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Abstract

In this thesis, the risk measures expected shortfall (ES), potential future exposure (PFE) and expected positive exposure (EPE) are studied in the context of counterparty exposure for a pay floating –receive fixed swap contract. The Heath-Jarrow-Morton framework for modelling interest rates is used to generate future market scenarios by Monte Carlo simulation. Further, the simulated future interest rates are used to price an interest rate swap at every simulated time step in the lifetime of the swap. Finally, the collection of simulated swap prices is used to generate values for the counterparty exposure, represented by the risk measures ES, PFE, and EPE.

The performance of the measures were tested during different periods between 2005 and 2019, with interest rate data going back to 2002. Results show that the ES measure performs better than PFE, but none of them are fully able to capture the actual exposure during periods of market stress, such as the financial crisis of 2008.

A shift toward ES as the primary risk measure for swaps is discussed from a regulatory standpoint, thus following the trend seen for other asset classes in the aftermath of the financial crisis.

Sammendrag

I denne oppgaven har risikomålene Expected Shortfall (ES), Potential Future Exposure (PFE) og Expected Positive Exposure (EPE) blitt studert for å undersøke motpartseksponering for en rentebytteavtale hvor flytende rente betales og en fast rente mottas. Heath-Jarrow-Morton-rammeverket for rentemodellering har blitt brukt for å generere fremtidige scenarioer for markedsutvikling ved bruk av Monte Carlo-simulering. Videre har de simulerte fremtidige rentene blitt brukt til å verdsette rentebytteavtaler ved hvert simulerte tidssteg innenfor rentebytteavtalens levetid. Til slutt er samlingen av de simulerte prisene på rentebytteavtalene brukt til å generere verdier for motpartseksponeringen, representert ved riskikomålene ES, PFE og EPE.

Risikomålene har blitt vurdert ut i fra hvilken grad de har evnet å forutse motpartseksponeringen som ville ha oppstått ved reelle rentebytteavtaler i forskjellige perioder mellom 2005 og 2019. Resultatene viser at ES presterer bedre enn PFE, men ingen av dem klarer i tilstrekkelig grad å forutse den virkelige eksponeringen i perioder hvor finansmarkedene er presset, slik som under finanskrisen i 2008.

Et skifte mot ES som det primære risikomålet for rentebytteavtaler har blitt diskutert fra et regulatorisk ståsted, slik at det dermed også følger trenden for andre aktivaklasser i etterkant av finanskrisen.

Preface

This thesis concludes the M.Sc program in Physics and Mathematics at the Norwegian University of Science and Technology (NTNU). The work was carried out at the Department of Mathematical Sciences during the first half of 2019.

I would like to thank my supervisor Jacob Laading for contributing with insightful feedback for this thesis patience throughout the semester. I would also like to thank DNB for supplying the interest rate data that has been used for the analysis. Finally, I want to express my warmest gratitude to Angela Maiken Johnsen for proofreading.

Eirik L. Vestgren,
Trondheim, July 2019

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Chapter 1

Introduction

The global markets for financial derivatives totalled a notional amount of more than 595 trillion (10^{12}) USD at the end of June 2018 [1]. Comparing this to the market capitalization of publicly traded companies globally, which totals approximately 80 trillion USD [2], it is clear that the derivatives markets are huge, even though it is worth noting that market value of all outstanding derivatives are currently estimated to be around 10 trillion USD [1]. In any case, it is evident that the consequences of losses in the derivatives markets have the potential to cause serious damage to the whole financial system.

During the financial crisis of 2008, counterparty risk turned out to be a major issue for many actors in the financial markets. Several of the largest banks were unable to fulfill their obligations agreed to in derivative contracts, causing the collapse of the investment bank Lehman Brothers and the U.S. government rescuing several others to avoid a collapse of the entire financial system. Since counterparty risk had suddenly become such a major issue, hedged positions involving more than one counterparty now carried significant risk since a default would lead to the position being unhedged.

In the wake of these events, regulatory actions were taken to reduce the systemic risk caused by over-the counter (OTC) derivative instruments [3]. These regulations demanded that standardized OTC derivatives be cleared through a central counterparty (CCP) as a general rule. This results in both parties of the contract having the CCP as their only counterparty. Since CCPs are only concerned with handling such risks, this is considered adding robustness to the system. New regulations were also imposed to the trading books of banks, being closer to the regulations that were already existing for the banking books. These regulations marked

[1] This value may change drastically in event of market stress.

a shift towards using the risk measure Expected Shortfall (ES) instead of the up-until-then benchmark Value at Risk (VaR), because of significant shortcomings in the latter to capture tail risk [4]. A similar move when in terms of counterparty risk is discussed in this thesis.

Modelling counterparty risk is fairly complex, requiring two things; frameworks for fair and consistent pricing of derivative contracts given the current information available in the markets and the generation of possible future market scenarios and changes in factors affecting the counterparties' ability to fulfill their obligations. In this thesis, a framework combining these tasks to estimate counterparty risk has been implemented. The main focus of the thesis is on aspects considering such a model.

To model the evolution of interest rates, the framework introduced in 1992 by David Heath, Robert A. Jarrow and Andrew Morton in their article "Bond Pricing and the Term Structure of Interest Rates: A New Methodology" [5] has been used. This model marked a revolution within interest rate modelling, incorporating the full term structure of interest rates as opposed to the existing models at the time, which only had one or a few sources of randomness.

The interest rates modelled by the Heath-Jarrow-Morton model is used to model counterparty exposure for swap contracts. The model has been tested during different periods after 2002 to discover its strengths and shortcomings of the model. Since all models are simplified representations of the real world, they only capture a certain perception of the real world [6]. Thus, following the model in all situations may lead to significant errors during situations where the model is unable to represent the real world accurately. Consequently, knowing the limitations of models is as crucial as knowing their strengths.

The contract known as interest rate swaps are studied within the implemented framework. This is the most widespread interest rate derivative, accounting for almost 80 % of the total market for interest rate derivatives totalling 326 trillion USD in notional amounts [1]. Acting mainly as insurance instruments, protecting against unfavorable movements in the interest rate markets, swaps are used by companies in all industries to provide increased financial visibility and stability.

The thesis is constructed in the following way. Chapter 2 introduces basic financial assumptions and mathematical concepts that are integral for the models in this thesis. This chapter also introduces some general techniques that are important parts of the models implemented for this thesis.

Chapter 3 provides an introduction to financial derivatives and the pricing of such instruments. Concepts such as arbitrage and risk neutrality are discussed and put into the context of derivative pricing.

Further, in chapter 4, interest rate modelling is introduced. The chapter starts off by providing some background on stochastic interest rate modelling, and explaining the development of these models. The chapter touches into a few important concepts in interest rate modelling and some basic interest rate models. Finally, the Heath-Jarrow-Morton model for interest rates is introduced and described in detail.

Chapter 5 considers financial risk, focusing on counterparty risk. The chapter first defines the concept of counterparty risk and outlines important concepts of credit risk. Then, some risk measures are introduced, before regulations and ways to manage and mitigate risk is discussed.

Chapter 6 describes the data used for the analysis, and describes the methods in more detail. The results are then presented and discussed in chapter 7, before some concluding remarks are finally presented in chapter 8.

Chapter 2

Theory

2.1 Financial Preliminaries

In the theory of mathematical finance, certain conditions about the nature of the market are assumed to hold. The pure financial assumptions are presented below.

Absence of Arbitrage

The assumption of an arbitrage-free market is central to the theory within mathematical finance. Arbitrage can be defined in various ways, in more or lesser mathematical terms, and the definition below is in purely financial terms.

Definition 2.1 (Arbitrage)**.** *Arbitrage is a strategy yielding an instantaneous profit above the risk-free rate with carrying zero risk of loss, or equivalently, with an initial investment of zero.*

Informally, the no-arbitrage theorem is often stated as ”there is no free lunch”. A more detailed treatment of arbitrage, with more mathematical focus is presented in chapter 3.

Efficient Markets

Markets are assumed to be efficient, meaning that all asset prices are correct given the information available in the market at any given time. Prices are assumed to react instantaneously to news in the market carrying information that could move prices. This assumption is closely related to the no-arbitrage assumption.

Time Value of Money

Financial theory assumes that a unit of money received at the current time t is worth more than a unit of money received at time $T > t$. A heuristic argument for this assumption is that the money received today can be invested in something yielding a positive return at a later time, giving back the initial investment in addition to the return. At least, in any case, just keeping the money has provided some optionality in the meantime, carrying some value.

This assumption leads to the assumption of the existence of a risk-free asset yielding positive return. Such an asset will be referred to as the numéraire for the remainder of this thesis.

Other Assumptions

Some other assumptions are also required to ensure consistency in the theory. These are easily interpretable and will only be stated without further discussion.

- Markets are liquid, meaning that assets can be sold at any time to market price.
- Financial assets are divisible and can be traded in fractions.
- Transactions are small compared to market depth, and does not move market prices.
- There are no transaction costs.

2.2 Mathematical Preliminaries

Much of the theory on financial modelling and derivative pricing is based on elementary results from measure theory and stochastic calculus. Readers unfamiliar with these subjects are referred to the appendix, respectively sections A and B, for a treatise on some elementary concepts relevant for the topics presented in this thesis.

The market is assumed to be a probability space $(\Omega, \mathcal{F}, P)$. Here Ω is the sample space, representing all the potential outcomes in the market, and $\mathcal{F}$ is a σ-algebra representing measurable events. Measurable events are also often referred to as random variables. The market is enabled with the filtration $\mathbb{F} = \{\mathcal{F}_t\},\ t \in [0, T]$, with T being the time horizon in which the model operates. P is a probability measure, assigning a non-negative probability to each outcome ω in the countable set of potential outcomes, Ω. The market model also assumes the existence of a non-dividend paying numéraire asset defined in the following manner.

Definition 2.2 (Numéraire)**.** *A numéraire is a price process* $(\beta(t)), t \in [0, T]$, *which is almost surely strictly positive for all* t.

The numéraire is a price process acting as a discount factor or deflation asset for other assets. Future cash flows are discounted by the numéraire, since this asset represents the risk free rate. The existence of a numéraire ensures that asset prices be martingales when discounted, which is a desirable property. Further, an important notion considering numéraires, is the concept of equivalent martingale measures.

Definition 2.3 (Equivalent Martingale Measure)**.** *A probability measure* $\mathbb{P}^*$ *on* $(\Omega, \mathcal{F}_T)$ *being equivalent to* $\mathbb{P}$ *is called an (equivalent) martingale measure for a price process* $\tilde{S}$ *if* $\tilde{S}$ *follows a* $\mathcal{P}^*$*-martingale with respect to the filtration* $\mathbb{F}$.

2.2.1 Change of Measure and Girsanov's Theorem

The technique concerning change of measures is especially useful in derivative pricing, enabling to change into a risk-neutral measure instead of the real-world measure. Risk-neutrality will be properly introduced later in chapter 3.

Theorem 2.1 (Girsanov)**.** *Let* $X(t)$ *be a stochastic process in the interval* $[0, T]$ *and the standard probability space and let* $\mathbb{P}$ *and* $\mathbb{Q}$ *be two measures with Radon-Nikodým derivative* $d\mathbb{Q}/d\mathbb{P}$.

Consider the k*-dimensional standard Brownian motion* $W(t), t \in [0, T]$. *Let* γ *be an* $\mathbb{R}^k$*-valued process adapted to* $\{\mathcal{F}_t^W\}$, *satisfying*

$$\int_0^t ||\gamma(u)||^2 du < \infty$$

almost surely for all t. *Also, let*

$$X(t) = \exp\left(-\frac{1}{2}\int_0^t ||\gamma(u)||^2 du + \int_0^t \gamma(u) dW(u)\right). \tag{2.1}$$

If $E_{\mathbb{P}}[X(T)] = 1$, *then* $\{X(t), t \in [0, T]\}$ *is a martingale and the measure* $\mathbb{Q}$ *defined*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = X(T)$$

is equivalent to $\mathbb{P}$. *Under this measure* $\mathbb{Q}$, *the process*

$$W^{\mathbb{Q}}(t) \equiv W(t) - \int_0^t \gamma(u) du, \quad t \in [0, T] \tag{2.2}$$

is a standard Brownian motion with respect to $\{\mathcal{F}_t^W\}$.

