# SOME RESULTS ON HARD LEFSCHETZ CONDITION

ADRIANO TOMASSINI AND XU WANG

Abstract. We discuss the Hard Lefschetz Condition on various cohomology groups and verify them for the Nakamura manifold of completely solvable type and the Kodaira-Thurston manifold. A general Demailly-Griffiths-Kähler identity is also given.

Contents



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## 1. Introduction

A special class of symplectic manifolds is represented by those ones satisfying the *Hard Lefschetz Condition* (shortly HLC), i.e., those compact $2n$-dimensional symplectic manifolds $(X, \omega)$ for which the maps

$$[\omega]^k : H^{n-k}_{dR}(X, \mathbb{R}) \to H^{n+k}_{dR}(X, \mathbb{R}), \quad 0 \leq k \leq n$$

are isomorphisms. In particular, a classical result states that if $(X,\, \omega,\, J)$ is a compact Kähler manifold, then $(X,\, \omega)$ satisfies the HLC (see e.g., [8]) and the de Rham complex $(\Omega^*(X), d)$ is a formal DGA in the sense of Sullivan (see [6]); furthermore, HLC symplectic manifolds have some of the cohomological properties of a Kähler manifold (e.g., the odd Betti numbers $b_{2k+1}(X)$ are even, $b_k(X) \leq b_{k+2}(X)$ $, 0 \leq k < n-1$, $b_{2k}(X) > 0$).
On any almost symplectic manifold $(X, \omega)$, i.e., $X$ is a $2n$-dimensional manifold endowed with a non-degenerate 2-form $\omega$, it is defined a symplectic codifferential operator $d^\Lambda : \Omega^k(X) \to \Omega^{k-1}(X)$, by using the symplectic star operator. If $\omega$ is closed, for such an operator the following basic symplectic identity holds

$$[d, \Lambda] = d^\Lambda,$$

where $\Lambda$ is the symplectic adjoint of the Lefschetz operator $L$. Furthermore, in the symplectic case, $(\Omega^*(X), d, d^\Lambda)$ is a differentiable Gerstenhaber-Batalin-Vilkovisky (dGBV) algebra, that is integrable (i.e., the $dd^\Lambda$-lemma holds), if and only if $(X, \omega)$ satisfies the HLC (see [13], [11], [2], [18]).

In the present paper, we will generalize such an identity to the context of almost symplectic and almost complex manifolds. Then we will give a notion of $\partial\overline{\partial}^\Lambda$-*Lemma* on special complex manifolds. First of all, starting with a *Lefschetz space* $(A, L)$, where $A = \oplus_{k=0}^{2n} A^k$ is a direct sum of complex vector spaces and $L \in \mathrm{End}(A)$ satisfies $L(A^l) \subset A^{l+2}$ for every $0 \leq l \leq 2(n-1)$, $L(A^{2n-1}) = L(A^{2n}) = 0$ and $L^k : A^{n-k} \to A^{n+k}$ is an isomorphism for every $0 \leq k \leq n$, we prove the following general Demailly-Griffiths-Kähler identity (see page 307 in [7] and Theorem 4.6 in [16])

**Theorem A** (see Theorem 3.6)**.** *Let $(A, L)$ be a Lefschetz space. Let $d$ be a $\mathbb{C}$-linear endomorphism of $A$ such that $d(A^l) \subset A^{l+1}$. Let us define*

$$d^\Lambda := (-1)^{k+1} *_s d *_s$$

*on $A^k$. Assume that $[L, [d, L]] = 0$. Then*

$$[d^\Lambda, L] = d + [\Lambda, [d, L]], \qquad [d, \Lambda] = d^\Lambda + [[\Lambda, d^\Lambda], L].$$

As a direct consequence (see Theorem 5.2), if $(X, \omega)$ is a symplectic manifold, $A = \oplus_{k=0}^{2n} \Omega^k(X)$ and $L := \omega\wedge$, then we recover the basic symplectic identity above. Furthermore, for an almost symplectic manifold, by applying the Theorem above, we derive the following identity (see Theorem 5.1)

$$[d^\Lambda, L] = d + [\Lambda, [d, L]], \quad [d, \Lambda] = d^\Lambda + [[\Lambda, d^\Lambda], L].$$

Then we show that the fundamental form of a compact almost Kähler manifold $(X, \omega, J, g)$ restricted to $\ker \square_d \cap \ker \square_{d^c}$ satisfies the Hard Lefschetz Condition (see Theorem 6.2). We will provide explict computations on two non-Kähler manifolds: the *Kodaira-Thurston manifold* and the *Nakamura manifold* of completely solvable type.

In Section 9, we will define the notion of $\partial\overline{\partial}^\Lambda$-*Lemma* on special complex manifolds. Namely, we consider a complex manifold $(X, J)$ endowed with a symplectic form $\omega$ such that $J$ is $\omega$-symmetric, or equivalently, $\omega$ is a symplectic form of $(1,1)$-type with respect to the decomposition induced by $J$. In this situation, by using the symplectic Hodge operator $*_s$, the symplectic adjoint $\Lambda$ of the Lefschetz operator $L$ and the $\overline{\partial}$ operator, one can define $\overline{\partial}^\Lambda = (-1)^{k+1} *_s \overline{\partial} *_s$.
By applying Theorem 3.6 we obtain complex symplectic identities (see Corollary 9.4). In particular,

$\overline{\partial}^2 = (\overline{\partial}^\Lambda)^2 = 0$ and $\overline{\partial}\overline{\partial}^\Lambda + \overline{\partial}^\Lambda\overline{\partial} = 0$, so that it is natural to consider the complex $(\Omega^\bullet(X), \overline{\partial}, \overline{\partial}^\Lambda)$; by definition, $(X, J, \omega)$ is said to satisfy the $\overline{\partial}\,\overline{\partial}^\Lambda$-*Lemma* if

$$\ker\overline{\partial} \cap \ker\overline{\partial}^\Lambda \cap (\operatorname{Im}\overline{\partial} + \operatorname{Im}\overline{\partial}^\Lambda) = \operatorname{Im}\overline{\partial}\overline{\partial}^\Lambda.$$

We show that any compact Kähler manifold satisfies the $\overline{\partial}\,\overline{\partial}^\Lambda$-Lemma and we provide a family of non Kähler manifolds satisfying the $\overline{\partial}\,\overline{\partial}^\Lambda$-Lemma. Finally, we apply our construction to holomorphic vector bundles over special complex manifolds (section 8.6). Our results on the $\overline{\partial}\,\overline{\partial}^\Lambda$-Lemma include

**Theorem B.** *Let $(X, J)$ be a compact complex manifold with a $J$-symmetric symplectic structure $\omega$. Then we have*

*(1) The Dolbeault cohomology satisfies the HLC if and only if the $\overline{\partial}\,\overline{\partial}^\Lambda$-Lemma holds (special case of Theorem 3.12);*

*(2) Nakamura manifold of completely solvable type (see Example 2 and the appendix) satisfies the $\overline{\partial}\,\overline{\partial}^\Lambda$-Lemma (see section 8.3) and the $\partial\overline{\partial}$-Lemma (see* [1]*);*

*(3) The holomorphic parallelizable Nakamura manifold in section 8.4 satisfies the $\overline{\partial}\,\overline{\partial}^\Lambda$-Lemma, but it does not satisfy the $\partial\overline{\partial}$-Lemma;*

*(4) The Kodaira-Thurston manifold in section 8.5 does not satisfy the $\overline{\partial}\,\overline{\partial}^\Lambda$-Lemma.*

## 2. Preliminaries

2.1. **Hard-Lefschetz-theorem on symplectic vector spaces.** Let $V$ be an $N$-dimensional real vector space. Let $\omega$ be a bilinear form on $V$. We call $\omega$ a *symplectic form* if $\omega$ is non-degenerate and $\omega \in \wedge^2 V^*$, i.e., $\omega(u, v) = -\omega(v, u)$, $\forall\ u, v \in V$. We have the following well-known proposition

**Proposition 2.1.** *Assume that there is a symplectic form $\omega$ on $V$. Then $N = 2n$ for some integer $n$ and there exists a base, say $\{e_1^*, f_1^*; \cdots; e_n^*, f_n^*\}$, of $V^*$ such that*

$$\omega = \sum_{j=1}^{n} e_j^* \wedge f_j^*.$$

One may use $\omega$ to define a bilinear form, say $\omega^{-1}$, on $V^*$ such that

$$\omega^{-1}(f_j^*, e_k^*) = -\omega^{-1}(e_k^*, f_j^*) = \delta_{jk},\ \omega^{-1}(f_j^*, f_k^*) = \omega^{-1}(e_j^*, e_k^*) = 0.$$

**Remark**: In [15], the bilinear form on $V^*$ is defined to be $-\omega^{-1}$.

Let $T_\omega : V \to V^*$ be the linear isomorphism defined by

$$T_\omega(u)(v) = \omega(v, u),\ \forall\ u, v \in V.$$

Then we have

$$T_\omega^{-1} = T_{\omega^{-1}},$$

thus the definition of $\omega^{-1}$ does not depend on the choice of bases in the above proposition. We shall also use $\omega^{-1}$ to denote the following bilinear form on $\wedge^p V^*$, defined on simple elements as

$$\omega^{-1}(\mu, \nu) := \det(\omega^{-1}(\alpha_i, \beta_j)),\ \mu = \alpha_1 \wedge \cdots \wedge \alpha_p,\ \nu = \beta_1 \wedge \cdots \wedge \beta_p, \tag{2.1}$$

and then extended linearly.
Then we can have

**Definition 2.2.** *The* symplectic star operator $*_s : \wedge^p V^* \to \wedge^{2n-p} V^*$ *is defined by*

$$\mu \wedge *_s \nu = \omega^{-1}(\mu, \nu)\frac{\omega^n}{n!}. \tag{2.2}$$

The following theorem is well known, see [16].

**Theorem 2.3** (Hard Lefschetz theorem). *For each* $0 \leq k \leq n$,

$$u \mapsto \omega^{n-k} \wedge u, \ u \in \wedge^k V^*,$$

*defines an isomorphism between* $\wedge^k V^*$ *and* $\wedge^{2n-k} V^*$.

**Definition 2.4.** *We call* $u \in \wedge^k V^*$ *a* primitive form *if* $k \leq n$ *and* $\omega^{n-k+1} \wedge u = 0$.

The following Lefschetz decomposition theorem follows directly from Theorem 2.3.

**Theorem 2.5** (Lefschetz decomposition formula). *Every* $u \in \wedge^k V^*$ *has a unique decomposition as follows:*

$$u = \sum \omega_r \wedge u^r, \ \omega_r := \frac{\omega^r}{r!}, \tag{2.3}$$

*where each* $u^r$ *is a primitive* $(k-2r)$*-form.*

By the above theorem, it is enough to study the symplectic star operator on $\omega_r \wedge u$, where $u$ is primitive, see [16].

**Theorem 2.6.** *If* $u$ *is a primitive* $k$*-form then* $*_s(\omega_r \wedge u) = (-1)^{k+\cdots+1} \omega_{n-k-r} \wedge u$.

**Definition 2.7.** *We call* $\{L, \Lambda, B\}$ *the* $sl_2$-triple *on* $\oplus_{0\leq k\leq 2n} \wedge^k V^*$, *where*

$$Lu := \omega \wedge u, \ \Lambda := *_s^{-1} L *_s, \ B := [L, \Lambda].$$

We have

$$\omega^{-1}(Lu, v) = \omega^{-1}(u, \Lambda v).$$

Hence $\Lambda$ is the adjoint of $L$. Put

$$L_r := L^r/r!, \ L_0 := 1, \ L_{-1} := 0.$$

We have:

**Proposition 2.8.** *If* $u$ *is a primitive* $k$*-form then*

$$\Lambda(L_r u) = (n-k-r+1) L_{r-1} u, \ B(L_r u) = (k+2r-n) L_r u,$$

*for every* $0 \leq r \leq n-k+1$.

**Definition 2.9.** *We call a linear map* $J : V \to V$ *an* almost complex structure *on* $V$ *if* $J(Ju) = -u$ *for every* $u \in V$.

**Definition 2.10.** *An almost complex structure* $J$ *is said to be* tamed *by* $\omega$ *if*

$$\omega(u, Ju) > 0,$$

*for every non-zero* $u \in V$. $J$ *is said to be* symmetric *with respect to* $\omega$ *if*

$$\omega(u, Jv) = \omega(v, Ju),$$

*for every* $u, v \in V$. *We say* $J$ *is* $\omega$-compatible *if it is both taming and symmetric.*

If $J$ is an almost complex structure on $V$ then

$$J(v)(u) := v(Ju), \ \forall\, u \in V, \ v \in V^*,$$

defines an almost complex structure on $V^*$.

