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Silting modules, silting complexes and their correspondence with (co-) t-structures.

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Abstract

We present the theory of τ-tilting over finite dimensional algebras and show how silting modules over arbitrary rings is a generalization, in particular we prove that silting modules coincide with support τ-tilting modules over finite dimensional algebras. Quasitilting modules will play an important role in the concept of silting modules, as they classify torsion classes which provide left approximations with Ext-projective cokernel. Furthermore, we will show how equivalence classes of silting complexes in the derived categpry correspond bijectively certain t-structures and co-t-structures. These correspondences are then adjusted for equivalence classes of 2-term silting complexes which are in bijection with equivalence classes of silting modules.

Sammendrag

Vi presenterer teorien om τ-tilting over endelig dimensjonale algebraer og viser hvordan silte moduler over vilkårlige ringer er en generalisering, spesielt viser vi at silte moduler sammenfaller med støtte τ-tilte moduler over endelig dimensjonale algebraer. Kvasitilte moduler vil ha en viktig rolle i konseptet for silte moduler, siden de klassifiserer torsjonsklasser som gir venstre tilnærminger med Ext-projektiv kokjerne. Videre vil vi vise hvordan ekvivalensklasser av silte komplekser i den deriverte kategorien er i bijeksjon med visse t-strukturer og ko-t-strukturer. Disse forbindelsene vil bli justert for ekvivalensklasser av 2-ledds silte komplekser som er i bijeksjon med ekvivalensklasser av silte moduler.

Contents

Introduction

The definition of tilting modules over finite-dimensional algebras (artin algebras) is due to Brenner and Butler, [BB80] . The foundation of tilting theory lies there, as does many of the classical results concerning tilting modules. Brenner and Butler are also due credit for the naming of tilting modules. The definition of tilting modules was later relaxed a bit by Happel and Ringel in [HR82]. That is also where much of the modern approach to tilting originates from, particularly, as it relates to this thesis, the connection between tilting modules and certain pairs of categories called torsion pairs. That is, let T be a tilting Λ-module, $\mathcal{T}(T)$ the category of all Λ-modules generated by T and $\mathcal{F}(T)$ the category of all Λ-modules X such that $\mathrm{Hom}_\Lambda(T, X) = 0$, then $(\mathcal{T}(T), \mathcal{F}(T))$ is a torsion pair. An important result in tilting theory due to Bongartz is that for any partial tilting module M there exists a *complement* X such that $M \oplus X$ is a tilting module, commonly known as Bongartz' completion [Bon81]. Over a finite dimensional k-algebra Λ over a field k, a partial tilting module M is called an *almost complete tilting module* provided that the number of non-isomorphic indecomposable direct summands of M is one less than the number of isomorphism classes of simple modules in Λ. A well known fact is that over such an algebra, an almost complete tilting module has precisely either one or two non-isomorphic complements. Tilting theory has also been studied extensively in the "large" module categories $\mathrm{Mod}(A)$ over a ring A.

The notion of τ-tilting was introduced by Adachi, Iyama and Reiten in [AIR14], and in particular the τ-tilting and support τ-tilting modules over Λ. They rely on the existence of the Auslander-Reiten translation τ, and thus τ-tilting is only applicable when Λ is a finite-dimensional algebra over a field k. In the sense of tilting completion, τ-tilting has stronger results than classical tilting. In particular, an almost complete support τ-tilting module has exactly two complements. They also show that support τ-tilting modules are in bijection with certain 2-term complexes in $K^b(\mathrm{proj}(\Lambda))$ called *silting* or *semi-tilting.*

This brings us to the concept of *silting modules* introduced by Hügel, Marks and Vitória in [HMV15], which is the main focus of this thesis. As a generalization of support τ-tilting modules over Λ, silting modules are defined over $\mathrm{Mod}(A)$ for arbitrary unitary rings A, where the definition is heavily motivated by certain key properties of both tilting modules and support τ-tilting modules. In particular, silting modules coincide with support τ-tilting modules over $\mathrm{mod}(\Lambda)$. An important feature of silting modules is that they are always *finendo quasitilting modules*, which correspond to torsion classes providing left approximations with Ext-projective cokernel. There is also an analog of Bongartz completion for silting modules, which relies on the existence of left approximations. Furthermore, silting modules are in bijection with 2-term silting complexes in $K^b(\mathrm{Proj}(A))$, similarly to support τ-tilting modules.

The structure of this thesis will be roughly as follows. In section 2.1 we give a few introductory results on tilting as it is the foundation of the entire theory mentioned above.

Sections 2.2 and 2.3 are dedicated to τ-tilting theory, of which we will give a condensed introduction to provide some technical backstory and understanding of the generalization to silting modules. Theorem 2.37 is of particular interest to us as it provides generalized descriptions of key properties in τ-tilting, which both motivate the definition of silting modules and provide the means to show that silting modules in fact generalize support τ-tilting modules. We will also give a fairly comprehensive description of how support τ-tilting modules relate to 2-term silting complexes in $K^b(\mathrm{proj}(\Lambda))$.

Then we present the theory of silting modules in section 3, starting with the quasitilting modules. They correspond to torsion classes providing left-approximations, which is an important property as both tilting and support τ-tilting modules provide approximations sequences. Section 3.2 is dedicated to the theory of silting modules. It turns out that tilting modules are always silting modules, and that silting modules are always finendo quasitilting. We also prove an analog of Bongartz completion for silting modules, and conclude the section by proving that silting modules coincide with support τ-tilting modules over Λ.

In section 4 we give an introduction to the morphism category $\mathrm{Mor}(A)$ and show in particular that (partial) silting A-modules correspond bijectively to (partial) tilting objects in $\mathrm{Mor}(A)$.

Section 5.1 is dedicated to silting complexes and their relationship with (co)-t-structures in the derived category of A. The main theorem of that section is theorem 5.21, proving a bijection between silting complexes and certain t-structures and (co-)t-structures. In section 5.2 we give the theory linking silting modules and 2-term silting complexes. Theorem 5.21 is then adjusted in theorem 5.28 to the case of 2-term silting complexes which then also provides bijections with silting modules.

1 Preliminaries and notation

Throughout this thesis, A will be any unitary ring. We will denote by $\mathrm{Mod}(A)$ (respectively $\mathrm{mod}(A)$) the category of (finitely generated) left R-modules, and $\mathrm{Proj}(A)$ (respectively $\mathrm{proj}(A)$) the subcategories of (finitely generated) projective modules. The unbounded derived (respectively homotopy) category of $\mathrm{Mod}(A)$ is denoted by $D(A)$ (respectively $K(A)$). The restriction of these categories to right bounded or bounded complexes will be denoted by the superscripts $^-$ and ^{b} respectively. The bounded homotopy category of complexes of (respectively finitely generated) projective A-modules will be denoted by (respectively $K^b(\mathrm{proj}(A))$) $K^b(\mathrm{Proj}(A))$. For an object X in some category, we will denote the identity morphism on X by 1_X, and sometimes simply by 1 when no confusion can arise. We will always use the term *subcategory* to mean a strictly full subcategory.

For an A-module T, we define the following subcategories of $\mathrm{Mod}(A)$

$\mathrm{Sub}(T)$: the category consisting of all submodules of arbitrary direct sums of copies of T.

$\mathrm{Add}(T)$: the category consisting of all A-modules isomorphic to a direct summand of an arbitrary direct sum of copies of T.

$\mathrm{Gen}(T)$: the category consisting of all A-modules X such that there exists a set I, an A-module $S \in \mathrm{Add}(T)$ and a short exact sequence $0 \to S \to T^{(I)} \to X \to 0$. We say that X is T*-generated* or *generated by* T. The category $\mathrm{Gen}(T)$ is also often denoted by $\mathrm{Fac}(T)$ as all objects in $\mathrm{Gen}(T)$ are factors of $T^{(I)}$ for some set I.

$\mathrm{Pres}(T)$: the category consisting of all A-modules X such that there exists a right exact sequence $T_1 \to T_0 \to X \to 0$ where $T_1, T_0 \in \mathrm{Add}(T)$. Such a presentation is sometimes called an $\mathrm{Add}(T)$-presentation, and we say that X is T*-presented* or *presented by* T.

Note that $\mathrm{Pres}(T) \subseteq \mathrm{Gen}(T)$, indeed if $X \in \mathrm{Pres}(T)$ and $T_0 \to X$ appears as the epimorphism in the $\mathrm{Add}(T)$-presentation of X, then it induces a short exact sequence $0 \to T_0 \to T^{(I)} \to X \to 0$ for some set I. Also $\mathrm{Add}(T) \subseteq \mathrm{Gen}(T)$, for if $X \in \mathrm{Add}(T)$ such that $X \oplus Y \cong T^{(I)}$, then there is a short exact sequence $0 \to Y \to T^{(I)} \to X \to 0$.

Some familiarity with triangulated categories will be assumed. For a triangulated category D, we denote the suspension functor by $[1] : D \to D$. We will in particular use that $K^b(\mathrm{Proj}(A))$ and $D(A)$ are triangulated without mention. Sometimes we refer to the axioms for triangulated categories, they will be denoted by $(TR1), (TR2), (TR3), (TR4)$.

For a subcategory $\mathcal{C}$ of $D(A)$, we define the following subcategories

$\mathcal{C}^{\perp_{>0}}$: consists of all objects D in $D(A)$ such that $\mathrm{Hom}_{D(A)}(\mathcal{C}, D[i]) = 0$ for all $i > 0$.

$\mathcal{C}^{\perp_{<0}}$: consists of all objects D in $D(A)$ such that $\mathrm{Hom}_{D(A)}(\mathcal{C}, D[i]) = 0$ for all $i < 0$.

$\mathcal{C}^{\perp_0}$: consists of all objects D in $D(A)$ such that $\mathrm{Hom}_{D(A)}(\mathcal{C}, D) = 0$.

Should the subcategory have only one object C, we write the subcategories above as $C^{\perp_{>0}}$, $C^{\perp_{<0}}$ and $C^{\perp_0}$.

Let $\mathcal{B}$ be a subcategory of some category $\mathcal{C}$. A homomorphism $f : B \to C$ for some B in $\mathcal{B}$ and C in $\mathcal{C}$ is a left $\mathcal{C}$-approximation of B if for any C' in $\mathcal{C}$, the map $\mathrm{Hom}_{\mathcal{C}}(f, C')$ is surjective. Similarly, a homomorphism $f : C \to B$ is a right $\mathcal{C}$-approximation of B if for any C' in $\mathcal{C}$, the map $\mathrm{Hom}_{\mathcal{C}}(C', f)$ is surjective.

Let $\mathcal{A} \subseteq \mathcal{B}$ be two subcategories of some category $\mathcal{C}$. We say that $\mathcal{A}$ is

covariantly finite in $\mathcal{B}$ if every object B in $\mathcal{B}$ admits a left $\mathcal{A}$-approximation.

contravariantly finite in $\mathcal{B}$ if every object B in $\mathcal{B}$ admits a right $\mathcal{A}$-approximation.

functorially finite in $\mathcal{B}$ if it is both covariantly finite and contravariantly finite in $\mathcal{B}$.

