories $\mathcal{C}(k Q_m) \to \mathcal{J}_m Q_{g,h}$ from [4], see Theorem 3.4. The category $\mathcal{C}(k Q_m)$ is the k-category of Q_m subject to the relations $(a) - (g)$. As we have mentioned before, we do not consider the homogeneous tubes, so we will use the equivalence $\mathcal{C}(k Q'_m) \to \mathcal{J}'_m Q_{g,h}$, where we take the full subcategories without the homogeneous tubes.

Now we define $\mathcal{C}(k \overline{\Gamma})$ to be the k-category determined by $\overline{\Gamma}$ (modulo relations) and $\mathcal{C}(k \overline{\Gamma}_m)$ the k-category determined by $\overline{\Gamma}_m$ (modulo relations) Now, clearly $\mathcal{C}(k \overline{\Gamma}_m)$ is a full subcategory of $\mathcal{C}(k \overline{\Gamma})$. Since in $\overline{\Gamma}_m$ the relations (not involving the homogeneous tubes) are satisfied, we have the following theorem.

Theorem 6.4 *Let Q'_m be the full subquiver of Q_m consisting of all objects not in a homogeneous tube. Then the category $\mathcal{C}(k \overline{\Gamma}_m)$ is equivalent to $\mathcal{C}(k Q'_m)$ and $\mathcal{J}'_m Q_{g,h}$.*

7 Relations and Their Geometric Interpretation

In this section, we give the geometric interpretation of the relations we have defined for $\overline{\Gamma}$, using lifts of arcs in $\mathbb{U}$.

Recall that $\overline{\Gamma}$ is the quiver obtained from $\Gamma = \mathrm{AR}(\mathrm{mod}\, k Q_{g,h})$ by adding arrows corresponding to long moves, subject to additional relations (Sections 5.4, 5.5 and 5.6). We first consider commuting squares. Let

$$
\begin{array}{ccc}
 & Y_1 & \\
{\scriptstyle f_1}\nearrow & & \searrow{\scriptstyle g_1} \\
X & & Z \\
{\scriptstyle f_2}\searrow & & \nearrow{\scriptstyle g_2} \\
 & Y_2 &
\end{array}
$$
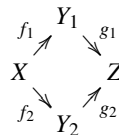
be any diamond in $\overline{\Gamma}$. Such a diamond commutes in the following cases:

a) All the arrows in this diamond are elementary moves and hence all four vertices belong to the same connected component of Γ, with $\tau(Z) = X$.

b) Two opposite arrows are elementary moves and the other two are long moves. In the latter case, X and Y_1 (or Y_2) are in the same component (preprojective or a tube) of Γ and Y_2 (Y_1, respectively) and Z are in a common component of Γ (a tube or the preinjective component).

In both cases, the relations $g_1 \circ f_1 = g_2 \circ f_2$ are imposed on $\overline{\Gamma}$. We will illustrate the meshes in Section 7.1. They are all instances of the so-called Ptolemy relation: the two end terms X and Z of the diamond are viewed as the two different diagonals of a quadrilateral, the AR translate is the exchange of one of those diagonals by the other. This is called a flip.

Then we give the geometric interpretation of the new relations for diamonds in 7.2. We show that the new relations can also be viewed as Ptolemy relations by describing the quadrilateral in which the flip takes place.

## 7.1 Geometric Interpretation of Mesh Relations: Diamonds

The geometric interpretation of a mesh relation in type A is well known, cf. [6, Section 2] (for cluster categories of type A).
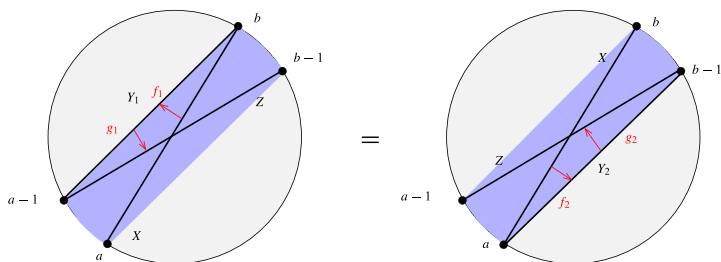
The geometric reason for the commutativity is the Ptolemy relation: The two arcs Y_1 and Y_2 can be viewed as opposite boundary edges of a quadrilateral, with X and Z the two diagonals of this quadrilateral. Then the diagonals X and Z are related by a flip inside this polygon.

In Γ, we have four cases: meshes within $\mathcal{P}$ or $\mathcal{I}$, and meshes within the tubes. These meshes are all instances of the Ptolemy relation, we illustrate all cases briefly.

