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Optimal Turbine Re-Investment Strategies in Hydropower

Master's thesis in Industriell Økonomi og Teknologiledelse
Supervisor: Maria Lavrutich
June 2019

Master's thesis

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Preface

This thesis was written during the spring of 2019 and serves as the concluding part of our Master's degree programme at the Norwegian University of Science and Technology (NTNU). The degree is a specialization in Financial Engineering at the Department of Industrial Economics and Technology Management.

We would like to thank our supervisor, Associate Professor Maria Lavrutich, for excellent guidance and invaluable feedback during the work on this thesis. We have taken great lessons from her knowledge within the field of real options and she has been a true inspirational source throughout the work. We would also like to thank our co-supervisor, Professor Stein-Erik Fleten, for stimulating discussions regarding how to link the technical and economic aspects that are embedded in the model. We highly appreciate their contribution and involvement. Further, we would like to thank Eivind Solvang and Thomas Welte at SINTEF for providing valuable insights into the industry of hydropower and for good advice on the technical parts of the thesis.

Trondheim, 10th June 2019

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Abstract

This thesis studies the investment behaviour of a hydropower firm that faces a deteriorating turbine efficiency. The firm can choose between two strategies: (i) replacing the turbine directly, and (ii) upgrading, or extending the lifetime of the turbine by changing the arc of the future degradation before an eventual replacement. The thesis aims to show what incentives the firm may have to instigate either of the two strategies and under which conditions it chooses to do so. Our main goal is to provide a tractable model that helps to navigate the firm in an uncertain environment to make the optimal re-investment decisions. In this setting, the firm faces an optimal stopping problem of when to make the investment. We solve this by using the real options approach. As opposed to traditional models, we find that the investment region in some cases becomes dichotomous, and that both strategies are optimal in different regions of the state-space. If this is not the case, direct replacement is the uniformly dominating strategy.

In our model, the optimal investment environment depends on the relative attractiveness of the renovating strategies. Because the replacement option is embedded in both strategies, it is the upgrading option that mainly determines the optimal decision. When upgrading becomes more favourable due to an increase in the relative value of a lifetime-extension of the turbine, we find that the dichotomous investment environment is more likely to be dominant. If, however, a turbine upgrade is found to have a smaller impact on the overall value, the opposite holds. This is especially notable when we investigate the effect of operating with a low turbine efficiency. In this case, the payoff from boosting the efficiency to a new starting level outweighs the benefits from improving the future trajectory of the degradation process. As a result, direct replacement is the dominating strategy in this situation.

As a possible extension of our model, we discuss the implications of including the risk of a turbine failure. We argue that the firm would be incentivized to invest earlier in both of the strategies if the failure rate decreases when the turbine is upgraded or replaced. The reason for this, is that the firm should be more willing to pay the sunk investment cost in order to reap the benefit of a reduced possibility of failures.

Samandrag

Denne oppgåva tek føre seg investeringvala til eit vasskraftselskap som har ein turbin med avtakande verknadsgrad. Selskapet kan velje mellom to strategiar: (i) å erstatte turbinen direkte, eller (ii) å oppgradere den eksisterande turbinen, altså forlenge levetida ved å endre utviklinga i verknadsgradskurva, for deretter å erstatte turbinen på eit seinare tidspunkt. Denne oppgåva ønsker å vise kva for nokre incentiv selskapet har til å iverksetje ein strategi, og under kva omstende det vel å investere. Vårt hovudmål er å utvikle ein brukarvennleg modell som kan hjelpe selskapet med å navigere i eit usikkert landskap. På denne måten tilbyr vi eit verktøy som kan brukast for å avgjere dei beste vala knytt til reinvesteringar. For å finne den optimale investeringsstrategien brukar vi realopsjonsmetoden. I motsetnad til tradisjonelle modellar finn vi at investeringsregionen kan vere dikotom, og at begge strategiane då er optimale i ulike delar av løysingsrommet. I dei tilfella der investeringsregionen ikkje er dikotom, er det optimalt å berre erstatte turbinen.

Vi finn at det optimale investeringsmiljøet i stor grad er avhengig av den relative appellen til dei to strategiane. Sidan å erstatte turbinen inngår i begge strategiane, er det i hovudsak oppgraderinga som avgjer investeringsmiljøet. Når det blir meir gunstig å oppgradere grunna ein høgare relativ verdi av å forlenge levetida har det dikotome investeringsmiljøet eit høgare sannsyn for å vere dominerande. Utfallet vert derimot det motsette om ei oppgradering ikkje har like stor innverknad på den overordna verdien. Denne skilnaden er spesielt tydeleg dersom ein ser på konsekvensen av å ha ein turbin med lav verknadsgrad. I dette tilfellet vil det å auke verknadsgrada direkte vere mykje meir verdifullt enn å auke den framtidige utviklinga til den eksisterande verknadsgradskurva. Difor er det i denne situasjonen den strategien som berre erstattar turbinen som er dominant.

Som ei mogleg utviding av modellen diskuterer vi kva implikasjonar det vil ha for resultata dersom ein inkluderer moglegheita for turbinsvikt. Vi argumenterer for at selskapet vil ha eit ekstra incentiv til å investere tidlegare i begge strategiane dersom både oppgradering og erstatting av turbinen reduserer sviktraten. Grunnen til dette er at selskapet burde vere villegare til å betale investeringskostnaden for å redusere sjansen for svikt.

Problem description:

What is the optimal re-investment policy for a hydropower firm that faces a deteriorating turbine efficiency when having the option to choose between two mutually exclusive investment strategies?

Contents

1. Introduction

The EU's Renewable Energy directive aims to increase the share of renewable energy sources in the EU to 27% by 2030 (European Commission, 2014). The majority of the increase is likely to come from intermittent energy sources, such as windmills and photovoltaic cells. Although these energy sources have a low environmental footprint, the availability of energy from these is inherently dependent on current meteorological conditions. This poses a challenge, and creates an increasing need for balancing power. Hydropower has the potential to fill this need, since it to a large extent can be controlled independent of current meteorological conditions.

For this reason, the future increase of intermittent energy sources in the energy mix is likely to change the use of hydropower plants (Gaudard and Romerio, 2014). The standard today is to use production scheduling tools that calculate the optimal price and time to produce, preferably at the best operating point (BOP), where the efficiency is at its highest level. If hydropower is going to be used as a source of balancing power, the production at BOP is likely to decrease, while the number of starts and stops will increase. This does not only change the operating pattern and associated costs, but also accelerates the rate at which the turbine degrades (Bakken and Bjorkvoll, 2002). With the alleged change in operational conditions, it will become more important for hydropower operators to optimally time their refurbishment decisions so as to maximize their overall net profit, reduce re-investments and extend the component lifetimes (Kristiansen, 2017).

Our goal is to propose a general framework that can be used by different hydropower producers to determine their optimal re-investment policy, and when these investments should be undertaken. We use real options theory to model this, and apply it to a Norwegian context in a numerical example. For decades, Norway's primary energy source has been hydropower. In 2017, hydropower accounted for 95.8% of domestic energy production (Statistisk Sentralbyrå, 2018). As most of the possibilities for hydropower production in Norway have already been developed, the capacity of existing plants must either be increased or utilized in a more efficient manner to increase the total output. For hydropower producers, realization of this potential becomes relevant when faced with the opportunity to make investments in their own power plants. Examples of such investments are replacement of the turbine or overhauling of the existing turbine. The reason for making these investments is largely due to the degradation of existing equipment. During the lifetime of a hydropower turbine, the efficiency will decrease due to mechanical fatigue, cavities, cracks and erosion. This reduces the maximum possible output of the turbine and thereby also the profits.

In Norway, the majority of hydropower plants were developed between 1950 and 1990[1]. Many of these power plants still use the original turbine, which means that there will be a growing need for renovation and turbine re-investments in the relatively near future, as shown in Figure 1.1. This poses the following problem: given the state of the current turbine and the choice between replacing or lifetime extension, what will the choice be and when will it be acted upon? Since different operating conditions and operating patterns may cause the state of turbines of similar age to vary greatly, some turbines may exceed their technical lifetime without interference, while others will need a more serious premature intervention. Examples of such interventions are refurbishment, upgrading and replacement. In the hydropower industry, replacement of the turbine runner is a common procedure in order to increase the output of the hydropower plant. Some hydropower plants which have already undergone such re-investments in recent years are Trollheimen[2] and Tussa[3].

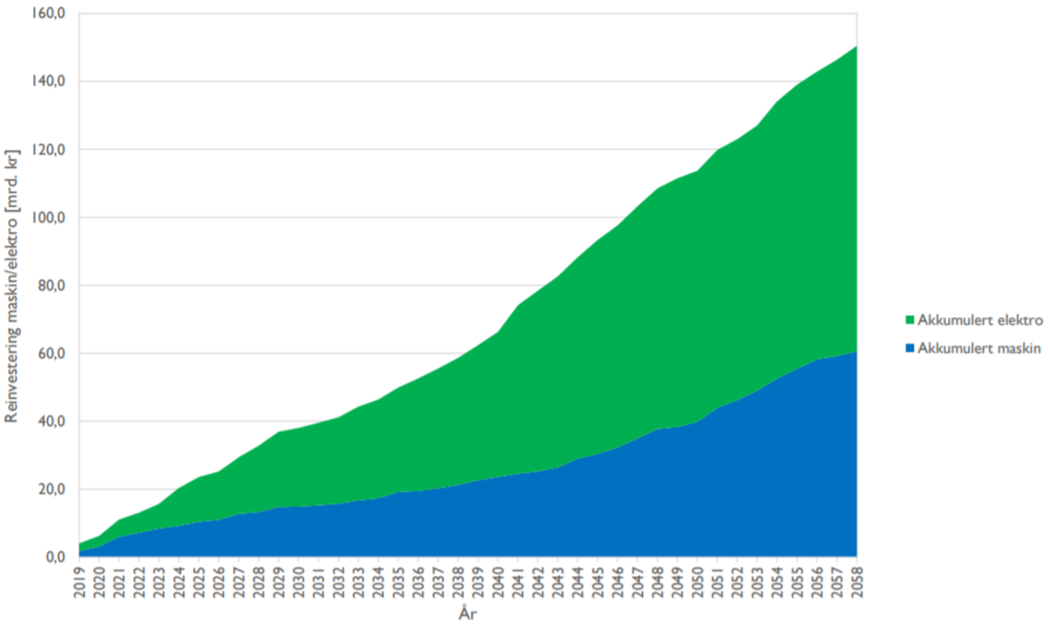

Figure 1.1: *Estimate for future re-investment needs in Norwegian hydropower (NVE, 2019)*

In this thesis, we aim to investigate the re-investment problem by utilizing two different renovating options corresponding to the choices that are available to the hydropower operator when facing an ageing turbine. The two investment options that we investigate are upgrading and replacement. Upgrading is defined as an improvement of the existing turbine in order to increase the expected remaining economic lifetime. Different methods of this include overhauling, coating and changing the operational pattern. Upgrading is a less costly alternative to replacing the turbine, both in terms of investment costs and associated production interruptions. In many cases, it is therefore used as an alternative in order to delay the larger investment cost of replacement (Goldberg and Espeseth Lier, 2011). Upon replacement, on the other hand, the old turbine is disposed of in favour

[1] https://www.nve.no/energiforsyning/vannkraft/reinvesteringsbehov-opprusting-og-utvidelse/?ref=mainmenu
[2] https://energiteknikk.net/2017/03/turbinjobber-rainpower
[3] https://www.tussa.no/om-oss/aktuelt-og-media/oppgraderer-tussa-kraftverk

of a new one. The new turbine's fit is often improved relative to the flow of water and the expected future operating patterns by changing the maximum capacity and/or the efficiency's distribution relative to its load. These improvements, combined with the efficiency decay, implies that it is still attractive for hydropower producers to renew their equipment even though technological development has stagnated.

Within the hydropower industry, choices related to re-investments often have large economic consequences. Turbines are costly to replace, and their impact on the overall profit flow is substantial (Ruud, 2017). When replacing or upgrading the turbine, production stops are required, which means that the timing of such actions has a high impact on the total cost. Also, not taking actions on a sufficiently degraded turbine comes with a high alternative cost in terms of forgone profits. Thus, the hydropower producer faces a trade-off between increased future profits, incurring the investment cost, current and future maintenance costs, forgone profits during production stops and a sub-optimal existing turbine.

The contribution of the thesis is two-fold. From the theoretical standpoint, we provide a novel modelling framework that encapsulates the managerial flexibility that is present when a firm faces a deteriorating technical sub-system. More specifically, the thesis expands the literature related to the choice between two mutually exclusive strategies where the investment policy is not merely a simple trigger strategy, but may instead be governed by an investment region that is no longer connected. Our goal is to provide a tractable and flexible model which is easy to implement and that enables the decision-maker to explore the landscape related to renovation and refurbishment. This leads to our second contribution of practical nature, which is to apply the model as a decision-making tool within the hydropower industry. Current practice usually involves calculating operational profits for a given production system by simulating operation. Yet, these models normally do not consider technical information, such as the impact of system operation on failure rates and component efficiencies (Kristiansen, 2017). We propose a method that improves this by accounting for a declining profitability through a continuously degrading turbine efficiency and later discusses how possible turbine failures could affect the results.

We find that if the hydropower firm has the choice between two investment strategies, the investment region may become dichotomous. Because the option to replace the turbine is embedded in both strategies, we find that it is the value of the option to upgrade that mainly determines the optimal investment environment. We also provide a discussion where we contemplate the different aspects of including the risk of a turbine failure. Here we argue that the firm should be incentivized to be more proactive with its refurbishment activities in order to reduce the expected cost of failure at an earlier stage.

The remainder of the thesis is organised as follows: Chapter 2 describes the chosen valuation method and literature related to our problem, Chapter 3 presents the model and the comparative statics for this, whereas Chapter 4 discusses a possible extension of including the risk of failure to our model. Finally, Chapter 5 concludes and presents ideas for further research.

2. Background

2.1 Method for Valuation

The most commonly used method when valuing an investment opportunity is net present value (NPV) analysis. In a study based on more than 350 large companies in Scandinavia, Horn et al. (2015) found that 74% of the respondents utilized NPV analysis in their companies. One of the major strengths of this method is its simplicity and ease of calculation. However, the simplicity comes with several drawbacks (Mun, 2006). First, due to the uncertainty inherent to long-term projects, it is often challenging to estimate the cash flows accurately. Second, it is challenging to approximate the risk-adjusted discount rate for the project, as it is often time-dependent. Third, NPV analysis assumes that the investment is either completely irreversible or completely reversible, which is often not the case. Finally, the method assumes that every investment opportunity is a now or never decision. This implies that it does not take into account the choice of postponing the investment to a later stage in order to collect valuable information along the way.

A valuation method that has the potential to remedy some of the shortcomings of NPV analysis is real options analysis (ROA). A real option can be seen as the real-world counterpart of a financial option, where you have the right, but not the obligation, to trade an underlying asset at a predetermined price. Thus, it is the right, but not the obligation, to act on an investment opportunity. Dixit et al. (1994) present three fundamental elements that must be present in a project in order to call it a real option:

1. Uncertainty related to the profit of the project
2. Irreversibility of the decision to invest
3. Managerial flexibility

The uncertainty in the underlying gives rise to a value in waiting as the firm can observe the development of the different uncertainty factors. However, there is a trade-off between information uncertainty, forgone dividends and the decision to spend a sunk investment cost. In ROA, the value of waiting is taken into account, which is one of its main advantages over NPV analysis. Overall, since NPV analysis is a special case of ROA, ROA is a more robust framework. The simplifications made in NPV analysis could also, according to Dixit et al. (1994), lead to significant decision-making errors and severe undervaluation of investment opportunities. Examples of such undervaluations can be seen when comparing NPVs and ROA values in e.g. Weibel and Madlener (2015) and Fertig et al. (2014).

Triantis and Borison (2001) find that real options theory is more commonly used in industries that exhibit features of high upfront cost and price uncertainty. This is typical for engineering-driven industries which have been subject to major structural changes due to the use of highly sophisticated analytical tools. In the power generating industry, the majority of the investment occurs when building or making alterations to the plant, whereas the cost of operations is small in comparison. For renewable energy plants, such as water and wind, there is no cost of input, which increases the importance of the upfront cost. We therefore find the real options approach to be more suitable in our case. The higher usage rate of ROA as a decision-making tool in this industry is also shown in the study by Horn et al. (2015), where 24% of the respondents in the energy sector answered that they apply ROA. For all respondents, however, the usage rate was only 6%. Part of the reason for this is that ROA is a more complex tool than NPV analysis and that the benefit of implementing it in industries with lower uncertainty and upfront costs does not outweigh the additional complexity.

Dixit et al. (1994) present two methodologies for solving real option problems, namely dynamic programming and contingent claims analysis. Contingent claims analysis is based on ideas from financial economics where an artificially constructed portfolio of financially traded assets exactly matches the cash flows, or at least the variations in cash flows, of the underlying asset of the option. By no arbitrage arguments, the replicating portfolio and the underlying must share return and risk characteristics, and thereby also have the same market value. The advantage of this method is that the parameters are either determined within the model specifications or can be observed or estimated from the market. Dynamic programming, on the other hand, takes a more general approach by assuming an exogenous discount rate. This method was originally developed by Bellman (1956), and is particularly useful when it is challenging to find a replicating portfolio that spans the risk of the project. Its usefulness is also manifested in the simplification of reducing a whole sequence of decisions into a binary world where the process is either stopped or continued until the next period. Because of the challenge in determining an appropriate replicating portfolio, we will use the dynamic programming approach.

Dixit et al. (1994) present the optimal stopping problem as a particular class of the dynamic programming problem. The optimal stopping problem makes use of the binary classification where the immediate decision results in a termination payoff, and the continuation of operation encapsulates the consequences of all future actions. In the formal formulation, let $\pi(x)$ denote the profit flow accrued by being in operation, and $\Omega(x)$ be the termination payoff. The Bellman equation describing the value of the project with respect to the investment can then be written as

$$F(x) = max\left\{\Omega(x),\ \pi(x) + \frac{1}{1+\rho}\,\mathbb{E}[F(x') \mid x]\right\}, \tag{2.1}$$

where x is the state variable, x' is the state in the subsequent period and ρ is the discount rate. The solution space of this problem can be divided into a stopping region, where it is optimal to invest, and a continuation region, where it is optimal to continue operations. In the standard investment models, these regions are separated by a threshold value x^*, marking the value at which the process moves from one area to the other. When $x < x^*$, it is optimal to wait, whereas for $x \geq x^*$, it is optimal to investment immediately[1].

2.2 Relevant Literature

In our model, we use the real options framework to examine two investment opportunities of a hydropower firm, namely replacement and upgrading. Because of the high costs and uncertainty associated with a replacement, ROA is a well-suited tool to handle such decisions. Investments to improve the existing turbine through upgrading, however, are often considered as maintenance and are incurred as corrective measures. We investigate the effects of considering these improvements to be preventive, and as a real option within the same framework as the replacement option.

When considering a decision to improve the state of the system in a capital intensive industry, there is a trade-off between the uncertainties in future profitability and renovating costs. In this thesis, we address how a firm can balance this trade-off. We analyze the behaviour of a hydropower producer with the opportunity to undertake strategic investments which alter the state or the future development of the equipment. This contrasts the vision of the firm as a passive bystander who is obliged to react to an exogenous market process. Our renovating options are based on changes in the underlying state of the system. Upgrading changes the underlying stochastic profit process, whereas replacement restarts the degradation process at a higher level.

The upgrading option in our model alters the parameters of the underlying stochastic process. This is similar to Kwon (2010), who presents a general application on this subject. In this study, a firm has the opportunity to innovate an ageing product while facing a declining profit stream. The profit is modelled as an arithmetic Brownian motion with a predefined change in drift that boosts the stream of profits if the firm chooses to innovate. The firm also has the opportunity to cease operations and exit the market if the conditions become too adverse. The one-time option to make an investment that boosts the project's profit rate in the presence of a declining profit stream is similar to the option to upgrade the turbine in order to slow down the degradation in our model. Unlike Kwon (2010), however, we have an additional option to replace the turbine alongside the option to upgrade. This option gives us a new strategic dimension that needs to be taken into account. Moreover, in the hydropower industry, the operating net profit is almost always positive due to very low operating costs, and therefore, unlike in Kwon (2010), the option to exit the market is not relevant in our case.

The second option that we examine is replacing the turbine. The replacement literature can be divided into two main categories, namely capital replacement of physical assets and asset renewals in general (Adkins and Paxson, 2006). The former focuses on minimizing the losses incurred by having an imperfect component, whereas the latter studies the problem of maximizing the net profit by balancing the revenue from the component with its operational and maintenance costs. Within the capital replacement stream of literature, Yilmaz (2001) studies a single investment optimal stopping problem of when to fix a partially defective component that is conditional on the profitability and expected lifetime.