We will at the end of every proof write the symbol $\square$ to mark the end. Similarly we use the symbol at the end of a result which we do not prove and to mark the end of examples.

2 Tilting theory

This section will provide the foundation for large parts of this thesis. We begin by defining tilting modules and torsion classes.

2.1 Tilting modules

This section on tilting in $\mathrm{Mod}(A)$ follows section 2.1 in [HMV15], but we provide a few additional results in order to create a more self-contained text. Let us first give the definition for (not necessarily finitely generated) tilting modules over A.

Definition 2.1. An A-module T is said to be **tilting** if it satisfies the following

(T1) $\mathrm{p.\,d.}(T) \leq 1$ (projective dimension less than or equal to 1).

(T2) $\mathrm{Ext}^1_A(T, T^{(I)}) = 0$ for any set I.

(T3) There is an exact sequence

$$0 \longrightarrow A \xrightarrow{\phi} T_0 \longrightarrow T_1 \longrightarrow 0$$

where $T_0, T_1 \in \mathrm{Add}(T)$ and ϕ is a left $\mathrm{Gen}(T)$-approximation.

The following proposition provides an alternative definition for tilting modules, which is easier to work with. For that reason, we will for the most part use the alternative definition throughout this thesis. See [CT95, Proposition 1.3(i)] for proof.

Proposition 2.2. *An A-module T is tilting if and only if $\mathrm{Gen}(T) = T^{\perp_1}$. The class $\mathrm{Gen}(T)$ is called a* ***tilting class****.* □

Recall that for an A-module T, the subring $\mathrm{Ann}(T) \subseteq A$ consists of all $a \in A$ such that $at = 0$ for all $t \in T$. An A-module T is called **faithful** if $\mathrm{Ann}(T) = 0$, or equivalently if multiplication by elements of the ring A induce unique endomorphisms of T. An easy observation is that tilting modules are always faithful.

Lemma 2.3. *Let T be a tilting A-module, then it is faithful.*

Proof. Since T is tilting, there is an exact sequence

$$0 \longrightarrow A \xrightarrow{\phi} T_0 \longrightarrow T_1 \longrightarrow 0$$

where $T_0, T_1 \in \mathrm{Add}(T)$ and ϕ is a left $\mathrm{Gen}(T)$-approximation. Let $a \in \mathrm{Ann}(T)$, then $a \in \mathrm{Ann}(T_0)$ since $T_0 \in \mathrm{Add}(T)$. But then $\phi(a) = a\phi(1) = 0$, so $a \in \mathrm{Ker}(\phi) = 0$, so T is faithful. □

Torsion classes play a critical role in tilting theory, and will be important also for the theory of silting modules. We adopt the definition from [BR07].

Definition 2.4. Let $\mathcal{A}$ be an abelian category, and $\mathcal{T}, \mathcal{F}$ two subcategories of $\mathcal{A}$. We say that $(\mathcal{T}, \mathcal{F})$ is a **torsion pair** if it satisfies the following:

(1) $\operatorname{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0$.

(2) For any $M \in \mathcal{A}$, there is a short exact sequence

$$0 \longrightarrow T \longrightarrow M \longrightarrow F \longrightarrow 0$$

where $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

If $(\mathcal{T}, \mathcal{F})$ is a torsion pair, we call $\mathcal{T}$ the **torsion class** and $\mathcal{F}$ the **torsion-free class**. We will also say "$\mathcal{T}$ is a torsion class" if it appears as the first argument in some torsion pair, and dually for a torsion-free class. It follows directly that $\mathcal{F} = \mathcal{T}^{\perp_0}$ and $\mathcal{T} = {}^{\perp_0}\mathcal{F}$.

We say that an object $M \in \mathcal{T}$ is **Ext-projective** in $\mathcal{T}$ if $M \in \mathcal{T}^{\perp_1}$.

Lemma 2.5. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in an abelian category $\mathcal{A}$. Then the following hold.*

(1) $\mathcal{T}$ is closed under factors, extensions and coproducts.

(2) $\mathcal{F}$ is closed under subobjects, extensions and products.

Proof. (1) : Let $T \in \mathcal{T}$, and consider a short exact sequence

$$0 \longrightarrow M \longrightarrow T \longrightarrow M' \longrightarrow 0$$

Applying $\operatorname{Hom}_{\mathcal{A}}(-, \mathcal{F})$ to the sequence, one sees that $M' \in \mathcal{T}$, i.e. it is closed under factors. Now, assume instead that $M, M' \in \mathcal{T}$. The same argument as above shows that then $T \in \mathcal{T}$, i.e. it is closed under extensions.

Finally, let $\{T_i\}_{i \in I}$ be a family of objects in $\mathcal{T}$ and consider the coproduct $\bigoplus_{i \in I} T_i$. The functor $\operatorname{Hom}_{\mathcal{A}}(-, \mathcal{F})$ takes coproducts to products, so we have

$$\operatorname{Hom}_{\mathcal{A}}(\bigoplus_{i \in I} T_i, \mathcal{F}) \cong \prod_{i \in I} \operatorname{Hom}_{\mathcal{A}}(T_i, \mathcal{F}) = 0$$

i.e. it is closed under coproducts.

(2) : The proof is dual to that of (1). □

Remark 2.6. For a torsion class $\mathcal{T}$, one can always form the torsion pair $(\mathcal{T}, \mathcal{T}^{\perp_0})$, and dually for a torsion-free class. By [AK96, Section 1.2], a subcategory $\mathcal{T}$ of $\mathcal{A}$ which satisfies property (1) in lemma 2.5 is a torsion class . Dually for torsion-free classes. Moreover, an equivalent definition for torsion pairs is that $\operatorname{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0$ and that they are maximal with respect to that property.

The notion of left (or right) approximations will turn up frequently throughout this thesis. Torsion classes $\mathcal{T}$ which provide left $\mathcal{T}$-approximations will be of particular interest later on. We now give an easy proof that they provide right $\mathcal{T}$-approximations by default (and dually for torsion-free classes $\mathcal{F}$).

Lemma 2.7. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in an abelian category $\mathcal{A}$. Then $\mathcal{T}$ (respectively $\mathcal{F}$) is contravariantly finite (respectively covariantly finite) in $\mathcal{A}$.*

Proof. Let $M \in \mathcal{A}$, then it fits in a short exact sequence

$$0 \longrightarrow T \xrightarrow{f} M \xrightarrow{g} F \longrightarrow 0$$

where $T \in \mathcal{T}$ and $F \in \mathcal{F}$. Let $T' \in \mathcal{T}$ and consider $h \in \mathrm{Hom}_{\mathcal{A}}(T', M)$. Clearly, $gh = 0$, so h factors through T via f, thus f is a right $\mathcal{T}$-approximation of M.

Similarly, one shows that g is a left $\mathcal{F}$-approximation of M. □

The following definition of partial tilting modules is from [CT95].

Definition 2.8. An A-module T is said to be **partial tilting** if it satisfies:

(PT1) $T^{\perp_1}$ is a torsion class.

(PT2) $T \in T^{\perp_1}$.

Remark 2.9. Note that if (PT1) holds for an A-module T, then condition (PT2) is equivalent to $\mathrm{Gen}(T) \subseteq T^{\perp_1}$ since $\mathrm{Gen}(T)$ consist of factors of direct sums of copies of T and $T^{\perp_1}$ is closed under factors and coproducts by lemma 2.5.

The following lemma shows that partial tilting modules satisfy some of the axioms of tilting modules, which is not directly clear from the definition. The proof is based on [Trl92, Lemma 1.2]

Lemma 2.10. *Let T be a partial tilting A-module. Then it satisfies axioms $(T1)$ and $(T2)$ in definition 2.1.*

Proof. $(T1)$: Let $I = \mathrm{Hom}_A(A, T)$ and consider the map $u : A^{(I)} \to T$, which is surjective because for all $t \in T$ there exists a unique map $f : A \to T$ with $f(1) = t$. We will show that $K := \mathrm{Ker}(u)$ is projective. The sequence

$$0 \longrightarrow K \longrightarrow A^{(I)} \xrightarrow{u} T \longrightarrow 0$$

is exact. Let $N \in \mathrm{Mod}(A)$ and apply the functor $\mathrm{Hom}_A(-, N)$ to the sequence above to get the long exact sequence

$$\begin{array}{l} 0 \longrightarrow \mathrm{Hom}_A(T,N) \longrightarrow \mathrm{Hom}_A(A^{(I)},N) \longrightarrow \mathrm{Hom}_A(K,N) \\ \hookrightarrow \mathrm{Ext}^1_A(T,N) \longrightarrow \mathrm{Ext}^1_A(A^{(I)},N) \longrightarrow \mathrm{Ext}^1_A(K,N) \\ \hookrightarrow \mathrm{Ext}^2_A(T,N) \longrightarrow \mathrm{Ext}^2_A(A^{(I)},N) \longrightarrow \dots \end{array}$$

Since $A^{(I)}$ is projective we have $\mathrm{Ext}^1_A(A^{(I)}, N) = 0 = \mathrm{Ext}^2_A(A^{(I)}, N)$. Thus it follows from the long exact sequence that $\mathrm{Ext}^1_A(K, N) \cong \mathrm{Ext}^2_A(T, N)$.

Next, let $I(N)$ be the injective envelope of N, then the sequence

$$0 \longrightarrow N \longrightarrow I(N) \longrightarrow I(N)/N \longrightarrow 0$$

is exact. Apply the functor $\mathrm{Hom}_A(T, -)$ to the sequence above to get the long exact sequence

$$\begin{array}{l} 0 \longrightarrow \mathrm{Hom}_A(T, N) \longrightarrow \mathrm{Hom}_A(T, I(N)) \longrightarrow \mathrm{Hom}_A(T, I(N)/N) \\ \longrightarrow \mathrm{Ext}^1_A(T, N) \longrightarrow \mathrm{Ext}^1_A(T, I(N)) \longrightarrow \mathrm{Ext}^1_A(T, I(N)/N) \\ \longrightarrow \mathrm{Ext}^2_A(T, N) \longrightarrow \mathrm{Ext}^2_A(T, I(N)) \longrightarrow \dots \end{array}$$

Since $I(N)$ is injective we have $\mathrm{Ext}^1_A(T, I(N)) = 0 = \mathrm{Ext}^2_A(T, I(N))$. Thus it follows from the long exact sequence that $\mathrm{Ext}^1_A(T, I(N)/N) \cong \mathrm{Ext}^2_A(T, N)$.