Girsanov's theorem makes sure that when the measures change, the volatility remains the same through this change [7]. In derivative pricing this has great importance, since prices are dependent on volatility of the underlying instrument. Derivative pricing is discussed in chapter 3 where the importance of Girsanov's theorem will be made even clearer.

2.3 Monte Carlo Simulation

A popular method for simulation for problems involving many dimensions, is Monte Carlo simulation. The method generates a given number, n, different scenarios, representing n realizations of the real-world evolution of some system. Monte Carlo simulation is based on random number generation, and it is assumed that there exists a method to generate random numbers [2]. In finance, Monte Carlo simulation is especially efficient for simulation when dealing with path-dependent contracts, as these in reality are problems of very high dimensionality.

The principles of Monte Carlo simulation are quite simple, and is easily illustrated by the integral

$$\alpha = \int_0^1 f(x)dx.$$

Now, consider this integral as an expectation, such that $\alpha = \mathrm{E}[f(U)]$, where $U \sim \mathrm{unif}(0,1)$. Assuming the existence of a method to generate independent realizations from this distribution, $U_1, \ldots, U_n$, evaluating the function f at each U_i leads to the Monte Carlo estimate for α. This estimate is denoted $\hat{\alpha}$ and is given by

$$\hat{\alpha} = \frac{1}{n}\sum_{i=1}^{n} f(U_i).$$

If f is indeed integrable on the interval $[0,1]$, the strong law of large numbers ensures that

$$\hat{\alpha} \to \alpha \quad \text{as} \quad n \to \infty$$

with probability 1.

It is evident that for the purpose of evaluating simple integrals such as the one above, Monte Carlo is an inefficient method. Even for more complicated functions f, there exist much more efficient numerical methods. The rate of convergence of the Monte Carlo method is $\mathcal{O}(n^{-1/2})$. The benefits of Monte Carlo are, however, apparent when the dimensionality of the problem increases, since the convergence rate is independent of the number of dimensions [7].

[2] True random number generation is not possible with current computers, as they produce deterministic sequences of numbers based on a seed. However, pseudo-random numbers generated by R or other software, can be considered truly random for all practical purposes.

Monte Carlo Error Estimation

Consider, as before, the function f, which is supposed to be square integrable. Now, define

$$\sigma_f^2 = \int_0^1 (f(x) - \alpha)^2 dx.$$

The error in the Monte Carlo estimate, $\hat{\alpha} - \alpha$, is normally distributed with mean 0 and standard deviation $\sigma_f / \sqrt{n}$. As the true value of α in the general case is unknown, meaning that σ_f is also unknown. However, an estimate can be provided by the standard error of the sample, given by

$$s_f = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (f(U_i) - \hat{\alpha}_n)^2}.$$

This estimator is unbiased for σ_f, and it is easily verified that the precision of the Monte Carlo estimate indeed converges by a factor of $1/n^{1/2}$. It is also observed that the number of dimensions is not a parameter in any of the estimates, and the convergence rate is indeed independent of dimensionality.

2.4 Principal Component Analysis

Principal component analysis (PCA) is a well-known technique for reduction of dimensionality of data. The technique consists of transforming a set of dependent, correlated variables into independent uncorrelated ones. For each new variable, called the *principal components*, the goal is to describe as much variance as possible not explained by the previous principal components. The derivation in this section is mostly based on [8].

To derive the theory on PCA, consider a matrix containing the observed data. Let this matrix be denoted $\mathbf{X}$, with dimension $n \times p$, with n being the number of observations and p being the number of random variables. The goal now, is to construct a new vector $\mathbf{a}_1$ such that $\mathbf{a}_1^\top \mathbf{X}$ has the maximum variance. Then, construct another vector $\mathbf{a}_2$, orthogonal to $\mathbf{a}_1$, where $\mathbf{a}_2^\top \mathbf{X}$ has the maximum variance. Continue in this manner until a sufficient amount of the variance has been explained, or until the desired number of principal components has been found.

Further, let Σ be the covariance matrix of $\mathbf{X}$. Now, for each $k = 1, \ldots, p$, the k'th principal component is determined by the eigenvector $\mathbf{z}_k$ corresponding to the k'th largest eigenvalue of Σ, which will be denoted λ_k.

Having found the desired principal components, it is natural to ask how much of the variance in the data that could be explained by each principal component.

Let q_i denote the fraction of the total variance explained by the t'th principal component. This fraction is given by the formula

$$q_i = \frac{\lambda_i}{\sum_{\ell=1}^{p} \lambda_\ell}. \tag{2.3}$$

There is much more that could be said about PCA, but a full treatment on the intricacies of PCA is not the scope of this thesis. Since further theory about the calculations in detail is not necessary to understand the applications of PCA in this thesis, it is not included either. A more detailed treatment can be found in [8].

Chapter 3

Derivative Pricing

Pricing derivatives is an important issue in mathematical finance, initially introduced by Fisher Black and Myron Scholes in their 1973 article ”The Pricing of Options and Corporate Liabilities” [9]. This article is the benchmark of derivative pricing, which the field of derivative pricing is based on. A different approach to derivative pricing, called arbitrage pricing, is presented in this chapter, but the same results are ultimately arrived upon.

3.1 Outlining Derivative Pricing

First, before presenting the theory behind derivative pricing, a precise definition of a financial derivative might be useful for readers unfamiliar with the concept.

Definition 3.1 (Derivative)**.** *A derivative, also commonly referred to as contingent claim, is a financial contract whose payoff depends entirely on the value*[3] *of another financial asset, called the underlying, at the time of expiry,* T*.*

The underlying instrument may be a stock, an interest rate, a commodity or even the occurrence of events such as hurricanes or earthquakes [10]. Derivatives in this thesis, however, are linked to interest rates. The specific contract that is used for the analysis in this thesis, is called a *swap* contract.

Definition 3.2 (Swap)**.** *A swap contract is a financial contract between two parties, in which they agree to switch cash flows at certain predetermined future dates until the maturity date,* T*.*

[3]Here, the value may also include the path taken between the time of agreement and expiry of the contract.

Being the most common interest rate derivative, swaps have an integral position in the corporate world, for companies in all industries, not only in the financial sector. In addition to swaps, many other derivatives exist, each of them having their own properties and payoff structures, requiring them to be priced in a specific way. In principle, derivatives can be constructed off any view that anyone may have about the markets.

3.2 Arbitrage Pricing

The concept of arbitrage, defined in chapter 2, is a cornerstone when pricing contingent claims. The no-arbitrage condition is assumed to hold, and the market is the same as defined earlier, containing $d + 1$ assets, including the numéraire. This means that d assets carry risk. Now, define a trading strategy

$$\varphi(t) = (\varphi_0, \varphi_1, \ldots, \varphi_d), \quad t \in [0, T].$$

Here, each φ_i represent the holdings of one asset in the market, whose weights may be negative. It is assumed that the expectations of these functions exist. At time t the value of the trading portfolio is given by

$$V_\varphi(t) = \varphi(t) \cdot S(t), \quad t \in [0, T],$$

where S is the price processes of the d assets. V_φ is called the wealth process of φ. Further, the earnings of the trading strategy in the interval $[0, t]$ is given by

$$G_\varphi(t) = \int_0^t \varphi(u) dS(u),$$

called the gains process of φ. The trading strategy is said to be self-financing if $V_\varphi(t)$ satisfies

$$V_\varphi(t) = V_\varphi(0) + G_\varphi(t) \; \forall t \in [0, T].$$

Now, it is possible to define an arbitrage opportunity in new terms.

Definition 3.3 (Arbitrage Opportunity)**.** *A self-financing trade strategy is an arbitrage opportunity if*

$$V_\varphi(0) = 0, \quad P(V_\varphi(T) \geqslant 0) = 1, \quad \text{and} \quad P(V_\varphi(T) > 0) > 0.$$

Conversely, if the market allows for no arbitrage opportunities, as is assumed, then the following is true.

Theorem 3.1 (No Arbitrage)**.** *If the market is arbitrage-free, equivalent martingale measures $\mathbb{P}^*$ exist.*

This theorem has the further implication for the prices of assets in the arbitrage-free market that the discounted asset prices are martingales under the equivalent martingale measure $\mathbb{P}^*$.

3.3 Risk-Neutral Pricing

Risk neutrality is a central concept in derivative pricing theory, which, to many may seem contrary to classical finance where an investor requires additional compensation for taking on extra risk. A risk-neutral investor does not have such a preference, and thus disregards the volatility of the asset from the equation. Since all inherent risk can be hedged away for most derivatives, the level of risk is irrelevant, and the absence of arbitrage also makes it clear that the expected return will be that of the numéraire.

Now, let a contingent claim be denoted by X. If there exists a trading strategy φ whose value process $V_\varphi(T) = X$, the claim is said to be attainable. If a contingent claim is attainable, then arbitrage considerations make it clear that the price of the contingent claim must be equal to the price of the replicating strategy. A consequence of this is stated in the theorem below. A proof is found in [11].

Theorem 3.2. *Given an arbitrage-free market, $\mathcal{M}$. Then, any attainable contingent claim X is uniquely replicated in $\mathcal{M}$.*

This property leads to an important definition in arbitrage pricing, called an arbitrage price process.

Definition 3.4 (Arbitrage Price Process)**.** *Let X be an attainable contingent claim in an arbitrage-free market. Then the arbitrage price process $\pi_X(t),\ 0 \leqslant t \leqslant T$ of X is given by the wealth process of any replicating strategy φ for X.*

This is according to the risk-neutrality described above. Since the contingent claims can be hedged, the risk in the individual contingent claim does not matter, and the price is just the discounted expected payoff with respect to the risk-neutral equivalent martingale measure $\mathbb{P}^*$, which, in the section above was found to exist in an arbitrage-free market.

Theorem 3.3. *For an attainable contingent claim X, the arbitrage price process is given by the risk-neutral valuation formula*

$$\pi_X(t) = \beta(t)^{-1}\mathrm{E}^*(X\beta(T)|\mathcal{F}_t) \quad \forall\, T \in [0, T],$$

where E^* is the expectation with respect to the equivalent martingale measure $\mathbb{P}^*$, and β is the numéraire.

These are the basics of risk-neutral pricing theory, laying the foundation for the next section.

3.4 Complete Markets

While theorem 3.3 gives an explicit formula for calculating the fair price of an attainable contingent claim, markets are often complicated. Hence, knowing whether a claim is attainable or not is not immediately clear. In order to make this problem more manageable, the concept of a complete market is useful.

Definition 3.5 (Complete Market)**.** *A market is said to be complete if every contingent claim is attainable.*

This definition provides no further idea of whether the contingent claims are attainable or not. The theorem below, however, provides more general conditions for when a market is complete.

Theorem 3.4 (Completeness of Markets)**.** *A market is complete if either of the following equivalent statements are true.*

1. *There exists a unique equivalent martingale measure* $\mathbb{P}^*$*.*
2. *Let* d *be the number of sources of randomness in the market. Then the number of traded assets, including the numéraire asset, is* $d + 1$.

Further, the notions of no-arbitrage and complete markets lead to a central theorem when pricing assets, referred to as the *fundamental theorem of asset pricing*, which is stated below.

Theorem 3.5 (Fundamental Theorem of Asset Pricing)**.** *In an arbitrage-free complete market, there exists a unique equivalent martingale measure,* $\mathbb{P}^*$

An observation that is clear throughout this entire chapter is the infrequency of the real-world measure, $\mathbb{P}$. The risk-neutral measure, $\mathbb{P}^*$, however, appears frequently, and is much more important in terms of derivative pricing. While investors generally assign widely different probabilities to different events, using this measure would not provide any consistent prices of derivatives. The models used, however assume that the investors agree on the volatility in the market. By theorem 2.1, the volatility remains the same after a change of measure, enabling pricing of derivatives based on the real-world volatility which does not change in a risk-neutral setting. Another prerequisite for pricing under the risk-neutral measure is that the two measures agree on which events that are assigned a probability zero, since that is a requirement for equivalence in the first place.