[1] Remark that the waiting and the investment regions are defined for a call option.

An analytical solution is obtained using a geometric Brownian motion (GBM) as a proxy for lost revenues, a Poisson process to represent the malfunction risk and a deterministically declining equipment state. This is an example of a tractable model that provides a range of opportunities for further development, which we will elaborate on below.

Richardson et al. (2013) also make use of a cost-based approach, but expands the setting of Yilmaz (2001) to provide a way to tackle long and uncertain lead times with repeated investments. This moves the centre of interest from when to replace to when to order because the incorporation of lead times introduces an "option-less" period where the firm awaits the new equipment. That is, once the decision to invest is made, the option to utilize new information that comes at hand until the new asset arrives is forfeited. Also Adkins and Paxson (2011) consider repeated investments, but expands the setting of Yilmaz (2001) in a different direction. They include two correlated GBMs, representing revenue and cost, to investigate the decision of when to renew the equipment. Since the model has two sources of uncertainty, the boundary solution is no longer unambiguous and clear-cut. Owing to the complexity of the functional terms, only a quasi-analytical solution for the function discriminating between the choice of continuance and renewal is obtainable. Lange et al. (2019) have later refuted the approach adopted by Adkins and Paxson (2011), whose method may cause a large approximation error and incorrect optimal policies.

As opposed to a cost-based approach, Reindorp and Fu (2011) aim to maximize the profits of a company within an analytical real options model. In this framework, the firm has the option to renew its obsolete subsystems in an infinite series of investments. While the aforementioned papers also examine replacement decisions in a real options framework, they either assume that some key profit or cost measure remains constant, or that the fault does not return after fixing. Reindorp and Fu (2011) argue that these assumptions do not fit well with the long time horizons that are typical for asset renewal problems. To remedy these limitations, Reindorp and Fu (2011) model the operating profits in an exogenous price process, while the profitability is modelled in a separate, uncorrelated GBM with negative drift. Together, these processes determine the investment cost which is assumed to be stochastic. The authors acknowledge that further work on the stochastic processes is important for advancing the real options approach within the field of replacement investment analysis.

Taking into account the literature mentioned above, real option models often consider either a single investment (as Yilmaz (2001)) or an infinite sequence of investments (as Richardson et al. (2013) and Reindorp and Fu (2011)). A middle ground is to constrain the problem to a finite set of options which are exercised in a sequential manner. Although the time horizon can still be infinite, the length of the sequence is bounded. The main advantage of such an approach is the increased flexibility to make the stages heterogeneous by incorporating different characteristics. The literature in this strand of real options theory is wide and cover many disparate topics. For example, Dixit et al. (1994) show that sequential investment is really no different than ordinary investment, conditioning on that the projects take no time to complete and that there are no other impediments to investment. This is demonstrated with a two-stage investment problem where the firm will undertake both stages simultaneously. Bar-Ilan and Strange (1998) extend the view of Dixit et al. (1994) by incorporating time to completion and the opportunity to suspend investments during a two-stage project. This suspension is essential, because without it, the first-stage trigger is always above the second-stage trigger (Bar-Ilan and Strange, 1998).

In contrast to sequential options, a firm can be offered multiple mutually exclusive options. Dixit et al. (1994) also present a solution procedure for this case. The procedure is based on a separate evaluation of the options, and the optimal investment strategy is simply to choose the investment with the highest option value. Décamps et al. (2006) challenge the premises put forward by Dixit et al. (1994), where the dominant incongruity is that the optimal investment strategy is not necessarily an unambiguous trigger strategy, but can instead be dictated by the existence of a dichotomous investment region. This implies that the investment region may no longer be connected. In the study by Décamps et al. (2006), a decision-maker has to choose, under price uncertainty, among two alternative projects of different sizes and with different investment costs. The model is thereafter developed to evaluate a sequential investment problem where the firm is allowed to switch from the smallest to the largest project at an arbitrary time. The key finding is that the investment region under certain conditions is dichotomous, and that there might exist a sub-set of regions where the decision-maker will wait to invest in either of the two projects, even though the current price is above one of the initial project thresholds.

Similar to Décamps et al. (2006), we find that the investment region may become dichotomous under certain conditions. However, where Décamps et al. (2006) study a switch between similar projects of different sizes, we study options which affect the underlying profit flow in different ways. In addition, we apply our model to a specific industry. Because of this, some additional factors have to be addressed. Since a turbine is subject to degradation over time, it will have an adverse effect on the hydropower producer's profit. This may influence the resulting investment choices. Furthermore, because there is a direct relationship between the profit stream in a hydropower plant and its production, the scheduling of power production can have a large impact on the resulting profit flow.

Within the strand of literature that more specifically focuses on hydropower investments, production scheduling tools are commonly included. For instance, Fertig et al. (2014) study the optimal investment timing and capacity choice for a pumped hydropower storage system in Norway. This study makes use of a production scheduling model based on daily optimization of the next week's production to obtain the profit flow. Other versions of production scheduling tools are used in Brøndbo et al. (2019) and Andersson et al. (2014). These tools often include factors such as long-term climate development, inflow, and long- and short-term production management of the reservoir. It also enables the hydropower producer to modify the model to their specific plant. Even though production scheduling as an input to the real options framework results in a more realistic profit flow, it often requires numerical solutions. Among analytical alternatives is the framework developed by Ernstsen and Boomsma (2018) to value renewable energy plants, including hydropower. Although this framework requires additional assumptions to be made, it provides a tractable and user-friendly model. Yet, it is important to mention that their model is a pure value-setting tool, so combining this with an investment problem would result in a highly complex model. The aforementioned papers do not, however, incorporate the possibility for the firm to affect its profitability. This is a common assumption, but we argue that the uncertainty affecting the firm's internal affairs, such as the deterioration of the turbine efficiency, is endogenous and can to some extent be controlled. Our model tackles this opportunity by including a deterioration process that is affected by the investment choices.

Even though it is commonly known that the turbine efficiency degrades, the rate of degradation and its development over time is to a large extent unknown (Ruud, 2017). The reason is that measurement of the efficiency is a relatively seldom procedure, as it requires a full stop in the production and is, therefore, costly to undertake. Thus, the industry does not find it beneficial to pay such a high cost to know the exact efficiency of the turbine. In the future, however, this is likely to change. As the efficiency of hydropower turbines are closing in on their theoretical limits, researchers are shifting their main focus away from improving the equipment to learning more about how to flexibly adapt the operation to different market conditions. This implies that the modelling of efficiency degradation will become increasingly more important.

Since degradation processes usually are challenging to accurately observe, the literature offers a wide range of models. Markov Decision Processes (MDPs), which usually work in a discrete state-space, is one of the more common approaches. The Markov states divide the state-space into different intervals of degradation. Such intervals strongly relate to industry standards where 4 or 5 state levels often are used because of the challenge of accurately describing the true state of the equipment (Welte, 2008). MDPs are exemplified by Papakonstantinou and Shinozuka (2014*a*,*b*) and Welte (2008). The strength of MDPs lies in their transparency and the few parameters that need to be estimated. However, this comes at a cost of requiring the degradation rate to be constant at each step. As a remedy to this, the number of steps is often increased to get a more realistic degradation rate development. That, however, quickly increases the number of parameters and the computational complexity of the problem.

An alternative to MDPs are more theoretical and computer-based models. The success of such models is largely contingent on the user-friendliness of the framework (Welte, 2008). An early model developed by Lu (1995) assumes that the degradation follows a Gaussian process which is particularly applicable when lifetime data is sparse. Yet, the model has later encountered criticism because the Gaussian process is not strictly increasing and can become negative. As a remedy to this, Park and Padgett (2005) provide several accelerated stochastic degradation models, among them a GBM which by definition is always positive. Another approach that more specifically targets the degradation process per se, is proposed by Whitmore (1995). Here the degradation is assumed to follow a Wiener diffusion process whose parameters are based on empirical measured data. Compared to a GBM, it is assumed to represent a more realistic process, but this naturally necessitates a longer history of sampled efficiency data.

Ruud (2017) presents multiple examples of degradation processes in a study on the optimal time to replace the turbine runner of a hydropower plant. The study uses the state of the equipment as one of the determinants of the investment decision. Many of the degradation models employed by the industry are highly tractable, but are not made to handle such investment decisions. As a result, the effect of degradation is often overshadowed by the complexity in other components (Ruud, 2017). Examples of models that face this challenge are linear degradation curves, which assume a constant degradation rate, and polynomic and logistic models, which assume that the degradation rate increases with time. A model that attempts to overcome this issue is the VTG Revision model, developed by SINTEF in 1995. This model provides decision support for maintenance and revision of hydropower plants by incorporating an exponentially declining function approaching an asymptotic value for the decay in efficiency[2] (Ruud, 2017). In spite of the issues that these models present, many of them are still in use. However, the industry is on a constant look-out for new and improved models.

In this thesis, we provide a stylized model that allows hydropower producers to optimally choose between different re-investment strategies. The thesis is also a novel contribution to the literature because we combine the theoretical perspective of two mutually exclusive investment strategies with a practical application to the hydropower industry. In the model, we incorporate firm-specific influences on the future profit as it is incentivized to take action due to the declining profitability originating from the decaying turbine efficiency.

[2] The VTG Revision model is no longer commercially available and has been criticised to only be applicable to specific cases, see Ruud (2017).

3. Modelling Framework

In this chapter we present the modelling framework that we use to analyze the investment decisions of a hydropower producer considering several investment opportunities and facing a deteriorating turbine efficiency. The strategic approach is two-fold, where the question is if the turbine should be upgraded before it is replaced, or simply just replaced without upgrading it first. Thus, the objective of the firm is to maximize its value with respect to both the optimal renovating strategy, and the optimal investment timing. Our aim is to do this with a tractable model and in a general format in order to provide insights from the decision-making process. We start by building a structure of the different processes that will serve as the building blocks for our model. These building blocks are the gross operating profit (excluding the efficiency) and the turbine efficiency. Thereafter, we isolate the two turbine renovating strategies and solve them separately by applying the real options framework. Finally, the two strategies are embedded in the same framework to reveal under which circumstances one dominates the other.

Our modelling framework is based on a stochastic process that combines the uncertain processes of price and inflow into one single entity, namely the gross profit. This simplification contrasts the norm, which is to first use a production scheduling framework to obtain the profit, before using its results as input in a real options model. The most prominent feature of production scheduling is its ability to modify the model in relation to the particular hydropower plant, regarding, for example, topographical, technological and meteorological conditions. However, this comes at the cost of computational time, and the inability to generalize the results. By contrast, our approach allows for an analytical solution procedure which is considerably less computationally demanding. The model also provides a stylized and general decision-support tool. This is particularly valuable when considering long-term investment decisions where day-to-day operations of the hydropower plant has a negligible impact. If the time horizon is shorter, or if the choices made in the production scheduling framework are likely to have a significant impact on the solution, it is possible to solve a semi-analytical model given some simplifying assumptions. An example of this is the model of Ernstsen and Boomsma (2018).

In our model, both revenues and costs have to be integrated in the same, unique stochastic process, or else we would get a two-factor model similar to Lange et al. (2019), which requires numerical solution methods. With this in mind, we represent the operating profit as a stochastic process, gross of the turbine efficiency. The reason for separating the operating profit and the turbine efficiency is to disentangle their effects in the model such that it becomes intuitive to interpret the implications. A single process also maintains the generalizability of our model. In a shorter horizon, specific underlying factors of this process, for example inflow and O&M costs, might have to be modelled separately because of the change in model requirements. This increases the complexity of the calculations, and will not have a significant impact given our long-term perspective.

Unlike Richardson et al. (2013) and Yilmaz (2001), our model is based on maximizing the expected profit, and can therefore not be purely cost-based. A cost-based approach would shift the focus towards maintenance and inspection, which would lead to a cost minimizing problem. Nonetheless, a profit maximizing approach does not exclude the possibility of including the relevant costs. Fixed costs related to maintenance and labour are easily incorporated by subtracting a constant term from the gross operating profit. Variable costs, on the other hand, such as the cost of obtaining water in the reservoir is negligible. Together, this results in a quite stable cost structure conditioning on that no high-impact events occur[1]. These events are usually assumed to occur with a very low probability, and therefore we disregard them in our model.

The gross profit flow of a hydropower plant equals its revenue less its costs. As the costs are relatively predictable, the main uncertainty in profit flow will stem from the revenue. The revenue is made up of two components, usually presented by the following simple relationship:

$$Revenue = price \cdot volume\ sold.$$

With a long-term perspective, the total accumulated volume tends to be quite predictable for a given hydropower plant. This relies on an exclusion of the consequences that climate change may have on the development of the inflow pattern in the future, which is outside the scope of this thesis. Regarding the development of price, this process is likely to have a much higher influence on the parameters in our model due to its idiosyncratic and erratic nature (Escribano et al., 2011). With the structural preliminaries for the processes behind us, we can now formally introduce the two building blocks for our model. We start by discussing the specifics of the profit flow before we present our proposed process for the development of the turbine efficiency.

3.1 Defining the Profit Flow, Gross of the Efficiency

We assume a continuous-time framework where the gross profit for the hydropower plant can be described by a stochastic Itô-process. This is because the profit flow of the hydropower plant is typically subject to temporal uncertainty and is dynamically dependent on its current state. We define the profit as P_0, which is the time-dependent profit received from being in active operation before adjusting for any deteriorating processes. We assume that P_0 can be properly represented by a GBM, expressed by

$$dP_0 = \alpha P_0 dt + \sigma P_0 dZ_t, \tag{3.1}$$

[1] Examples of high-impact events are turbine failures, long-term power outage, damage or flooding of the dam and destruction of transmission lines due to bad weather.

where α is the deterministic drift and dZ_t is the increment of a Wiener process, representing the stochastic component. The volatility of the stochastic process is denoted by σ.

Our argument for choosing GBM as a proxy for the profit flow is two-fold. First, since there are virtually no variable costs, the operating profit flow will always be positive. In the previous section, we mentioned that our model can be adjusted to include the fixed costs, but as their name suggests, they are inherently non-stochastic. Hence, we are able to separate them from the profit flow without affecting our model. We also assume that these costs are unaffected by the investment decisions that we examine. As a result, these costs are excluded from the model. Second, having a long-term perspective enables us to downplay the intricacies at the micro level, such as capturing all the relevant features for the electricity price and the inflow. The long-term perspective also decreases the potential of model misspecifications by using an integrated, stochastic profit process rather than the combined production scheduling price-dynamical framework. This argumentation is supported by Fleten et al. (2007), who state that a stochastic description of short-term deviations is more important if the focus lies in the investigation of the operational flexibility that facilitates a opportunistic market strategy. Similarly, Dixit et al. (1994) claim that applying a GBM for the price of a commodity will not lead to large errors when considering long-term investments.

In order to apply a GBM, we have to assume that both α and σ from Eq. (3.1) are constant. According to Fleten et al. (2007), an investment in a renewable power generating unit should be regarded as a long-term investment where short-term deviations, which are normally time-dependent, only have minor influence on the resulting investment decisions. If the perspective were to have a shorter horizon, a constant σ would not be as suitable. This is because, in the short-term, the price process is likely to display signs of time-dependence, spikes, volatility clustering, seasonality and mean-reversion, which all contradict a constant volatility (Swider and Weber, 2007; Geman and Roncoroni, 2006). Also the production volume is unlikely to follow a GBM in the short-term because of strong signs of seasonality. This is not likely over the long-course, however, due to natural limitations[2].

The main motivation for modelling P_0 as a single exogenous process originates from the objective of finding a good balance between a model that provides a realistic representation and tractable and analytic results. By assuming that the horizon of the investment opportunities stretches over decades, applying a GBM for the gross profit flow is considered to be reasonable. If the perspective is more balanced towards the short-term, a different profit model would be recommended. One example is to use the two-factor model developed by Schwartz and Smith (Schwartz and Smith, 2000), which takes into account both long-term and short-term components.

[2] The production has to stay within the reservoir limitations and is mainly affected by precipitation and a relatively stable market demand in the long-term.

3.2 Defining the Turbine Efficiency

As highlighted in the previous chapter, it is common knowledge that the turbine efficiency decays over time, yet it is very hard to accurately determine the true state of the equipment. Reliable data is often scarce and expensive to obtain (Welte et al., 2006; Ruud, 2017). Many of the degradation models that are applied in the industry are rather complex, such as MDPs and Gaussian, GBM and complicated gamma processes. Some of these models are classified as stochastic, others are classified as deterministic. We use a deterministic process, which fits the tractable framework that we aim for. If a stochastic process is used, the complexity increases significantly due to multiple sources of uncertainty. To avoid spurious relationships resulting from the model, we choose the fairly tractable exponential function to represent the degradation process. This function is relatively transparent, and is commonly used as a first modelling approach in quality engineering (Rausand, 2004). The degradation process can therefore be written as:

$$Q_t = Q_0 e^{-\gamma t}, \tag{3.2}$$

where the change in state can be formulated in the following way:

$$dQ_t = -\gamma Q_0 dt. \tag{3.3}$$

In this exponentially declining process, $\gamma \in \mathbb{R}_{\geq 0}$ is the shape parameter that defines the speed of deterioration, Q_t is the technical condition of the system at time t, and Q_0 is the initial state of the system. Note that Q_0 can be specified in both absolute and relative terms. In relative terms, it represents the current efficiency relative to the BOP-peak-efficiency of the existing turbine at time 0. In absolute terms, it represents the actual level of efficiency at all times. In this context, the latter approach is more intelligible because it provides a more transparent relationship between the efficiency and the profit flow. Hence, we use an absolute interpretation of Q_0.

Most of the stochastic models that are employed by the industry are multiparametric, whereas the exponential function only has one parameter. According to the Norwegian Water Resources and Energy Directorate (NVE), the annual rate of efficiency decay is typically in the magnitude of 0,1% (NVE, 2017). This favours a parametrically lean model. The scarce amount of available data also favours a model with fewer parameters. Another property of the exponential function is that the rate of decay only depends on the current state of the efficiency (see Eq. (3.3)). Upon linking the efficiency and the gross profit in a common framework, these properties are advantageous.

3.3 Two Baseline Strategies

In the subsequent sections we introduce the two strategies the firm can implement. These are (1) the option to simply replace the turbine at an optimal time in order to restart the degradation process, meaning that the efficiency recoups its lost potential, and (2) the option to refurbish the existing turbine and stagnate the degradation process before later replacing it. The former is a standard optimal stopping problem, whereas the latter is a constrained sequential optimal stopping problem. In both (1) and (2), we define γ_0 to be the status quo shape parameter of the degradation process. Moreover, we define γ_U to be the new shape parameter after refurbishment, which is only relevant for (2). For this to be a viable option, it follows that $\gamma_0 > \gamma_U$. This means that the upgraded turbine degrades slower. In both strategies, the efficiency is reset to a new level Q_R after replacement, with γ_0 as the prevailing degradation rate. The latter premise has no implications for (1), but for (2) it means that the degradation process reverts back to its original decay.

In what follows, we assume the hydropower producer to be both price-taking and profit maximizing. Furthermore, we assume that the investments are made instantaneously without the existence of neither investment lag, nor lead time uncertainty. This implies that the hydropower plant is always active in the market. Including lead time uncertainty should not change the investment incentives, but would complicate the solution procedure so that the added complexity does not weigh up for the added benefit of a more realistic model. To ensure that the generated profit has a unique one-to-one relationship with the turbine that is studied, we also assume that the hydropower plant only consists of one turbine, and that it is not connected to any other hydropower plants. Moreover, the lifetimes of both the currently operating turbine, the upgraded turbine and the renewed turbine are assumed to be infinite. This enables us to maintain an analytical framework, as direct time-dependence is avoided in the optimal stopping problem. Finally, we assume that the decision-maker discounts the future profit at a constant exogenous rate, $\rho > \alpha - \gamma$. This allows us to ignore the situation where it would never be optimal to exercise either of the options, as the expected growth would exceed the discount factor.

Symbol	**Description**
$P_0(t)$	Gross profit process, excluding degradation
Q_0	Starting efficiency of the existing turbine
Q_R	Starting efficiency of a new turbine
γ_0	Degradation rate of a turbine which has not been upgraded
γ_U	Degradation rate of an upgraded turbine
k	Change in profit level due to a changed operating pattern after upgrading
$\pi_0(t)$	Net profit process of a non-upgraded turbine, including degradation
$\pi_U(t)$	Net profit process of an upgraded turbine, including degradation
I_U	Investment cost of upgrading the turbine
I_R	Investment cost of replacing the turbine
ρ	Discount rate
σ	Volatility of gross profit
α	Growth rate of gross profit
μ_0	Redefinition of $\rho - (\alpha - \gamma_0)$, for ease of notation
μ_U	Redefinition of $\rho - (\alpha - \gamma_U)$, for ease of notation
F_R	Option value of the basic replacement option
G_R	Option value of replacement in the sequential strategy
G_U	Option value of upgrading in the sequential strategy

Table 3.1: *Nomenclature*

Having made the necessary modelling assumptions and defined the gross profit and the efficiency degradation, we can define the full set of parameters that will be used. The nomenclature is found in Table 3.1.

