HOCHSCHILD COHOMOLOGY OF SOME QUANTUM COMPLETE INTERSECTIONS

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Abstract. We compute the Hochschild cohomology ring of the algebras $A = k\langle X, Y\rangle/(X^a, XY - qYX, Y^a)$ over a field k where $a \geq 2$ and where $q \in k$ is a primitive a-th root of unity. We find the the dimension of $\mathrm{HH}^n(A)$ and show that it is independent of a. We compute explicitly the ring structure of the even part of the Hochschild cohomology modulo homogeneous nilpotent elements.

1. Introduction

Let k be a field, and let $0 \neq q \in k$. Quantum complete intersections originate from work of Manin [8]. Here we focus on the algebras

$$A_q = k\langle X, Y\rangle/(X^a, XY - qYX, Y^a).$$

Such algebras have provided several examples giving answers to homological conjectures and questions. Perhaps most spectacular amongst these is Happel's question. In [6] Happel asked whether an algebra whose Hochschild cohomology is finite-dimensional, must have finite global dimension. The main result of [3] gave a negative answer: It shows that the Hochschild cohomology of the quantum complete intersection A_q as above, when $a = 2$ and q not a root of unity, is finite-dimensional. However the algebra A_q is selfinjective, hence has infinite global dimension. Already earlier, R. Schulz discovered unusual properties for these algebras A_q, see [11] and [10].

Furthermore, there is a theory of support varieties in terms of Hochschild cohomology provided the algebra satisfies suitable finite generation properties, known as condition (Fg) (see [5] and [13]). For A_q, this condition is satisfied precisely when q is a root of unity. The general theory of these support varieties has now been well established in several papers. However, in order to actually compute the varieties over a given algebra, one needs to determine the ring structure of the Hochschild cohomology, or at least modulo homogeneous nilpotent elements.

The results in this paper will be a contribution towards this goal. We determine the ring structure of the even part of $\mathrm{HH}^*(A_q)$ (which will be denoted $\mathrm{HH}^{2*}(A_q)$) modulo the ideal of homogeneous nilpotent elements for A_q when q is a primitive a-th root of unity. The proofs are quite technical, but this illustrates the typical difficulties and computations one is faced with when trying to compute Hochschild cohomology.

First we present an unpublished result by P. Bergh and K. Erdmann which determines the dimensions of the Hochschild cohomology groups; this is done via exploiting Hochschild homology. Surprisingly, the answer is independent of a (see Theorem 3.1 and Corollary 3.2). This suggests that perhaps also the ring structure might not depend too much on the parameter a. We determine explicit bases of $\mathrm{HH}^{2*}(A)$ (see Section 5.2).

Furthermore, we compute the algebra structure of $\mathrm{HH}^{2*}(A)$ modulo the largest homogeneous nilpotent ideal. We show that it is $\mathbb{Z}_2$-graded, with degree zero part isomorphic to the polynomial ring in two variables, generated in degree 2. The explicit description is given in 4.2 when $a = 2$, and in 5.4 when $a \geq 3$.

Date: November 1, 2017.
2010 *Mathematics Subject Classification.* Primary 16E40; Secondary 16U80; 16S80; 81R50.
Key words and phrases. Hochschild Cohomology; Quantum complete intersections.

An explicit description when $a = 2$ was also given in [3, Section 3.4]. We include this case (in Section 4), as it shows that it is part of the general pattern.

S. Oppermann gave also a description of the Hochschild cohomology and homology of more general quantum complete intersections in [9]. The products are, however, not computed completely explicitly, though it discusses a more general setting. However, in this paper we calculate products explicitly by liftings along a minimal projective resolution (which will be discussed in Section 2). This illustrates techniques that might be of independent interest.

In a larger context, there is even more structure in some classes of Hochschild cohomology than the well known Gerstenhaber algebra structure. In [7] T. Lambre, G. Zhou and A. Zimmermann prove that the Hochschild cohomology ring of quantum complete intersections is a so called Batalin–Vilkovisky algebra (Corollary 5.8). Roughly speaking a Batalin–Vilkovisky algebra is a Gerstenhaber algebra with an additional operation $\Delta : \mathrm{HH}^n \to \mathrm{HH}^{n-1}$ which squares to zero and which, together with the cup product, can express the Lie Bracket.

2. Preliminaries

More generally, let A be any finite-dimensional algebra over a field k, and let $A^e = A \otimes_k A^{\mathrm{op}}$ denote the *enveloping algebra*. We view bimodules over A as left modules over A^e. In this setting, the *Hochschild cohomology* of A can be taken as $\mathrm{HH}^n(A) = \mathrm{Ext}^n_{A^e}(A, A)$, the n-th cohomology of the complex $\mathrm{Hom}_{A^e}(\mathbb{P}, A)$, i.e.

$$\mathrm{Ext}^n_{A^e}(A, A) = \ker d^*_{n+1} / \mathrm{im}\, d^*_n, \tag{2.1}$$

where $d^*_n = \mathrm{Hom}_{A^e}(d_n, A)$ and where d_n are the maps in a minimal projective resolution:

$$\mathbb{P} : \cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\mu} A \to 0. \tag{2.2}$$

Then the *Hochschild cohomology*

$$\mathrm{HH}^*(A) = \mathrm{Ext}^*_{A^e}(A, A) \tag{2.3}$$

is a k-algebra which is graded-commutative. There are various equivalent ways to define the product; here we will work with the Yoneda product.

We specialize now to the quantum complete intersections. Let a be an integer such that $a \geq 2$. We also let $q \in k$ be a primitive a-th root of unity, and A is the k-algebra defined by

$$A = k\langle X, Y\rangle / (X^a, XY - qYX, Y^a). \tag{2.4}$$

We write x and y for the residue classes of X and Y, respectively.

In [2], for arbitrary parameter $q \neq 0$, an explicit minimal projective bimodule resolution $\mathbb{P}$ as in (2.2) was constructed. The nth bimodule in $\mathbb{P}$ is

$$P_n = \bigoplus_{i=0}^{n} A^e f^n_i, \tag{2.5}$$

the free A^e-module of rank $n+1$ having generators $\{f^0_n, f^1_n, ..., f^n_n\}$. For each $s \geq 0$ define the following four elements of A^e:

$$\tau_1(s) = q^s(1 \otimes x) - (x \otimes 1) \tag{2.6}$$

$$\tau_2(s) = (1 \otimes y) - q^s(y \otimes 1) \tag{2.7}$$

$$\gamma_1(s) = \sum_{j=0}^{a-1} q^{js}(x^{a-1-j} \otimes x^j) \tag{2.8}$$

$$\gamma_2(s) = \sum_{j=0}^{a-1} q^{js}(y^j \otimes y^{a-1-j}) \tag{2.9}$$

The maps $d_n : P_n \to P_{n-1}$ in $\mathbb{P}$ are given by

$$d_{2t} : f_i^{2t} \mapsto \begin{cases} \gamma_2\left(\frac{ai}{2}\right) f_i^{2t-1} + \gamma_1\left(\frac{2at-ai}{2}\right) f_{i-1}^{2t-1} & \text{for } i \text{ even} \\ -\tau_2\left(\frac{ai-a+2}{2}\right) f_i^{2t-1} + \tau_1\left(\frac{2at-ai-a+2}{2}\right) f_{i-1}^{2t-1} & \text{for } i \text{ odd} \end{cases} \tag{2.10}$$

$$d_{2t+1} : f_i^{2t+1} \mapsto \begin{cases} \tau_2\left(\frac{ai}{2}\right) f_i^{2t} + \gamma_1\left(\frac{2at-ai+2}{2}\right) f_{i-1}^{2t} & \text{for } i \text{ even} \\ -\gamma_2\left(\frac{ai-a+2}{2}\right) f_i^{2t} + \tau_1\left(\frac{2at-ai+a}{2}\right) f_{i-1}^{2t} & \text{for } i \text{ odd} \end{cases} \tag{2.11}$$

where the convention $f_{-1}^n = f_{n+1}^n = 0$ has been used. So far, q is arbitrary. Later in our setting we will simplify these expressions.

We will wish to identify nilpotent elements of Hochschild cohomology. This can be done by exploiting the following result of N. Snashall and Ø. Solberg, see Proposition 4.4 in [12].

