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INFINITY-HARMONIC POTENTIALS AND THEIR STREAMLINES

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Abstract. We consider certain solutions of the Infinity-Laplace Equation in planar convex rings. Their ascending streamlines are unique while the descending ones may bifurcate. We prove that bifurcation occurs in the generic situation and as a consequence, the solutions cannot have Lipschitz continuous gradients.

1. Introduction.

The solutions of the celebrated ∞-Laplace Equation

$$\Delta_\infty u \equiv \sum_{i,j} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j} = 0,$$

which is the formal limit of the p-Laplace Equations

$$\Delta_p u \equiv \nabla\cdot(|\nabla u|^{p-2}\nabla u) = 0$$

as $p \to \infty$, have many fascinating properties. The solutions provide the best Lipschitz extension of their boundary values (see [1]) and the equation appears even in Stochastic Game Theory (see [20]).

A characteristic feature for classical solutions is that *the speed* $|\nabla u|$ *is constant along a streamline*, which is a useful property for applications to image processing, see [5]. Indeed, along the streamline $x = x(t)$ with the equation

$$\frac{dx}{dt} = \nabla u(x(t))$$

we should have

$$\frac{d}{dt}|\nabla u(x(t))|^2 = 2\Delta_\infty u(x(t)) = 0$$

so that

$$|\nabla u(x(t))| = \text{constant}.$$

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However, the calculation requires second partial derivatives. We shall see that this interpretation of constant speed often fails.

The solutions of the ∞-Laplace Equation, the so-called *∞-harmonic functions*, are defined in the viscosity sense as in [11], [13] and [22]. They are continuous and even differentiable. O. Savin [22] has proved that in the plane their gradient is continuous and even locally Hölder continuous, according to [8]. Thus the solutions are of class $C^{1,\alpha}_{loc}$ in the two dimensional case. In [16] the speed $|\nabla u|$ is shown to belong to a Sobolev space. In higher dimensions the gradient exists (in the classical sense) at every point by a result of L. Evans and Ch. Smart, cf. [9]. At the moment of writing, the C^1_{loc}-property is not known in higher dimensions. This unsettled urgent question is the reason for why we restrict our exposition to two dimensions. In the plane the equation reads

$$\Big(\frac{\partial u}{\partial x_1}\Big)^2\frac{\partial^2 u}{\partial x_1^2} + 2\,\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2}\frac{\partial^2 u}{\partial x_1 \partial x_2} + \Big(\frac{\partial u}{\partial x_2}\Big)^2\frac{\partial^2 u}{\partial x_2^2} = 0$$

as in G. Aronsson's work [2] about the streamlines.

Notation. We fix some notation. Suppose that Ω is a *convex* bounded domain in the plane $\mathbb{R}^2$ containing a compact *convex* set K with boundary $\Gamma = \partial K$. The case when K reduces to a single point is of special interest. The domain $G = \Omega \setminus K$ is a "convex ring"; it has the outer boundary $\partial\Omega$ and the inner boundary Γ. The object of our work is the Dirichlet boundary value problem

$$\begin{cases} \Delta_\infty u = 0 & \text{in} \quad G \\ \quad\; u = 0 & \text{on} \quad \partial\Omega \\ \quad\; u = 1 & \text{on} \quad \Gamma. \end{cases} \tag{1}$$

The unique solution, say V_∞, attains the boundary values in the classical sense (this holds for all domains, whether they are convex or not). Hence

$$V_\infty \in C(\overline{G}) \quad \text{where} \quad \overline{G} = \partial\Omega \cup G \cup \Gamma.$$

Some properties. By the Maximum Principle, $0 < V_\infty < 1$ in G. (It is convenient to put $V_\infty = 1$ in K and $= 0$ outside Ω.) The gradient $\nabla V_\infty \in C^\alpha_{loc}(G)$ for some small α, cf. [8]. We use some fundamental properties valid in convex rings, which are due to J. Lewis [18]. See also [10]. We need the following

- The level sets $\{V_\infty(x) > c\}$ are convex, $0 \le c < 1$.
- $\Delta_p V_\infty \equiv \nabla\cdot\big(|\nabla V_\infty|^{p-2}\nabla V_\infty\big) \le 0$ when $p \ge 2$.
- $\nabla V_\infty \neq 0$ in G.

We interpret the inequality $\Delta_p V_\infty \le 0$ in the viscosity sense. This is equivalent to the usual definition of p-superharmonic functions, cf. [13], [12]. In particular "$\Delta V_\infty \le 0$" and so V_∞ *is an ordinary superharmonic function.*

Streamlines. Let us return to the *ascending* streamlines $x = x(t)$. They are the trajectories of the gradient flow

$$\begin{cases} \dfrac{dx}{dt} = \nabla V_\infty(x(t)), \quad t > t_0, \\ x(t_0) = x_0 \in G \cup \partial\Omega \end{cases} \tag{2}$$

and intersect the convex level curves orthogonally. (If the initial point $x_0 \in \partial\Omega$ and $\nabla V_\infty(x(t_0)) = 0$, some special care is needed.) By Peano's Existence Theorem, *there exists at least one solution starting at x_0*. Since $\nabla V_\infty \neq 0$, the trajectory cannot terminate inside G. In fact, $x(t) \in G$ when $t_0 \leq t < T$ for some finite T and $x(T) \in \Gamma$. One of our main results is that the solution is unique.

Theorem 1 (Ascending uniqueness)**.** *The solution to the equation (2) of the ascending gradient flow is unique and terminates at* Γ*.*

Despite uniqueness, two trajectories, starting at different points, can meet and join. But the trajectories cannot cross. *The first point at which two streamlines meet* (after which they become a joint trajectory) is here called a Cl-point. Notice that uniqueness is not valid for the usual *descending* streamlines coming from the equation

$$\frac{dx}{dt} = -\nabla V_\infty(x(t))$$

with a *minus* sign! They allow bifurcation. The proof of the uniqueness theorem is delicate, since the Picard-Lindelöf Theorem is not applicable, when ∇V_∞ is not Lipschitz continuous. (Mere Hölder continuity is not sufficient.) We base our reasoning on the expedient inequality

$$\oint_{\partial D} |\nabla V_\infty|^{p-2} \langle \nabla V_\infty, \mathbf{n} \rangle \, ds \leq 0, \qquad p \geq 2, \tag{3}$$

valid for any domain $D \subset\subset G$ with Lipschitz boundary ∂D. Here $\mathbf{n}$ denotes the outer unit normal. The proof given in Proposition 1 requires several regularizations so that the inequality $\Delta_p V_\infty \leq 0$ can be used pointwise as in [12]. The difficulty is the absence of second derivatives.