Now we have

$$\mathrm{Ext}^1_A(K, N) \cong \mathrm{Ext}^2_A(T, N) \cong \mathrm{Ext}^1_A(T, I(N)/N).$$

Since $I(N)$ is injective, we have $I(N) \in T^{\perp_1}$, and since $T^{\perp_1}$ is closed under factors we also have $I(N)/N \in T^{\perp_1}$. Thus $\mathrm{Ext}^1_A(K, N) = 0$ for all A-modules N, so K is projective. Then the following is a projective presentation of T of length 1, finishing the proof

$$0 \longrightarrow K \longrightarrow A^{(I)} \xrightarrow{u} T \longrightarrow 0$$

$(T2)$: Since $T \in T^{\perp_1}$ and $T^{\perp_1}$ is closed under coproducts, we have $T^{(I)} \in T^{\perp_1}$ for any set I. □

We give a short proof showing that taking the direct sum of a partial tilting module T with certain modules T' preserve the partial tilting property.

Lemma 2.11. *Let T be an A-module. If T is partial tilting and T' a projective-injective A-module, then $\tilde{T} = T \oplus T'$ is partial tilting.*

Proof. Let X be an A-module, then $\mathrm{Ext}^1_A(T', X) = 0$ since T' is projective. Therefore we have

$$\mathrm{Ext}^1_A(\tilde{T}, X) \cong \mathrm{Ext}^1_A(T, X) \oplus \mathrm{Ext}^1_A(T', X) = \mathrm{Ext}^1_A(T, X)$$

so $\tilde{T}^{\perp_1} = T^{\perp_1}$. Then we have that $\tilde{T}^{\perp_1}$ is a torsion class since $T^{\perp_1}$ is a torsion class.

Finally, since T' is projective-injective and T is partial tilting, we have

$$\mathrm{Ext}^1_A(\tilde{T}, \tilde{T}) \cong \mathrm{Ext}^1_A(T, T) \oplus \mathrm{Ext}^1_A(T, T') \oplus \mathrm{Ext}^1_A(T', T) \oplus \mathrm{Ext}^1_A(T', T') = 0$$

so $\tilde{T} \in \tilde{T}^{\perp_1}$. □

By remark 2.9 we have $\operatorname{Gen}(T) \subseteq T^{\perp_1}$ for a partial tilting A-modules T. The class $T^{\perp_1}$ is then by definition a torsion class, and the following lemma shows that then so is $\operatorname{Gen}(T)$.

Lemma 2.12. *If an A-modules T satisfies $\operatorname{Gen}(T) \subseteq T^{\perp_1}$, then $(\operatorname{Gen}(T), T^{\perp_0})$ is a torsion pair.*

Proof. Clearly we have $\operatorname{Hom}_A(\operatorname{Gen}(T), T^{\perp_0}) = 0$. By remark 2.6 it is then sufficient to show that

$$\operatorname{Gen}(T) = {}^{\perp_0}(T^{\perp_0}) \tag{2.1}$$

$$(\operatorname{Gen}(T))^{\perp_0} = T^{\perp_0} \tag{2.2}$$

First we show (2.2). Let $N \in (\operatorname{Gen}(T))^{\perp_0}$, and since $T \in \operatorname{Gen}(T)$ we have $N \in T^{\perp_0}$. Conversely, if $N \in T^{\perp_0}$, then clearly $N \in (\operatorname{Gen}(T))^{\perp_0}$ as $\operatorname{Gen}(T)$ consists of A-modules of the form $T^{(I)}/K$ for some $K \in \operatorname{Add}(T)$.

We are then left to show (2.1). First we show $\operatorname{Gen}(T) \subseteq {}^{\perp_0}(T^{\perp_0})$. Let $M \in \operatorname{Gen}(T)$, then there is a surjection $\phi : T^{(I)} \to M$ for some set I and an induced short exact sequence

$$0 \longrightarrow \operatorname{Ker}(\phi) \longrightarrow T^{(I)} \xrightarrow{\phi} M \longrightarrow 0$$

Let $N \in T^{\perp_0}$ and apply $\operatorname{Hom}_A(-, N)$ to the sequence above, yielding

$$0 \longrightarrow \operatorname{Hom}_A(M, N) \longrightarrow \operatorname{Hom}_A(T^{(I)}, N) \longrightarrow \operatorname{Hom}_A(\operatorname{Ker}(\phi), N) \longrightarrow \dots$$

Since $\operatorname{Hom}_A(T^{(I)}, N) \cong \prod_I \operatorname{Hom}_A(T, N) = 0$, we have that $\operatorname{Hom}_A(M, N) = 0$, and thus $M \in {}^{\perp_0}(T^{\perp_0})$.

We now show the inclusion ${}^{\perp_0}(T^{\perp_0}) \subseteq \operatorname{Gen}(T)$. Let $M \in {}^{\perp_0}(T^{\perp_0})$, and consider the following exact sequence where $\operatorname{Tr}_T(M) = \sum_{f:T\to M} \operatorname{Im}(f)$ denotes the **trace of T in M**

$$0 \longrightarrow \operatorname{Tr}_T(M) \longrightarrow M \xrightarrow{\pi} M/\operatorname{Tr}_T(M) \longrightarrow 0$$

Apply $\operatorname{Hom}_A(T, -)$ to the sequence to get a long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}_A(T, \operatorname{Tr}_T(M)) \arrow[r] & \operatorname{Hom}_A(T, M) \arrow[r, "\pi_*"] & \operatorname{Hom}_A(T, M/\operatorname{Tr}_T(M)) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- (Z) [near end]\tikztonodes -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\ & \operatorname{Ext}^1_A(T, \operatorname{Tr}_T(M)) \arrow[r] & 0 & \end{tikzcd}$$

Clearly, $\operatorname{Tr}_T(M) \in \operatorname{Gen}(T)$, so $\operatorname{Ext}^1_A(T, \operatorname{Tr}_T(M)) = 0$. Furthermore, because π_* is then surjective, any map $f : T \to M/\operatorname{Tr}_T(M)$ factors through $\pi : M \to M/\operatorname{Tr}_T(M)$ via some map $f' : T \to M$. That is, there exists an f' such that the following diagram commutes.

$$\begin{tikzcd} & T \arrow[dl, dashed, "f'"'] \arrow[d, "f"] \\ M \arrow[r, "\pi"] & M/\operatorname{Tr}_T(M) \end{tikzcd}$$

But $\mathrm{Im}(f') \subseteq \mathrm{Tr}_T(M)$, so $\pi f' = 0 = f$ for all such maps f. Therefore we have

$$\mathrm{Hom}_A(T, M/\mathrm{Tr}_T(M)) = 0$$

that is $M/\mathrm{Tr}_T(M) \in T^{\perp_0}$. Since $M \in {}^{\perp_0}(T^{\perp_0})$, we have

$$\mathrm{Hom}_A(M, M/\mathrm{Tr}_T(M)) = 0.$$

In particular, $\pi = 0$ and so $\mathrm{Tr}_T(M) \cong M$ which means $M \in \mathrm{Gen}(T)$. □

We will, without reference, frequently use the fact that $\mathrm{Gen}(T)$ is a torsion class in the later proofs given that $\mathrm{Gen}(T) \subseteq T^{\perp_1}$.

An important result due to Bongartz states that a partial tilting module $P \in \mathrm{mod}(\Lambda)$ for some finite dimensional k-algebra Λ, is in fact a direct summand of a tilting module T, which is known as the Bongartz completion of P. It also holds true in $\mathrm{Mod}(A)$, which was proven in [CT95]. Later in this thesis we will present some lemmas with similar statements, but for other sorts of modules, specifically completing τ-rigid Λ-modules to τ-tilting Λ-modules and completing partial silting A-modules to silting A-modules. For context, we present the theorem for $\mathrm{Mod}(A)$, see [CT95, Theorem 1.9] for the proof.

Lemma 2.13. *Let $P \in \mathrm{Mod}(A)$. Then P is a partial tilting A-module if and only if P is a summand of a tilting A-module T such that $P^{\perp_1} = T^{\perp_1}$.*

2.2 τ-tilting modules

In this section we give a condensed introduction to τ-tilting which was introduced in [AIR14]. Most of the results presented here are from [AIR14], and we will explicitly say so when we include results from other sources. As τ-tilting is not the main focus of this thesis, not all presented results will be proven. Rather, we give the proofs when they contain important constructions. Throughout this section, Λ will be a finite dimensional algebra over an algebraically closed field k.

The main purpose of this section is to provide the motivation for the concept of silting modules. The properties of silting modules will in many cases be analogous to certain properties of τ-rigid and support τ-tilting modules, therefore it is beneficial to be somewhat familiar with τ-tilting theory before consider the silting modules. In particular, theorem 2.37 shows that the conditions for $T \in \mathrm{mod}(\Lambda)$ to be τ-rigid or support τ-tilting will have equivalent conditions in $\mathrm{Mod}(A)$, thus making a rather explicit motivation for the definition of silting modules.

Let $M \in \mathrm{mod}(\Lambda)$, we denote by $\mathrm{add}(M)$ (respectively $\mathrm{gen}(M)$ and $\mathrm{sub}(M)$) the subcategory of $\mathrm{mod}(\Lambda)$ consisting of all direct summands (respectively factors and submodules) of finite direct sums of copies of M. For a subcategory $\mathcal{T}$ of $\mathrm{mod}(\Lambda)$, we denote by $P(\mathcal{T})$ the direct sum of one copy of each of the indecomposable objects in $\mathcal{T}$ which are *proj* Ext-projective in $\mathcal{T}$.

First, we give the construction of the AR-translation τ. From the dualities

$$D := \operatorname{Hom}_k(-, k) : \operatorname{mod}(\Lambda) \to \operatorname{mod}(\Lambda)^{\mathrm{op}}$$
$$(-)^* := \operatorname{Hom}_\Lambda(-, \Lambda) : \operatorname{proj}(\Lambda) \to \operatorname{proj}(\Lambda)^{\mathrm{op}}$$

we construct the **Nakayama functor**

$$\nu := D(-)^* : \operatorname{proj}(\Lambda) \to \operatorname{inj}(\Lambda)$$

The Nakayama functor plays an important role in representation theory. In particular, it is necessary for the theory of τ-tilting as we shall see later. First we prove a lemma showing the existence of an important isomorphism (invertible natural transformation in fact) involving ν.

Lemma 2.14. *Let $P \in \operatorname{proj}(\Lambda)$, then*

$$\operatorname{Hom}_\Lambda(X, \nu P) \cong D \operatorname{Hom}_\Lambda(P, X)$$

for all $X \in \operatorname{Mod}(\Lambda)$.