3.3.1 The Single Replacement Option

When the existing turbine is replaced, the efficiency will be reset to a higher level relative to the turbine it replaces. As the best available technology will not decrease with time, $Q_R > Q_0$ will always hold. With this notation, Q_0 represents the initial efficiency of the original turbine at time 0, while Q_R is the efficiency of the new turbine at the time of replacement[3]. We assume that the degradation rate of the new turbine is equal to that of the old one. This is a reasonable assumption in an environment where the equipment has not been through any revolutionary changes in the last decades. Also, the mechanical equipment in the hydropower industry has reached a technological level which closes in on the theoretical limits[4]. Thus, we find it appropriate to say that the hydropower industry is an industry where the technology has reached maturity, which also gives rise to the assumption that technological development is negligible. Therefore, we assume that Q_R will not change over time.

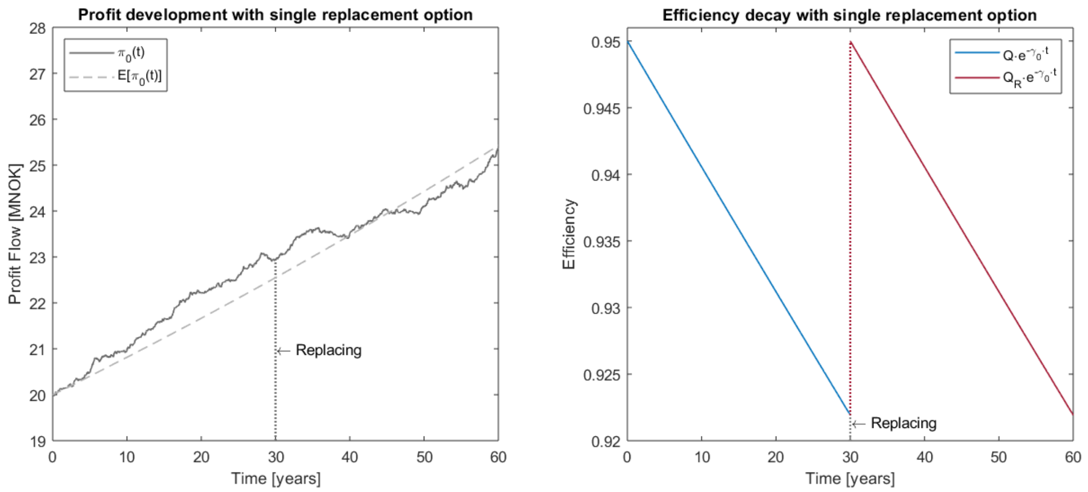

Figure 3.1: *Illustrative profit flow and efficiency development with single replacement option*

Figure 3.1 illustrates the evolution of the profit flow and the development of the efficiency for a firm that chooses to replace the turbine after 30 years. From the first part of the figure, it is obvious that the profit flow closely trails its expected growth. In the second part of the figure, we see that the firm experiences a significant boost in efficiency at the time of replacement.

[3]The values of Q_0 and Q_R are assumed to be slightly lower than the efficiency at the BOP. This is because, in the long run, a hydropower producer is unlikely to always operate under optimal conditions. If the BOP were to be used, it would overestimate the total production and thereby also the operating profits.

[4]In current turbines, the efficiency can be as high as 96% (Store Norske Leksikon, 2019), and the technology is said to be relatively mature. A theoretical limit to the efficiency exists below 100%, as extracting 100% of the kinetic energy in the water means that the water must come to a complete stop.

Formally, the firm needs to solve the following optimal stopping problem:

$$
\begin{aligned}
F_R(P_0) = \sup_{\tau} \mathbb{E}\Bigg[\int_0^{\tau} & P_0(t) Q_0 e^{-\gamma_0 t} e^{-\rho t} \mathrm{d}t - I_R e^{-\rho\tau} \\
& + \int_{\tau}^{\infty} P_0(t) Q_R e^{-\gamma_0 (t-\tau)} e^{-\rho t} \mathrm{d}t \;\Bigg|\; P_0(0) = p_0 \Bigg].
\end{aligned}
\tag{3.4}
$$

In Eq. (3.4), we adjust for the elapsed degradation until the turbine is replaced by multiplying with a factor $e^{\gamma_0 \tau}$, where τ denotes the time of replacement. In this way, the efficiency decline of the old turbine that has happened up until this point will not affect the performance of the new turbine. Moreover, to simplify notation, we incorporate the efficiency decay into the profit flow according to $\pi_0 = P_0 e^{-\gamma_0 t}$. This, as shown in Appendix A.1, can be written as a new stochastic process

$$
d\pi_0 = (\alpha - \gamma_0)\pi_0 dt + \sigma \pi_0 dZ_t. \tag{3.5}
$$

The problem can now be rewritten to yield the following formulation:

$$
F_R(\pi_0) = \sup_{\tau} \mathbb{E}\left[\int_0^{\tau} \pi_0(t) Q_0 e^{-\rho t} \mathrm{d}t - I_R e^{-\rho\tau} + \int_{\tau}^{\infty} \pi_0(t) Q_R e^{\gamma_0 \tau} e^{-\rho t} \mathrm{d}t \;\middle|\; \pi_0(0) = \pi_0 \right]. \tag{3.6}
$$

Proposition 3.3.1 below gives the value of the option to replace the turbine.

Proposition 3.3.1 *It is optimal for the firm to replace its turbine as soon as the process π_0 reaches the optimal threshold, given by*

$$
\pi_{0,R}^* = \frac{\beta_1}{\beta_1 - 1} \cdot \frac{\mu_0}{Q_R - Q_0} I_R, \tag{3.7}
$$

where

$$
\beta_1 = \frac{1}{2} - \frac{\alpha - \gamma_0}{\sigma^2} + \sqrt{\left(\frac{\alpha - \gamma_0}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}}. \tag{3.8}
$$

Thus, the value of the option to replace the existing turbine is given by

$$
F_R(\pi_0) = \begin{cases} A_1 \pi_0^{\beta_1} + \dfrac{Q_0 \pi_0}{\mu_0} & \text{if } \pi_0 < \pi_{0,R}^*, \\ \dfrac{Q_R \pi_0}{\mu_0} - I_R & \text{if } \pi_0 \geq \pi_{0,R}^*, \end{cases} \tag{3.9}
$$

where

$$A_1 = \frac{I_R}{\beta_1 - 1}\left[\frac{\beta_1 - 1}{\beta_1} \cdot \frac{Q_R - Q_0}{\mu_0} \cdot \frac{1}{I_R}\right]^{\beta_1}. \tag{3.10}$$

We observe that the values and thresholds in Proposition 3.3.1 are independent of all the parameters that are related to upgrading. This is as expected because the option does not include an upgrade of the turbine. Furthermore, since this option only represents a single investment opportunity, it closely resembles the solution of a standard real options problem, e.g. as in Dixit et al. (1994).

3.3.2 The Sequential Investment Option

We now consider the strategy where the firm has the option to extend the lifetime of its existing turbine before replacing it. In this sequence, the second option has the same implications as the single replacement option, but is conditional on a different initial state. The exercise order of the options is constrained to this particular sequence, because rearranging it would change the perspective and, as a result, it will not be possible to make a comparison with the single replacement option. Hence, the problem is now reduced to finding the optimal stopping times of the two options conditioning on that upgrade takes place before replacement. The total investment problem can therefore be formulated as follows:

$$\begin{aligned} G(P_0) = \sup_{\tau_1, \tau_2 > \tau_1} \mathbb{E}\Bigg[&\int_0^{\tau_1} P_0(t) Q_0 e^{-\gamma_0 t} e^{-\rho t} \mathrm{d}t - I_U e^{-\rho \tau_1} \\ &+ \int_{\tau_1}^{\tau_2} P_0(t) k Q_0 e^{-\gamma_0 \tau_1} e^{-\gamma_U (t - \tau_1)} e^{-\rho t} \mathrm{d}t - I_R e^{-\rho \tau_2} \\ &+ \int_{\tau_2}^{\infty} P_0(t) Q_R e^{-\gamma_0 (t - \tau_2)} e^{-\rho t} \Bigg]. \end{aligned} \tag{3.11}$$

To simplify calculations, we reformulate the problem using the following notation: $\pi_U = P_0(t) e^{-\gamma_U t}$ and π_0 as in Eq. (3.5). The process for π_U, as shown in Appendix A.1, is represented by

$$d\pi_U = (\alpha - \gamma_U)\pi_U dt + \sigma \pi_U dZ_t. \tag{3.12}$$

Inserting the formulations into Eq. (3.11) gives the following optimal stopping problem:

$$\begin{aligned} G(\pi_0) = \sup_{\tau_1, \tau_2 > \tau_1} \mathbb{E}\Bigg[&\int_0^{\tau_1} \pi_0(t) Q_0 e^{-\rho t} \mathrm{d}t - I_U e^{-\rho \tau_1} + \int_{\tau_1}^{\tau_2} \pi_U(t) k Q_0 e^{-\tau_1 (\gamma_0 - \gamma_U)} e^{-\rho t} \mathrm{d}t \\ - I_R e^{-\rho \tau_2} + &\int_{\tau_2}^{\infty} \pi_0(t) Q_R e^{\gamma_0 \tau_2} e^{-\rho t} \; \Bigg| \; \pi_0(0) = \pi_0, \pi_U(\tau_1) = \pi_0(\tau_1), \pi_0(\tau_2) = \pi_U(\tau_2) \Bigg]. \end{aligned} \tag{3.13}$$

The optimal stopping problem in Eq. (3.13) is visualized in Figure 3.2, including both the profit development and the efficiency decay. The figure illustrates a firm which upgrades

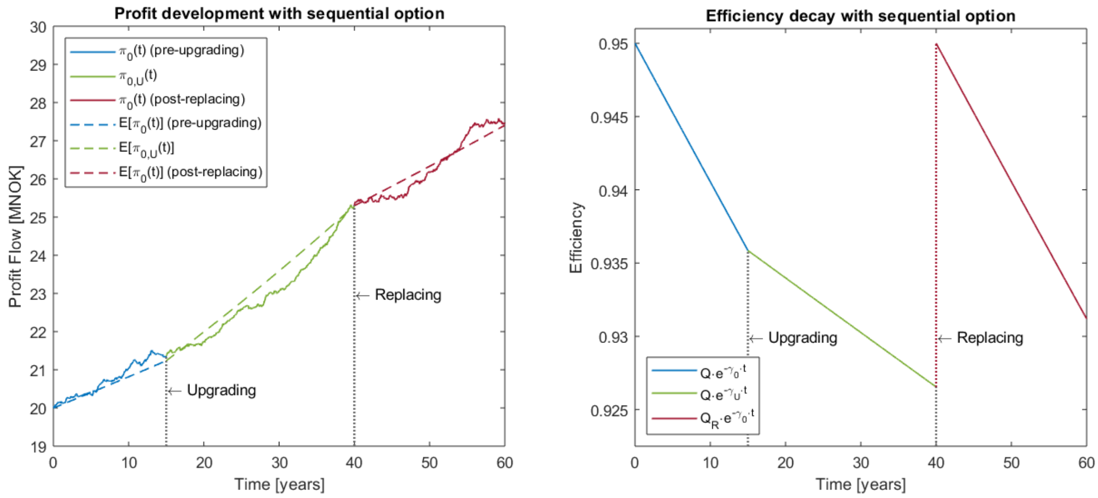

Figure 3.2: *Illustrative profit flow and efficiency development with sequential option. Note that the change in drift is exaggerated for illustrative purposes.*

the turbine after 15 years, and then replaces the upgraded turbine after 40 years. In the figures, the increase in the growth rate of the profit flow caused by the reduction in efficiency decay after upgrading is apparent.

In solving the investment problem, several factors have to be taken into consideration. Choosing to exercise the first option will not exclude the use of the remaining option. It is also important to note that because of the irreversibility property of real options, an exercised option cannot be reclaimed (Dixit et al., 1994). Additionally, when upgrading first, the new turbine that subsequently will be installed can never be upgraded. When considering a real-life situation, this is a rather strong assumption. However, since our main focus is on the strategic planning related to the existing turbine, upgrading the new turbine would make us unable to perform a reasonable comparison with the single replacement option, and is, therefore, out of the scope of this thesis[5]. Finally, as in all investment problems, the investment timing has to be optimally chosen to maximize the value to the hydropower plant owner.

3.3.2.1 Replacement Option After Upgrading

The solution procedure in sequential investment problems is normally to solve the problem backwards with the procedure described in Olsen (2018). That is, first the value of the second-stage option is obtained, and then this is used to find the value of the first-stage investment problem. We shall therefore start by calculating the value of the replacement option before the value of the upgrading option is found.

Proposition 3.3.2 *It is optimal for the firm to replace its turbine after upgrading as soon as the process π_U reaches the optimal threshold given by*

[5]The model could be extended to consider multiple sequential options, but this would require numerical methods and make the model much more challenging and non-transparent.

$$\pi_U^* = I_R \frac{\phi_2}{\phi_2 - 1} \cdot \frac{\mu_0 \mu_U}{Q_R \mu_U - k Q_0 \mu_0}, \tag{3.14}$$

where

$$\phi_2 = \frac{1}{2} - \frac{\alpha - \gamma_U}{\sigma^2} + \sqrt{\left(\frac{\alpha - \gamma_U}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}}. \tag{3.15}$$

Thus, the value of the option to replace the existing turbine after having upgraded it is given by

$$G_R(\pi_U) = \begin{cases} B_2 \pi_U^{\phi_2} + \dfrac{k Q_0 \pi_U}{\mu_U} & \text{if } \pi_U < \pi_U^*, \\ \dfrac{Q_R \pi_U}{\mu_0} - I_R & \text{if } \pi_U \geq \pi_U^*, \end{cases} \tag{3.16}$$

where

$$B_2 = \frac{I_R}{\phi_2 - 1} \left[\frac{\phi_2 - 1}{\phi_2} \cdot \frac{Q_R \mu_U - k Q_0 \mu_0}{\mu_0 \mu_U} \cdot \frac{1}{I_R} \right]^{\phi_2}. \tag{3.17}$$

The sequential replacement option in Proposition 3.3.2 has a solution that is very similar to the single replacement option, but in this case the profit flow in the continuation region is affected by the upgrading option being exercised beforehand. This implies that the parameters related to upgrading are now affecting the problem.

3.3.2.2 Upgrading Option

There are several interpretations of what it means to upgrade a turbine. In this thesis, upgrading is defined to be the same as undergoing a refurbishment process, i.e that future degradation of the turbine is reduced and that the technical lifetime is extended (Goldberg and Espeseth Lier, 2011). In practice, there are different ways to implement such a procedure. We distinguish between two different approaches: (i) where the hydropower firm opts to change its prevailing operating pattern in order to mitigate the damaging effects inflicted on the turbine, and (ii) where the the turbine undergoes a surface treatment, e.g. by hard coating[6]. The former approach implies that the firm trades off a more optimal operating pattern, and thus higher profits, against a less severe deterioration process[7]. In our model, this is accounted for by marginally reducing the profits after upgrading through multiplication with a factor $k < 1$. This factor represents the new profitability relative to the non-upgraded turbine. The latter approach, on the other hand, results in a turbine with other deterioration properties than before upgrading (Welte, 2008), and normally involves a relatively large sunk-cost, I_U. To make our model as general as possible, we

[6] This is typically only relevant for the turbine blades. The coating makes the turbine more resistant against wear, such as erosion and cavitation (Welte, 2008).

[7] https://www.ntnu.edu/hydrocen/2.4-turbine-and-generator-lifetime

account for both (i) and (ii). Since only one of the two cases will be relevant in a practical context, (i) can be excluded by setting $k = 1$, whereas (ii) can be excluded by setting the investment cost $I_U \approx 0$. In the extreme event when $I_U = 0$, no trade-off exists because the cost of reducing the degradation rate is zero. Thus, the problem reduces to a simple NPV decision where the firm chooses to do what yields the highest NPV.

To find the value of the upgrading option, we first have to distinguish between two cases. If the first investment threshold, $\pi^*_{0,U}$, is smaller than the second, π^*_U, the firm obtains the second-stage option to replace the upgraded turbine upon exercising the first-stage option. On the other hand, if the first investment threshold is larger, the firm would, according to Dixit et al. (1994), undertake both stages concurrently, and receive the expected present value of the cash flows from exercising both options. However, by undertaking both stages simultaneously, the firm is unable to reap the economic gains from operating with an improved turbine until eventually replacing it. As a result, a simultaneous investment within our framework simply means that the immediate replacement would be the preferred strategy, and the value of the firm is given by Proposition 3.3.1. The relation that determines whether $\pi^*_{0,U}$ is smaller than π^*_U is presented in Proposition 3.3.3.

Proposition 3.3.3 *If the following inequality holds, the dominant strategy will be to replace only*

$$\frac{I_R}{\phi_2 - 1} \cdot \frac{\phi_1 - \phi_2}{\phi_1} + \frac{\phi_1 - 1}{\phi_1} \cdot \frac{kQ_0\mu_0 - Q_0\mu_U}{Q_R\mu_U - kQ_0\mu_0} I_R - I_U \leq 0. \tag{3.18}$$

We note that when I_U increases, or I_R decreases, the inequality is more likely to hold. That is, if upgrading becomes relatively more expensive compared to replacing, the single replacement strategy will eventually become the only practical alternative. Given that Proposition 3.3.3 does not hold, the value of the option to invest in the first stage, i.e to upgrade the turbine, is presented in Proposition 3.3.4.

Proposition 3.3.4 *It is optimal for the firm to upgrade its existing turbine as soon as the process reaches the optimal threshold $\pi^*_{0,U}$ which implicitly solves the equation given by*

$$B_2 \frac{\phi_1 - \phi_2}{\phi_1} \pi^{*\,\phi_2}_{0,U} + \frac{\phi_1 - 1}{\phi_1} \cdot \frac{Q_0\left(k\mu_0 - \mu_U\right)}{\mu_0\mu_U} \pi^*_{0,U} - I_U = 0, \tag{3.19}$$

where

$$\phi_1 = \frac{1}{2} - \frac{\alpha - \gamma_0}{\sigma^2} + \sqrt{\left(\frac{\alpha - \gamma_0}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}}. \tag{3.20}$$

Thus, the value of the option is given by

$$G_U(\pi_0) = \begin{cases} B_1\pi_0^{\phi_1} + \dfrac{Q_0\pi_0}{\mu_0} & \textit{if } \pi_0 < \pi_0^*, \\ B_2\pi_0^{\phi_2} + \dfrac{kQ_0\pi_0}{\mu_U} - I_U & \textit{if } \pi_0 \geq \pi_0^*, \end{cases} \tag{3.21}$$

where

$$B_1 = B_2\frac{\phi_2}{\phi_1}\pi_{0,U}^{*\ \phi_2-\phi_1} + \frac{Q_0}{\phi_1}\cdot\frac{k\mu_0 - \mu_U}{\mu_0\mu_U}\pi_{0,U}^{*\ 1-\phi_1}. \tag{3.22}$$

It is worth pointing out that the value in the stopping region in Eq. (3.21) is not a linear function of profit, reflecting the fact that it is an option itself. This option value is represented by the $B_2\pi_U^{\phi_2}$-term in Eq. (3.16). We also note that the solution to the characteristic equation, ϕ_1, differs from ϕ_2 in Proposition 3.3.2 due to the change in degradation rate. Since ϕ_2 is governed by the smallest degradation rate, it follows that $\phi_1 > \phi_2$.

3.4 Results

In this section, we compare the results of the two renovating strategies and study what the optimal strategy is, given that a hydropower producer has to make a decision of which strategy to pursue. The strategies can therefore be considered as mutually exclusive investment opportunities. According to Dixit et al. (1994), the solution to this problem relies on a simple adaptation of the single investment case studied by McDonald and Siegel (1986). More specifically, Dixit et al. (1994) argue that each project or strategy should be evaluated separately. This leads to two option values and corresponding optimal stopping thresholds. The solution to the investment problem thus boils down to the following two cases:

1. If the initial net operating profit is below both of the thresholds, $\pi_{0,R}^*$ and $\pi_{0,U}^*$, it is optimal to invest in the strategy that has the highest option value at the first time the corresponding threshold is reached.

2. For values of the net operating profit above this optimal threshold, it is optimal to choose the strategy with the highest stopping value and immediately undertake the investment for the respective strategy.

