Some new results on the preservation of the NBUE and NWUE aging classes under the formation of coherent systems

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May 3, 2019

Abstract

We consider the classical problem of whether certain classes of lifetime distributions are preserved under the formation of coherent systems. Under the assumption of i.i.d. component lifetimes, we consider the NBUE *(new better than used in expectation)* and NWUE (*new worse than used in expectation)* classes. First, a necessary condition for a coherent system to preserve the NBUE class is given. Sufficient conditions are then obtained for systems satisfying this necessary condition. The sufficient conditions are satisfied for a collection of systems which includes all parallel systems, but the collection is shown to be strictly larger. We also prove that no coherent system preserves the NWUE class. As byproducts of our study, we obtain the following results for the case of i.i.d. component lifetimes: (i) the DFR *(decreasing failure rate)* class is preserved by no coherent systems other than series systems, and (ii) the IMRL *(increasing mean residual life)* class is not preserved by any coherent systems. Generalizations to the case of dependent component lifetimes are briefly discussed.

Keywords: coherent system; reliability polynomial; mean residual life; failure rate; copula

1 Introduction

1.1 Classes of life distributions based on aging properties

Motivated by studies of maintenance policies, Frank Proschan, Richard Barlow and several co-authors introduced, in the 1960s, classes of lifetime distributions with relevance to replacement policies. An early reference is Barlow and Proschan [2]. Much of this work was later presented in the monographs [3] and [4] by Barlow and Proschan.

The following main classes of lifetime distributions are considered in [4], where each of them comes with a "deteriorating" version and an "improving" version: IFR (DFR) – *increasing (decreasing) failure rate*; IFRA (DFRA) – *increasing (decreasing) failure rate average*; NBU (NWU) – *new better (worse) than used*; NBUE (NWUE) – *new better (worse) than used in expectation*. Marshall and Proschan [8] later introduced the classes DMRL (IMRL) – *decreasing (increasing) mean residual life*.

The relationships given below are by now well known to reliability scientists, see [4] for background.

$$IFR \Rightarrow \begin{array}{c} IFRA \Rightarrow NBU \\ DMRL \end{array} \Rightarrow NBUE \tag{1}$$

and

$$DFR \Rightarrow \begin{array}{c} DFRA \Rightarrow NWU \\ IMRL \end{array} \Rightarrow NWUE \tag{2}$$

1.2 Motivation for the study

Since the pioneering papers mentioned above, a large number of studies have been devoted to the problem of preservation of the various classes of lifetime distributions under the formation of convolutions and mixtures, and in particular under the formation of coherent systems. It is the purpose of the present paper to complement the existing theory, with emphasis on preservation properties of particular classes, under formation of coherent systems with independent and identically distributed (i.i.d.) component lifetimes. An extension to the dependent case will be briefly discussed in the concluding section.

The engineering motivation for the interest in preservation results is simple. It is reasonable to consider the properties of the i.i.d. component lifetimes to be subject to the control of the engineer; the quality of a batch of like components can be selected by the engineer from among the varied component types available for use. When the components are used in a specific system, the reliability of the system is often a complex function of the component reliability, and the stochastic behavior of the system may be difficult to describe. When it is possible to establish the fact that a particular property of component lifetimes is preserved in a system in which these components are used, the user stands to benefit from a more detailed understanding of the system's expected performance.

It is well known that certain classes (e.g., the IFRA and NBU classes) of lifetime distributions are preserved under the formation of coherent systems ([4, Chapter 4, p. 85 and Chapter 6, p. 182]). On the other hand, such an implication does not hold for IFR component lifetimes, even in the case of i.i.d. component lifetimes ([5],[13]). The present work is dedicated to a careful examination of the larger NBUE class (with a similar treatment of the NWUE class) which is not necessarily preserved under the formation of coherent systems without additional assumptions. Our results provide some clarity in the latter area, as they identify specific conditions under which the NBUE and NWUE properties of component lifetimes ensure the same properties for the system's lifetime.

Given the relationships in (1) and (2), preservation results for the NBUE and NWUE classes are of special interest, as they correspond to the weakest restriction made among those defining the nonparametric classes mentioned above. Thus, this restriction results in the largest class of life distributions among these varied classes. When used as a modeling assumption for observed experimental data (e.g., Whitaker and Samaniego [16]), the restriction under discussion permits treatment of a larger class of data sources and experimental circumstances than is possible under standard nonparametric models.

Unlike the smaller IFRA and NBU classes for which preservation results under the formation of coherent systems are known, the preservation results for the NBUE class are not universal ([4, Chapter 6, p. 183]), that is, they hold only under additional conditions. The need to specify such conditions may be the reason for the time lapse between the present results and the preservation results established for the smaller classes.