We consider the preprojective component, the preinjective case is completely analogous. Let $X = [i_\partial, j_{\partial'}]$, $Y_1 = [(i-1)_\partial, j_{\partial'}]$, $Y_2 = [i_\partial, (j+1)_{\partial'}]$ and $Z = [(i-1)_\partial, (j+1)_{\partial'}]$.
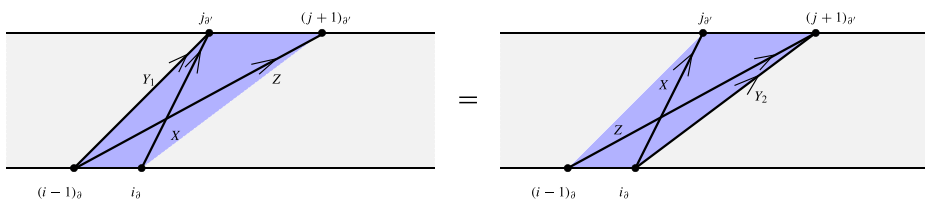
Next we consider the regular tubes. W.l.o.g. let X and Y be indecomposable objects of the tube $\mathcal{T}_g$. Let $X = [i_\partial, j_\partial]$, with $j \geq i + 3$ $Y_1 = [(i-1)_\partial, j_\partial]$, $Y_2 = [i_\partial, (j-1)_\partial]$ and $Z = [(i-1)_\partial, (j-1)_\partial]$.
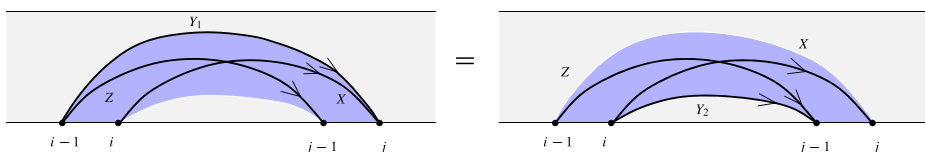
7.2 Diamonds Involving Long Moves

As before, long moves are indicated by dashed arrows.

We have four cases:

A) X and Y_1 belong to $\mathcal{P}$, Y_2 and Z to $\mathcal{T}_g$,
B) X and Y_2 belong to $\mathcal{P}$, Y_1 and Z to $\mathcal{T}_h$,
C) X and Y_2 belong to $\mathcal{T}_g$, Y_1 and Z to $\mathcal{I}$,
D) X and Y_1 belong to $\mathcal{T}_g$, Y_2 and Z to $\mathcal{I}$.

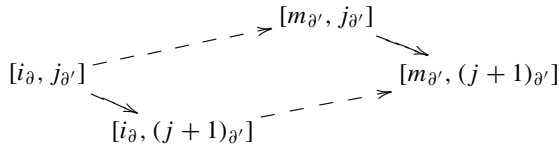We will illustrate each of them by giving the quadrilateral of the Ptolemy relation.

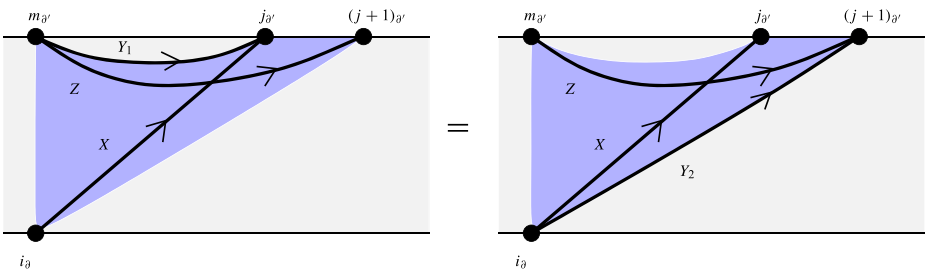Case A): $\mathcal{P}$ to $\mathcal{T}_g$.

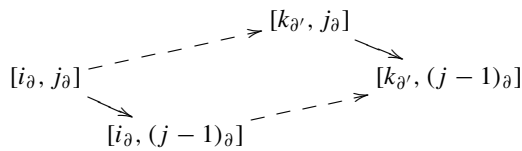$$[(i-1)_\partial, j_{\partial'}]$$

$$[i_\partial, j_{\partial'}] \qquad\qquad [(i-1)_\partial, k_\partial]$$

$$[i_\partial, k_\partial]$$

with $k \geq i+2$.

Case B): $\mathcal{P}$ to $\mathcal{T}_h$.

$$[m_{\partial'}, j_{\partial'}]$$

$$[i_\partial, j_{\partial'}] \qquad\qquad [m_{\partial'}, (j+1)_{\partial'}]$$

$$[i_\partial, (j+1)_{\partial'}]$$
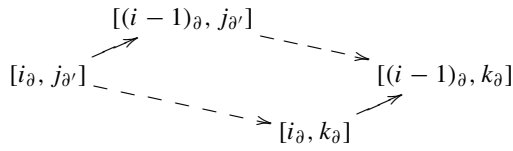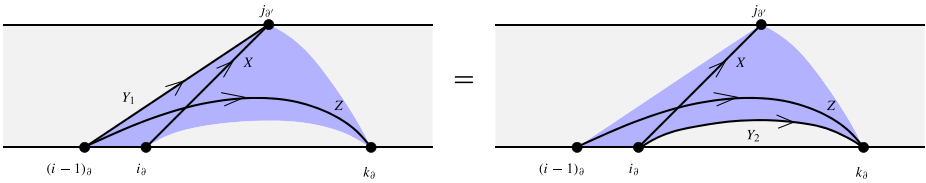
with $m \leq j-2$.

