A REMARK ON THE ALEXANDROV-FENCHEL INEQUALITY

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ABSTRACT. In this article, we give a complex-geometric proof of the Alexandrov-Fenchel inequality without using toric compactifications. The idea is to use the Legendre transform and develop the Brascamp-Lieb proof of the Prékopa theorem. New ingredients in our proof include an integration of Timorin's mixed Hodge-Riemann bilinear relation and a mixed norm version of Hörmander's L^2-estimate, which also implies a non-compact version of the Khovanskiĭ-Teissier inequality.

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1. INTRODUCTION

The classical Brunn-Minkowski inequality is an inequality on the volumes of convex bodies in $\mathbb{R}^n$. It plays an important role in many branches of mathematics, to quote from Gardner's survey article [20]: "In a sea of mathematics, the Brunn-Minkowski inequality appears like an octopus, tentacles reaching far and wide...". A far reaching generalization of it is the Alexandrov-Fenchel inequality, which has many different proofs (see section 20.3 in [12]). In 1936, Alexandrov found a combinatorial proof and an analytic proof. The later is a generalization of Hilbert's 1910 proof ("Minkowskis Theorie von Volumen und Oberfläche") of the Brunn-Minkowski inequality. A simple algebraic proof (see [26] and [27]) based on the Bernstein-Kushnirenko theorem and the intersection theory on quasi-projective variety was given by Kaveh and Khovanskiĭ around 2008. For other interesting proofs and related results, see [22], [30], [18] and [13], to cite only a few. The Brunn-Minkowski inequality also has a functional version, i.e. the Prékopa theorem [31] for convex functions, which was found by Prékopa in 1973. In 1976 [11], Brascamp and Lieb gave another proof of the Prékopa theorem, the main idea is to use the Brascamp-Lieb lemma (see Lemma 4.2) to reduce the Prékopa theorem to a weighted L^2-estimate of Hörmander type [23] (so called the Brascamp-Lieb inequality) for the minimal solution u of

$$du = v.$$

In 1998, by a magic way of using Hörmander's $\overline{\partial}$-L^2 estimate [23], Berndtsson [3] proved a complex version of the Prékopa theorem for plurisubharmonic functions. In 2005, inspired by [1], Cordero-Erausquin [15] discovered the relation between Berndtsson's work and the Brascamp-Lieb proof. Shortly after that, a very general and useful theory (so called the complex Brunn-Minkowski theory) [6, 5] behind the Brascamp-Lieb proof and Maitani-Yamaguchi's result [29]

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was established by Berndtsson. The main result in that theory is a deep and beautiful curvature formula for a certain direct image bundle, which has found many highly non-trivial applications in Kähler geometry and algebraic geometry, see [6, 9, 8, 7, 4] and references therein. Inspired by [34] and Berndtsson's theory, in this paper we obtain a new complex-geometric proof of the Alexandrov-Fenchel inequality. The main idea is that the Brascamp-Lieb lemma (see Lemma 4.2) reduces the Alexandrov-Fenchel inequality to an L^2-estimate $||u|| \leq ||\theta||$ on $\mathbb{R}^n \times (\mathbb{R}^n/\mathbb{Z}^n)$ for the minimal solution of

$$du = (d^c)^*\theta, \ d^c := i\overline{\partial} - i\partial,$$

with respect to Timorin's mixed norm (see [33] and [35]). The main advantage of this approach is that *we can prove the L^2-estimate $||u|| \leq ||\theta||$ directly, without using the compactification theory.* In fact, by Hörmander's L^2-theory [24, 17], it is enough to construct a special complete Kähler metric on $\mathbb{R}^n \times (\mathbb{R}^n/\mathbb{Z}^n)$ (Lemma 7.1). Another advantage is that the L^2-estimate $||u|| \leq ||\theta||$ is true on a large class of non-compact manifolds, not only on $\mathbb{R}^n \times (\mathbb{R}^n/\mathbb{Z}^n)$. In [21] (p 21), Gromov suggested to study non-compact generalizations of the Khovanskiĭ-Teissier inequality. Our approach generalizes the Khovanskiĭ-Teissier inequality to the following:

Theorem 1.1. *Let $(X, \hat{\omega})$ be an n-dimensional complete Kähler manifold with finite volume. Let $\alpha_1, \cdots, \alpha_n$ be smooth d-closed semi-positive $(1,1)$-forms such that $\alpha_j \leq \hat{\omega}$ on X for every $1 \leq j \leq n$. Assume that $n \geq 2$. Put*

$$T := \alpha_3 \wedge \cdots \wedge \alpha_n, \ T := 1, \ \textit{if}\ n = 2.$$

Then

$$\left(\int_X \alpha_1 \wedge \alpha_2 \wedge T\right)^2 \geq \left(\int_X \alpha_1^2 \wedge T\right)\left(\int_X \alpha_2^2 \wedge T\right).$$

Remark: The above theorem can be seen as a special case of our main result (Theorem 3.1). Recall that a Hermitian manifold $(X, \hat{\omega})$ is said to be *complete* if there exists a smooth function, say

$$\rho : X \to [0, \infty),$$

such that $\rho^{-1}([0, c])$ is compact for every $c > 0$ and

$$|d\rho|_{\hat{\omega}}(x) \leq 1, \ \forall\, x \in X.$$

In order to deduce the classical Alexandrov-Fenchel inequality from Theorem 1.1, we construct a special complete Kähler metric on $\mathbb{R}^n \times (\mathbb{R}^n/\mathbb{Z}^n)$ in Lemma 7.1. The whole paper is organized as follows.

Contents

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2. PRELIMINARIES

2.1. Basic notions in convex geometry.

(1) A set Ω in $\mathbb{R}^n$ is said to be *convex* if the line segment between any two points in Ω lies in Ω.
(2) We call a *compact convex set, say A, with non-empty interior*, say A°, in $\mathbb{R}^n$ a *convex body*.

Let A_0, A_1 be two convex bodies in $\mathbb{R}^n$. We call

$$A_0 + A_1 := \{a_0 + a_1 \in \mathbb{R}^n : a_0 \in A_0,\ a_1 \in A_1\},$$

the *Minkowski sum* of A_0 and A_1. The Brunn-Minkowski theorem (see [20] for a nice survey) reads as follows:

Theorem 2.1 (Brunn-Minkowski inequality). $|A_0 + A_1|^{1/n} \geq |A_0|^{1/n} + |A_1|^{1/n}$, *where the absolute value of a convex body means its volume (Lebesgue measure).*

Remark: The Brunn-Minkowski inequality is also true for compact *non-convex* sets with non-empty interior, see [28].

We will also need the following notion in convex geometry.

Definition 2.1 (Legendre transform). *Let A be a convex body. Let ψ be a smooth real-valued function on A°. ψ is said to be strictly convex if the Hessian matrix (ψ_{jk}) is positive definite at every point in A°. We call*

$$\psi^*(y) := \sup_{x\in A^\circ} x\cdot y - \psi(x),\ x\cdot y := \sum_{j=1}^n x^j y^j,$$

the Legendre transform of ψ (with respect to A°).

Proposition 2.2. *Let ψ be a smooth strictly convex function that tends to infinity at the boundary of a convex body A. Then its Legendre transform ψ^* is also smooth, strictly convex, moreover the gradient map of ψ^**

$$\nabla\psi^* : y \mapsto x = \nabla\psi^*(y) := (\partial\psi^*/\partial y^1, \cdots, \partial\psi^*/\partial y^n), \tag{2.1}$$

defines a diffeomorphism from $\mathbb{R}^n$ onto A°.

Proof. It is enough to prove that the gradient map of ψ defines a diffeomorphism from A° to $\mathbb{R}^n$, ψ^* is smooth and $\nabla\psi^*$ is the inverse of $\nabla\psi$.