**Definition 2.11.** *We call*

$$J(v_1 \wedge \cdots \wedge v_k) := J(v_1) \wedge \cdots \wedge J(v_k),$$

*the* Weil operator *on* $\oplus_{0\leq k\leq 2n} \wedge^k V^*$.

Since the eigenvalues of $J$ are $\pm i$, its eigenvectors lie in $\mathbb{C}\otimes V^*$. Put

$$E_i := \{u \in \mathbb{C}\otimes V^* : J(u) = iu\}, \quad E_{-i} := \{u \in \mathbb{C}\otimes V^* : J(u) = -iu\},$$

we know that

$$E_i = \{u - iJu : u \in V^*\}, \quad E_{-i} = \{u + iJu : u \in V^*\}.$$

and $\mathbb{C}\otimes V^* = E_i \oplus E_{-i}$. Put

$$\wedge^{p,q}V^* := (\wedge^p E_i) \wedge (\wedge^q E_{-i}).$$

Then we have

$$\mathbb{C}\otimes(\wedge^k V^*) = \wedge^k(\mathbb{C}\otimes V^*) = \oplus_{p+q=k} \wedge^{p,q} V^*,$$

and

$$Ju = i^{p-q}u, \ \forall\ u \in \wedge^{p,q}V^*.$$

We call $\wedge^{p,q}V^*$ the space of $(p,q)$-forms.

**Proposition 2.12.** *An almost complex structure $J$ on $(V,\omega)$ is compatible with $\omega$ iff*

$$(\alpha,\beta) := \omega^{-1}(\alpha, J\bar{\beta}),$$

*defines a Hermitian inner product structure on $\wedge^{p,q}V^*$, $0 \le p,q \le n$.*

**Definition 2.13.** *The* Hodge star operator $* : \wedge^{p,q}V^* \to \wedge^{n-q,n-p}V^*$ *is defined by*

$$u \wedge *\bar{v} = (u,v)\omega_n.$$

The above proposition gives

$$* = *_s \circ J = J \circ *_s.$$

## 3. Hard Lefschetz Condition and the $dd^\Lambda$-Lemma

In this section, we shall introduce the *Hard Lefschetz Condition* on a general (can be infinite dimensional) linear space and the general *$dd^\Lambda$-Lemma.*

### 3.1. **Lefschetz spaces.**

**Definition 3.1.** *Let $A = \oplus_{k=0}^{2n} A^k$ be a direct sum of complex vector spaces. We say that $L \in \mathrm{End}(A)$ satisfies the* Hard Lefschetz Condition *and $(A,L)$ is a* Lefschetz space *if*

$$L(A^l) \subset A^{l+2}, \ \forall\ 0 \le l \le 2(n-1), \ L(A^{2n-1}) = L(A^{2n}) = 0,$$

*and each $L^k : A^{n-k} \to A^{n+k}$, $0 \le k \le n$, is an isomorphism.*

**Definition 3.2.** *Let $(A,L)$ be a Lefschetz space. We call $u \in A^k$ a* primitive form *if $k \le n$ and $L^{n-k+1}u = 0$.*

The Hard Lefschetz Condition implies the following Lefschetz decomposition theorem (see [16] for the proof).

**Theorem 3.3.** *Let $(A,L)$ be a Lefschetz space. Then every $u \in A^k$ has a unique decomposition as follows:*

$$u = \sum L_r u^r, \ L_r := \frac{L^r}{r!}. \tag{3.1}$$

*where each $u^r$ is a primitive form in $A^{k-2r}$.*

**Definition 3.4.** *We call the following $\mathbb{C}$-linear map $*_s : A \to A$ defined by*

$$*_s(L_r u) := (-1)^{k+\cdots+1} L_{n-r-k} u,$$

*where $u \in A^k$ is primitive, the* Lefschetz star operator *on $A$.*

Notice that $*_s^2 = 1$. We know from the last section that the Lefschetz star operator is a generalization of the symplectic star operator.

**Definition 3.5.** *Put $\Lambda = *_s^{-1} L *_s$, $B := [L,\Lambda]$. We call $(L,\Lambda,B)$ the $sl_2$-*triple *on $(A,L)$ (Proposition 2.8 is also true for general Lefschetz space).*

3.2. **General Demailly-Griffiths-Kähler identity.** We shall use the following general Demailly-Griffiths-Kähler identity [7], see also Theorem 3.1 in [16].

**Theorem 3.6.** *Let $(A, L)$ be a Lefschetz space. Let $d$ be a $\mathbb{C}$-linear endomorphism of $A$ such that $d(A^l) \subset A^{l+1}$. Let us define*

$$d^\Lambda := (-1)^{k+1} *_s d *_s$$

*on $A^k$. Assume that $[L, [d, L]] = 0$. Then*

$$[d^\Lambda, L] = d + [\Lambda, [d, L]], \qquad [d, \Lambda] = d^\Lambda + [[\Lambda, d^\Lambda], L].$$

*Proof.* We shall follow the proof of Theorem 3.1 in [16]. By the Lefschetz decompostion theorem, it suffices to prove the theorem for $L_r u$, where $u \in A^k$ is primitive. Put

$$\theta := [d, L].$$

*Step 1*: Since $[L, \theta] = 0$, we have

$$d(L^p u) = \theta L^{p-1} u + L d L^{p-1} u = 2L^{p-1}\theta u + L^2 d L^{p-2} u = \cdots = pL^{p-1}\theta u + L_p du. \tag{3.2}$$

Thus

$$0 = d(L_{n-k+1}u) = L_{n-k}(\theta u + \frac{Ldu}{n-k+1}).$$

Put

$$v := \theta u + \frac{Ldu}{n-k+1}.$$

$L_{n-k}v = 0$ implies that the primitive decomposition of $v$ contains at most three terms. Thus we can write

$$v = v_0 + Lv_1 + L^2 v_2,$$

where $v_0, v_1, v_2$ are primitive. Moreover, since $L_{n-k+1}\theta u = \theta L_{n-k+1} u = 0$, we can write

$$\theta u = e + Lf + L^2 g + L^3 h.$$

where $e, f, g, h$ are primitive. Thus

$$v_0 = e, \ du = (n-k+1)(v_1 - f + L(v_2 - g) - L^2 h).$$

Let us write

$$du = a + Lb + L^2 c,$$

where $a, b, c$ are primitive and

$$c = -(n-k+1)h. \tag{3.3}$$

*Step 2*: Notice that

$$[d^\Lambda, L] = (-1)^{k+1}(*_s d *_s L - L *_s d *_s),$$

on $A^k$. Using $*_s \Lambda = L *_s$, we get

$$[d^\Lambda, L] = (-1)^{k+1} *_s (d\Lambda - \Lambda d) *_s,$$

on $A^k$. Now

$$(d\Lambda - \Lambda d) *_s (L_r u) = (-1)^{k+\cdots+1}(d\Lambda - \Lambda d)(L_{n-r-k}u).$$

Put

$$m := n - r - k.$$

By Proposition 2.8, we have

$$\begin{aligned}
d\Lambda(L_m u) &= (r+1)dL_{m-1}u = (r+1)(L_{m-2}\theta u + L_{m-1}du) \\
&= (r+1)\big[L_{m-2}e + (m-1)L_{m-1}f + (m-1)mL_m g + (m-1)m(m+1)L_{m+1}h \\
&\quad + L_{m-1}a + mL_m b + m(m+1)L_{m+1}c\big],
\end{aligned}$$

and

$$\begin{aligned}\Lambda d(L_m u) &= \Lambda(L_{m-1}\theta u + L_m du) = (r-1)L_{m-2}e + rmL_{m-1}f \\ &\quad + (r+1)m(m+1)L_m g + (r+2)m(m+1)(m+2)L_{m+1}h \\ &\quad + rL_{m-1}a + (r+1)(m+1)L_m b + (r+2)(m+1)(m+2)L_{m+1}c.\end{aligned}$$

By (3.3), we have

$$\begin{aligned}(d\Lambda - \Lambda d)(L_m u) &= 2L_{m-2}e + (m-r-1)L_{m-1}f - 2(r+1)mL_m g \\ &\quad + 2(r+1)(r+2)(m+1)L_{m+1}h + L_{m-1}a - (r+1)L_m b.\end{aligned}$$

Thus

$$\begin{aligned}[d^\Lambda, L](L_r u) &= -2L_{r-1}e + (m-r-1)L_r f + 2(r+1)mL_{r+1}g \\ &\quad + 2(r+1)(r+2)(m+1)L_{r+2}h + L_r a + (r+1)L_{r+1}b.\end{aligned}$$

*Step 3*: Since

$$d(L_r u) = L_{r-1}\theta u + L_r du,$$

and

$$\theta\Lambda(L_r u) = (m+1)L_{r-1}\theta u.$$

We have

$$(d - \theta\Lambda)(L_r u) = L_r du - mL_{r-1}\theta u.$$

Notice that

$$L_r du = L_r a + (r+1)L_{r+1}b + (r+1)(r+2)L_{r+2}c,$$

and

$$L_{r-1}\theta u = L_{r-1}e + rL_r f + r(r+1)L_{r+1}g + r(r+1)(r+2)L_{r+2}h.$$

Moreover, since

$$\Lambda\theta(L_r u) = \Lambda L_r(e + Lf + L^2 g + L^3 h),$$

by Proposition 2.8, we have

$$\begin{aligned}\Lambda\theta(L_r u) &= (m-2)L_{r-1}e + (m-1)(r+1)L_r f \\ &\quad + (r+1)(r+2)mL_{r+1}g + (r+1)(r+2)(r+3)(m+1)L_{r+2}h.\end{aligned}$$

Thus

$$(d + [\Lambda, \theta])(L_r u) = L_r du - mL_{r-1}\theta u + \Lambda\theta(L_r u),$$

can be written as

$$\begin{aligned}&L_r a + (r+1)L_{r+1}b + (r+1)(r+2)L_{r+2}c \\ &-2L_{r-1}e + (m-r-1)L_r f \\ &+ 2m(r+1)L_{r+1}g + (r+1)(r+2)(r+3m+3)L_{m+1}h,\end{aligned}$$

which is equal to $[d^\Lambda, L](L_r u)$ by *Step 2* and (3.3). Thus

$$[d^\Lambda, L] = d + [\Lambda, [d, L]].$$

By definition of $d^\Lambda$ and $\Lambda$, we know that $[d^\Lambda, L] = d + [\Lambda, [d, L]]$ is equivalent to $[d, \Lambda] = d^\Lambda + [[\Lambda, d^\Lambda], L]$. Thus the proof is complete. $\square$

**Remark**: In case $[d, L] = 0$, then the above theorem is just the general Kähler identity and its proof is much simpler. The general Kähler identity implies the following result.

**Theorem 3.7.** *Let $(A, L)$ be a Lefschetz space. Let $d$ be a $\mathbb{C}$-linear endomorphism of $A$ such that $d(A^l) \subset A^{l+1}$. If $[d, L] = 0$ then*

1. $(\ker d \cap \ker d^\Lambda, L)$ *and* $(\ker d \cap \ker d^\Lambda, \Lambda)$ *are Lefschetz;*
2. $(\operatorname{Im} d + \operatorname{Im} d^\Lambda, L)$ *and* $(\operatorname{Im} d + \operatorname{Im} d^\Lambda, \Lambda)$ *are Lefschetz;*
3. *Assume further that $d^2 = 0$. Then* $(\operatorname{Im} dd^\Lambda, L)$ *and* $(\operatorname{Im} dd^\Lambda, \Lambda)$ *are Lefschetz.*

*Proof.* Notice that $\ker d \cap \ker d^\Lambda$ is $*_s$ invariant and $\Lambda = *_s L *_s$. $(\ker d \cap \ker d^\Lambda, \Lambda)$ is Lefschetz if $(\ker d \cap \ker d^\Lambda, L)$ is Lefschetz. Now let us prove that $(\ker d \cap \ker d^\Lambda, L)$ is Lefschetz. Since $(A, L)$ is Lefschetz, it suffices to prove that the primitive decomposition preserves $(\ker d \cap \ker d^\Lambda, L)$. Thus it is enough to show

$$L(\ker d \cap \ker d^\Lambda) \subset \ker d \cap \ker d^\Lambda,$$

and

$$\Lambda(\ker d \cap \ker d^\Lambda) \subset \ker d \cap \ker d^\Lambda,$$

which follows from

$$[d, L] = 0, \ [d^\Lambda, L] = d, \ [d, \Lambda] = d^\Lambda, \ [\Lambda, d^\Lambda] = 0.$$

Thus (1) follows from general Kähler identity. (2) and (3) can be proved by a similar argument. □

### 3.3. $dd^\Lambda$-Lemma for a general Lefschetz complex.

**Definition 3.8.** *Let $(A, L)$ be a Lefschetz space. Let $d$ be a $\mathbb{C}$-linear endomorphism of $A$ such that $d(A^l) \subset A^{l+1}$. We call $(A, L, d)$ a* Lefschetz complex *if $d^2 = 0$.*

Let $(A, L, d)$ be a Lefschetz complex. In case $[d, L] = 0$, Theorem 3.6 implies that

$$[d, d^\Lambda] = 0,$$

thus $(A, d, d^\Lambda)$ is a double-complex.