Proposition 2.1. *Assume k is a field and A is a finite-dimensional k-algebra. Suppose η is a map into A representing an element of $\mathrm{HH}^n(A)$. If $\mathrm{im}(\eta)$ is in the radical of A then η is nilpotent in $\mathrm{HH}^*(A)$.*

3. Dimensions of Hochschild cohomology groups

We recall an unpublished result by Petter A. Bergh and Karin Erdmann which determines the dimensions.

By viewing A as a left A^e-module, it follows from [4, p. VI.5.3] that $D(\mathrm{HH}^*(A,A))$ is isomorphic to $\mathrm{Tor}_*^{A^e}(D(A),A)$ as a vector space, where D denotes the usual k-dual i.e. $D(-) := \mathrm{Hom}_k(-,k)$. In particular, we see that $\dim \mathrm{HH}^n(A) = \dim \mathrm{Tor}_n^{A^e}(D(A),A)$ for all $n \geq 0$. Moreover, it follows from [2] that A is a Frobenius algebra with Nakayama automorphism $\nu : A \to A$ defined by

$$\nu : \begin{cases} x \mapsto q^{1-a}x \\ y \mapsto q^{a-1}y. \end{cases} \tag{3.1}$$

The bimodules $D(A)$ and ${}_\nu A_1$ are isomorphic; here the left action on ${}_\nu A_1$ is taken as $a \cdot m := \nu(a)m$. Consequently the dimensions of the Hochschild cohomology of A are given by

$$\dim \mathrm{HH}^n(A) = \dim \mathrm{Tor}_n^{A^e}({}_\nu A_1, A) \tag{3.2}$$

for all $n \geq 0$.

To compute $\mathrm{Tor}_n^{A^e}({}_\nu A_1, A)$, we tensor the deleted projective bimodule resolution $\mathbb{P}$ with the right A^e-module ${}_\nu A_1$. We then obtain an isomorphism

$$\begin{array}{ccccccccc} \cdots \longrightarrow & {}_\nu A_1 \otimes_{A^e} P_{n+1} & \xrightarrow{1\otimes d_{n+1}} & {}_\nu A_1 \otimes_{A^e} P_n & \xrightarrow{1\otimes d_n} & {}_\nu A_1 \otimes_{A^e} P_{n-1} & \longrightarrow \cdots \\ & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & \\ \cdots \longrightarrow & \oplus_{i=0}^{n+1}({}_\nu A_1)e_i^{n+1} & \xrightarrow{\delta_{n+1}} & \oplus_{i=0}^{n}({}_\nu A_1)e_i^{n} & \xrightarrow{\delta_n} & \oplus_{i=0}^{n-1}({}_\nu A_1)e_i^{n-1} & \longrightarrow \cdots \end{array}$$

of complexes, where $\{e_0^n, e_1^n, \ldots, e_n^n\}$ is the standard generating set of $n+1$ copies of ${}_\nu A_1$. Now given an element $\alpha \in k$ and a positive integer t, define an element $K_t(\alpha) \in k$ by

$$K_t(\alpha) := \sum_{j=0}^{t-1} \alpha^j. \tag{3.3}$$

The map δ_n is then given by

$$\delta_{2t} : y^u x^v e_i^{2t} \mapsto$$

$$\begin{cases} qK_a(q^{v+1})y^{u+a-1}x^v e_i^{2t-1} + K_a(q^{u+1})y^u x^{v+a-1} e_{i-1}^{2t-1} & \text{for } i \text{ even} \\ [q^{v+1} - q^{a-1}]y^{u+1}x^v e_i^{2t-1} + [q^{u+2} - 1]y^u x^{v+1} e_{i-1}^{2t-1} & \text{for } i \text{ odd} \end{cases} \tag{3.4}$$

$$\delta_{2t+1} : y^u x^v e_i^{2t+1} \mapsto$$

$$\begin{cases} [q^{a-1} - q^v]y^{u+1}x^v e_i^{2t} + K_a(q^{u+2})y^u x^{v+a-1} e_{i-1}^{2t} & \text{for } i \text{ even} \\ -qK_a(q^{v+2})y^{u+a-1}x^v e_i^{2t} + [q^{u+1} - 1]y^u x^{v+1} e_{i-1}^{2t} & \text{for } i \text{ odd} \end{cases} \tag{3.5}$$

where we use the convention $e_{-1}^n = e_{n+1}^n = 0$. This was proved in [2] in a more general setting, and by specializing q and using that x, y have the same nilpotency index, we obtain the above formulae (correcting an unimportant sign error in [2]).

For the following result we use this complex to compute the Hochschild cohomology of our algebra A, in the case when q is a primitive a-th root of unity. The result shows that the dimensions of the cohomology groups do not depend on the characteristic of the field, except that the characteristic of k does not divide a since k contains a primitive a-th root of unity.

Theorem 3.1. *If q is a primitive a-th root of unity, then* $\dim_k \mathrm{HH}^n(A) = 2n + 2$ *for all* $n \geq 0$.

Proof. Since $\mathrm{HH}^0(A)$ is isomorphic to the centre of A, we see immediately that $\mathrm{HH}^0(A)$ is 2-dimensional. To find the dimension of $\mathrm{HH}^n(A)$ for $n > 0$, we compute $\ker \delta_{2t}$ for $t \geq 1$ and $\ker \delta_{2t+1}$ for $t \geq 0$.

Since k contains a primitive q-th root of unity, the characteristic of k does not divide a. The equalities $0 = 1 - (q^m)^a = (1 - q^m)K_a(q^m)$, valid for any integer m, show that $K_a(q^m) = 0$ if and only if m is not divisible by a. We will use this fact throughout.

We first compute $\ker \delta_{2t}$ for $t \geq 1$. By the previous observation, $K_a(q^{v+1}) = 0$ if and only if $0 \leq v \leq a - 2$, whereas $K_a(q^{u+1}) = 0$ if and only if $0 \leq u \leq a - 2$. Therefore

$$\delta_{2t}(y^u x^v e_i^{2t}) = 0 \Leftrightarrow \begin{cases} u \in \{1, 2, \ldots, a-1\},\ v \in \{1, 2, \ldots, a-1\},\ \ i \text{ even}, ,\ 0 \leq i \leq 2t \\ u \in \{0, 1, \ldots, a-2\},\ v = 0,\ i \text{ even},\ 0 \leq i \leq 2t \\ u = 0,\ v \in \{0, 1, \ldots, a-2\},\ i \text{ even}, ,\ 0 \leq i \leq 2t \\ u = 0,\ v = a-1,\ i = 2t \\ u = a-1,\ v = 0,\ i = 0 \\ u = a-2,\ v = a-2,\ i \text{ odd}, 1 \leq i \leq 2t-1 \\ u = a-1,\ v = a-1,\ i \text{ odd}, 1 \leq i \leq 2t-1 \end{cases} \tag{3.6}$$

and there are $a^2 t + a^2$ such elements.

Let $\mathcal{B} := \{y^u x^v e_i^{2t} : 0 \leq u, v \leq a-1, 0 \leq i \leq 2t\}$, a basis for ${}_\nu A_1 \otimes_{A^e} P_{2t}$. We split this basis into three parts. Let $\mathcal{X}$ be the set of basis vectors which are in the kernel of δ_{2t}, so that $\mathcal{X}$ is given by the above list. Next, let

$$\mathcal{Y} := \{y^u x^v e_i^{2t} :\ i \text{ odd}, ,\ 0 \leq u, v < a-2\}$$

One checks directly that $\ker(\delta_{2t}) \cap \mathrm{Sp}(\mathcal{Y}) = \{0\}$. Let $\mathcal{Z} := \mathcal{B} \setminus (\mathcal{X} \cup \mathcal{Y})$. We find that $\mathcal{Z}$ is equal to

$$\{x^{a-1} e_{2j}^{2t} : 0 \leq j < t\} \cup \{y^{a-1} e_{2j}^{2t} : 0 < j \leq t\}$$
$$\cup \{y^{a-2} x^{a-1} e_{2j+1}^{2t} : 0 \leq j < t\} \cup \{y^{a-1} x^{a-2} e_{2j+1}^{2t} : 0 \leq j < t\}.$$

The map δ_{2t} takes each member of $\mathcal{Z}$ to a non-zero scalar multiple of $y^{a-1} x^{a-1} e_i^{2t-1}$, and each of these occurs, for $0 \leq i \leq 2t-1$. Hence the image of δ_{2t} restricted to the span of $\mathcal{Z}$ has dimension $2t$, and the size of $\mathcal{Z}$ is $4t$. This means that the restriction of δ_{2t} to the span of $\mathcal{Z}$ has kernel of dimension $2t$, by the rank-nullity formula. Hence we get $2t$ further linearly independent elements of the kernel of δ_{2t}. In total, this shows that $\dim_k \ker \delta_{2t} = (a^2 + 2)t + a^2$.