Step 1: $\nabla\psi$ is a diffeomorphism from A° to $\mathbb{R}^n$. Since ψ is smooth and strictly convex, we know that $\nabla\psi$ is a local diffeomorphism.

1. $\nabla\psi$ is injective: assume that $\nabla\psi(x_1) = \nabla\psi(x_2) = y_0$, consider

$$\psi^{y_0}(x) := \psi(x) - y_0\cdot x, \tag{2.2}$$

we know that ψ^{y_0} is smooth, strictly convex and

$$\nabla\psi^{y_0}(x_1) = \nabla\psi^{y_0}(x_2) = 0. \tag{2.3}$$

Consider the restriction, say g, of ψ^{y_0} to the line determined by x_1 and x_2, then g is convex with critical points x_1 and x_2. Thus g is a constant on the line segment from x_1 to x_2, moreover, strict convexity of g implies $x_1 = x_2$. Thus $\nabla\psi$ is injective.

2. $\nabla\psi(A^0) = \mathbb{R}^n$: fix $y\in\mathbb{R}^n$, since ψ^y tends to infinity at the boundary of A, strict convexity of ψ implies that ψ^y has a unique minimum point, say $x\in A^\circ$. Thus

$$0 = \nabla\psi^y(x) = \nabla\psi(x) - y.$$

Step 2: ψ^* is smooth. Notice that

$$\psi^*(\nabla\psi(x)) = \nabla\psi(x)\cdot x - \psi(x). \tag{2.4}$$

Thus $\psi^*\circ\nabla\psi$ is a smooth, which implies that ψ^* is smooth on $\mathbb{R}^n$.

Step 3: $\nabla\psi^*$ is the inverse of $\nabla\psi$. Apply the differential to (2.4), we get that

$$(\nabla\psi^*\circ\nabla\psi(x))\cdot(\psi_{jk}) = x\cdot(\psi_{jk}),\ \forall\, x\in A^\circ. \tag{2.5}$$

Since (ψ_{jk}) is an invertible matrix function, the above formula gives $\nabla\psi^*\circ\nabla\psi = Id$. □

Remark: Put $\phi = \psi^*$. We know from the above proposition that $\nabla\phi$ is a diffeomorphism from $\mathbb{R}^n$ onto the interior of A, thus

$$|A| = \int_A dy = \int_{\mathbb{R}^n} MA(\phi)\, dx,\ dx := dx^1 \wedge \cdots \wedge dx^n,\ dy := dy^1 \wedge \cdots \wedge dy^n. \tag{2.6}$$

where $MA(\phi) := \det(\phi_{jk})$ denotes the determinant of the Hessian of ϕ. In case A is the convex hull of a finite set, say $\{p_j\}_{1\le j\le N} \subset \mathbb{R}^n$, one may choose

$$\phi(x) = \log\left(\sum_{j=1}^{N} e^{p_j\cdot x}\right).$$

For more results on convex function of the above type, see [36] and [21], see also [2] and [16] for the canonical choice of such ϕ.

The following proposition is a generalization of (2.6).

Proposition 2.3. *Let $\phi_1, \cdots, \phi_N$ be smooth strictly convex functions such that each $\nabla\phi_j$ is a diffeomorphism from $\mathbb{R}^n$ onto the interior of a convex body A_j. Then we have*

$$|t_1A_1 + \cdots + t_NA_N| = \int_{\mathbb{R}^n} MA(t_1\phi_1 + \cdots + t_N\phi_N)\, dx,\ t_j > 0,\ \forall\, 1 \le j \le N. \tag{2.7}$$

Proof. By induction on N, it suffices to show that

$$\nabla(\phi_1 + \phi_2)(\mathbb{R}^n) = A_1^\circ + A_2^\circ, \tag{2.8}$$

where A° denotes the interior of A. Obviously we have $\nabla(\phi_1 + \phi_2)(\mathbb{R}^n) \subset A_1^\circ + A_2^\circ$. Thus it is enough to show that for every $y_1 \in A_1^\circ$ and every $y_2 \in A_2^\circ$, there exists $x_0 \in \mathbb{R}^n$ such that $\nabla(\phi_1 + \phi_2)(x_0) = y_1 + y_2$. Consider $\phi_j^{y_j}$ instead of ϕ_j, one may assume that $y_1 = y_2 = 0$. Choose x_1 and x_2 such that

$$\nabla\phi_1(x_1) = \nabla\phi_2(x_2) = 0. \tag{2.9}$$

Since ϕ_j is convex, we know that each x_j is the minimum point of ϕ_j. Thus strict convexity of ϕ_j implies that

$$\phi_j(x) \to \infty,\ \text{as } |x| \to \infty, \tag{2.10}$$

i.e. each ϕ_j is proper. Thus $\phi_1 + \phi_2$ is also proper. Hence there exists a unique minimum point, say x_0, of $\phi_1 + \phi_2$. Thus $\nabla(\phi_1 + \phi_2)(x_0) = 0$. The proof is complete. □

Remark: The above proposition implies that

$$p(t) := |t_1A_1 + \cdots + t_nA_n|, \tag{2.11}$$

is a polynomial of degree n. We call the coefficient of $t_1 \cdots t_n$ in the polynomial $p(t)$, i.e.

$$V(A_1, \cdots, A_n) := \frac{\partial^n |t_1A_1 + \cdots + t_nA_n|}{\partial t_1 \cdots \partial t_n},$$

the *mixed volume* of $A_1, \cdots, A_n$.

2.2. Alexandrov-Fenchel inequality.

Theorem 2.4 (Alexandrov-Fenchel inequality). *Let $A_1, \cdots, A_n$ be convex bodies in $\mathbb{R}^n$. Assume that $n \geq 2$. Then*

$$V(A_1, \cdots, A_n)^2 \geq V(A_1, A_1, A_3, \cdots, A_n) V(A_2, A_2, A_3, \cdots, A_n).$$

The following lemma can be used to find equivalent forms of the Alexandrov-Fenchel inequality.

Lemma 2.5. *Let f be a positive smooth function on an open convex cone, say $\mathcal{K}$, in $\mathbb{R}^N$. Assume that f is 1-homogeneous, i.e.*

$$f(tx) \equiv tf(x), \ \forall\, t > 0, \ x \in \mathcal{K}.$$

Then the following statements are equivalent:
$A1$: $f(x+y) \geq f(x) + f(y), \ \forall\, x, y \in \mathcal{K}$;
$A2$: $-f$ is convex;
$A3$: $-\log f$ is convex;
$A4$: For every $x', y' \in \mathcal{K}$, $t \mapsto -\log f(tx' + (1-t)y')$ is convex on $(0,1)$.

Proof. Since f is 1-homogeneous, $A1$ implies

$$f(tx + (1-t)y) \geq tf(x) + (1-t)f(y). \tag{2.12}$$

Thus $A1 \Rightarrow A2$. Since

$$(-\log f)_{\xi\xi} = \frac{-f_{\xi\xi}}{f} + \frac{(f_\xi)^2}{f^2}, \ f_\xi = \sum \xi^j f_{x_j}, \tag{2.13}$$

we know $A2 \Rightarrow A3$. Since $A3 \Rightarrow A4$ is trivial, it is enough to show $A4 \Rightarrow A1$: notice that $A4$ implies

$$f(tx' + (1-t)y') \geq f(x')^t f(y')^{1-t}. \tag{2.14}$$

Take

$$x' = \frac{x}{f(x)}, \ y' = \frac{y}{f(y)}, \ t = \frac{f(x)}{f(x)+f(y)}, \tag{2.15}$$

we get $A1$. The proof is complete. □

Apply the above lemma to the following function

$$f(x) = V(A_x, A_x, A_3, \cdots, A_n)^{1/2}, \ A_x := x_1 A_1 + x_2 A_2, \tag{2.16}$$

on $\mathcal{K} := \mathbb{R}^2_+$. Notice that the square of

$$f(x+y) \geq f(x) + f(y), \tag{2.17}$$

is equivalent to

$$V(A_x, A_y, A_3, \cdots, A_n)^2 \geq V(A_x, A_x, A_3, \cdots, A_n) V(A_y, A_y, A_3, \cdots, A_n).$$

By the above lemma, we have

Proposition 2.6. *The Alexandrov-Fenchel inequality is equivalent to the convexity of*

$$t \mapsto -\log V(A_t, A_t, A_3, \cdots, A_n),\ A_t := tA_1 + (1-t)A_2,$$

on $(0,1)$.