**Definition 3.9.** *Let $(A, L, d)$ be a Lefschetz complex. Assume that $[d, L] = 0$. We say that $(A, L, d)$ satisfies the $dd^\Lambda$-*Lemma *if*

$$\ker d \cap \ker d^\Lambda \cap (\operatorname{Im} d + \operatorname{Im} d^\Lambda) = \operatorname{Im} dd^\Lambda,$$

*on each $A^k$, $0 \le k \le 2n$.*

**Definition 3.10.** *Let $(A, L, d)$ be a Lefschetz complex. We shall define*

$$H_d = \oplus_{k=0}^{2n} H_d^k, \qquad H_d^k := \frac{\ker d \cap A^k}{\operatorname{Im} d \cap A^k},$$

*and*

$$H_{d^\Lambda} = \oplus_{k=0}^{2n} H_{d^\Lambda}^k, \qquad H_{d^\Lambda}^k := \frac{\ker d^\Lambda \cap A^k}{\operatorname{Im} d^\Lambda \cap A^k}.$$

The following theorem is due to Mathieu [12] and Yan [18], we will follow the proof in [18].

**Theorem 3.11.** *Let $(A, L, d)$ be a Lefschetz complex. Assume that $[d, L] = 0$. Then the following facts are equivalent:*

*(1) $\ker d \cap \ker d^\Lambda \to H_d$ is surjective;*
*(2) For each $0 \le k \le n$, $L^k : H_d^{n-k} \to H_d^{n+k}$ is surjective;*
*(3) $\ker d \cap \ker d^\Lambda \to H_{d^\Lambda}$ is surjective;*
*(4) For each $0 \le k \le n$, $\Lambda^k : H_{d^\Lambda}^{n+k} \to H_{d^\Lambda}^{n-k}$ is surjective.*

*Proof.* By Theorem 3.7, we know that for each $0 \le k \le n$,

$$L^k : (\ker d \cap \ker d^\Lambda) \cap A^{n-k} \to (\ker d \cap \ker d^\Lambda) \cap A^{n+k},$$

is an isomorphism. Thus (1) implies (2). The same proof gives that (3) implies (4). Since $\ker d \cap \ker d^\Lambda$ is $*_s$-invariant and $*_s$ defines an isomorphism from $H_d^{n-k}$ to $H_{d^\Lambda}^{n+k}$, we know that (1) is equivalent to (3) and (2) is equivalent to (4). Thus it is enough to prove that (2) implies (1), which follows directly from the argument in the proof of Theorem 0.1 in [18] (the idea is: (2) implies that each class in $H_d$ has a Lefschetz decomposition and the primitive class has a primitive representative which lies in $\ker d \cap \ker d^\Lambda$). □

Now we can prove the following result:

**Theorem 3.12.** *Let $(A, L, d)$ be a Lefschetz complex. Assume that $[d, L] = 0$. Then the followings are equivalent:*

*(1)* $(A, L, d)$ *satisfies the* $dd^\Lambda$*-Lemma;*
*(2)* *the natural map* $\ker d \cap \ker d^\Lambda \to H_d$ *is surjective and the* $sl_2$*-triple* $(L, \Lambda, B)$ *on* $\ker d \cap \ker d^\Lambda$ *induces the* $sl_2$*-triple on* $H_d$*;*
*(3)* $(H_d, L)$ *satisfies the Hard Lefschetz Condition;*
*(4)* $(H_{d^\Lambda}, \Lambda)$ *satisfies the Hard Lefschetz Condition.*

*Proof.* (1) implies (2): If $u \in \ker d$ then $[d, d^\Lambda] = 0$ implies that $d(d^\Lambda u) = 0$. Thus the $dd^\Lambda$-Lemma implies

$$d^\Lambda u \in \operatorname{Im} dd^\Lambda.$$

Let us write $d^\Lambda u = dd^\Lambda v$. Thus

$$u + dv \in \ker d^\Lambda,$$

which implies that $\ker d \cap \ker d^\Lambda \to H_d$ is surjective. Notice that

$$*_s(\ker d^\Lambda \cap \operatorname{Im} d) = \ker d \cap \operatorname{Im} d^\Lambda.$$

Thus $dd^\Lambda$-Lemma gives

$$*_s(\ker d^\Lambda \cap \operatorname{Im} d) \subset \operatorname{Im} dd^\Lambda \subset \operatorname{Im} d,$$

which implies that $*_s$ is well defined on $H_d$ (using representatives in $\ker d \cap \ker d^\Lambda$). Now we can define $\Lambda := *_s L *_s$ on $H_d$. Thus (1) implies (2).

(2) implies (3) is well known (see [9]). The fact that (3) and (4) are equivalent follows from that for each $k$, $*_s$ defines an isomorphism from $H^k_d$ to $H^{2n-k}_{d^\Lambda}$ and $\Lambda = *_s^{-1} L *_s$.

Now it suffices to show (3) implies (1). By Theorem 3.7, we only need to prove the $dd^\Lambda$-Lemma on the primitive space $P$, i.e.,

$$P \cap \ker d \cap (\operatorname{Im} d + \operatorname{Im} d^\Lambda) \subset \operatorname{Im} dd^\Lambda. \tag{3.4}$$

We shall follow the proof by Merkulov (see page 4 in [13]). First, let us prove (3.4) is true on $P^0$. Let $u \in P^0 \cap \ker d \cap \operatorname{Im} d^\Lambda$, we know that $L^n[u] = 0$. Thus $u = [u] = 0 = P^0 \cap \operatorname{Im} dd^\Lambda$. In general, we shall prove that

$$P \cap \ker d \cap (\operatorname{Im} d + \operatorname{Im} d^\Lambda) = P \cap \operatorname{Im} d. \tag{3.5}$$

Assume that $u \in P^k$, $1 \leq k \leq n$. If $u \in \ker d \cap (\operatorname{Im} d + \operatorname{Im} d^\Lambda)$ then

$$L^{n-k}[u] = 0.$$

Thus $[u] = 0$ by the HLC-condition, which gives (3.5). Now we know that (3.4) is equivalent to

$$P \cap \operatorname{Im} d \subset \operatorname{Im} dd^\Lambda. \tag{3.6}$$

Let us first prove that (3.6) is true on $P^1$. In fact, since $A^0 \subset \ker d^\Lambda$, Theorem 3.11 implies that for every $u \in A^0$, there exists $a \in A^1$ and $b \in \ker d \cap \ker d^\Lambda$ such that

$$u = d^\Lambda a + b.$$

Thus

$$du = dd^\Lambda a,$$

which implies that (3.6) is true on $P^1$. Assume that the $dd^\Lambda$-Lemma is true on $A^k$, let us prove that (3.6) is true on $P^{k+2}$. Take $u = da \in P^{k+2}$. Primitivity of $u$ implies that $u \in \operatorname{Im} d^\Lambda$. Thus $d^\Lambda a \in \ker d$. Now the $dd^\Lambda$-Lemma on $A^k$ implies that there exists $b \in A^k$ such that

$$d^\Lambda a = d^\Lambda d\, b.$$

By Theorem 3.11, we know that there exists $e \in A^{k+1}$ and $f \in \ker d \cap \ker d^\Lambda$ such that

$$a - db = d^\Lambda e + f,$$

which implies that $u = dd^\Lambda e$. Thus (3.6) is true on $P^{k+2}$. The proof is complete. □

## 4. Kähler identity for arbitrary degree operators

4.1. **$\mathfrak{su}(2)$-representation.** In this section, we shall follow Wells' book [17]. It is known that (see page 172 in [17]) the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ of the special linear group $\mathrm{SL}(2,\mathbb{C})$ is generated by

$$X=\begin{bmatrix}0&1\\0&0\end{bmatrix},\ Y=\begin{bmatrix}0&0\\1&0\end{bmatrix},\ H=\begin{bmatrix}1&0\\0&-1\end{bmatrix},$$

which satisfy

$$[X,Y]=H,\ [H,X]=2X,\ [H,Y]=-2Y.$$

Let $(A,L)$ be a Lefschetz complex with $sl_2$-triple $(L,\Lambda,B)$. Then we know that

$$\rho(X)=L,\ \rho(Y)=\Lambda,\ \rho(H)=B,$$

defines an $\mathfrak{sl}(2,\mathbb{C})$-action on $(A,L)$. It is also known that the Lie-algebra $\mathfrak{su}(2)$ of the special unitary group is a real form of $\mathfrak{sl}(2,\mathbb{C})$, i.e.

$$\mathfrak{sl}(2,\mathbb{C})=\mathfrak{su}(2)\otimes_{\mathbb{R}}\mathbb{C}.$$

Moreover, $\mathfrak{su}(2)$ is generated by

$$iH,\quad X-Y,\quad i(X+Y).$$

Put

$$W(t)=e^{it(X+Y)},\qquad \#(t)=\rho(W(t))=e^{it(L+\Lambda)}.$$

Then we have the following formula (same as the proof in page 187 in [17]):

**Proposition 4.1.** *$\#(\frac{\pi}{2})u=i^{k^2+n}*_s$, for every $u\in A^k$, where $*_s$ denotes the Lefschetz star operator.*

**Definition 4.2.** *Let $(A,L)$ be a Lefschetz space. We call $D$ a degree $p$ map if $D$ is a $\mathbb{C}$-linear endomorphism of $A$ such that $D(A^l)\subset A^{l+p}$ for each $l$. We shall define $D^{\#}:=\#(-\frac{\pi}{2})\,D\,\#(\frac{\pi}{2})$.*

**Remark**: It is easy to check that

$$D^{\#}u=i^{p^2}(-1)^{p(k+1)}*_s D *_s u,$$

for every $u\in A^k$ if $D$ is degree $p$.

**Definition 4.3.** *If $D_1$ is degree $p_1$ and $D_2$ is degree $p_2$ then we shall write*

$$\mathrm{ad}_{D_1}D_2=[D_1,D_2]:=D_1D_2-(-1)^{p_1p_2}D_2D_1,$$

*and*

$$[\mathrm{ad}_{D_1},\mathrm{ad}_{D_2}]=\mathrm{ad}_{D_1}\mathrm{ad}_{D_2}-(-1)^{p_1p_2}\mathrm{ad}_{D_2}\mathrm{ad}_{D_1}.$$

**Remark**: The super Jacobi identity is equivalent to the following formula

$$[\mathrm{ad}_{D_1},\mathrm{ad}_{D_2}]=\mathrm{ad}_{[D_1,D_2]}.$$

We shall use the following lemmas:

**Lemma 4.4.** *If $D$ is degree $p$ then $\mathrm{ad}_BD=p\cdot D$.*

*Proof.* For every $u\in A^k$, we have

$$\mathrm{ad}_BD\,(u)=[B,D]u=(p+k-n)\,Du-(k-n)\,Du=p\cdot Du,$$

which gives our formula. □

It is convenient to introduce the following definition:

**Definition 4.5.** $(\mathrm{ad}_\Lambda)_k:=(\mathrm{ad}_\Lambda)^k/\,k!$, $(\mathrm{ad}_L)_k:=(\mathrm{ad}_L)^k/\,k!$.

We have the following generalization of Lemma 4.4.