When it comes to pricing derivatives in this thesis, the evolution of the market is simulated through Monte-Carlo simulation. This involves the simulation of a numéraire for the entire lifetime of the derivative, which then discounts the cash flow generated at maturity time T, making the discounted prices martingales. The risk-neutral drift is regarded to be the risk-free rate, which the numeraire is regarded to be. The price process of the underlying asset is also simulated through the risk-neutral measure, generating a payoff for each iteration of the Monte Carlo simulation in addition to the simulation for the evolution of the risk-free rate. The initial price of the derivative is then given by

$$V(0) = \mathrm{E}^*(\exp\left(\int_0^T r(t)dt\right) f(S,T)), \tag{3.1}$$

where $f(S,T)$ is the payoff function of the derivative at maturity.

Chapter 4

Interest Rate Modelling

4.1 Stochastic Interest Rate Models

Many different models exist for predicting the future development of interest rates, and as interest rates are involved in some way in nearly every aspect of finance, a lot of effort has been put into developing such models. The approach where interest rates are regarded as stochastic processes is common, and there exist a wide variety of *stochastic interest rate models*. Each of these models have their own strengths and weaknesses, each capturing certain features of the interest rate evolution, while missing out on others. They do, however, have one thing in common which is the general form of the stochastic differential equation (SDE) describing the evolution of the interest rates. This SDE is given by

$$dr = a(r,t)dt + b(r,t)dW(t) \tag{4.1}$$

where a and b are arbitrary functions chosen to fit to each individual model, while dt is a deterministic time increment and dW is a standard Brownian motion. Apart from this equation, there are differences between the different interest rate models, but they can roughly be divided into a few categories.

4.1.1 The Bond Pricing Equation and Market Price of Risk

Even though fixed-income instruments such as American Treasury bills (T-bills) in most cases are considered a risk-free investment, the value of T-bills are still turning out to show volatility. This is because the T-bills, as well as other bonds, in reality are priced based on the ”underlying” interest rate, which vary. Since there is no way to directly invest in the interest rate, an investment in a bond cannot be hedged by taking a short position in the interest rate. This makes hedging the position more complicated than if there existed a traded underlying asset. To hedge this position, a short position has to be taken in a bond with a different maturity. The value of the portfolio containing the two bonds is then

$$\Pi = B_1 - \Delta B_2,$$

where the subscripts denote the different times of maturity. The value of this portfolio in a small timestep dt changes according to

$$\begin{aligned} d\Pi = & \frac{\partial B_1}{\partial t}dt + \frac{\partial B_1}{\partial r}dr + \frac{1}{2}b^2\frac{\partial^2 B_1}{\partial r^2}dt \\ & -\Delta\left(\frac{\partial B_2}{\partial t}dt + \frac{\partial B_2}{\partial r}dr + \frac{1}{2}\frac{\partial^2 B_2}{\partial r^2}dt\right), \end{aligned} \tag{4.2}$$

This equation is obtained by applying Itô's lemma to functions of r and t. Define the quantity Δ in such a way that a small change in the price of the bond does not change the value of the portfolio. This is given by

$$\Delta = \frac{\partial B_1}{\partial r}\bigg/\frac{\partial B_2}{\partial r}.$$

Acknowledging that the return of a risk-free portfolio is not higher than the risk-free rate, assures that the return in equation (4.2) is $r\Pi dt$. By inserting the right quantities and collecting all terms with B_1 and B_2 at each side of the equation, the expression

$$\frac{\frac{\partial B_1}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 V_1}{\partial r^2} - rB_1}{\frac{\partial B_1}{\partial r}} = \frac{\frac{\partial B_2}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 B_2}{\partial r^2} - rB_2}{\frac{\partial B_2}{\partial r}}, \tag{4.3}$$

is obtained. This is only true if both sides are independent of the maturity times of the bonds, T_1 and T_2. Let either side equal the coefficient

$$k(r,t) = b(r,t)\Lambda(r,t) - a(r,t).$$

The quantity Λ is called the market price of risk, and is the amount an investor requires for taking on an extra unit of risk in the bond. The market price of risk varies with time, and during uncertain times where the market volatility is high, this quantity tends to increase. Finally, an equation that is also obtained in this process is the bond price equation, which is given by

$$\frac{dB}{dt} + \frac{1}{2}b^2\frac{d^2B}{dr^2} + (a - \Lambda b)\frac{dB}{dr} - rB = 0. \tag{4.4}$$

4.2 One-Factor Interest Rate Models

The simplest stochastic interest rate models are the one-factor models where one single source of randomness describes the evolution of the interest rates. These models primarily aim to model the short rate, the interest rate with the shortest maturity available. Then, using the modelled short rate as a basis, the rest of the yield curve describing interest rates with other maturities is derived.

When modelling interest rates, even though their evolution is considered stochastic, certain properties are considered desirable. Different named models incorporate these properties to a varying degree. These desirable properties are the following:

- **Positive interest rates:** Interest rates are generally positive, and this is motivated by the time value of money. Negative interest rates are, however, observed in many parts of the world, especially in Europe and Japan, as of 2019.

- **Mean reversion:** Interest rates are generally thought of to be rooted to some mean level, and the deterministic drift part of the equation should ideally have a form ensuring that the interest rates will move towards the mean with time. The mean level might be time dependent.

- **Non-attainable lower bound:** Many models would get stuck at the lower bound (such as 0) if it were attainable. This is solved by making the bound non-attainable.

Among the most widely known one-factor interest rate models are the Vasicek model introduced in 1977. The Vasicek model was introduced in [12] and follows the equation

$$dr = (\eta - \gamma r)dt + \beta dW$$

where η is the long term mean, γ is a parameter describing the speed of mean reversion and β a volatility parameter. This model is mean-reverting but it has no lower bound, and can simulate negative rates.

However, with the term structure of interest rates clearly not being a one dimensional object, and consequently cannot be explained fully by a single factor, the models quickly run out of sync with market data. The single factor implies that any shift in the yield curve is a parallel shift, thus causing all modelled rates to be perfectly correlated. After a quick look at the market, it is obvious that this is not the case. Because of this, one-factor models must frequently be calibrated to market data, to avoid too large discrepancies between real and modelled prices [13]. This serious shortcoming makes it hard to model interest rates for longer periods of time, motivating the need for more sophisticated models.

4.2.1 Multi-Factor Models

As argued, the one-factor models briefly discussed above have some serious shortcomings, and a natural step further to deal with these shortcomings would be extending the model. Multi-factor extensions of the one-factor models presented above exist, modelling at least one longer rate, with an independent source of randomness, in addition to the short rate modelled by the one-factor models. The governing SDE for multi-factor models is

$$dr_i = a_i(r,t) + b_i(r,t)dW_i, \quad i = 1, 2, \dots, N$$

where N is the number of factors, or sources of randomness, in the model. These models can capture more complex structures in the term structure of interest rates than a parallel shift and consequently provide simulations closer what is observed in real markets.

4.3 The Heath-Jarrow-Morton Framework

The Heath-Jarrow-Morton (HJM) framework was introduced in the 1992 article *Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation* by David Heath, Robert Jarrow and Andrew Morton [5]. This framework provided a new methodology for modelling interest rates, using the whole term structure instead of using only a fixed number of factors or driving sources of randomness. In principle, this makes the model an infinite-dimensional model, since the term structure is an infinite-dimensional object, with, in principle, infinitely many times of maturity. However, in practice, the interest rates in the market only exist with a finite number of maturities and the HJM model is thus implemented with a finite number of factors, usually quite low, between 3 and 5 [7]. For this thesis, an implementation of the HJM framework has been used to generate scenarios for interest rate development. The model implemented is in accordance with the method described by Glasserman in [7].

The HJM framework is a forward rate model and uses the instantaneous forward rates to describe the evolution of the interest rates. Forward rates differ from ordinary interest rates, and are defined the in following way.

Definition 4.1 (Forward Rate). *A forward rate, denoted $F(t, T_1, T_2)$, is the interest rate that can be guaranteed at time t for investing money in the interval $[T_1, T_2]$, where $t \leqslant T_1 \leqslant T_2$.*

While the model is specified for forward rates in the original article, using other rates observed in the markets, such as swap rates of zero-coupon bond rates, are equivalent to using the forward rates. There are, however one good reason to use the forward rates, since it can be argued that these reflect a somewhat more basic description of the term structure of interest rates [7].

The HJM framework specifies an ultimate maturity date, which is denoted by T^* as the final point of simulation. Hence the evolution in the forward curve is at every point in time, t, simulated for the interval $[t, T, T^*]$, $0 \leqslant t \leqslant T \leqslant T^*$, where T are the maturity dates of the forward rates observed in the market. Generally, T^* is the longest maturity available in the market. As the time increases, the maturities of the interest rates are adjusted to fit the remaining time to maturity of the given interest rate.

From the bond pricing equation, the equation

$$B(t,T) = \exp\left(-\int_t^T f(t,u)du\right) \tag{4.5}$$

can be deduced. This equation related the bond prices, B, and the forward rates, f. As a consequence of this relation, the forward rates can always be calculated from the prices the bonds are traded at in the market, and are given by

$$f(t,T) = -\frac{\partial}{\partial T}\log B(t,T). \tag{4.6}$$

The shortest available interest rate is used to generate the short rate, which in the case of the HJM model is the instantaneous forward rate with shortest available maturity time. In symbolic terms, this is denoted $r(t) = f(t,t)$. Evolution of the forward rates in the HJM framework are described through the stochastic differential equation

$$df(t,T) = \mu(t,T)dt + \sigma(t,T)^\top dW(t). \tag{4.7}$$

When operating in the HJM setting, the differential df is with respect to the current time t, not the maturity argument T. W is a d-dimensional standard brownian motion, and d is the number of random sources, or driving forces, of the model. Generally, this number is quite low, often in the range between one and three. A higher number of factors enables the model to capture more advanced movements in the yield curve than a model with fewer factors. For the model implemented for this thesis, the drift and volatility factors μ and σ are deterministic functions of the time arguments t and $T \geqslant t$. These functions could also in principle be stochastic [14], and although this is a highly interesting topic, such models will not be used for the interest rate models used in this thesis. Equation (4.7) represents the evolution of the forward curve under a risk-neutral measure. The absence of arbitrage in the markets implies that asset prices be martingales when divided by the numéraire

$$\beta(t) = \exp\left(\int_0^t r(u)du\right). \tag{4.8}$$

That being said, interest rates are not assets, and the implications of imposing risk-neutral dynamics on interest rates are not immediately clear, and will hence be discussed in the subsequent paragraphs.