Our next theorem provides a tricky device for detecting Cl-points.

Theorem 2. *Let* $\xi_0 \in \partial\Omega$ *and denote*

$$\alpha = \limsup_{x \to \xi_0} |\nabla V_\infty(x)|.$$

Assume that

$$\beta \leq \liminf_{x \to \xi} |\nabla V_\infty(x)| \quad \textit{whenever} \quad \xi \in \Gamma.$$

If $\beta > \alpha$*, then there exists a neighborhood of* ξ_0 *such that every pair of streamlines starting there will meet before reaching* Γ*.*

In general, we have not succeeded in proving that the speed $|\nabla V_\infty(x(t))|$ is nondecreasing along the streamline. Thus the use of the theorem is somewhat elaborate. Let us mention some immediate consequences. First, the fact that two streamlines meet means that the *descending* gradient flow does not have unique solutions. By the Picard-Lindelöf Theorem the function $-V_\infty$ cannot therefore belong to the class $C^{1,1}_{loc}(G)$ in the presence of Cl-points. By general theory, the *descending* gradient flow $\frac{dx}{dt} = -\nabla u(x)$ has a unique solution if u is locally semiconvex . It follows that our V_∞ cannot be locally semi*convex*[1]. (Neither can $\psi(V_\infty)$ be for a smooth strictly monotone function ψ, since V_∞ and $\psi(V_\infty)$ have the same level sets.)

To apply the theorem we notice that it is always possible to choose $\beta > 0$, see Lemma 7. Thus, if we can find a point $\xi_0 \in \partial\Omega$ yielding $\alpha = 0$, we have obtained the inequality $\beta > \alpha$. According to a result in [19] the following holds in convex domains in the plane: if the boundary has an irregular boundary point which is a corner with interior angle less than π, then $|\nabla V_\infty| = 0$ at the corner. This provides an $\alpha = 0$.

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[1] A function f is semiconvex if $f(x) + C|x|^2$ is convex for some constant $C > 0$.

Theorem 3. *If $\partial\Omega$ has a corner with angle less than π, then there are streamlines that meet in G before reaching Γ. In particular, V_∞ is not of class $C^{1,1}_{loc}(G)$.*

For a special kind of domains the distance function $\operatorname{dist}(x, \partial\Omega)$ is the ∞-potential. A *stadium* is a domain where the distance function attains its maximum value at all its singular points. These sets have a simple characterization in the plane. Namely,

$$H = \{x |\ \operatorname{dist}(x, \partial\Omega) = \| \operatorname{dist}(x, \partial\Omega)\|_\infty\},$$
$$\Omega = \{x |\ \operatorname{dist}(x, H) < \| \operatorname{dist}(x, \partial\Omega)\|_\infty\}.$$

See Theorem 6 in [3]. The set H is called the *High Ridge*. The simplest example of a stadium is the unit disk:

$$H = \{0\}, \quad \Omega = \{x |\, 0 < |x| < 1\}.$$

In a stadium, when Γ is the High Ridge, the solution is smooth and no streamlines meet. We argue that all other convex rings have Cl-points. If Γ is a single point we have the following theorem.

Theorem 4. *Assume that Γ is a single point. If Ω is not a disk centered at Γ, then there are streamlines that meet. In fact, all streamlines that are not entirely inside the closed disk with radius $\operatorname{dist}(\Gamma, \partial\Omega)$ centered at Γ have Cl-points. In particular, V_∞ is not $C^{1,1}_{loc}(G)$.*

That V_∞ is not of class $C^2_{loc}(G)$ has been proved before, see Corollary 1.2 in [23]. See also Corollary 23 in [4] for a related result. Also the case when Γ is a subset of the *High Ridge* (though the domain is not necessarily a stadium) is accessible.

Theorem 5. *Suppose Γ is a subset of the High Ridge of Ω. Unless Ω is a stadium and Γ its High Ridge, there are streamlines that meet. In particular, V_∞ is not $C^{1,1}_{loc}(G)$.*

We also mention that Theorem 4 reveals a queer *instability* for the ∞-Laplace equation. Indeed, the solution of (1) in the disk $0 < |x| < 1$ is smooth, while the corresponding solution in an ellipse exhibits points where the second order derivatives are not bounded. After a coordinate transformation, this implies that in a disk the solution of (1) with Δ_∞ replaced by the operator

$$u_x^2 u_{xx} + 2(1+\delta) u_x u_y u_{xy} + (1+\delta)^2 u_y^2 u_{yy}$$

exhibits this kind of singularites for any $\delta > 0$, but not for $\delta = 0$. A similar instability occurs if the midpoint of the disk is perturbed.

We conclude our work with some remarks about a square. This is a challenging example, indeed. Now the domain Ω is a square and Γ is its midpoint. In this case the gradient ∇V_∞ is continuous also on the sides, but $\nabla V_\infty = 0$ at the four corners (and only there), which gives an $\alpha = 0$ for free in Theorem 2. By symmetry the diagonals are streamlines, so are the medians. It seems as if all the streamlines, except the four medians, would join a diagonal before reaching the midpoint (see Figure 1). We record three results.

First, we show that there are infinitely many Cl-points near the corners. Second, we show that also near the origin there are are infinitely many Cl-points. Finally, we argue that all the streamlines, except the medians, do have infinitely many Cl-points. (It seems as if all points on the diagonals were Cl-points and that these are the only Cl-points.) It is likely that the ∞-harmonic potential function is related to the ∞-eigenvalue problem, introduced in [15]. Indeed, this resemblance was the starting point of our investigation.

The reader is supposed to be familiar with the ∞-Laplacian. For the concept of viscosity solutions we refer to [17] and [6]. We use standard notation. We restrict ourselves to the plane, but most of our exposition is valid even in higher dimensions provided that the gradient ∇V_∞ be continuous.

2. **Preliminaries.** A fundamental tool is inequality (3) for line integrals. For smooth functions it comes from an integration by parts. We shall use the method in [12].

Proposition 1. *Let $p \geq 2$ and assume that $D \subset\subset G$ has a Lipschitz boundary ∂D. Then*

$$\boxed{\oint_{\partial D} |\nabla V_\infty|^{p-2}\langle \nabla V_\infty, \mathbf{n}\rangle \, ds \;\leq\; 0} \tag{4}$$

where $\mathbf{n}$ is the outer unit normal.