**Lemma 4.6.** *If $D$ is degree $p$ then*

$$[\mathrm{ad}_L, (\mathrm{ad}_\Lambda)_k]\, D = (p-k+1)(\mathrm{ad}_\Lambda)_{k-1} D,$$

*and*

$$[\mathrm{ad}_\Lambda, (\mathrm{ad}_L)_k]\, D = (-p-k+1)(\mathrm{ad}_L)_{k-1} D,$$

*for every $k \geq 1$.*

*Proof.* Follows directly by induction on $k$ and the following formula

$$[\mathrm{ad}_{D_1}, (\mathrm{ad}_{D_2})^k] = \mathrm{ad}_{[D_1,D_2]}(\mathrm{ad}_{D_2})^{k-1} + \mathrm{ad}_{D_2}[\mathrm{ad}_{D_1}, (\mathrm{ad}_{D_2})^{k-1}],$$

for even degree maps. □

4.2. **A List of formulas.** Put

$$A_{jk} := (\mathrm{ad}_L)_j (\mathrm{ad}_\Lambda)_k D,\ B_{jk} := (\mathrm{ad}_\Lambda)_j (\mathrm{ad}_L)_k D,\ T := \mathrm{ad}_L + \mathrm{ad}_\Lambda.$$

Then Lemma 4.6 gives

$$T(A_{jk}) = (j+1)A_{(j+1)k} + (k+1)A_{j(k+1)} + (2k-j+1-p)A_{(j-1)k}, \tag{4.1}$$

and

$$T(B_{jk}) = (j+1)B_{(j+1)k} + (k+1)B_{j(k+1)} + (2k-j+1+p)B_{(j-1)k}. \tag{4.2}$$

**Remark**: By induction on $k$, Lemma 4.6 also gives

$$B_{kk} = \sum_{j=0}^{k} C^j_{p-1+j}(-1)^j A_{(k-j)(k-j)}, \quad A_{kk} = \sum_{j=0}^{k} C^j_{-p-1+j}(-1)^j B_{(k-j)(k-j)}, \tag{4.3}$$

where

$$C^0_m := 1,\ C^j_{m+j} := \frac{(m+j)(m+j-1)\cdots(m+1)}{j!},\ \forall\, m \in \mathbb{Z},\ j \geq 1.$$

Since $(k+1)B_{(k+1)k} = \Lambda B_{kk}$, (4.3) gives

$$B_{(k+1)k} = \sum_{j=0}^{k} C^j_{p-2+j}(-1)^j A_{(k-j)(k-j+1)}. \tag{4.4}$$

By a similar argument, we also have

$$A_{(k+1)k} = \sum_{j=0}^{k} C^j_{-p-2+j}(-1)^j B_{(k-j)(k-j+1)}. \tag{4.5}$$

In this paper we will not use (4.3), (4.4) and (4.5). A direct consequence of (4.1) is:

**Lemma 4.7.** *If $D$ is degree $p$ then we can write*

$$T^l D = \sum_{j+k \leq 2l} a_{jk} A_{jk},$$

*where $a_{jk}$ are integer constants that only depend on $l$ and $p$.*

Notice that

$$e^{t(\mathrm{ad}_L + \mathrm{ad}_\Lambda)} D = \sum_{l \geq 0} \frac{t^l}{l!} T^l D = \sum_{j,k} \left( \sum_{l \geq 0} \frac{t^l}{l!} a_{jk} \right) A_{jk}.$$

Since for every $N \geq n$, we have

$$A_{jk} \equiv 0, \qquad \text{if } \max\{j,k\} > N.$$

Let us fixed $N \geq n$ and define

$$a^N_{jk} = a_{jk}, \quad \text{if } \max\{j,k\} \leq N, \quad a^N_{jk} = 0 \ \text{ if } \max\{j,k\} > N.$$

Then we have

$$e^{t(\mathrm{ad}_L+\mathrm{ad}_\Lambda)}D=\sum_{j,k\leq N} b_{jk}^N A_{jk}, \quad b_{jk}^N:=\sum_{l\geq 0}\frac{t^l}{l!}\,a_{jk}^N.$$

**Lemma 4.8.** *Each $b_{jk}^N$ defines a holomorphic function on $\mathbb{C}$.*

*Proof.* By definition of $a_{jk}^N$, we have

$$T^l D=\sum_{j+k\leq 2l} a_{jk}^N A_{jk}.$$

Thus (4.1) gives

$$a_{jk}^N(l,p)=j\,a_{(j-1)k}^N(l-1,p)+k\,a_{j(k-1)}^N(l-1,p)+(2k-j-p)a_{(j+1)k}^N(l-1,p). \tag{4.6}$$

Put

$$M_l:=\sup\{|a_{jk}^N(l,p)|\}.$$

Then (4.6) gives

$$M_l\leq (4N+p)\,M_{l-1},$$

which implies

$$\sum_{l\geq 0}\frac{|t|^l}{l!}\,|a_{jk}^N|\leq M_0\,e^{(4N+p)|t|}.$$

Thus the Lemma follows. □

Our Key Lemma is the following:

**Lemma 4.9** (Derivative of exponential map)**.** *Let $X$ be a finite sum of even degree maps. Then for every $\mathbb{C}$-linear endomorphism $D$ of $A$, we have*

$$e^{tX}D\,e^{-tX}=e^{t\cdot\mathrm{ad}_X}D, \qquad \forall\; t\in\mathbb{R}.$$

*Proof.* Put

$$f(t)=e^{tX}D\;e^{-tX}, \quad g(t)=e^{t\cdot\mathrm{ad}_X}D.$$

Then we have

$$f(0)=g(0),\ \frac{dg}{dt}=\mathrm{ad}_X g(t).$$

Moreover, since $X$ is a sum of even degree maps, we have

$$\frac{df}{dt}=Xf(t)-f(t)X=[X,f(t)]=\mathrm{ad}_X f(t).$$

Thus $f$ and $g$ satisfy the same equation, which gives $f=g$. □

4.3. **Main theorem.** Apply the above Lemma to

$$X=(L+\Lambda), \quad t=-\frac{\pi i}{2},$$

we get the following universal version of the Kähler identity.

**Theorem 4.10** (Main theorem)**.** *For each $p\in\mathbb{Z}$ there exists a sequence $\{c_j^p\}_{j\geq 0}$ such that*

$$D^{\#}=\sum_{j,j+p\leq n} c_j^p\,(\mathrm{ad}_L)_j(\mathrm{ad}_\Lambda)_{j+p}D,$$

*if $D$ is degree $p$.*

*Proof.* Follows from Lemma 4.8 and the fact that $D^{\#}$ is degree $-p$. □

**Remark**: Consider $B_{jk}$ instead of $A_{jk}$, we know that there also exists a sequence $\{a_j^p\}_{j\geq 0}$ such that

$$D^{\#} = \sum_{j, j+p\leq n} a_j^p\, (\mathrm{ad}_\Lambda)_{j+p} (\mathrm{ad}_L)_j D, \tag{4.7}$$

if $D$ is degree $p$. One may compute $a_j^p$ and $c_j^p$ by taking special $D$. In case $D$ is degree zero, our main theorem implies the following generalization of the main theorem in [16]:

**Theorem 4.11.** *Let $(A, L)$ be a Lefschetz space. If $D$ is degree zero and $(\mathrm{ad}_L)^3 D = 0$ then*

$$D^{\#} = *_s D *_s = (1 - \mathrm{ad}_L \mathrm{ad}_\Lambda + (\mathrm{ad}_L)_2 (\mathrm{ad}_\Lambda)_2)\ D.$$

*Proof.* Since $D$ is degree zero, (4.3) gives

$$(\mathrm{ad}_L)^k (\mathrm{ad}_\Lambda)^k D = (\mathrm{ad}_\Lambda)^k (\mathrm{ad}_L)^k D.$$

Now $(\mathrm{ad}_L)^3 D = 0$ implies that it suffices to compute $c_j^0$ for $j = 0, 1, 2$. Take $D = 1$, we get

$$c_0^0 = 1.$$

Take $D = B$, we get

$$D^{\#} = -B = B + c_1^0 \cdot (2B),$$

Thus

$$c_1^0 = -1.$$

Take $D = B^2$, we know that

$$\mathrm{ad}_\Lambda (B^2) = 4\Lambda + 4B\Lambda,\ (\mathrm{ad}_\Lambda)_2 (B^2) = 4\Lambda^2,$$

Thus

$$D^{\#} = (-B)^2 = B^2 - \mathrm{ad}_L (4\Lambda + 4B\Lambda) + c_2^0 \cdot (4 (\mathrm{ad}_L)_2 (\Lambda^2)).$$

Since

$$(\mathrm{ad}_L)_2 (\Lambda^2) = \mathrm{ad}_L (\Lambda + B\Lambda),$$

we get

$$c_2^0 = 1.$$

The proof is complete. □

## 5. Kähler identities on almost symplectic manifolds

An *almost symplectic manifold* $(X, \omega)$ is a smooth manifold $X$ with a non-degenerated 2-form $\omega$. Denote by $*_s$ the *symplectic star operator* with respect to $\omega$. Let $d$ be the usual exterior derivative on $X$ and denote by $\Omega^k(X)$ the space of $k$-forms on $X$. By applying Theorem 3.6 to

$$A = \bigoplus_{k=0}^{2n} \Omega^k(X),\ L := \omega \wedge,$$

where $\dim X = 2n$, we get

**Theorem 5.1.** $[d^\Lambda, L] = d + [\Lambda, [d, L]], \quad [d, \Lambda] = d^\Lambda + [[\Lambda, d^\Lambda], L].$

The above theorem implies the following well known Kähler identities on symplectic manifolds, see e.g., [18, 4].

**Theorem 5.2.** *If $d\omega = 0$ then $[d^\Lambda, L] = d$ and $[d, \Lambda] = d^\Lambda$.*

Now let $J$ be an *almost complex structure* on $X$. We shall also use $J$ to denote the associated *Weil-operator*. Denote by $\Omega^{p,q}(X)$ the space of smooth $(p,q)$-forms on $X$. Then

$$Ju = i^{p-q}u, \ \forall \ u \in^{p,q} .$$

We shall define

$$d^c := J^{-1}dJ,$$

where $J$ is the Weil-operator. If $J$ is compatible with $\omega$ then $J$ commutes with $*_s$ and we call

$$\star := *_s \circ J,$$

the *Hodge star operator* on $X$. If $J$ is compatible with $\omega$ then $(\omega, J)$ defines a pointwise Hermitian inner product structure, say $(\cdot,\cdot)$, on the space of differential forms such that

$$(u,v)(x)\omega_n(x) = u \wedge \star\bar{v}, \quad \omega_n := \omega^n/n!.$$

Assume that $X$ is compact. Denote by $d^*$ and $(d^c)^*$ the adjoint of $d$ and $d^c$ with respect to the following inner product

$$(u,v) := \int_X (u,v)(x)\omega_n(x).$$

By integration by parts, we have

$$d^* = -\star d\star, \ (d^c)^* = -\star d^c \star .$$

Thus

$$d^* = -*_s JdJ*_s = -*_s J^2d^c*_s = (-1)^k *_s d^c*_s,$$

on $\Omega^k(X)$ and

$$(d^c)^* = -\star d^c\star = (-1)^{k+1} *_s d*_s = d^\Lambda,$$

on $\Omega^k(X)$. Moreover, since $JJ^* = 1$, we have $(d^c)^* = J^{-1}d^*J$. Thus we get

**Theorem 5.3.** *If $J$ is compatible with $\omega$ then $[(d^c)^*, L] = d + [\Lambda, [d,L]]$ and*

$$[L, d^*] = d^c + [\Lambda, [d^c, L]].$$

*Assume further that $d\omega = 0$. Then*

$$[(d^c)^*, L] = d, \ [L, d^*] = d^c.$$

## 6. Hard Lefschetz Condition on almost complex manifolds

Let $(X, \omega, J, g)$ be a compact almost symplectic manifold with a compatible almost complex structure $J$, where the Riemannian metric $g$ is defined by

$$g(u,v) = \omega(u, Jv).$$

Set

$$H^\bullet_d(X) = \frac{\ker d}{\operatorname{Im} d}, \ H^\bullet_{d^c}(X) = \frac{\ker d^c}{\operatorname{Im} d^c}.$$

By the Hodge theory, we have

**Theorem 6.1.** *Set $\Box_d = dd^* + d^*d$, $\Box_{d^c} = d^c(d^c)^* + (d^c)^*d^c$, then*

$$\mathcal{H}^\bullet_{\Box_d} := \ker \Box_d \simeq H^\bullet_d(X), \ \mathcal{H}^\bullet_{\Box_{d^c}} := \ker \Box_{d^c} \simeq H^\bullet_{d^c}(X).$$

We now study the Hard Lefschetz property of $\ker \Box_d$. Let us start from the following example.