The risk-neutral dynamics of assets are known, and hence, starting with assets is natural for deriving the implications of risk-neutrality on interest rates, more specifically a bond, called B. Then, the discounted bond price $B(t,T)/\beta(t)$ must be a positive martingale and the bond price returns are given by

$$\frac{dB(t,T)}{B(t,T)} = r(t)dt + \nu(t,T)^\top dW(t), \quad 0 \leqslant t \leqslant T \leqslant T^*. \tag{4.9}$$

Here, ν denotes the bond volatility, and is a function of the bond prices, or equivalently through equation (4.6), the forward rates. ν is an unwanted factor in this equation because the pricing is performed under risk neutrality. To eliminate this factor, Itô's lemma is applied to (4.9), and this yields

$$d\log B(t,T) = \left[r(t) - \frac{1}{2}\nu(t,T)^\top \nu(t,T)\right] dt + \nu(t,T)^\top dW(t). \tag{4.10}$$

Further, differentiation with respect to the maturity argument T, and changing the order of differentiation gives the dynamics for the forward curve when inserted into (4.6). The equation obtained is

$$\begin{aligned} df(t,T) &= -\frac{\partial}{\partial T} d\log B(t,T) \\ &= -\frac{\partial}{\partial T}\left[r(t) - \frac{1}{2}\nu(t,T)^\top \nu(t,T)\right] dt - \frac{\partial}{\partial T}\nu(t,T)^\top dW(t). \end{aligned}$$

Now, the desired expressions for the risk-neutral drift and volatility parameters, μ and σ, can be extracted from inserting this expression into the governing stochastic differential equation (4.7). These are given by

$$\sigma(t,T) = -\frac{\partial}{\partial T}\nu(t,T) \tag{4.11}$$

and

$$\mu(t,T) = \left(\frac{\partial}{\partial T}\nu(t,T)\right)^\top \nu(t,T). \tag{4.12}$$

Clearly, the bond volatilities are still present in (4.12), but inserting (4.3) yields the final expression for the risk-neutral drift in an arbitrage-free world, which is explicitly given by

$$\mu(t,T) = \sigma(t,T)^\top \int_t^T \sigma(t,u)du. \tag{4.13}$$

Now, all the terms of the expression for the forward rate under risk-neutral dynamics are found and the governing equation (4.7) in the risk-neutral world becomes

$$df(t,T) = \left(\sigma(t,T)^\top \int_t^T \sigma(t,u)du\right) dt + \sigma(t,T)^\top dW(t). \tag{4.14}$$

4.3.1 Discretization of the HJM Framework

As briefly mentioned, the HJM framework described in the above section regards the forward curve as a continuous function. This fact has the consequence that simulation from the continuous model is not possible, except for certain, highly specific situations. Hence, a discretization scheme for f is necessary in order to generate simulations from (4.14).

The forward curve $f(t,T)$ is a function of both the current time t and the time to maturity T, and a discretization is needed for both these arguments. Start by fixing a time grid $0 = t_0 < t_1 < \ldots < t_M$ for the time argument t. Even when keeping the time argument fixed at t_i, representing the forward curve in a continuous manner is still infeasible. Hence, a grid is fixed for the maturity argument T as well, and the two grids are assumed to be equal for the rest of this thesis even though they in principle could be different. However, having them equal simplifies notation a lot, and the implementation of the model in this thesis also assumes that the two grids coincide.

In the continuation, a circumflex is used to distinguish discretized variables from their corresponding continuous version. The discretized bond prices, $\hat{B}(t_i, t_j)$ are, analogously to equation (4.5), given by

$$\hat{B}(t_i, t_j) = \exp\left(-\sum_{\ell=i} j - 1\hat{f}(t_i, t_j)[t_{\ell+1} - t_\ell]\right). \tag{4.15}$$

In order to minimize discretization error, the initial values of the discretized bonds, $\hat{B}(0, t_j)$ are calibrated so that they coincide with the values of the market bond prices, $B(0, t_j)$ for all maturities in the discrete grid. By comparing (4.15) and (4.5) is is clear that this holds when

$$\sum_{\ell=0}^{j-1} \hat{f}(0, t_\ell)[t_{\ell+1} - t_\ell] = \int_0^{t_j} f(0,u)du. \tag{4.16}$$

Stated for each component of the discretized forward rate, this becomes, equivalently,

$$\hat{f}(0, t_\ell) = \frac{1}{t_{\ell+1} - t_\ell} \int_{t_\ell}^{t_{\ell+1}} f(0,u)du = \frac{1}{t_{\ell+1} - t_\ell} \log \frac{B(0, t_\ell)}{B(0, t_{\ell+1})}, \tag{4.17}$$

for $\ell = 0, 1, \ldots, M-1$. From this it is observed that the initial values of the discretized forward rates should be set to the average level in the intervals $[t_\ell, t_{\ell+1}]$.

Now, a simulation of the forward rate will evolve according to the equation

$$\hat{f}(t_i, t_j) = \hat{f}(t_{i-1}, t_j) + \hat{\mu}(t_{i-1}, t_j)[t_i, t_{i-1}] + \sum_{k=1}^{d} \hat{\sigma_k}(ti-1, t_j)\sqrt{t_i - t_{i-1}} Z_{ik},$$

where the Z_{ik}'s are iid random variables following a $N(0,1)$-distribution. The discrete drift parameter is approximated to best fit to the continuous drift derived in (4.13), as well as preserving the martingale property of discounted bond prices under risk-neutral measures. These drift parameters are given by

$$\hat{\mu}_k(t_{i-1}, t_j)[t_{j+1} - t_j] =$$
$$\frac{1}{2}\left(\sum_{\ell=i}^{j} \hat{\sigma}_k(t_{i-1}, t_\ell)[t_{\ell+1} - t_\ell]\right)^2 - \frac{1}{2}\left(\sum_{\ell=i}^{j-1} \hat{\sigma}_k(t_{i-1}, t_\ell)[t_{\ell+1} - t_\ell]\right)^2.$$

Here, $\hat{\sigma}_k$ denotes the k'th component of the d-dimensional vector $\hat{\sigma}$. Further, this gives the total drift, which is given by

$$\hat{\mu}(t_{i-1}, t_j)[t_{j+1} - t_j] = \sum_{k=1}^{d} \hat{\mu}_k(t_{i-1}, t_j). \tag{4.18}$$

4.3.2 Volatility in the HJM Model

The volatility parameter, σ, is an important quantity in the implementation of the HJM framework performed for this thesis. Since both the deterministic drift parameters, μ as well as the random deviations depend on this parameter, a good specification of the volatility is crucial for obtaining reliable simulations.

Since the HJM framework includes a wide variety of forward rates of all maturities within the final time of consideration T^*, it seems more natural to refer to the volatility parameter as a volatility structure. This structure is found through principal component analysis, which is presented in chapter 2.

This volatility structure, σ, is calibrated to the market prices of bonds at the time of initialization of the model.

Chapter 5

Counterparty Credit Risk

Counterparty credit risk is the risk that a counterparty in a contract is unable fulfill its obligations agreed upon at initialization of the contract. In this chapter, an outline is first provided on the basics of credit risk in general. Then, some elementary theory on measuring risk in finance is presented. Further, the theory is extended to cope with the more specific counterparty risk, and finally some background is provided on the regulations that apply to the field. The material in this chapter is mainly based on the books by Cesari et. al. [15] and Jorion [16].

5.1 Outlining Credit Risk

Credit risk is the risk that an issuer of a debt instrument, such as a bond, fails to meet its contractual obligations, i.e. defaults the contract. When estimating credit risk, especially three metrics are fundamental for the estimation.

1. **Probability of Default (PD):** The probability that the issuer of the instrument is not able to fulfill its contractual obligations.

2. **Loss Given Default (LGD):** When a counterparty defaults on a contract, one can normally recover parts of the claim from the counterparty's assets. Hence the loss is rarely the full notional amount of the defaulted claim.

3. **Exposure At Default (EAD):** When a default occurs, what is the exposure of the defaulting counterparty. This is especially important for contracts where the exposure varies with the levels of interest rates.

This thesis will focus on the part considering exposure at default, given different scenarios of development in the markets. The other two parts of the credit risk, requires more specific data about the counterparty, such as credit ratings, capital structure, and future development of these factors among others. A complete framework for estimating credit risk related to a transaction, would incorporate all these factors.

Further, frameworks can also be extended to cover portfolio risk, in where the following two risk elements also have to be covered [11]

1. **Default and Credit Quality Correlation:** To which extent the defaults or credit migrations of one counterparty are correlated to similar events for other counterparties.

2. **Risk Contribution and Credit Concentration:** How much of the total exposure of a portfolio depends on a single counterparty or a single risk factor.

These may also require data from external sources to provide good estimates, but while constructing a portfolio, an estimate of the concentration of the exposure due to a single counterparty or risk factor can be provided without the use of external data, in many cases. These parts of the credit risk process are not explicitly part of this thesis, but is useful to keep in mind for the analysis.

5.2 Risk Measures

According to Holton [6], risk is a product of exposure and the uncertainty of outcomes. While the credit risk part in this thesis concerns the exposure part of the equation, and what consequences that may arise in unfavorable outcomes such as defaults, one can also look at the losses that will occur to a portfolio in the event of unfavorable development of the markets. As part of the revisions of the regulations of banks and other financial institutions in the wake of the financial crisis in 2008 , banks and other financial institutions are required to report the market risk of their trading portfolios [4]. Two risk measures that are particularly well-known are the measures Value-at-Risk (VaR) and Expected Shortfall (ES). They are defined the following way.

Definition 5.1 (Value-at-Risk)**.** *A VaR measure is specified with a level of confidence, α and a time-horizon $\mathcal{T}$. Let L be a loss of the portfolio. The VaR estimate for given values of these parameters are then*

$$P(L > \mathrm{VaR}_\alpha^{\mathcal{T}}) \leqslant 1 - \alpha. \tag{5.1}$$

Definition 5.2 (Expected Shortfall)**.** *Given a confidence level and time-horizon as above. Then the corresponding ES estimate of a portfolio is*

$$\mathrm{ES}_\alpha^{\mathcal{T}} = \mathrm{E}\left[L | L \geqslant \mathrm{VaR}_\alpha^{\mathcal{T}}\right] \tag{5.2}$$

A VaR estimate thus only contains the loss which occurs during the worst fraction of periods, while ES better captures the real losses when these periods happen. The risk estimates from the two measures may differ quite significantly, especially when portfolio returns show signs of leptokurtosis.

5.2.1 Coherent Risk Measures

Risk can be measured in variety of different ways, each measure providing different results and capturing different aspects of the riskiness of an investment, but up until now there has not been a way of comparing a risk measure to another. Artzner et. al. proposed, in the article *"Coherent Measures of Risk*, a framework for estimating the quality of a risk measure. The article proposes four desirable properties that a high quality risk measure should have. A risk measure satisfying all these properties are called *coherent*.

Definition 5.3 (Coherent Risk Measure)**.** *A risk measure, R, on a portfolio return P is called a coherent risk measure if it satisfies all the following properties.*

1. *Monotonicity.* $P_1 \leqslant P_2 \implies R(P_1) \geqslant R(P_2)$. *If the returns of portfolio 1 is systematically lower than the returns of portfolio 2, for all states of the world, then the risk is greater in portfolio 1.*

2. *Translation invariance. Let K be an amount of cash. Then $R(P + K) = R(P) - K$. The addition of cash to a portfolio reduces risk by that amount.*

3. *Homogeneity. Let a be an arbitrary factor. Then, $R(aP) = aR(P)$. Increasing the portfolio, by a factor, causes the risk to increase proportionally.*

4. *Sub-additivity.* $R(P_1 + P_2) \leqslant R(P_1) + R(P_2)$. *Adding two portfolios together cannot increase the total risk. Equality holds if the returns are independent.*

From this definition, an important conclusion that can be drawn is that ES is a coherent risk measure, while VaR is not, because it is not sub-additive [17].

5.3 Basics of Counterparty Exposure

As argued previously, counterparty exposure, or exposure at default is an important part in the process of evaluating credit risk. When estimating counterparty exposure, multiple risk measures exist, providing different information about the risk associated with the contract. Two of the most notable are stated below.

Definition 5.4 (Potential Future Exposure (PFE)). *The* PFE *of a contract is given with a confidence level* α. *Then* PFE^{α} *is the exposure that will not be exceeded in a proportion* α *of scenarios.*

Definition 5.5 (Expected Positive Exposure (EPE)). *The* EPE *of a contract is the mean of the positive part of the distribution of the exposure.*

One thing worth noting about the PFE measure, is that it is in fact a VaR measure, as introduced in section 5.2 , without the time horizon being specified explicitly. The EPE is not a measure capturing the tail of the distribution of future values, and is more to be thought of as a measure for a development that is likely to occur. The EPE thus reflects a very likely scenario, and exposure equal to the EPE should not be thought of as worrisome requiring extraordinary intervention through hedging or similar.