1.3 Recent literature

The distribution classes mentioned above, as well as several modifications, extensions and new classes, have recently been studied by a number of authors, see, e.g. Franco et al. [6] and Nanda et al. [9] and references therein. These authors studied preservation properties under formation of coherent systems with various assumptions. More recently, Navarro et al. [11] extended the scope of these studies by considering the case of dependent, identically distributed (d.i.d.) component lifetimes, with dependency modeled by a copula. A key feature of this approach, is a generalization of the reliability polynomial.

In this generalized framework, Navarro [10] gave sufficient conditions for preservation of the DMRL and IMRL classes, extending work by Abouammoh and El-Neweihi [1]. The latter authors proved that parallel systems preserve both the DMRL class and the NBUE class in the case of i.i.d. component lifetimes. These results had earlier been anticipated by Klefsjö [7].

1.4 Outline and contribution of the paper

The main purpose of the present paper is to study in more detail the preservation properties of the NBUE class and the NWUE class under formation of coherent systems for the i.i.d. case, with the aim of obtaining some new results. For the NBUE class, we first give a necessary condition for a coherent system to preserve this class. In particular, this condition is not satisfied for series systems. Sufficient conditions obtained next, are shown to be satisfied for a collection of systems which includes all parallel systems. As will be pointed out in the final section of the paper, our results on the NBUE class can be extended to the d.i.d. case.

We next prove the somewhat surprising result that, in the i.i.d. case, no coherent system preserves the NWUE property of component distributions. From this, we finally obtain the following consequences for the case of i.i.d. component lifetimes: (i) the DFR class is not preserved by any coherent systems other than series systems, and (ii) no coherent system preserves the IMRL class.

The paper is organized as follows. In Section 2 we give, for later use, some preliminary definitions and results on coherent systems and classes of lifetime distributions. The NBUE class is studied in Section 3. Both a necessary condition and a sufficient condition for preservation of this class are given. Section 4 is devoted to the NWUE class, for which, as already mentioned, no coherent system preserves the class. Section 5 presents results for the DFR and IMRL classes, which are implied by the results and, in particular, by the proof of Theorem 3 in Section 4. In Section 6, we conclude the present study with a brief discussion of the d.i.d. case.

2 Preliminaries

2.1 Coherent systems

2.1.1 Component and system lifetimes

We consider coherent systems with n binary components, with component lifetimes denoted by $X_1, \ldots, X_n$, and with T denoting the system lifetime. The key reference to coherent systems and their properties is the classical monograph by Barlow and Proschan [4]. We assume that $X_1, \ldots, X_n$ are i.i.d. with distribution F, where we let the cumulative distribution function and survival (reliability) function be denoted, respectively F and $\bar{F}$, with $\bar{F} = 1 - F$. We will assume, in addition, that F has finite expectation, denoted by μ. If F is absolutely continuous, we will denote the hazard rate by λ, where $\lambda = F'/\bar{F}$. For the system lifetime T, the corresponding functions are given the subscript T.

2.1.2 Minimal path set representation of T

A *minimal path set* $\mathcal{P}$ is a subset of the component set $\{1, 2, \ldots, n\}$ such that the system works when all components in $\mathcal{P}$ are working, but the system fails if only a proper subset of components in $\mathcal{P}$ are working (with all components in the complementary set $\mathcal{P}'$ assumed to have failed). The system functions if and only if all components in at least one minimal path set are working. Thus, if the minimal path sets are denoted $\mathcal{P}_1, \ldots, \mathcal{P}_m$, the system lifetime is

$$T = \max_{j=1,\ldots,m} \left(\min_{i \in \mathcal{P}_j} X_i \right). \tag{3}$$

If the system is coherent, then no minimal path set can be a subset of another, and their union is the full component set $\{1, 2, \ldots, n\}$, see [4, Chapter 1, p. 15].

2.1.3 Reliability polynomial

Barlow and Proschan [4, Chapter 2, p. 21] define the reliability function $h(p_1, \ldots, p_n)$ to be the probability that the system functions if the components function independently of each other, with p_i being the probability that component i functions. Thus, if the p_i are equal, with value p, the probability that the system functions is a polynomial in p, which we shall denote by $h(p)$, referred to as the *reliability polynomial.* The coefficients of the reliability polynomial depend only on the structure of the system. If the component lifetimes are i.i.d. with survival function $\bar{F}$, then the system lifetime T has survival function

$$\bar{F}_T(t) = h(\bar{F}(t)). \tag{4}$$

The reliability polynomial of a system can be derived in alternative ways, for example using minimal path sets or minimal cut sets of the system, see [4, Chapter 2, p. 24]. The reliability polynomial can also be given in terms of the *signature* $\mathbf{s} = (s_1, \ldots, s_n)$ of the system, see Samaniego [14, p. 84]:

$$h(p) = \sum_{j=1}^{n} a_j \binom{n}{j} p^j (1-p)^{n-j},$$

where

$$a_j = \sum_{i=n-j+1}^{n} s_i \quad \text{for } j = 1, \ldots, n.$$

2.2 Distribution classes

Let F be a lifetime distribution with finite expectation and let $\tau = \sup\{t : \bar{F}(t) > 0\}$. Then for a lifetime X with distribution F, we define

$$m(t) = E[X - t | X > t] = \frac{\int_t^\infty \bar{F}(u)du}{\bar{F}(t)} \text{ for } 0 \leq t < \tau. \tag{5}$$

The definition of the NBUE and NWUE classes, which are the main subjects of the present paper, can be given in terms of the mean residual life function (5). This also applies to the closely related DMRL and IMRL classes. For completeness, these classes are defined below.