Next we compute $\ker \delta_{2t+1}$ for $t \geq 0$, recall that the characteristic of k does not divide a. We see that

$$\delta_{2t+1}(y^u x^v e_i^{2t+1}) = 0 \Leftrightarrow \begin{cases} u = a-1,\ v \in \{0,\dots,a-1\}, & i \text{ arbitrary} \\ u \in \{0,\dots,a-2\},\ v = a-1, & i \text{ arbitrary} \end{cases} \tag{3.7}$$

and there are $(2a-1)(2t+2)$ such elements.

Let $\mathcal{B} := \{y^u x^v e_i^{2t+1} : 0 \leq u, v \leq a-1, 0 \leq i \leq 2t+1\}$, a basis for ${}_\nu A_1 \otimes_{A^e} P_{2t+1}$. We split this basis into three parts. Let $\mathcal{X}$ be the set of basis vectors which are in the kernel of δ_{2t+1}, that is $\mathcal{X}$ is given by the above list. Next, consider

$$\begin{aligned} \mathcal{Y} = &\ \{y^{a-2} x^v e_i^{2t+1} : i \text{ even}, 0 \leq v \leq a-2\} \cup \{y^u e_i^{2t+1} : i \text{ even}, 0 \leq u \leq a-3\} \\ &\cup \{y^u x^{a-2} e_i^{2t+1} : i \text{ odd}, 0 \leq u \leq a-2\} \cup \{x^v e_i^{2t+1} : i \text{ odd}, 0 \leq v \leq a-3\}. \end{aligned}$$

One checks that $\mathrm{Sp}(\mathcal{Y}) \cap \mathrm{Ker}(\delta_{2t+1}) = \{0\}$. Now let $\mathcal{Z} := \mathcal{B} \setminus (\mathcal{X} \cup \mathcal{Y})$. This is the disjoint union of two sets, $\mathcal{Z} = \mathcal{Z}_e \cup \mathcal{Z}_o$ where

$$\mathcal{Z}_e := \{y^u x^v e_i^{2t+1} : i \text{ even},\ 0 \leq u \leq a-3,\ 1 \leq v \leq a-2\}$$

$$\mathcal{Z}_o := \{y^u x^v e_i^{2t+1} :\ i \text{ odd},\ 1 \leq u \leq a-2,\ 0 \leq v \leq a-3\}$$

both of size $(a-2)^2(t+1)$. Then $\delta_{2t+1}(k\langle \mathcal{Z}_e \rangle) = k\langle \tilde{\mathcal{Z}} \rangle$ and $\delta_{2t+1}(k\langle \mathcal{Z}_o \rangle) = k\langle \tilde{\mathcal{Z}} \rangle$ where

$$\tilde{\mathcal{Z}} := \{y^u x^v e_j^{2t} : j \text{ even}, 1 \leq u \leq a-2, 1 \leq v \leq a-2\}$$

which also has size $(a-2)^2(t+1)$. By the rank-nullity formula, the kernel of δ_{2t+1} restricted to $\mathcal{Z}$ has dimension $(a-2)^2(t+1)$. (Note that, if $a = 2$, then $\mathcal{Z} = \emptyset$). In total we get that $\dim_k \ker \delta_{2t+1} = (a^2+2)(t+1)$.

We have now computed $\ker \delta_{2t}$ for $t \geq 1$ and $\ker \delta_{2t+1}$ for $t \geq 0$. Using the equalities

$$\dim_k \mathrm{im}\, \delta_n + \dim_k \ker \delta_n = \dim_k \oplus_{i=0}^n ({}_\nu A_1) e_i^n = (n+1)a^2, \tag{3.8}$$

we see that $\dim_k \mathrm{im}\, \delta_{2t+1} = \dim_k \mathrm{im}\, \delta_{2t+2} = (a^2-2)(t+1)$. Consequently

$$\dim_k \mathrm{HH}^{2t+1}(A) = \dim_k \ker \delta_{2t+1} - \dim_k \mathrm{im}\, \delta_{2t+2} \tag{3.9}$$

$$= 4t+4 \tag{3.10}$$

$$\dim_k \mathrm{HH}^{2t+2}(A) = \dim_k \ker \delta_{2t+2} - \dim_k \mathrm{im}\, \delta_{2t+3} \tag{3.11}$$

$$= 4t+6 \tag{3.12}$$

for $t \geq 0$, and the proof is complete. □

This result implies immediately the following:

Corollary 3.2. *The dimension of the cohomology groups* $\mathrm{HH}^n(A)$ *is independent of* a.

4. Hochschild cohomology when $a = 2$

In this section we let $a = 2$ and $q = -1$ (and $\mathrm{char}(k) \neq 2$), so we have that

$$A = k\langle X, Y \rangle / (X^2, XY + YX, Y^2). \tag{4.1}$$

We write x, y again for the images of X, Y in A. We also mention related work by P. A. Bergh in [1] where the main objective is to compute the homology and cohomology of A with coefficients in the twisted bimodule ${}_1\Lambda_\phi$ for any k-linear automorphism ϕ of the algebra Λ.

We will simplify the differentials of the minimal projective resolution, before studying the even part of cohomology ring for this case.

4.1. **Minimal projective resolution when $a = 2$.** We introduce the following notation:

$$\beta_y = (1 \otimes y) + (y \otimes 1) \qquad \beta_x = (1 \otimes x) + (x \otimes 1) \tag{4.2}$$

$$\alpha_y = (1 \otimes y) - (y \otimes 1) \qquad \alpha_x = (1 \otimes x) - (x \otimes 1) \tag{4.3}$$

Now we can rewrite the differentials for the minimal projective resolution $\mathbb{P}$ in Equation 2.10 and 2.11; we get:

$$d_n(f_i^n) = \begin{cases} (-1)^i(\beta_y f_i^{n-1} + \beta_x f_{i-1}^{n-1}) & \text{when } n \text{ is even} \\ (-1)^i(\alpha_y f_i^{n-1} - \alpha_x f_{i-1}^{n-1}) & \text{when } n \text{ is odd .} \end{cases} \tag{4.4}$$

4.2. **Description of cohomology groups.** In Section 3 we have seen that $\dim \mathrm{HH}^n(A) = 2n + 2$. Knowing this, we will determine a basis for $\mathrm{HH}^n(A)$ for arbitrary even degrees n. We write δ_{ir} as usual for the Kronecker symbol.

Lemma 4.1. *Let $n = 2t$. For $r = 0, 1, \ldots, 2t$ define maps $\xi_r, \eta_r : P_{2t} \to A$ as follows.*

$$\xi_r(f_i^{2t}) = \delta_{ir} \cdot 1_A, \qquad \eta_r(f_i^{2t}) = \delta_{ir} \cdot xy. \tag{4.5}$$

(a) The classes of these maps form a basis for $\mathrm{HH}^{2t}(A)$.
(b) The classes of the η_r give nilpotent elements in $\mathrm{HH}^*(A)$.

Proof. Part (b) will follow from Proposition 2.1. We prove now part (a). Note that these are $2n + 2$ elements, so we only have to show that the maps are in the kernel of d_{2t+1}^*, and that they are linearly independent modulo the image of d_{2t}^*.