Proof. Due to the lack of second derivatives we use two regularizations.

Step 1. Let $V_{\infty,\varepsilon}$ be the infimal convolution

$$V_{\infty,\varepsilon}(x) \;=\; \inf_{y\in G}\Big\{V_\infty(y) + \frac{|x-y|^2}{2\varepsilon}\Big\}.$$

By standard theory $V_{\infty,\varepsilon} \nearrow V_\infty$ locally uniformly in G and

$$\Delta_p V_{\infty,\varepsilon} \;\leq\; 0 \qquad \text{in} \qquad D \tag{5}$$

in the viscosity sense, when $\varepsilon > 0$ is small enough. The fact that $\Delta_p V_\infty \;\leq\; 0$ implies this. Furthermore, the function

$$V_{\infty,\varepsilon}(x) \;-\; \frac{|x|^2}{2\varepsilon}$$

is concave. Therefore it has second derivatives in the sense of Alexandroff a.e. So does $V_{\infty,\varepsilon}$. It follows that inequality (5) holds almost everywhere, when the second derivatives are taken in Alexandroff's sense. At almost every $x \in D$

$$V_{\infty,\varepsilon}(y) \;=\; V_{\infty,\varepsilon}(x) + \langle \nabla V_{\infty,\varepsilon}(x), y-x\rangle$$
$$+\frac{1}{2}\langle y-x, \mathbb{D}^2 V_{\infty,\varepsilon}(x)(y-x)\rangle + o(|x-y|^2)$$

as $y \to x$. Here $\mathbb{D}^2 V_{\infty,\varepsilon}$ is the Hessian matrix of second Alexandroff derivatives.

Step 2. We claim that

$$\nabla V_{\infty,\varepsilon} \to \nabla V_\infty$$

a.e. in D, as $\varepsilon \to 0$. Since $V_{\infty,\varepsilon}$ is Lipschitz continuous, it is differentiable almost everywhere. Fix a point $x \in D$ at which $\nabla V_{\infty,\varepsilon}(x)$ exists. The infimum is attained at a point x_ε in G:

$$V_{\infty,\varepsilon}(x) \;=\; V_\infty(x_\varepsilon) + \frac{|x-x_\varepsilon|^2}{2\varepsilon}.$$

It is easy to see that

$$\nabla V_{\infty,\varepsilon}(x) \;=\; \nabla V_\infty(x_\varepsilon). \tag{6}$$

Indeed,

$$V_{\infty,\varepsilon}(x+h) - V_{\infty,\varepsilon}(x) \leq V_\infty(y) + \frac{|x+h-y|^2}{2\varepsilon} - V_\infty(x_\varepsilon),$$

provided that $x+h$ and y are in G. The choice $y = x_\varepsilon + h$ yields

$$V_{\infty,\varepsilon}(x+h) - V_{\infty,\varepsilon}(x) \leq V_\infty(x_\varepsilon + h) - V_\infty(x_\varepsilon).$$

Write $h = t\mathbf{e}$, $t > 0$, where $\mathbf{e}$ is a unit vector. Divide by t and let $t \to 0^+$ to see that

$$\langle \nabla V_{\infty,\varepsilon}(x), \mathbf{e}\rangle \leq \langle \nabla V_\infty(x_\varepsilon), \mathbf{e}\rangle.$$

Since $\mathbf{e}$ was arbitary, (6) follows. The convergence at x now follows from

$$\begin{aligned} |\nabla V_{\infty,\varepsilon}(x) - \nabla V_\infty(x)| &= |\nabla V_\infty(x_\varepsilon) - \nabla V_\infty(x)| \\ &\leq C_D |x - x_\varepsilon|^\alpha \\ &\leq C_D \varepsilon^{\alpha/2} \to 0, \end{aligned} \tag{7}$$

as $\varepsilon \to 0$, upon renaming the constant, since ∇V_∞ is locally Hölder continuous in G. Thus (7) holds at a.e. point x.

We also note that

$$\nabla V_{\infty,\varepsilon}(x) = \frac{x - x_\varepsilon}{\varepsilon} = \nabla V_\infty(x_\varepsilon)$$

necessarily holds at a point of differentiability. Therefore, x_ε is unique at such a point.

From (6) we also get the uniform bound

$$\|\nabla V_{\infty,\varepsilon}\|_{L^\infty(D)} \leq \|\nabla V_\infty\|_{L^\infty(G)},$$

which will be needed.

Step 3. To obtain second derivatives we define the convolution

$$V_{\infty,\varepsilon,j} = V_{\infty,\varepsilon} \star \rho_j$$

where ρ_j is a standard mollifier. Since (7) holds a.e., the following estimate follows from a standard argument

$$\|\nabla V_{\infty,\varepsilon,j}(x) - \nabla V_{\infty,j}(x)\|_{L^\infty(D)} \leq C\varepsilon^{\alpha/2}, \tag{8}$$

for some $\alpha > 0$.

By the proof of Alexandroff's Theorem in [7]

$$\mathbb{D}^2 V_{\infty,\varepsilon} = \lim_{j\to\infty} \big(D^2(V_{\infty,\varepsilon} \star \rho_j)\big)$$

almost everywhere. Thus

$$\lim_{j\to\infty} \Delta_p V_{\infty,\varepsilon,j} = \Delta_p V_{\infty,\varepsilon}$$

almost everywhere in D; the second derivatives are in the sense of Alexandroff. The convolution preserves concavity:

$$D^2 V_{\infty,\varepsilon,j} \leq \frac{I_2}{\varepsilon}, \qquad \Delta V_{\infty,\varepsilon,j} \leq \frac{2}{\varepsilon}$$

where I_2 is the identity matrix. It is immediate that

$$|\nabla V_{\infty,\varepsilon,j}| \leq \|\nabla V_{\infty,\varepsilon}\|_{\infty,D} \leq \|\nabla V_\infty\|_{\infty,G} = C.$$

Together, these inequalities yield the bound

$$-\Delta_p V_{\infty,\varepsilon,j} \geq -C^{p-2}\,\frac{2 + (p-2)}{\varepsilon}.$$

Thus we can use Fatou's Lemma to obtain

$$\begin{aligned}&\liminf_{j\to\infty}\iint_D(-\Delta_p V_{\infty,\varepsilon,j})\,dx_1dx_2\\ &\geq\iint_D\liminf_{j\to\infty}(-\Delta_p V_{\infty,\varepsilon,j})\,dx_1dx_2\\ &=\iint_D(-\Delta_p V_{\infty,\varepsilon})\,dx_1dx_2\geq\iint_D 0\,dx_1dx_2\;=\;0,\end{aligned}\tag{9}$$

where inequality (5) was used at the end.