2.2.1 NBUE and NWUE

A lifetime distribution F is NBUE (NWUE) if

$$m(t) \leq (\geq)\ \mu \ \text{ for } 0 \leq t < \tau, \tag{6}$$

where $\mu < \infty$ is the expected value of F.

Abouammoh and El-Neweihi [1] noted that condition (6) is equivalent to

$$\left(\int_t^\infty \bar{F}(u)du \right) \frac{1 - \bar{F}(t)}{\bar{F}(t)} \leq (\geq) \int_0^t \bar{F}(u)du. \tag{7}$$

2.2.2 DMRL and IMRL

A lifetime distribution F is DMRL (IMRL) if $m(t)$ is decreasing (increasing) in $[0, \tau)$. Clearly, $m(0) = \mu$ and hence DMRL (IMRL) implies NBUE (NWUE) by (6) above.

Navarro [10] recently studied preservation properties of the DMRL and IMRL distribution classes.

2.2.3 IFR and DFR

We also discuss some connections to the IFR and DFR distribution classes and therefore recall their definitions below.

A lifetime distribution F is IFR (DFR) if $\bar{F}$ is *logconcave (logconvex)*, see Shaked and Shanthikumar [15, p. 1]. If F is absolutely continuous, then this is equivalent to the hazard rate $\lambda(t)$ being increasing (decreasing) in t. Esary and Proschan [5] showed that for a system with reliability polynomial $h(p)$ and component lifetimes with absolutely continuous distribution F, one has

$$\left.\frac{ph'(p)}{h(p)}\right|_{p=\bar{F}(t)} = \frac{\lambda_T(t)}{\lambda(t)}. \tag{8}$$

They concluded from (8) that if F is IFR (DFR), then the system lifetime T has an IFR (DFR) distribution provided $ph'(p)/h(p)$ is decreasing (increasing) in p on $(0,1]$. In fact, it has been noted (Samaniego [13], Navarro et al. [11]) that the converse also holds, that is, a coherent system with reliability polynomial $h(p)$ preserves the IFR (DFR) class *if and only if* $ph'(p)/h(p)$ is decreasing (increasing) for $0 \leq p \leq 1$. Hence these two classes cannot be preserved at the same time in a coherent system, except for series systems (see Proposition 2.1 in [11]).

3 Preservation of the NBUE class

Barlow and Proschan [4, Chapter 6, p. 183] showed that a series system of two components with i.i.d. lifetime distributions does not preserve the NBUE class. Theorem 1 below provides a necessary condition for preservation of the NBUE class. The condition is clearly not satisfied for any series system, but in fact excludes a much larger collection of systems. The theorem is followed by Theorem 2, which provides a sufficient condition for the preservation of the NBUE class by adding a condition to that of Theorem 1.

Theorem 1 *Let a coherent system with i.i.d. component lifetimes have reliability polynomial $h(p)$. Then a necessary condition for the system to preserve the NBUE class is that $h(p) \geq p$ for all $0 \leq p \leq 1$.*

Proof: Consider a coherent system with reliability polynomial $h(p)$. Suppose that there is $0 < p_0 < 1$ such that $h(p_0) < p_0$. We will now prove that the system does not preserve the NBUE class, a fact that implies the desired result. With the given p_0, let the component lifetime distribution F give probability $1 - p_0$ to $t = 1$ and probability p_0 to $t = a$, where

$$a = 1 + 1/(1 - p_0).$$

Thus

$$\bar{F}(t) = \begin{cases} 1 & \text{for } 0 \leq t < 1 \\ p_0 & \text{for } 1 \leq t < a \\ 0 & \text{for } t \geq a \end{cases},$$

while the expected value of F is $\mu = 1-p_0+ap_0 = 1+(a-1)p_0 = 1/(1-p_0)$. That F is NBUE is seen from (5) and (6) and by calculating

$$\frac{\int_t^\infty \bar{F}(x)dx}{\bar{F}(t)} = \begin{cases} 1-t+(a-1)p_0 & \text{for } 0 \le t < 1 \\ a-t & \text{for } 1 \le t < a. \end{cases} \tag{9}$$

Indeed, (6) obviously holds for $0 \le t < 1$, while for $1 \le t < a$ we have

$$a - t \le a - 1 = 1/(1-p_0) = \mu.$$

Thus, F is NBUE.