Case C): $\mathcal{T}_g$ to $\mathcal{I}$.

$$[k_{\partial'}, j_\partial]$$

$$[i_\partial, j_\partial] \qquad\qquad [k_{\partial'}, (j-1)_\partial]$$

$$[i_\partial, (j-1)_\partial]$$

with $j \geq i+3$.

Case D): $\mathcal{T}_h$ to $\mathcal{I}$.

$$[(i+1)_{\partial'}, j_{\partial'}]$$

$$[i_{\partial'}, j_{\partial'}] \qquad [(i+1)_{\partial'}, k_\partial]$$

$$[i_{\partial'}, k_\partial]$$
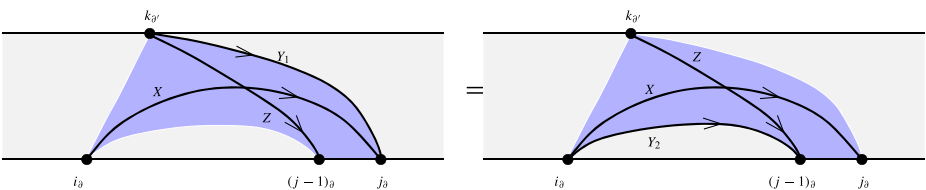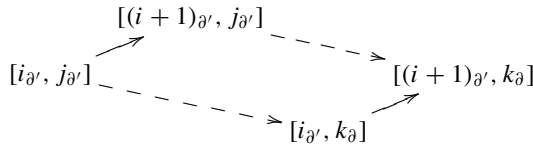
with $j \geq i + 3$.
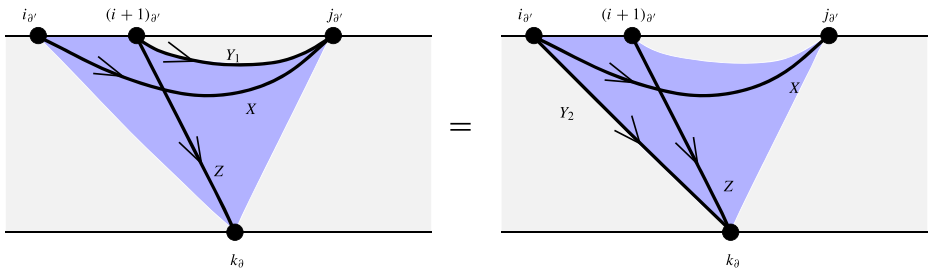
7.3 Commuting Triangles Involving Long Moves, Geometrically

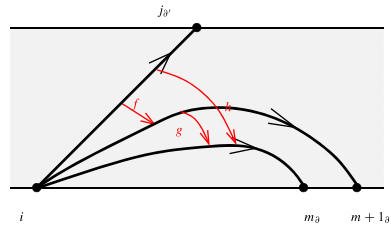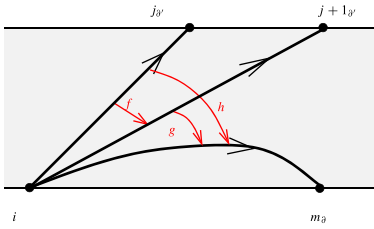In $\overline{\Gamma}$, there are also commuting triangles, involving two long moves and an elementary move.
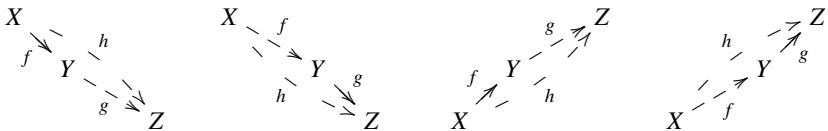
Recall that arrows for moves fixing the endpoint of an arc go up, arrows for moves fixing the starting point go down. There are twice four cases: two from $\mathcal{P}$ to $\mathcal{T}_g$ or to $\mathcal{T}_h$, two from $\mathcal{T}_g$ or from $\mathcal{T}_h$ to $\mathcal{I}$.