5.3.1 Simulating Counterparty Exposure

In its essence, there are two elements of computing counterparty exposure of a derivative. The first part is to generate scenarios for the underlying process, which in this thesis are the interest rates. This is done by Monte Carlo simulation, which is described in chapter 2. Then, the second part is to evaluate the given derivative at each simulated point in the simulation of the underlying. For a swap, this means that the forward curve for the entire time until maturity needs to be estimated for each time in the time grid used for simulation, in order to generate an estimate for the exposure given a certain realization of the simulation.

Further, as a final step, risk measures are applied to the collection of simulations of the counterparty exposure, to generate an estimate of the risk associated with the contract.

5.4 Regulations on Counterparty Credit Risk

Regulations on counterparty risk was revised in the aftermath of the financial crisis. During this crisis, even the largest banks and insurance providers globally were considered to carry significant amounts of counterparty risk, requiring governmental aid to avoid bankruptcy. Since banks are highly leveraged institutions, involved at ll levels in the financial system, failures of large banks can destabilize the whole financial system. As a consequence of this, banks are subject to a variety of regulations, ensuring the stability of the financial system. These regulations were revised after significant weaknesses had been exposed in the crisis.

The Bank for International Settlements proposed new regulations, stating that standardized contracts should be cleared through a central counterparty. The required margin should correspond to the VaR at a 99 % significance level on a 5 day time horizon for centrally cleared derivatives and a 10 day horizon for non-cleared derivatives [3] [18]. This is, as earlier stated, equivalent to a potential future exposure with the same significance level.

For the most standard OTC derivatives, the new regulations demanded that these be cleared through a central counterparty (CCP), thus transferring the risk to the CCP. A CCP has the sole responsibility of handling the risks associated with the contract. Thus, the CCP has a net zero position, and both counterparties have the CCP as their only counterparty. This has led to lower risk in the derivatives sector in banks, and a more stable financial system in general. However, as very few entities are responsible for very large amounts of the clearing, a new systemic weakness has emerged, potentially causing grave damage to the financial system if risk management is performed poorly [19].

In a note written by Rama Cont [18] for the International Swaps and Derivatives Association (ISDA), some changes in the regulations are proposed. The article argues that the existing regulations do not take into consideration the market depth and concentration of positions, causing significant liquidity risk for larger position in thin markets. Further, a longer horizon for the the VaR measure is proposed, to reflect a longer liquidation horizon.

Other regulations that are imposed to financial institutions concern topics such as maximum exposure to single counterparties, frequency of settlement of collateral and reserve capital. These regulations are of less importance for the discussion in this thesis.

5.5 Managing, Mitigating and Hedging Counterparty Risk

The regulations described in the section above assigns higher margin requirements to higher notional amounts. This incentivizes minimization of the positions. Since many CCPs handle large amounts of transactions, and market participants are split in their view of the market, CCPs handle many positions with opposite exposure. A way to decrease the overall margin requirements and also reduce total risk, is by netting these opposite positions to reduce the overall exposure from the derivatives. The process of netting out positions is called *trade compression*. Trade compression has been a significant explanatory factor for the reduction in the total outstanding amounts in the derivative markets in recent years [1]. However, multilateral netting may lead to changes in which counterparties the exposure is towards, requiring that the CCP do its risk management correctly in order to avoid systemic risks to arise [20].

As mentioned at several places earlier in this chapter, derivative contracts come with certain margin requirements. This is in itself a way to reduce the overall risk, since the margin works as a security against unfavorable development in the market. This collateral is usually posted in cash or other liquid assets, such as government bonds. Posting collateral for parts of the exposure reduces the potential loss by the same same amount as the collateral. Generally, collateral is split in two parts, one part which is paid when entering the agreement, and a variable part paid during the lifetime of the agreement depending on the current exposure.

In cases where a CCP is not handling the risks associated to the derivatives, a party entering into a derivative agreement may need to manage the counterparty risk themselves. Obviously this involves getting an overview of the potential exposure to the counterparty, which this thesis largely focus on. In addition, estimates for the probability of default and recovery rate given default are required.

A normal way to hedge the counterparty exposure is through purchasing credit default swaps (CDS), which is a derivative generating a payout in case of a counterparty's default. CDSs work as insurance against counterparty default, but it is merely a transfer of risk to the issuer of the CDS. Generally, the CDS prices should reflect the risk of default for the given counterparty [16].

For small or high-risk counterparties, CDSs may not be available for purchase, and consequently hedging has to be done through other means. In such situations where CDS purchasing is impossible or impractical taking a short position in the underlying debt is among the ways to hedge the exposure. This position will develop positively in an event of default. In many cases it is also possible to identify market factors that are specific for the counterparty having a high correlation to their specific area of business. This strategy carries other types of risk, but may still be a useful tool for managing counterparty risk.

Chapter 6

Data and Methodology

For this thesis, the Heath-Jarrow-Morton framework for interest rates was implemented to generate future scenarios for the development of interest rates. The simulation was done using the Monte Carlo approach.

Further, counterparty exposure for different swap contracts have been estimated, based on the HJM simulations. To quantify the exposure, the measures that have been used are Potential Future Exposure and Expected Positive Exposure.

6.1 Supplied Data

The data used for the analysis were supplied by DNB and are the Norwegian Inter-Bank Offered Rates (NIBOR) and Norwegian swap rates from 16.08.2002 to 11.03.2019. The maturities of the NIBOR rates are 3 months and 6 months, and the maturities of the swap rates are 1, 2, 3, 4, 5, 7, 10, 15 and 20 years. Due to small deviations in which days these rates were quoted, data from dates where at least one of the rates were not quoted has been removed from the data set. This problem could also have been solved by other means, for instance by interpolation of the missing points. However, the number of missing observations was small, and consequently, the effect of removing the observations is also limited. The missing observations were also fairly evenly distributed in the data set, causing a smaller impact on the data than if the missing observations had been clustered around certain periods. The NIBOR and swap interest rates are considered risk free, with probability of default equal to 0. Hence they represent the risk free rate with different maturities.

The period the data is taken from was characterized by certain events in the financial markets. The beginning of the period, the years around 2002, was characterized by the burst of the dot-com bubble. Further, in 2008, the most profound financial crisis since the great depression in 1929 took place. This crisis had a huge impact on the financial markets and led to a sharp decrease in interest rates. In the aftermath of this financial crisis, interest rates have been kept low, and negative interest rates, generally regarded as a very bad property for interest rate models, have been the reality in most of Europe and Japan.

In figure 6.1, the swap rates of all the different maturities are shown as a function of the yield they generate at any given time. From the graph, it is evident that, even though the evolution of the interest roughly follows each other, the spreads are not constant and vary with time. As argued in chapter 4, one factor models, which only describe a parallel shift in the yield curve, are not accurate, and this is clearly shown in the figure. A table showing some essential summary statistics of the interest rates can be found in table 6.1.

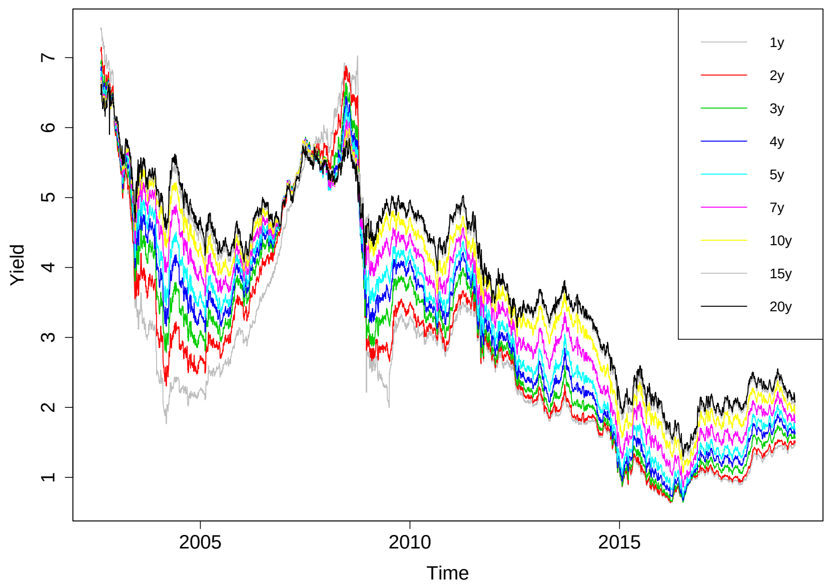

Figure 6.1: The swap rates in the period from 2002 to 2019 used for the analysis. The different colors denote different times of maturity.

Interest rate	Average yield (%)	Annualized volatility	Daily volatility
3m NIBOR	2.612	0.7105	0.0448
6m NIBOR	2.608	0.6945	0.0437
1y Swap	2.760	0.6643	0.0418
2y Swap	2.896	0.6374	0.0402
3y Swap	3.044	0.6402	0.0403
4y Swap	3.183	0.6411	0.0404
5y Swap	3.310	0.6338	0.0399
7y Swap	3.516	0.6246	0.0393
10y Swap	3.724	0.6281	0.0396
15y Swap	3.879	0.6898	0.0435
20y Swap	3.914	0.6845	0.0431

Table 6.1: Some summary statistics about initial interest rate supplied. The volatilities are calculated in absolute terms (not relative). The data has been collected from the time period between 16.08.2002 and 11.03.2019.

From table 6.1 it is easily observed that the interest rates with longer maturity times tend to be at higher levels than interest rates with shorter maturity. This is to be expected from financial theory, as investors will normally require higher compensation for tying up their money for longer periods of time. An important observation from figure 6.1 is that the general level of the interest rates tend to decrease across the period, with a slight increase during the most recent years.

The average volatility across the entire period of observation is also shown in table 6.1, in daily and annual terms. It is worth noting that the volatility is seen to be far from constant in the period, showing significant signs of clustering at certain periods. Hence, the volatility in these interest rates varies greatly throughout the period, with long periods with significantly lower volatility, as well as periods with significantly higher levels of volatility.

6.1.1 Interpolation of data

Since the maturities of the interest rates are unevenly distributed, rates are interpolated to create a grid of evenly distributed rates for the whole term structure in the interval $[0, T]$, where $T = 20$ years. Interpolation of rates to fit to predefined maturities, such as whole years is common market practice, and is done in the yields provided daily by the U.S. Treasury [21]. This interpolation is mostly due to the fact that there are generally no existing outstanding bond with the exact remaining maturity of, say, 10 years. This would require debt to be issued every day, which would not be suitable. It would also require trading in the instrument every day, at a certain volume, to make the prices trustworthy. Neither of these requirements are realistic, hence interpolation is performed to provide an approximate value based on the assets that are traded, having remaining maturity closest to the desired time.

Since the interest rate data supplied for this thesis only contained data for the maturities stated above, intermediary maturities have been interpolated linearly such that there exist annual maturities in the entire period of consideration, which is 20 years.

6.1.2 Transformation to Forward Rates

Since the HJM framework requires forward rates, and the rates supplied for the analysis are NIBOR and swap rates, meaning that a transformation has to be performed in order to obtain the rates as forward rates. The interest rates are transformed according to equation (4.17), to obtain the instantaneous forward rates for all the original times of maturity as well as the interpolated ones.

6.2 Volatility Structure

The volatility structure is specified from principal component analysis, as briefly mentioned in chapter 4. For this specification, the data with interest rates transformed into forward rates is used. Using the entire dataset, the principal components are calculated, and in table 6.2 below, the proportions of the variance explained by each principal component is found, as well as the cumulative proportion of the variance explained by the PCs $\lambda_1, \ldots, \lambda_i$.

PC	Explained variance	Cumulative explained variance
λ_1	62.79 %	62.79 %
λ_2	17.29 %	80.09 %
λ_3	7.47 %	87.56 %
λ_4	6.07 %	93.63 %
λ_5	3.11 %	96.74 %
λ_6	1.41 %	98.15 %
λ_7	0.58 %	98.73 %

Table 6.2: Proportion of explained variance and cumulative explained variance by each PC in the transformed data.

From the table it is seen that more than 95 % of the variance in the data can be explained from the first five PCs, which is assumed to be sufficient in this case. Consequently, five PCs are used for the HJM algorithm in the further analysis of this thesis.