Now, let $h(p)$ be given as in the beginning of the proof. The corresponding survival function of the system is $\bar{F}_T(t) = h(\bar{F}(t))$, and hence by (9),

$$\frac{\int_t^\infty \bar{F}_T(x)dx}{\bar{F}_T(t)} = \begin{cases} 1-t+(a-1)h(p_0) & \text{for } 0 \le t < 1 \\ a-t & \text{for } 1 \le t < a \end{cases} . \tag{10}$$

The expected value μ_T of F_T is found by letting $t = 0$ in (10), so that

$$\mu_T = 1 + (a-1)h(p_0) < 1 + (a-1)p_0 = 1/(1-p_0) = a - 1.$$

Thus, by (10), $\int_t^\infty \bar{F}_T(x)dx/\bar{F}_T(t) > \mu_T$ for $t = 1$, so (6) does not hold for the system with reliability polynomial $h(p)$. Hence this system is not NBUE, so the system with reliability polynomial $h(p)$ does not preserve the NBUE property.

Remark 1 *It is clear that this theorem rules out any series system as a system that preserves the NBUE property. By (4), the necessary condition of the theorem implies $\bar{F}_T(t) \ge \bar{F}(t)$ for all t, which means that the system lifetime stochastically dominates the component lifetimes.*

Theorem 2 *Let $h(p)$ be the reliability polynomial of a coherent system with i.i.d. component lifetimes, such that $h(p) \ge p$ for all $0 \le p \le 1$, and such that*

$$\frac{p}{h(p)} \cdot \frac{1-h(p)}{1-p} \cdot \sup_{r\in(0,p]} \frac{h(r)}{r} \;\le\; 1 \quad \text{for all } 0 < p < 1. \tag{11}$$

Then the system preserves the NBUE class.

Remark 2 *If $h(r)/r$ decreases in r, then (11) simplifies to*

$$\frac{p}{h(p)} \cdot \frac{1-h(p)}{1-p} \cdot h'(0) \;\le\; 1 \quad \text{for all } 0 < p < 1. \tag{12}$$

Remark 3 *Since $h(p)$ is a polynomial without constant term, it is clear that $h(p)/p$ is always a polynomial. Further, since $1 - h(1) = 0$, it follows that $1 - h(p)$ is divisible by $1 - p$, i.e., it follows that $(1 - h(p))/(1 - p)$ is also a polynomial.*

Proof of Theorem 2: Suppose that F satisfies (6) (and hence (7)), and suppose that $h(p) \geq p$ for $0 \leq p \leq 1$. Now, considering the left-hand side of (7) with $h(\bar{F}(t))$ replacing $\bar{F}(t)$ and noting that $h(p) \geq p$ for all $0 \leq p \leq 1$, we have

$$
\begin{aligned}
&\int_t^\infty h(\bar{F}(u))du \cdot \frac{1 - h(\bar{F}(t))}{h(\bar{F}(t))} \\
&= \frac{\int_t^\infty h(\bar{F}(u))du}{\int_t^\infty \bar{F}(u)du} \cdot \int_t^\infty \bar{F}(u)du \cdot \frac{1 - h(\bar{F}(t))}{h(\bar{F}(t))} \\
&= \frac{\int_t^\infty h(\bar{F}(u))du}{\int_t^\infty \bar{F}(u)du} \cdot \int_t^\infty \bar{F}(u)du \cdot \frac{1 - h(\bar{F}(t))}{h(\bar{F}(t))} \\
&\leq \sup_{r \in (0, \bar{F}(t)]} \frac{h(r)}{r} \cdot \int_0^t \bar{F}(u)du \cdot \frac{\bar{F}(t)}{1 - \bar{F}(t)} \cdot \frac{1 - h(\bar{F}(t))}{h(\bar{F}(t))} \\
&\leq \sup_{r \in (0, \bar{F}(t)]} \frac{h(r)}{r} \cdot \frac{\bar{F}(t)}{1 - \bar{F}(t)} \cdot \frac{1 - h(\bar{F}(t))}{h(\bar{F}(t))} \cdot \int_0^t h(\bar{F}(u))du.
\end{aligned}
$$

Hence (7) holds for $h(\bar{F}(t))$ under the assumption of the theorem, so the system with reliability polynomial $h(p)$ preserves the NBUE class.