Step 4. By the Divergence Theorem

$$\oint_{\partial D}|\nabla V_{\infty,\varepsilon,j}|^{p-2}\langle\nabla V_{\infty,\varepsilon,j},\mathbf{n}\rangle\,ds\;=\;\iint_D\Delta_p V_{\infty,\varepsilon,j}\,dx_1dx_2.$$

By (8),

$$\oint_{\partial D}|\nabla V_{\infty,\varepsilon,j}|^{p-2}\langle\nabla V_{\infty,\varepsilon,j},\mathbf{n}\rangle\,ds=\oint_{\partial D}|\nabla V_{\infty,j}|^{p-2}\langle\nabla V_{\infty,j},\mathbf{n}\rangle\,ds+\mathcal{O}(\varepsilon^{\alpha/2}).$$

Therefore, since $\nabla V_{\infty,j}\to\nabla V_\infty$ uniformly,

$$\begin{aligned}\oint_{\partial D}|\nabla V_\infty|^{p-2}\langle\nabla V_\infty,\mathbf{n}\rangle\,ds+\mathcal{O}(\varepsilon^{\alpha/2})&\leq\limsup_{j\to\infty}\oint_{\partial D}|\nabla V_{\infty,\varepsilon,j}|^{p-2}\langle\nabla V_{\infty,\varepsilon,j},\mathbf{n}\rangle\,ds\\ &\leq 0,\end{aligned}$$

by (9). Since ε is arbitrary, the proposition follows.

□

The function

$$W_\infty\;=\log(V_\infty)$$

is often more convenient. It has the same level curves and streamlines as V_∞. Under the same assumptions as in Proposition 1 we have

$$\boxed{-(p-1)\iint_D|\nabla W_\infty|^p\,dx_1dx_2\;\geq\;\oint_{\partial D}|\nabla W_\infty|^{p-2}\langle\nabla W_\infty,\mathbf{n}\rangle ds.}$$

The proof is similar, since

$$\Delta_p v+(p-1)|\nabla v|^p\;=\;\frac{\Delta_p u}{u^{p+1}},\qquad v=\log(u)$$

holds for smooth functions $u>0$.

3. **Estimates for the Gradient.**

Lemma 6. *We have*

$$0<|\nabla V_\infty(x)|\leq\frac{1}{\operatorname{dist}(\Gamma,\partial\Omega)}\quad\text{when}\quad x\in G.$$

Proof. That $\nabla V_\infty \neq 0$ is proved in [18], see also [10]. This is a simple consequence of the convexity of the level curves.[2]

Since V_∞ is an optimal extension of its boundary values,

$$\|\nabla V_\infty\|_{\infty,G} \;\leq\; \|\nabla v\|_{\infty,G}$$

for every Lipschitz function $v \in C(\overline{G})$ with the same boundary values as V_∞. The distance function

$$v(x) \;=\; \min\left\{1, \frac{\operatorname{dist}(x,\partial\Omega)}{\delta}\right\}, \quad \text{where} \quad \delta \;=\; \operatorname{dist}(\Gamma,\partial\Omega)$$

will do. Now $|\nabla v| = 1/\delta$ almost everywhere. The upper bound follows. □

Lemma 7. *We have*

$$\liminf_{x\to\xi} |\nabla V_\infty(x)| \;\geq\; \beta \;>\; 0 \quad \textit{whenever} \quad \xi \in \Gamma$$

where the constant $\beta = \operatorname{diam}(\Omega)^{-1}$.

Proof. A simple geometric reasoning provides this. Since the level curves are convex, a level set always lies entirely on one side of the tangent lines. This makes it possible to construct a linear function which lies above V_∞ in that part of Ω which is on the outer side of a tangent and which coincides with $V_\infty(\xi)$ at the tangent point ξ. The slope of the plane can be taken to be $\leq V_\infty(\xi)/\operatorname{diam}(\Omega)$ and now $V_\infty(\xi) = 1$. (The reader may wish to draw a picture.) Then the comparison principle yields the estimate. □

Proposition 2. *Let* Γ *be a single point, say* $\Gamma = \{0\}$. *Then*

$$\lim_{x\to 0} |\nabla V_\infty(x)| \;=\; \sup_G\{|\nabla V_\infty|\}.$$

Proof. By Theorem 1 in [23]

$$\lim_{x\to 0} \frac{V_\infty(x) - 1 + c|x|}{|x|} \;=\; 0, \qquad c \;=\; \sup_G\{|\nabla V_\infty|\},$$

and $c > 0$. Let $\varepsilon > 0$. Writing $x = r(y+z)$ where $r > 0$, $|y| < 1$, and $|z| = 1$, we have

$$\big|V_\infty\big(r(y+z)\big) - 1 - cr|y+z|\big| \;\leq\; \varepsilon r|y+z| \;<\; 2\varepsilon r$$

for $0 < r < r_\varepsilon$ (= some number < 1). Keep $|z| = 1$ fixed. Dividing out r we get

$$\sup_{B(z,1)} \left|\frac{V_\infty(rx) - 1}{r} - c|x|\right| \;<\; 2\varepsilon \tag{10}$$

when $0 < r < r_\varepsilon$. According to Theorem 2 in [23], inequality (10) implies that for any $\delta > 0$ we can find an ε_δ such that

$$\left|\nabla\Big(\frac{V_\infty(rx) - 1}{r}\Big) - \nabla(c\,|x|)\right|_{x=z} \;<\; \delta \quad \text{when} \quad 0 < \varepsilon < \varepsilon_\delta$$

which is equivalent to

$$\left|\nabla V_\infty(rz) - c\frac{z}{|z|}\right| \;<\; \delta.$$

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[2] Actually, one has

$$|\nabla V_\infty(x)| \;\geq\; |V_\infty(x)|\operatorname{diam}(\Omega)^{-1}.$$

This holds for all $0 < r < r_\varepsilon$ where $0 < \varepsilon < \varepsilon_\delta$ and hence it follows as $r \to 0$ that

$$\lim_{x\to 0} |\nabla V_\infty(x)| \;=\; c > 0,$$

as desired. □

Corollary 1. *Under the same assumptions as in Proposition 2,*

$$\lim_{x\to 0} |\nabla V_\infty(x)| \;=\; \frac{1}{\operatorname{dist}(\Gamma,\partial\Omega)}.$$

Proof. Since the function V_∞ is an optimal Lipschitz extension of its boundary data, it follows from Proposition 2 that

$$\|\nabla V_\infty\|_{\infty(\Omega)} = \|V_\infty\|_{\mathrm{Lip}(\Gamma\cup\partial\Omega)} = \frac{1}{\operatorname{dist}(\Gamma,\Omega)}.$$

□

If Γ is part of the High Ridge, it must be a point or a segment of a straight line. Corollary 1 can be extended to this case.