A generalized form of the Alexandrov-Fenchel inequality is also true.

Theorem 2.7. *Let* $A_1, A_2, A_{m+1}, \cdots, A_n$*,* $2 \leq m \leq n$*, be convex bodies in* $\mathbb{R}^n$*. Then the following function is convex on* $(0,1)$

$$t \mapsto -\log V(\underbrace{A_t, \cdots, A_t}_{m}, A_{m+1}, \cdots, A_n),\ A_t := tA_1 + (1-t)A_2.$$

The above theorem is in fact equivalent to the Alexandrov-Fenchel inequality (see Theorem 7.4.5 in [32]).

2.3. **Khovanskiĭ-Teissier inequality.** We will use the following complex geometry interpretation of the volume function in Proposition 2.3.

Lemma 2.8. *Let* $\phi_1, \cdots, \phi_N$ *be smooth strictly convex functions such that each* $\nabla\phi_j$ *is a diffeomorphism from* $\mathbb{R}^n$ *onto the interior of a convex body* A_j*. Let us look at*

$$\phi := \sum_{j=1}^{N} t_j \phi_j,$$

as a function on

$$\mathbb{R}^n \times \mathbb{T}^n = \mathbb{C}^n / i\mathbb{Z}^n,\ \mathbb{T} := \mathbb{R}/\mathbb{Z},\ i := \sqrt{-1},$$

i.e. $\phi(x+iy) := \sum_{j=1}^{N} t_j \phi_j(x)$*. Then we have*

$$\int_{\mathbb{R}^n} MA(\phi)\, dx = \int_{\mathbb{R}^n \times \mathbb{T}^n} \frac{(dd^c\phi)^n}{n!},\ d^c := i\bar{\partial} - i\partial.$$

Proof. Since

$$dd^c\phi = 2i\partial\bar{\partial}\phi = \frac{i}{2} \sum_{j,k=1}^{n} \phi_{jk}\, dz^j \wedge d\bar{z}^k,\ z^j := x^j + iy^j,$$

where $\phi_{jk} := \partial^2\phi / \partial x^j \partial x^k$, we have

$$\frac{(dd^c\phi)^n}{n!} = \det(\phi_{jk})\,(dx^1 \wedge dy^1) \wedge \cdots \wedge (dx^n \wedge dy^n),$$

thus the lemma follows from the Fubini theorem and $\int_{\mathbb{T}^n} dy = 1$. □

The above lemma implies

Lemma 2.9. *Let* $\phi_1, \cdots, \phi_n$ *be smooth strictly convex functions such that each* $\nabla\phi_j$ *is a diffeomorphism from* $\mathbb{R}^n$ *onto the interior of a convex body* A_j*. Then we have the following mixed volume formula*

$$V(A_1, \cdots, A_n) = \int_{\mathbb{R}^n \times \mathbb{T}^n} dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n.$$

Proof. The previous lemma gives

$$|\sum_{j=1}^{n} t_j A_j| = \int_{\mathbb{R}^n\times\mathbb{T}^n} \frac{(dd^c\phi)^n}{n!},\ t_j>0,\ \forall\, 1\leq j\leq n.$$

Notice that

$$\frac{(dd^c\phi)^n}{n!} = \sum_{\alpha_1+\cdots+\alpha_n=n} \frac{t_1^{\alpha_1}\cdots t_n^{\alpha_n}}{\alpha_1!\cdots\alpha_n!}\,(dd^c\phi_1)^{\alpha_1}\wedge\cdots\wedge(dd^c\phi_n)^{\alpha_n},$$

and each term $(dd^c\phi_1)^{\alpha_1}\wedge\cdots\wedge(dd^c\phi_n)^{\alpha_n}$ is a positive (n,n)-form, thus

$$|\sum_{j=1}^{n} t_j A_j| < \infty \Rightarrow \int_{\mathbb{R}^n\times\mathbb{T}^n} (dd^c\phi_1)^{\alpha_1}\wedge\cdots\wedge(dd^c\phi_n)^{\alpha_n} < \infty.$$

Now we have

$$|\sum_{j=1}^{n} t_j A_j| = \sum_{\alpha_1+\cdots+\alpha_n=n} \frac{t_1^{\alpha_1}\cdots t_n^{\alpha_n}}{\alpha_1!\cdots\alpha_n!}\int_{\mathbb{R}^n\times\mathbb{T}^n} (dd^c\phi_1)^{\alpha_1}\wedge\cdots\wedge(dd^c\phi_n)^{\alpha_n},$$

and the lemma follows. □

By the above lemma, we know that Theorem 2.7 is equivalent to the following:

Theorem 2.10. *Let $\phi_1,\phi_2,\ \phi_{m+1},\cdots,\phi_n$, $2\leq m\leq n$, be smooth strictly convex functions such that each $\nabla\phi_j$ is a diffeomorphism from $\mathbb{R}^n$ onto the interior of a convex body A_j. Then the following function is convex on $(0,1)$*

$$t\mapsto -\log\int_{\mathbb{R}^n\times\mathbb{T}^n} \frac{\omega^m}{m!}\wedge T,$$

where

$$\omega := t dd^c\phi_1 + (1-t) dd^c\phi_2, \qquad T := dd^c\phi_{m+1}\wedge\cdots\wedge dd^c\phi_n.$$

Let us recall the following Khovanskiĭ-Teissier theorem.

Theorem 2.11 (Khovanskiĭ-Teissier inequality)**.** *Let $\omega_1,\cdots,\omega_n$ be Kähler forms on a compact Kähler manifold X. Assume that $n\geq 2$. Put*

$$T := \omega_3\wedge\cdots\wedge\omega_n, \qquad T := 1,\ \textit{if}\ n=2.$$

Then

$$\left(\int_X \omega_1\wedge\omega_2\wedge T\right)^2 \geq \left(\int_X \omega_1^2\wedge T\right)\left(\int_X \omega_2^2\wedge T\right).$$

By Lemma 2.5, we know that the Khovanskiĭ-Teissier inequality is equivalent to the ($m=2$ case) convexity of

$$t\mapsto -\log\int_X \frac{\omega^m}{m!}\wedge T, \qquad \omega := t\omega_1 + (1-t)\omega_2,\ T := \omega_{m+1}\wedge\cdots\wedge\omega_n.$$

Thus Theorem 2.10 can be seen as a Khovanskiĭ-Teissier inequality for $\mathbb{R}^n\times\mathbb{T}^n$.

Remark: The above equivalent description of the Khovanskiĭ-Teissier inequality was first used by Graham in his proof of the convexity of the interpolating function, see [19]. There are also other descriptions of the Khovanskiĭ-Teissier inequality. A very nice intersection theory description of its algebraic version can be found in [25] and [26]. In the Hodge theory description, the Khovanskiĭ-Teissier inequality is a direct application of the *mixed generalization of the classical Hodge-Riemann bilinear relation* (MHRR) for $(1,1)$-forms. MHRR for general (p,q)-forms on a compact Kähler manifold was first proved by Dinh-Nguyên in [18] based on Timorin's result [33] for the torus case, see also [13] for another approach that applies to general polarized Hodge-Lefschetz modules.