Décamps et al. (2006) agree with (1), but find (2) questionable. Their counter-argument goes as follows. Assume that there are two similar projects of different sizes, project 1 and project 2, with optimal thresholds p_1 and p_2, respectively. Further, suppose that project 1 has a higher option value than the second project. This normally occurs if project 1 generates only a marginally lower profit, but entails a significantly lower investment cost. If the threshold of project 1 is reached, the decision-maker is strictly better off by investing in project 1 rather than in project 2, such that p_1 is below the indifference point where the projects are equally valuable to undertake. Décamps et al. (2006) show that there exists a region around the indifference point of the two projects, i.e above p_1 and below p_2, where it is optimal to wait with investment in either of the two projects. In effect, this causes the optimal investment region to be dichotomous.

We can also regard our investment problem as two different projects, namely upgrading and replacement. The only premise is that the last action the decision-maker makes is replacement because it is the largest project; i.e if it is optimal to upgrade the turbine, it must happen before it can be replaced, and thus it becomes a constrained sequence. The analogy is accordingly that project 1 corresponds to upgrading, while project 2 corresponds to replacement. Our investment problem is therefore two-fold: (1) if the profit of the firm is below both thresholds, should the firm simply replace or upgrade the turbine as soon as the first threshold is hit? (2) If the profit is in between the threshold for upgrading and replacement, should the firm wait until one of the thresholds are hit, or invest immediately in the strategy showing the largest value?

The argument that Décamps et al. (2006) use to reaffirm the existence of a dichotomous investment region, namely by saying that project 1 only generates a marginally smaller profit than project 2, but entails a significantly lower investment cost, has a slightly different interpretation in our case. As the upgrading strategy eventually will lead to a replacement of the turbine, the trade from going from an upgraded state to a replaced state of the turbine must entail a net positive increase in the operating profits between the two states. If not, the option to replace after upgrading would have no value. This precondition can be specified as follows:

$$\frac{kQ_0\pi}{\mu_U} < \frac{Q_R\pi}{\mu_0} \implies kQ_0\mu_0 < Q_R\mu_U, \tag{3.23}$$

which is similar to Décamps et al. (2006) saying that the second project has a higher profitability than the first. This is assumed to be true in Eqs. (3.14) and (3.17). In addition, I_U must be significantly lower than I_R, i.e $I_U < I_R$. In the hydropower industry, this is a natural assumption to make, since investment costs related to replacement are greater than re-investment costs on an already existing turbine.

3.4.1 Strategy Choice

In order to find the optimal investment strategy, we must investigate the different forces at play and the resulting incentives when both strategies are embedded in the same framework. Since the firm must choose one of the two strategies at some point in time, the ultimate goal is to optimize the following expression for the entire profit state-space:

$$H = \sup_{\tau \in \mathcal{T}^{\mathcal{S}}} \mathbb{E}\left[e^{-\rho\tau} \cdot \max\left\{G_U^{\mathcal{S}}, F_R^{\mathcal{S}}\right\}\right], \tag{3.24}$$

where $G_U^{\mathcal{S}}$ is the stopping value of upgrading in the sequential strategy[8] (see Eq. (3.21)), and $F_R^{\mathcal{S}}$ is the stopping value of the single replacement strategy (see Eq. (3.9)). Recall

[8] It is worth pointing out that the value of stopping in G_U, i.e $G_U^{\mathcal{S}}$, equals the value of continuing in G_R, i.e $G_R^{\mathcal{C}}$, less the investment cost of upgrading.

that $G_U^{\mathcal{S}}$ is not linear as opposed to $F_R^{\mathcal{S}}$. Let us now denote by $\mathcal{S} = \{\pi_0 > 0 \mid \Omega(\pi_0)\}$ the stopping region for Eq. (3.24), and by $\mathcal{T}^{\mathcal{S}} = \inf\{\tau > 0 \mid \pi_0 \in \mathcal{S}\}$ the associated stopping time. Using the same notations as before, the expression in Eq. (3.24) can be expanded to yield

$$\begin{aligned} H = \sup_{\tau_1,\tau_2 \geq \tau_1 \in \mathcal{T}^{\mathcal{S}}} \mathbb{E}\Bigg[\Bigg[\mathbb{1}_{\{\tau_1<\tau_2\}}\Bigg[\int_0^{\tau_1} \pi_0(t) Q_0 e^{-\rho t} \mathrm{d}t - I_U e^{-\rho\tau_1} \\ + \int_{\tau_1}^{\tau_2} \pi_U(t) k Q_0 e^{-\tau_1(\gamma_0-\gamma_U)} e^{-\rho t}\mathrm{d}t - I_R e^{-\rho\tau_2} + \int_{\tau_2}^{\infty} \pi_0(t) Q_R e^{\gamma_0\tau_2} e^{-\rho t}\mathrm{d}t\Bigg]\Bigg] \\ + \mathbb{E}\Bigg[\mathbb{1}_{\{\tau_1=\tau_2\}}\Bigg[\int_0^{\tau_2} \pi_0(t) Q_0 e^{-\rho t}\mathrm{d}t - I_R e^{-\rho\tau_2} + \int_{\tau_2}^{\infty} \pi_0(t) Q_R e^{\gamma_0\tau_2} e^{-\rho t}\mathrm{d}t\Bigg]\Bigg]\Bigg]. \end{aligned} \tag{3.25}$$

Here, τ_1 denotes the time at which the decision-maker invests in the upgrade, and $\tau_2 \geq \tau_1$ denotes the time at which he invests in replacement. The first part of Eq. (3.25) originates from the sequential optimal stopping problem, i.e Eq. (3.13), whereas the last part stems from the single replacement optimal stopping problem, i.e Eq. (3.6).

In order to present the solution to this problem in an intelligible format, the solution space is divided into distinct regions with different investment strategies. Table 3.2 gives an overview of the nomenclature that results from these divisions. In the solution procedure, the first step is to determine whether single replacement is the dominant strategy in the entire state-space. Proposition 3.4.1 gives the mathematical expression for this. If this proposition holds, the value of the option will be as illustrated in Figure 3.3. The proposition is based on the option values in the continuation region for the single replacement strategy and the first stage for the sequential strategy. If the single replacement strategy is more valuable in this area, its option value will always dominate that of the sequential strategy.

Dominant Strategy	**Symbol**	**Description**
Single replacement	$\pi_{0,R}^*$	Optimal stopping threshold
Depends on the value and development of π_0 (dichotomous investment region)	$\pi_{0,U}^*$	Lower optimal stopping threshold for the sequential strategy
	π_{0,W_L}^*	Upper optimal stopping threshold for the sequential strategy
	$\widetilde{\pi}_0^*$	Indifference point
	π_{0,W_U}^*	Optimal stopping threshold for single replacement
	π_U^*	Optimal stopping threshold for replacement in the sequential strategy
	H	Value of the optimal stopping problem where both strategies are embedded

Table 3.2: *Thresholds and option value*

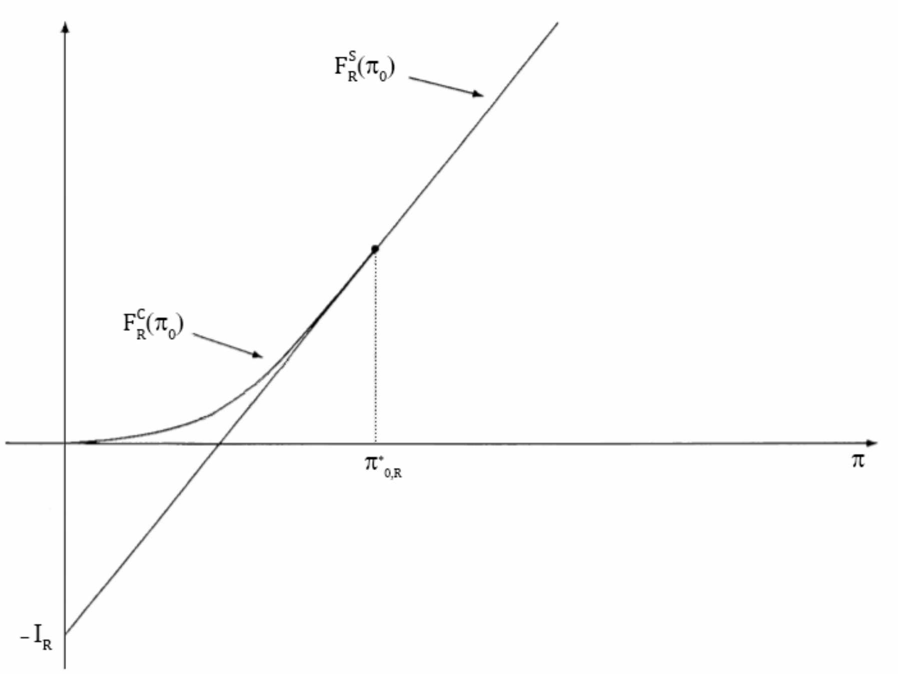

Figure 3.3: *Value plot when the single replacement strategy is dominant*

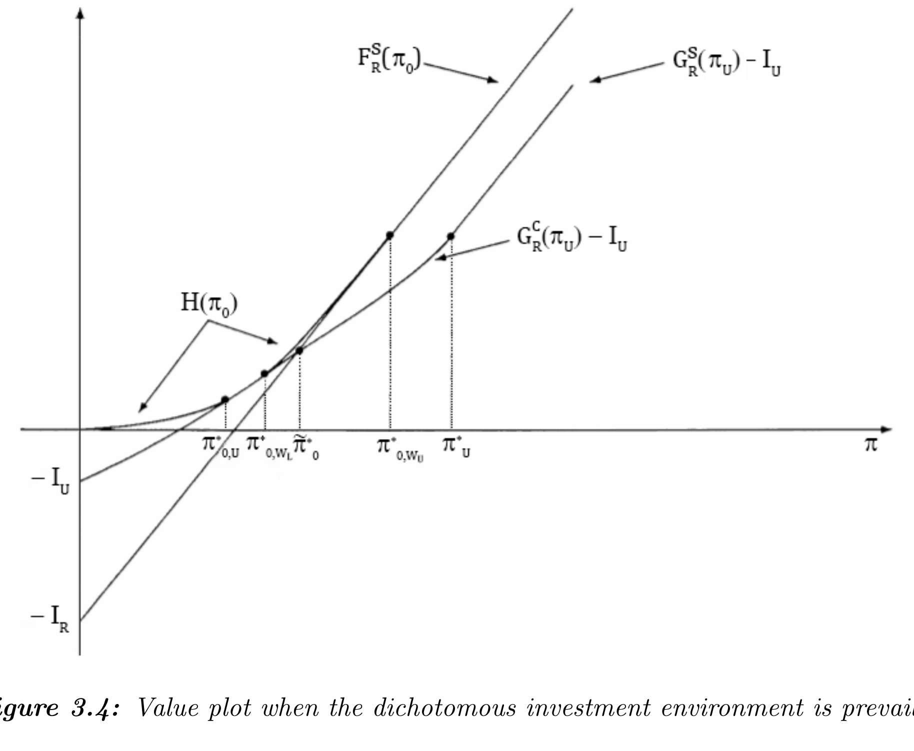

Figure 3.4: *Value plot when the dichotomous investment environment is prevailing*

Proposition 3.4.1 *It will be optimal to replace only if the following holds:*

$$I_R \frac{\beta_1}{\beta_1 - 1} \left[\frac{\beta_1 - 1}{\beta_1} \cdot \frac{Q_R - Q_0}{I_R \mu_0} \pi^*_{0,U} \right]^{\beta_1} - B_2 \phi_2 \pi^{*\ \phi_2}_{0,U} - Q_0 \frac{k\mu_0 - \mu_U}{\mu_0 \mu_U} \pi^*_{0,U} \geq 0, \tag{3.26}$$

where B_2 and $\pi^*_{0,U}$ are given by Eqs. (3.17) *and* (3.19) *respectively.*

However, if the inequality in Proposition 3.4.1 does not hold, both strategies might be optimal, and the strategy will depend on the current value of π_0. Under these circumstances, derivation of the optimal investment strategy will closely mimic the procedure laid forward by Décamps et al. (2006). The solution space is, thus, divided into four different regions. The first region runs from 0 to $\pi^*_{0,U}$, the second from $\pi^*_{0,U}$ to π^*_{0,W_L}, the third between π^*_{0,W_L} and π^*_{0,W_U} and the fourth for values above π^*_{0,W_U}. The value of $\pi^*_{0,U}$ stems from Eq. (3.14), whereas the remaining thresholds will be defined below. The first and third regions are inaction regions, whereas the second and fourth make up the investment regions of the problem. The division of the regions can be summed up by Figure 3.4. We will now explain the reasoning behind the different regions.

3.4.1.1 First Inaction Region, $[0, \pi^*_{0,U})$

In this region, shown in Figure 3.5, the option value for the sequential option is most valuable. This follows from Proposition 3.4.1, where, in this case, the inequality does not hold. Thus, the value of H is given by

$$H = B_1 \pi_0^{\phi_1} + \frac{\pi_0 Q_0}{\mu_0},$$

where B_1 is given by Eq. (3.22). In this region, the optimal strategy is to wait until the investment threshold $\pi^*_{0,U}$ is reached, and then to upgrade the turbine. This is in fact the same continuation region as in the standard model proposed by Dixit et al. (1994).

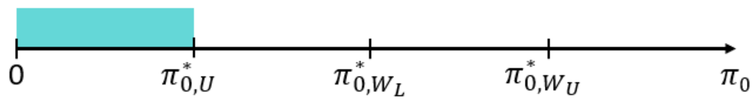

Figure 3.5: *First inaction region*

3.4.1.2 Second Inaction Region, $(\pi^*_{0,W_L}, \pi^*_{0,W_U})$

When the option value of the sequential strategy is higher than that of the single replacement strategy in the first inaction region, the optimal investment region is no longer connected. Consequently, there are two thresholds, π^*_{0,W_L} and π^*_{0,W_U}, that form an intermediate region of inaction around the indifference point, $\widetilde{\pi}^*_0$. The indifference point is given by Proposition 3.4.2, and is always part of the inaction region where it is optimal for the hydropower operator to wait to decide in which strategy to invest.

Proposition 3.4.2 *The indifference point never belongs to any of the stopping regions and will always be located between π^*_{0,W_L} and π^*_{0,W_U}. The point is implicitly given by the following equation:*

$$B_2 \widetilde{\pi}_0^{*\ \phi_2} + \frac{kQ_0\mu_0 - Q_R\mu_U}{\mu_0\mu_U}\widetilde{\pi}_0^* - (I_U - I_R) = 0, \tag{3.27}$$

where ϕ_2 is given by Eq. (3.15) *and B_2 is given by Eq.* (3.17)*.*

Note that in the inaction region $(\pi^*_{0,W_L}, \pi^*_{0,W_U})$, shown in Figure 3.6, the difference between the option value H and the sequential investment value $G_U^{\mathcal{S}}$ is at its maximal at the indifference point $\widetilde{\pi}_0^*$. This leads to a region of inaction implying that the firm is better off by waiting for more information rather than investing in either of the two strategies. As a result, when the profit lies within this region, two situations can occur. Either the profit first raises to π^*_{0,W_U} or it first falls to π^*_{0,W_L}. The optimal strategy in each of the two instances is summarised by the following:

- If the profit falls to the lower threshold π^*_{0,W_L}, it is optimal to immediately invest in the sequential strategy. In this case, the value of the firm is given by G_R in Eq. (3.16) less the investment cost I_U.
- If the profit rises to the upper threshold π^*_{0,W_U}, it is optimal to immediately invest in the single replacement strategy. In this case, the value of the firm is given by the stopping value of F_R, i.e $F_R^{\mathcal{S}}$, in Eq. (3.9).

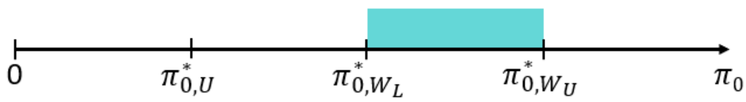

Figure 3.6: *Second inaction region*

The consequence of the two possible scenarios is that the choice of strategy is path-dependent and that the associated values of the firm differ depending on the choices made. Since this region may lead to two different outcomes, it contains two separate options. Hence, it follows that H on the interval $(\pi^*_{0,W_L}, \pi^*_{0,W_U})$, is of the form $C\pi_0^{\beta_1} + D\pi_0^{\beta_2} + \frac{Q_0\pi_0}{\mu_0}$. The first two terms represent the value of waiting without having made any irreversible decisions yet. More specifically, the first term represents the option to invest in the single replacement strategy should the profit increase to π^*_{0,W_U}, whereas the second term represents the option to invest in the sequential strategy should the profit decrease to π^*_{0,W_L}. $\beta_1 > 1$ and $\beta_2 < 0$ are therefore the solutions to the fundamental quadratic equation[9]

[9]Note that this equation is identical to the one which gives ϕ_1 in Eq. (3.20). This is because both strategies share the same fundamental quadratic equation before any action has taken place.

$$\frac{1}{2}\beta(\beta-1)\sigma^2+\beta(\alpha-\gamma_0)-\rho=0.$$
$$\Longrightarrow \beta_{1,2}=\frac{1}{2}-\frac{\alpha-\gamma_0}{\sigma^2}\pm\sqrt{\left(\frac{\alpha-\gamma_0}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2\rho}{\sigma^2}}. \tag{3.28}$$

The coefficients C and D, as well as the optimal stopping thresholds π^*_{0,W_L} and π^*_{0,W_U} can be found by solving the value matching and smooth pasting conditions given by Proposition 3.4.3. Since it is not straightforward to find suitable initial values for the optimal thresholds in this proposition, we propose an algorithm in Appendix A.2.7.1 that explains how this can be done.

Proposition 3.4.3 *The values of π^*_{0,W_L}, π^*_{0,W_U}, C and D are given as the solution to the following set of equations:*

$$C\pi^{*\ \beta_1}_{0,W_L}+D\pi^{*\ \beta_2}_{0,W_L}+\frac{Q_0\pi^*_{0,W_L}}{\mu_0}=\frac{kQ_0\pi^*_{0,W_L}}{\mu_U}+B_2\pi^{*\ \phi_2}_{0,W_L}-I_U, \tag{3.29}$$

$$C\pi^{*\ \beta_1}_{0,W_U}+D\pi^{*\ \beta_2}_{0,W_U}+\frac{Q_0\pi^*_{0,W_U}}{\mu_0}=\frac{Q_R\pi^*_{0,W_U}}{\mu_0}-I_R, \tag{3.30}$$

$$\beta_1C\pi^{*\ \beta_1-1}_{0,W_L}+\beta_2D\pi^{*\ \beta_2-1}_{0,W_L}+\frac{Q_0}{\mu_0}=\frac{kQ_0}{\mu_U}+\phi_2B_2\pi^{*\ \phi_2-1}_{0,W_L}, \tag{3.31}$$

$$\beta_1C\pi^{*\ \beta_1-1}_{0,W_U}+\beta_2D\pi^{*\ \beta_2-1}_{0,W_U}+\frac{Q_0}{\mu_0}=\frac{Q_R}{\mu_0}, \tag{3.32}$$

where ϕ_2 is given by Eq. (3.15) *and B_2 is given by Eq.* (3.17). *Due to the non-linear form of the equations, the solution can only be found numerically.*

A striking feature that follows from the existence of the inaction region $(\pi^*_{0,W_L}, \pi^*_{0,W_U})$ is that it can be optimal for the hydropower firm to undertake an investment even though the associated profit flow of this strategy falls. This contrasts most standard real option models which would characterise this as an exit option. The reason for why it is optimal to invest in the upgrading strategy when the profit falls to π^*_{0,W_L} is composed of two elements: (i) π^*_{0,W_L} is higher than $\pi^*_{0,U}$ above which it would be optimal to invest in the upgrading strategy if that was the only option available, and (ii) it is too costly to wait until the profit reaches the upper threshold π^*_{0,W_U} and then invest in the single replacement strategy due to the time value of money. The prerogative to choose between the two different strategies instead of being confined to either one of them also increases the demand for information and creates an additional incentive to delay investment. Thus, in this particular region, it is optimal to delay investment even though it would be optimal to invest if only the sequential strategy was available. We also note that the investment threshold $\pi^*_{0,R}$, which corresponds to the situation where single replacement is the uniformly dominating strategy, must be strictly less than π^*_{0,W_U}. This is because π^*_{0,W_U} incorporates the additional value of waiting that is present in the dichotomous investment environment.

Let us now determine the value of H in the the two regions where the firm actively decides to invest in one of the strategies.

3.4.1.3 First Investment Region, $[\pi^*_{0,U}, \pi^*_{0,W_L}]$

Between the two regions of inaction, there exist a region where it is optimal for the firm to invest immediately. Result 3.4.1 describes this region.