Proof. From tensor-hom adjunction, we have that

$$\operatorname{Hom}_\Lambda(X, \nu P) = \operatorname{Hom}_\Lambda(X, D \operatorname{Hom}_\Lambda(P, \Lambda)) \cong D(\operatorname{Hom}_\Lambda(P, \Lambda) \otimes_\Lambda X) \tag{2.3}$$

To complete the proof we prove that the following morphism is an isomorphism.

$$\Phi : \operatorname{Hom}_\Lambda(P, \Lambda) \otimes_\Lambda X \to \operatorname{Hom}_\Lambda(P, X)$$
$$f \otimes x \mapsto f(-)x$$

Suppose that $\Phi(f \otimes x)(p) = f(p)x = 0$ for all $p \in P$, then $f = 0$ or $x = 0$ in which case $f \otimes x = 0$. The map Φ is therefore injective. To prove that it is surjective, we need that P is a finitely generated projective module. Let $P = \langle p_1, p_2, \ldots, p_n \rangle$ and $f : \Lambda^n \to P$ a surjective map such that $f(\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{i=1}^n \lambda_i p_i$. Since P is projective and f surjective, there also exists a map $f' : P \to \Lambda^n$ such that $1_P = ff'$. We have $p = \sum_{i=1}^n \lambda_i p_i$ for all $p \in P$, and $f'(p) = \sum_{i=1}^n \lambda_i$. We also have component maps $f'_i : P \to \Lambda$ such that $f'_i(p) = \lambda_i$. Thus, for any $p \in P$ we have $p = \sum_{i=1}^n f'_i(p) p_i$.

Let $g \in \operatorname{Hom}_\Lambda(P, X)$, and consider the element $\sum_{i=1}^n (f'_i \otimes g(p_i)) \in \operatorname{Hom}_\Lambda(P, \Lambda) \otimes_\Lambda X$. Then we have

$$\Phi\big(\sum_{i=1}^n (f'_i \otimes g(p_i))\big)(p) = \sum_{i=1}^n f'_i(p) g(p_i) = \sum_{i=1}^n \lambda_i g(p_i) = \sum_{i=1}^n g(\lambda_i p_i) = g(p)$$

so the map Φ is surjective and thus an isomorphism. Picking up from equation 2.3, we have

$$\operatorname{Hom}_\Lambda(X, \nu P) \cong D(\operatorname{Hom}_\Lambda(P, \Lambda) \otimes_\Lambda X) \cong D \operatorname{Hom}_\Lambda(P, X)$$

and we are done. $\square$

Remark 2.15. The isomorphism above is usually proved for $X \in \operatorname{mod}(\Lambda)$, but it only depends on P being finitely generated and projective. Additionally, we need the isomorphism for large modules $X \in \operatorname{Mod}(\Lambda)$ later in this section, so we stated the lemma in that way.

Let $M \in \operatorname{mod}(\Lambda)$ and $P_{-1} \xrightarrow{p_{-1}} P_0 \xrightarrow{p_0} M \to 0$ be a minimal projective presentation of M. Applying the duality $(-)^*$ to p_{-1} yields a map $p_{-1}^* : P_0^* \to P_{-1}^*$. The module $\operatorname{Cok}(f^*)$ is called the **transpose** of M and denoted by $\operatorname{Tr} M$.

The following proposition gives the existence of a particularly important functor, which relies on the functor ν as mentioned earlier.

Proposition 2.16. *Let $P_{-1} \xrightarrow{p_{-1}} P_0 \xrightarrow{p_0} M \to 0$ be a minimal projective presentation of a non-projective Λ-module M. Then there is an exact sequence.*

$$0 \longrightarrow D \operatorname{Tr} M \longrightarrow \nu(P_{-1}) \xrightarrow{\nu(p_{-1})} \nu(P_0) \xrightarrow{\nu(p_0)} \nu(M) \longrightarrow 0$$

Proof. Apply $(-)^*$ to the sequence $P_{-1} \xrightarrow{p_{-1}} P_0 \xrightarrow{p_0} M \to 0$ to get an exact sequence

$$0 \longrightarrow M^* \xrightarrow{p_0^*} P_0^* \xrightarrow{p_{-1}^*} P_{-1} \longrightarrow \operatorname{Tr} M \longrightarrow 0$$

Then, applying D to the sequence yields the desired exact sequence. $\square$

We following functors are called the **Auslander-Reiten translations**, shortened to AR-translations.

$$\tau := D \operatorname{Tr}(-) : \underline{\operatorname{mod}}(\Lambda) \to \overline{\operatorname{mod}}(\Lambda)$$
$$\tau^- := \operatorname{Tr}(-)D : \overline{\operatorname{mod}}(\Lambda) \to \underline{\operatorname{mod}}(\Lambda)$$

where $\underline{\operatorname{mod}}(\Lambda)$ denotes the stable category modulo projectives and $\overline{\operatorname{mod}}(\Lambda)$ the costable category modulo injectives. They have the same objects as $\operatorname{mod}(\Lambda)$, but the vector spaces $\underline{\operatorname{Hom}}_\Lambda(X, Y)$ (respectively $\overline{\operatorname{Hom}}_\Lambda(X, Y)$) are quotients of $\operatorname{Hom}_\Lambda(X, Y)$ by all morphisms $X \to Y$ which factor through a projective (respectively injective) Λ-module. There also exists isomorphisms known as the **Auslander-Reiten formulae**, see [ASS06, Theorem 2.13]

$$\operatorname{Ext}^1_\Lambda(X, Y)) \cong D\underline{\operatorname{Hom}}_\Lambda(\tau^- Y, X) \cong D\overline{\operatorname{Hom}}_\Lambda(Y, \tau X) \tag{2.4}$$

Let's look at an easy example of the use of τ in the representation theory of quivers.

Example 2.17. Let k be an algebraically closed field, Γ be the quiver

$$1 \xrightarrow{\alpha} 2$$

and let $\Lambda = k\Gamma$ be the path algebra over Γ. There are three indecomposable modules over Λ which corresponds to the following representations

$$P_2 = 0 \longrightarrow k \qquad P_1 = k \longrightarrow k \qquad I_1 = k \longrightarrow 0$$

and $P_1 = I_2$. Now, I_1 admits the minimal projective presentation

$$0 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow I_1 \longrightarrow 0$$

and so

$$\tau I_1 = \mathrm{Ker}(\nu P_2 \to \nu P_1) = \mathrm{Ker}(I_1 \to I_2) = P_2$$

by proposition 2.16.

The following lemma will be useful later.

Lemma 2.18. *Let $M \in \mathrm{mod}(\Lambda)$. Then $\mathrm{Tr}\, DM = \tau^- M$ has no non-trivial projective summands and $D\,\mathrm{Tr}\, M = \tau M$ has no non-trivial injective summands.*

Proof. We prove that $\mathrm{Tr}\, DM$ has no non-trivial projective summands. The proof for $D\,\mathrm{Tr}\, M$ is dual.

Let $M \in \mathrm{mod}(\Lambda)$ and $P_{-1} \xrightarrow{p_{-1}} P_0 \to M \to 0$ a minimal projective presentation of M. Then $\mathrm{Tr}\, M = \mathrm{Cok}(p_{-1}^*)$ where $p_{-1}^* : P_0^* \to P_{-1}^*$. If $\mathrm{Tr}\, M$ has a non-trivial projective summand Q^*, then $0 \to Q^*$ is a summand of $P_0^* \to P_{-1}^*$. But since $(-)^* = \mathrm{Hom}_\Lambda(-, \Lambda)$ is an equivalence taking finitely generated left Λ-modules to finitely generated right Λ-modules, it follows that $Q \to 0$ is a summand of $P_{-1} \to P_0$, which contradicts the fact that it is a minimal projective presentation of M. Thus, $\mathrm{Tr}\, M$ has no non-trivial projective summands. □

We now define the main objects under consideration in this section.

Definition 2.19. Let M in $\mathrm{mod}(\Lambda)$. We say that M is

- **τ-rigid** if $\mathrm{Hom}_\Lambda(M, \tau M) = 0$.
- **τ-tilting** (respectively **almost complete τ-tilting**) if M is τ-rigid and $|M| = |\Lambda|$ (respectively $|M| = |\Lambda| - 1$)
- **support τ-tilting** if there exists an idempotent $e \in \Lambda$ such that M is τ-tilting as a $(\Lambda/\langle e \rangle)$-module.

The following definition and proposition allows for easier arguments later on.

Definition 2.20. Let $M \in \mathrm{mod}(\Lambda)$ and $P \in \mathrm{proj}(\Lambda)$, then the pair (M, P) is called a

(1) τ-rigid pair if M is τ-rigid and $\mathrm{Hom}_\Lambda(P, M) = 0$.

(2) support τ-tilting (respectively, almost complete support τ-tilting) pair if (M, P) is τ-rigid and $|M| + |P| = |\Lambda|$ (respectively, $|M| + |P| = |\Lambda| - 1$).

A τ-rigid or support τ-tilting pair does in fact correspond bijectively to τ-rigid or support τ-tilting modules, as one might expect. We include the following proposition from [AIR14, Proposition 2.3(a)] as a reference point, but we will not present the proof as it relies on several properties which we have not considered.

Proposition 2.21. *Let (M, P) be a pair as above and $e \in \Lambda$ some idempotent such that* $\operatorname{add}(P) = \operatorname{add}(\Lambda e)$. *Then (M, P) is a τ-rigid (respectively, support τ-tilting, almost complete support τ-tilting) pair if and only if M is a τ-rigid (respectively, τ-tilting, almost complete τ-tilting) module over $\Lambda/\langle e\rangle$.* □

We also include the following proposition from [Jas15, Proposition 2.14] which states that support τ-tilting modules are precisely those which admit certain left approximation sequences. The proof is excluded as it would take us outside of our scope, but the result provides some motivation for the studying of quasitilting modules in section 3.1.

Proposition 2.22. *Let $T \in \operatorname{mod}(\Lambda)$ be τ-rigid. Then T is a support τ-tilting Λ-module if and only if there exists an exact sequence*

$$\Lambda \xrightarrow{f} T_0 \xrightarrow{g} T_1 \longrightarrow 0$$

where $T_0, T_1 \in \operatorname{add}(T)$ and f is a left $\operatorname{add}(T)$-approximation of Λ. □

The following proposition collects some useful properties from [Sma84, Theorem].

Proposition 2.23. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\operatorname{mod}(\Lambda)$. Then the following are equivalent*

(1) $\mathcal{T}$ is functorially finite in $\operatorname{mod}(\Lambda)$.

(2) $\mathcal{T} = \operatorname{gen}(X)$ for some X in $\operatorname{mod}(\Lambda)$.

(3) $P(\mathcal{T})$ is a tilting $(\Lambda/\operatorname{ann}(\mathcal{T}))$-module.

(4) $\mathcal{T} = \operatorname{gen}(P(\mathcal{T}))$.

Proof. The first three conditions are equivalent by [Sma84].

(4) $\Rightarrow$ (2): It is clear.

(3) $\Rightarrow$ (4): We first show $\operatorname{gen}(P(\mathcal{T})) \subseteq \mathcal{T}$. We have by definition $P(\mathcal{T}) \subseteq \mathcal{T}$. The category $\operatorname{gen}(P(\mathcal{T}))$ consists of factor modules of $P(\mathcal{T})^n$ for some $n \in \mathbb{N}$. Since $P(\mathcal{T}) \subseteq \mathcal{T}$ and $\mathcal{T}$ is closed under coproducts and factors, we have that $\operatorname{gen}(P(\mathcal{T})) \subseteq \mathcal{T}$.