3. Main theorem

Theorem 3.1. *Let $(X, \hat{\omega})$ be an n-dimensional complete Kähler manifold with finite volume. Let $\alpha_1, \alpha_2, \alpha_m, \cdots, \alpha_n$, $2 \leq m \leq n$, be smooth d-closed semi-positive $(1,1)$-forms such that each $\alpha_j \leq \hat{\omega}$ on X. Then the following function is convex on $(0,1)$*

$$t \mapsto -\log \int_X \frac{\omega^m}{m!} \wedge T, \qquad \omega := t\alpha_1 + (1-t)\alpha_2,$$

where $T := \alpha_{m+1} \wedge \cdots \wedge \alpha_n$, $T := 1$, if $n = m$.

By Lemma 2.5, in case $m = 2$, our main theorem is equivalent to Theorem 1.1, which is a non-compact generalization of the Khovanskiĭ-Teissier inequality.

About the proof of the main theorem. Put

$$f(t) = -\log \int_X \frac{\omega^m}{m!} \wedge T.$$

Consider $\alpha_j + \epsilon\hat{\omega}$ instead of α_j and denote by f^ϵ the associated function. Then we have

$$f = \lim_{\epsilon \to 0} f^\epsilon.$$

Thus it suffices to show that each f^ϵ is convex on $(0,1)$, i.e. one may assume that

$$\frac{\hat{\omega}}{C} \leq \alpha_j \leq C\hat{\omega}, \tag{3.1}$$

for every j in Theorem 3.1, where C is a fixed positive constant. Then Theorem 3.1 follows from the following three lemmas.

Lemma 3.2. *Assume that* (3.1) *is true. Define G on X such that*

$$\frac{d}{dt}\left(\frac{\omega^m}{m!} \wedge T\right) = -G\,\frac{\omega^m}{m!} \wedge T.$$

Then

$$f_{tt} := \frac{d^2 f}{dt^2} = \int_X \left(G_t - (G - E_\mu(G))^2\right) d\mu,$$

where

$$d\mu := \frac{\frac{\omega^m}{m!} \wedge T}{\int_X \frac{\omega^m}{m!} \wedge T}, \; E_\mu(G) := \int_X G\, d\mu.$$

Lemma 3.3. *Assume that* (3.1) *is true. Then*

$$\int_X G_t \, d\mu = e^f ||\theta||^2_{T,\omega}, \qquad \theta := \frac{d}{dt}\omega = \alpha_1 - \alpha_2, \tag{3.2}$$

and

$$\int_X (G - E_\mu(G))^2 \, d\mu = e^f ||G - E_\mu(G)||^2_{T,\omega}, \tag{3.3}$$

where $||\cdot||_{T,\omega}$ *denotes the* T*-Hodge theory norm (see Definition 5.6). Moreover,*

$$T \wedge G = -\Lambda(T \wedge \theta), \tag{3.4}$$

where Λ *denotes the adjoint of* $\omega \wedge \cdot$ *in* T*-Hodge theory.*

Lemma 3.4. *Assume that* (3.1) *is true. Then* $T \wedge (E_\mu(G) - G)$ *is the* L^2*-minimal solution of*

$$d(\cdot) = (d^c)^*(T \wedge \theta),$$

with respect to the T*-Hodge theory norm and*

$$||G - E_\mu(G)||_{T,\omega} \le ||\theta||_{T,\omega}.$$

4. Brascamp-Lieb lemma

We shall use the Brascamp-Lieb lemma to prove Lemma 3.2.

4.1. **Brascamp-Lieb proof of the Prékopa theorem.** The following Prékopa theorem was found by Prékopa around 1973.

Theorem 4.1 (Prékopa's theorem [31])**.** *Let* ϕ *be a smooth, strictly convex function of* (t, x) *in* $\mathbb{R}^{n+1}$*. Then*

$$t \mapsto -\log \int_A e^{-\phi(t,x)} \, d\lambda(x), \tag{4.1}$$

is strictly convex on $\mathbb{R}$*, where* A *is a fixed convex body in* $\mathbb{R}^n$ *and* $d\lambda(x)$ *denotes the Lebesgue measure.*

The Brascamp-Lieb proof in [11] contains three steps.

Step 1: The second order derivative of function (4.1) can be written as

$$\int_A \phi_{tt} - (\phi_t - E_\nu(\phi_t))^2 \, d\nu, \tag{4.2}$$

where

$$d\nu := \frac{e^{-\phi(t,x)} \, d\lambda(x)}{\int_A e^{-\phi(t,x)} \, d\lambda(x)}, \; E_\nu(\phi_t) := \int_A \phi_t \, d\nu. \tag{4.3}$$

Step 2: Prove the following Brascamp-Lieb inequality:

$$\int_{\mathbb{R}^n} (\phi_t - E_\nu(\phi_t))^2 \, d\nu \le \int_{\mathbb{R}^n} \sum_{j,k=1}^n \phi_{tj} \phi^{jk} \phi_{tk} \, d\nu,$$

where (ϕ^{jk}) denotes the inverse matrix of (ϕ_{jk}).

Step 3: Use strict convexity of ϕ to prove $\phi_{tt} > \sum_{j,k=1}^n \phi_{tj}\phi^{jk}\phi_{tk}$.

Remark: The first step follows from the following lemma (take $dV = e^{-\phi}\, d\lambda$). Since

$$\phi_t - E_\nu(\phi_t)$$

is the (weighted) L^2-minimal solution of $d(\cdot) = d(\phi_t)$, an Hörmander type L^2-estimate gives step 2, see also [11] for a direct proof. For step 3, let $D_{t,x}$ be the determinant of the full hessian matrix of ϕ, let D_x be the determinant of the hessian matrix of ϕ as a function of x, then

$$\frac{D_{t,x}}{D_x} = \phi_{tt} - \sum_{j,k=1}^{n} \phi_{tj}\phi^{jk}\phi_{tk}.$$

Strict convexity of ϕ implies $D_{t,x} > 0$ and $D_x > 0$. Thus Step 3 follows.

Lemma 4.2 (Brascamp-Lieb lemma)**.** *Let A be a relatively compact open set in a smooth manifold X. Let $\{dV(t)\}_{t\in\mathbb{R}}$ be a smooth family of smooth volume forms on X. Let us define G such that*

$$\frac{d}{dt}dV(t) = -G(t,x)\, dV(t), \qquad (t,x) \in \mathbb{R}\times X.$$

Then

$$\frac{d^2}{dt^2}\left(-\log \int_A dV(t)\right) = \int_A \left(G_t - (G - E_\mu(G))^2\right)\, d\mu,$$

where

$$d\mu := \frac{dV}{\int_A dV}, \qquad E_\mu(G) := \int_A G\, d\mu.$$

Proof. Since A is relatively compact, we have

$$\frac{d}{dt}\left(-\log \int_A dV(t)\right) = \int_A G\, d\mu.$$

Apply the differential again, we get

$$\frac{d^2}{dt^2}\left(-\log \int_A dV(t)\right) = \int_A G_t\, d\mu + G\frac{d}{dt}d\mu.$$

A direct computation gives

$$\frac{d}{dt}d\mu = -G\, d\mu + E_\mu(G)\, d\mu,$$

which implies $\int_A G\frac{d}{dt}d\mu = -\int_A (G - E_\mu(G))^2\, d\mu$. Thus the lemma follows. □

Remark: In [6], Berndtsson proved that the Brascamp-Lieb lemma is essentially a subbundle curvature formula associated to a certain direct image bundle. Our main theorem can also be proved along this line, see [35, 34]. Other interesting formulas for the second order derivative of $-\log \int dV$ can be found in [1].