*Example 1*: **Kodaira-Thurston manifold.** On $\mathbb{R}^4$ with coordinate $x^1, \ldots, x^4$ consider the following product: given any $a = (a^1, \ldots, a^4), b = (b^1 \ldots, b^4) \in \mathbb{R}^4$, set

$$a * b = (a^1 + b^1, a^2 + b^2, a^3 + a^1b^2 + b^3, a^4 + b^4).$$

Then $(\mathbb{R}^4, *)$ is a Lie group and $\Gamma = \{(\gamma^1, \ldots, \gamma^4) \in \mathbb{R}^4 \mid \gamma_j \in \mathbb{Z}, j = 1, \ldots, 4\}$ is a lattice in $(\mathbb{R}^4, *)$, so that $M = \Gamma\backslash\mathbb{R}^4$ is a 4-dimensional compact manifold. Then,

$$e^1 = dx^1, \quad e^2 = dx^2, \quad e^3 = dx^3 - x^1dx^2, \quad e^4 = dx^4,$$

are $\Gamma$-invariant 1-forms on $\mathbb{R}^4$, and, consequently, they give rise to a gobal coframe on $M$. It is immediate to check that $de^3 = -e^1 \wedge e^2$, the other differential vanishing. Define an almost Kähler structure on $M$, by setting:

$$Je^1 = -e^3, \quad Je^2 = -e^4, \quad Je^3 = e^1, \quad Je^4 = e^2,$$

and

$$\omega = e^{13} + e^{24},$$

where $e^{ij} = e^i \wedge e^j$ and so on. Then $\omega$ is a symplectic structure on $M$ and $J$ is an $\omega$-compatible non integrable almost complex structure so that $(M, J, \omega)$ is an almost Kähler manifold. Set $g = \sum_{i=1}^4 e^i \otimes e^i$. Then, a direct computation gives

$$\begin{aligned} \mathcal{H}^1_{\Box_d} &= Span_{\mathbb{R}} < e^1, e^2, e^4 > \\ \mathcal{H}^2_{\Box_d} &= Span_{\mathbb{R}} < e^{13}, e^{24}, e^{14}, e^{23} > \\ \mathcal{H}^3_{\Box_d} &= Span_{\mathbb{R}} < e^{234}, e^{134}, e^{123} > \end{aligned}$$

and

$$\begin{aligned} \mathcal{H}^1_{\Box_{d^c}} &= Span_{\mathbb{R}} < e^2, e^3, e^4 > \\ \mathcal{H}^2_{\Box_{d^c}} &= Span_{\mathbb{R}} < e^{13}, e^{24}, e^{14}, e^{23} > \\ \mathcal{H}^3_{\Box_{d^c}} &= Span_{\mathbb{R}} < e^{123}, e^{134}, e^{124} > . \end{aligned}$$

Therefore,

$$\ker \Box_d \cap \ker \Box_{d^c} = \mathbb{R}\langle 1 \rangle \oplus \mathbb{R}\langle e^2, e^4 \rangle \oplus \mathbb{R}\langle e^{13}, e^{14}, e^{23}, e^{24} \rangle \oplus \mathbb{R}\langle e^{123}, e^{134} \rangle \oplus \mathbb{R}\langle 1 \rangle$$

and $\omega$ restricted to $\ker \Box_d \cap \ker \Box_{d^c}$ satisfies the Hard Lefschetz Condition. As a generalization of the above fact, we have

**Theorem 6.2.** *Let $(X, \omega, J, g)$ be a compact symplectic manifold with a compatible almost complex structure $J$. Then $\omega$ restricted to $\ker \Box_d \cap \ker \Box_{d^c}$ satisfies the Hard Lefschetz Condition.*

*Proof.* As usual, put $L := \omega \wedge$. It suffifes to show that

$$[\Box_d + \Box_{d^c}, L] = 0.$$

In fact, by the Jocobi identity, we have

$$[L, [d, d^*]] + [d, [d^*, L]] - [d^*, [L, d]] = 0,$$

and

$$[L, [d^c, (d^c)^*]] + [d^c, [(d^c)^*, L]] - [(d^c)^*, [L, d^c]] = 0.$$

Since $[L, d] = [L, d^c] = 0$, $[d^*, L] = -d^c$, $[(d^c)^*, L] = d$ and

$$[d, d^c] = [d^c, d] = dd^c + d^c d,$$

we have

$$[L, [d, d^*]] + [L, [d^c, (d^c)^*]] = 0,$$

which gives $[\Box_d + \Box_{d^c}, L] = 0$. □

**Remark 6.3.** *The above proof gives*

$$[L, \Box_d] = [d, d^c].$$

*Since $[d, d^c] = 0$ if and only if $J$ is integrable. We know that, in the above theorem, $X$ is Kähler if and only if*

$$[L, \Box_d] = 0.$$

*The following example tells us the above identity is strictly stronger than the Hard Lefschetz condition on $(\ker \Box_d, L)$ in general.*

*Example 2*: **Completely solvable Nakamura manifolds.** Let $\mathfrak{g}$ be the 6-dimensional Lie algebra whose dual space has a basis $\{e^i\}_{i\in\{1,\ldots,6\}}$ satisfying the following Maurer-Cartan equations:

$$\begin{cases} de^1 = 0, & de^2 = 0, \quad de^3 = e^{13}, \\ de^4 = -e^{14}, & de^5 = e^{15}, \quad de^6 = -e^{16}. \end{cases} \tag{6.1}$$

Then it turns out that the connected and simply-connected Lie group $G$ whose Lie algebra is $\mathfrak{g}$ admits a lattice $\Gamma$ such that $M = \Gamma\backslash G$ is a compact solvmanifold of completely solvable type.

Then $(J, \omega, g)$ is defined respectively as

$$\begin{cases} Je^1 := -e^2, \\ Je^3 := -e^4, \\ Je^5 := -e^6, \end{cases} \tag{6.2}$$

$$\omega := e^{12} + e^{34} + e^{56} \tag{6.3}$$

and $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$ give rise to an almost Kähler structure on $M$. It turns out that $b_0(M) = 1$, $b_1(M) = 2$, $b_2(M) = 5$, $b_3(M) = 8$, $b_4(M) = 5$, $b_5(M) = 2$ and $b_6(M) = 1$. Then, a straightforward computation yields to:

$$\begin{aligned}
\mathcal{H}^1_{\Box_d} &= Span_{\mathbb{R}} < e^1, e^2 > \\
\mathcal{H}^2_{\Box_d} &= Span_{\mathbb{R}} < e^{12}, e^{34}, e^{56}, e^{36}, e^{45} > \\
\mathcal{H}^3_{\Box_d} &= Span_{\mathbb{R}} < e^{134}, e^{156}, e^{136}, e^{145}, e^{234}, e^{256}, e^{236}, e^{245} > \\
\mathcal{H}^4_{\Box_d} &= Span_{\mathbb{R}} < e^{3456}, e^{1256}, e^{1234}, e^{1245}, e^{1236} > \\
\mathcal{H}^5_{\Box_d} &= Span_{\mathbb{R}} < e^{23456}, e^{13456} >
\end{aligned}$$

and,

$$\begin{aligned}
\mathcal{H}^1_{\Box_{d^c}} &= Span_{\mathbb{R}} < e^1, e^2 > \\
\mathcal{H}^2_{\Box_{d^c}} &= Span_{\mathbb{R}} < e^{12}, e^{34}, e^{56}, e^{36}, e^{45} > \\
\mathcal{H}^3_{\Box_{d^c}} &= Span_{\mathbb{R}} < e^{134}, e^{156}, e^{136}, e^{145}, e^{234}, e^{256}, e^{236}, e^{245} > \\
\mathcal{H}^4_{\Box_{d^c}} &= Span_{\mathbb{R}} < e^{3456}, e^{1256}, e^{1234}, e^{1245}, e^{1236} > \\
\mathcal{H}^5_{\Box_{d^c}} &= Span_{\mathbb{R}} < e^{23456}, e^{13456} >,
\end{aligned}$$

that is $\ker \Box_d = \ker \Box_{d^c}$ and $(M, \omega)$ satisfies the Hard Lefschetz Condition.

As a generalization of the above fact, we have

**Theorem 6.4.** *Let $(X, \omega, J, g)$ be a compact symplectic manifold with an compatible almost complex structure $J$. Then the followings are equivalent:*

*(1)* $\ker \Box_d = \ker \Box_{d^c}$*;*
*(2) Hard Lefschetz condition on* $(\ker \Box_d, L)$*;*
*(3) Hard Lefschetz condition on* $(\ker \Box_{d^c}, L)$.

*Proof.* We already know that (1) implies (2) and (3). Since

$$d^c = (-1)^k *_s d^* *_s, \ (d^c)^* = (-1)^{k+1} *_s d *_s,$$

one the space of $k$-forms, we have

$$\ker \Box_d = *_s \ker \Box_{d^c}.$$

Now (2) implies $*_s \ker \Box_d = \ker \Box_d$ thus (2) implies (1). A similar argument gives that (1) is equivalent to (3). $\square$

**Remark 6.5.** *It is easy to see that the Hard Lefschetz condition on* $(\ker \Box_d, L)$ *implies the Hard Lefschetz condition on* $(H^\bullet_d, L)$. *But in general, we don't know whether they are equivalent or not.*

## 7. Hard Lefschetz Condition for other cohomology groups

7.1. **Dolbeault cohomology groups.** Let $(X, J, g, \omega)$ be a compact Kähler manifold of dimension $n$. Let $(E, h_E)$ be a holomorphic vector bundle on $X$ with smooth Hermitian metric $h^E$ along the fibres. Denote by

$$D^E := \overline{\partial} + \partial^E,$$

the Chern connection on $(E, h_E)$. Let

$$\Theta^E := (D^E)^2,$$

be the Chern curvature. Then we have the following Kähler identity

$$[\overline{\partial}^*, L] = i\partial^E,$$

which implies the following Bochner-Kodaira-Nakano identity

$$\Box_{\overline{\partial}} - \Box_{\partial^E} = [i\Theta^E, \Lambda], \ \Box_{D^E} = \Box_{\overline{\partial}} + \Box_{\partial^E},$$

where

$$\Box_{\overline{\partial}} := \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}, \ \Box_{\partial^E} := \partial^E(\partial^E)^* + (\partial^E)^*\partial^E, \ \Box_{D^E} = (D^E)^*D^E + D^E(D^E)^*.$$

We have the following theorem:

**Theorem 7.1.** *Let $(X, J, g, \omega)$ be a compact Kähler manifold of dimension $n$. Let $(E, h_E)$ be a holomorphic vector bundle on $X$ with smooth Hermitian metric $h^E$ along the fibres. Then* $(\ker \Box_{\overline{\partial}} \cap \ker \Box_{\partial^E}, L)$ *satisfies the hard Lefschetz condition. Moreover, the followings are equivalent:*

*(1)* $\ker \Box_{\overline{\partial}} = \ker \Box_{\partial^E}$*;*
*(2) Hard Lefschetz condition on* $(\ker \Box_{\overline{\partial}}, L)$*;*
*(3) Hard Lefschetz condition on* $(\ker \Box_{\partial^E}, L)$*.*

*Proof.* (1) $\Leftrightarrow$ (2) follows from $[L, \Box_{\overline{\partial}} + \Box_{\partial^E}] = 0$; $\ker \Box_{\overline{\partial}} = \star \ker \Box_{\partial^E}$ gives (2) $\Leftrightarrow$ (3). □

7.2. **Symplectic cohomology groups and its analogies.** Let $(X, \omega)$ be a compact symplectic manifold. From [15], we know that

$$H^\bullet_{d+d^\Lambda}(X) := \frac{\ker d \cap \ker d^\Lambda}{\operatorname{Im} dd^\Lambda}, \ H^\bullet_{dd^\Lambda}(X) := \frac{\ker dd^\Lambda}{\operatorname{Im} d \cup \operatorname{Im} d^\Lambda}$$

always satisfy the Lefschetz property. Let $(X, J, g, \omega)$ be a compact Kähler manifold. Since

$$\partial^* = i[\Lambda, \overline{\partial}],$$

gives $[\overline{\partial}, \partial^*] = 0$, one may consider the following analogies of the above symplectic cohomology groups

$$H^{\bullet,\bullet}_{\overline{\partial}+\partial^*}(X) = \frac{\ker \partial^* \cap \ker \overline{\partial}}{\operatorname{Im} \partial^*\overline{\partial}}, \ H^{\bullet,\bullet}_{\overline{\partial}\partial^*}(X) = \frac{\ker \overline{\partial}\partial^*}{\operatorname{Im} \partial^* \cup \operatorname{Im} \overline{\partial}}.$$

Then

**Lemma 7.2.** *Let $(X, J, g, \omega)$ be a compact Kähler manifold. Then the following natural maps*

$$H^{\bullet,\bullet}_{\overline{\partial}+\partial^*}(X) \to H^{\bullet,\bullet}_{\overline{\partial}}(X), \ H^{\bullet,\bullet}_{\overline{\partial}}(X) \to H^{\bullet,\bullet}_{\overline{\partial}\partial^*}(X),$$

*are bijective.*

*Proof.* Let $\alpha$ be a smooth $(p, q)$-form on $X$ such that

$$\overline{\partial}\alpha = 0, \qquad \partial^*\alpha = 0 \tag{7.1}$$

Assume that $\alpha = \overline{\partial}\beta$. Taking the Hodge decomposition of $\beta$ with respect to $\Box_\partial$, we may write

$$\beta = \beta_H + \partial\lambda + \partial^*\mu\,.$$

Since $X$ is Kähler, we have $\Box_\partial = \Box_{\overline{\partial}}$; consequently,

$$\alpha = \overline{\partial}\partial\lambda + \overline{\partial}\partial^*\mu.$$

By (7.1), we get $\partial^*\partial\overline{\partial}\lambda = 0$. Therefore,

$$0 = \langle \partial^*\partial\overline{\partial}\lambda, \overline{\partial}\lambda\rangle = |\partial\overline{\partial}\lambda|^2,$$

which implies $\partial\overline{\partial}\lambda = 0$. Hence $\alpha = \overline{\partial}\partial^*\mu$, that is the natural map

$$H^{\bullet,\bullet}_{\overline{\partial}+\partial^*}(X) \to H^{\bullet,\bullet}_{\overline{\partial}}(X), \qquad \alpha + \operatorname{Im}\partial^*\overline{\partial} \mapsto \alpha + \operatorname{Im}\overline{\partial}$$

is injective. On the other hand, since $\square_\partial = \square_{\overline{\partial}}$, we know the $\overline{\partial}$-harmonic representative of a class in $H^{\bullet,\bullet}_{\overline{\partial}}(X)$ is always $\partial^*$-closed. Thus the above map is also surjective.