The term structure of interest rates is a complex function, and the principal components provides a way to explain this complex structure in fewer dimensions than the original data. In figure 6.2 the first three principal components are shown as a function of the maturity of the interest rates. Accordingly, these PCs represent the three most important movements in the yield curve. Only the first three PCs are shown, since these are more interpretable than the rest.

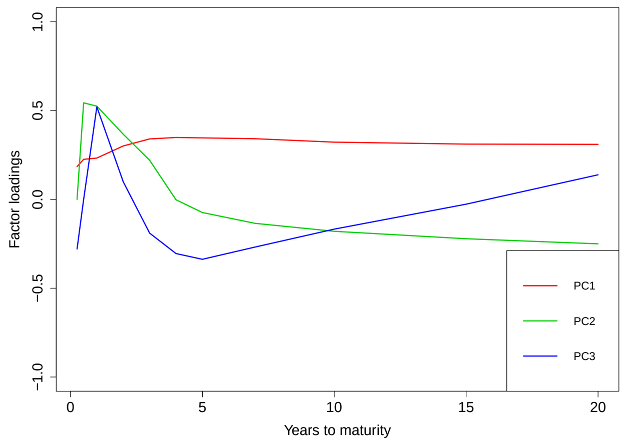

Figure 6.2: The factor loadings of the first three principal components in the interest rate data, as a function of the maturity time.

It is seen from the figure that the first PC represents a near parallel shift in the yield curve, where the loadings from all maturities are almost equal.. This is the most important movement in the yield curve, accounting for more than 60 % of the variability in the data. The second PC represent movements where the short end of the yield curve move upwards, while the longer rates fall. This is a movement that is generally present during yield curve inversions, seen ahead of many financial crises. The third PC represents situations of increasing rates in the very short term, while interest rates are falling for longer maturities. The yields are decreasing only at the middle of the yield curve, while short and the longest rates increase slightly.

6.2.1 Swap specification

Swaps were briefly defined in definition 3.2, but this definition is very basic, and different types of swaps exist. Swaps in this thesis have interest rates as the underlying. It is common market practice that swaps are initialized to have zero value at $t = 0$. For this thesis it is assumed that swaps are specified in such a way that a floating rate is paid, while a fixed rate is received. This is a normal exposure for banks, when offering fixed rates to corporations or private persons. The opposite is also possible, resulting in the exact opposite exposure profile. Further, annual settlements are assumed, and the received interest rate at each settlement date is the short rate. A maturity of five years is chosen for the swaps. Shorter maturities will to a limited degree provide good exposure profiles, while longer maturities are a bit less common.

6.3 Simulation Procedure

First, simulation of the future evolution of the interest rates is done through the HJM framework described in chapter 4. The framework is implemented in accordance with the algorithm described in *Monte Carlo Methods in Financial Engineering* by Paul Glasserman [7].

By using the simulated forward curves obtained from the HJM algorithm, swaps are priced at every point in the specified time grid, which in this case is annual. The swaps are initialized to have zero exposure at the beginning of the simulation, in accordance to market practice. For each simulated scenario, the fair swap price is obtained at every point in the time grid. The price of this swap at every time point is then regarded as the level of counterparty exposure in each scenario.

Then, by using the intermediate forward curves generated by the HJM algorithm, swaps are priced. The fair price obtained from this procedure at every step of the simulation is the exposure in the given scenario. Finally, for each of the periods simulated, the risk measures PFE, EPE and ES are applied to the exposures obtained from the previous step. Both lower and upper bounds for the exposure are estimated, with a significance level of 95 % for the upper bounds and 5 % for the lower bounds.

6.3.1 Backtesting Methodology

To perform the backtesting procedure, the data are divided into periods. As the model requires some data to generate reliable results, the first period is only to provide the model with some data. Longer periods provide the model with more data to estimate the volatility structure, leading to more precise estimates for exposure of the swaps. The data were divided into five three-year periods, rounded to the next trading day if the initial endpoint was not a trading day. However, since the lifetime of the swaps that are analysed were five years, and exposure curves are desired for the entire lifetime of the swaps, exposure is estimated for the next five years, resulting in some overlap between the different swaps. Since there were no data for the full five years for the last period, the exposure is terminated at the latest data point for that period.

Further, each period has been analysed in two ways. First, an analysis was conducted by using only the volatility data for the preceding period to generate the exposure, while the second approach used all available data from previous periods.

This causes the results from the first approach are more directly comparable to each other than the results from the second approach, as more data is expected to yield better results. A further discussion of the results and the validity of them will be conducted in the next chapter.

Chapter 7

Results and Discussion

7.1 Results

In this section the results obtained from the analysis is presented. First, the convergence of the results obtained from the algorithm that generates possible scenarios is presented. Then, the results obtained for each of the periods in the backtest are presented.

7.1.1 Convergence of Swap Price

To check the correctness of the model and assess the question of ho many iterations that are required to obtain satisfactory results from the swap pricing procedure, a convergence test is performed. First, a number is selected for how many times each experiment is to be performed. This number is chosen to be 50, which is well above the roughly 30 that is required according to the central limit theorem is sufficient to make the result well approximated by the normal distribution [22]. Further, for each of these 50 replications, a swap is priced a number of times. Finally for each number of iterations that is performed, the mean and standard deviation of the 50 replications is estimated. The results from this convergence test are found in the figures 7.1 and 7.2.

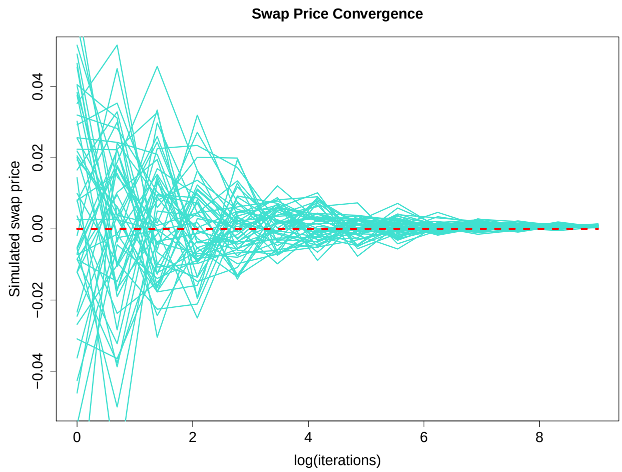

Figure 7.1: The average price of each of the 50 replication as a function of the number of iterations. The dotted red line shows the theoretical price.

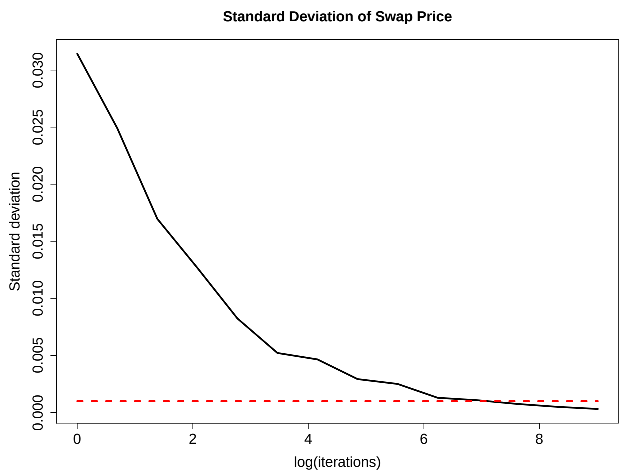

Figure 7.2: The standard deviation of the price of the swaps, as a function of the number of iterations. The dotted line shows the tolerance limit of 0.001.

From figure 7.1, it is clearly seen that the prices of the simulated converges towards zero, which is the theoretically correct price. It is also seen that the standard deviation decreases according to the theory, proportionally to the square root of the number of iterations. From figure 7.2, it is seen that the estimated standard deviation crosses the tolerance limit of 0.001 for the error in the swap price at around e^8, corresponding to roughly 5000 iterations. The standard deviations for 4096 iterations is 0.0005, well below the tolerance limit. For the remainder of this thesis, 5000 iterations are performed to price swaps when estimating the counterparty exposure. This provides a good margin with respect to the tolerance, without requiring excessive computational power.

7.1.2 Results from Backtest

For each of the three-year periods in the backtest, results of the estimated counterparty exposure is found in the sections below. Each period is taken from August the initial year of the period, and lasts to August five years later. A figure of the estimated exposure is found in the section for each period. In these figures, the actual exposure (AE) of a swap entered into at the beginning of the period is shown, and the value of this swap is marked to market every trading day, thus having a much finer granularity than the estimated quantities, which are only estimated at each point in the time grid. The figures also show the EPE for the swap in each period as well as the 95 % ES and PFE. If the exposures were to go negative, the 5 % PFE and ES[4] are also shown for each period.

2005 - 2010

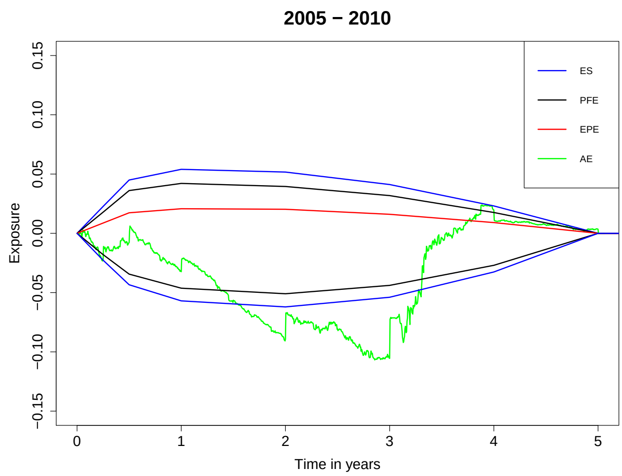

Figure 7.3: The exposure in the period 2005 - 2010.

[4]This ES represent the 5% of cases having the most negative exposure, not the 95 % with the least negative as original definition could indicate.

From figure 7.3 it is seen that during the period of rapid rate hikes which took place in the period before the financial crisis of 2008, the exposure profiles generated by the model is not able to capture the AE from the swap contract studied. Since the exact same volatility data is used for the model using all previous available data, the result from this model generates the exact same exposure profiles. As expected, the lower ES boundary is closest to capturing the negative exposure that is seen in a period where interest rates increased rapidly, but the AE still moves significantly outside the 5 % ES.

2008 - 2013

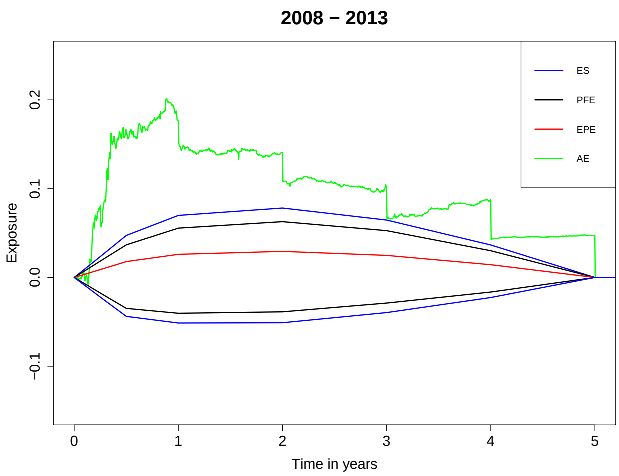

Figure 7.4: The exposure in the period 2011 - 2016.

This period contains the financial crisis at the very beginning of the period, where interest rates plummeted. The massive increase in volatility, reaching record highs, causes the estimated exposure to be far outside all exposure estimates, including the ES. As expected, when more payments are settled, the AE moves closer to the 95 % ES boundary since fewer remaining payments naturally leads to lower exposure.