Result 3.4.1 *When the current value of π_0 is in the range $[\pi^*_{0,U}, \pi^*_{0,W_L}]$, it is optimal to immediately upgrade the existing turbine in order to slow down the degradation rate and obtain the option to replace it. In this case, the value of H is given by $G_R^{\mathcal{C}} - I_U$, where $G_R^{\mathcal{C}}$ is specified by Eq.* (3.16).

In the investment region given by Result 3.4.1, and which is shown in Figure 3.7, the value of investing in an upgrade of the turbine is worth more than both the single investment option and the value of waiting. Hence, the firm should upgrade immediately. For the sake of completeness, Result 3.4.1 follows from Eq. (3.24) and Proposition 3.4.3. We also note that for values of profit below π^*_{0,W_L}, the solution of Dixit et al. (1994) remains valid. More specifically, if the initial profit flow either reaches $\pi^*_{0,U}$ from below or is already located in the first investment region, it is optimal to invest in the sequential strategy by exercising the option to upgrade the turbine.

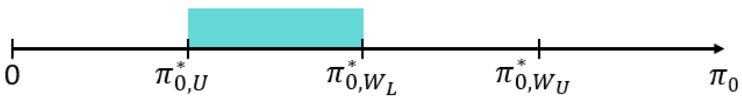

Figure 3.7: *First investment region*

Once the upgrading investment is undertaken, the optimal strategy moving forward is constrained to that of the sequential option. It is worth emphasizing that in this particular region, the profit flow changes and is governed by the process in Eq. (3.12). Now the option value is irrevocably determined by $G_R^C(\pi_U) - I_U$ in Figure 3.4. Thus, when the threshold π^*_U, given by Eq. (3.14) is reached, it is optimal to replace the upgraded turbine. As a result, the turbine will eventually be replaced, independent of which investment strategy is chosen in the first stage. Also note that the replacement threshold in the sequential strategy, π^*_U, is strictly larger than that in the single replacement strategy, $\pi^*_{0,R}$. This directly follows from comparing Eqs. (3.16) and (3.9), in the case when Proposition 3.4.1 does not hold. Intuitively, $\pi^*_U > \pi^*_{0,R}$ because the expected discounted gain from going from an upgraded turbine to a replaced one is less than the expected discounted gain from directly investing in the single replacement strategy, if the replacement cost is the same.

3.4.1.4 Second Investment Region, $[\pi^*_{0,W_U}, \infty)$

Above the second region of inaction, the firm enters the second investment region. This is described in Result 3.4.2.

Result 3.4.2 *When the current value of π_0 is equal to or above the threshold π^*_{0,W_U}, the optimal strategy is to replace immediately.*

The result above stems from a comparison of the option values above π^*_{0,W_U} for the two strategies. From the proof of Proposition 3.4.2, we know that for all values above $\widetilde{\pi_0}^*$, the stopping value of single replacement exceeds the first-stage stopping value of the sequential option. Combined with the fact that the value functions for both of the strategies above all thresholds are parallel[10], single replacement becomes the dominant strategy for high values of π_0. Hence, single replacement may be the optimal investment strategy in spite of not being dominant in the entire state-space.

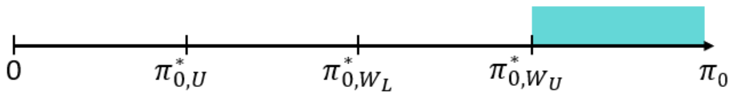

Figure 3.8: *Second investment region*

3.4.2 Summary of Results

Based on the different investment environments, a set of decision rules can be made. A summary of these decision rules is given by Figure 3.9. First, the firm has to check whether the single replacement strategy is uniformly dominant. If it is not, the investment region becomes dichotomous and the decision tree branches into several sub-cases. The single replacement strategy might still end up being the preferred strategy, but this is no longer unambiguous. The two strategies now each have their own unique investment region. As a result, the choice of strategy might depend on the development of the profit flow from its current state, making it path dependent. The first and lowermost investment region favours the sequential strategy, whereas the second and uppermost favours the single replacement strategy. The reason for this is that the stakes are higher for replacement of the turbine as this entails a larger sunk investment cost compared to the upgrade. In general, higher risk systematically leads to a decreased incentive to invest since the value of waiting becomes comparatively more worth (Dixit et al., 1994). Accordingly, the lower investment region allows the firm to postpone the relatively bigger investment in replacement by first committing to the upgrade, and hence the stakes are not as high.

The two inaction regions, on the other hand, are a natural implication of the dichotomous investment region. In the first inaction region, the solution is similar to that of Dixit et al. (1994) as if there only existed one strategy, meaning that investment in the sequential strategy is triggered by the lowest threshold, $\pi^*_{0,U}$. However, in the second region of inaction, the solution is more complex. Here the firm chooses to delay investment although it would be optimal to invest immediately if only the sequential strategy was available. This demonstrates the interaction between the two investment options, namely that the firm is willing to delay further investment in the single replacement strategy because it is aware of the option to invest in the sequential strategy should the profit deteriorate too much. Then, if the optimal investment strategy is to invest in the sequential option, the turbine

[10]Comparing F_R and $G_R - I_U$ in their respective stopping regions gives that the value of single replacement is I_U greater than that of doing sequential investment.

will be replaced when the profit reaches the threshold π_U^*. Ultimately, the final stage will be a new turbine, independent of the investment strategy.

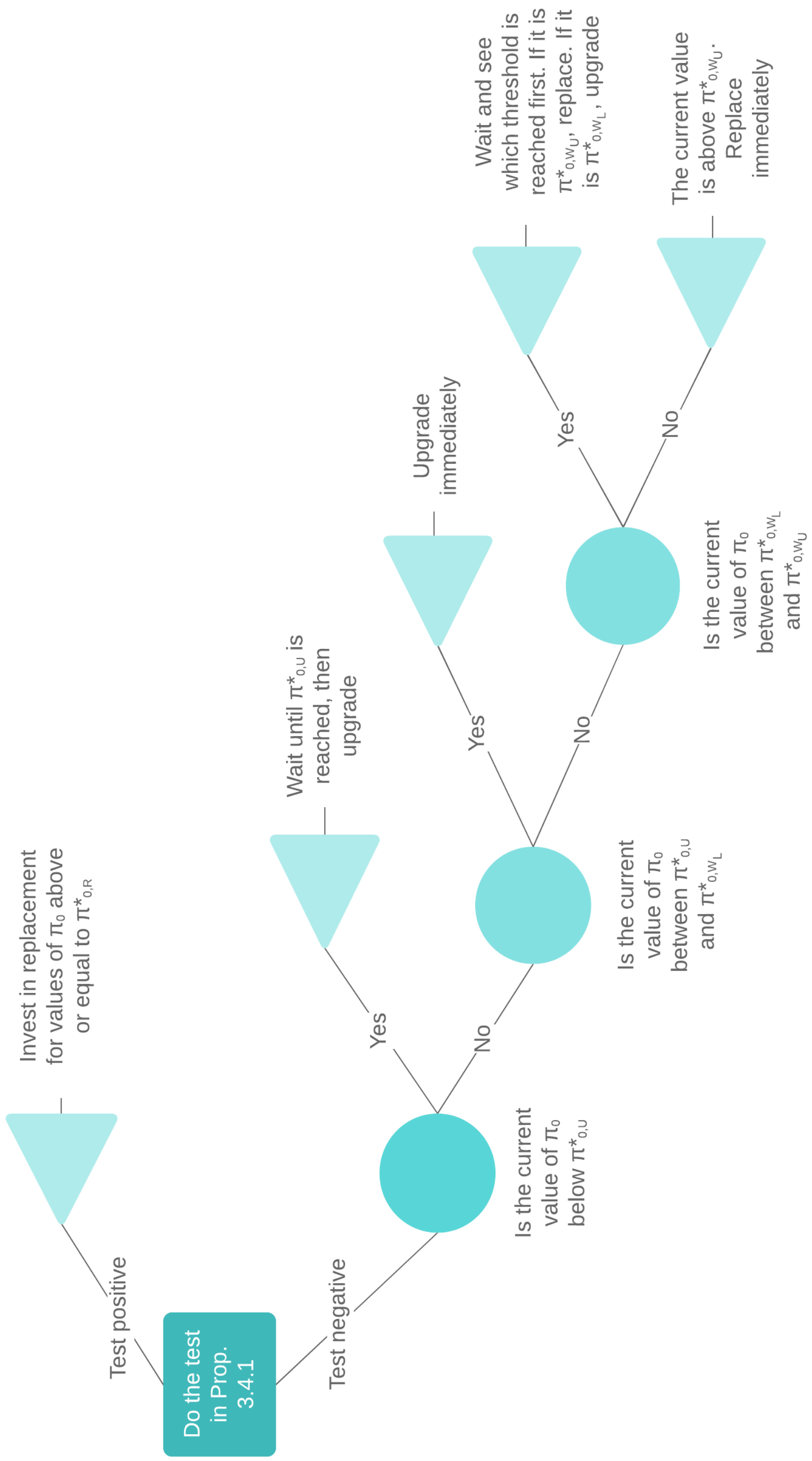

Figure 3.9: *Decision tree for optimal investment strategy*

3.5 Model Implications

In this section, we examine the implications of our model and how they are affected by the parameter values. Our baseline parameter values are given within a Norwegian hydropower context based on general industry standards. They are not based on a specific plant, but can easily be changed to accommodate the conditions of any specific hydropower plant. To obtain insights about the optimal decisions when the operating environment of the plant is changed, either due to external or internal causes, we run a comparative statics analysis.

As outlined in the previous section, the firm can choose between two strategies. In an isolated case, the sensitivity analysis for the single replacement strategy can be performed analytically, but if this strategy is not dominant, the procedure becomes more complex and requires numerical calculations. An extensive numerical analysis is therefore conducted to examine how the investment thresholds and the optimal strategy change with the model parameters when analytic results are hard to obtain. The main focus is to examine the conditions under which the optimal strategy transitions from the dichotomous case, where both strategies can be optimal, to when the single replacement strategy is dominant over the entire state-space, or vice versa.

3.5.1 Baseline Parameter Values

Parameter Description	**Symbol**	**Baseline Value**
Discount rate	ρ	0.06
Starting efficiency of the existing turbine	Q_0	0.91
Starting efficiency of a new turbine	Q_R	0.95
Degradation rate of a non-upgraded turbine	γ_0	0.001
Degradation rate of an upgraded turbine	γ_U	0.0005
Investment cost of upgrading the turbine	I_R	30
Investment cost of replacing the turbine	I_U	5 or 0.75
Change in profit level after upgrading	k	1 or 0.99
Volatility of gross profit	σ	0.2
Growth rate of gross profit	α	0.025

Table 3.3: *Baseline parameter values*

Table 3.3 presents the baseline parameter values which are used in the analysis. Because there are two modus operandi when it comes to the upgrading process, we have two sets of values for k and I_U: the first set corresponds to when the cost of upgrading is a sunk investment cost. The second set represents the cost of upgrading when it predominantly stems from a change in the operational pattern. Rather than incurring a significant sunk cost, this upgrading scheme reduces the overall profit level of the hydropower company. We refer to Section 3.3.2.2 for a more elaborate explanation of the differences in the decision to upgrade. In the analysis, we only use the first set of values to represent the cost of upgrading. The second set will show the same effect, and to highlight this we include the two interpretations when we examine the degradation rates. However, in the remaining part of the analysis we stick to the first set.

The discount rate for a certain hydropower plant, ρ, can vary significantly, both depending on the plant's risk characteristics and financing. To provide a general case, we have chosen a value based on the industry average. Andersson et al. (2014) argue for a discount rate of 7% on an investment with risk characteristics similar to our model. This provides a good starting point. However, in recent years the discount rate has trended downwards, resulting from a more stable overall economy. A survey performed by the consulting and accounting firm Grant Thornton (2018) proposes a guiding discount rate of 5.75% for a levered hydropower firm. Their results were obtained after consulting incumbents in the Nordic hydropower industry. As investment projects in this industry are seldom self-financed by equity exclusively, we will use this as our other basis point when determining the discount rate. Since the latter study is more up to date, we will put more emphasis on this. Hence, we choose a discount rate of 6%.

When considering the efficiency of the existing turbine, we consider a semi-old turbine that has experienced some efficiency decay, but is still some time from reaching its economical lifetime. Based on the numerical values of Ruud (2017), 0.91 is set as a baseline value. The efficiency of a new turbine reflects the state of the art for the turbine efficiency. This parameter varies depending on the type of turbine, and also on how the turbine is designed to operate with different loads. According to Ruud (2017), a suitable value for Q_R is 0.95, which also gives a realistic difference between Q_0 and Q_R for a semi-old turbine[11].

The degradation rate for turbines in the hydropower industry is quite low compared to other energy generating industries. In the appraisal of applications from hydropower producers, NVE use a guiding degradation rate of 0.00087 [12]. As this number is used to calculate the potential improvement of a hydropower plant, it is likely to be a conservative number. We therefore adjust this to 0.001 in our baseline values. A suitable value for γ_U is significantly harder to find because of the lack of empirical studies on the subject. Thus, we opt for a value which gives an obvious reduction in the degradation rate so that the firm might be willing to upgrade the turbine. Still, the reduction cannot be too large as this would mean that the turbine virtually does not degrade, which contradicts industry observations. With this in mind, we set the value for γ_U equal to 0.0005. That is a reduction of 50% relative to a non-upgraded turbine.

The investment costs, I_R and I_U, are highly dependent on the specific hydropower plant due to the high level of idiosyncrasy. However, some general characteristics of the relationship between the two do exist. First, the value of I_U should be significantly lower than I_R. This is because of the difference in the comprehensiveness of the two investments, and also in the physical characteristics. A replacement requires a brand new turbine to be made, whereas an upgrade is a significantly less extensive procedure. Second, if the upgrade entails a physical change of the turbine, such as hard coating, it is a one-time process which has a significant cost, but much lower than that of replacing altogether. On the other hand, if the upgrade is a change in the operational pattern, I_U represents the cost of re-planning and switching expenses. This cost should be much lower than the one-time process upgrade. To quantify the suitable cost levels, we have consulted several experts on the area. Based on these discussions, and taking the limitations above into account, we have set I_R equal to 30 MNOK. If the upgrade entails a physical change of the turbine, I_U is set to 5 MNOK

[11] The numerical values for Q_0 and Q_R are based on a Francis turbine subject to Norwegian conditions.
[12] https://www.nve.no/Media/5330/veileder-elsertifikater-ou_vannkraftverk_09-02-2017.pdf

with no change in the profit flow, that is, $k = 1$. For the other upgrading mode, I_U is set equal to 0.75, while k is set to be 0.99. This means that the profit flow is reduced by 1% due to the change in operational pattern. These numbers also comply with Eq. (3.19), which means that the existence of a dichotomous investment region is not excluded.

When it comes to the parameters σ and α, the estimation is often quite extensive and demands an in-depth analysis of different economical and site specific factors. Therefore, we use the work of Andersson et al. (2014) and Ruud (2017) as a basis for σ, and Norli (2017) to estimate α. For the volatility, we choose $\sigma = 0.2$, whereas the drift rate, α, is set to be 0.025. The value of α is mainly based on the inflation target presented by Norli (2017). It is important to emphasize that this is not a case study, and therefore our focus is merely to get the range correct. If the model is used for a specific plant, the parameters should be calculated based on available empirical data and future estimates.

3.5.2 Analysis

To investigate when the optimal investment strategy changes, we alter several of the parameters. Recall that when the single replacement strategy is dominant over the entire state-space, the investment threshold is given by $\pi^*_{0,R}$. This investment environment is represented in Figure 3.3. If, on the other hand, the investment region is dichotomous, $\pi^*_{0,U}$ is the threshold which demarcates the first investment region where upgrading is optimal. This region lasts until π^*_{0,W_L} where the second inaction region is initiated. Between π^*_{0,W_L} and π^*_{0,W_U}, it is optimal to wait, whereas for values above π^*_{0,W_U}, single replacement is dominant. This investment environment is represented in Figure 3.4. In addition to these thresholds, the tables in this section also include the indifference point, denoted by $\widetilde{\pi_0}^*$. This serves as a robustness check to validate that the thresholds comply with Proposition 3.4.2 which states that the indifference point should always be located between π^*_{0,W_L} and π^*_{0,W_U}. To make clear what the dominant investment environment is, we use two colour codes. Turquoise represents the dominant environment which the investor should base his or her decisions on. Grey, on the other hand, represents the dominated environment. In addition, when the dichotomous environment is not dominant, the required conditions for the sequential option may not be satisfied. Therefore we do not include the numerical values for the dichotomous investment thresholds and the indifference point when single replacement is dominant.

We start by examining the effect of σ. Table 3.4 shows that all thresholds are increasing with uncertainty. This is in line with Dixit et al. (1994), who postulate that increasing uncertainty elevates the value of waiting, thus raising the thresholds. Furthermore, there are particularly two features that are interesting to point out. First, when the uncertainty becomes high enough, the single replacement strategy is always dominating. Thus, by holding the other parameters of the model fixed, a relatively low volatility is required for the dichotomous environment to be dominant. This is because the option value increases when the volatility in profit increases (Dixit et al., 1994). While this holds for both investment environments, the value of the single replacement option increases with a higher rate than that of the sequential option. As a result, the option to invest in a replacement of the turbine becomes more valuable as the option value of upgrading is too small when the

σ	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
$\pi^*_{0,R}$	52.69	60.28	69.52	80.31	92.63	106.5	122.0	139.2
$\pi^*_{0,U}$	27.17	31.03	35.72	41.19	47.43	-	-	-
π^*_{0,W_L}	39.51	40.29	42.26	45.26	49.20	-	-	-
$\widetilde{\pi_0}^*$	46.58	51.70	58.06	65.58	74.26	-	-	-
π^*_{0,W_U}	53.39	61.17	70.36	80.94	92.95	-	-	-
π^*_U	78.48	89.74	103.5	119.5	137.9	-	-	-

Table 3.4: *Effect of varying volatility*

volatility in profit is high. In other words, the payoff from upgrading does not overcome the long-term benefit of operating with a new and more efficient turbine. Second, when the investment region is dichotomous, the second inaction region $(\pi^*_{0,W_L}, \pi^*_{0,W_U})$ becomes larger as the uncertainty increases. This emphasizes the fact that when the environment is more uncertain, the firm demands more information before undertaking either of the two renovating strategies.

The effect of changing the initial efficiency of the existing turbine, Q_0, is shown in Table 3.5. In contrast to the volatility, a lower efficiency of the turbine, i.e the gap between Q_R and Q_0 increases, makes the single replacement strategy dominant. This is because, if the efficiency is already at quite a low level, the payoff from replacing and restarting the degradation process dominates that of upgrading and improving the future trajectory of the degradation process, even though the cost is higher. In addition, we observe that when the dichotomous strategic environment is prevailing, all thresholds except $\pi^*_{0,U}$ experience a significant increase when Q_0 approaches Q_R. The logic behind this phenomenon is that when the net benefit of replacing the turbine becomes small, the firm requires a drastically higher profit level before it is profitable to replace it. However, the same effect has little influence on the threshold to upgrade in the sequential strategy; $\pi^*_{0,U}$. This can be explained by two contradicting incentives. On one hand, the firm has an incentive to invest earlier in an upgrade because reducing the degradation rate on a turbine with higher efficiency extends the turbine's lifetime more substantially, and hence delays the subsequent replacement. On the other hand, the threshold is indirectly affected by the replacement option through the implicit equation (3.19). This gives the hydropower producer an incentive to delay the investment because a replacement is no longer as imminent with such a high efficiency of the initial turbine. The dominating effect is the former, thus the slight reduction

Q_0	0.88	0.89	0.90	0.91	0.92	0.93
$\pi^*_{0,R}$	39.73	46.35	55.62	69.52	92.70	139.1
$\pi^*_{0,U}$	-	-	-	35.72	35.56	35.39
π^*_{0,W_L}	-	-	-	42.26	79.74	217.7
$\widetilde{\pi_0}^*$	-	-	-	58.06	92.61	228.7
π^*_{0,W_U}	-	-	-	70.36	104.5	239.6
π^*_U	-	-	-	103.5	165.1	407.6

Table 3.5: *Effect of varying pre-investment turbine efficiency*

in the threshold. We do, however, note that if upgrading is assumed to be a change in the operational pattern, so that $k = 0.99$ and $I_U = 0.75$, the second effect would dominate and the threshold would increase slightly. This is caused by a lower incentive to invest in the upgrade, as the cost of upgrading now rises in parallel to the increasing efficiency, which contrasts the constant sunk investment cost in the other case.