For the second isomorphism, let $u$ be a smooth $(p,q)$-form on $X$ such that $\overline{\partial}u = 0$. Asume that $u = \partial^* v$. Taking the Hodge decomposition of $v$ with respect to $\square_{\overline{\partial}}$, we may write

$$v = v_H + \overline{\partial}a + \overline{\partial}^* b.$$

Since $X$ is Kähler, we have $\square_\partial = \square_{\overline{\partial}}$; consequently,

$$u = \partial^*\overline{\partial}a + \partial^*\overline{\partial}^* b.$$

Now $\overline{\partial}u = 0$ gives $\overline{\partial}\partial^*\overline{\partial}^* b = 0$, thus

$$0 = \langle \overline{\partial}\partial^*\overline{\partial}^* b, \partial^* b\rangle = -|\partial^*\overline{\partial}^* b|^2,$$

which implies $\partial^*\overline{\partial}^* b = 0$. Thus $u = -\overline{\partial}\partial^* a$ and the second map is injective. In order to prove the surjectivity, let $\phi$ be a smooth $(p,q)$-form on $X$ such that $\overline{\partial}\partial^*\phi = 0$. Taking the Hodge decomposition of $\phi$ with respect to $\square_{\overline{\partial}}$, we may write

$$\phi = \phi_H + \overline{\partial}\psi + \overline{\partial}^*\varphi.$$

Since $X$ is Kähler, we have $\square_\partial = \square_{\overline{\partial}}$; consequently,

$$0 = \overline{\partial}\partial^*\phi = \overline{\partial}\partial^*\overline{\partial}^*\varphi,$$

by the same argument, we get $\partial^*\overline{\partial}^*\varphi = 0$. Thus we can write

$$\overline{\partial}^*\varphi = \theta_H + \partial^*\sigma, \qquad \theta_H \in \ker\square_\partial.$$

Now we know that $\rho := \phi - \partial^*\sigma$ is $\overline{\partial}$-closed. Thus the second map is surjective. □

**Corollary 7.3.**

$$\dim_{\mathbb{C}} H^{\bullet,\bullet}_{\overline{\partial}+\partial^*}(X) = \dim_{\mathbb{C}} H^{\bullet,\bullet}_{\overline{\partial}}(X) = \dim_{\mathbb{C}} H^{\bullet,\bullet}_{\overline{\partial}\partial^*}(X) < \infty.$$

**Corollary 7.4.** *Let $X$ be a compact Kähler manifold. Then $X$ satisfies the $\overline{\partial}\partial^*$-Lemma, that is*

$$\ker\overline{\partial} \cap \ker\partial^* \cap (\operatorname{Im}\overline{\partial} + \operatorname{Im}\partial^*) = \operatorname{Im}\overline{\partial}\partial^*.$$

**Remark**: It is well known that if $X$ is compact Kähler then $H^{\bullet,\bullet}_{\overline{\partial}}(X)$ satisfies the Hard Lefschetz Condition and $\partial^* = i[\Lambda, \overline{\partial}]$. Thus the above corollary also follows from Theorem 3.12.

In general, let $(E, h_E)$ be a holomorphic vector bundle over a compact Kähler manifold $(X, \omega, J)$. The Kähler identity $(\partial^E)^* = i[\Lambda, \overline{\partial}]$ gives $[\overline{\partial}, (\partial^E)^*] = 0$, which suggests us to define

$$H^{\bullet,\bullet}_{BC}(X, E, L) = \frac{\ker(\partial^E)^* \cap \ker\overline{\partial}}{\operatorname{Im}(\partial^E)^*\overline{\partial}}, \; H^{\bullet,\bullet}_{A}(X, E, L) = \frac{\ker\overline{\partial}(\partial^E)^*}{\operatorname{Im}(\partial^E)^* \cup \operatorname{Im}\overline{\partial}}.$$

The following theorem is a generalization of Theorem 3.11 and Theorem 3.22 in [15].

**Theorem 7.5.** *Let $(E, h_E)$ be a holomorphic vector bundle over a compact Kähler manifold $(X, \omega, J)$. Then $H^{\bullet,\bullet}_{BC}(X, E, L)$ and $H^{\bullet,\bullet}_{A}(X, E, L)$ satisfies the* Hard Lefschetz Condition *(for a bigraded spaces, it means the Hard Lefschetz Condition on its associated graded space $H^\bullet$, where $H^k := \oplus_{p+q=k} H^{p,q}$).*

*Proof.* Applying the elliptic operator theory (see the remark below), we have

$$H^{\bullet,\bullet}_{BC}(X,E,L) \simeq \ker \Box_{BC}, \qquad \Box_{BC} := \overline{\partial}^*\overline{\partial} + \partial^E(\partial^E)^* + (\partial^E)^*\overline{\partial}\overline{\partial}^*\partial^E.$$

It is easy to check that $[\Box_{BC}, L] = 0$. Thus $H^{\bullet,\bullet}_{BC}(X,E,L)$ satisfies the hard Lefschetz condition. For $H^{\bullet,\bullet}_{A}(X,E,L)$, we have

$$H^{\bullet,\bullet}_{A}(X,E,L) \simeq \ker \Box_{A}, \qquad \Box_{A} := \overline{\partial}\overline{\partial}^* + (\partial^E)^*\partial^E + \overline{\partial}^*\partial^E(\partial^E)^*\overline{\partial},$$

and $[\Box_A, L] = 0$, which implies that $H^{\bullet,\bullet}_{A}(X,E,L)$ satisfies the hard Lefschetz condition. □

**Remark 7.6.** *In general, we have the following natural maps*

$$H^{\bullet,\bullet}_{BC}(X,E,L) \to H^{\bullet,\bullet}_{\overline{\partial}}(X,E) \oplus H^{\bullet,\bullet}_{(\partial^E)^*}(X,E),$$

*and*

$$H^{\bullet,\bullet}_{\overline{\partial}}(X,E) \oplus H^{\bullet,\bullet}_{(\partial^E)^*}(X,E) \to H^{\bullet,\bullet}_{A}(X,E,L).$$

*The elliptic operator theory implies that both $H^{\bullet,\bullet}_{BC}(X,E,L)$ and $H^{\bullet,\bullet}_{A}(X,E,L)$ are finite dimensional. For instance, we have*

$$\ker \Box_A = \ker \triangle_A, \ \ \triangle_A := \Box_A + (\partial^E)^*\overline{\partial}^*\overline{\partial}\partial^E + (\partial^E)^*\overline{\partial}\overline{\partial}^*\partial^E + \overline{\partial}\partial^E(\partial^E)^*\overline{\partial}^*.$$

*It is easy to check that the principal symbol of $\triangle_A$ equals to that of $(\Box_{\overline{\partial}})^2$, which is elliptic. Thus $H^{\bullet,\bullet}_{A}(X,E,L)$ is finite dimensional.*

## 8. Complex surfaces

**Proposition 8.1.** *Let $(X, J, \omega, g)$ be a compact Hermitian manifold. Assume that*

$$[\overline{\partial}, \partial^*] = 0.$$

*Then*

$$\Box_d = \Box_{\overline{\partial}} + \Box_{\partial} = \Box_{d^c}.$$

*In particular, $\Box_d$ preserves the bi-degree, $\ker \Box_d$ is $J$-invariant and the first Betti number of $X$ is even.*

*Proof.* The conjugate of $[\overline{\partial}, \partial^*]$ is $[\partial, \overline{\partial}^*]$. Thus if $[\overline{\partial}, \partial^*] = 0$ then

$$[d, d^*] = [\overline{\partial} + \partial, \overline{\partial}^* + \partial^*] = [\overline{\partial}, \overline{\partial}^*] + [\partial, \partial^*].$$

Similar proof for $\Box_{d^c}$. $\Box_d$ preserves the bi-degree since $\Box_{\overline{\partial}}$ and $\Box_\partial$ do. The fact that $b_1(X)$ is even follows from

$$\mathcal{H}^1_{\Box_d} = \mathrm{Span}_{\mathbb{R}}\langle e_1, \cdots, e_j; Je_1, \cdots, Je_j\rangle.$$

□

**Corollary 8.2.** *Let $X$ be a compact complex surface. Then $X$ is Kähler if and only if there is a Hermitian metric on $X$ such that $[\overline{\partial}, \partial^*] = 0$.*

**Remark 8.3.** *The above proof also implies: if $[\overline{\partial}, \partial^*] = 0$ then*

$$\Box_{d^\lambda} = \Box_{\overline{\partial}} + |\lambda|^2 \Box_\partial,$$

*where $d^\lambda := \overline{\partial} + \lambda\partial$, $\lambda \in \mathbb{C}$.*

## 9. The $\overline{\partial}\,\overline{\partial}^{\Lambda}$-Lemma on special complex manifolds

9.1. **Taming and symmetric almost complex structures.** Recall the following definition (see Definition 2.10):

**Definition 9.1.** *Let $X$ be a $2n$-dimensional manifold with an almost symplectic form $\omega$. Let $J$ be an almost complex structure on $X$. Then $J$ is said to be* tamed *by $\omega$ if*

$$\omega(u, Ju) > 0,$$

*for every non-zero vector $u$; $J$ is said to be* symmetric *with respect to $\omega$, or $\omega$-*symmetric *if*

$$\omega(u, Jv) = \omega(v, Ju),$$

*for every vectors $u, v$. We call $J$ an $\omega$-*compatible *almost complex structure if it is both taming and symmetric.*

9.2. $\overline{\partial}\,\overline{\partial}^{\Lambda}$**-Lemma.** Let us define $\overline{\partial}^{\Lambda}$ first.

**Definition 9.2.** *Let $(X, J)$ be a complex manifold. Let $\omega$ be an almost symplectic form on $X$. Put*

$$\Lambda = *_s L *_s, \qquad L := \omega \wedge \cdot,$$

*where $*_s$ denotes the symplectic star operator. Set*

$$\overline{\partial}^{\Lambda} u := (-1)^{k+1} *_s \overline{\partial} *_s u,$$

*for every $k$-form $u$.*

Apply Theorem 3.6 to $(\oplus\Omega^k(X), \omega\wedge, \overline{\partial})$, we get

**Theorem 9.3.** *Let $(X, J)$ be a complex manifold with an almost symplectic form $\omega$. Then*

$$[\overline{\partial}^{\Lambda}, L] = \overline{\partial} + [\Lambda, [\overline{\partial}, L]], \ [\overline{\partial}, \Lambda] = \overline{\partial}^{\Lambda} + [[\Lambda, \overline{\partial}^{\Lambda}], L]. \tag{9.1}$$

As a consequence of the previous Theorem, we get the following

**Corollary 9.4.** *Let $(X, J)$ be a complex manifold with a symplectic form $\omega$. Assume that $J$ is symmetric with respect to $\omega$, then we have*

$$[\overline{\partial}^{\Lambda}, L] = \overline{\partial}, \qquad [\overline{\partial}, \overline{\partial}^{\Lambda}] = 0.$$

*Proof.* Since $J$ is symmetric with respect to $\omega$, we know that $\omega$ is degree $(1, 1)$. Thus $d\omega = 0$ is equivalent to $\overline{\partial}\omega = 0$. Hence $[\overline{\partial}, L] = 0$ and it is enough to apply the previous theorem. □

**Definition 9.5.** *Let $(X, J)$ be a complex manifold with a symplectic form $\omega$. Assume that $J$ is symmetric with respect to $\omega$, so that the differential operators $\overline{\partial}$ and $\overline{\partial}^{\Lambda}$ satisfy*

$$\overline{\partial}^2 = 0, \qquad (\overline{\partial}^{\Lambda})^2 = 0, \qquad \overline{\partial}\overline{\partial}^{\Lambda} + \overline{\partial}^{\Lambda}\overline{\partial} = 0.$$

*Then $X$ is said to satisfy the $\overline{\partial}\,\overline{\partial}^{\Lambda}$-*Lemma *if*

$$\ker \overline{\partial} \cap \ker \overline{\partial}^{\Lambda} \cap (\operatorname{Im} \overline{\partial} + \operatorname{Im} \overline{\partial}^{\Lambda}) = \operatorname{Im} \overline{\partial}\overline{\partial}^{\Lambda}.$$

**Remark**: $\overline{\partial}\,\overline{\partial}^{\Lambda}$-*Lemma* is a generalization of the $\overline{\partial}\partial^*$-Lemma on a compact Kähler manifold. In fact, since $\partial^* = -i\overline{\partial}^{\Lambda}$, we know that Corollary 7.4 is equivalent to:

**Theorem 9.6.** *Let $(X, J, \omega)$ be a compact Kähler manifold. Then $X$ satisfies the $\overline{\partial}\,\overline{\partial}^{\Lambda}$-Lemma.*

The following examples also suggest to study the $\overline{\partial}\,\overline{\partial}^{\Lambda}$-Lemma on non-Kähler manifolds.