We now show the converse. Since $P(\mathcal{T})$ is tilting over $\Lambda/\operatorname{ann}(\mathcal{T})$, there is an exact sequence

$$0 \longrightarrow \Lambda/\operatorname{ann}(\mathcal{T}) \xrightarrow{\phi} T_0 \longrightarrow T_1 \longrightarrow 0$$

where $T_0, T_1 \in \operatorname{add}(P(\mathcal{T}))$. Take any X in $\mathcal{T}$ and a surjection $f : (\Lambda/\operatorname{ann}(\mathcal{T}))^k \to X$. Then because $\operatorname{Ext}^1_\Lambda(P(\mathcal{T})^k, X) = 0$ we also have $\operatorname{Ext}^1_\Lambda(T_1^k, X) = 0$. Applying $\operatorname{Hom}_\Lambda(-, X)$ to the sequence above we get an exact sequence

$$0 \longrightarrow \operatorname{Hom}_\Lambda(T_1^k, X) \longrightarrow \operatorname{Hom}_\Lambda(T_0^k, X) \longrightarrow \operatorname{Hom}_\Lambda((\Lambda/\operatorname{ann}(\mathcal{T}))^k, X) \longrightarrow 0$$

and we see that the map f factors through ϕ^k via a surjection $f' : T_0^k \to X$, thus $X \in \operatorname{gen}(P(\mathcal{T}))$. □

This next lemma from [AS81, Proposition 5.8] shows that for two Λ-modules M, N we have $N \in {}^{\perp_0}(\tau M)$ if and only if $\mathrm{gen}(N) \subseteq M^{\perp_1}$.

Proposition 2.24. *Let $M, N \in \mathrm{mod}(\Lambda)$, then the following are equivalent.*

(1) $\mathrm{Ext}^1_\Lambda(M, N') = 0$ *for all factor modules N' of N.*

(2) $\overline{\mathrm{Hom}}_\Lambda(N', \tau M) = 0$ *for all factor modules N' of N.*

(3) $\mathrm{Hom}_\Lambda(N, \tau M) = 0$.

Proof. (1) $\iff$ (2) : By the Auslander-Reiten formulae, we have

$$\overline{\mathrm{Hom}}_\Lambda(X, \tau Y) \cong D\,\mathrm{Ext}^1_\Lambda(Y, X)$$

for all $X, Y \in \mathrm{mod}(\Lambda)$. Since D is a duality, in particular $\mathrm{Hom}_\Lambda(X, Y) \to D\,\mathrm{Hom}_\Lambda(X, Y)$ is a bijection, it follows that $D\,\mathrm{Ext}^1_\Lambda(Y, X) = 0$ if and only if $\mathrm{Ext}^1_\Lambda(Y, X) = 0$. The claim then follows.

(3) $\Rightarrow$ (2) : If $\mathrm{Hom}_\Lambda(N, \tau M) = 0$, then $\mathrm{Hom}_\Lambda(N', \tau M) = 0$ for all factor modules N' of N and clearly also $\overline{\mathrm{Hom}}_\Lambda(N', \tau M) = 0$.

(2) $\Rightarrow$ (3) : Suppose that $f : N \to \tau M$ is a non-zero map, then $N' = \mathrm{Im}(f)$ is a factor module of N which induces a monomorphism $g : N' \to \tau M$. Let $I(N')$ be an injective envelope of N', and suppose that g factors through an injective module I. It then follows that g factors through $I(N')$ as indicated by the dashed arrow in the following diagram

$$\begin{tikzcd}
N' \arrow[rr, "i"] \arrow[dr, "h_1"] \arrow[dd, "g"'] & & I(N') \arrow[dl, dashed, "j"] \\
& I \arrow[dl, "h_2"] & \\
\tau M & &
\end{tikzcd}$$

Let $h = h_2 j$ so that we have the following commutative diagram

$$\begin{tikzcd}
N' \arrow[r, "i"] \arrow[d, "g"] & I(N') \arrow[dl, "h"] \\
\tau M &
\end{tikzcd}$$

Now, if $\mathrm{Ker}(h) = 0$, then h is a monomorphism starting in an injective, so it splits and $I(N')$ is a direct summand of τM, which is a contradiction by lemma 2.18, so $\mathrm{Ker}(h) \neq 0$. Since N' is an essential submodule of $I(N')$, we have that $N' \cap \mathrm{Ker}(h) \neq 0$. But then, the composition $hi = g$ is not a monomorphism, which is another contradiction. Therefore, g does not factor through an injective module.

Thus, if $\overline{\mathrm{Hom}}_\Lambda(N', \tau M) = 0$ for all factor modules N' of N, then we have $\mathrm{Hom}_\Lambda(N, \tau M) = 0$. $\square$

We collect a few results from [AS81] which are useful when studying τ-tilting theory, specifically as it characterizes τ-rigid modules. We do not present the proofs.

Proposition 2.25. *Let $X, \in \mathrm{mod}(\Lambda)$. Then the following are equivalent.*

(1) $\mathrm{gen}(X)$ *is closed under extensions.*

(2) $\mathrm{Ext}^1_\Lambda(X, X') = 0$ *for all factor modules X' of X.*

(3) X is τ-rigid.

Proof. (1) $\iff$ (2) : See [AS81, Proposition 5.5].
(2) $\iff$ (3) : Follows directly from proposition 2.24. □

Note that (2) says $\mathrm{gen}(X) \subseteq X^{\perp_1}$. The following theorem from [AS81, Theorem 5.10] is similar to lemma 2.12, but stronger in that $\mathrm{gen}(X)$ is actually a functorially finite torsion class.

Theorem 2.26. *Let $X \in \mathrm{mod}(\Lambda)$. If any of the equivalent conditions in proposition 2.25 hold, then* $\mathrm{gen}(X)$ *is a functorially finite torsion class in* $\mathrm{mod}(\Lambda)$ *and* $X \in \mathrm{add}(P(\mathrm{gen}(X)))$.

Proof. The first part of the statement is precisely [AS81, Theorem 5.10]. Concerning the last statement, X is a Λ-module such that $\mathrm{Ext}^1_\Lambda(X, \mathrm{gen}(X)) = 0$. We have $X = \bigoplus_{i=1}^n X_i$ as a sum of indecomposable Λ-modules X_i, and let $Y \in \mathrm{gen}(X)$. Then since $\mathrm{Ext}^1_\Lambda(-, Y)$ commutes with finite direct sums, we have

$$\mathrm{Ext}^1_\Lambda(X, Y) = \mathrm{Ext}^1_\Lambda(\bigoplus_{i=1}^n X_i, Y) \cong \bigoplus_{i=1}^n \mathrm{Ext}^1_\Lambda(X_i, Y) = 0.$$

Clearly we have $X_i \in \mathrm{gen}(X)$, so then we have $X_i \in P(\mathrm{gen}(X))$ and thus $X \in \mathrm{add}(P(\mathrm{gen}(X)))$. □

The following is a direct consequence of the previous theorem.

Proposition 2.27. *Let X, Y be τ-rigid Λ-modules. Then the following hold.*

(1) $|X| \leq |\Lambda|$ *and* $|Y| \leq |\Lambda|$.

(2) If $X \in \mathrm{add}(Y)$ *and* $|X| \geq |Y|$, *then* $\mathrm{add}(X) = \mathrm{add}(Y)$.

Proof. (1) : By theorem 2.26, if X is τ-rigid then $X \in \mathrm{add}(P(\mathrm{gen}(X)))$ which implies that $|X| \leq |P(\mathrm{gen}(X))|$. Furthermore, since $P(\mathrm{gen}(X))$ is a functorially finite torsion class in $\mathrm{mod}(\Lambda)$, we have by proposition 2.23 that $P(\mathrm{gen}(X))$ is a tilting $\Lambda/\mathrm{ann}(\mathrm{gen}(X))$-module. That is,

$$|P(\mathrm{gen}(X))| = |\Lambda/\mathrm{ann}(\mathrm{gen}(X))| \leq |\Lambda|.$$

It then follows that $|X| \leq |\Lambda|$. Similarly for Y.

(2) : Surely, if $X \in \mathrm{add}(Y)$ then also $\mathrm{add}(X) \subseteq \mathrm{add}(Y)$. In particular, every indecomposable direct summand of X is also an indecomposable direct summand of Y. Therefore $|X| \leq |Y|$, and so we get $|X| = |Y|$. We conclude then that $\mathrm{add}(X) = \mathrm{add}(Y)$. □

The Λ-module $P(\mathcal{T})$ for a subcategory $\mathcal{T}$ will play an important part in one of the main theorems of [AIR14]. We have already seen in proposition 2.23 that if $\mathcal{T}$ is functorially finite then $P(\mathcal{T})$ is tilting over $\Lambda/\operatorname{ann}(\mathcal{T})$ and that $P(\mathcal{T})$ generates $\mathcal{T}$. The next proposition gives another property of $P(\mathcal{T})$, but does not require $\mathcal{T}$ to be functorially finite.

Proposition 2.28. *If $\mathcal{T}$ is a torsion class in $\operatorname{mod}(\Lambda)$, then $P(\mathcal{T})$ is a τ-rigid Λ-module.*

Proof. Let $\mathcal{T}$ be a torsion class. Then since $P(\mathcal{T})$ is a direct sum of modules $X \in \mathcal{T}$ and $\mathcal{T}$ is closed under coproducts, we have $P(\mathcal{T}) \in \mathcal{T}$. Furthemore, since $\operatorname{Ext}^1_\Lambda(-, \mathcal{T})$ turns coproducts into products, and each summand X of $P(\mathcal{T})$ satisfies $\operatorname{Ext}^1_\Lambda(X, \mathcal{T}) = 0$ we have $\operatorname{Ext}^1_\Lambda(P(\mathcal{T}), \mathcal{T}) = 0$. Let $Y \in \operatorname{gen}(P(\mathcal{T}))$, so there is a surjection $\phi : P(\mathcal{T})^n \to Y$ for some $n \in \mathbb{N}$, then $Y \in \mathcal{T}$ since $\mathcal{T}$ is closed under factors. Then we have $\operatorname{Ext}^1_\Lambda(P(\mathcal{T}), Y) = 0$ for all $Y \in \operatorname{gen}(P(\mathcal{T}))$. By proposition 2.25 we have that $P(\mathcal{T})$ is a τ-rigid Λ-module. $\square$

In light of theorem 2.26, we begin to see a connection between (functorially finite) torsion classes and τ-rigid modules. The follows proposition gives a more concrete approach to determining if a Λ-module X is τ-rigid, as well as when $\operatorname{Hom}_\Lambda(Y, \tau X) = 0$ for some Λ-module Y.