In the last column of Table 3.5, we observe that the threshold $\pi^*_{0,R}$ lies within the first investment region in the dichotomous environment, $[\pi^*_{0,U}, \pi^*_{0,W_L}]$. This implies that in the part of this investment region above $\pi^*_{0,R}$, the firm would replace immediately if this was the only alternative. However, if upgrading is considered within the same option framework, it is optimal to upgrade and then wait for a higher profit before incurring the bigger cost of replacing. This shows the importance of simultaneously considering all means that help to improve the current operating conditions. Similar observations are found elsewhere in situations where upgrading is very profitable, for example when the difference between I_U and I_R increases. These parameters are the next to be examined.

In Table 3.6, the value of I_U is fixed at three different levels, while the value of I_R is varied. It is immediately clear that increasing the replacement cost increases all of the thresholds. On the other hand, the effect of increasing the cost of upgrading has different implications. When the sunk upgrading cost is low, the dichotomous investment environment dominates for all values of I_R, as presented in Table 3.6a. Under these conditions upgrading becomes increasingly more profitable compared to when the upgrading cost is high, implying that the dichotomous investment environment is dominant regardless of the value of I_R[13]. We also observe that for all of the levels of I_U in Table 3.6, increasing I_R has the same effect as increasing Q_0 on the thresholds because they both reduce the net gain from replacement. Consequently, a higher investment cost requires a higher expected value before investing. This is because the value of investing must exceed the direct investment cost plus the opportunity cost of exercising either of the two options (Dixit et al., 1994). On the other hand, as I_U increases, the dominance of the dichotomous investment environment diminishes. We see this transition from Table 3.6a to Table 3.6b, and further on to Table 3.6c. When I_U is high, as presented in Table 3.6c, the single replacement strategy is the uniformly preferred strategy. Thus, for our values, there exist a cut-off point of $I_U \approx 5.4$ [MNOK] where the single replacement strategy becomes the preferred choice independent of the value of I_R. The underlying reason for this is that the value gained from upgrading before an eventual replacement is not high enough compared to directly replacing the turbine when the cost of upgrading is large.

[13]If the value of I_R is increased to a unreasonable high level, this is not the case. However, because this requires such an unrealistically high investment cost, we consider this point to be an upper end cut-off value.

I_R	15	20	25	30	35	40
$\pi^*_{0,R}$	34.76	46.35	57.94	69.52	81.11	92.70
$\pi^*_{0,U}$	7.182	7.190	7.196	7.200	7.203	7.205
π^*_{0,W_L}	33.12	47.44	62.09	76.98	92.04	107.3
$\widetilde{\pi_0}^*$	37.20	52.15	67.37	82.79	98.35	114.0
π^*_{0,W_U}	40.60	56.05	71.73	87.56	103.5	119.5
π^*_U	51.74	68.99	86.24	103.5	120.7	138.0

(a) $I_U = 1$ [MNOK]

I_R	15	20	25	30	35	40
$\pi^*_{0,R}$	34.76	46.35	57.94	69.52	81.11	92.70
$\pi^*_{0,U}$	-	-	-	35.72	35.76	35.79
π^*_{0,W_L}	-	-	-	42.26	55.97	69.38
$\widetilde{\pi_0}^*$	-	-	-	58.06	71.54	85.28
π^*_{0,W_U}	-	-	-	70.36	84.24	98.48
π^*_U	-	-	-	103.5	120.7	138.0

(b) $I_U = 5$ [MNOK]

I_R	15	20	25	30	35	40
$\pi^*_{0,R}$	34.76	46.35	57.94	69.52	81.11	92.70
$\pi^*_{0,U}$	-	-	-	-	-	-
π^*_{0,W_L}	-	-	-	-	-	-
$\widetilde{\pi_0}^*$	-	-	-	-	-	-
π^*_{0,W_U}	-	-	-	-	-	-
π^*_U	-	-	-	-	-	-

(c) $I_U = 9$ [MNOK]

Table 3.6: *Effect of varying investment costs*

Table 3.7 presents the effect of changing the discount rate, ρ. The results show that increasing ρ effectively devalues the sequential strategy, making the single investment strategy dominant for higher values of ρ. A higher discount rate mitigates the relative importance of the change in drift when the turbine is in an upgraded state, so the sequential strategy loses its attractiveness, and thus the gain of a lifetime extension is discounted too much to be a viable strategy for the firm. Further, we see that $\pi^*_{0,R}$ increases with ρ. Intuitively, this follows from the time value of money, where increasing the cost of capital reduces the value of the expected future cash flows from replacement relative to the expected future cash flows from continuing current operations. However, in the dichotomous case it is not as clear-cut. From Table 3.7 we observe that when ρ approaches the drift rate, i.e $\alpha - \gamma$, the upgrading option becomes more valuable and $\pi^*_{0,U}$ decreases, whereas π^*_U increases significantly. The latter effect stems from the fact that when the discount rate is low, it is more profitable to operate with an upgraded turbine for a longer period of time due to the favourable drift rate, and, thus, investment in replacement is delayed. The opposite

ρ	0.04	0.05	0.06	0.07	0.08	0.10
$\pi^*_{0,R}$	49.77	59.76	69.52	79.10	88.54	107.1
$\pi^*_{0,U}$	11.25	22.12	35.72	-	-	-
π^*_{0,W_L}	97.03	51.18	42.26	-	-	-
$\widetilde{\pi_0}^*$	101.3	59.89	58.06	-	-	-
π^*_{0,W_U}	105.6	67.95	70.36	-	-	-
π^*_U	192.4	109.7	103.5	-	-	-

Table 3.7: *Effect of varying discount rate*

is true when ρ increases and approaches the cut-off point where the optimal investment environment changes. Now, the discounting effect dominates and the firm is incentivized to replace the turbine earlier, making it the more valuable option. It is also worth mentioning that the inaction regions given by $[0, \pi^*_{0,U})$ and $(\pi^*_{0,W_L}, \pi^*_{0,W_U})$ are expanding with ρ. This is caused by an increased value in the option to invest in either of the two strategies and hence increases the opportunity cost of investing immediately. As a result, π^*_{0,W_U} does not behave monotonically as opposed to the other thresholds. The reason is that while the value of replacing increases, upgrading becomes even more profitable because operating with a lower degradation rate becomes more important. This elevates the indifference point and results in a rebound of π^*_{0,W_U} towards higher values.

Lastly, we have studied the effects of changing the degradation rates. Table 3.8 also differentiates between the two different ways to upgrade the turbine in the sequential strategy. In Table 3.8a and Table 3.8b, upgrading is undertaken as a surface treatment, while in Table 3.8c and Table 3.8d upgrading leads to a change in the operating pattern (see section 3.3.2.2 for a more elaborate explanation). In addition, for each of the two alternative cases, we have run one set of values where the post-upgrading degradation rate is low, and another where the post-upgrading rate is high. The former implies that the reduction in efficiency decay after upgrading is quite large, whereas in the latter, the effect of upgrading is downplayed. First, we see that for both interpretations of the upgrading cost, increasing γ_0 has the same effect on the development of the thresholds. The same can be seen when transitioning from a high to a low rate of γ_U, as also here the benefit from upgrading increases. More specifically, we observe that when γ_U is low, as presented in Table 3.8a and Table 3.8c, the dichotomous environment dominates. In contrast, when the degradation after upgrading is high, as presented in Table 3.8b and Table 3.8d, the single replacement strategy dominates for a wider range of values. Thus, the firm experiences a lower incentive to invest in the sequential strategy when the difference between γ_0 and γ_U decreases, as is expected.

A prominent feature in Table 3.8 is that $\pi^*_{0,R}$ behaves identically in all of the sub-tables. The reason for this is that the single replacement threshold is independent of both k, I_U and γ_U. These parameters only relate to the upgrading option in the sequential strategy. We also see that $\pi^*_{0,R}$ is quite insensitive to changes in γ_0. This is most likely due to the drift rate being dominated by α, which is at least one order of magnitude greater than γ_0. Concurrently, independent of whether γ_U is high or low, a recurring theme is that when the benefit of upgrading increases, the thresholds for replacement in the dichotomous environment are elevated. This applies to both strategies, i.e π^*_{0,W_U} and π^*_U increase. At

γ_0	0.08%	0.09%	0.10%	0.12%	0.15%
$\pi^*_{0,R}$	69.45	69.49	69.52	69.59	69.70
$\pi^*_{0,U}$	29.57	25.42	22.29	17.90	13.82
π^*_{0,W_L}	53.17	63.32	74.74	107.3	247.7
$\widetilde{\pi_0}^*$	64.46	72.27	82.17	112.8	251.8
π^*_{0,W_U}	74.47	80.66	89.34	118.3	255.8
π^*_U	115.0	129.0	146.6	201.3	449.3

(a) *$k = 1$, $I_U = 5$, $\gamma_U = 0.02\%$*

γ_0	0.08%	0.09%	0.10%	0.12%	0.15%
$\pi^*_{0,R}$	69.45	69.49	69.52	69.59	69.70
$\pi^*_{0,U}$	-	-	-	35.96	22.67
π^*_{0,W_L}	-	-	-	42.04	73.71
$\widetilde{\pi_0}^*$	-	-	-	58.01	81.30
π^*_{0,W_U}	-	-	-	70.37	88.58
π^*_U	-	-	-	103.3	144.8

(b) *$k = 1$, $I_U = 5$, $\gamma_U = 0.07\%$*

γ_0	0.08%	0.09%	0.10%	0.12%	0.15%
$\pi^*_{0,R}$	69.45	69.49	69.52	69.59	69.70
$\pi^*_{0,U}$	10.87	7.785	6.062	4.202	2.879
π^*_{0,W_L}	59.83	67.97	76.22	96.11	147.1
$\widetilde{\pi_0}^*$	68.97	74.84	81.76	100.1	149.9
π^*_{0,W_U}	74.87	79.86	86.13	103.6	152.6
π^*_U	83.48	90.58	98.96	121.2	181.5

(c) *$k = 0.99$, $I_U = 0.75$, $\gamma_U = 0.02\%$*

γ_0	0.08%	0.09%	0.10%	0.12%	0.15%
$\pi^*_{0,R}$	69.45	69.49	69.52	69.59	69.70
$\pi^*_{0,U}$	-	-	-	18.87	6.235
π^*_{0,W_L}	-	-	-	49.20	75.51
$\widetilde{\pi_0}^*$	-	-	-	63.80	81.18
π^*_{0,W_U}	-	-	-	71.11	85.62
π^*_U	-	-	-	77.18	98.19

(d) *$k = 0.99$, $I_U = 0.75$, $\gamma_U = 0.07\%$*

Table 3.8: *Effect of varying the degradation rates*

the same time, $\pi^*_{0,U}$ decreases drastically under the dichotomous regime. Both of these changes can be explained by the attractiveness of operating in the upgraded state when the relative difference between γ_0 and γ_U escalates. By upgrading earlier, the benefit is reaped sooner and the time until a replacement is required is prolonged due to the decelerated degradation rate. The attractiveness of operating in the upgraded state is also shown in the partition between the respective investment and inaction regions. When γ_0 increases, the second inaction region, $(\pi^*_{0,W_L}, \pi^*_{0,W_U})$, shrinks, while the first investment region, $[\pi^*_{0,U}, \pi^*_{0,W_L}]$, expands rapidly.

4. Possible Extension with Risk of Failure

Every year, large sums are invested in maintenance activities and rehabilitation in order to sustain operations in Norwegian hydropower plants. These investments contribute to maintaining the total production volume, and also reduce the probability of failures as knowledge about the current system state is obtained. If a failure should occur, the consequences that follow are huge and can have a significant impact on the economic position of the firm. In order to improve the decision making process, the constant risk of experiencing a failure could therefore be included in the analysis. As a result, the economic utility value of incorporating failure in the analysis may have a significant influence on the maintenance decisions of the firm.

The practical relevance of failures is also irrefutable. It is said that Norwegian hydropower turbines have had a hard time adapting to the new energy market that is unfolding in Europe at the moment[1]. Several of the biggest turbines have experienced some sort of trouble in recent years, among them Svartisen[2]. The fact that hydropower enables a highly flexible production scheme that can exploit fluctuations in the market price, leaves the power plant especially vulnerable to demanding operating patterns. This was what culminated in the break-down of the turbine at Svartisen, where the power plant was forced out of operation for six months and consequently suffered a great economical loss. Because of the great consequences, it is clear that hydropower firms seek to improve the state of their turbines by optimizing the maintenance schedules and re-investment activities to minimize the likelihood of experiencing a failure.

A failure may have several root causes, but can normally be separated into two distinct types (Amari and McLaughlin, 2004). Soft faults occur because the turbine is degraded over time until it finally fails, whereas hard faults are unpredictable and occur instantaneously. The probabilities of these failures modes also differ. Hard faults are caused by external factors and are consequently independent of the system state (Amari and McLaughlin, 2004). Therefore, they have a relatively stable failure rate over time. Soft faults, on the other hand, grow gradually with time, and have a failure rate that increases with time (Amari and McLaughlin, 2004). This means that a turbine that has been in operation for 25 years has a higher failure rate compared to the same turbine 10 years earlier.

Similar to degradation, the numerical values and development of the failure rate is to a large extent unknown. The reasons are much the same. First, turbines are uniquely

[1] https://www.byggfakta.no/taper-store-summer-pa-turbinhavari-85502/nyhet.html
[2] https://gemini.no/2015/04/hvorfor-sprekker-vannturbinene/

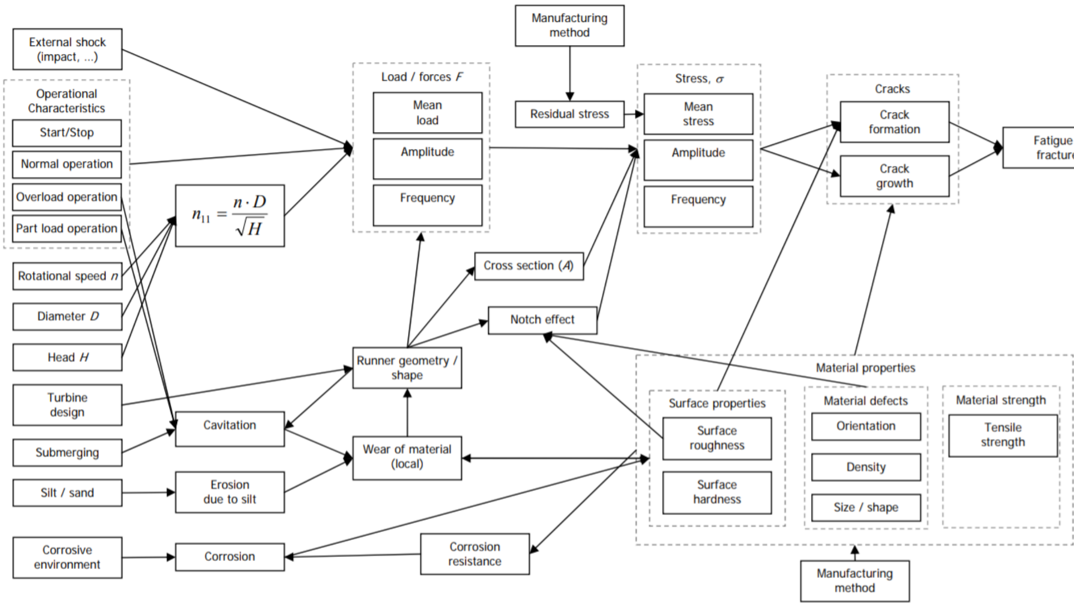

Figure 4.1: *Different turbine failure modes (Welte, 2008)*

engineered to fit specific locations and operating patterns. Second, failures are relatively rare. Therefore, a sufficient sample size is challenging to obtain, and the low failure rates will therefore carry a relatively high uncertainty. Finally, different turbine designs have many different failure modes, with failure rates and consequences that also differ (Welte et al., 2006). A wide range of these failure modes are shown in Figure 4.1. Together, these challenges indicate that modelling failure in a specific hydropower plant is an intricate matter. Therefore, failure models are often rather complex and computationally heavy to solve. The scarcity of empirical data makes it particularly difficult to determine the failure rate for a specific hydropower plant. Therefore, expert opinions are often the best available method to parameterize a model (Welte et al., 2007). While experts have vast knowledge, it can be challenging to translate their expertise into numerical values in order to use the models.

In the literature concerning maintenance, failure risk has been modelled in various ways. Soft faults are often the main focus of such models as they are, to a great extent, possible to monitor. A commonly used method is to link failure to the degradation process. With this procedure, degradation eventually leads to failure unless preventive measures are taken. Such measures are usually in the form of maintenance, and the models are therefore often coupled with maintenance scheduling. Amari and McLaughlin (2004) present a condition-based maintenance model for a deteriorating system. The model is based on a Markov chain where maintenance actions bring the system back to the "as good as new"-state. In their work, only soft faults are included and closed-form solutions are presented when assuming that the duration of each deterioration stage is exponentially distributed.

A possible extension of traditional Markov chains is to make use of non-exponential distributions to represent the duration of each state. These models are claimed to be more realistic, especially as they are able to vary the failure rate through time. Welte (2009) discusses some of the limitations of traditional Markov chains, and highlights that in a

Markov chain, "an inspection, followed by the decision to do nothing, has no influence on the system state" (Welte, 2009, Ch. IV, B). In real life, inspections will provide information that affects future decisions. This contradicts the memoryless property of the exponential distribution and results in more flexible distributions being suggested, such as Gamma and Weibull distributions.

Following these lines, Welte et al. (2006) present a Markov state model where the duration of each state is represented by a Gamma distribution. As opposed to an exponential distribution, the Gamma distribution allows the deterioration process to be dependent on previous states and to be time-inhomogeneous (Grall et al., 2002). Although Gamma and Weibull distributions come with more flexibility, they are also considerably harder to solve. Therefore, approximations are often used to simplify the solution method. Welte et al. (2007) approximates a semi-Markov process with Gamma distributed step durations to a Markov chain by utilising the Erlang distribution[3]. Weibull distributions are not as easy to approximate since no direct relationship to the exponential distribution exists. Welte et al. (2007) present possible simplifications of the Weibull distribution by partial distribution fits. Though a partial fit is considerably easier to solve, it tends to perform poorly in the tails of the distribution.

Our model has a strong focus on maintaining tractability so that it can easily be applied to a broad range of different circumstances and individual hydropower plants. Since the modelling of failures necessarily will require a numerical implementation, it would be hard to legitimize the results of comparing such a model with our proposed modelling framework. As our goal has been to investigate the big picture related to what the optimal overall renovation strategy should be, the specific modelling of failures would not be compatible with this approach. Instead, we encourage the scientific community to extend the model in a direction where the dimension of failures is implemented as an integral part of the framework, and where inferences can be made in relation to the optimal renovating strategies. Furthermore, as the risk of failures is a multi-faceted problem, it would require a solid base of empirical data and specific information about age, design and technical conditions to model it accurately. We will now present a short discussion which aims to highlight the effects that the risk of experiencing a turbine failure would have for the model implications.

Whether the failure rates are represented by a time-homogeneous process such as the exponential process, or a time-inhomogeneous process such as Gamma and Weibull processes, it is natural to assume that both upgrading and replacing would affect the failure rate positively. That is, if the turbine is either upgraded or replaced, the failure rate just after investment should be lower than that just before investment. It is also expected that replacement has a larger effect on the failure rate than upgrading because the new turbine has not experienced any mechanical wear.

When considering how including a failure would affect the model implications, there are mainly two factors that need to be taken into account. One is the cost of repairing a failure, and the other is the difference in failure rates and how they develop through time. In the following discussion, we focus on the intuitive effects that these factors should have on the

[3] This approximation is only valid if the shape parameter of the Gamma distribution has an integer value. If this is not the case, more complex approximations have to be used (Welte et al., 2007).

model results, and also how the implications would change if they were to be varied. To simplify the understanding, we assume that there is only one kind of failure, so that there is only one cost and one failure rate at each point in time.

By including the new element of turbine failure to the model, both the value functions and the respective thresholds are likely to be altered. Depending on the way that failure is modelled, these alterations can have various manifestations. However, some general consequences can be determined independently of the choice of failure model. First, the value of the options should decrease because of the added cost. As a potential failure will reduce the profit flow, this has a negative impact on the value of both the firm and the options. However, the investment options also provide an upside in that the firm has the possibility to reduce the failure probability. Therefore, the thresholds are expected to decrease because of the added incentive that the firm has to reduce the failure probability as soon as possible.

When the cost of failure increases, the expected discounted loss of experiencing a failure also increases. Therefore, the firm should be incentivized to invest sooner in either of the strategies. In other words, when the firm faces the possibility of a more unfavourable expected cost, it would be more eager to invest in a strategy which reduces the probability of this happening. Since this incentive is similar in both of the stages in the sequential strategy and in the single replacement investment, it is fairly one-dimensional. The same eagerness to invest should be present if the initial failure rate increases. However, whereas an increased cost incentivizes the firm to invest in order to reduce the higher expected cost, an increased initial failure rate incentivizes the firm to invest in order to reduce the probability of experiencing a failure.