2011 - 2016

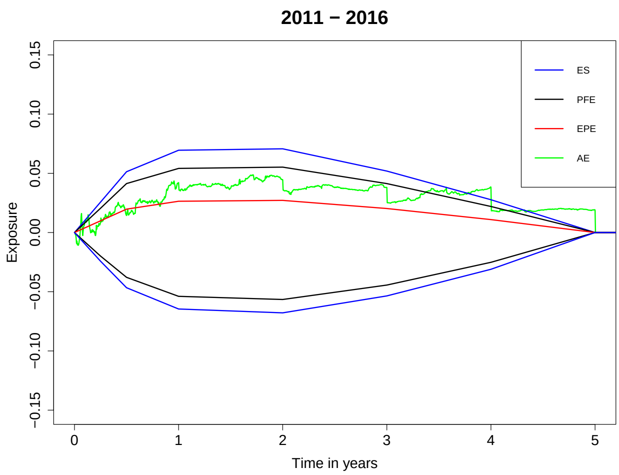

Figure 7.5: The exposure in the period 2011 - 2016.

During this period, the volatility has decreased to more normal levels, even though they are still decreasing fairly rapidly. For the majority of the period, the AE stays within the PFE boundary, but toward the end of the period, the PFE and ES go below the AE. The linear decrease of the ES and PFE before the last settlement is unrealistic, since defaults happen at settlement dates, and thus the exposure stays the same until the payment is completed.

2014 - 2019

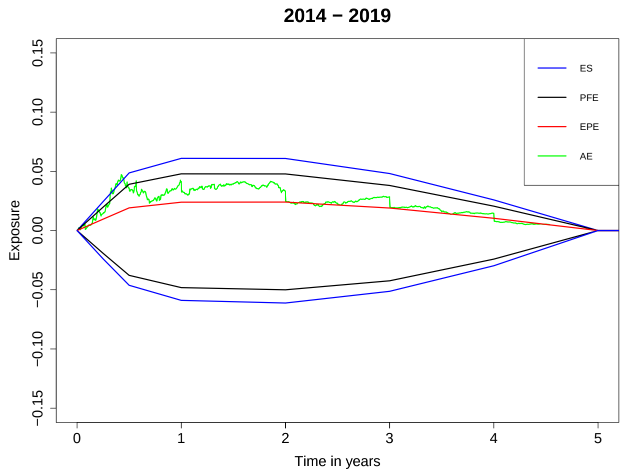

Figure 7.6: The exposure in the period 2014 - 2019.

In this period, the interest rates continue to decrease, but a bit less aggressively that in the previous periods, and they even increase a bit from the middle of the period toward the end. The AE stays inside the PFE boundary for the entire period, except for a short time at the beginning of the period. For the very end of the period, there is no available data.

The exposure generated by the analysis using all previous available data estimate somewhat higher exposures, but still fail to capture the dramatic fall in the interest rates during and after the financial crisis by a good margin. Thus, the results from such an analysis is surprisingly equal to the results presented here. However, including high-volatility periods in the data may lead to the model being more robust to more volatile periods in the time going forward. Plots showing the exposure curves generated from the analysis using all available data are shown in the appendix.

Distribution of the swap exposures

In figure 7.7 the distributions of the exposures after three years for the four periods are shown.

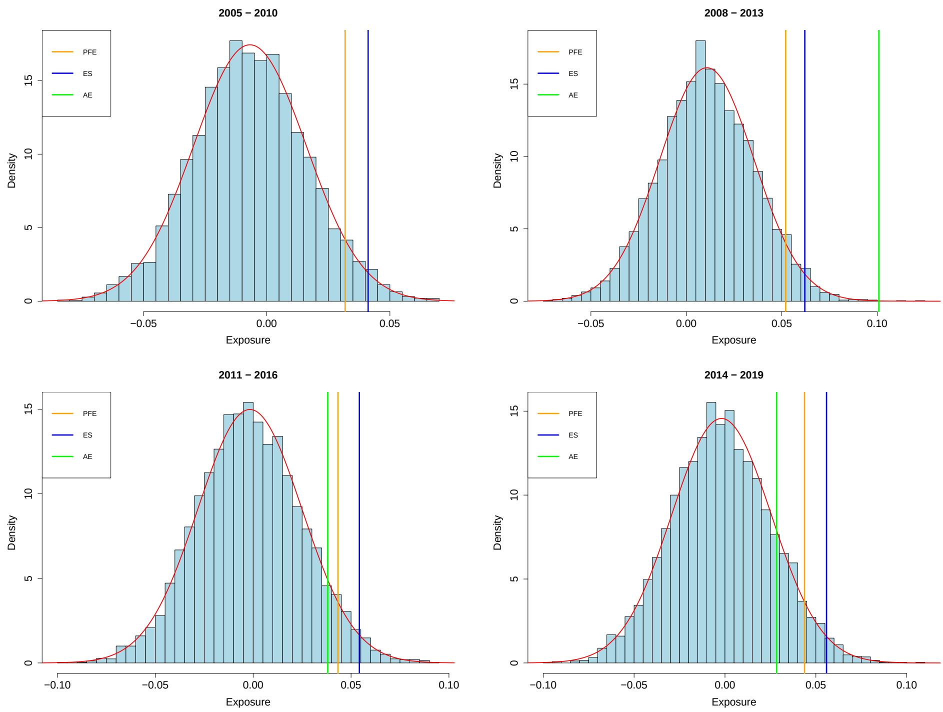

Figure 7.7: The simulated exposure profiles after 3 years for each of the four periods. The red line shows the normal distribution, and the vertical lines show ES, PFE and AE as indicated by the legends.

For the first period, the AE is negative to a degree that it falls outside of the window shown in the figure. As for the second, more interesting period, it is seen that AE exceeds the simulated measures by a lot. This is even after the interest rates are normalized a bit, thus bringing the AE down from the values observed earlier in the period. Examining the profile a little further, there are some simulations giving exposure higher than the AE at this point. For the third and fourth periods the AE of the swaps 3 years out in the simulation are well within what would be expected, well below both the PFE and ES boundaries. Interest rates during these two periods have been historically low, representing a low-rate environment unlike anything observed previously, even though the volatility has been fairly low across these periods.

7.2 Discussion

Since interest rates have been decreasing for most of the period that is analysed, a pay floating – receive fixed swap have caused positive exposure profiles for all except the first period analysed. While carrying significant exposure in all these periods, it is worth pointing out that this means that a positive counterparty exposure is a positive development of the investment, thus representing earnings on the contract. Until the payments are settled, however, the earnings cannot be considered risk free, and in the case of a counterparty defaulting, losses could amount to the level of exposure to the counterparty in default.

Observing interest rates in developed markets since the mid-1980's, rates have had a downward trend in general, with lower highs and lower lows. This has left holders of the kind of swaps studied in this thesis, which in many cases are banks, in a favorable position for an extended period of time.

7.2.1 Negative interest rates

The HJM framework allows for negative interest rates, which, as mentioned earlier traditionally has been regarded as a very bad property. However, negative interest are at the time of writing this thesis the reality in major economies in Europe and Japan. Generally, in the few cases where interest rates would turn negative, one would expect that only the shorter interest rates turn negative, but at the time of writing this, examples of 10-year government bonds with negative yields is not rare [5]. This may indicate that allowing interest rates to go negative is not a bad property after all. Examining the simulated interest rates generated by the HJM algorithm, the simulated interest rates tend to go more negative than the real ones, providing a bit more strength to the original theory, or at least that there should still be a lower bound for how low interest rates may go, even though this bound may be negative.

7.2.2 Normally Distributed Interest Rates

The interest rate evolution simulated by the HJM algorithm are normally distributed. However, there is widespread evidence that this is not the case, and that especially large negative outcomes are much more frequent in real markets than in the normal distribution [23]. As is also observed from the analysis in this thesis, the simulated exposures fail spectacularly when facing events such as the financial crisis in 2008. The normal distribution is, however, a convenient one,

[5] German 10-year government bonds yield -0.36 % at the time of writing (05.07.19). Source: https://www.bundesbank.de/resource/blob/772220/9498171a60ac9532503ffce5a89c13b9/mL/rendbund-data.pdf

representing the majority of events fairly well, and also having desirable statistical properties, making it easy to sample from and easily understandable. These desirable properties are also important to account for when discussing the shortcomings of normally distributed interest rates. An alternative to the normal distribution is discussed below.

7.2.3 Volatility

The volatility structure used for this thesis is based on PCA, preserving the general structure of the correlation between interest rates of different maturity. This is good for many purposes, but the level of volatility is severely underestimated during times of market stress, such as 2008. In addition to volatility being significantly higher, there is also evidence that historical market correlations tend to break down during such periods [24]. This may also lead to the exposure profiles being wrong not only in terms of level of exposure, but also in shape, during stressful periods.

Using the volatility structure from only the most recent three years might be a suitable strategy for estimating exposure under the current market situation. Thus, using the most updated data would better capture the correlation structure that is seen at the exact point of initialization of a contract. However, as risk measurement is much about expecting the unexpected, one could argue that a longer period where abnormal periods are included would be more suitable for capturing the risk when things actually go wrong. This is also seen in the estimates in this thesis, where the exposure generated with all available estimated higher exposure, although the difference was not very large.

7.2.4 Risk Measures

Examining the estimated exposure profiles generated with ES and PFE, it is observed that the model using all the available data tend to provide a bit wider estimates for the exposure. This is expected, since including periods with higher volatility will result in the model estimating a higher general level of volatility.

It seems that ES provides better estimates than PFE, but none of them produce satisfactory results. However, it is worth pointing out that all these exposure profiles are generated with normally distributed interest rates. By incorporating a leptokurtic distribution such as Student's t-distribution, one can observe that the differences between the two measures would increase dramatically. Table 7.1 shows how the quantiles of the standard normal distribution and Student's t-distribution as the observations go toward more and more extreme observations.

It is clearly seen that the distance between the normal and the t-distribution increases drastically with higher quantiles. This indicates that a leptokurtic distribution would be able capture abnormal events better than the normal distribution.

Quantile	N(0,1)	t_6	t_4
95 %	1.64	1.94	2.13
97.5 %	1.96	2.45	2.78
99 %	2.33	3.14	3.75
99.9 %	3.09	5.21	7.17

Table 7.1: Quantiles of standard normal and Student's t-distribution with 6 and 4 degrees of freedom.

However, many issues are also related to such distributions, being more complex and lacking the convenient properties of the normal distribution. Still, a leptokurtic distribution would lead to a larger distance between PFE and ES, making the latter even better at capturing the exposure that may arise during periods of market stress.

Financial institutions need to balance their profits and robustness. Incorporating risk measures generating higher exposure will also naturally lead to higher capital requirements, reducing profitability. Thus, measuring risk most correctly is the goal, resulting in more understandable capital requirements, as well as more solid financial institutions. An ES measure could thus lead to lower safety margins on top of the measured exposure.

A shift toward ES has been seen in the banks' trading books after the financial crisis [4]. These regulations concern exchange traded instruments such as stocks and bonds. For standard derivatives, traded on exchanges, central clearing is standard, and the CCP is responsible for handling counterparty risk. This leaves the OTC derivatives, where the risk is only required measured through a VaR measure [18]. A shift towards measurement of risk in non-centrally cleared through ES as well could thus lead to a more stable financial system in total, reducing financial institutions' vulnerability to larger market movements than expected.

With none of the risk measures providing satisfactory results for the exposure during stressful periods, capital held by financial institutions should exceed the limits posed by the ES if models similar to this is used. One solution could be a constant factor times the ES, chosen carefully to balance profitability and risk reduction. Hedging against certain risk factors, such as increased volatility, is also a viable solution for many financial institutions, but hardly anything that can be required through regulations.

Chapter 8

Concluding Remarks

8.1 Conclusion

In this thesis counterparty exposure has been estimated for a pay floating – receive fixed swap contract with five years maturity and annual settlements. The exposure was estimated using the risk measures ES and PFE. The future evolution of interest rates was simulated with the HJM framework for interest rates. The swap contract was analysed for four different five year period, using historical data of at least three years for each period.

ES was found to outperform the PFE measure in terms of capturing the real exposure of a swap during these periods. However, none of them provided sufficiently wide boundaries to capture the actual exposure during the financial crisis, and neither during the fairly aggressive interest rate hike in the period preceding the crisis of 2008. For the more recent periods with historically low rates, the actual exposure was well below the boundaries posed by both these risk measures.