9.3. **Nakamura manifold of completely solvable type.** We shall show that the Nakamura manifold of completely solvable type $M = \Gamma\backslash G$ in Example 2 satisfies the $\overline{\partial}\,\overline{\partial}^{\Lambda}$-Lemma. Consider the almost complex structure $J^M$ on $M$ defined by requiring that a co-frame for the space of complex $(1,0)$-forms is given by

$$\begin{cases} \varphi^1 := \frac{1}{2}(e^1 + ie^2), \\ \varphi^2 := (e^3 + ie^5), \\ \varphi^3 := (e^4 + ie^6). \end{cases} \tag{9.2}$$

We know that $J^M$ is integrable. Indeed,

$$\begin{cases} d\varphi^1 &= 0 \\ d\varphi^2 &= \varphi^1 \wedge \varphi^2 - \varphi^2 \wedge \overline{\varphi^1} \\ d\varphi^3 &= -\varphi^1 \wedge \varphi^3 + \varphi^3 \wedge \overline{\varphi^1} \end{cases}$$

and, consequently,

$$\begin{cases} \partial\varphi^1 &= 0 \\ \partial\varphi^2 &= \varphi^1 \wedge \varphi^2 \\ \partial\varphi^3 &= -\varphi^1 \wedge \varphi^3 \end{cases}$$

and

$$\begin{cases} \overline{\partial}\varphi^1 &= 0 \\ \overline{\partial}\varphi^2 &= -\varphi^2 \wedge \bar{\varphi}^1 \\ \overline{\partial}\varphi^3 &= \varphi^3 \wedge \bar{\varphi}^1 \end{cases}.$$

According to Kasuya (see section 5.1 (C) in [10]), the Lie group $G$ admits a lattice $\Gamma$ such that the Dolbeault cohomology is given by

$$\begin{aligned}
H^0_{\overline{\partial}}(M) &= Span_{\mathbb{C}}\langle 1\rangle\ , \\
H^1_{\overline{\partial}}(M) &= Span_{\mathbb{C}}\left\langle \varphi^1, \overline{\varphi^1}\right\rangle\ , \\
H^2_{\overline{\partial}}(M) &= Span_{\mathbb{C}}\left\langle \varphi^{23},\, \varphi^{1\bar{1}},\, \varphi^{2\bar{3}},\, \varphi^{3\bar{2}},\, \varphi^{\bar{2}\bar{3}}\right\rangle\ , \\
H^3_{\overline{\partial}}(M) &= Span_{\mathbb{C}}\left\langle \varphi^{123},\, \varphi^{12\bar{3}},\, \varphi^{13\bar{2}},\, \varphi^{23\bar{1}},\, \varphi^{1\bar{2}\bar{3}},\, \varphi^{2\bar{1}\bar{3}},\, \varphi^{3\bar{1}\bar{2}},\, \varphi^{\bar{1}\bar{2}\bar{3}}\right\rangle\ , \\
H^4_{\overline{\partial}}(M) &= Span_{\mathbb{C}}\left\langle \varphi^{1\bar{1}\bar{2}\bar{3}},\, \varphi^{23\bar{2}\bar{3}},\, \varphi^{13\bar{1}\bar{2}},\, \varphi^{12\bar{1}\bar{3}},\, \varphi^{123\bar{1}}\right\rangle\ , \\
H^5_{\overline{\partial}}(M) &= Span_{\mathbb{C}}\left\langle \varphi^{23\bar{1}\bar{2}\bar{3}},\, \varphi^{123\bar{2}\bar{3}}\right\rangle\ , \\
H^6_{\overline{\partial}}(M) &= Span_{\mathbb{C}}\left\langle \varphi^{123\bar{1}\bar{2}\bar{3}}\right\rangle\ ,
\end{aligned}$$

where $\varphi^{i\bar{j}} = \varphi^i \wedge \overline{\varphi^j}$ and so on. Consider the symplectic form $\omega$ on $M$ defined by (6.3), then we know that

$$\omega = 2i\varphi^1 \wedge \overline{\varphi^1} + \frac{1}{2}\varphi^2 \wedge \overline{\varphi^3} + \frac{1}{2}\overline{\varphi^2} \wedge \varphi^3.$$

In particular, we know that $J^M$ is $\omega$-symmetric. Then a direct computation shows that

$$\omega^k : H^{3-k}_{\overline{\partial}}(M) \to H^{3+k}_{\overline{\partial}}(M) \ \ \text{for}\, 0 \le k \le 3$$

is an isomorphism. Thus, by Theorem 3.12, it follows that $M$ satisfies the $\overline{\partial}\,\overline{\partial}^{\Lambda}$-Lemma.

9.4. **Holomorphically parallelizable Nakamura manifold.** On $\mathbb{C}^3$ with coordinates $(z_1, z_2, z_3)$ consider the following product $*$

$$(w_1, w_2, w_3) * (z_1, z_2, z_3) = (w_1 + z_1, e^{-w_1} z_2 + w_2, e^{w_1} z_3 + w_3).$$

Then $G = (\mathbb{C}^3, *)$ is a solvable Lie group, which is the semidirect product $\mathbb{C} \ltimes \mathbb{C}^2$, admitting a uniform discrete subgroup $\Gamma = \Gamma' \ltimes \Gamma''$, where $\Gamma' \subset \mathbb{C}$ is given by $\Gamma' = \lambda\mathbb{Z} \oplus i2\pi\mathbb{Z}$ and $\Gamma''$ is a lattice in $\mathbb{C}^2$; thus $N := \Gamma\backslash\mathbb{C}^3$ is a compact complex 3-dimensional manifold (see the appendix and [14, case (III)(b)]). It turns out that $h^{0,1}(N) = 3$ ([14, p.90]). It is immediate to check that

$$\psi^1 = dz_1, \qquad \psi^2 = e^{z_1} dz_2, \qquad \psi^3 = e^{-z_1} dz_3$$

are $G$-invariant holomorphic 1-forms on $\mathbb{C}^3$, so that they induce holomorphic 1-forms on $N$, namely $\{\psi^1, \psi^2, \psi^3\}$ is a global holomorphic co-frame on $N$ and the complex manifold $N$ is holomorphically parallelizable. We have

$$d\psi^1 = 0, \qquad d\psi^2 = \psi^1 \wedge \psi^2, \qquad d\psi^3 = -\psi^1 \wedge \psi^3.$$

By the construction of $N$, it follows that $e^{\frac{z_1 - \bar{z}_1}{2}}$ is a well-defined complex-valued smooth function on $N$. Let

$$\omega = \frac{i}{2}\psi^1 \wedge \overline{\psi^1} + \frac{1}{2} e^{z_1 - \bar{z}_1}\, \overline{\psi^2} \wedge \psi^3 + \frac{1}{2} e^{-z_1 + \bar{z}_1}\, \psi^2 \wedge \overline{\psi^3}.$$

Then

$$\overline{\omega} = \omega, \qquad \omega^3 = -\frac{3}{4}(i dz_1 \wedge d\bar{z}_1) \wedge (i dz_2 \wedge d\bar{z}_3) \wedge (i dz_3 \wedge d\bar{z}_3) < 0,$$

and explicitly,

$$\omega = \frac{i}{2} dz_1 \wedge d\overline{z_1} + \frac{1}{2} d\bar{z}_2 \wedge dz_3 + \frac{1}{2} d\bar{z}_3 \wedge dz_2\,,$$

so that $d\omega = 0$ and the complex structure on $N$ is $\omega$-symmetric. By noting that

$$e^{z_1 - \overline{z_1}}\, \overline{\psi^2} = e^{z_1} d\bar{z}, \qquad e^{-z_1 + \overline{z_1}}\, \overline{\psi^3} = e^{-z_1} d\bar{z}_3,$$

in view of [1, p. 86], we have

$$\begin{aligned}
H^0_{\overline{\partial}}(N) = \quad & Span_{\mathbb{C}} \langle 1 \rangle\ , \\
H^1_{\overline{\partial}}(N) = \quad & Span_{\mathbb{C}} \langle dz_1, e^{z_1} dz_2, e^{-z_1} dz_3, d\bar{z}_1, e^{z_1} d\bar{z}_2, e^{-z_1} d\bar{z}_3 \rangle\ , \\
H^2_{\overline{\partial}}(N) = \quad & Span_{\mathbb{C}} \big\langle e^{z_1} dz_{12}, e^{-z_1} dz_{13}, dz_{23}, dz_{1\bar{1}}, e^{z_1} dz_{1\bar{2}}, e^{-z_1} dz_{1\bar{3}}, e^{2z_1} dz_{2\bar{2}}, dz_{2\bar{3}}, e^{-z_1} dz_{3\bar{1}}, dz_{3\bar{2}}, \\
& e^{-2z_1} dz_{3\bar{3}}\, e^{z_1} dz_{\bar{1}\bar{2}}, e^{-z_1} dz_{\bar{1}\bar{3}}, dz_{\bar{2}\bar{3}} \big\rangle\ , \\
H^3_{\overline{\partial}}(N) = \quad & Span_{\mathbb{C}} \big\langle dz_{123}, e^{z_1} dz_{12\bar{1}}, e^{2z_1} dz_{12\bar{2}},\ dz_{12\bar{3}}, e^{-z_1} dz_{13\bar{1}}, dz_{13\bar{2}}, e^{-2z_1} dz_{13\bar{3}}, dz_{13\bar{3}}, dz_{23\bar{1}}, \\
& e^{z_1} dz_{23\bar{2}}, e^{-z_1} dz_{23\bar{3}}, dz_{3\bar{1}\bar{2}}, dz_{2\bar{1}\bar{3}}, e^{z_1} dz_{1\bar{1}\bar{2}}, e^{-z_1} dz_{1\bar{1}\bar{3}}, e^{2z_1} dz_{2\bar{1}\bar{2}}, e^{z_1} dz_{2\bar{2}\bar{3}}, \\
& e^{-2z_1} dz_{3\bar{1}\bar{3}}, e^{-z_1} dz_{3\bar{2}\bar{3}}, dz_{\bar{1}\bar{2}\bar{3}} \big\rangle\ , \\
H^4_{\overline{\partial}}(N) = \quad & Span_{\mathbb{C}} \big\langle dz_{123\bar{1}}, e^{z_1} dz_{123\bar{2}}, e^{-z_1} dz_{123\bar{3}}, e^{2z_1} dz_{12\bar{1}\bar{2}}, dz_{12\bar{1}\bar{3}}, e^{z_1} dz_{12\bar{2}\bar{3}}, dz_{13\bar{1}\bar{2}}, e^{-2z_1} dz_{13\bar{1}\bar{3}}, \\
& e^{-z_1} dz_{13\bar{2}\bar{3}}, e^{z_1} dz_{23\bar{1}\bar{2}}, e^{-z_1} dz_{23\bar{1}\bar{3}}, dz_{23\bar{2}\bar{3}}, dz_{1\bar{1}\bar{2}\bar{3}}, e^{z_1} dz_{2\bar{1}\bar{2}\bar{3}}, e^{-z_1} dz_{3\bar{1}\bar{2}\bar{3}} \big\rangle\ , \\
H^5_{\overline{\partial}}(N) = \quad & Span_{\mathbb{C}} \langle e^{z_1} dz_{123\bar{1}\bar{2}}, e^{-z_1} dz_{123\bar{1}\bar{3}}, dz_{123\bar{2}\bar{3}}, e^{z_1} dz_{12\bar{1}\bar{2}\bar{3}}, e^{-z_1} dz_{13\bar{1}\bar{2}\bar{3}}, dz_{23\bar{1}\bar{2}\bar{3}} \rangle\ , \\
H^6_{\overline{\partial}}(N) = \quad & Span_{\mathbb{C}} \langle dz_{123\bar{1}\bar{2}\bar{3}} \rangle\ ,
\end{aligned}$$

where $dz_{h\bar{k}} = dz_h \wedge d\bar{z}_k$ and so on. A straightforward computation shows that

$$\omega^k : H^{3-k}_{\overline{\partial}}(N) \to H^{3+k}_{\overline{\partial}}(N) \quad \text{for } 0 \le k \le 3$$

is an isomorphism. Thus, by Theorem 3.12, it follows that $N$ satisfies the $\overline{\partial}\,\overline{\partial}^{\Lambda}$-Lemma.