(1) We apply ξ_r to $d_{2t+1}(f_i^{2t+1})$, this gives

$$\xi_r[(-1)^i(\alpha_y f_i^{2t} - \alpha_x f_{i-1}^{2t})] = (-1)^i[\alpha_y[\delta_{ir} \cdot 1_A] - \alpha_x[\delta_{i-1,r} \cdot 1_A]] = 0 \tag{4.6}$$

(we view A as a left A^e module, and $\alpha_y \cdot 1_A = 0 = \alpha_x \cdot 1_A$). Similarly we apply η_r to $d_{2t+1}(f_i^{2t+1})$ and get

$$\eta_r[(-1)^i(\alpha_y f_i^{2t} - \alpha_x f_{i-1}^{2t})] = (-1)^i[\alpha_y[\delta_{ir} \cdot xy] - \alpha_x[\delta_{i-1,r} \cdot xy]] = 0 \tag{4.7}$$

(since xy is in the socle of A we see that $\alpha_y \cdot xy = 0$ and $\alpha_x \cdot xy = 0$).

(2) Let $c_r, d_r \in K$ and $\rho : P_{2t-1} \to A$ such that

$$\sum_{r=0}^{2t} c_r\xi_r + d_r\eta_r = \rho \circ d_{2t} \in \mathrm{im}(d_{2t}^*). \tag{4.8}$$

We must show that $c_r = 0 = d_r$ for all r. Write $\rho(f_i^{2t-1}) = p_i = a_i + b_i x + c_i y + d_i xy \in A$. Then we have

$$\rho \circ d_{2t}(f_i^{2t}) = (-1)^i[\beta_y p_i + \beta_x p_{i-1}] = (-1)^i[2a_i y + 2a_{i-1}x] \tag{4.9}$$

which are elements in A. On the other hand if we apply the map given by the sum to f_i^{2t} then we get

$$c_i \ + \ d_i xy, \tag{4.10}$$

also elements in A. We assume these are equal, and it follows that all scalars are zero.

□

4.3. **Products in even degrees of** $\mathrm{HH}^*(A)$**.** Recall that the even part $\mathrm{HH}^{2*}(A)$ is a subring of the Hochschild cohomology, and it is commutative. The aim of this section is to prove the following:

Theorem 4.2. *Let k be a field with $char(k) \neq 2$, and let $A = k\langle X, Y\rangle/(X^2, XY + YX, Y^2)$. Assume*

$$R = \mathrm{Sp}\{\xi_i^{2t} : t \geq 0, \text{ and } 0 \leq i \leq 2t\}. \tag{4.11}$$

Then R is a subalgebra of $\mathrm{HH}^{2*}(A)$. It is $\mathbb{Z}_2$-graded, with

$$R_0 := k\langle \xi_i^{2t} : i \text{ even} \rangle, \text{ and } \qquad R_1 := k\langle \xi_i^{2t} : i \text{ odd} \rangle. \tag{4.12}$$

We have $\xi_l^{2m}\xi_r^{2t} = \xi_{l+r}^{2m+2t}$. The subalgebra R_0 is isomorphic to the polynomial ring $k[z_0, z_1]$ where we identify ξ_0^2 with z_0 and ξ_2^2 with z_1. Moreover, $R_1 = R_0\xi_1^2$ and $\xi_1^2 \cdot \xi_1^2 = \xi_2^4$.

Corollary 4.3. *Let $\mathcal{N}$ be the largest homogeneous nilpotent ideal of $\mathrm{HH}^{2*}(A)$. Then*

$$\mathrm{HH}^{2*}(A)/\mathcal{N} \cong R. \tag{4.13}$$

We fix a degree $2t$, and we will compute the product of a general element ξ of degree $2t$ with an element χ of degree $2m$, and we let $2m$ vary. We take representatives $\xi : P_{2t} \to A$ and $\chi : P_{2m} \to A$ which are k-linear combinations of the basis. Let

$$\xi(f_i^{2t}) = p_i \in A \qquad \text{with } 0 \leq i \leq 2t \tag{4.14}$$

$$\chi(f_i^{2m}) = \bar{p}_i \in A \qquad \text{with } 0 \leq i \leq 2m. \tag{4.15}$$

By (4.5), the elements p_i and $\bar{p}_i$ are then in the centre of A, we will use this freely.

Definition 4.4. The Yoneda product $\chi \bullet \xi$ is the residue class of $\chi \circ h_{2m}$ where the family (h_s) with $h_s : P_{2t+s} \to P_s$ is a lifting of ξ. That is, we have the following diagram:

$$\begin{array}{ccccccccccccccc}
P_{2t+2m} & \xrightarrow{d_{2t+2m}} & P_{2t+2m-1} & \longrightarrow \cdots \longrightarrow & P_{2t+s} & \xrightarrow{d_{2t+s}} & P_{2t+s-1} & \longrightarrow \cdots \longrightarrow & P_{2t+1} & \xrightarrow{d_{2t+1}} & P_{2t} & & \\
\downarrow h_{2m} & & \downarrow h_{2m-1} & & \downarrow h_s & & \downarrow h_{s-1} & & \downarrow h_1 & & \downarrow h_0 & \searrow \xi & \\
P_{2m} & \xrightarrow{d_{2m}} & P_{2m-1} & \longrightarrow \cdots \longrightarrow & P_s & \xrightarrow{d_s} & P_{s-1} & \longrightarrow \cdots \longrightarrow & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\mu} & A \\
\downarrow \chi & & & & & & & & & & & & \\
A & & & & & & & & & & & &
\end{array}$$

where $\xi = \mu \circ h_0$ and where all squares commute. We define maps h_s $(0 \leq s \leq 2m)$, and will show that they are a lifting.

$$h_s(f_i^{2t+s}) = \begin{cases} \sum_{j=0}^{s} p_{i-j} f_j^s & \text{when } s \text{ even} \\ (-1)^i \left(\sum_{j=0}^{s} (-1)^j p_{i-j} f_j^s\right) & \text{when } s \text{ odd} \end{cases} \tag{4.16}$$

Proposition 4.5. *The maps h_s for $0 \leq s \leq 2m$ make the lifting diagram commutative, that is $d_s \circ h_s = h_{s-1} \circ d_{2t+s}$.*

Proof. When $2t$ is fixed the proof of this result is an examination when s is even and when s odd, and the result follows from explicit calculations.

Case s even: We have

$$(d_s \circ h_s)(f_i^{2t+s}) = d_s\left(\sum_{j=0}^{s} p_{i-j} f_j^s\right) = \sum_{j=0}^{s} p_{i-j} d_s(f_j^s) \tag{4.17}$$

$$= \sum_{j=0}^{s} p_{i-j}(-1)^j \left(\beta_y f_j^{s-1} + \beta_x f_{j-1}^{s-1}\right).$$

$$\begin{aligned}(h_{s-1} \circ d_{2t+s})(f_i^{2t}) &= h_{s-1}\left((-1)^i \left(\beta_y f_i^{2t+s-1} + \beta_x f_{i-1}^{2t+s-1}\right)\right) \\ &= \beta_y h_{s-1}(f_i^{2t+s-1}) + \beta_x h_{s-1}(f_{i-1}^{2t+s-1}) \\ &= \sum_{j=0}^{s-1}(-1)^j \beta_y p_{i-j} f_j^{s-1} + (-1)^{i-1}\sum_{j=0}^{s}(-1)^j \beta_x p_{i-(j+1)} f_j^{s-1} \\ &= \sum_{j=0}^{s}(-1)^j p_{i-j}(\beta_y f_j^{s-1} + \beta_x f_{j-1}^{s-1}).\end{aligned} \tag{4.18}$$

We observe that the expressions are equal, hence $d_s \circ h_s = h_{s-1} \circ d_{2t+s}$.
Case s odd: We calculate

$$(d_s \circ h_s)(f_i^{2t+s}) = d_s\left((-1)^i \sum_{j=0}^{s}(-1)^j p_{i-j} f_j^s\right) = \sum_{j=0}^{s}(-1)^j p_{i-j} d_s f_j^s \tag{4.19}$$

$$= (-1)^i \sum_{j=0}^{s}(-1)^j p_{i-j}(-1)^j(\alpha_y f_j^{s-1} - \alpha_x f_{j-1}^{s-1})$$

$$= (-1)^i \sum_{j=0}^{s} p_{i-j}(\alpha_y f_j^{s-1} - \alpha_x f_{j-1}^{s-1}).$$

$$\begin{aligned}(h_{s-1} \circ d_{2t+s})(f_i^{2t}) &= h_{s-1}\left((-1)^i \left(\alpha_y f_i^{2t+s-1} - \alpha_x f_{i-1}^{2t+s-1}\right)\right) \\ &= (-1)^i \sum_{j=0}^{s}(-1)^j p_{i-j}(\alpha_y f_j^{s-1} - \alpha_x f_{j-1}^{s-1}).\end{aligned} \tag{4.20}$$

The expressions are equal, which deals with the odd case. In total, these show that $(h_s)_{s\geq 0}$ defines a lifting map. □

4.4. **Description of Yoneda products.** In Section 4.2 we have described a basis for $\mathrm{HH}^{2t+2m}(A)$. Now we compute the Yoneda product of $\xi \in \mathrm{HH}^{2t}(A)$ and $\chi \in \mathrm{HH}^{2m}(A)$.