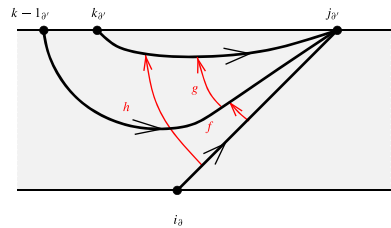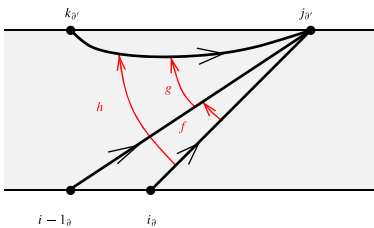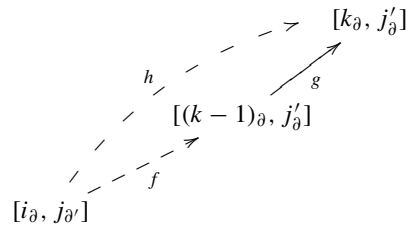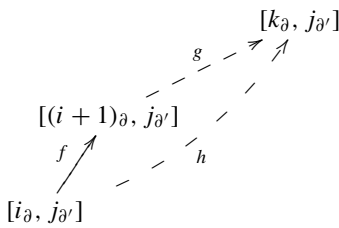#### 7.3.1 Commuting Triangles with Long Maps from $\mathcal{P}$ to $\mathcal{T}_g$

Let $[i_\partial, j_{\partial'}]$ be a preprojective arc. Then for every $m \geq i + 2$ there are two cases of commuting triangles $g \circ f = h$ involving two long and one elementary move.
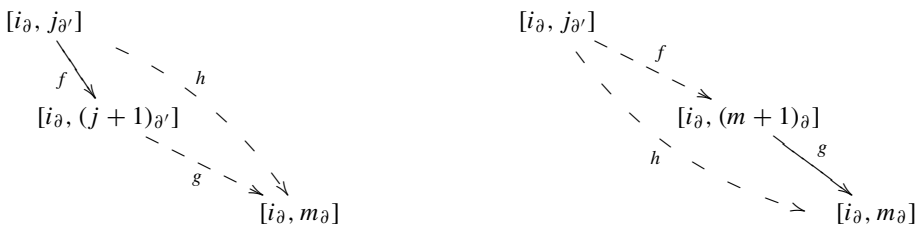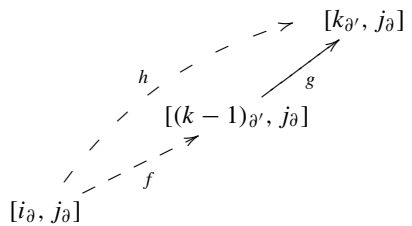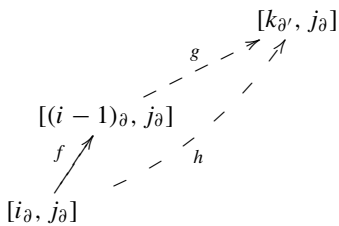
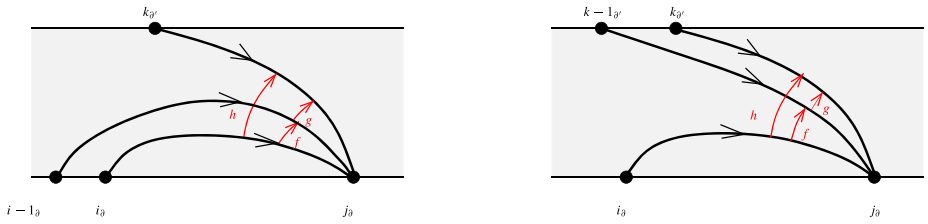### 7.3.2 Commuting Triangles from $\mathcal{P}$ to $\mathcal{T}_h$

Let $[i_\partial, j_{\partial'}]$ be a preprojective arc. Then for every $k \leq j - 2$, there are two kind of commuting triangles $g \circ f = h$ involving one elementary and two long moves.

7.3.3 Commuting Triangles from $\mathcal{T}_g$ to $\mathcal{I}$
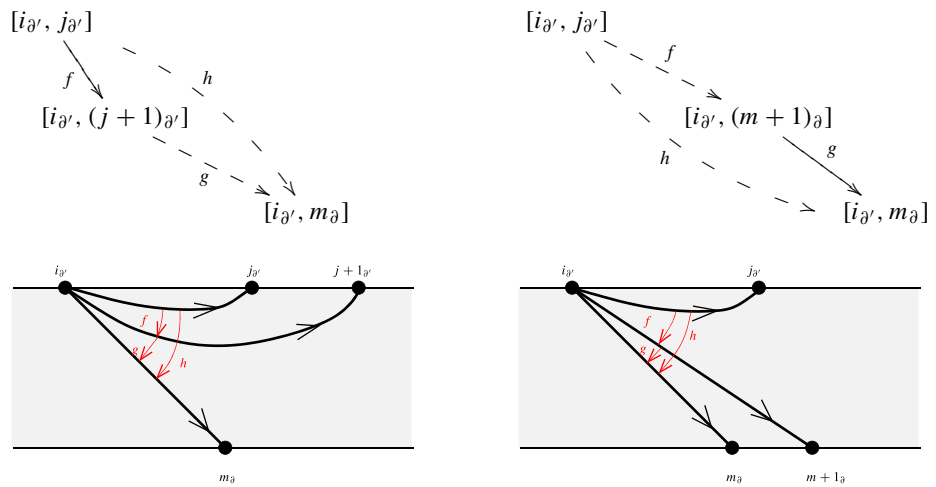
Let $[j_\partial, i_\partial]$ be a peripheral arc (i.e. $j \leq i - 2$). The two kind of commuting triangles $g \circ f = h$ involving elementary and long moves end at an arbitrary preinjective arc $[k_{\partial'}, i_\partial]$:

7.3.4 Commuting Triangles from $\mathcal{T}_h$ to $\mathcal{I}$

Let $[i_{\partial'}, j_{\partial'}]$ be a peripheral arc (i.e. $i \leq j - 2$). The two kind of commuting triangles $g \circ f = h$ involving elementary and long moves end at an arbitrary preinjective arc $[i_{\partial'}, m_{\partial'}]$:

7.4 Zero Relations, Geometrically

It is well known that there are zero relations at the mouth of the tubes. Geometrically, they can be viewed as special cases of the diamond relations in the tubes, namely the case where Y_2 (or Y_1) becomes zero, i.e. where the vertex X is of the form $[i_\partial, (i+2)_\partial]$ or $[i_{\partial'}, (i+2)_{\partial'}]$ see Section 7.1

8 Application to the Cluster Category of Affine Type A

The cluster category of type $\tilde{A}$ has been introduced by [3]. It is by definition the orbit category $D^b(\text{mod } \tilde{A})/\tau^{-1} \circ [1]$. The effect of taking such a quotient is most visible on the AR quiver: the cluster category has a transjective component, arising from the components $\mathcal{P}$ and $\mathcal{I}$ of the module category.

Using unoriented versions of our arcs, we can describe the AR-quiver of the cluster category. The only missing ingredient is the slice linking the preprojective component with the preinjective component. We also write τ for the translation map on unoriented arcs, induced by $i \mapsto i + 1$ on ∂ and $i \mapsto i - 1$ on ∂'.

If $\alpha \in \overline{\Gamma}$ is an oriented arc, let $\underline{\alpha}$ be the unoriented version of α. Furthermore, let $\underline{\eta_0}$, $\underline{\eta_1}$, ..., $\underline{\eta_n}$ be the arcs $\tau^{-1}(\gamma_i) = \tau(\beta_i)$.

Let $\underline{\Gamma}$ be the quiver whose vertices are the $\underline{\alpha}$, $\alpha \in \overline{\Gamma}$, together with the $\underline{\eta_i}$. Its arrows are the arrows of $\overline{\Gamma}$ together with the obvious arrows from the γ_i to the η_i and from the η_i to the β_i, subject to the relations from $\overline{\Gamma}$ and the additional mesh relations around the new slice. The new quiver $\underline{\Gamma}$ is a stable translation quiver. Up to the additional slice of vertices, the quiver $\underline{\Gamma}$ looks like $\overline{\Gamma}$, the latter is isomorphic to a full subquiver of $\underline{\Gamma}$. The quiver $\underline{\Gamma}$ has components $\underline{\Gamma}^0$ and $\underline{\Gamma}^\infty$ consisting of peripheral arcs and the transjective component of the arcs from $\overline{\Gamma}^P \cup \overline{\Gamma}^I$ together with $\{\underline{\eta_i} \mid i = 0 \dots, n\}$. Write $\underline{\Gamma}_m^{Tr}$ for the full subquiver on the vertices of $\overline{\Gamma}^P \cup \overline{\Gamma}^I$ and the $\{\underline{\eta_i} \mid i = 0 \dots, n\}$.

Definition 8.1 We define $\underline{\Gamma}_m$ to be the full subquiver of $\underline{\Gamma}$ consisting of the vertices in $\underline{\Gamma}_m^{Tr}$, $\underline{\Gamma}_m^0$ and $\underline{\Gamma}_m^\infty$.

Let $\mathcal{J}_m'(\mathcal{C}_{\tilde{A}})$ be the full subcategory of the cluster category $\mathcal{C}_{\tilde{A}}$ of type $Q_{g,h}$ consisting of all objects who correspond to vertices in $\underline{\Gamma}_m$. Then the following is a direct consequence of Theorems 6.3 and 6.4.

Corollary 8.2 *The k-category of $\underline{\Gamma}_m$ is equivalent to $\mathcal{J}_m'(\mathcal{C}_{\tilde{A}})$.*

Appendix: Relations appearing in Theorem 3.4

We use the geometric interpretation of relations to visualize the relations in Theorem 3.4

Relations (a) and (b) of the theorem are the usual mesh relations, they have already been described above.