Example 1: Parallel system. Abouammoh and El-Neweihi [1] showed that the NBUE class is preserved under the formation of parallel systems with i.i.d. components. Note that this is also implied by Theorem 2, which can be seen as follows. Let $h(p) = 1 - (1 - p)^n$. We first show that $h(p)/p$ is decreasing in p. Let $q = 1 - p$. We will instead show that $(1 - q^n)/(1 - q)$ is increasing in q. This follows from the fact that

$$\frac{1 - q^n}{1 - q} = 1 + q + q^2 + \ldots q^{n-1}. \qquad (13)$$

Thus we can use condition (12). Noting that $h'(0) = n$ and using (13), the left hand side of (12) is

$$\frac{np(1 - p)^n}{(1 - p)(1 - (1 - p)^n)} = \frac{n(1 - q)q^n}{q(1 - q^n)} = \frac{nq^{n-1}}{1 + q + q^2 + \ldots q^{n-1}}$$

which is clearly ≤ 1.

Example 2: A system that preserves the NBUE class but not the IFR class. Consider the system illustrated in Figure 1. As noted by Samaniego [13], this system does not preserve the IFR class. We shall see, however, that the NBUE class is preserved.

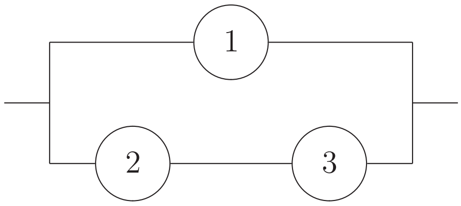

Figure 1: System with 3 components that preserves the NBUE class but not the IFR class.

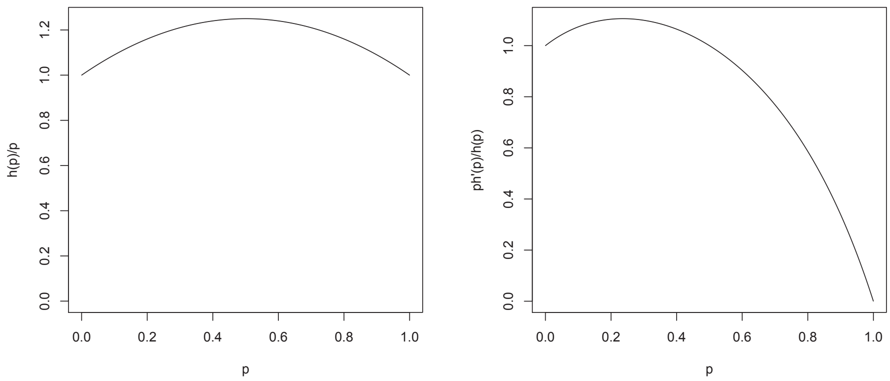

Figure 2: System that preserves the NBUE class but not the IFR class. Left: The function $h(p)/p$. Right: The function $ph'(p)/h(p)$.

In this example, $h(p) = p+p^2-p^3$, so $h(p)/p = 1+p-p^2$ with a maximum of $5/4$ at $p = 1/2$ (see left panel of Fig. 2). Hence

$$\sup_{r\in(0,p]} \frac{h(r)}{r} = \begin{cases} \frac{h(p)}{p} & \text{for } 0 < p \leq 1/2 \\ \frac{5}{4} & \text{for } 1/2 < p \leq 1 \end{cases}$$

Substituting this in the left hand side of (11), we get a decreasing function with a maximum value of 1 at $p = 0$. This shows that the system preserves the NBUE class.

The fact that the IFR class is not preserved by the system in Fig. 1 is illustrated in the right panel of Fig. 2, in which the curve $ph'(p)/h(p)$ is seen not to be decreasing.

Example 3: A system that preserves the IFR class but not the NBUE class. Consider now the system displayed in Figure 3.

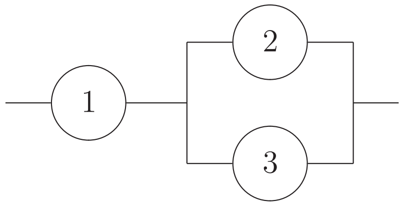

Figure 3: System with 3 components that preserves the IFR class but not the NBUE class.

In this case, $h(p) = 2p^2 - p^3$, so it is seen that $h(p) < p$ for all $0 < p < 1$. Thus Theorem 1 shows that this system does not preserve the NBUE property.

By a simple calulation, we have that

$$\frac{ph'(p)}{h(p)} = \frac{4 - 3p}{2 - p}$$

which is a decreasing function, implying that the system preserves the IFR property.

4 The NWUE class

It is shown in [4, Chapter 6, p. 183] that the NWUE class is not preserved under the formation of coherent systems. Specifically, it is shown that the lifetime of a parallell system of two components with i.i.d. standard exponential distributions is not NWUE. In this section, we show that *no coherent system*, regardless of its size, preserves the NWUE class in the case of i.i.d. components. This is the result of Theorem 3 below. The proof is divided into two cases, one for non-series systems and one for series systems. In the former case, it is shown that for any coherent non-series system with i.i.d. exponential component lifetimes, the system lifetime is not NWUE. In the case of series systems, we present a component lifetime distribution in the NWUE class for which no series system of two or more components will have an NWUE distribution.