Corollary 4.6. *Let $\xi(f_r^{2t}) = p_r \in A$ and $\chi(f_r^{2m}) = \bar{p}_r \in A$. Then*

$$\chi \circ h_{2m}(f_i^{2t+2m}) = \sum_{0\leq j\leq 2m \text{ and } 0\leq i-j\leq 2t} p_{i-j}\bar{p}_j. \tag{4.21}$$

In particular if we let ξ_u^{2m} and ξ_v^{2t} denote the basis elements of Lemma 4.1 then we have

$$\xi_u^{2m} \cdot \xi_v^{2t} = \xi_{u+v}^{2t+2m} \tag{4.22}$$

showing that R is closed under multiplication.

Proof. We apply the lifting formula and obtain the first part directly. If we take $\chi = \xi_u^{2m}$ and $\xi = \xi_v^{2t}$ then $p_v = 1$ and $p_r = 0$ for $r \neq v$ and similarly $\bar{p}_u = 1$ and $\bar{p}_r = 0$ otherwise. So we get that the image of f_ν^{2t+2m} is 1 if $\nu = u + v$ and is zero otherwise. The last part follows. □

4.5. **Completing the proof of Theorem 4.2.** We are left to show that R_0 is isomorphic to the polynomial ring $k[z_0, z_1]$, the rest follows from (4.22). We define $z_0 \mapsto \xi_0^2$ and $z_1 \mapsto \xi_2^2$; this extends to an algebra map (recall that R_0 is commutative). This takes $z_0^r z_1^s$ to $(\xi_0^2)^r(\xi_2^2)^s = \xi_0^{2r}\xi_{2s}^{2s} = \xi_{2s}^{2r+2s}$. The map is bijective: namely a general basis vector ξ_{2r}^{2t}, where we must have $r \leq t$, factorizes uniquely as

$$\xi_{2r}^{2t} = \xi_0^{2(t-r)}\xi_{2r}^{2r}.$$

Corollary 4.3 is a direct consequence: The intersection of R with $\mathcal{N}$ is zero, and as we have observed, any element in the span of maps η_r is in $\mathcal{N}$. □

5. Cohomology for $a \geq 3$

Now we study the cohomology when $a \geq 3$. Still let q be an a-th root of unity and assume the algebra is

$$A = k\langle X, Y\rangle/(X^a, XY - qYX, Y^a). \tag{5.1}$$

We write again x, y for the images of X, Y in A.

5.1. **Differentials.** We assume $a \geq 3$, then we can simplify the differentials defined in 2.10 and 2.11. We observe that the elements in A introduced in 2.6 to 2.9 depend only on the parity of s modulo a, and the arguments in 2.10 and 2.11 make only use of the cases where $s \equiv 0$ or $s \equiv 1$ modulo a. Using this the differentials take the following form which we will use from now:

$$d_{2t} : f_i^{2t} \mapsto \begin{cases} \gamma_y(0)f_i^{2t-1} + \gamma_x(0)f_{i-1}^{2t-1} & \text{for } i \text{ even} \\ -\tau_y(1)f_i^{2t-1} + \tau_x(1)f_{i-1}^{2t-1} & \text{for } i \text{ odd} \end{cases} \tag{5.2}$$

$$d_{2t+1} : f_i^{2t+1} \mapsto \begin{cases} \tau_y(0)f_i^{2t} + \gamma_x(1)f_{i-1}^{2t} & \text{for } i \text{ even} \\ -\gamma_y(1)f_i^{2t} + \tau_x(0)f_{i-1}^{2t} & \text{for } i \text{ odd} \end{cases} \tag{5.3}$$

where we have replaced

$$\tau_1 = \tau_x \qquad \tau_2 = \tau_y \qquad \gamma_1 = \gamma_x \qquad \gamma_2 = \gamma_y \tag{5.4}$$

5.2. **A basis for $\mathrm{HH}^{2t}(A)$ for $a \geq 3$.** As observed the dimension of the degree $2t$ part is always $4t + 2$ which is independent of a. We therefore expect that there should be a basis when $a \geq 3$ which is not so different from the one we had for $a = 2$.

Definition 5.1. Let $\zeta_j : P_{2t} \to A$ be the map

$$\zeta_j(f_i^{2t}) = \begin{cases} 1 & i = j \\ 0 & \text{else.} \end{cases} \tag{5.5}$$

Let j be even, then define

$$\eta_j^+(f_i^{2t}) = \begin{cases} x^{a-1}y^{a-1} & i = j \\ 0 & \text{else.} \end{cases} \tag{5.6}$$

Now let j be odd, then define

$$\eta_j^-(f_i^{2t}) = \begin{cases} xy & i = j \\ 0 & \text{else.} \end{cases} \tag{5.7}$$

Lemma 5.2. *We fix a degree $2t$.*

(a) The classes of the elements ζ_i and $\eta_j^{\pm}$ as defined above form a basis of $\mathrm{HH}^{2t}(A)$.

(b) The classes of the elements $\eta_j^{\pm}$ give nilpotent elements of $\mathrm{HH}^*(A)$.

Proof. Part (b) will follow again from Proposition 2.1. We prove now part (a). These are in total $4t+2$ maps, so we only have to show that the maps lie in the kernel of d_{2t+1}^*, and that they are linearly independent modulo the image of d_{2t}^*.

(1) Let ξ be one these maps. We write $\xi(f_i^{2t}) = p_i \in A$, so that p_i is either 0 or 1 or one of $x^{a-1}y^{a-1}$ or xy depending on the parity of i. We need to check that $\xi(d_{2t+1}(f_i^{2t+1})) = 0$.

(a) Assume i is even, then this is equal to

$$\xi(d_{2t+1}(f_i^{2t+1})) = \xi(\tau_y(0)f_i^{2t} + \gamma_x(1)f_{i-1}^{2t}) = \tau_y(0)p_i + \gamma_x(1)p_{i-1}. \tag{5.8}$$

This has to be calculated in A which is viewed as an A^e left module. We have

$$\tau_y(0)p_i = p_i y - y p_i \tag{5.9}$$

This is zero if $p_i = 1$. Otherwise since i is even we only need to consider $p_i = x^{a-1}y^{a-1}$ and then $p_i y = 0$ and $y p_i = 0$. Next, if $p_{i-1} = 1$ then

$$\gamma_x(1)p_{i-1} = \sum_{j=0}^{a-1} q^j x^{a-1-j} \cdot 1 \cdot x^j = (\sum_{j=0}^{a-1} q^j)x^{a-1} \tag{5.10}$$

and this is zero, note that $1 + q + \ldots + q^{a-1} = 0$ since q is an a-th root of 1. Otherwise $p_{i-1} = xy$ and then

$$\gamma_x(1)p_{i-1} = \sum_{j=0}^{a-1} q^j (x^{a-1-j} xy x^j) \tag{5.11}$$

and this is a scalar multiple of $x^a y$ and hence is zero.

(b) Let i be odd, we get

$$\xi(-\gamma_y(1)f_i^{2t} + \tau_x(0)f_{i-1}^{2t}) = -\gamma_y(1)p_i + \tau_x(0)p_{i-1}. \tag{5.12}$$

By calculations similar to part (a) we see that this is zero in all cases to be considered.