Proposition 2.29. *Let X lie in $\operatorname{mod}(\Lambda)$ and let $P_{-1} \xrightarrow{p_{-1}} P_0 \xrightarrow{p_0} X \to 0$ be a minimal projective presentation of X, then the following hold.*

(1) For any $Y \in \operatorname{mod}(\Lambda)$, there exists an exact sequence

$$0 \longrightarrow \operatorname{Hom}_\Lambda(Y, \tau X) \longrightarrow D\operatorname{Hom}_\Lambda(P_{-1}, Y) \xrightarrow{D(p^*_{-1})} D\operatorname{Hom}_\Lambda(P_0, Y) \xrightarrow{D(p^*_0)} D\operatorname{Hom}_\Lambda(X, Y) \longrightarrow 0$$

(2) For any $Y \in \operatorname{mod}(\Lambda)$, $\operatorname{Hom}_\Lambda(Y, \tau X) = 0$ if and only if the morphism $\operatorname{Hom}_\Lambda(P_0, Y) \xrightarrow{p^*_{-1}} \operatorname{Hom}_\Lambda(P_{-1}, Y)$ is surjective.

(3) X is τ-rigid if and only if $\operatorname{Hom}_\Lambda(P_0, X) \xrightarrow{p^*_{-1}} \operatorname{Hom}_\Lambda(P_{-1}, X)$ is surjective.

Proof. (1) : By proposition 2.16, there is an exact sequence

$$0 \longrightarrow \tau X \longrightarrow \nu(P_{-1}) \xrightarrow{\nu(p_{-1})} \nu(P_0)$$

Applying $\operatorname{Hom}_\Lambda(Y, -)$ to the sequence, and using lemma 2.14 we have the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_\Lambda(Y, \tau X) \arrow[r] & \operatorname{Hom}_\Lambda(Y, \nu(P_{-1})) \arrow[r, "\nu(p_{-1})_*"] \arrow[dl, "\cong"'] & \operatorname{Hom}_\Lambda(Y, \nu(P_0)) \arrow[dl, "\cong"'] & \\
& D\operatorname{Hom}_\Lambda(P_{-1}, Y) \arrow[r, "D(p^*_{-1})"] & D\operatorname{Hom}_\Lambda(P_0, Y) \arrow[r, "D(p^*_0)"] & D\operatorname{Hom}_\Lambda(X, Y) \arrow[r] & 0
\end{tikzcd}$$

which completes the proof.

(2) : It follows from the exact sequence in (1). Indeed, if $\operatorname{Hom}_\Lambda(Y, \tau X) = 0$, then

$$D\operatorname{Hom}_\Lambda(P_{-1}, Y) \xrightarrow{D(p_{-1}^*)} D\operatorname{Hom}_\Lambda(P_0, Y)$$

is injective, which means that

$$\operatorname{Hom}_\Lambda(P_0, X) \xrightarrow{p_{-1}^*} \operatorname{Hom}_\Lambda(P_{-1}, X)$$

is surjective. Similarly one shows the converse.

(3) : It follows from (2) replacing Y with X. □

Denote by $\mathrm{s}\tau\text{-tilt}\,\Lambda$ the set of isomorphism classes of basic support τ-tilting Λ-modules, and $\text{f-tors}\,\Lambda$ the set of functorially finite torsion classes in $\operatorname{mod}(\Lambda)$. We conclude this section with a theorem from [AIR14, theorem 2.7], giving an explicit bijection between $\mathrm{s}\tau\text{-tilt}\,\Lambda$ and $\text{f-tors}\,\Lambda$.

Theorem 2.30. *There is a bijection*

$$\mathrm{s}\tau\text{-tilt}\,\Lambda \longleftrightarrow \text{f-tors}\,\Lambda$$

given by

$$\mathrm{s}\tau\text{-tilt}\,\Lambda \ni T \mapsto \operatorname{gen}(T) \in \text{f-tors}\,\Lambda$$
$$\text{f-tors}\,\Lambda \ni \mathcal{T} \mapsto P(\mathcal{T}) \in \mathrm{s}\tau\text{-tilt}\,\Lambda$$

Proof. Let $\mathcal{T}$ be a functorially finite torsion class in $\operatorname{mod}(\Lambda)$. By proposition 2.28 we have that $T = P(\mathcal{T})$ is a τ-rigid Λ-module. Let $e \in \Lambda$ be an idempotent which is maximal such that $\mathcal{T} \subseteq \operatorname{mod}(\Lambda/\langle e\rangle)$. Then we have

$$|\Lambda/\langle e\rangle| = |\Lambda/\operatorname{ann}(\mathcal{T})|$$

and since T is a tilting $\Lambda/\operatorname{ann}(\mathcal{T})$-module by proposition 2.23 we have that $|\Lambda/\operatorname{ann}(\mathcal{T})| = |T|$. Then $(T, \Lambda e)$ is a τ-rigid pair since T is τ-rigid and $\operatorname{Hom}_\Lambda(\Lambda e, T) = 0$. Furthermore, it is a support τ-tilting pair since

$$|T| + |\Lambda e| = |\Lambda/\operatorname{ann}(\mathcal{T})| + |\operatorname{ann}(\mathcal{T})| = |\Lambda|$$

and so T is a support τ-tilting Λ-module. Furthermore, by proposition 2.23(4) we have $\mathcal{T} \cong \operatorname{gen}(P(\mathcal{T}))$.

Conversely, let T be a support τ-tilting Λ-module, i.e. T is a τ-tilting $\Lambda/\langle e\rangle$-module for some idempotent $e \in \Lambda$. In particular T is τ-rigid, so by theorem 2.26 we have that $\operatorname{gen}(T)$ is a functorially finite torsion class in $\operatorname{mod}(\Lambda/\langle e\rangle)$ such that $T \in \operatorname{add}(P(\operatorname{gen}(T)))$. Then $P(\operatorname{gen}(T))$ is a τ-rigid $\Lambda/\langle e\rangle$-module by proposition 2.28. By proposition 2.27 we have

$$|P(\operatorname{gen}(T))| \leq |\Lambda/\langle e\rangle| = |T|$$

and $\operatorname{add}(T) = \operatorname{add}(P(\operatorname{gen}(T)))$. Thus, $T \cong P(\operatorname{gen}(T))$. □

Given a τ-rigid Λ-module U, there exists a τ-tilting Λ-module T such that U is a direct summand of T, i.e. it's possible to complete U to a τ-tilting module. This is the analog of Bongartz completion for τ-tilting modules, see [AIR14, Theorem 2.10]. We state the theorem without proof, and instead give an example.

Theorem 2.31. *Let PU be a τ-rigid Λ-module. Then $\mathcal{T} := {}^{\perp}(\tau U)$ is a sincere functorially finite torsion class and $T := P(\mathcal{T})$ is a τ-tilting Λ-module satisfying $U \in \operatorname{add}(T)$ and ${}^{\perp}(\tau T) = \operatorname{gen}(T)$.* □

Example 2.32. Let k be an algebraically closed field, Γ the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

and $\Lambda = k\Gamma$ the path algebra over Γ. Then the AR-quiver of $\Lambda = k\Gamma$ is

$$\begin{tikzcd}
 & & P_1 \arrow[dr] & & \\
 & P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ll, dashed, "\tau"'] \arrow[dr] & \\
P_3 \arrow[ur] & & S_2 \arrow[ll, dashed, "\tau"'] \arrow[ur] & & S_1 \arrow[ll, dashed, "\tau"']
\end{tikzcd}$$

Let $U = P_1 \oplus I_2$, which is τ-rigid. Then $\tau U = P_2$, and let $\mathcal{T} = {}^{\perp_0}(\tau U) = \operatorname{add}(\{P_1, I_2, S_2, S_1\})$. We show that the Λ-module $P(\mathcal{T})$ is τ-tilting. We have $\operatorname{Ext}^1_\Lambda(P_1, \mathcal{T}) = 0$ since $P_1 \in \mathcal{T}$ is projective, and so P_1 is a direct summand of $P(\mathcal{T})$. We have the following projective presentations of I_2, S_2, S_1 respectively.

$$0 \to P_3 \to P_1 \to I_2 \to 0 \tag{2.5}$$
$$0 \to P_3 \to P_2 \to S_2 \to 0 \tag{2.6}$$
$$0 \to P_2 \to P_1 \to S_1 \to 0 \tag{2.7}$$

Applying $\operatorname{Hom}_\Lambda(-, S_2)$ to the sequences (2.5) and (2.6) we get the following exact sequences respectively.

$$0 \to \operatorname{Hom}_\Lambda(I_2, S_2) \to \operatorname{Hom}_\Lambda(P_1, S_2) \to \operatorname{Hom}_\Lambda(P_3, S_2) \to \operatorname{Ext}^1_\Lambda(I_2, S_2) \to 0 \tag{2.8}$$
$$0 \to \operatorname{Hom}_\Lambda(S_2, S_2) \to \operatorname{Hom}_\Lambda(P_2, S_2) \to \operatorname{Hom}_\Lambda(P_3, S_2) \to \operatorname{Ext}^1_\Lambda(S_2, S_2) \to 0 \tag{2.9}$$

Since $\operatorname{Hom}_\Lambda(P_3, S_2) = 0$, sequence (2.8) gives us that $\operatorname{Ext}^1_\Lambda(I_2, S_2) = 0$. Furthermore, $\operatorname{Ext}^1_\Lambda(I_2, \{P_1, I_2, S_1\}) = 0$ since P_1, I_2, S_1 are all injective. Thus, I_2 is a direct summand of $P(\mathcal{T})$. Similarly, one gets that S_2 is a direct summand of $P(\mathcal{T})$ by the sequence (2.9). Finally, applying $\operatorname{Hom}_\Lambda(-, S_2)$ to sequence (2.7) one sees that $\operatorname{Ext}^1_\Lambda(S_1, S_2) \neq 0$, so S_1 is not a direct summand of $P(\mathcal{T})$. Then, we conclude that $P(\mathcal{T}) = P_1 \oplus I_2 \oplus S_2$ is the Bongartz completion of $U = P_1 \oplus I_2$. Clearly it is τ-tilting.

Note that $P_1 \oplus I_2 \oplus S_1$ is also a τ-tilting module, but not the Bongartz completion of U. One can obtain this second τ-tilting module from the first by a process known as mutation, which is another topic in [AIR14]. □

We conclude this section with an important theorem which will be crucial when we consider silting modules in section 3. The aim is to generalize the notions of τ-rigid, τ-tilting and support τ-tilting to arbitrary rings A and arbitrary A-modules. Since the Auslander-Reiten translation τ does not exist in $\mathrm{Mod}(A)$ for arbitrary rings, we need descriptions of the aforementioned notions which does not rely on τ. There are three particular results we need, which we will present here without proof as they take us somewhat outside of our scope. First, we need a more general version of the Auslander-Reiten formula. We state a proposition from [Kra03, Corollary 2 on p.269] which provides that.