**Remark 9.7.** *It has to be noted that the Nakamura manifold of Example 2 satisfies the $\partial\overline{\partial}$-Lemma (see* [1]*). Indeed, from the table (*[10, Section 5.1 (C)]*) recalled in subsection 9.3,*

$$H^k(M; \mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}_{\overline{\partial}}(M) \qquad H^{q,p}_{\overline{\partial}}(M) =\simeq \overline{H^{p,q}_{\overline{\partial}}(M)}$$

*so that $M$ satisfies the $\partial\overline{\partial}$-Lemma. On the contrary, the holomorphically parallelisable Nakamura manifold $N$ does not satisfy the $\partial\overline{\partial}$-Lemma.*

9.5. **Kodaira-Thurston manifold.** Similar computations can be performed also for the Kodaira-Thurston manifold. Consider the almost complex structure $J^M$ on $M$ assigning a complex co-frame of $(1,0)$-forms by

$$\begin{cases} \varphi^1 := (e^1 + ie^2), \\ \varphi^2 := (e^3 + ie^4). \end{cases} \tag{9.3}$$

Then

$$\begin{cases} d\varphi^1 = 0, \\ d\varphi^2 = -\frac{i}{2}\varphi^1 \wedge \overline{\varphi^1}. \end{cases} \tag{9.4}$$

Thus $J^M$ is integrable. Consider the symplectic form $\omega$ on $M$ defined by

$$\omega = \frac{1}{2}(\varphi^1 \wedge \overline{\varphi^2} + \overline{\varphi^1} \wedge \varphi^2);$$

then $J^M$ is $\omega$-symplectic.

We can prove that $(M, J^M, \omega)$ does not satisfy the $\overline{\partial}\,\overline{\partial}^{\Lambda}$-Lemma. In fact, assume that it satisfies the $\overline{\partial}\,\overline{\partial}^{\Lambda}$-Lemma. Then Theorem 3.12 implies that $H_{\overline{\partial}}$ satisfies the Hard-Lefschetz property. Notice that

$$[\omega \wedge \overline{\varphi^1}]_{\overline{\partial}} = \frac{1}{2}[\varphi^1 \wedge \overline{\varphi^2} \wedge \overline{\varphi^1}]_{\overline{\partial}}.$$

Since $\varphi^1 \wedge \overline{\varphi^1}$ is $\overline{\partial}$-exact and $\overline{\varphi^2}$ is $\overline{\partial}$-closed, we know that $[\omega \wedge \overline{\varphi^1}]_{\overline{\partial}} = 0$. Thus the Hard-Lefschetz property implies $[\overline{\varphi^1}]_{\overline{\partial}} = 0$, which gives (use $\overline{\partial}(\varphi^1 \wedge \overline{\varphi^2} \wedge \varphi^2) = 0$) the following contradiction

$$2[\omega^2]_{\overline{\partial}} = [\varphi^1 \wedge \overline{\varphi^2} \wedge \overline{\varphi^1} \wedge \varphi^2]_{\overline{\partial}} = 0.$$

Thus $(M, J^M, \omega)$ does not satisfy the $\overline{\partial}\,\overline{\partial}^{\Lambda}$-Lemma.

9.6. $\overline{\partial}_E\,\overline{\partial}_E^{\Lambda}$**-Lemma.**

**Definition 9.8.** *Let $(X, J)$ be a complex manifold. Let $E$ be a holomorphic vector bundle on $X$. We shall write the $\overline{\partial}$-operator on $E$ as $\overline{\partial}_E$. Let $\omega$ be an almost symplectic form on $X$. Put*

$$\Lambda = *_s^{-1} L *_s, \qquad L := \omega \wedge \cdot,$$

*where $*_s$ denotes the symplectic star operator. We shall define the* symplectic adjoint *of $\overline{\partial}_E$ as*

$$\overline{\partial}_E^{\Lambda} u := (-1)^{k+1} *_s \overline{\partial}_E *_s u,$$

*where $u$ is a smooth $E$-valued $k$-form.*

Theorem 3.12 implies:

**Theorem 9.9.** *Let $(X, J)$ be a complex manifold with an almost symplectic form $\omega$. Let $E$ be a holomorphic vector bundle on $X$. Then*

$$[\overline{\partial}_E^{\Lambda}, L] = \overline{\partial}_E + [\Lambda, [\overline{\partial}_E, L]], \; [\overline{\partial}_E, \Lambda] = \overline{\partial}_E^{\Lambda} + [[\Lambda, \overline{\partial}_E^{\Lambda}], L]. \tag{9.5}$$

As a consequence of the previous Theorem, we get the following

**Corollary 9.10.** *Let $(X, J)$ be a complex manifold with a symplectic form $\omega$. Let $E$ be a holomorphic vector bundle on $X$. Assume that $J$ is symmetric with respect to $\omega$, then we have*

$$[\overline{\partial}_E^{\Lambda}, L] = \overline{\partial}_E, \qquad [\overline{\partial}_E, \overline{\partial}_E^{\Lambda}] = 0.$$

*Proof.* Since $J$ is symmetric with respect to $\omega$, we know that $\omega$ is degree $(1,1)$. Thus $d\omega = 0$ is equivalent to $\overline{\partial}\omega = 0$. Hence $[\overline{\partial}_E, L] = 0$ and it is enough to apply the previous theorem. $\square$

**Remark**: Consider the complex $(\Omega^\bullet(X,E), \overline{\partial}_E, \overline{\partial}_E^\Lambda)$ of smooth $E$-valued $(k)$-forms on $X$, where the differential operators $\overline{\partial}_E$ and $\overline{\partial}_E^\Lambda$ satisfy

$$\overline{\partial}_E^2 = 0, \qquad (\overline{\partial}_E^\Lambda)^2 = 0, \qquad \overline{\partial}_E \overline{\partial}_E^\Lambda + \overline{\partial}_E^\Lambda \overline{\partial}_E = 0,$$

**Definition 9.11.** *Let $(X,J)$ be a complex manifold with a symplectic form $\omega$. Let $E$ be a holomorphic vector bundle on $X$. Assume that $J$ is symmetric with respect to $\omega$. Then $X$ is said to satisfy the $\overline{\partial}_E\,\overline{\partial}_E^\Lambda$-*Lemma *if*

$$\ker \overline{\partial}_E \cap \ker \overline{\partial}_E^\Lambda \cap (\operatorname{Im} \overline{\partial}_E + \operatorname{Im} \overline{\partial}_E^\Lambda) = \operatorname{Im} \overline{\partial}_E \overline{\partial}_E^\Lambda.$$

Theorem 3.12 implies

**Theorem 9.12.** *Let $(X,J)$ be a complex manifold with a symplectic form $\omega$. Let $E$ be a holomorphic vector bundle on $X$. Assume that $J$ is symmetric with respect to $\omega$. Then $X$ satisfies the $\overline{\partial}_E\,\overline{\partial}_E^\Lambda$-*Lemma *if and only if its Dolbeault cohomology $H_{\overline{\partial}_E}$ satisfies the Hard Lefschetz Condition.*

Thus by the Hodge theory, we get

**Theorem 9.13.** *Let $(X,J,\omega)$ be a compact Kähler manifold. Let $E$ be a holomorphic vector bundle on $X$. Assume that there exists a smooth Hermitian metric $h_E$ on $E$ such that the Chern curvature $\Theta(E,h_E) \equiv 0$. Then $X$ satisfies the $\overline{\partial}_E\,\overline{\partial}_E^\Lambda$-*Lemma.

## 10. Appendix

We briefly recall the construction of completely solvable Nakamura manifolds (see e.g., [14], [5] and [3]). Let $A \in SL(2,\mathbb{Z})$ have two real positive distinct eigenvalues

$$\mu_1 = e^\lambda, \quad \mu_2 = e^{-\lambda}.$$

Set

$$\Lambda = \begin{pmatrix} e^{-\lambda} & 0 \\ 0 & e^\lambda \end{pmatrix}$$

and let $P \in M_{2,2}(\mathbb{R})$ be such that

$$\Lambda = PAP^{-1}$$

Consider $\Gamma := P\mathbb{Z}^2 + iP\mathbb{Z}^2$; then $\Gamma$ is a lattice in $\mathbb{C}^2$. Let $\mathbb{T}^2_{\mathbb{C}} = \mathbb{C}^2/\Gamma$ be a 2-dimensional complex torus.

Then the map

$$\Phi : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$$

$$\Phi(z) = \Lambda z, \quad \text{where} \quad z = (z^1, z^2)^t,$$

induces a biolomorphism of $\mathbb{T}^2_{\mathbb{C}}$ by setting $\tilde{\Phi}([z]) = [\Phi(z)]$.

First of all, $\tilde{\Phi}$ is well-defined, since if $z'$ and $z$ are equivalent, i.e., $z' = z + P(\gamma_1 + i\gamma_2)$, with $\gamma_1$, $\gamma_2 \in \mathbb{Z}^2$, then

$$\begin{aligned}
\Phi(z') &= \Lambda z' = \Lambda z + \Lambda P(\gamma_1 + i\gamma_2) \\
&= \Lambda z + PAP^{-1}P(\gamma_1 + i\gamma_2) \\
&= \Lambda z + PA(\gamma_1 + i\gamma_2) \\
&= \Lambda z + P(\lambda_1 + i\lambda_2) \\
&= \Phi(z) + P(\lambda_1 + i\lambda_2) \quad \text{with} \quad \lambda_1, \lambda_2 \in \mathbb{Z}^2,
\end{aligned}$$

so that $\Phi(z') \sim \Phi(z)$. Furthermore $\tilde{\Phi}^{-1}([z]) = [\Phi^{-1}(z)]$.
We identify $\mathbb{R} \times \mathbb{C}^2$ with $\mathbb{R}^5$ by $(s, z^1, z^2) \longmapsto (s, x^1, x^2, x^3, x^4)$, where $z^1 = x^1 + ix^3$, $z^2 = x^2 + ix^4$,

and consider

$$T_1 : \mathbb{R}^5 \longrightarrow \mathbb{R}^5$$
$$T_1(s, x^1, x^2, x^3, x^4) = (s + \lambda, e^{-\lambda}x^1, e^{\lambda}x^2, e^{-\lambda}x^3, e^{\lambda}x^4),$$

then

$$T_1(s, x^1, x^2, x^3, x^4) = T_1(s, z^1, z^2) = (s + \lambda, \Phi(z^1, z^2)) .$$

Hence $T_1$ induces a transformation of $\mathbb{R} \times \mathbb{T}^2_{\mathbb{C}}$, by setting

$$T_1(s, [(z^1, z^2)]) = (s + \lambda, [\Phi(z^1, z^2)]).$$

Define

$$M := \frac{\mathbb{R}}{b\mathbb{Z}} \times \frac{\mathbb{R} \times \mathbb{T}^2_{\mathbb{C}}}{< T_1 >}.$$

Then, we obtain a family of compact 6-dimensional solvmanifold of completely solvable type $M$, called *Nakamura manifolds.*

We give a numerical example. Let

$$A = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}$$

$A \in SL(2, \mathbb{Z})$. Then $\mu_{1,2} = \frac{3 \pm \sqrt{5}}{2}$. We set

$$\mu_1 = \frac{3 - \sqrt{5}}{2} = e^{-\lambda} \quad \text{and} \quad \mu_2 = \frac{3 + \sqrt{5}}{2} = e^{\lambda},$$

i.e., $\lambda = \log(\frac{3+\sqrt{5}}{2})$. Then

$$P^{-1} = \begin{pmatrix} \frac{3-\sqrt{5}}{2} & \frac{3+\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix},$$

and

$$P = \begin{pmatrix} 1 & -\frac{3+\sqrt{5}}{2} \\ -1 & \frac{3-\sqrt{5}}{2} \end{pmatrix}$$

and the lattice $\Gamma$ is given by

$$\Gamma = Span_{\mathbb{Z}} < \begin{bmatrix} -\frac{\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{5+3\sqrt{5}}{10} \\ \frac{5-3\sqrt{5}}{10} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \frac{-\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \frac{5+3\sqrt{5}}{10} \\ \frac{5-3\sqrt{5}}{10} \end{bmatrix} > .$$

**Remark 10.1.** *Note that, according to Kasuya* [10], *the Dolbeault cohomology of* $(M, J^M)$ *depends on* $b$. *In particular, if* $b \neq n\pi$, *then the Dolbeault groups have been computed in* [10, Section 5.1 (C)], *see Section 6, Example 2 for the list of the explicit representatives.*

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Dipartimento di Scienze Matematiche, Fisiche ed Informatiche Unità di Matematica e Informatica, Università di Parma, Parco Area delle Scienze 53/A, 43124, Parma, Italy
*E-mail address*: `adriano.tomassini@unipr.it`

Department of Mathematical Sciences, Norwegian University of Science and Technology, NO-7491 Trondheim, Norway
*E-mail address*: `xu.wang@ntnu.no`