(2) We consider a linear combination of the above elements and assume that it lies in the image of d_{2t}^*. Explicitly let

$$\sum_{j=0}^{2t} c_j \zeta_j + \sum_{j \text{ even}} s_j^+ \eta_j^+ + \sum_{j \text{ odd}} s_j^- \eta_j^- = \xi \circ d_{2t} \tag{5.13}$$

where $\xi : P_{2t-1} \to A$, with c_j and $s_j^{\pm}$ in k. We must show that this is only possible, as ξ varies, with all c_j and $s_j^{\pm}$ equal to zero.

(a) Apply the LHS to f_i^{2t} with i even, this gives

$$c_i + s_i^+(x^{a-1}y^{a-1}). \tag{5.14}$$

On the other hand,

$$\xi \circ d_{2t}(f_i^{2t}) = \gamma_y(0)\xi(f_i^{2t-1}) + \gamma_x(0)\xi(f_{i-1}^{2t-1}). \tag{5.15}$$

This is an element in A viewed as an A^e left module. For any element $z \in A$, $\gamma_x(0)z$ or $\gamma_y(0)z$ can never have a non-zero constant term since $\gamma_x(0)$ and $\gamma_y(0)$ are in the radical of A^e. Hence the Equation (5.15) does never have a non-zero constant term and $c_i = 0$.

We claim that we also cannot get a term which is a multiple of $x^{a-1}y^{a-1}$. Namely if so this could only come from either $\gamma_y(0)x^{a-1}$ or from $\gamma_x(0)y^{a-1}$. Now,

$$\gamma_y(0)x^{a-1} = \sum_{j=0}^{a-1} y^j x^{a-1} y^{a-1-j} = \sum_{j=0}^{a-1} (q^{-1})^{j(a-1)} x^{a-1}y^{a-1} = 0 \tag{5.16}$$

since $\sum_{j=0}^{a-1} q^j = 0$. Hence $s_i^+ = 0$. Similarly one sees that $\gamma_x(0)y^{a-1} = 0$.

(b) Apply the LHS to f_i^{2t} with i odd, this gives

$$c_i + s_i^-(xy). \tag{5.17}$$

On the other hand,

$$\xi \circ d_{2t}(f_i^{2t}) = -\tau_y(1)\xi(f_i^{2t-1}) + \tau_x(1)\xi(f_{i-1}^{2t-1}). \tag{5.18}$$

As before, since $\tau_y(1)$ and $\tau_x(1)$ are in the radical of A^e, this cannot have non-zero constant terms. Hence $c_i = 0$.

We must check that we cannot get xy. If xy should occur in $\tau_y(1)$ this can only come from $\tau_y(1)x$ but this is equal to $xy - qyx = 0$. Similarly $\tau_x(1)y = 0$ and we do not get xy. Hence $s_i^- = 0$.

We have proved that the $4t + 2$ maps are linearly independent modulo the image of d_{2t}^*. By dimensions, they are a basis of $\mathrm{HH}^{2t}(A)$. □

The aim of this section is to prove the following.

Theorem 5.3. *Let k be a field, $a \geq 3$ an integer, $q \in k$ a primitive a-th root of unity, and A the quantum complete intersection $k\langle X, Y\rangle/(X^a, XY - qYX, Y^a)$. Assume*

$$R := \mathrm{Sp}\{\zeta_i^{2t} : \ t \geq 0 \ \text{and} \ 0 \leq i \leq 2t\}. \tag{5.19}$$

Then R is a subalgebra of $\mathrm{HH}^{2*}(A)$. It is $\mathbb{Z}_2$-graded with $R_0 := k\langle \zeta_i^{2t} : i \text{ even} \rangle$ and $R_1 := k\langle \zeta_i^{2t} : i \text{ odd} \rangle$. Moreover

$$\zeta_l^{2m} \cdot \zeta_r^{2t} = \begin{cases} 0 & l, r \text{ odd} \\ \zeta_{l+r}^{2m+2t} & \text{otherwise.} \end{cases} \tag{5.20}$$

As for the case $a = 2$ we can see:

Corollary 5.4. *The even part R_0 of R is isomorphic to the polynomial ring in two variables.*

Corollary 5.5. *Assume A is as in the Theorem, and let $\mathcal{N}$ be the largest homogeneous nilpotent ideal of $\mathrm{HH}^{2*}(A)$. Then $\mathrm{HH}^{2*}(A)/\mathcal{N}$ is isomorphic to R_0.*

5.3. **Lifting.** We compute the Yoneda product $\chi \bullet \xi$ where χ, ξ are k-linear combinations of maps ζ_j as in Definition 5.1.

For ξ in the span of the ζ_j, the values of ξ are scalars and therefore they commute with elements of A^e. Luckily, we are only interested in the even Hochschild cohomology modulo homogeneous nilpotent elements.

Similar as for the case where $a = 2$ we use liftings along the minimal projective resolution to define the Yoneda products in the cohomology ring. Let $\xi : P_{2t} \to A$ where

$$\xi(f_i^{2t}) := p_i \tag{5.21}$$

and we assume p_i is a scalar multiple of 1, for all i. Consequently the values p_i commute with all elements in A^e. As usual we set $p_i = 0$ if $i > 2t$ or if $i < 0$.

The map $h_0 : P_{2t} \to P_0$ is defined by

$$h_0(f_i^{2t}) := p_i f_0^0 \qquad (0 \leq i \leq 2t). \tag{5.22}$$

Moreover, we search explicit formulae for maps

$$h_s : P_{2t+s} \to P_s. \tag{5.23}$$

For $s \geq 1$ we require

$$h_{s-1} \circ d_{2t+s} = d_s \circ h_s. \tag{5.24}$$

If so, then $(h_s)_{s \geq 0}$ lifts ξ along the minimal projective resolution.

5.3.1. *Some formulae in A^e.* In order to define such lifting maps h_s for $s > 0$ we establish some formulae in A^e. Let

$$c_i = 1 + q + \ldots + q^i \qquad \text{for } 0 \le i \le a-2. \tag{5.25}$$

Definition 5.6. For an integer s we define

$$\beta_x(s) = \sum_{i=0}^{a-2} c_i q^{si}(x^{a-2-i} \otimes x^i) \tag{5.26}$$

$$\beta_y(s) = \sum_{i=0}^{a-2} c_i q^{si}(y^i \otimes y^{a-2-i}) \tag{5.27}$$

Recall now the elements in A^e which occur in the definition of the differentials:

$$\gamma_y(s) = \sum_{j=0}^{a-1} q^{js}(y^j \otimes y^{a-1-j}) \tag{5.28}$$

$$\gamma_x(s) = \sum_{j=0}^{a-1} q^{js}(x^{a-1-j} \otimes x^j) \tag{5.29}$$

At the end we will only need $s = 0$ and $s = 1$. Recall also

$$\tau_y(1) = (1 \otimes y) - q(y \otimes 1) \tag{5.30}$$

$$\tau_x(1) = q(1 \otimes x) - (x \otimes 1) \tag{5.31}$$

$$\tau_y(0) = (1 \otimes y) - (y \otimes 1) \tag{5.32}$$

$$\tau_x(0) = (1 \otimes x) - (x \otimes 1). \tag{5.33}$$

With this notation, we will define maps $h_s : P_{2t+s} \to P_s$, defined on the generators f_i^{2t+s} of the free A^e module P_{2t+s}, and we will show below that they lift ξ:

Definition 5.7.
Assume s is even. For an integer i we define the following elements in the algebra,

$$\omega(j) = \begin{cases} \beta_x(-1)\beta_y(1) & j \text{ odd} \\ 1 & j \text{ even} \end{cases} \tag{5.34}$$

With this, we define for s even

$$h_s(f_i^{2t+s}) := \begin{cases} \sum_{j=0}^{s} p_{i-j}\omega(j) f_j^s & i \text{ even} \\ \sum_{j=0}^{s} p_{i-j} f_j^s & i \text{ odd.} \end{cases} \tag{5.35}$$

Now assume s is odd. Here we need two parameters in A^e, one for x and one for y. We set

$$\varepsilon_x(j) = \begin{cases} -\beta_x(0) & j \text{ odd} \\ 1 & j \text{ even} \end{cases} \qquad \varepsilon_y(j) = \begin{cases} 1 & j \text{ odd} \\ -\beta_y(0) & j \text{ even.} \end{cases} \tag{5.36}$$

With these, we define for s odd,

$$h_s(f_i^{2t+s}) := \begin{cases} \sum_{j=0}^{s} p_{i-j}\varepsilon_x(j) f_j^s & i \text{ even} \\ \sum_{j=0}^{s} p_{i-j}\varepsilon_y(j) f_j^s & i \text{ odd.} \end{cases} \tag{5.37}$$

We will show that $(h_s)_{s\ge 0}$ is a lifting for ξ. For this, we need some formulae.