The normally distributed interest rates generated by the HJM model was discussed, and found to be reason for the ES not capturing the actual exposure during volatile periods. To accommodate this problem, using a simulation method with a leptokurtic distribution such as Student's t-distribution was discussed. A such distribution would lead to a higher difference between the PFE and the ES, causing the ES boundary to be higher, thus capturing the tail risk experienced in volatile periods.

It was briefly discussed whether the volatility structure used to simulate the interest rates was sufficiently well formulated to capture periods of market stress. Not only does such periods have much higher levels of volatility, but there is also observed a correlation breakdown during such periods, changing the traditional correlation patterns. In light of this, incorporating a stochastic volatility model was briefly discussed.

As for the current markets, interest rates are very low, to a degree that could be regarded as ”uncharted territory”. No extreme events have occurred for a significant period of time, and market volatility is low. The implications of this low rate environment is unknown, and one cannot be sure whether this is a new normal, or if interest rates and volatility will return to higher levels again. In any case, loosening the regulatory framework and capital requirements is something to handle with utmost care. Crises have a tendency to occur when nobody expects them, having unexpected consequences. It is also worth noting that the existing regulations have many more considerations than what is done in this thesis, many which are not taken into account at all.

Risk measures that are incorporated into regulations for financial institutions are expected to provide accurate representations of the real risk associated with the financial activities of the institution. Regulatory bodies as well as financial institutions should be aware of the strengths and, especially, shortcomings of the regulations that are effective at any time, enabling them to take action when the situation require measures to be taken.

8.2 Further Work

A natural path to explore for further work, is to perform similar analysis to the one in this thesis with other modern interest rate models, such as the LIBOR market model (also called the Brace-Gatarek-Musiela model). An outline of this model can be found in [13] or [7]. Apart from historical interest rate data, no market data has been used for the analysis in this thesis, and comparing the prices of derivatives generated by the HJM framework to real market prices could be useful for checking whether the simulated prices are in fact accurate or far off from reality.

To further examine the claims made about leptokurtic distributions in this thesis, it is also natural to continue working with incorporating a framework that produces simulations according to such distributions. Then, examining the claims made here further could also be done.

Another step that could contribute to establishing confidence in the models and/or capital requirements proposed as a consequence of the model is to construct scenarios of market stress, and simulating counterparty exposure under these these conditions. Accurate generation of realistic scenarios of stress would be a major challenge if this exercise were to be performed.

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Appendix

A Measure theory

This section presents some elementary background from measure theory, necessary to fully understand the theory presented in chapter 2. The contents of this section is based on the books by Bingham & Kiesel [11] and Klenke [25].

For the remainder of this section, let Ω be a set.

Definition A.1 (Algebra)**.** *A collection of subsets A_0 of Ω is called an algebra on Ω if*

1. $\Omega \in A_0$,
2. $A \in A_0 \Rightarrow A^c = \Omega \backslash A \in A_0$,
3. $A, B \in A_0 \Rightarrow A \cup B \in A_0$.

A more general version of an algebra is the σ-algebra, defined the following way.

Definition A.2 (σ-algebra)**.** *An algebra A of subsets of Ω is called a σ-algebra on Ω if for any sequence $A_n \in A$, additionally*

$$\bigcup_{n=1}^{\infty} A_n \in A$$

The pair formed by the set Ω and the σ-algebra is called a measurable space. In order to make this measurable space into a more useful concept, functions, or more accurately measures, will be defined on the measurable space.

Definition A.3 (Measure)**.** *A countably additive map, μ, on the measurable space (Ω, A), such that*

$$\mu : A \to [0, \infty],$$

is called a measure on that measurable space. Further, (Ω, A, μ) is called a measure space.

When the measure on the measurable space is a probability measure assigning values in $[0, 1]$ to each event, the measure space is called a probability space.

A subclass of σ-algebras that is important in derivative pricing theory is the filtration. It is defined the following way. [25]

Definition A.4 (Filtration)**.** *Let* $\mathbb{F} = (\mathcal{F}_t, t \in \mathbb{R})$ *be a family of* σ*-algebras, where* $\mathcal{F}_t \subset \mathcal{F}$ *for every* $t \in \mathbb{R}$*. If* $\mathcal{F}_s \subset \mathcal{F}_t$ *for all* $s \leqslant t$*,* $\mathcal{F}$ *is called a filtration.*

Another concept from measure theory having significant importance in the theory of derivative pricing is the equivalence of measures, enabling market participants to disagree on the probabilities of an event, yet agreeing on the price of derivatives. A few results are necessary here.

Definition A.5. *A measures* $\mathbb{P}$ *is absolutely continuous with respect to a measure* $\mathbb{Q}$ *defined on the same* σ*-algebra* $\mathcal{F}$ *if* $\mathbb{P}(A) = 0$ *whenever* $\mathbb{Q}(A) = 0, A \in \mathcal{F}$*. This is denoted*

$$\mathbb{P} << \mathbb{Q}.$$

Theorem A.1 (Radon-Nikodým)**.** $\mathbb{P} << \mathbb{Q}$ *iff there exists a* $\mathcal{F}$*-measurable function* f *s.t.*

$$\mathbb{P}(A) = \int_A f d\mathbb{Q}, \quad \text{for all } A \in \mathcal{F}$$

The Radon-Nikodým theorem is an important prerequisite for the vital *Radon-Nikodým derivative* in derivative pricing. Analogous with ordinary calculus, $d\mathbb{P}/d\mathbb{Q}$ for a function f is written if

$$\int_A d\mathbb{P} = \int_A \frac{d\mathbb{P}}{d\mathbb{Q}} d\mathbb{Q}, \forall A \in \mathcal{F}.$$

Shorthand, when $\mathbb{P} << \mathbb{Q}$ this is written

$$d\mathbb{P} = \frac{d\mathbb{P}}{d\mathbb{Q}} \mathbb{Q}.$$

Definition A.6 (Radon-Nikodým derivative)**.** *The measurable function* $d\mathbb{P}/d\mathbb{Q}$ *is called the Radon-Nikodým derivative of* $\mathbb{P}$ *with respect to* $\mathbb{Q}$*.*

Definition A.7 (Equivalence of measures)**.** *The two measures are* $\mathbb{P}$ *and* $\mathbb{Q}$ *are said to be equivalent if* $\mathbb{P} << \mathbb{Q}$ *and* $\mathbb{Q} << \mathbb{P}$*.*

Essentially, this means that equivalent measures agree on which events that have probability zero. The concept of equivalent measures is especially important for the theory on risk-neutrality, being a cornerstone in the theory of derivative pricing.

B Stochastic Calculus

Many models in mathematical finance rely heavily on results from stochastic calculus. This section presents the theory within this field being relevant for the models in this thesis.

Definition B.1 (Standard Brownian Motion). *A continuous-time stochastic process W_t is a Standard Brownian motion, or a Wiener process, if the following three statements are true.*

1. $W_0 = 0$;
2. $\{W_t\ , t \geqslant 0\}$ *has stationary and independent increments;*
3. *For every* $t > 0,\ W_t \sim N(0, \sigma^2 t)$.

The Standard Brownian motion is a widely used process, which is central in many financial models, mainly used for modelling the evolution of financial assets, including stock prices and interest rates.

Definition B.2 (Adapted Stochastic Process). *A stochastic process $X(t), t \geqslant 0$, is adapted to the filtration $\mathcal{F} = \mathcal{F}_t, t \geqslant 0$ if X_n is $\mathcal{F}_t$-measurable for all t.*

In other words, the value of an adapted stochastic process, $X(t)$,is known at time t.

A class of stochastic processes having great importance in finance, is the martingales. Martingales describe "fair games" which is typically the case for many processes in finance.

Definition B.3 (Martingale). *A stochastic process, X_t, is said to be a Martingale if*

1. *X is adapted, and* $\mathrm{E}[|X_t|] < \infty$ *for all* $t \geqslant 0$
2. $\mathrm{E}[X_t|\mathcal{F}_s] = X_s$ *for all* $0 \leqslant s < t < \infty$

where $\mathcal{F}_s$ is a filtration on X.

Further, a property that is implied by the efficient market hypothesis is the Markov property. This property states that the future development of stochastic processes only depend on the current state [26]. Such stochastic processes are called Markov Chains.

Definition B.4 (Markov Property). *A stochastic process, X, has the Markov property if its next state only depends on the current state, independently of how the current state is reached.*

The Markov property is useful for simplifying many problems by disregarding all earlier information about a process. This causes the complexity of many problems of decrease drastically.

Trading in financial markets takes in reality place in discrete takes, but the intervals are very small, and assuming that trading takes place in continuous time (when markets are open) is normal. When going from discrete time to continuous time, as the time increment $dt \to 0$, the result known as Itô's lemma is useful. Itô's lemma plays an important role in stochastic calculus, comparable to Taylor's theorem in ordinary calculus [26].

Theorem B.1 (Ito's Lemma)**.** *Suppose that the random function, G, is governed by a stochastic differential equation of the form*

$$dG = A(G,t)dX + B(G,t)dt, \tag{1}$$

where A and B are arbitrary functions. Given the function $f(G)$, Itô's lemma says about the differential df that

$$df = A\frac{df}{dG}dX + \left(B\frac{df}{dG} + \frac{1}{2}A^2\frac{d^2 f}{dG^2}\right)dt. \tag{2}$$

C Additional Figures

In this section, the figures obtained from the analysis done with all the available data prior to each period are shown. Since they barely differ from from the ones shown in the analysis, they are discussed in little detail. As the results from the period 2005 - 2010 is exactly equal to the ones presented in the main part, the figure is omitted.

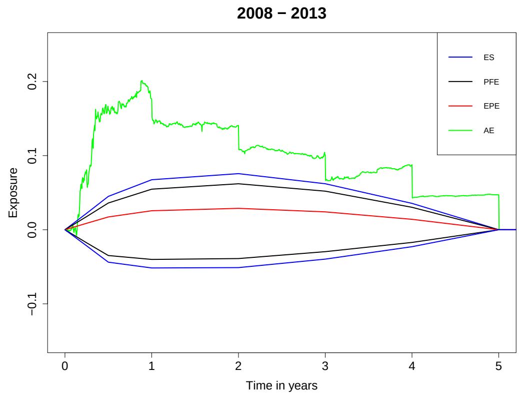

Figure C.1: The exposure using all data in the period 2008 - 2013.

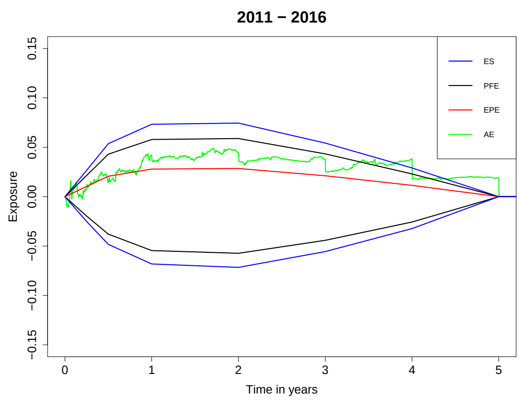

Figure C.2: The exposure using all data in the period 2011 - 2016.

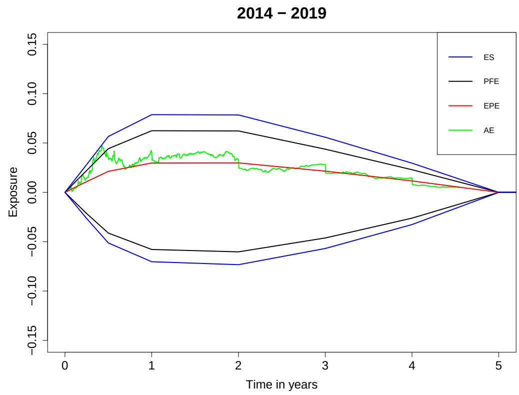

Figure C.3: The exposure using all data in the period 2014 - 2019.