Proposition 3. *Let Γ be a segment on the High Ridge of Ω. Then*

$$\lim_{x\to \xi} |\nabla V_\infty(x)| \;=\; \frac{1}{\operatorname{dist}(\Gamma,\partial\Omega)}, \qquad \xi\in\Gamma.$$

Proof. We normalize the geometry so that Γ is the closed segment joining the points $(\pm a, 0)$ on the x_1-axis and $\operatorname{dist}(\xi,\partial\Omega) = 1$ whenever $\xi\in\Gamma$. Now $V_\infty(x) \le \operatorname{dist}(x,\partial\Omega)$. Construct the largest stadium S with Γ as its High Ridge which is contained in Ω. That is,

$$S = \{x\in\Omega |\ \operatorname{dist}(x,\Gamma) < 1\}.$$

It follows by comparison that

$$\operatorname{dist}(x,\partial S) \;\le\; V_\infty(x) \;\le \operatorname{dist}(x,\partial\Omega), \qquad x\in S,$$

because $\operatorname{dist}(x,\partial S)$ is ∞-harmonic in $S\setminus\Gamma$.

In particular, since the domain is convex, these functions coincide on a rectangle:

$$\operatorname{dist}(x,\partial S) \;=\; V_\infty(x) \;= \operatorname{dist}(x,\partial\Omega) = 1 - |x_2| \tag{11}$$

for $-a \;\le\; x_1 \;\le a$ and $-1 \;\le x_2 \;\le 1$. (Draw the unit discs with centers $(\pm a, 0)$ to see that the points $(\pm a, \pm 1)$ are the corners of a rectangle in Ω.)

As we shall see, V_∞ is glued together of three pieces (inspired by the example in Section 5 of [14]). Let u_L be the solution of (1) with $\Gamma = \{(-a, 0)\}$. Similarly, we define u_R with $\Gamma = \{(a, 0)\}$. Now Corollary 1 implies

$$\lim_{x\to(-a,0)} |\nabla u_L(x)| \;=\; 1, \qquad \lim_{x\to(+a,0)} |\nabla u_R(x)| \;=\; 1.$$

We claim that in Ω

$$V_\infty(x) \;=\; \begin{cases} u_R(x), & x_1 \ge a \\ \operatorname{dist}(x,\partial\Omega), & a \ge x_1 \ge -a \\ u_L(x) & x_1 \le -a. \end{cases}$$

First, it is continuous. Second, it is ∞-harmonic in $\Omega\cap\{|x_1| \;>\; a\}$ and when $|x_1| \;\le\; a$ the function coincides with V_∞ by (11). The desired result follows by comparison. □

4. **Proofs of the Theorems.**

Proof of Theorem 1. Assume that two streamlines $x_1(t)$ and $x_2(t)$ for the ascending gradient flow in equation (2) emerge at a point $x_{Cl} \in G$. If they intersect some level curve at the points y_1 and y_2 and $y_1 \neq y_2$, then we apply the fundamental inequality (4) to the domain D bounded by parts of the three curves $x_1(t), x_2(t)$, and the level curve. Only the arcs with endpoints: x_{Cl}, y_1, and y_2 count. (One may think of a curved triangle). By inequality (4)

$$0 \geq \oint_{\partial D} |\nabla V_\infty|^{p-2}\langle \nabla V_\infty, \mathbf{n}\rangle\, ds \,=\, \int_{y_1}^{y_2} |\nabla V_\infty|^{p-1}\, ds$$

since naturally $\langle \nabla V_\infty, \mathbf{n}\rangle = 0$ along the streamlines and

$$\mathbf{n} \,=\, +\frac{\nabla V_\infty}{|\nabla V_\infty|}$$

is the outer unit normal along the level curve between the points y_1 and y_2. Since ∇V_∞ is continuous, it must be identically 0 along this level curve. This contradicts the fact that $\nabla V_\infty \neq 0$ in G. Hence we must have $y_1 = y_2$ and so the streamlines coincide: $x_1(t) \equiv x_2(t)$. $\square$

For a curved quadrilateral bounded by the arcs of two level curves and of two streamlines we have a convenient comparison for the supremum norm of ∇V_∞ on the level arcs. The result indicates that such quadrilaterals cannot always exist, not if the level difference is too big.

Lemma 8. *Assume that*

- *the points x_1 and x_2 are on the same level curve $V_\infty = a$,*
- *the points y_1 and y_2 both are on the higher level curve $V_\infty = b > a$,*
- *ascending streamlines join x_1 with y_1 and x_2 with y_2.*

Then

$$\|\nabla V_\infty\|_{\infty,\overline{y_1y_2}} \leq \|\nabla V_\infty\|_{\infty,\overline{x_1x_2}}, \tag{12}$$

that is, the lower level curve has the larger maximum norm for the gradient.

Proof. Use inequality (4) on the boundary of the domain D bounded by the four arcs. The streamlines do not contribute to the line integral. Along the level arcs the outer normal has the directions $\pm\nabla V_\infty$, the minus sign being for the lower arc between x_1 and x_2. This yields

$$\int_{y_1}^{y_2} |\nabla V_\infty|^{p-1}\, ds \leq \int_{x_1}^{x_2} |\nabla V_\infty|^{p-1}\, ds.$$

Taking the $p-1^{\text{th}}$ roots and sending p to ∞, we arrive at inequality (12). $\square$

Proof of Theorem 2. The theorem follows from the above lemma. Indeed, let $\varepsilon > 0$ be very small. There is a strip near Γ, say $\operatorname{dist}(x,\Gamma) < l_\varepsilon$, where $|\nabla V_\infty| > \beta - \varepsilon$. This strip contains all sufficiently high level curves. In a neighborhood of ξ_0 we have $|\nabla V_\infty| < \alpha + \varepsilon$. If two different streamlines, starting at the same level curve in this neighborhood reach the strip without joining, then it follows from inequality (12) that we must have

$$\beta - \varepsilon \leq \alpha + \varepsilon,$$

which for a small ε contradicts the assumption $\beta > \alpha$. Therefore the streamlines must have joined before reaching the top level. □

We now prove a localized version of Theorem 2, which is Corollary 2. In order to do that, we need the following equicontinuity of streamlines.