Lemma 5.8. *We have that the following relations hold:*

(a) $$\beta_y(1)\tau_y(1) = \gamma_y(2)$$

(b) $$\beta_x(-1)\gamma_y(2) = \gamma_y(0)\beta_x(0)$$

(c) $$\beta_x(-1)\beta_y(1) = \beta_y(-1)\beta_x(1)$$

(d) $$\beta_x(1)\tau_x(1) = -\gamma_x(2)$$

(e) $$\beta_y(-1)\gamma_x(2) = \gamma_x(0)\beta_y(0)$$

(f) $$\beta_y(0)\tau_y(0) = \gamma_y(1)$$

(g) $$\beta_x(0)\tau_x(0) = -\gamma_x(1)$$

(h) $$\tau_y(1)\beta_y(0) = \gamma_y(0)$$

(i) $$\tau_x(0)\beta_x(-1) = -\gamma_x(-1)$$

(j) $$\gamma_x(-1)\beta_y(1) = \beta_y(0)\gamma_x(1)$$

(k) $$\tau_y(0)\beta_y(-1) = \gamma_y(-1)$$

(l) $$\tau_x(1)\beta_x(0) = -\gamma_x(0)$$

(m) $$\beta_x(0)\gamma_y(1) = \gamma_y(-1)\beta_x(1)$$

Proof. We prove (a) and (b), and the other relations follows from the same kind of reasoning. Start with (a), we have

$$\beta_y(1)\tau_y(1) = \left(\sum_{i=0}^{a-2} c_i q^i (y^i \otimes y^{a-2-i})\right)((1\otimes y) - q(y\otimes 1)) \tag{5.38}$$

$$= \sum_{i=0}^{a-2} \left(c_i q^i (y^i \otimes y^{a-1-i}) - c_i q^{i+1}(y^{i+1}\otimes y^{a-2-i})\right) \tag{5.39}$$

$$= c_0(1\otimes y^{a-1}) + c_1 q(y\otimes y^{a-2}) + \cdots + c_{a-2}q^{a-2}(y^{a-2}\otimes y) \tag{5.40}$$

$$- c_0 q(y\otimes y^{a-2}) - \cdots - c_{a-3}q^{a-2}(y^{a-2}\otimes y) - c_{a-2}q^{a-1}(y^{a-1}\otimes 1) \tag{5.41}$$

$$= c_0(1\otimes y^{a-1}) + q(c_1 - c_0)(y\otimes y^{a-2}) + q^2(c_2-c_1)(y^2\otimes y^{a-3}) + \tag{5.42}$$

$$\cdots + q^{a-2}(c_{a-2} - c_{a-3})(y^{a-2}\otimes y) - q^{a-1}c_{a-2}(y^{a-1}\otimes 1) \tag{5.43}$$

where we have that $c_0 = 1$, $c_1 - c_0 = 1 + q - 1 = q$, ...

$$c_{i+1} - c_i = (1 + q + \cdots + q^{i+1}) - (1 + q + \cdots + q^i) = q^{i+1} \tag{5.44}$$

We also observe

$$c_{a-2} = 1 + q + \cdots + q^{a-2} = -q^{a-1} \tag{5.45}$$

since a is a root of unity and hence $1 + q + \cdots + q^{a-2} + q^{a-1} = 0$. Then we have,

$$\beta_y(1)\tau_y(1) = (1\otimes y^{a-1}) + q^2(y\otimes y^{a-2}) + \cdots + q^{2(a-1)}(y^{a-1}\otimes 1) = \gamma_y(2) \tag{5.46}$$

For the relation (b) we inspect a typical element in this sum:

$$c_i q^{-i}(x^{a-2-i}\otimes x^i)q^{2j}(y^j\otimes y^{a-1-j}) = c_i q^{-i} q^{2j}(x^{a-2-i}y^j \otimes x^i * y^{a-1-j}) \tag{5.47}$$

(where $*$ denotes the multiplication in A^{op}). Now we recall that $xy = qyx$ (and $x * y = q^{-1}y * x$) hence $x^{a-2-i}y^j = q^{j(a-2-i)}y^j x^{a-2-i}$ and $x^i * y^{a-1-j} = q^{-i(a-1-j)}y^{a-1-j} * x^i$. We get

$$c_i q^{-i} q^{2j} q^{j(a-2-i)} q^{-i(a-1-j)}(y^j x^{a-2-i}\otimes y^{a-1-j} * x^i) = (y^j \otimes y^{a-2-j})c_i(x^{a-2-i}\otimes x^i) \tag{5.48}$$

which is the most typical element in the sum $\gamma_y(0)\beta_x(0)$. □

The relations (a) to (m) in Lemma 5.8 can be used to prove that the maps h_s are liftings for the given map ξ:

Proposition 5.9. *The lifting formulas make the suggested squares commutative, that is* $h_{s-1} \circ d_{2t+s} = d_s \circ h_s$ *when* $s \geq 1$ *and* $\xi = \mu \circ h_0$.

Proof. We give details when s and i are even, the other cases are similar. The strategy is to apply both sides to f_i^{2t+s} and express the answer in terms of the basis $\{f_j^{s-1}\}$, with coefficients in A^e and then show that the coefficients of the f_j^{s-1} in the two expressions are equal.

We have

$$(d_s \circ h_s)(f_i^{2t+s}) = d_s \left(\sum_{j=0}^{s} p_{i-j}\omega(j) f_j^s \right) \tag{5.49}$$

$$= \sum_{j \text{ even},\ 0 \leq j \leq s} p_{i-j}\omega(j) \left[\gamma_y(0) f_j^{s-1} + \gamma_x(0) f_{j-1}^{s-1} \right] \tag{5.50}$$

$$+ \sum_{j \text{ odd},\ 0 \leq j \leq s} p_{i-j}\omega(j) \left[-\tau_y(1) f_j^{s-1} + \tau_x(1) f_{j-1}^{s-1} \right] \tag{5.51}$$

We split each of the two sums, and when the index is $j-1$ we change variables, setting $l = j-1$ so that $j = l+1$ and noting that l has opposite parity as j. As well we recall $\omega(j) = 1$ for j even. Then this becomes

$$= \sum_{j \text{ even},\ 0 \leq j \leq s} p_{i-j}\gamma_y(0) f_j^{s-1} + \sum_{l \text{ odd},\ -1 \leq l \leq s-1} p_{i-l-1}\gamma_x(0) f_l^{s-1} \tag{5.52}$$

$$+ \sum_{j \text{ odd},\ 0 \leq j \leq s} -p_{i-j}\omega(j)\tau_y(1) f_j^{s-1} + \sum_{l \text{ even},\ -1 \leq l \leq s-1} p_{i-l-1}\omega(l+1)\tau_x(1) f_l^{s-1} \tag{5.53}$$

The range of summation can be unified since $f_j^s = 0$ for $j = -1$ or $j = s$. We write this now as a combination in the A^e-basis f_j^{s-1} for $0 \leq j \leq s-1$, (writing j for l) and we get

$$= \sum_{j \text{ even},\ 0 \leq j \leq s-1} [p_{i-j}\gamma_y(0) + p_{i-j-1}\omega(j+1)\tau_x(1)] f_j^{s-1}$$

$$+ \sum_{j \text{ odd},\ 0 \leq j \leq s-1} [p_{i-j-1}\gamma_x(0) - p_{i-j}\omega(j)\tau_y(1)] f_j^{s-1} \tag{5.54}$$

On the other hand

$$(h_{s-1} \circ d_{2t+s})(f_i^{2t+s}) = h_{s-1}(\gamma_y(0) f_i^{2t+s-1} + \gamma_x(0) f_{i-1}^{2t+s-1}) \tag{5.55}$$

$$= \gamma_y(0)[\sum_{j=0}^{s-1} p_{i-j}\varepsilon_x(j) f_j^{s-1}] + \gamma_x(0)[\sum_{j=0}^{s-1} p_{i-1-j}\varepsilon_y(j) f_j^{s-1}] \tag{5.56}$$

$$= \sum_{j=0}^{s-1} [p_{i-j}\gamma_y(0)\varepsilon_x(j) + p_{i-1-j}\gamma_x(0)\varepsilon_y(j)] f_j^{s-1}. \tag{5.57}$$

We must show that for each j the coefficients of f_j^{s-1} in (5.54) and in (5.57) are equal.