Theorem 3 *No coherent systems with i.i.d. components preserve the NWUE class.*

Proof of Theorem 3 for non-series systems: Let $h(p)$ be the reliability polynomial of a system which is not a series system. Suppose the component lifetimes are i.i.d. with the standard exponential distribution. The survival function of the system lifetime is then $h(e^{-t})$, and by (5) and (6), the system lifetime is NWUE provided that

$$\frac{\int_t^\infty h(e^{-x})dx}{h(e^{-t})} \geq \int_0^\infty h(e^{-x})dx \quad \text{for all } t \geq 0. \tag{14}$$

Substitution of $p = e^{-x}$ in the integrals gives

$$\int_t^\infty h(e^{-x})dx = \int_0^{e^{-t}} \frac{h(p)}{p} dp.$$

Hence, by defining $c(r) = \int_0^r \frac{h(p)}{p} dp$ for $0 \leq r \leq 1$, (14) is equivalent to

$$\frac{c(r)}{h(r)} \geq c(1) \quad \text{for all } 0 < r \leq 1. \tag{15}$$

Further, writing $h(p) = \sum_{i=1}^n a_i p^i$, it is seen that

$$c(r) = \sum_{i=1}^n \frac{a_i}{i} r^i,$$

and that

$$\lim_{r \to 0} \frac{c(r)}{h(r)} = \frac{1}{k},$$

where $k = \min\{i : a_i \neq 0\}$. Moreover, k is the number of components in a minimal path set of the system containing the fewest number of components.

It can hence be concluded, by letting r tend to 0 in (15), that the system does not preserve the NWUE property if

$$c(1) \equiv E[T] > \frac{1}{k}, \tag{16}$$

where T is the lifetime of the system.

We now show that (16) holds, recalling that the system under consideration is not a series system. To see this, recall from subsection 2.1.2 that

$$T = \max_{j=1,\ldots,m} \left(\min_{i \in \mathcal{P}_j} X_i \right),$$

where $\mathcal{P}_1, \ldots, \mathcal{P}_m$ are the minimal path sets of the system. Without loss of generality we can assume that $\mathcal{P}_1$ is a minimal path set of minimum size, k. Then if $T_1 = \min_{i \in \mathcal{P}_1} X_i$, we clearly have $T \geq T_1$ with probability 1, while $E[T_1] = 1/k$, since T_1 is the minimum of k independent standard exponentially distributed lifetimes. It thus suffices to show that $E[T] > E[T_1]$. Below, we first show that we can find $w > 0$ and $\alpha > 0$ such that $P(T - T_1 > w) \geq \alpha$. This is seen as follows. Since the system under consideration is coherent and is not a series system, it must have at least two minimal path sets, and no minimal path set can be contained in another. Thus, if we fix one of the other minimal path sets, say $\mathcal{P}_2$, the set $\mathcal{P}_1$ contains at least one component that is not in $\mathcal{P}_2$. The set $\mathcal{P}_1 \setminus \mathcal{P}_2 \equiv \mathcal{P}_1 \cap \mathcal{P}_2^c$ is hence nonempty. Let $w > 0$ be arbitrarily chosen, and let $0 < u < v$ be such that $w = v - u$. Then, with T and T_1 defined as above, we have

$$\begin{aligned}
P(T - T_1 > w) &\geq P(\min_{i \in \mathcal{P}_2} X_i - \min_{i \in \mathcal{P}_1} X_i > w) \\
&\geq P(\{X_i < u, \; \forall i \in \mathcal{P}_1 \setminus \mathcal{P}_2\} \cap \{X_i > v, \; \forall i \in \mathcal{P}_2\}) \\
&\equiv \alpha > 0.
\end{aligned}$$

But then

$$\begin{aligned}
E[T] - E[T_1] &= E[T - T_1] \\
&= E[(T - T_1) I(T - T_1 > w)] + E[(T - T_1) I(T - T_1 \leq w)] \\
&\geq w P(T - T_1 > w) \\
&= w\,\alpha > 0,
\end{aligned}$$

where $I(A)$ is the indicator function of the event A. Hence $E[T] > E[T_1] = 1/k$ and consequently the considered system (an arbitrary non-series system) does not preserve the NWUE class.