If the effect that the investment opportunities have on the failure rates is altered, it is likely to affect the dominant investment strategy. When the effect that upgrading has on the failure rate decreases, the attractiveness of the sequential strategy relative to the single replacement strategy is also reduced. The firm is therefore more hesitant to invest in the sequential strategy because the net gain from upgrading decreases. As a consequence, the investment thresholds for the sequential strategy should increase, and the first investment region, where investing in this strategy is optimal, should narrow. When the effect of upgrading becomes sufficiently small, it is expected that the single replacement strategy will take over as the dominating strategy for the firm's renovating decisions.

As opposed to the effect of upgrading, the effect of replacement will affect both of the investment strategies. This is because both strategies will eventually lead to the installation of a new turbine, so that the benefit of replacement does not have the same discriminating effect as that of upgrading. It is therefore challenging to determine how an alteration of this benefit will affect the resulting investment strategy, as it is also likely to depend on the effect of upgrading. However, it is fair to say that the benefit of upgrading is likely to have a larger impact on the optimal strategy. Finally, it is important to note that all of the effects above are considered in isolation. If multiple parameters were to be changed simultaneously, interdependencies would make it increasingly harder to anticipate the resulting consequences.

In this section, we have discussed the possible consequences that including the risk of failure would have for the model implications. In the literature concerning maintenance, a variety of different models have been proposed to model the occurrence of failures. Some of these are highly complex and computationally heavy. Still, we propose that some overall implications should be similar, regardless of the choice of modelling framework. First, we argue that the incentive to invest should be increased due to the great economic consequences that follow in the wake of a turbine failure. Furthermore, we hypothesize that the benefit of upgrading will continue to be the main determinant of the optimal investment environment. This should be fairly easy to verify or refute within a possible extension of our modelling framework.

5. Conclusion

This paper examines the decisions of a hydropower firm concerning a potential upgrade or replacement of the turbine within the real options framework. We study the optimal strategy of the firm when it faces two alternative investment opportunities; a direct turbine replacement or a lifetime-extension before a subsequent replacement. We focus on building a tractable model where we examine the conditions for when it is optimal to switch from one strategy to the other.

This thesis contributes to the literature on mutually exclusive strategies within the real options framework where the investment region may no longer be connected. From a theoretical standpoint, it can be viewed as an extension of the model developed by Décamps et al. (2006) by including a declining profitability and the options to change either its future path or its current value. The decline in profitability stems from the deteriorating efficiency of an ageing hydropower turbine. Compared to related models that examine investment decisions in hydropower, our model is considerably more tractable and easy to implement. This means that the model is able to incorporate several highly relevant technical factors within the same stylized framework. Our study provides two main insights.

First, we find that there is a possibility that the investment region is dichotomous. That is, the investment region is no longer a connected set, but includes multiple regions. We find that this environment is more likely to be present when the upgrading option is more valuable. Examples of such real-life situations are when the initial efficiency of the turbine closes in on the efficiency of a new turbine. In such an event, an upgrade is very attractive in order to keep the turbine in operation for a longer period of time. If, however, the initial efficiency is considerably lower than that of a new turbine, it is optimal to replace directly.

Second, we consider two different interpretations in relation to the turbine upgrade. The first one considers the upgrade as a one-time investment cost, whereas the second considers the upgrade as a change in the operating pattern. We find that the interpretations do not have very different implications for our model. This is because both of them affect the optimal strategy and the thresholds in the same way. Therefore, from an investment point of view, the difference between the two is small. However, the practical differences between them are considerable. The second interpretation will yield a given percentage reduction the in the overall profit, whereas the first one entails a larger upfront investment cost.

We also present a discussion that highlights the expected changes to the model implications if the risk of failure is incorporated as an aspect. In this discussion, we assume that the failure rate is decreased when investments are made, and thereafter argue that the firm should be incentivized to invest earlier due to the great consequences that follow in the wake of a turbine failure. Furthermore, we hypothesize that the main determinant of the optimal investment environment will still follow from the benefit of upgrading.

Our aim has been to build a model which is applicable to a wide range of different hydropower turbines and locations. The high degree of idiosyncrasy between hydropower plants also implies that the model can be interpreted in a more general manner. For instance, the model should be a good fit for other power generating industries, such as wind power. In order for the model to fit this particular industry, the numerical values will have to be adjusted. Additionally, some minor alterations to the model would be suggested. It would, for example, be advantageous to include innovation in the efficiency of the new turbine since wind power is a less mature industry. The opportunity to utilize the model in similar industries is an advantage of our stylized setting which underscores its generalizability.

5.1 Suggestions for Further Research

In this section we outline possibilities for further research. First, our model relies on the relatively tractable exponentially declining function to represent the degradation process. We defined this deterministic process to be all-encompassing, accounting for all the different sub-processes that affect the efficiency. However, a real-life degradation process is not deterministic and, as Welte (2008) describes it, is likely to resemble a grey-box model where the underlying principles are known, but where the manifestation of the dynamical process is stochastic. Thus, typical examples of grey-box models include stochastic processes which are empirically founded on a measurable quantity, indicating a time-dependency. More specifically, this could for example be a Gamma or Weibull process, which are regarded to conform to the hydropower industry requirements relatively well (Welte, 2008). These processes can also be applied to the risk of failure, and since the processes of failure and deterioration are so intimately coupled, it could be beneficial to merge the two aspects into one comprehensive model. Welte (2008) does this by applying a continuous time semi-Markov process with a discretized state-space. The reason for the discretization is to enable a dynamically changing process where it is relatively easy to define and verify the state of the system. If the main objective is to develop a maintenance and inspection policy, this becomes increasingly relevant.

The real options model can also be extended to include investment lag and uncertainty in lead time and investment costs. All of these factors were omitted in our model to simplify the calculations. However, in the hydropower industry, lead time uncertainty is a prominent issue as each turbine has to be custom-made to comply with the geographical, economical and operating conditions at the specific site. This can make the ordering process quite extensive and initiates an "option-less" period, similar to Richardson et al. (2013). In the "option-less" period, the hydropower firm has placed an order and is therefore refrained from making any decisions regarding the turbine. The main costs associated with this period are comprised of lower revenues and the potential of being out of operation if a major failure occurs. The uncertainty in these costs can be incorporated in the model by relating the value of the firm across the stochastic "option-less" lead time interval in the value matching conditions. This, however, is not expected to have a large impact on the results, but complicates the solution procedure, and may require numerical methods.

Our model only allows for a specific upgrade, which does not affect the current value of the turbine's efficiency. However, when renovating the turbine, the efficiency will often experience a small positive jump in its value. The size of the upgrade can also be varied by the hydropower producer. The model presented in this thesis could therefore be extended to include a choice in investment size, where the impact on the efficiency varies with the magnitude of investment. This could be done similar to what is presented in Olsen (2018). Alternatively, a set of different upgrade opportunities could be presented, where some alternatives could represent changes in operational patterns, whereas others represent physical alterations to the turbine. In that case, the hydropower producer has to make a choice regarding which option to exercise.

As mentioned earlier, it is fairly difficult to model the failure of a hydropower turbine if all the complex real-life phenomena are to be included. Welte (2009) argues that a suitable approach is to use a time-dependent distribution which enables the encapsulation of several characteristics of a hydropower plant. An extension of our model could, therefore, be to include a stochastic process representing failures, where failures may occur randomly. Such a process could possibly utilize a fairly flexible distribution, e.g a Gamma or a Weibull distribution, for the failure rate in order to incorporate time-dependencies. It is also possible to include multiple failure modes with different associated costs by having several failure processes for each state. One way to do this is to create an event tree where the branches symbolize different groups of failure modes, and also the specific failures within each failure mode. Then the associated costs related to each of these failures could be represented in the tree along with the failure mode. This can be implemented with differing complexity, from grouping the similar failure modes together to form larger "families" of failures, to creating separate failure processes associated with each individual failure mode.

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A. Appendix

A.1 Derivations of the π-processes

Here we show the derivation of the stochastic processes describing π_0 and π_U, which follow from Eq. (3.5) and Eq. (3.12), respectively. Since the derivation is the same in both cases, we show the derivation of a general process π.

First, we define

$$\pi(t) = P_0(t)e^{-\gamma t}, \tag{A.1}$$

where dP_0 is defined by Eq. (3.1). From this, we find the following partial derivatives:

$$\frac{\partial \pi}{\partial t} = -\gamma P_0 \pi e^{-\gamma t}, \qquad \frac{\partial \pi}{\partial P_0} = e^{-\gamma t}, \qquad \frac{\partial^2 \pi}{\partial P_0^2} = 0. \tag{A.2}$$

Applying Itô's lemma to expand $d\pi(t)$, we get

$$d\pi = \frac{\partial \pi}{\partial t}dt + \frac{\partial \pi}{\partial P_0}dP_0 + \frac{1}{2}\frac{\partial^2 \pi}{\partial P_0^2}\left(dP_0\right)^2. \tag{A.3}$$

Inserting the expression for dP_0 and the partial derivatives of π, we obtain

$$\begin{aligned} d\pi &= -\gamma P_0 e^{-\gamma t}dt + e^{-\gamma t}(\alpha P_0 dt + \sigma P_0 dZ_t) \\ &= (\alpha - \gamma)P_0 e^{-\gamma t}dt + \sigma P_0 e^{-\gamma t}dZ_t. \end{aligned} \tag{A.4}$$

Now, by substituting Eq. (A.1) into the equation above, we get

$$d\pi = (\alpha - \gamma)\pi dt + \sigma \pi dZ_t. \tag{A.5}$$

This derivation holds for both π_0 and π_U. The difference between the two is that, in the former, the degradation process is characterized by γ_0, whereas in the latter it is characterized by γ_U.

A.2 Proof of Propositions

A.2.1 Proposition 3.3.1

In the stopping region, the option has the following value:

$$F_R(\pi_0) = \left[\int_0^\infty \pi_0(t) Q_R e^{-\rho t} \mathrm{d}t - I_R \,\middle|\, \pi_0(0) = \pi_0\right] = \frac{Q_R \pi_0}{\mu_0} - I_R. \tag{A.6}$$

To find the value in the continuation region, the Bellman equation must hold. This equation states that

$$\rho F_R(\pi_0) dt = \mathbb{E}[dF_R] + Q_0 \pi_0 dt. \tag{A.7}$$

Applying Itô's lemma to expand the dF_R-term gives

$$dF_R = \frac{\partial F_R}{\partial \pi_0} d\pi_0 + \frac{1}{2}\frac{\partial^2 F_R}{\partial \pi_0^2}(d\pi_0)^2. \tag{A.8}$$

By using $d\pi_0 = (\alpha - \gamma_0)\pi_0 dt + \sigma \pi_0 dZ_t$, combined with the the approximations $dt^2 \approx 0$ [1], $dZ_t \cdot dt \approx 0$ and $dZ_t^2 = dt$, we get that $d\pi_0^2 = \sigma^2 \pi_0^2 dt$. Inserting this into Eq. (A.8), yields

$$dF_R = \Big((\alpha - \gamma_0)\pi_0 dt + \sigma \pi_0 dZ_t\Big)\frac{\partial F_R}{\partial \pi_0} + \frac{1}{2}\frac{\partial^2 F_R}{\partial \pi_0^2}\sigma^2 \pi_0^2 dt. \tag{A.9}$$

Inserting this in Eq. (A.7) and using that $\mathbb{E}[dZ_t] = 0$, gives the following ODE:

$$(\alpha - \gamma_0)\pi_0 \frac{\partial F_R}{\partial \pi_0} + \frac{1}{2}\sigma^2 \pi_0^2 \frac{\partial^2 F_R}{\partial \pi_0^2} - \rho F_R = -Q_0 \pi_0. \tag{A.10}$$

Solving this differential equation yields one homogeneous and one particular solution. i.e $F_R = F_h + F_p$. We start by solving the homogeneous equation, namely

$$(\alpha - \gamma_0)\pi_0 \frac{\partial F_h}{\partial \pi_0} + \frac{1}{2}\sigma^2 \pi_0^2 \frac{\partial^2 F_h}{\partial \pi_0^2} - \rho F_h = 0. \tag{A.11}$$

We proceed by guessing a functional form

$$F_h = A\pi_0^\beta. \tag{A.12}$$

[1] Note that the expression with dt to a power of more than unity are approximated as zero, because in the limit $\lim_{dt \to 0}$ these terms go faster to zero.

Inserting this into the homogeneous ODE, yields

$$\beta(\beta-1)A\pi_0^{\beta-2}\frac{\sigma^2\pi_0^2}{2}+\beta A\pi_0^{\beta-1}(\alpha-\gamma_0)\pi_0-\rho A\pi_0^{\beta}=0. \tag{A.13}$$

Solving for β gives

$$\begin{aligned} &\frac{1}{2}\beta(\beta-1)\sigma^2+\beta(\alpha-\gamma_0)-\rho=0,\\ \implies &\beta_{1,2}=\frac{1}{2}-\frac{\alpha-\gamma_0}{\sigma^2}\pm\sqrt{\left(\frac{\alpha-\gamma_0}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2\rho}{\sigma^2}}. \end{aligned} \tag{A.14}$$

Hence, we get that the solution to the homogeneous part is $F_h(\pi_0)=A_1\pi_0^{\beta_1}+A_2\pi_0^{\beta_2}$. We also observe that $\beta_1>1$ and $\beta_2<0$. Following the lines of Dixit et al. (1994), we can infer that the option to refurbish is worthless if the profit flow becomes zero, i.e $F_R\to 0$ if $\pi_0\to 0$. Since $\beta_2<0$, we must have that $A_2=0$ in order to exclude the possibility of an infinite option value if $\pi_0\to 0$. This means that we can rewrite the expression for the homogeneous solution to

$$F_h=A_1\pi_0^{\beta_1}. \tag{A.15}$$

Next, we need to find a particular solution to the inhomogeneous part in Eq. (A.10). We assume that the functional form is

$$F_p=aQ_0\pi_0+b. \tag{A.16}$$

Inserting this in Eq. (A.10) and rearranging, yields

$$Q_0\pi_0(\alpha a-\gamma_0 a+1-\rho a)-\rho b=0. \tag{A.17}$$

This expression needs to hold for all values of π_0, which intuitively requires b to be equal to zero. Then, solving for a gives the following result

$$\begin{aligned} &\alpha a-\gamma_0 a+1-\rho a=0,\\ &a=\frac{1}{\rho-(\alpha-\gamma_0)}=\frac{1}{\mu_0}. \end{aligned} \tag{A.18}$$

Combining the expressions for the homogeneous and particular solutions of F_R, yields

$$F_R(\pi_0)=F_h(\pi_0)+F_p(\pi_0)=A_1\pi_0^{\beta_1}+\frac{Q_0\pi_0}{\mu_0}. \tag{A.19}$$

To find the optimal stopping value, $\pi^*_{0,R}$, the value matching and smooth pasting conditions must hold. These are given by the following expressions:

Value matching:

$$A_1 \pi^{*\,\beta_1}_{0,R} + \frac{Q_0 \pi^*_{0,R}}{\mu_0} = \frac{Q_R \pi^*_{0,R}}{\mu_0} - I_R. \tag{A.20}$$

Smooth pasting:

$$\beta_1 A_1 \pi^{*\,\beta_1 - 1}_{0,R} + \frac{Q_0}{\mu_0} = \frac{Q_R}{\mu_0}. \tag{A.21}$$

Solving these two equations for A_1 and $\pi^*_{0,R}$, yields

$$\pi^*_{0,R} = I_R \cdot \frac{\beta_1}{\beta_1 - 1} \cdot \frac{\mu_0}{Q_R - Q_0}, \tag{A.22}$$

$$A_1 = \frac{I_R}{\beta_1 - 1} \left[\frac{\beta_1 - 1}{\beta_1} \cdot \frac{1}{I_R} \cdot \frac{Q_R - Q_0}{\mu_0} \right]^{\beta_1}. \tag{A.23}$$

Thus, the value of the option to replace the existing turbine is given by

$$F_R(\pi_0) = \begin{cases} A_1 \pi_0^{\beta_1} + \dfrac{\pi_0 Q_0}{\mu_0} & \text{if } \pi_0 < \pi^*_{0,R}, \\ \dfrac{Q_R \pi_0}{\mu_0} - I_R & \text{if } \pi_0 \geq \pi^*_{0,R}. \end{cases} \tag{A.24}$$

A.2.2 Proposition 3.3.2

The stopping value is given by

$$G_R(\pi_U) = \left[\int_0^\infty \pi_U(t) Q_R e^{-\rho t} \mathrm{d}t - I_R \;\middle|\; \pi_U(0) = \pi_U \right] = \frac{Q_R \pi_U}{\mu_0} - I_R. \tag{A.25}$$

In the continuation region, the problem is almost the same as in Proposition 3.3.1. The differences between the two are that the current profit development process is given by π_U, and that it is multiplied by the factor k. Thus, the value of the replacement option in the continuation region is given by

$$G_R(\pi_U) = B_2 \pi_U^{\phi_2} + \frac{k Q_0 \pi_U}{\mu_U}, \tag{A.26}$$

where

$$\phi_2 = \frac{1}{2} - \frac{\alpha - \gamma_U}{\sigma^2} + \sqrt{\left(\frac{\alpha - \gamma_U}{\sigma^2} - \frac{1}{2} \right)^2 + \frac{2\rho}{\sigma^2}}. \tag{A.27}$$

To find the optimal stopping value, π_U^*, the value matching and smooth pasting conditions must be met. These are given by the following expressions:

Value matching:

$$B_2 {\pi_U^*}^{\phi_2} + \frac{kQ_0\pi_U^*}{\mu_U} = \frac{Q_R\pi_U^*}{\mu_0} - I_R. \tag{A.28}$$

Smooth pasting:

$$B_2\phi_2 {\pi_U^*}^{\phi_2-1} + \frac{kQ_0}{\mu_U} = \frac{Q_R}{\mu_0}. \tag{A.29}$$

Solving these equations to find π_U^* and B_2, yields

$$\pi_U^* = \frac{\phi_2}{\phi_2 - 1} \cdot \frac{\mu_0\mu_U}{Q_R\mu_U - kQ_0\mu_0} \cdot I_R, \tag{A.30}$$

$$B_2 = \frac{I_R}{\phi_2 - 1}\left[\frac{\phi_2 - 1}{\phi_2} \cdot \frac{Q_R\mu_U - kQ_0\mu_0}{\mu_0\mu_U} \cdot \frac{1}{I_R}\right]^{\phi_2}. \tag{A.31}$$

Thus, the value of the option to replace in the sequential strategy is given by

$$G_R(\pi_U) = \begin{cases} B_2\pi_U^{\phi_2} + \dfrac{kQ_0\pi_U}{\mu_U} & \text{if } \pi_U < \pi_U^*, \\ \dfrac{Q_R\pi_U}{\mu_0} - I_R & \text{if } \pi_U \geq \pi_U^*. \end{cases} \tag{A.32}$$

A.2.3 Proposition 3.3.3

To determine if π_U^* lies below $\pi_{0,U}^*$ we insert the expression for π_U^* from Eq. (3.14) into Eq. (3.19). As shown in A.2.4.1, this implicit equation has a unique solution. This means that for any values of the profit flow below the investment threshold $\pi_{0,U}^*$, the inequality below will hold, and single replacement is the dominant strategy. The inequality is given by:

$$\begin{aligned} &B_2\frac{\phi_1 - \phi_2}{\phi_1}\left[\frac{\phi_2}{\phi_2 - 1} \cdot \frac{\mu_0\mu_U}{Q_R\mu_U - kQ_0\mu_0} \cdot I_R\right]^{\phi_2} \\ &+\frac{\phi_1 - 1}{\phi_1} \cdot \frac{Q_0(k\mu_0 - \mu_U)}{\mu_0\mu_U}\left[\frac{\phi_2}{\phi_2 - 1} \cdot \frac{\mu_0\mu_U}{Q_R\mu_U - kQ_0\mu_0} \cdot I_R\right] - I_U \leq 0. \end{aligned} \tag{A.33}$$

This can be simplified to

$$\frac{I_R}{\phi_2 - 1} \cdot \frac{\phi_1 - \phi_2}{\phi_1} + \frac{\phi_1 - 1}{\phi_1} \cdot \frac{kQ_0\mu_0 - Q_0\mu_U}{Q_R\mu_U - kQ_0\mu_0} I_R - I_U \leq 0. \tag{A.34}$$

If the inequality above does not holds, we know that π_U^* is greater than $\pi_{0,U}^*$.