(a) Assume first j is even. We require

$$p_{i-j}\gamma_y(0) + p_{i-j-1}\omega(j+1)\tau_x(1) = p_{i-j}\gamma_y(0)\varepsilon_x(j) + p_{i-j-1}\gamma_x(0)\varepsilon_y(j) \tag{5.58}$$

For j even, $\varepsilon_x(j) = 1$ and the first terms agree. The second terms agree provided

$$\omega(j+1)\tau_x(1) = \gamma_x(0)\varepsilon_y(j) \tag{5.59}$$

Consider the LHS, by identities (c), (d) and (e) it is equal to

$$\beta_y(-1)\beta_x(1)\tau_x(1) = -\beta_y(-1)\gamma_x(2) = -\gamma_x(0)\beta_y(0) = \gamma_x(0)\varepsilon_y(j) \tag{5.60}$$

from the definition of $\varepsilon_y(j)$ in this case. Hence the second terms agree as well.

(b) Now assume j is odd. We require

$$p_{i-j-1}\gamma_x(0) - p_{i-j}\omega(j)\tau_y(1) = p_{i-j}\gamma_y(0)\varepsilon_x(j) + p_{i-1-j}\gamma_x(0)\varepsilon_y(j) \tag{5.61}$$

For j odd, $\varepsilon_y(j) = 1$ and the terms with p_{i-j-1} agree. For the other two terms to agree we need

$$\gamma_y(0)\varepsilon_x(j) = -\omega(j)\tau_y(1) \tag{5.62}$$

We have using the definition and identities (a) and (b) that

$$-\omega(j)\tau_y(1) = -\beta_x(-1)\beta_y(1)\tau_y(1) = -\beta_x(-1)\gamma_y(2) = -\gamma_y(0)\beta_x(0) = \gamma_y(0)\varepsilon_x(j) \tag{5.63}$$

as required.

□

Similar as for the case $a = 2$ we define the Yoneda product of the residue classes represented by ξ of degree $2t$ and χ of degree $2m$ to be the residue class represented by the composition

$$\xi \bullet \chi = \chi \circ h_{2m}. \tag{5.64}$$

5.4. **Description of Yoneda products of basis elements when $a \geq 3$.** In the definition 5.7 of the lifting maps, we have the term $\omega(j) = \beta_x(-1)\beta_y(1) \in A^e$ (for j odd). When this is evaluated in A, it becomes $\omega(j) \cdot 1_A$. We claim that this is always zero, in fact $\beta_y(1) \cdot 1_A = 0$.

Namely, we must view A as an A^e bimodule and then

$$\beta_y(1) \cdot 1_A = \sum_{i=0}^{a-2} c_i q^i y^{a-2} = \left(\sum_{i=0}^{a-2} c_i q^i\right) y^{a-2} \tag{5.65}$$

The following shows that this is zero:

Lemma 5.10. *Let q be a primitive a-th root of unity for $a \geq 3$. Let $c_i = 1 + q + \ldots + q^i$ for $i \geq 0$, then*

$$\sum_{i=0}^{a-2} c_i q^i = 0 \tag{5.66}$$

Proof. Set also $c_{-1} := 0$. Then we have for $i \geq 0$ that $c_i - c_{i-1} = q^i$. We get

$$\sum_{i=0}^{a-2} c_i q^i = \sum_i c_i(c_i - c_{i-1}) \tag{5.67}$$

Therefore (all summations are from $i = 0$ to $a - 2$)

$$\begin{aligned}(1+q)(\sum_i c_i q^i) =& \sum_i c_i q^i + \sum_i c_i q^{i+1}\\ =& \sum_i c_i(c_i - c_{i-1}) + \sum_i c_i(c_{i+1} - c_i)\\ =& \sum_i (c_i c_{i+1} - c_i c_{i-1})\\ =& c_{a-2}c_{a-1} - c_0 c_{-1}\\ =& 0\end{aligned}$$

since $c_{a-1} = 1 + q + \ldots + q^{a-1} = 0$ and $c_{-1} = 0$. But $q \neq -1$, so we can cancel by $(1 + q)$ and get the claim. □

We analyse now the products, and this will complete the proof of Theorem 5.3. Define

$$R = \mathrm{Sp}\{\zeta_i^{2t} : t \geq 0, 0 \leq i \leq 2t\}. \tag{5.68}$$

We compute products of elements in R.

Let χ be of degree $2m$ and ξ of degree $2t$, both in R. Let $\xi(f_i^{2t}) = p_i \in K$ for $0 \leq i \leq 2t$ and $\chi(f_j^{2m}) = \bar{p}_j \in K$ for $0 \leq j \leq 2m$. As before we set $p_i = 0$ for $i < 0$ or $i > 2t$, and similarly we define $\bar{p}_j$ for any $j \in \mathbb{Z}$.

Then $\chi \bullet \xi$ is the class of $\chi \circ h_{2m}$ where (h_s) is a lifting of ξ, with the formula computed above. Note that we only need the case when $s = 2m$ is even. We have

$$\chi \circ h_s(f_i^{2t+s}) = \chi(\sum_{j=0}^{s} p_{i-j}\omega(j) f_j^s) = \begin{cases} \sum_{j=0}^{s} p_{i-j}\omega(j)\bar{p}_j & i \text{ even} \\ \sum_{j=0}^{s} p_{i-j}\bar{p}_j & i \text{ odd.} \end{cases} \tag{5.69}$$

Now assume $\chi = \zeta_l$ for some $0 \leq l \leq s$, so $\bar{p}_l = 1$ and $\bar{p}_j = 0$ otherwise. Then the above simplifies to

$$f_i^{2t+s} \mapsto \begin{cases} p_{i-l}\omega(l) \cdot 1 & i \text{ even} \\ p_{i-l} \cdot 1 & i \text{ odd} \end{cases} \tag{5.70}$$

Now take $\xi = \zeta_r$ for some $0 \leq r \leq 2t$. Then $p_{i-l} = 1$ if $i - l = r$, and 0 otherwise.

Note that $\omega(l) \cdot 1_A$ is zero for l odd and is equal to 1 otherwise. The zero occurs precisely when l is odd and $i = l + r$ is even, i.e. if both l, r are odd. So we get

$$\zeta_l^{2m} \cdot \zeta_r^{2t} = \begin{cases} \zeta_{l+r}^{2m+2t} & l, r \text{ not both odd} \\ 0 & l, r \text{ odd.} \end{cases} \tag{5.71}$$

As for the case $a = 2$ we see that R_0 is isomorphic to the polynomial ring in two variables.

Furthermore, we see that elements in R_1 are nilpotent. The subalgebra R_0 intersects the largest homogeneous nilpotent ideal $\mathcal{N}$ trivially, and the span of the $\eta^{\pm}$ is contained in $\mathcal{N}$.

Acknowledgements. Both authors thank Petter A. Bergh for his joint notes with Karin Erdmann. The second author would like to thank the first author a lot for the invitation to Oxford and is very grateful for spending these months on this wonderful place. Thanks for valuable discussions on this project and on various topics in general, and for including me into the mathematical community in Oxford. Thanks also to Petter for arranging and help with applications. The Research Council of Norway supports the PhD work of the second author through the research project *Triangulated categories in algebra* (NFR 221893). The Research Council of Norway has also supported the second author with an overseas grant for the stay in Oxford. Both authors thank our anonymous referee for helpful comments.

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