Relations (c1),(c2) Relation (c1) has on one side h compositions of elementary moves between preprojective arcs, followed by a long move and then g elementary moves in the tube of rank g, we can write it as

$$X_0 \xrightarrow{\alpha} X_1 \xrightarrow{\alpha} \cdots \xrightarrow{\alpha} X_h \overset{\iota_0(0)}{\dashrightarrow} Y_g \xrightarrow{\pi} Y_{g-1} \xrightarrow{\pi} \cdots \xrightarrow{\pi} Y_0$$

dropping the subscripts on the α and π. The arcs corresponding to these vertices all have lifts starting at 0_∂, the X_i end at ∂' and the Y_i correspond to peripheral arcs. More precisely, we can choose a lift for X_0, $X_0 = [-ghm_\partial, ghm_{\partial'}]$. Under the arrows α, the arcs get lengthened by elementary moves fixing the ending point $-ghm_\partial$, up to the lift $[-ghm_\partial, (ghm + h)_{\partial'}] X_h$.

To prove that in our geometric set-up the first relation of (c1) holds, we can use the commuting triangle on the left hand side of Section 7.3.2 to see that $\iota_0(0) \circ \alpha$ is a long move μ_{h-1} from X_{h-1} to Y_g. Using this relation again, we see that $\alpha \circ \mu_{h-1}$ is a long move μ_{h-2} from X_{h-2} to Y_g, etc. This proves that the composition $\iota_0(0) \circ \alpha_P^g$ is equal to a long move $\mu_0 : X_0 \to Y_g$. Now we use the commuting triangle on the right hand side of Section 7.3.2 h times to iteratively replace the long move from X_0 composed with π by another long move from X_0 to Y_i, $i = g-1, \ldots, 0$.

The other three relations in (c1) and (c2) work completely analogously, using Sections 7.3.2, 7.3.3 and 7.3.4 respectively.

The relations in (e) rely on relation (f), so we will first consider the latter.

Relation (f) For this, we need more work. Consider a tube $\mathcal{T}_g$ of rank g. We want to describe the effect of a composition of $2g$ elementary moves from an indecomposable object X of $\mathcal{T}_g$ to itself.

Let $[a_\partial, b_\partial]$ be the arc (viewed in $\mathbb{U}$), $b \geq a + 2$. There are (at most) two elementary moves on $[a_\partial, b_\partial]$.

Going down (up) from a peripheral arc in $\mathbb{U}$ corresponds to moving the endpoint (resp. the starting point) of the corresponding arc one step to the left, thus making it shorter (longer). We write f_d for the elementary move downwards, $f_d : [a_\partial, b_\partial] \mapsto [a_\partial, (b-1)_\partial]$ and f_u for the one going up, $f_u : [a_\partial, b_\partial] \mapsto [(a-1)_\partial, b_\partial]$. Note that for $b = a+2$, the image under f_d is a boundary segment and hence zero.

For $g \geq 1$ we write $f^{\downarrow g}$ for the composition of g elementary moves downwards and $f^{\uparrow g}$ for g consecutive elementary moves upwards. We then abbreviate the composition of g downwards with g upwards elementary moves by $f^{\downarrow \uparrow g}$:

$$f^{\downarrow \uparrow g}([a_\partial, b_\partial]) := f^{\uparrow g} \circ f^{\downarrow g}([a_\partial, b_\partial])$$

We will use the notations for these compositions of elementary moves within the universal cover $\mathbb{U}$, and also in the annulus.

The effect of a composition of g downwards with g upwards moves on peripheral arcs at the lower boundary is the following:

$$f^{\downarrow \uparrow g}[i_\partial, j_\partial] = \begin{cases} [(i-g)_\partial, (j-g)_\partial] & \text{if } j - i > g + 1 \\ 0 & \text{else.} \end{cases}$$

For arcs of $\mathcal{T}_h$, we define $f^{\uparrow \downarrow h}$ analogously, the effect of h elementary moves up followed by h elementary moves down. Since for endpoints at ∂', elementary moves increase an endpoint by $+1$, we get

$$f^{\uparrow \downarrow h}[i_\partial, j_\partial] = \begin{cases} [(i+h)_\partial, (j+h)_\partial] & \text{if } j - i > h + 1 \\ 0 & \text{else.} \end{cases}$$

In $P_{g,h}$, this shift by g or by $+h$ is not visible, the effect of the compositions $f^{\downarrow \uparrow g}$ and $f^{\uparrow \downarrow h}$ on arcs in $P_{g,h}$ is to move both endpoints of a peripheral arc around the boundary once (or zero, if the arc is close to the mouth of the tube):

$$f^{\downarrow \uparrow g} \pi[i_\partial, j_\partial] = \begin{cases} \pi[i_\partial, j_\partial] & \text{if } j - i \geq g + 2 \\ 0 & \text{else} \end{cases}$$

$$f^{\uparrow \downarrow h} \pi[k_{\partial'}, l_{\partial'}] = \begin{cases} \pi[k_{\partial'}, l_{\partial'}] & \text{if } l - k \geq h + 2 \\ 0 & \text{else} \end{cases}$$

Note that it follows from the (geometric) mesh relations, that the effect of g downwards moves combined with g upwords moves in $\mathcal{T}_g$ (h upwards moves with h downwards moves in $\mathcal{T}_h$) is independent of the order in which this moves are done.