4.2. **Proof of Lemma 3.2.** Notice that the Brascamp-Lieb lemma gives Lemma 3.2 if X is compact. In case X is non-compact we can not directly apply the Brascamp-Lieb lemma. In our case the main point is that

$$e^{-f} = \int_X \frac{\omega^m}{m!} \wedge T,$$

is a polynomial of degree m. The reason is that we can write

$$\frac{\omega^m}{m!} \wedge T = \sum_{j=1}^{m} t^j \Omega_j.$$

Then (3.1) implies that each $\int_X \Omega_j$ is finite and

$$e^{-f} = \sum_{j=1}^{m} \left(\int_X \Omega_j \right) t^j.$$

Thus in our case, $\int_X$ commutes with $\frac{d}{dt}$ and the Brascamp-Lieb lemma applies.

5. Timorin's T-Hodge theory

We shall use Timorin's T-Hodge theory to prove Lemma 3.3. The motivation comes from the Brunn-Minkowski case, i.e. $T = 1$ and $X = \mathbb{R}^n \times \mathbb{T}^n$ (recall $\mathbb{T} := \mathbb{R}/\mathbb{Z}$).

5.1. **Brunn-Minkowski inequality.** By Lemma 2.5, we know that the Brunn-Minkowski inequality is equivalent to the convexity of

$$f : t \mapsto -\log |A_t|, \ A_t := tA_1 + (1-t)A_2,$$

on $(0, 1)$. Let ϕ_1 and ϕ_2 be smooth strictly convex functions that tend to infinity at the boundary of A_1 and A_2 respectively. Put

$$\psi_1 := \phi_1^*, \ \psi_2 := \phi_2^*.$$

Proposition 2.2 gives

$$\nabla\psi_1(\mathbb{R}^n) = A_1^\circ, \ \nabla\psi_2(\mathbb{R}^n) = A_2^\circ.$$

Thus by Proposition 2.3 we have

$$|A_t| = \int_{\mathbb{R}^n} \det(\phi_{jk})\, dx, \ \phi := t\psi_1 + (1-t)\psi_2.$$

Apply the Brascamp-Lieb lemma to

$$dV = \det(\phi_{jk})\, dx,$$

we get

$$f_{tt} = \int_{\mathbb{R}^n} G_t - (G - E_\mu(G))^2 \, d\mu, \tag{5.1}$$

where

$$d\mu := \frac{\det(\phi_{jk})\, d\lambda(x)}{\int_{\mathbb{R}^n} \det(\phi_{jk})\, d\lambda(x)}, \ E_\mu(G) := \int_{\mathbb{R}^n} G\, d\mu.$$

Lemma 5.1. $G = -\sum_{j,k=1}^{n} \phi_{tjk}\phi^{jk}$.

Proof. We use the fact that if $M(t)$ is a smooth family of positive definite matrices then

$$(\log\det M)_t = \mathrm{Trace}(M^{-1}M_t).$$

Consider $M = (\phi_{jk})$ then $G = -\mathrm{Trace}(M^{-1}M_t)$ and the lemma follows. □

Lemma 5.2. $G_t = \sum_{j,k,l,m=1}^{n} \phi_{tjk}\phi_{tlm}\phi^{jl}\phi^{km}$.

Proof. If $M(t)$ is a smooth family of positive definite matrices then

$$(M^{-1})_t = -M^{-1}M_tM^{-1}.$$

Apply the above fact, we get

$$(\phi^{jk})_t = -\sum_{l,m=1}^{n} \phi_{tlm}\phi^{jl}\phi^{km}.$$

Moreover, Lemma 5.1 implies $G_t = -\sum_{j,k=1}^{n}\phi_{tjk}(\phi^{jk})_t$, thus the lemma follows. □

By Lemma 2.8, we have

$$f = -\log\int_{\mathbb{R}^n\times\mathbb{T}^n}\frac{(dd^c\phi)^n}{n!}.$$

Consider $\omega = dd^c\phi$. The above two lemmas give

$$G = -\Lambda\theta,\ G_t = |\theta|^2_{\omega},$$

thus Lemma 3.3 is true in case $T = 1$ and $X = \mathbb{R}^n\times\mathbb{T}^n$.

5.2. **T-Hodge theory.** In this subsection, we will introduce the T-Hodge theory behind the proof of Lemma 3.3. The T-Hodge theory is an integration of Timorin's work in [33], see the author's notes [35] for a systematic study of the T-Hodge theory.

Denote by $V^{p,q}$ the space of smooth (p,q)-forms on an n-dimensional complex manifold X. Put

$$V := \oplus_{0\le p,q\le n}V^{p,q},\ V^k := \oplus_{p+q=k}V^{p,q}.$$

Definition 5.1. *Let*

$$T = \alpha_{m+1}\wedge\cdots\wedge\alpha_n,$$

be a finite wedge product of smooth positive $(1,1)$-forms on X. We call the Hodge theory on $V_T := \{T\wedge u : u\in V\}$ the T-Hodge theory.

For bidegree reason, we have

$$V_T = \oplus_{0\le p,q\le m}V_T^{p,q},$$

where $V_T^{p,q}$ denotes the space of forms that can be written as $T\wedge u$, where u is a smooth (p,q)-form on X. Fix a smooth positive $(1,1)$-form ω on X. The L operator

$$L : T\wedge u \mapsto \omega\wedge T\wedge u,$$

is well defined and maps $V_T^{p,q}$ to $V_T^{p+1,q+1}$.

Theorem 5.3 (Timorin's mixed hard-Lefschetz theorem). *Put $V_T^k=\oplus_{p+q=k}V_T^{p,q}$ then*

$$L^{m-k}: T\wedge u\mapsto T\wedge u\wedge \omega^{m-k},\ 0\leq k\leq m,$$

defines an isomorphism from V_T^k to V_T^{2m-k}.

Proof. By Theorem 4.2 in [35], we know that

$$A: u\mapsto T\wedge u\wedge\omega^{m-k},$$

defines an isomorphism from V^k to V^{2n-k}. Hence $V^{2n-k}=V_T^{2m-k}$ and the following map

$$f_T: u\mapsto T\wedge u,\ u\in V^k,$$

is injective. Thus f_T defines an isomorphism from V^k to V_T^k. Hence $L^{m-k}=A\circ f_T^{-1}$ is an isomorphism from V_T^k to V_T^{2m-k}. □

Definition 5.2. *We call $T\wedge u\in V_T^k$ a primitive k-form if $k\leq m$ and $L^{m-k+1}(T\wedge u)=0$.*

Theorem 5.3 implies:

Theorem 5.4. *Every $T\wedge u\in V_T^k$ has an Lefschetz decomposition as follows:*

$$T\wedge u=\sum_{r=0}^{j}L^r(T\wedge u_r),\qquad \text{for some } 0\leq j\leq m, \tag{5.2}$$

where each $T\wedge u_r$ is zero or primitive in V_T^{k-2r}. If $T\wedge u=0$ then $T\wedge u_r=0$ for every r.