Example 4: A system close to a series system. This example is meant to illustrate the main features of the above first part of the proof of Theorem 3. Consider the system illustrated in Figure 4, where $n \geq 2$. If n is large, we

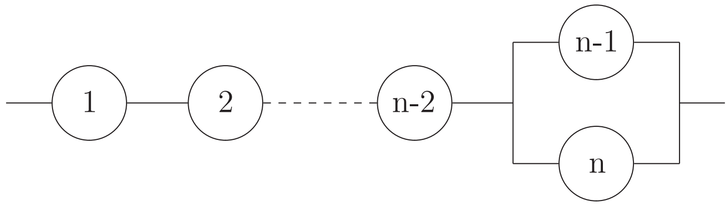

Figure 4: System with n components and $k = n - 1$.

can think of this system as "almost" a series system. For this system, we have $h(p) = 2p^{n-1} - p^n$, and it is readily shown that

$$c(r) = r^{n-1}\left[\frac{2}{n-1} - \frac{r}{n}\right]. \tag{17}$$

We thus obtain

$$\lim_{r\to 0} \frac{c(r)}{h(r)} = \frac{1}{n-1}.$$

This is in accordance with the developments in the above proof of Theorem 3, since the minimal path sets of the system are $\mathcal{P}_1 = \{1, 2, \ldots, n-2, n-1\}$ and $\mathcal{P}_2 = \{1, 2, \ldots, n-2, n\}$, which both have size $n-1$. Note also that $\mathcal{P}_1 \setminus \mathcal{P}_2 = \{n-1\}$. We complete this example by noting that by (17) we have, for all $n \geq 2$,

$$E[T] = c(1) = \frac{2}{n-1} - \frac{1}{n} = \frac{n+1}{n} \cdot \frac{1}{n-1} > \frac{1}{n-1} = E[T_1].$$

Following the proof of the theorem, this shows that the system in Fig. 4 does not preserve the NWUE class.

Proof of Theorem 3 for series systems: We first present a distribution F which is NWUE (in fact it is IMRL), and then show that the NWUE property does not hold for the lifetime of a series system with $n \geq 2$ components with component lifetime distribution F.

Let F have the survival function

$$\bar{F}(t) = \begin{cases} e^{-t} & \text{for } 0 \le t < 2 \ , \\ e^{-1.3t+0.6} & \text{for } 2 \le t < 4 \ , \\ e^{-0.2t-3.8} & \text{for } t \ge 4 \ . \end{cases} \tag{18}$$

Writing c for $1/13 \approx 0.769$ for simplicity, one may calculate

$$m(t) = \frac{\int_t^\infty \bar{F}(u)du}{\bar{F}(t)} = \begin{cases} 1 + e^t[(5-c)e^{-4.6} - (1-c)e^{-2.0}] & \text{for } 0 \le t < 2, \\ c + (5-c)e^{1.3t-5.2} & \text{for } 2 \le t < 4, \\ 5 & \text{for } t \ge 4. \end{cases}$$

The function $m(t)$ is increasing in t, and thus the distribution F is IMRL and hence NWUE. The mean residual life function $m(t)$ is shown in the left panel of Fig. 5.

Consider now a series system of $n \ge 2$ components, having F as component lifetime distribution. We shall show that such a system is neither IMRL nor NWUE.

The survival function of the lifetime T of this series system is $(\bar{F}(t))^n$, and the corresponding mean residual life function (5) is hence given by

$$m_n(t) = \frac{1}{(\bar{F}(t))^n} \int_t^\infty (\bar{F}(u))^n du.$$

Substituting $\bar{F}$ from (18) we obtain, for $n \ge 1$, the particular values

$$\begin{aligned} m_n(0) &= \frac{1}{n}\left\{1 + (5-c)e^{-4.6n} - (1-c)e^{-2.0n}\right\}, \\ m_n(2) &= \frac{1}{n}\left\{c + (5-c)e^{-2.6n}\right\}. \end{aligned}$$

Since $n = 1$ corresponds to the component distribution F itself, one may easily check that $m_1(0) - m_1(2) < 0$. However, for all $n \ge 2$, we have $m_n(0) - m_n(2) > 0$. This is seen in the right panel of Fig. 5 when $n = 2$. For the general case, this can be proven by differentiating $d(n) = m_n(0) - m_n(2)$ with respect to n and observing that the derivative is positive for $n \ge 1$. Thus, the expected lifetime of a new series system is larger than the expected remaining lifetime of one which has reached the age 2. It follows from (6) that the system lifetime T is not NWUE. This completes the proof of Theorem 3 in the series system case.

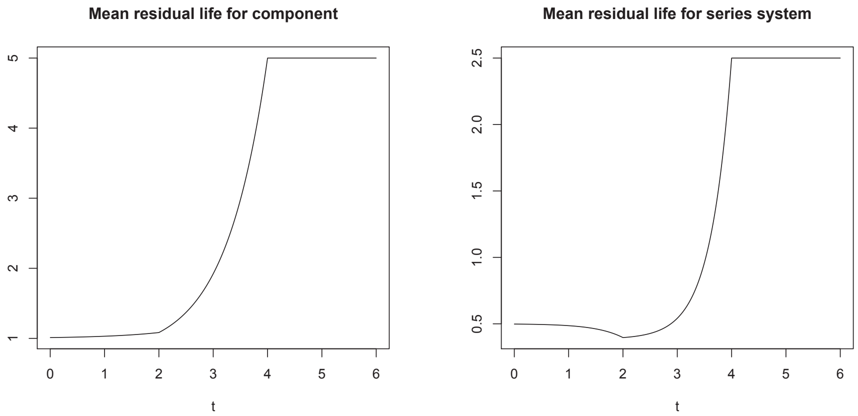

Figure 5: Left: The mean residual life function $m(t)$ for the component distribution F. Right: The mean residual life function $m_2(t)$ for a series system with $n = 2$ components.