Relation (f) involves the four "top" vertices in Q_m and arrows corresponding to long moves between them. We will first describe these vertices giving one lift in $\mathbb{U}$ for each of them. First recall that $(ghm, 0)_P = \tau^{-ghm}(0, 0)_P$, $(ghm, 0)_I = \tau^{ghm}(0, 0)_I$ and observe that $(0, g)_g$ has as lift $[0_\partial, 2_\partial]$, $(0, 0)_h$ has as lift $[-2_{\partial'}, 0_{\partial'}]$ (cf. Figure 2 for $m = 2$ to determine the latter).

vertices of Q_m	lifts in $\mathbb{U}$
$(ghm, 0)_P$	$[-ghm_\partial, (h + ghm)_{\partial'}]$
$(2hm(n + 1) + h, 0)_h$	$[-2_{\partial'}, (2hm(n + 1) + h)_{\partial'}]$
$(2gm(n + 1) + g, g)_g$	$[0_\partial, (2gm(n + 1) + g + 2)_\partial]$
$(ghm, 0)_I$	$[(-2 - ghm)_{\partial'}, (2 + ghm)_\partial]$

In terms of lifts in $\mathbb{U}$, the four arrows $\iota_*(0)$, $\kappa_*(0)$ of Q_m can be described as long moves moving vertically between the two boundary components composed with a sequence of elementary moves.

A.1 $\iota_\infty(0)$: $(ghm, 0)_P \to (2hm(n + 1) + h, 0)_h$

This arrow is a long move from an arc $\partial \to \partial'$ to a peripheral arc at ∂'. In geometric terms it is a rotation around the common ending point on ∂' of the involved arcs. In terms of lifts in $\mathbb{U}$ with a common ending point:

$$[-ghm_\partial, (h + ghm)_{\partial'}] \xrightarrow{(v_r)_r \mu} [(-2 - h(m(n + 1) + hm))_{\partial'}, (h + ghm)_{\partial'}]$$

The effect of $\iota_\infty(0)$ is to first use a long move μ, changing the starting point from ghm_∂ by sending it vertically across to $hhm_{\partial'}$ and then to use $2 + hm(n + 1)$ elementary moves ν_r still fixing the ending point on ∂' to send this starting point along ∂' to the left by subtracting $2 + hm(n + 1)$.

A.2 $\iota_0(0)$: $(ghm, 0)_P \to (2gm(n + 1) + g, g)_g$

The arrow $\iota_0(0)$ is a long move around a common starting point on ∂. In terms of lifts:

$$[-ghm_\partial, (h + ghm)_{\partial'}] \xrightarrow{(v_r)_r \mu} [-ghm_\partial, (2 + g(m(n + 1) + gm + 1))_\partial]$$

The effect of $\iota_0(0)$ is to first send the ending point of the arc from $(h + ghm)_{\partial'}$ to $(g(1 + gm))_\partial$ (long move μ) and then to send this point along ∂ to the right by adding $2 + gm(n + 1)$ (composition of $2 + gm(n + 1)$ elementary moves ν_r around the same starting point).

A.3 $\kappa_\infty(0)$: $(2hm(n + 1) + h, 0)_h \to (ghm, 0)_I$

This arrow corresponds to a rotation around the common starting point on ∂'. In $\mathbb{U}$,

$$[(-2 - h(m(n + 1) + hm))_{\partial'}, (h + ghm)_{\partial'}] \xrightarrow{(v_r)_r \mu} [(-2 - h(m(n + 1) + hm)_{\partial'}, 2 - ghm_\partial]$$

we can describe the effect of $\kappa_\infty(0)$. First, the endpoint of the arc on ∂' is sent vertically across to $(g + g^2m)_\partial$ under the long move μ and then under $2 - g - g(m(n + 1))$ elementary moves μ_r, the endpoint is sent to the right to obtain $[(-2 - h(m(n + 1) + hm)_{\partial'}, 2 - ghm_\partial]$.

A.4 $\kappa_0(0)$: $(2gm(n + 1) + g, g)_g \rightarrow (ghm, 0)_l$

$\kappa_0(0)$ corresponds to a rotation around the common ending point on ∂. In $\mathbb{U}$

$$[-ghm_\partial, (2+g(m(n+1)+gm+1))_\partial] \xrightarrow{(v_r)_r\mu} [(-2+ghm+h)_{\partial'}, (2+g(m(n+1)+gm+1))_\partial]$$

We first have a long move μ fixing the endpoints, sending the starting point from $-ghm$ on ∂ across to $-h^2m$ on ∂'. This is then composed with $-2+h+hm(n+1)$ elementary moves μ_r to send the new starting point to the right and get the desired result.