Proof. By the isomorphism in Theorem 5.3, one may assume that $0\leq k\leq m$. Notice that all forms in V_T^0 and V_T^1 are primitive. Assume that $2\leq k\leq m$, Theorem 5.3 gives $\hat{u}\in V^{k-2}$ such that

$$L^{m-k+2}(T\wedge\hat{u})=L^{m-k+1}(T\wedge u).$$

Put $u_0=u-L\hat{u}$, then $T\wedge u_0$ is primitive and

$$T\wedge u=T\wedge u_0+L(T\wedge\hat{u}).$$

Consider $\hat{u}$ instead u, the Lefschetz decomposition of $T\wedge u$ follows by repeating the above argument. If $T\wedge u=\sum_{r=0}^{j}L^r(T\wedge u_r)=0$ then primitivity of $T\wedge u_r$ for $0\leq r<j$ implies

$$0=L^{m-k+j}(\sum_{r=0}^{j}L^r(T\wedge u_r))=L^{m-k+2j}(T\wedge u_j),$$

which gives $T\wedge u_j=0$ by Theorem 5.3. By induction on j, we get $T\wedge u_r=0$ for every r. □

Definition 5.3. *If $T\wedge u\in V_T^k$ is primitive then we define*

$$*_s(L_r(T\wedge u)):=(-1)^{[k]}L_{m-r-k}(T\wedge u),$$

where

$$L_p:=\frac{L^p}{p!},\qquad [k]:=1+\cdots+k=\frac{k(k+1)}{2}.$$

$*_s$ extends to a $\mathbb{C}$-linear map $*_s: V_T\to V_T$, we call it the Lefschetz star operator on V_T.

The Lefschetz star operator above is a generalization of the symplectic star operator, see [35] for the background.

Definition 5.4. *Put $\Lambda = *_s^{-1} L *_s$, $B := [L, \Lambda]$. We call (L, Λ, B) the sl_2-triple on V_T.*

Definition 5.5. *We call $* := *_s \circ J$ the Hodge star operator on V_T, where J is the Weil-operator defined by $Ju = i^{p-q} u$ if $u \in V_T^{p,q}$.*

Timorin's mixed Hodge-Riemann bilinear relation [33] gives:

Theorem 5.5. *For every non-zero $u \in V^k$, $0 \le k \le m$,*

$$\int_X u \wedge \overline{*(T \wedge u)} > 0,$$

where $*$ denotes the Hodge star operator on V_T.

Proof. Let $T \wedge u = \sum_{r=0}^{j} L_r (T \wedge u_r)$ be the Lefschetz decomposition of $T \wedge u$. By our assumption, the degree of u is no bigger than m, thus Theorem 4.2 in [35] implies

$$u = \sum_{r=0}^{j} L_r u_r.$$

Now primitivity of $T \wedge u_r$ gives

$$u \wedge \overline{*(T \wedge u)} = \sum_{r=0}^{j} (-1)^{[k-2r]} L_r L_{m+r-k} (T \wedge u_r) \wedge \overline{J(u_r)}.$$

By Theorem 4.1 in [35], if u_r is not zero then

$$(-1)^{[k-2r]} L_r L_{m+r-k} (T \wedge u_r) \wedge \overline{J(u_r)} > 0,$$

as a positive (n, n)-form. Thus the theorem follows. □

Let us define

$$||T \wedge u||^2 := ||u||_{T,\omega}^2 := \int_X u \wedge \overline{*(T \wedge u)},\ u \in V^k,\ 0 \le k \le m.$$

Definition 5.6. *We call $||T \wedge u|| = ||u||_{T,\omega}$ the T-Hodge theory norm on V_T^k.*

5.3. **Proof of Lemma 3.3.** (3.3) follows directly from the definition of the T-Hodge theory norm. For (3.2), notice that

$$\frac{d}{dt}\left(\frac{\omega^m}{m!} \wedge T\right) = \theta \wedge \frac{\omega^{m-1}}{(m-1)!} \wedge T,$$

gives

$$(\theta + G\frac{\omega}{m}) \wedge \frac{\omega^{m-1}}{(m-1)!} \wedge T = 0. \tag{5.3}$$

Definition 5.7. $\theta_0 := \theta + G\frac{\omega}{m}$, $\theta_1 := -\frac{G}{m}$, $\theta' := -\theta_0 \wedge \frac{\omega^{m-2}}{(m-2)!} + \theta_1 \wedge \frac{\omega^{m-1}}{(m-1)!}$.

We have $\theta = \theta_0 + \theta_1\omega$. (5.3) implies that $T \wedge \theta_0$ is primitive. Thus we have

$$T \wedge \theta' = *(T \wedge \theta) = \overline{*(T \wedge \theta)}. \tag{5.4}$$

Apply the derivative of (5.3) with respect to t, we get

$$(G_t \frac{\omega}{m} + G \frac{\theta}{m}) \wedge \frac{\omega^{m-1}}{(m-1)!} \wedge T + \theta_0 \wedge \theta \wedge \frac{\omega^{m-2}}{(m-2)!} \wedge T = 0,$$

thus

$$\begin{aligned} G_t \frac{\omega^m}{m!} \wedge T &= \theta_1 \theta \wedge \frac{\omega^{m-1}}{(m-1)!} \wedge T - \theta_0 \wedge \theta \wedge \frac{\omega^{m-2}}{(m-2)!} \wedge T \\ &= \theta \wedge \theta' \wedge T = \theta \wedge \overline{*(T \wedge \theta)}, \end{aligned}$$

which gives (3.2). Now it suffices to prove (3.4). Notice that Definition 5.4 gives

$$\Lambda(T \wedge \theta) = *_s^{-1}(\omega \wedge T \wedge \theta') = T \wedge m\theta_1 = -T \wedge G.$$

Thus (3.4) is true.

6. HÖRMANDER L^2-ESTIMATE IN T-HODGE THEORY

Notation: In this paper, d^* and $(d^c)^*$ denote the adjoint of d and d^c with respect to the T-Hodge theory norm.

Theorem 6.1. *Let $(X, \hat{\omega})$ be an n-dimensional complete Kähler manifold. Let*

$$T := \alpha_{m+1} \wedge \cdots \wedge \alpha_n, \ 2 \leq m \leq n,$$

be a finite wedge product of Kähler forms on X such that (3.1) *is true. Let θ be a smooth d-closed 2-form on X. Assume that the T-Hodge theory norm $||T \wedge \theta||$ is finite. Then there exists a smooth solution of*

$$d(T \wedge u) = (d^c)^*(T \wedge \theta)$$

such that $||T \wedge u|| \leq ||T \wedge \theta||$.

Proof. The proof contains two steps.

Step 1: "A prior estimate"

$$|(T \wedge \alpha, (d^c)^*(T \wedge \theta))|^2 \leq ||T \wedge \theta||^2 Q(\alpha, \alpha), \tag{6.1}$$

for every smooth 1-form α with compact support in X, where

$$Q(\alpha, \alpha) := ||d(T \wedge \alpha)||^2 + ||d^*(T \wedge \alpha)||^2.$$

Proof of Step 1: Since

$$(T \wedge \alpha, (d^c)^*(T \wedge \theta)) = (d^c(T \wedge \alpha), T \wedge \theta),$$

it suffices to show the following T-geometry version of the Bochner-Kodaira-Nakano identity

$$||d(T \wedge \alpha)||^2 + ||d^*(T \wedge \alpha)||^2 = ||d^c(T \wedge \alpha)||^2 + ||(d^c)^*(T \wedge \alpha)||^2,$$

which is a special case of Theorem 4.8 in [35].

Step 2: By *Step 1*, we know that

$$F : \alpha \mapsto (T \wedge \alpha, (d^c)^*(T \wedge \theta)),$$

is Q-bounded by $||T \wedge \theta||$. Thus F extends to a bounded linear functional on the Q-completion, say H, of the space of smooth 1-forms with compact support in X. The Riesz representation theorem gives $\beta \in H$ with

$$Q(\beta, \beta) \leq ||T \wedge \theta||^2, \tag{6.2}$$

such that

$$Q(\alpha, \beta) = F(\alpha) = (T \wedge \alpha, (d^c)^*(T \wedge \theta)), \tag{6.3}$$

for every smooth 1-form α with compact support in X, where

$$Q(\alpha, \beta) = (d(T \wedge \alpha), d(T \wedge \beta)) + (d^*(T \wedge \alpha), d^*(T \wedge \beta)). \tag{6.4}$$

Since H is a subspace of the space of currents, we have

$$Q(\alpha, \beta) = (T \wedge \alpha, (dd^* + d^*d)(T \wedge \beta)). \tag{6.5}$$

Thus (6.3) and (6.5) together give

$$(dd^* + d^*d)(T \wedge \beta) = (d^c)^*(T \wedge \theta),$$

in the sense of current. Let us define u such that $T \wedge u = d^*(T \wedge \beta)$. Since $dd^* + d^*d$ is elliptic, we know that β is smooth. Thus u is smooth. Notice that (6.2) gives

$$||T \wedge u|| \leq ||T \wedge \theta||,$$

Thus it suffices to prove the following identity. □

Lemma 6.2. $d^*d(T \wedge \beta) \equiv 0$.