Proposition 4 (Convergence)**.** *Suppose that a sequence of streamlines*

$$\gamma_k \,=\, \gamma_k(t), \quad 0 \le t \le T, \qquad (k = 1, 2, 3, ...)$$

in G is given. Then the family $\{\gamma_k\}$ is equicontinuous and bounded. Furthermore, if the initial points $\gamma_k(0)$ converge to a point $a \in G$, then the streamlines converge uniformly to the streamline via a.

Proof. Integrating the equation

$$\frac{d\gamma_k}{dt} \,=\, \nabla V_\infty(\gamma_k(t))$$

we see that

$$|\gamma_k(t_2) - \gamma_k(t_1)| \,=\, \left| \int_{t_1}^{t_2} \nabla V_\infty(\gamma_k(t))\, dt \right| \le C|t_2 - t_1|$$

by Lemma 6. Also

$$|\gamma_k(t)| \,=\, \left| \int_0^t \nabla V_\infty(\gamma_k(\tau))\, d\tau \right| \,\le\, Ct \,\le\, CT.$$

Hence the family is uniformly equicontinuous and bounded.

Thus we can apply Ascoli's Theorem to find a uniformly convergent subsequence, say

$$\gamma_{k_j} \to \gamma.$$

We may take the limit under the integral sign in

$$\gamma_{k_j}(t) - \gamma_{k_j}(0) \,=\, \int_0^t \nabla V_\infty(\gamma_{k_j}(\tau))\, d\tau$$

to arrive at

$$\gamma(t) - \gamma(0) \,=\, \int_0^t \nabla V_\infty(\gamma(\tau))\, d\tau.$$

Differentiating, we get

$$\frac{d\gamma}{dt} \,=\, \nabla V_\infty(\gamma(t)),$$

which means that the limit curve is a streamline and $\gamma(0) = a$.

This was for a subsequence, but using the uniqueness theorem (Theorem 1) one can deduce that also the full sequence γ_k converges. □

Corollary 2. *Suppose that a streamline γ joins the points a_0 and b_0 in G, where a_0 is on the lower level, i.e. $V_\infty(a_0) \le V_\infty(b_0)$. If*

$$|\nabla V_\infty(b_0)| \,>\, |\nabla V_\infty(a_0)|,$$

then there is a neighborhood of a_0 such that every streamline starting there joins the streamline γ before reaching the level curve of b_0.

Proof. By continuity, we can find a neighborhood of a_0 and a neighborhood of b_0 such that the strict inequality above holds extended to the neighborhoods. Consider a sequence of points a_k on the level curve of a_0 such that $a_k \to a_0$. By Proposition 4 the streamlines γ_k starting at a_k converge uniformly to γ. This implies that when the index k is big enough, the streamline starting at a_k must reach the level of b_0 at a point inside the upper neighborhood. By Theorem 2 this is possible only if the streamline has joined γ already before reaching the upper level. (It means that all these streamlines pass via the point b_0.) □

Proof of Theorem 4. We may assume that $\Gamma = \{0\}$ and $\operatorname{dist}(\Gamma, \partial\Omega) = 1$ so that

$$\lim_{x \to 0} |\nabla V_\infty(x)| = 1$$

by Theorem 2 and its Corollary. With this normalization $B = B(0,1)$ is the largest disk centered at 0 which is comprised in Ω. If $B \neq \Omega$, we can find a point $\xi \in \partial\Omega$ such that $\xi \notin \overline{B}$. Consider the streamline $x = x(t)$ from ξ to the origin. By Lemma 6 $|\nabla V_\infty| \leq 1$. We have two cases.

If $|\nabla V_\infty(x(t^*))| < 1$ at some point $x^* = x(t^*)$ then there is a neighborhood U^* of x^* where $|\nabla V_\infty| \leq \alpha < 1$ for some suitable α. Given a small $\varepsilon > 0$, there is a neighborhood of the top 0 in which $|\nabla V_\infty| > 1-\varepsilon$. If ε is so small that $\alpha < 1-\varepsilon$, the quadrilateral described in Lemma 8 cannot exist, since inequality (12) is violated. This means that any two streamlines passing via the neighborhood U^* must join before reaching the top.

We are left with the case $|\nabla V_\infty(x(t)| \equiv 1$. Using the arclength

$$s = \int_0^t |\nabla V_\infty(x(\tau))|\, d\tau, \qquad \frac{ds}{dt} = |\nabla V_\infty(x(t))|$$

as parameter we see that

$$\begin{aligned} 1 = V_\infty(0) - V_\infty(\xi) &= \int_0^T \frac{dV_\infty(x(t))}{dt}\, dt = \int_0^T \langle \nabla V_\infty(x(t)), \frac{dx(t)}{dt} \rangle\, dt \\ &= \int_0^T |\nabla V_\infty(x(t))|^2\, dt = \int_0^s \overbrace{|\nabla V_\infty(x(s))|}^{\equiv 1}\, ds = s \end{aligned}$$

Thus the length of the streamline from ξ to 0 is $= 1$. But that violates the requirement that $|\xi - 0| > 1$. Therefore this second case is impossible.

The proof reveals that all streamlines starting outside the inscribed disk $\overline{B}$ have Cl-points. □

Proof Theorem 5. The proofs follows the same lines as the proof of Theorem 4. The only difference is that we use Proposition 3 instead of Proposition 1. □

5. **The Streamlines in a Square.**

In this section, Ω is the square defined by

$$-1 < x_1 < 1, \ -1 < x_2 < 1$$

and Γ is the origin $(0,0)$. Thus $V_\infty(0,0) = 1$. In this case the ∞-potential V_∞ can be defined in the whole plane by reflection through the sides of the square. (The principle is the same as the Schwartz reflecion for harmonic functions.) The resulting function is ∞-harmonic except at the isolated points $(2m, 2n)$, $m, n = 0, \pm 1, \pm 2, \ldots$ The gradient ∇V_∞ is now continuous except at the aforementioned points. Moreover, at the corners $\nabla V_\infty(\pm 1, \pm 1) = 0$ since $V_\infty = 0$ on the sides of the square.