A.5 $\rho_\infty^h \pi_\infty^h$ and $f^{\uparrow\downarrow h}$

In terms of arcs, the effect of the path $\rho_\infty^h \pi_\infty^h$ is $f^{\uparrow\downarrow h}$ and the effect of $\pi_0^g \rho_0^g$ is $f^{\downarrow\uparrow g}$.

Path $\kappa_\infty(0)(\rho_\infty^h \pi_\infty^h)^j \iota_\infty(0)$ We use the lifts from above (Section A.1) describing $\iota_\infty(0)$ and compose with the remaining paths:

$$(\rho_\infty^h \pi_\infty^h)^j : [(-2 - h(m(n + 1) + hm))_{\partial'}, (h + hgm)_{\partial'}]$$
$$\longrightarrow [(-2 - h(m(n + 1) + hm - j))_{\partial'}, (hgm + hj + h)_{\partial'}]$$

Applying $\kappa_\infty(0)$ (Section A.3), this has the image

$$[(-2 - h(m(n + 1) + hm - j))_{\partial'}, 2 - ghm + jg)_\partial] \tag{10}$$

Path $\kappa_0(0)(\pi_0^g \rho_0^g)^{2m(n+1)+1-j} \iota_0(0)$ The arrow $\iota_0(0)$ is described in Section A.2, we compose with the remaining paths:

$$(\rho_0^g \pi_0^g)^{2m(n+1)+1-j} : [-ghm_\partial, (g(m(n + 1) + gm + 1) + 2)_\partial]$$
$$\longrightarrow [(-g(2m(n + 1) + hm + 1 - j))_\partial, (g(mh - j) + 2)_\partial]$$

Finally, applying $\kappa_0(0)$ (Section A.4), we get

$$[\underbrace{(-2 - ghm - hj)}_{:=x}{}_{\partial'}, \underbrace{(2 + ghm - gj)}_{:=y}{}_\partial] \tag{11}$$

With the projection to $P_{g,h}$ in mind, we translate the resulting arc in $\mathbb{U}$ so that the images (11) and (10) of the two paths have the same endpoint. To the endpoint y_∂ of (11) we add $2g(j - hm)$ (on boundary ∂). On ∂', the corresponding translation is by $+2h(j - hm)$:

$$\pi[x_{\partial'}, y_\partial] = [(-2 - ghm - hj + 2h(j - hm))_{\partial'}, (2 - ghm + gj)_\partial]$$
$$= [(-2 + hj - h(2hm + gm))_{\partial'}, (2 - ghm + gj)_\partial]$$
$$= [(-2 + hj - h(hm + m(n + 1)))_{\partial'}, (2 - ghm + gj)_\partial]$$

Hence the images of these arcs in $P_{g,h}$ under the projection map are the same and relation (f) is satisfied.

Relation (e) We only show the first claim $\kappa_0(0)\iota_0(0)(ghm, \alpha_n)_P = 0$, the second claim is completely analogous.

By (f), with $j = 2m(n + 1) + 1$, we know that $\kappa_0(0)\iota_0(0) = \kappa_\infty \rho_\infty^h \pi_\infty^h{}^h$. In terms of arcs in the annulus, $\rho_\infty^h \pi_\infty^h$ is the map $f^{\uparrow\downarrow h}$ from above, it sends any arc $\pi[i_{\partial'}, j_{\partial'}]$ with $j - i \geq h + 2$ to itself. Relation (e) concerns a path going through $(2m(n + 1)h, 0)_h$, the vertex corresponding to $\alpha := \pi[-2_{\partial'}, (2hm(n+1)-h)_{\partial'}]$ and this is high up in the tube $\overline{\Gamma}^\infty$,

hence the effect of $f^{\uparrow\downarrow h}$ is to send this arc to itself. Applying $f^{\uparrow\downarrow h}$ to the arc $2m(n+1)-1$ times still sends α to itself, however, the composition of all these elementary moves touches the mouth of the tube exactly once. In (e), this is precomposed with the elementary move corresponding to $(ghm, \alpha_n)_P$. But this means that we can use Case B) from Section 7.2 to replace the elementary move in $\overline{\Gamma}^P$ (and the long move from $\overline{\Gamma}^P$) by a long move to the arc $\pi[-2_{\partial'}, (2hm(n+1)+h-1)_{\partial'}]$ followed by an elementary move to α within the tube. Hence the path from α to itself, going all the way down to the mouth, is precomposed with one downwards move (in $\overline{\Gamma}^\infty$). Then we use the mesh relations within $\overline{\Gamma}^\infty$ to push this all the way to the mouth, hitting the tube a second time just to the left of the other vertex at the mouth. That means that in our path, there are two shortest peripheral arcs. Between them, there is only a boundary segment (hence a zero object) and an arc of the form $[i_{\partial'}, (i+3)_{\partial'}]$. The zero relation at the mouth proves the claim (Section 7.4).

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