5 Consequences for the DFR and IMRL classes

A closer look at the two parts of the proof of Theorem 3 shows that we can conclude as below in the case of i.i.d. components.

Corollary 1 *The following preservation properties hold for systems with i.i.d. component lifetimes:*

(i) *A coherent system preserves the DFR class if and only if it is a series system.*

(ii) *No coherent systems preserve the IMRL class.*

Proof: (i) Here the 'if'-part is well known; see, e.g., [11]. In the proof of the non-series case of Theorem 3, we showed that an exponential component distribution, which in particular is DFR, leads to a system lifetime which is not NWUE. But then it cannot be DFR, since the DFR class is contained in the NWUE class. This proves (i).

(ii) For non-series systems, we showed that an exponential component distribution, which is also IMRL, leads to a system lifetime which is not NWUE and hence not IMRL. Thus, non-series systems do not preserve the IMRL property. Further, in the proof of the series system part of Theorem 3, we showed that a component distribution, which was noted to be IMRL, led to a non-NWUE distribution of the lifetimes of series systems of any size. Since the IMRL class is contained in the NWUE class, this means that the IMRL class is not preserved for series systems, completing the proof of claim (ii).

6 Concluding remarks

In this paper, we have so far restricted the attention to coherent systems with i.i.d. component lifetimes, with the aim to complement existing knowledge. It should be noted, however, that there has been a significant interest in the independent and *non*-identically distributed case in the reliability literature; see, e.g., [4, Chapter 4]).

Recently, Navarro et al. [11] have made important advances also for the case of *dependent* component lifetimes. Specifically, these authors generalize the reliability polynomial to the case where the dependency of the component lifetimes is given by a copula, while the marginal distributions of the component lifetimes are specified separately from the system and the copula.

In particular, [11] considers coherent systems with component lifetimes $X_1, \ldots, X_n$ having identical marginal distributions given by a survival function $\bar{F}$, and a possible dependency among the component lifetimes represented by a survival copula $K(u_1, \ldots, u_n)$ (e.g., Nelsen [12]). In this case, which the authors name the *dependent and identically distributed* case (d.i.d.), the joint survival function for the component lifetimes can be written

$$P(X_1 > t_1, \ldots, X_n > t_n) = K(\bar{F}(t_1), \ldots, \bar{F}(t_n)).$$

From this, the authors demonstrate that the survival function of the system lifetime T can be written

$$\bar{F}_T(t) = q(\bar{F}(t))$$

for a continuous and increasing function $q(u)$ defined on $[0, 1]$, with $q(0) = 0$, $q(1) = 1$. A key feature is that the function q depends on the system structure and on the dependence copula K, but not on the component distribution F. It has been named the *domination function* or the *distortion function*. The i.i.d. case is now obtained as the special case where K is the independence copula given by $K(u_1, \ldots, u_n) = \prod_{i=1}^n u_i$, and q is in this case simply the reliability polynomial h as defined in Section 2.1.3.

It is easy to confirm that the results of Theorems 1 and 2 of the present paper will continue to hold for the d.i.d. case when the reliability function h is replaced by the distortion function d in the statements and in the proofs. Since the proof of Theorem 3 is strongly based on properties of the reliability polynomial for the i.i.d. case, it is, however, not clear whether the conclusion of the theorem holds in some form for the d.i.d. case. This also applies to the conclusions of Corollary 1.

In view of the latter result, it is however interesting to note that Navarro [10, Example 3.3] presents an example where the IMRL class is preserved for a series system with $n = 2$ d.i.d. components, with the dependency given by a Clayton-Oakes copula. This is in contrast to the i.i.d. case where no coherent system preserves this class. Another example in that paper shows, on the other hand, that the IMRL class is not preserved in parallel systems with independent but not identically distributed component lifetimes.

Further studies in these directions are of obvious interest for future research.

Acknowledgments

The authors would like to thank the Editor-in-Chief, the Associate Editor and two reviewers for their valuable comments and suggestions, which led to an improved paper. This work was done while the first author was visiting the Department of Statistics, University of California, Davis in the academic year 2017/2018. The kind hospitality of the faculty and staff is gratefully acknowledged. The second author's research was supported in part by grant W911NF-17-1-0381 from the US Army Research Office.

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