Comparison yields

$$1-|x| \,\le\, V_\infty(x) \,\le\, \operatorname{dist}(x,\partial\Omega)$$

so that V_∞ is a linear function on the medians (= the coordinate axes).

If $x_p = x_p(t)$ is a streamline for the p-harmonic function V_p with the same boundary values as V_∞ so that $V_p \to V_\infty$ as $p\to\infty$, then

$$\begin{aligned}\frac{d}{dt}V_p(x_p(t)) &= \langle \nabla V_p, \frac{dx_p}{dt}\rangle \,=\, |\nabla V_p(x_p(t))|^2\\ \frac{d^2}{dt^2}V_p(x_p(t)) &= \frac{d}{dt}|\nabla V_p(x_p(t))|^2 \,=\, 2\,\Delta_\infty V_p(x_p(t))\\ &= -\,\frac{1}{p-1}|\nabla V_p(x_p(t))|^2\Delta V_p \,\ge 0,\end{aligned}$$

since $\Delta V_p \le 0$ (superharmonic) by Lewis's theorem. Thus the functions

$$t \mapsto V_p(x_p(t))$$

are convex. Unfortunately, the streamlines usually move as $p\to\infty$, making the control of the process difficult. However, the diagonals are streamlines for all p. Thus the limit function

$$V_\infty(t,t) \quad \text{is convex when} \quad -1 \le t \le 0$$

on the diagonal from $(-1,-1)$ to $(0,0)$. Since the limit $V_\infty(t,t)$ has a continuous derivative with respect to t, it follows by Theorem 25.7 in [21], that on the diagonal even the derivatives of V_p converge uniformly.[3] It follows that the speed $|\nabla V_\infty|$ is non-decreasing along the diagonal.

We sum up a few properties:

1. From each point on the boundary $\partial\Omega$ a unique streamline starts and terminates at the origin. Through each point there passes at least one streamline.
2. A streamline has a continuous tangent.
3. The diagonals and medians are streamlines.
4. No streamline can join the medians.
5. The speed $|\nabla V_\infty|$ is non-decreasing on the diagonals.[4]
6. There are infinitely many Cl-points near the corners.
7. There are infinitely many Cl-points near the origin.
8. There are infinitely many Cl-points along any streamline except the medians.

This can be directly deduced from the previous results except for the three last points, which require some further explanation.

Proof of 6). The gradient is zero at the corners and the gradient is non-zero at all interior points. Therefore there must be infinitely many points a_0 and b_0 near the corners satisfying the assumptions of Corollary 2. This implies that there are infinitly many Cl-points near the corners. □

Proof of 7). We prove that in each disk around the origin, there is at least one Cl-point. The result follows from this. We assume towards a contradiction that there is $c \in (0,1)$ such that the set $\{V_\infty > c\}$ does not contain any such points. We apply Theorem 4 to the restriction of $w = (V_\infty - c)/(1-c)$ to the set $\{V_\infty \ge c\}$ to conclude that the set $\{V_\infty > c\}$ is a ball B. In particular, $|\nabla V_\infty| = 1$ in B.

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[3] Unfortunately, the uniform convergence $\nabla V_p \to \nabla V_\infty$ is not known to us.

[4] It is likely that this holds on all streamlines.

Denote by y_1 the intersection of B and the lower right diagonal. Let x_1 be the closest point to the midpoint $(0,-1)$ of the lower side, such that the streamline starting at x_1 passes through y_1.[5] We have two alternatives: 1) x_1 is the corner point $(1,-1)$ and 2) x_1 is not the corner point (it cannot be the midpoint).

In the case of 1), any streamline starting at a point x_2 to the left of the corner, intersects ∂B at a point $y_2 \neq y_1$ which is not on the diagonal. Since we may take x_2 as close as we wish to the diagonal, we may assume $|\nabla V_\infty| < \frac{1}{2}$ on the line between x_1 and x_2. Moreover, on the level set joining y_1 and y_2 (= the circle ∂B), we have $|\nabla V_\infty| = 1$. By applying Lemma 8 to the pair of points x_1, x_2 and y_1, y_2, we obtain

$$\|\nabla V_\infty\|_{\infty,\overline{x_1\,x_2}} \geq \|\nabla V_\infty\|_{\infty,\overline{y_1\,y_2}} = 1,$$

which is a contradiction.

In the case of 2), let x_2 be a point to the left of x_1 and y_2 the corresponding point on ∂B. By definition, $y_2 \neq y_1$. Take z_1 to be a point on the streamline from x_1 to y_1. Let z_2 be a point on the same level line as z_1 and on the streamline between x_2 to y_2. By Lemma 8 applied to the pair of points y_1, y_2 and z_1, z_2, we obtain that

$$\|\nabla V_\infty\|_{\infty,\overline{z_1\,z_2}} \geq 1.$$

Since the pair z_1, z_2 is arbitrary and since we may choose x_2 arbitrary close to x_1, this implies that $|\nabla V_\infty| = 1$ along the streamline starting at x_1. Since the distance between x_2 and the origin is strictly larger than 1, this is a contradiction. ☐

Proof of 8). Let x be a boundary point which is not a midpoint of a side. Then $|x| > 1$. Therefore, along any streamline starting at x, there must be a point y where $|\nabla V_\infty| < 1$. Since $|\nabla V_\infty|$ is continuous along the streamline, there must be infinitely many points a_0 and b_0 along this streamline satisfying the assumptions of Corollary 2 and therefore there are infinitly many Cl-points along this streamline. ☐

We conjecture that every streamline except the medians joins a diagonal before reaching the midpoint and that the only Cl-points are the points on the diagonals. This is also suggested by Figure 1.

Epilogue. One may wonder whether $|\nabla \log V_\infty| \geq 1$ in the square. This would show that V_∞ is the same function as the ∞-Ground State described in Section 4 of [14]. This is also suggested by numerics.

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[5] Here the notation $x = (x_1, x_2)$ is abandoned, the subindices referring to different points.