Proposition 2.33. *Let X be a finitely presented Λ-module and Y an arbitrary Λ-module. Then there is an isomorphism*

$$D\,\mathrm{Ext}^1_\Lambda(X,Y) \cong \overline{\mathrm{Hom}}_\Lambda(Y,\tau X).$$

$\square$

Note that Λ is both artinian and noetherian, and thus any finitely generated Λ-module is also finitely presented. Indeed, if there is a surjection $\Lambda^n \to M$, then Λ^n is notherian and so the kernel of the map is again finitely generated, thus M is finitely presented.

We also need the following proposition from [AIR14, Corollary 2.13]

Proposition 2.34. *Let (M,P) be a τ-rigid pair in $\mathrm{mod}(\Lambda)$. Then (M,P) is a support τ-tilting pair if and only if ${}^{\perp_0}(\tau M) \cap P^{\perp_0} = \mathrm{gen}(M)$.* $\square$

Remark 2.35. Recall that $P \cong \Lambda e$ for some idempotent $e \in \Lambda$ by proposition 2.21, and let $d : P_{-1} \to P_0$ be a minimal projective presentation of M. Since $\mathrm{Hom}_\Lambda(\Lambda e, M) = 0$, then also $d' : P_{-1} \oplus \Lambda e \to P_0$ is a projective presentation of M.

The category ${}^{\perp_0}(\tau M) \cap P^{\perp_0}$ then consists precisely of those finitely generated Λ-modules X such that $\mathrm{Hom}_\Lambda(d', X)$ is surjective. It follows easily by proposition 2.29.

We also need the following proposition from [Mar15, Proposition 7.4.2]

Proposition 2.36. *If $(\mathcal{T},\mathcal{F})$ is a torsion pair in $\mathrm{Mod}(\Lambda)$ and $\mathcal{T}_0 = \mathcal{T} \cap \mathrm{mod}(\Lambda)$ is a functorially finite torsion class (respectively if $\mathcal{F}_0 = \mathcal{F} \cap \mathrm{mod}(\Lambda)$ is a functorially finite torsion-free class) in $\mathrm{mod}(\Lambda)$. Then $(\mathcal{T},\mathcal{F})$ is the unique torsion pair in $\mathrm{Mod}(A)$ such that $\mathcal{T}_0 \subseteq \mathcal{T}$ and $\mathcal{F}_0 \subseteq \mathcal{F}$.* $\square$

Now we give the theorem from [HMV15, Theorem 2.5]. Note that we now use the notation $\mathrm{Gen}(T)$, meaning factors of direct sums of arbitrarily many copies of T.

Theorem 2.37. *Let $T \in \mathrm{mod}(\Lambda)$ and let $\sigma : P_{-1} \to P_0$ be a minimal projective presentation of T. Then the following hold.*

(1) Let $M \in \mathrm{Mod}(A)$, then $\mathrm{Hom}_\Lambda(M, \tau T) = 0$ if and only if the morphism of abelian groups $\mathrm{Hom}_\Lambda(\sigma, M)$ is surjective.

(2) $\mathrm{Hom}_\Lambda(T, \tau T) = 0$ if and only if $\mathrm{Gen}(T) \subseteq T^{\perp_1}$.

(3) T is support τ-tilting if and only if $\mathrm{Gen}(T)$ consists of the Λ-modules M such that $\mathrm{Hom}_\Lambda(\sigma', M)$ is surjective, where σ' is the following projective presentation of T

$$P_{-1} \oplus \Lambda e \xrightarrow{(\sigma \oplus 0)} P_0$$

for a suitable idempotent $e \in \Lambda$.

Proof. (1) : The proof is exactly the same as the proof for proposition 2.29(2), using lemma 2.14 and remark 2.15.

(2) : The proof is very similar to the proof of proposition 2.24 with the replacements $M = T$ and $N = T^{(I)}$ for some set I. The only different part is the first equivalence, which then uses the generalized Auslander-Reiten formula from proposition 2.33. We have

$$D\,\mathrm{Ext}^1_\Lambda(T, T') \cong \overline{\mathrm{Hom}}_\Lambda(T', \tau T)$$

for any $T' \in \mathrm{Gen}(T)$. Then $\mathrm{Ext}^1_\Lambda(T, T') = 0$ if and only if $\overline{\mathrm{Hom}}_\Lambda(T', \tau T) = 0$. The rest of the arguments are the same as in the proof of proposition 2.24 with the replacements above.

(3) : By proposition 2.34 and remark 2.35 we have that T is support τ-tilting if and only if

$$\mathrm{gen}(T) = \mathrm{Gen}(T) \cap \mathrm{mod}(\Lambda)$$

consists of the finitely generated Λ-modules M for which $\mathrm{Hom}_\Lambda(\sigma', M)$ is surjective.

If $\mathrm{Gen}(T) \subseteq \mathrm{Mod}(A)$ consists precisely of the Λ-modules M such that $\mathrm{Hom}_\Lambda(\sigma', M)$ is surjective, then T is support τ-tilting by the equation above.

We prove the converse, so suppose that T is support τ-tilting. Then T is τ-rigid, and by (2) we have $\mathrm{Gen}(T) \subseteq T^{\perp_1}$, which by lemma 2.12 implies that

$$(\mathrm{Gen}(T), T^{\perp_0}) \tag{2.10}$$

is a torsion pair in $\mathrm{Mod}(A)$. Then there is an induced torsion pair in $\mathrm{mod}(\Lambda)$ given by

$$(\mathrm{gen}(T), T^{\perp_0} \cap \mathrm{mod}(\Lambda)). \tag{2.11}$$

The next step is to show that the subcategory $\mathcal{T} \subseteq \mathrm{Mod}(A)$ consisting of all Λ-modules M such that $\mathrm{Hom}_\Lambda(\sigma', M)$ is surjective is a torsion class, giving rise to the following torsion pair in $\mathrm{Mod}(A)$.

$$(\mathcal{T}, \mathcal{F}) \tag{2.12}$$

Then because $\mathrm{gen}(T) = \mathcal{T} \cap \mathrm{mod}(\Lambda)$, we also have $T^{\perp_0} \cap \mathrm{mod}(\Lambda) = \mathcal{F} \cap \mathrm{mod}(\Lambda)$. Since torsion pair (2.11) is then contained in both torsion pairs (2.10) and (2.12), with respect to their torsion and torsion-free classes of course, we can invoke proposition 2.36 to conclude that $\mathrm{Gen}(T) = \mathcal{T}$. This finishes the proof.

To show that $\mathcal{T}$ is a torsion class, we have to show that it is closed under factors, extensions and coproducts by remark 2.6.

Proving that $\mathcal{T}$ is closed under factors: Let $M \in \mathrm{Mod}(A)$ such that $\mathrm{Hom}_\Lambda(\sigma', M)$ is surjective, and M' a factor of M with the canonical projection $\pi : M \to M'$. Let $(h_1', h_2') : P_{-1} \oplus \Lambda e \to M'$ be a morphism, then since $P_{-1} \oplus \Lambda e$ is projective and π surjective, it lifts to a morphism $(h_1, h_2) : P_{-1} \oplus \Lambda e \to M$ such that $(h_1', h_2') = \pi(h_1, h_2)$. But we have $\mathrm{Hom}_\Lambda(\Lambda e, M) = 0$ by assumption, and so therefore we get $h_2 = 0 = h_2'$, as indicated by the following diagram.

$$\begin{tikzcd}
 & P_{-1} \oplus \Lambda e \arrow[r, "{(\sigma,0)}"] \arrow[dl, dashed, "{(h_1,0)}"'] \arrow[d, "{(h_1',0)}"] & P_0 \\
M \arrow[r, "\pi"] & M' \arrow[r] & 0
\end{tikzcd}$$

Furthermore, since $\mathrm{Hom}_\Lambda(\sigma', M)$ is surjective, there is a morphism $p : P_0 \to M$ such that $(h_1, 0) = p(\sigma, 0)$, and then we also have

$$(h_1', 0) = \pi(h_1, 0) = \pi p(\sigma, 0) = (\pi p \sigma, 0)$$

so $\mathrm{Hom}_\Lambda(\sigma', M')$ is surjective as well.

Proving that $\mathcal{T}$ is closed under extensions: Let $M_1, M_2 \in \mathrm{Mod}(A)$ such that $\mathrm{Hom}_\Lambda(\sigma', M_i)$ is surjective for $i = 1, 2$. Consider a short exact sequence in $\mathrm{Mod}(A)$ and suppose there is a morphism from $P_{-1} \oplus \Lambda e$ to the middle term, as indicated in the following diagram

$$\begin{tikzcd}
 & & P_{-1} \oplus \Lambda e \arrow[r, "{(\sigma,0)}"] \arrow[d, "{(h_1,h_2)}"] & P_0 & \\
0 \arrow[r] & M_1 \arrow[r, "f"] & X \arrow[r, "g"] & M_2 \arrow[r] & 0
\end{tikzcd}$$

By assumption, the composition $g(h_1, h2)$ factors through $(\sigma, 0)$ via some morphism $p : P_0 \to M_2$. We then have

$$g(h_1, h_2) = p(\sigma, 0) = (p\sigma, 0).$$

In particular, $gh_2 = 0$ and so h_2 factors through f via some map $\Lambda e \to M_1$. Since $\mathrm{Hom}_\Lambda(\Lambda e, M_1) = 0$ by assumption, we have $h_2 = 0$.

Since P_0 is projective and g is surjective, there is a morphism $p' : P_0 \to X$ such that $p = gp'$, which gives us

$$g(h_1, 0) = p(\sigma, 0) = (gp'\sigma, 0)$$

and so $(h_{-1} - p'\sigma)$ factors through f via some morphism $q : P_{-1} \to M_1$, indicated by the following diagram

$$\begin{tikzcd}
 & & P_{-1} \oplus \Lambda e \arrow[r, "{(\sigma,0)}"] \arrow[dl, "{(q,0)}"'] \arrow[d, "{(h_1,0)}"] & P_0 \arrow[d, "p"] \arrow[dl, "p'"'] & \\
0 \arrow[r] & M_1 \arrow[r, "f"] & X \arrow[r, "g"] & M_2 \arrow[r] & 0
\end{tikzcd}$$

Finally, since $\mathrm{Hom}_\Lambda(\sigma', M_1)$ is surjective, the morphism $(q, 0)$ factors through $(\sigma, 0)$ via some morphism $q' : P_0 \to M_1$, which then gives us the following

$$h_1 = fq + p'\sigma = fq'\sigma + p'\sigma = (fq' + p')\sigma$$

and therefore we have that $\mathrm{Hom}_\Lambda(\sigma', X)$ is surjective.

Proving that $\mathcal{T}$ is closed under coproducts: Let $\{M_i\}_{i\in I} \in \mathrm{Mod}(A)$ be modules such that $\mathrm{Hom}_\Lambda(\sigma', M_i)$ is surjective for all $i \in I$. Then $\mathrm{Hom}_\Lambda(\sigma', \bigoplus_I M_i)$ is surjective if and only if we have

$$\mathrm{Hom}_\Lambda(P_{-1} \bigoplus \Lambda e, \bigoplus_{i\in I} M_i) \cong \bigoplus_{i\in I} \mathrm{Hom}_\Lambda(P_{-1} \bigoplus \Lambda e, M_i)$$

and

$$\mathrm{Hom}_\Lambda(P_0, \bigoplus_{i\in I} M_i) \cong \bigoplus_{i\in I} \mathrm{Hom}_\Lambda(P_0, M_i)$$

By lemma 3.13(2), the isomorphisms above hold, and so $\mathcal{T}$ is closed under coproducts.