Proof. The T-Kähler identity $(d^c)^* = [d, \Lambda]$ (see section 4 in [35]) implies that

$$d(d^c)^* + (d^c)^*d = 0.$$

Thus

$$d(d^c)^*(T \wedge \theta) = -(d^c)^*d(T \wedge \theta) = 0.$$

Now we have

$$dd^*d(T \wedge \beta) \equiv 0.$$

Since $\hat{\omega}$ is complete, there exists a smooth exhaustion function, say ρ, on X such that

$$|d\rho|_{\hat{\omega}} \leq 1. \tag{6.6}$$

Let $0 \leq \chi \leq 1$ be a smooth function on $\mathbb{R}$ such that $\chi \equiv 1$ on $(-\infty, 1)$ and $\chi \equiv 0$ on $(2, \infty)$. Then for each $\varepsilon > 0$, $\chi(\varepsilon\rho)$ is a smooth function with compact support. Since

$$(\chi^2(\varepsilon b)dd^*d(T \wedge \beta), d(T \wedge \beta)) = 0, \tag{6.7}$$

and

$$\chi^2(\varepsilon b)dd^*d(T \wedge \beta) = d(\chi^2(\varepsilon b)d^*d(T \wedge \beta)) - 2d(\chi(\varepsilon b)) \wedge \chi(\varepsilon b)d^*d(T \wedge \beta),$$

we have

$$||\chi(\varepsilon b)d^*d(T \wedge \beta)||^2 = 2(d(\chi(\varepsilon b)) \wedge \chi(\varepsilon b)d^*d(T \wedge \beta), d(T \wedge \beta)). \tag{6.8}$$

Thus Lemma 6.2 follows from the following estimate

$$\lim_{\varepsilon\to 0} ||d(\chi(\varepsilon b))\wedge \chi(\varepsilon b) d^* d(T\wedge\beta)|| = 0. \tag{6.9}$$

The above estimate is easily seen to be true in case $T = 1$, see [14]. The general case will be proved in the appendix. □

6.1. **Proof of Lemma 3.4.** By Lemma 3.3, we have

$$d\left(T\wedge (E_\mu(G)-G)\right) = d\Lambda(T\wedge\theta) = [d,\Lambda](T\wedge\theta),$$

By the Kähler identity in T-Hodge theory (section 4 in [35]), we have $[d,\Lambda] = (d^c)^*$, thus $T\wedge(E_\mu(G)-G)$ is a solution of

$$d(\cdot) = (d^c)^*(T\wedge\theta).$$

Notice that $T\wedge(E_\mu(G)-G)$ is perpendicular to $\ker d$, thus it is also the L^2-minimal solution. By (3.1), for every fixed $0<t<1$, $\omega = t\alpha_1 + (1-t)\alpha_2$ is complete. Apply Theorem 6.1 to the case $\hat\omega = \omega$, Lemma 3.4 follows.

7. Proof of the Alexandrov-Fenchel inequality

Lemma 7.1. *Put*

$$\psi(x) = \sum_{j=1}^n \log\frac{1}{1+(x^j)^2} + C\log(1+e^{x^j}),\ C := 4(1+e^{\sqrt3})^2 e^{\sqrt3}.$$

Then ψ is strictly convex on $\mathbb{R}^n$ and $\nabla\psi(\mathbb{R}^n)\subset(-1, C+1)^n$. Moreover, if we look at ψ as a function on $\mathbb{R}^n\times\mathbb{T}^n$ then $dd^c\psi$ is complete Kähler on $\mathbb{R}^n\times\mathbb{T}^n$.

Proof. A direct computation gives

$$\left(\log\frac{1}{1+(x^j)^2}\right)_{x^j} = \frac{-2x^j}{1+(x^j)^2}, \tag{7.1}$$

and

$$\left(\log\frac{1}{1+(x^j)^2}\right)_{x^jx^j} = \frac{2(x^j)^2-2}{(1+(x^j)^2)^2} \geq \frac{1}{1+(x^j)^2},\ \text{ if } (x^j)^2\geq 3. \tag{7.2}$$

Since $\log(1+e^x)$ is convex, the above inequality gives

$$\psi_{x^jx^j} \geq \frac{1}{1+(x^j)^2}\ \text{ if } (x^j)^2\geq 3.$$

We also have

$$\left(\log(1+e^{x^j})\right)_{x^jx^j} = \frac{e^{x^j}}{(1+e^{x^j})^2} \geq \frac{e^{-\sqrt3}}{(1+e^{\sqrt3})^2},\ \text{ if } (x^j)^2\leq 3. \tag{7.3}$$

Thus

$$C\left(\log(1+e^{x^j})\right)_{x^jx^j} \geq 4 \geq \frac{4}{1+(x^j)^2},\ \text{ if } (x^j)^2\leq 3, \tag{7.4}$$

which gives

$$\psi_{x^jx^j} \geq \frac{4}{1+(x^j)^2} + \frac{2(x^j)^2-2}{(1+(x^j)^2)^2} \geq \frac{2}{1+(x^j)^2} \ \text{ if } (x^j)^2 \leq 3.$$

Notice that $\psi_{x^jx^k} = 0$ if $j \neq k$. Thus ψ is strictly convex and

$$dd^c\psi \geq \sum_{j=1}^n \frac{1}{1+(x^j)^2} dx^j \wedge dy^j,$$

on $\mathbb{R}^n \times \mathbb{T}^n$. Denote by g the associated Riemannian metric of $dd^c\psi$, then we have

$$g \geq g_0 := \sum_{j=1}^n \frac{1}{1+(x^j)^2}(dx^j \otimes dx^j + dy^j \otimes dy^j).$$

Thus

$$|dx^j|_g \leq |dx^j|_{g_0} = \sqrt{1+(x^j)^2}.$$

Since $d\log(1+|x|^2) = \sum_{j=1}^n \frac{2x^j dx^j}{1+|x|^2}$, we have

$$|d\log(1+|x|^2)|_g \leq \sum_{j=1}^n \frac{2|x^j|}{1+|x|^2}|dx^j|_g \leq \sum_{j=1}^n \frac{2|x^j|}{1+|x|^2}\sqrt{1+(x^j)^2} \leq n.$$

Notice that $\log(1+|x|^2)$ is an exhaustion function on $\mathbb{R}^n \times \mathbb{T}^n$, the above inequality implies that $dd^c\psi$ is complete Kähler. $\nabla\psi(\mathbb{R}^n) \subset (-1, C+1)^n$ follows from

$$\psi_{x^j} = \frac{-2x^j}{1+(x^j)^2} + C\frac{e^{x^j}}{1+e^{x^j}}, \ 2|x_j| \leq 1+(x^j)^2, \ 0 < \frac{e^{x^j}}{1+e^{x^j}} < 1.$$

The proof is complete. □

We shall use our main theorem and the above lemma to prove Theorem 2.10, which implies the Alexandrov-Fenchel inequality.