A.2.4 Proposition 3.3.4

In the stopping region, one pays the investment cost to obtain the second option. Thus, the value of the option is given by

$$G_U(\pi_0) = B_2\pi_0^{\phi_2} + \frac{kQ_0\pi_0}{\mu_U} - I_U. \tag{A.35}$$

In the continuation region, the Bellman equation must hold. This equation is given by

$$\rho G_U dt = \mathbb{E}[dG_U] + Q_0\pi_0 dt. \tag{A.36}$$

Solving this equation for the homogeneous and the particular solution, similar to A.2.1, yields the following expression for the option value:

$$G_U = B_1\pi_0^{\phi_1} + \frac{Q_0\pi_0}{\mu_0}, \tag{A.37}$$

where

$$\phi_1 = \frac{1}{2} - \frac{\alpha - \gamma_0}{\sigma^2} + \sqrt{\left(\frac{\alpha - \gamma_0}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}}. \tag{A.38}$$

At the investment threshold, $\pi_{0,U}^*$, the following value matching and smooth pasting conditions must hold:

Value matching:

$$B_1{\pi_{0,U}^*}^{\phi_1} + \frac{Q_0\pi_{0,U}^*}{\mu_0} = B_2{\pi_{0,U}^*}^{\phi_2} + \frac{kQ_0\pi_{0,U}^*}{\mu_U} - I_U. \tag{A.39}$$

Smooth pasting:

$$B_1\phi_1{\pi_{0,U}^*}^{\phi_1-1} + \frac{Q_0}{\mu_0} = B_2\phi_2{\pi_{0,U}^*}^{\phi_2-1} + \frac{kQ_0}{\mu_U}. \tag{A.40}$$

The expression for $\pi_{0,U}^*$ cannot be solved analytically, but implicitly solves the following equation:

$${\pi_{0,U}^*}^{\phi_2} B_2\frac{\phi_1 - \phi_2}{\phi_1} + \pi_{0,U}^*\frac{\phi_1 - 1}{\phi_1} \cdot \frac{Q_0(k\mu_0 - \mu_U)}{\mu_0\mu_U} - I_U = 0. \tag{A.41}$$

Given the value of $\pi_{0,U}^*$, one can calculate the value of B_1 as

$$B_1 = B_2\frac{\phi_2}{\phi_1} \cdot {\pi_{0,U}^*}^{\phi_2-\phi_1} + \frac{Q_0}{\phi_1} \cdot \frac{k\mu_0 - \mu_U}{\mu_0 \cdot \mu_U}{\pi_{0,U}^*}^{1-\phi_1}. \tag{A.42}$$

Thus, the value of the option in the case where $\pi_{0,U}^* < \pi_U^*$ is given by

$$G_U(\pi_0) = \begin{cases} B_1\pi_0^{\phi_1} + \dfrac{Q_0\pi_0}{\mu_0} & \text{if } \pi_0 < \pi_{0,U}^*, \\ B_2\pi_0^{\phi_2} + \dfrac{kQ_0\pi_0}{\mu_U} - I_U & \text{if } \pi_0 \geq \pi_{0,U}^*. \end{cases} \tag{A.43}$$

A.2.4.1 Proof of Unique Solution for $\pi_{0,U}^*$

The implicit solution for $\pi_{0,U}^*$ in the sequential problem, given by Eq. (A.41), is of the following form:

$$\Psi(\pi_{0,U}^*) = A\pi_{0,U}^{*\ \phi_2} + B\pi_{0,U}^* - C = 0. \tag{A.44}$$

In order to prove the existence of a unique solution for $\pi_{0,U}^*$, we must prove that Eq. (A.44) only crosses the x-axis at a single point. We start by defining the domain of the above function, which is restricted to positive values only, i.e $\pi_{0,U}^* \in [0, \infty\rangle$. We also know that ϕ_2 is the positive root of the quadratic equation given by Eq. (3.15), and is thus greater than 1 (see Dixit et al. (1994)).

We can prove that the constants A, B and C are strictly positive. A consists of two terms, namely $(\frac{\phi_1-\phi_2}{\phi_1})$ and the constant B_2 defined by Eq. (3.17). First, we know that $\phi_1 > \phi_2$ due to the fact that $\gamma_0 > \gamma_U$. This means that $(\frac{\phi_1-\phi_2}{\phi_1})$ is always positive. In order for B_2 to be positive, we must assume that

$$Q_R\mu_U > kQ_0\mu_0. \tag{A.45}$$

This inequality signifies that the net benefit of replacing the turbine after first having upgraded it is positive. Combined, these two parts yield that A is always positive. Further, to assure a positive B, we require that

$$k\mu_0 \geq \mu_U. \tag{A.46}$$

This is the same as assuming that the net benefit from upgrading the pre-existing turbine is either zero or strictly positive, which must be true, otherwise the this option would have no intrinsic value. The last constant, C, represents the investment cost of upgrading and is by definition always strictly greater than zero. As we know that the constants are always positive, we can take the derivative of Eq. (A.44) to show that the function is monotonically increasing

$$\Psi'(\pi_{0,U}^*) = A\phi_2\pi_0^{*\ \phi-1} + B. \tag{A.47}$$

Since we have already confirmed that $\phi_2 > 1$, this is a monotonically increasing function for $\pi_{0,U}^* \in [0, \infty\rangle$. By applying the intermediate value theorem, we therefore know that Eq. (A.44) has a unique solution for $\pi_{0,U}^*$.

A.2.5 Proposition 3.4.1

To determine whether single replacement is the dominant strategy, the option values for the different regions need to be taken into account. If the single replacement option has a higher value than the sequential option in the region before any thresholds are reached, it can be shown that the option to replace only will always have the higher value.

We know that when all the thresholds are reached, the value functions for both strategies are parallel, where the value of the sequential option is shifted I_U to the right relative to the option to replace only. It is also known that the derivative of the option value in the stopping region of G_U is less than the derivative of the option value in the stopping region for F_R. From value matching and smooth pasting, we know that the values in the continuation regions will always converge towards the values in their respective stopping regions in terms of both values and derivatives. Using this, and the fact that the first derivatives of all option values in the continuation regions are strictly positive, it can be shown that the sequential option will first converge towards a less steep function and thereafter converge towards the right-shifted parallel line. It will therefore never cross the option value which converges towards the stopping value of single replacement.

Let us, therefore, consider the option values where both strategies are in the first inaction region. In the case where F_R is more valuable, the following inequality will hold:

$$F_R^{\mathcal{C}} - G_U^{\mathcal{C}} \geq 0 \tag{A.48}$$

Inserting the relevant expressions from Eq. (3.9) and Eq. (3.21), yields

$$\left[A_1\pi_0^{\beta_1} + \frac{\pi_0 Q_0}{\mu_0}\right] - \left[B_1\pi_0^{\phi_1} + \frac{\pi_0 Q_0}{\mu_0}\right] = A_1\pi_0^{\beta_1} - B_1\pi_0^{\phi_1} \geq 0. \tag{A.49}$$

A comparison of Eqs. (3.8) and (3.20) shows that $\beta_1{=}\phi_1$, so that the inequality can be simplified to

$$A_1 - B_1 \geq 0. \tag{A.50}$$

We now substitute these parameters by their expressions given in Eqs. (3.10) and (3.22)

$$\frac{I_R}{\beta_1 - 1}\left[\frac{\beta_1 - 1}{\beta_1}\cdot\frac{Q_R - Q_0}{\mu_0}\cdot\frac{1}{I_R}\right]^{\beta_1} - \left[B_2\frac{\phi_2}{\beta_1}\pi_{0,U}^{*\ \phi_2-\beta_1} + \frac{Q_0}{\beta_1}\cdot\frac{k\mu_0 - \mu_U}{\mu_0\mu_U}\pi_{0,U}^{*\ 1-\beta_1}\right] \geq 0. \tag{A.51}$$

By reformulation,

$$I_R\frac{\beta_1}{\beta_1 - 1}\left[\frac{\beta_1 - 1}{\beta_1}\cdot\frac{Q_R - Q_0}{I_R\mu_0}\pi_{0,U}^{*}\right]^{\beta_1} - B_2\phi_2\pi_{0,U}^{*\ \phi_2} - Q_0\frac{k\mu_0 - \mu_U}{\mu_0\mu_U}\pi_{0,U}^{*} \geq 0. \tag{A.52}$$

Thus, if Eq. (A.52) holds, single replacement will be the dominant strategy in the entire state space.

A.2.6 Proposition 3.4.2

When contemplating investment, the firm will select the strategy which generates the highest net expected profit, given the current profit flow π_0. The value of investment is therefore the highest stopping value of the two strategies, i.e $max\{G_U^{\mathcal{S}}, F_R^{\mathcal{S}}\}$. When the two strategies are equally valuable, it is called the indifference point. This point is given as the solution to

$$B_2 \widetilde{\pi}_0^{*\ \phi_2} + \frac{kQ_0\widetilde{\pi}_0^*}{\mu_U} - I_U = \frac{Q_R\widetilde{\pi}_0^*}{\mu_0} - I_R. \tag{A.53}$$

Rearranging, we get

$$B_2 \widetilde{\pi}_0^{*\ \phi_2} + \left[\frac{kQ_0\mu_0 - Q_R\mu_U}{\mu_0 \cdot \mu_U}\right]\widetilde{\pi}_0^* - (I_U - I_R) = 0. \tag{A.54}$$

For values of π_0 below the indifference point, the value of the sequential option exceeds that of the replace only option, and vice versa for values above the indifference point.

The full proof for why the indifference point never belongs to the stopping region is rather technical and out of scope of this thesis, hence we refer to Proposition 2.2 in the Appendix of Décamps et al. (2006) for the derivation. However, the intuition for this result is quite instructive and serves the purpose of this thesis. We start with the heuristic argument put forward by Dixit et al. (1994) to justify the smooth pasting condition. Suppose that the current profit is equal to the indifference point. Then, by waiting for a small time dt, the firm can observe the evolution of the profit without having to make any decisions. The intuitive idea is that by waiting a little longer, the firm can observe the next step of π_0 and choose to invest on either side of $\widetilde{\pi}_0$. The resulting average pay-off is thus greater than the payoff obtained by investing at the indifference point itself since the payoff at this point is not differentiable (see Figure A.1). This is an implication that follows directly from Jensen's inequality (McDonald, 2014), which states that, given a convex function, equally spaced changes in π_0 give rise to unequally spaced changes in $V(\pi_0)$. In particular, $V[\mathbb{E}(\pi_0)] \leq \mathbb{E}[V(\pi_0)]$, as illustrated in Figure A.1. This remains true even though the average payoff must be discounted because it occurs at a later time dt. The reason is that, for a Brownian motion, the movements are proportional to $\sqrt{dt}$, which is valid for the expected payoff. However, the cost due to discounting is of magnitude dt, and thus when dt is small, the $\sqrt{dt}$-term dominates. The result is that the firm is better off by waiting for more information, which gives rise to an inaction region. Thus, whenever the inequality given by Proposition 3.4.1 does not hold, in contrast to Dixit et al. (1994), the stopping region is dichotomous and the optimal investment decision is not governed by a simple trigger strategy.

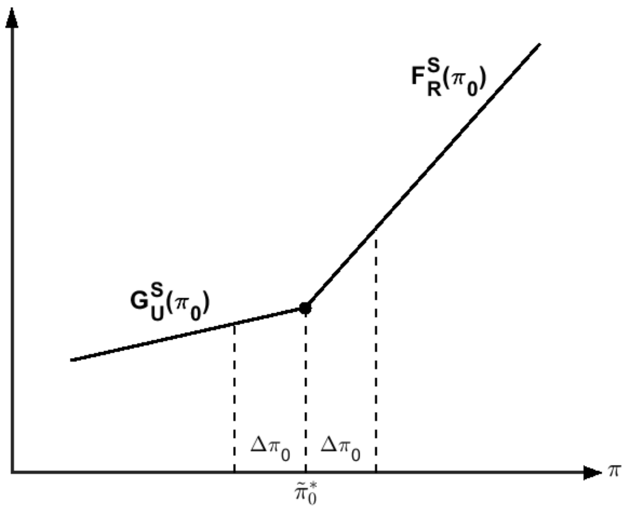

Figure A.1: *Smooth pasting condition at the indifference point. In reality, G_U^S is a convex function, but it can be approximated as a linear function for small values of delta.*

A.2.7 Proposition 3.4.3

In order to find the values for C, D, π^*_{0,W_L} and π^*_{0,W_U}, value matching and smooth pasting conditions must be met at the two thresholds. The conditions at π^*_{0,W_L} are given by

Value matching:

$$C\pi^{*\ \beta_1}_{0,W_L} + D\pi^{*\ \beta_2}_{0,W_L} + \frac{Q_0\pi^*_{0,W_L}}{\mu_0} = \frac{Q_0 k\pi^*_{0,W_L}}{\mu_U} + B_2\pi^{*\ \phi_2}_{0,W_L} - I_U. \tag{A.55}$$

Smooth pasting:

$$\beta_1 C\pi^{*\ \beta_1-1}_{0,W_L} + \beta_2 D\pi^{*\ \beta_2-1}_{0,W_L} + \frac{Q_0}{\mu_0} = \frac{Q_0 k}{\mu_U} + \phi_2 B_2\pi^{*\ \phi_2-1}_{0,W_L}. \tag{A.56}$$

Rearranging Eq. (A.56), we get

$$C = \left[\frac{Q_0 k}{\mu_U} + \phi_2 B_2\pi^{*\ \phi_2-1}_{0,W_L} - \frac{Q_0}{\mu_0} - \beta_2 D\pi^{*\ \beta_2-1}_{0,W_L}\right]\frac{\pi^{*\ 1-\beta_1}_{0,W_L}}{\beta_1}. \tag{A.57}$$

Inserting this into Eq. (A.55) and rearranging, yields

$$D = Q_0\frac{\beta_1 - 1}{\beta_1 - \beta_2}\cdot\frac{\mu_0 k - \mu_u}{\mu_0\mu_U}\pi^{*\ 1-\beta_2}_{0,W_L} + B_2\frac{\beta_1 - \phi_2}{\beta_1 - \beta_2}\pi^{*\ \phi_2-\beta_2}_{0,W_L} - I_U\frac{\beta_1}{\beta_1 - \beta_2}\pi^{*\ -\beta_2}_{0,W_L}. \tag{A.58}$$

By using the expression for D given by Eq. (A.58) in Eq. (A.57), we get

$$C = Q_0 \frac{\beta_2 - 1}{\beta_2 - \beta_1} \cdot \frac{\mu_0 k - \mu_U}{\mu_0 \mu_U} \pi^{*\ 1-\beta_1}_{0,W_L} + B_2 \frac{\beta_2 - \phi_2}{\beta_2 - \beta_1} \pi^{*\ 1-\beta_1}_{0,W_L} - I_U \frac{\beta_2}{\beta_2 - \beta_1} \pi^{*\ -\beta_1}_{0,W_L}. \tag{A.59}$$

On the other end of the interval, the conditions at π^*_{0,W_U} are given by

Value matching:

$$C \pi^{*\ \beta_1}_{0,W_U} + D \pi^{*\ \beta_2}_{0,W_U} + \frac{Q_0 \pi^*_{0,W_U}}{\mu_0} = \frac{Q_R \pi^*_{0,W_U}}{\mu_0} - I_R. \tag{A.60}$$

Smooth pasting:

$$\beta_1 C \pi^{*\ \beta_1 - 1}_{0,W_U} + \beta_2 D \pi^{*\ \beta_2 - 1}_{0,W_U} + \frac{Q_0}{\mu_0} = \frac{Q_R}{\mu_0}. \tag{A.61}$$

Rearranging Eq. (A.61), we get

$$C = \left[\frac{Q_R - Q_0}{\mu_0} - \beta_2 D \pi^{*\ \beta_2 - 1}_{0,W_U} \right] \frac{\pi^{*\ 1-\beta_1}_{0,W_U}}{\beta_1}. \tag{A.62}$$

Inserting this in Eq. (A.60) and solving for D, yields

$$D = \frac{\beta_1 - 1}{\beta_1 - \beta_2} \cdot \frac{Q_R - Q_0}{\mu_0} \pi^{*\ 1-\beta_2}_{0,W_U} - I_R \frac{\beta_1}{\beta_1 - \beta_2} \pi^{*\ -\beta_2}_{0,W_U}. \tag{A.63}$$

By using the expression for D given by Eq. (A.63) in Eq. (A.62), we get

$$C = \frac{\beta_2 - 1}{\beta_2 - \beta_1} \cdot \frac{Q_R - Q_0}{\mu_0} \pi^{*\ 1-\beta_1}_{0,W_U} - I_R \frac{\beta_2}{\beta_2 - \beta_1} \pi^{*\ -\beta_1}_{0,W_U}. \tag{A.64}$$

The expressions for C and D in both ends of the inaction region can be generalized by using the following expressions:

$$M_{i,j}(\pi_0) = \frac{\beta_i - 1}{\beta_i - \beta_j} \cdot \frac{Q_R - Q_0}{\mu_0} \pi_0^{1-\beta_j} - I_R \frac{\beta_i}{\beta_i - \beta_j} \pi_0^{-\beta_j}, \tag{A.65}$$

$$N_{i,j}(\pi_0) = Q_0 \frac{\beta_i - 1}{\beta_i - \beta_j} \cdot \frac{\mu_0 k - \mu_U}{\mu_0 \mu_U} \pi_0^{1-\beta_j} + B_2 \frac{\beta_i - \phi_2}{\beta_i - \beta_j} \pi_0^{\phi_2 - \beta_j} - I_U \frac{\beta_i}{\beta_i - \beta_j} \pi_0^{-\beta_j}. \tag{A.66}$$

By setting equal the two expressions for both constants, it is possible to rearrange the initial system to

For C:

$$N_{21}(\pi^*_{0,W_L}) = M_{21}(\pi^*_{0,W_U}). \tag{A.67}$$

For D:

$$N_{12}(\pi^*_{0,W_L}) = M_{12}(\pi^*_{0,W_U}). \tag{A.68}$$

These expressions can now be used to obtain the thresholds π^*_{0,W_L} and π^*_{0,W_U} by using a numerical solution procedure.

A.2.7.1 Suggested Solution Procedure for Proposition 3.4.3

We now present a solution procedure that helps to find suitable starting values for the thresholds π^*_{0,W_L} and π^*_{0,W_U} to further obtain the optimal values. The reason for doing this, is that there exist multiple invalid solutions which do not satisfy the assumptions we have made in the model, and this procedure helps us to find the correct solution. We can exploit the knowledge of how these functions look, and the fact that they are well-defined, in order to obtain suitable starting values. Also, we know that $\pi^*_{0,W_L} < \pi^*_{0,W_U}$. From this, we can narrow the search area such that optimal solutions to the thresholds are located in close proximity to the starting values. The way to do this is not trivial, so we will now describe the algorithm that we have used to solve the system of equations. Normally, our procedure is sufficient to find the correct solution, and also simplify the required computations.

The points where M_{12} and M_{21} are equal to zero are given by

$$M_{12}(\pi_0) = 0 \implies \pi_0 = \frac{\beta_1}{\beta_1 - 1} \cdot \frac{I_R \mu_0}{Q_R - Q_0}, \tag{A.69}$$

$$M_{21}(\pi_0) = 0 \implies \pi_0 = \frac{\beta_2}{\beta_2 - 1} \cdot \frac{I_R \mu_0}{Q_R - Q_0}. \tag{A.70}$$

Next, we find where the derivatives of these functions are equal to zero. That is where the derivatives change sign. Because of the shape of the functions, this point is the same for both. However, for M_{12} it is a minimum, whereas for M_{21} it is a maximum.

$$M'_{12}(\pi_0) = M'_{21}(\pi_0) = 0 \implies \pi_0 = \frac{\beta_1}{\beta_1 - 1} \cdot \frac{\beta_2}{\beta_2 - 1} \cdot \frac{I_R \mu_0}{Q_R - Q_0}. \tag{A.71}$$

The solution of the set of equations is found by iterating the initial value in the numerical solution procedure. In the first iteration, the values from Eqs. (A.69) and (A.70) are used to find values of π_0 when $M_{12}(\pi_0) = N_{12}(\pi_0)$ and $M_{21}(\pi_0) = N_{21}(\pi_0)$, respectively. The second iteration finds the threshold values and makes use of the results from the first iteration in determining suitable initial values. The initial value for π^*_{0,W_L} is given by the average values of when π_0 solves $N_{12}(\pi_0) = 0$, which is found by numerical methods, and when it solves $M_{12}(\pi_0) = N_{12}(\pi_0)$. The initial value for π^*_{0,W_U}, on the other hand, is given by the average values of when π_0 solves $M'_{12}(\pi_0) = M'_{21}(\pi_0) = 0$ and when it solves $M_{21}(\pi_0) = N_{21}(\pi_0)$. Using these initial values should yield the correct solution when solving the problem numerically.

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