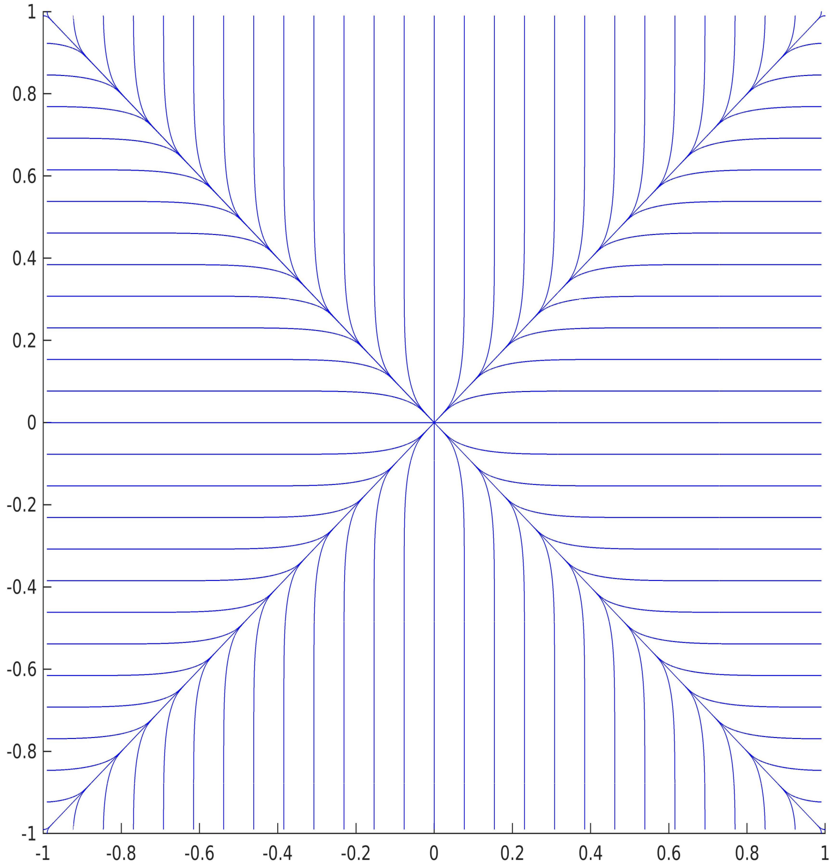

FIGURE 1. The streamlines of V_∞ when Ω is the square $-1 < x_1 < 1,\ -1 < x_2 < 1$.

REFERENCES

[1] G. Aronsson. Extension of functions satisfying Lipschitz conditions, *Arkiv för Matematik*, **6** (1967), 551–561.

[2] G. Aronsson. On the partial differential equation $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$, *Arkiv för Matematik*, **7** (1968), 397–425.

[3] G. Crasta, I. Fragalà. *On the characterization of some classes of proximally smooth sets*, *ESAIM: Control, Optimisation and Calculus of Variations*, **22** (2016) no. 3, 710–727.

[4] G. Crasta, I. Fragalà. *On the Dirichlet and Serrin problems for the inhomogeneous infinity Laplacian in convex domains: regularity and geometric results. Archive for Rational Mechanics and Analysis*, **218** (2015), no. 3, 1577–1607.

[5] V. Caselles, J-M. Morel, C. Sbert. *An axiomatic approach to image interpolation*, *IEEE Transactions on Image Processing*, **7** (1998), 376–386.

[6] M. Crandall, H. Ishii, P.-L. Lions. *User's guide to viscosity solutions of second order partial differential equations*, *Bulletin of the American Mathematical Society*, **27** (1992), 1–67.

[7] L. Evans, R. Gariepy. *Measure Theory and Fine Propeties of Functions*, CRC Press, Boca Raton 1992.

[8] L. Evans, O. Savin. $C^{1,\alpha}$ *regularity of infinite harmonic functios in two dimensions*, *Calculus of Variations and Partial Differential Equations*, **32** (2008), 325–347.

[9] L. Evans, Ch. Smart. *Everywhere differentiability of infinity harmonic functions*, *Calculus of Variations and Partial Differential Equations*, **42** (2011), 289–299.

[10] U. Janfalk. *Behaviour in the limit, as $p \to \infty$, of minimizers of functionals involving p-Dirichlet integrals, SIAM Journal on Mathematical Analysis*, **27** (1996) no.2, 341–360.

[11] R. Jensen. *Uniqueness of Lipschitz extension: minimizing the sup norm of the gradient*, *Archive for Rational Mechanics and Analysis*, **123** (1993), 51–74.

[12] V. Julin, P. Juutinen. *A new proof for the equivalence of weak and viscosity solutions for the p-Laplace equation*, *Communications on Partial Differential Equations*, **37** (2012) no. 5, 934–946.

[13] P. Juutinen, P. Lindqvist, J. Manfredi. *On the equivalence of viscosity solutions and weak solutions for a quasi-linear equation*, *SIAM Journal on Mathematical Analysis*, **33** (2001), 699–717.

[14] P. Juutinen, P. Lindqvist, J. Manfredi. *The infinity Laplacian: examples and observations*, *Papers on analysis, Rep. Univ. Jyväskylä Dep. Math. Stat. Univ. Jyväskylä*, , **83** (2001), 207–217.

[15] P. Juutinen, P. Lindqvist, J. Manfredi. *The ∞-eigenvalue problem*, *Archive for Rational Mechanics and Analysis*, **148** (1999) no. 2, 89–105.

[16] H. Koch, Y. Zhang, Y. Zhou. *An asymptotic sharp Sobolev regularity for planar infinity harmonic functions*, Preprint, `arXiv:1806.01982`

[17] S. Koike. A Beginner's Guide to the Theory of Viscosity Solutions. *MSJ Memoirs, Mathematical Society of Japan*, **13** (2004)

[18] J. Lewis. *Capacitory functions in convex rings*, *Archive for Rational Mechanics and Analysis*, **66** (1977) no. 3, 201–224.

[19] J. Manfredi, A. Petrosyan, H. Shahgholian. *A free boundary problem for ∞-Laplace equation*, *Calculus of Variations and Partial Differential Equations*, **14**, (2002) no. 3, 359–384.

[20] Y. Peres, O. Schramm, S. Sheffield, D. Wilson. *Tug-of-war and the infinity Laplacian*, *Journal of the American Mathematical Society*, **22** (2009), 167–210.

[21] R. Rockafeller. *Convex Analysis*, Princeton University Press, USA 1970.

[22] O. Savin. C^1 *regularity for infinity harmonic functions in two dimensions*, *Archive for Rational Mechanics and Analysis*, **176**, 2005 no.3, 351–361.

[23] O. Savin, C. Wang, Y. Yu. *Asymptotic behaviour of infinity harmonic functions near an isolated singularity*, *International Mathematical Research Notes*, **6** (2008).

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