7.1. **Proof of Theorem 2.10.** Put

$$\tilde{\phi} = \psi + \phi_1 + \phi_2 + \phi_{m+1} + \cdots + \phi_n.$$

The above lemma implies that $\hat{\omega} := dd^c\tilde{\phi}$ is complete on $\mathbb{R}^n \times \mathbb{T}^n$ and $dd^c\phi_j \leq \hat{\omega}$ for each j. Moreover, by the above lemma, $\nabla\psi(\mathbb{R}^n)$ is bounded, thus $\nabla\tilde{\phi}(\mathbb{R}^n)$ is bounded and $(X, \hat{\omega})$ has finite volume. We know that Theorem 2.10 follows from Theorem 3.1.

8. Appendix

8.1. **Compare the T-Hodge theory norm with the usual norm.** For every smooth k-form u, $0 \leq k \leq m$, on X, let us define $|u|^2_{T,\omega}$ such that

$$u \wedge \overline{*(T \wedge u)} = |u|^2_{T,\omega}\frac{\omega^m}{m!} \wedge T.$$

where $*$ denotes the Hodge star operator on V_T, see Definition 5.5.

Definition 8.1. *We call $|u|_{T,\omega}$ the pointwise T-norm of u.*

Lemma 8.1. *Let $|T\wedge u|_\omega$ be the usual pointwise norm of $T\wedge u$ with respect to ω. If $T=\omega^{n-m}$ then*

$$\frac{n!(n-m)!)}{m!}|u|^2_{T,\omega}\leq |T\wedge u|^2_\omega\leq \frac{(n!)^2}{(m!)^2}|u|^2_{T,\omega}.$$

Proof. By Definition 5.2, if $T=\omega^{n-m}$ then a form $T\wedge v\in V^k_T$ is primitive in T-Hodge theory if and only if v is primitive with respect to ω in the usual sense. Let

$$T\wedge u:=\sum_{r=0}^{j}L_r(T\wedge u_r)=\sum_{r=0}^{j}L_{n-m+r}u'_r,\ u'_r:=\frac{(n-m+r)!}{r!}u_r,$$

be the Lefschetz decomposition of $T\wedge u$. Then Definition 5.5 gives

$$*(T\wedge u)=\sum_{r=0}^{j}(-1)^{[k-2r]}L_{m-k+r}(T\wedge Ju_r).$$

Moreover,

$$\star(T\wedge u)=\sum_{r=0}^{j}(-1)^{[k-2r]}L_{m-k+r}(Ju'_r),$$

where $\star$ denotes the usual Hodge star operator. Recall that

$$T\wedge u\wedge\overline{\star(T\wedge u)}=|T\wedge u|^2_\omega\frac{\omega^n}{n!}.$$

Thus the lemma follows. □

For general $T=\alpha_{m+1}\wedge\cdots\wedge\alpha_n$, we have:

Lemma 8.2. *Assume that* (3.1) *is true. Then there exists a constant C_1 that only depends on C,n,m such that*

$$C_1^{-1}|u|_{T,\hat\omega}\leq|T\wedge u|_{\hat\omega}\leq C_1|u|_{T,\hat\omega}.$$

Proof. By Lemma 8.1, it suffices to compare $|u|^2_{T,\hat\omega}$ with $|u|^2_{T_0,\hat\omega}$, where $T_0:=\hat\omega^{n-m}$. Fix an arbitrary point, say z_0, in X, let us choose local coordinates, say $\{z^j\}$, near z_0 such that

$$\hat\omega(z_0)=i\sum_{j=1}^{n}dz^j\wedge d\bar z^j.$$

With respect to the local coordinates $\{z^j\}$, we can identify the space of positive $(1,1)$-forms at z_0 with the space of positive definite n by n Hermitian matrices. We know that every positive definite n by n Hermitian matrix can be written as

$$A=OBO^*,\ OO^*=I_n,$$

where O^* denotes the conjugate transpose of O, I_n is the identity matrix and B is a diagonal matrix with positive eigenvalues. Moreover,

$$\frac{\omega(z_0)}{C}\leq\hat\omega(z_0)\leq C\omega(z_0)$$

if and only if each eigenvalue of the associated matrix of $\omega(z_0)$ lies in $[1/C, C]$. Consider

$$V := U(n) \times [1/C, C]^n,$$

where $U(n) := \{O : OO^* = I_n\}$ is the unitary group. Every element, say $v = (O, \lambda_1, \cdots, \lambda_n)$, in V represents a positive $(1,1)$-form, say ω^v, at z_0 whose associated matrix is

$$O\text{Diag}\{\lambda_1, \cdots, \lambda_n\}O^*.$$

Consider the following map, say F, from

$$V^{n-m} := \underbrace{V \times \cdots \times V}_{n-m}$$

to the space of Hermitian norms on $\wedge^k(\mathbb{C} \otimes T^*_{z_0}X)$, $0 \leq k \leq m$, defined by

$$(v^{m+1}, \cdots, v^n) \mapsto |\cdot|_{T,\hat{\omega}(z_0)},\ T := \omega^{v^{m+1}} \wedge \cdots \wedge \omega^{v^n}.$$

The lemma follows since V^{n-m} is compact and connected. □

8.2. **Proof of estimate** (6.9)**.** Let us write $d^*d(T \wedge \beta)$ as $T \wedge \sigma$, where σ is a one-form. Then

$$||d(\chi(\varepsilon\rho)) \wedge \chi(\varepsilon\rho)d^*d(T \wedge \beta)||^2 = \int_X |d(\chi(\varepsilon\rho)) \wedge \chi(\varepsilon\rho)\sigma|^2_{T,\hat{\omega}} \frac{\hat{\omega}^m}{m!} \wedge T.$$

By Lemma 8.2, we have

$$|d(\chi(\varepsilon\rho)) \wedge \chi(\varepsilon\rho)\sigma|_{T,\hat{\omega}} \leq C_1 |d(\chi(\varepsilon\rho)) \wedge \chi(\varepsilon\rho)d^*d(T \wedge \beta)|_{\hat{\omega}}.$$

Since $|d\rho|_{\hat{\omega}} \leq 1$, we have

$$|d(\chi(\varepsilon\rho)) \wedge \chi(\varepsilon\rho)d^*d(\beta \wedge T)|_{\hat{\omega}} \leq (\varepsilon \sup|\chi'|)\, |\chi(\varepsilon\rho)d^*d(T \wedge \beta)|_{\hat{\omega}}.$$

Use Lemma 8.2 again, we get

$$|d(\chi(\varepsilon\rho)) \wedge \chi(\varepsilon\rho)\sigma|_{T,\hat{\omega}} \leq \left(\varepsilon C_1^2 \sup|\chi'|\right) |\chi(\varepsilon\rho)\sigma|_{T,\hat{\omega}},$$

which gives

$$||d(\chi(\varepsilon\rho)) \wedge \chi(\varepsilon\rho)d^*d(T \wedge \beta)|| \leq \left(\varepsilon C_1^2 \sup|\chi'|\right) ||\chi(\varepsilon\rho)d^*d(T \wedge \beta)||.$$

By (6.8), then we have

$$||\chi(\varepsilon\rho)d^*d(T \wedge \beta)||^2 \leq 2\left(\varepsilon C_1^2 \sup|\chi'|\right) ||\chi(\varepsilon\rho)d^*d(T \wedge \beta)|| \cdot ||T \wedge \theta||,$$

hence

$$||\chi(\varepsilon\rho)d^*d(T \wedge \beta)|| \leq \left(2\varepsilon C_1^2 \sup|\chi'|\right) ||T \wedge \theta||,$$

which gives

$$||d(\chi(\varepsilon\rho)) \wedge \chi(\varepsilon\rho)d^*d(T \wedge \beta)|| \leq 2(\varepsilon C_1^2 \sup|\chi'|)^2 ||T \wedge \theta||,$$

thus (6.9) follows.

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