Then $\mathcal{T}$ is a torsion class, and we are done. □

2.3 Support τ-tilting modules and 2-term silting complexes

In this section we present some of the main results from section 3 in [AIR14], showing correspondences between support τ-tilting in $\mathrm{mod}(\Lambda)$ and 2-term silting complexes in $K^b(\mathrm{proj}(\Lambda))$. It is of particular interest to us as silting modules in $\mathrm{Mod}(A)$ turn out to be in bijection with 2-term silting complexes in $K^b(\mathrm{Proj}(A))$, as we shall see in section 5.2.

Let Λ be a finite dimensional k-algebra. A complex σ in $K^b(\mathrm{proj}(\Lambda))$ will be called 2-term if it is concentrated in degrees $-1, 0$.

Definition 2.38. Let $P \in K^b(\mathrm{proj}(\Lambda))$.

(1) We call P **presilting** if $\mathrm{Hom}_{K^b(\mathrm{proj}(\Lambda))}(P, P[i]) = 0$ for all $i > 0$.

(2) We call P **silting** if it presilting and satisfies $\mathrm{thick}(P) = K^b(\mathrm{proj}(\Lambda))$, where $\mathrm{thick}(P)$ is the smallest subcategory of $K^b(\mathrm{proj}(\Lambda))$ which contains P and is closed under cones, $[\pm 1]$, direct summands and isomorphisms.

(3) We call P **partial silting** if it is a direct summand of a silting complex.

An example of a silting complex in $K^b(\mathrm{proj}(\Lambda))$ is the stalk complex Λ.

Proposition 2.39. *Let P be a basic silting complex in $K^b(\mathrm{proj}(\Lambda))$, then $|P| = |\Lambda|$.*

Proof. By [AI12, Theorem 2.28], any two silting objects T, T' satisfy $|T| = |T'|$. Now, since $\Lambda = \bigoplus_{i=1}^{n} P_i$ where each P_i is an indecomposable projective Λ-module and Λ is silting, then for any basic silting complex P we have $|P| = |\Lambda|$. $\square$

Here we present another analog of Bongartz completion, but for 2-term silting complexes.

Proposition 2.40. *Let P be a 2-term presilting complex in $K^b(\mathrm{proj}(\Lambda))$. Then the following hold*

(1) P is a direct summand of a 2-term silting complex.

(2) P is silting if and only if $|\Lambda| = |P|$.

Proof. (1) : We follow the proof in [Aih13, Proposition 2.16]. For the sake of simplicity, we write $K := K^b(\mathrm{proj}(\Lambda))$.

The stalk complex Λ is silting in $K^b(\mathrm{proj}(\Lambda))$, as is $\Lambda[1]$. First note that $P \in \Lambda^{\perp_{>0}}$ and $\Lambda[1] \in P^{\perp_{>0}}$. Now, take a triangle

$$Q \longrightarrow P' \xrightarrow{\phi} \Lambda[1] \longrightarrow Q[1]$$

where $P' \in \mathrm{add}(P)$ and ϕ is a right $\mathrm{add}(P)$-approximation of $\Lambda[1]$, and set $T := P \oplus Q$. We first prove that T is presilting in two steps, then verify that $\mathrm{thick}(T) = \mathrm{thick}(\Lambda[1])$, thus showing that T is silting.

(i) We first show that $\mathrm{Hom}_K(P, Q[i]) = 0$ for all $i > 0$. There is an exact sequence

$$\mathrm{Hom}_K(P, P'[i-1]) \xrightarrow{\phi[i]_*} \mathrm{Hom}_K(P, \Lambda[i]) \longrightarrow \mathrm{Hom}_K(P, Q[i]) \longrightarrow \underbrace{\mathrm{Hom}_K(P, P'[i])}_{=0}$$

The last term is 0 because P is presilting, and $\phi[i]_*$ is surjective because any map $P \to \Lambda[i]$ factors through the right $\mathrm{add}(P)$-approximation $\phi[i]$. Thus, we have $\mathrm{Hom}_K(P, Q[i]) = 0$ for all $i > 0$.

(ii) Next, we show that $\mathrm{Hom}_K(Q, T[i]) = 0$ for all $i > 0$.

There is an exact sequence

$$\underbrace{\mathrm{Hom}_K(P', T[i])}_{=0} \longrightarrow \mathrm{Hom}_K(Q, T[i]) \longrightarrow \mathrm{Hom}_K(\Lambda, T[i]) \longrightarrow \underbrace{\mathrm{Hom}_K(P'[-1], T[i])}_{=0}$$

By (i) we have $\operatorname{Hom}_K(P, Q[i]) = 0$ for all $i > 0$, then since P is presilting and $P' \in \operatorname{add}(P)$ we have

$$\operatorname{Hom}_K(P', Q[i]) = 0 = \operatorname{Hom}_K(P', P[i]) \quad \text{for all } i > 0.$$

Therefore, $\operatorname{Hom}_K(P', T[i]) = 0$ for all $i > 0$, so the first and last terms in the exact sequence above are 0. Then we get the isomorphism

$$\operatorname{Hom}_K(Q, T[i]) \cong \operatorname{Hom}_K(\Lambda, T[i]). \tag{2.13}$$

Furthermore, there is an exact sequence

$$\underbrace{\operatorname{Hom}_K(\Lambda[1], \Lambda[1+i])}_{=0} \longrightarrow \operatorname{Hom}_K(\Lambda[1], Q[i+1]) \longrightarrow \underbrace{\operatorname{Hom}_K(\Lambda[1], P'[i+1])}_{=0}$$

The first term is 0 because $\Lambda[1]$ is silting, and the last term is 0 because $P' \in \operatorname{add}(P)$ and $P \in \Lambda^{\perp_{>0}}$ as mentioned in the beginning of the proof. So then

$$\operatorname{Hom}_K(\Lambda[1], Q[i+1]) = 0 = \operatorname{Hom}_K(\Lambda[1], P[i+1]) \quad \text{for all } i > 0.$$

By the isomorphism 2.13, we then have $\operatorname{Hom}_K(Q, T[i]) = 0$ for all $i > 0$.

By combining the fact that P is presilting with the properties (i), (ii), we have

$$\operatorname{Hom}_K(T, T[i]) = \operatorname{Hom}_K(P, P[i]) \oplus \operatorname{Hom}_K(P, Q[i]) \oplus \operatorname{Hom}_K(Q, T[i]) = 0$$

so T is presilting.

Furthermore, since $T = P \oplus Q$ and $P' \in \operatorname{add}(P)$ clearly both $Q, P' \in \operatorname{thick}(T)$. Then by the first triangle, also $\Lambda[1] \in \operatorname{thick}(T)$. But $\Lambda[1]$ is silting, and since $\operatorname{thick}(\Lambda[1])$ is the smallest triangulated thick subcategory of $K^b(\operatorname{proj}(\Lambda))$ containing $\Lambda[1]$, we have

$$K^b(\operatorname{proj}(\Lambda)) = \operatorname{thick}(\Lambda[1]) \subseteq \operatorname{thick}(T).$$

But $\operatorname{thick}(T)$ is a subcategory of $K^b(\operatorname{proj}(\Lambda))$, so we then conclude that

$$\operatorname{thick}(T) = \operatorname{thick}(\Lambda[1]) = K^b(\operatorname{proj}(\Lambda))$$

and T is silting with P as a direct summand.

(2) : By proposition 2.39, if P is silting then $|P| = |\Lambda|$. Conversely, suppose that P is presilting with $|P| = |\Lambda|$. Then by (1), there exists a complex P' such that $P \oplus P'$ is silting. But then again by proposition 2.39, we have $|P \oplus P'| = |\Lambda| = |P|$. So in fact, $P' \in \operatorname{add}(P)$, so P is silting. □

The following lemma is useful in the context of τ-tilting as the concept of τ-rigid modules is correlated to presilting complexes. Furthermore, the equivalent statements translate very well to the concepts of silting modules and large silting complexes in sections 3 and 5.2.

Lemma 2.41. *Let $M, N \in \operatorname{mod}(\Lambda)$, and let $P_{-1} \xrightarrow{p_{-1}} P_0 \xrightarrow{p_0} M$ and $Q_{-1} \xrightarrow{q_{-1}} Q_0 \xrightarrow{q_0} N$ be minimal projective presentations of M and N respectively. Consider the 2-term complexes P and Q induced by the presentations. Then the following are equivalent.*

(1) $\operatorname{Hom}_\Lambda(P_0, N) \xrightarrow{p_{-1}^*} \operatorname{Hom}_\Lambda(P_{-1}, N)$ *is surjective.*

(2) $\operatorname{Hom}_\Lambda(N, \tau M) = 0$.

(3) $\operatorname{Hom}_{K^b(\operatorname{proj}(\Lambda))}(P, Q[1]) = 0$.

Proof. (1) $\iff$ (2) : It follows directly by proposition 2.29(1).

(2) $\Rightarrow$ (3) : Let $f \in \operatorname{Hom}_{K^b(\operatorname{proj}(\Lambda))}(P, Q[1])$, determined by $f \in \operatorname{Hom}_\Lambda(P_{-1}, Q_0)$. Then $q_0 f \in \operatorname{Hom}_\Lambda(P_{-1}, N)$, and since p_{-1}^* is surjective, there exists a map $g : P_0 \to N$ such that $q_0 f = g p_{-1}$, i.e. the following diagram commutes.

$$\begin{tikzcd} P_{-1} \arrow[r, "p_{-1}"] \arrow[d, "f"] & P_0 \arrow[d, dashed, "g"] \\ Q_0 \arrow[r, "q_0"] & N \end{tikzcd}$$

Since P_0 is projective, the map g lifts to some map $h_0 : P_0 \to Q_0$ such that $g = q_0 h_0$. Then we have

$$q_0 f = g p_{-1} = q_0 h_0 p_{-1} \quad \Rightarrow \quad q_0(f - h_0 p_{-1}) = 0$$

Therefore, there is some map $h_{-1} : P_{-1} \to Q_{-1}$ such that $f - h_0 p_{-1} = q_{-1} h_{-1}$, meaning f is null-homotopic, as indicated in the following diagram.

$$\begin{tikzcd} & P_{-1} \arrow[r, "p_{-1}"] \arrow[d, "f"] \arrow[dl, dashed, "h_{-1}"'] & P_0 \arrow[d, "g"] \arrow[dl, dashed, "h_0"'] \\ Q_{-1} \arrow[r, "q_{-1}"] & Q_0 \arrow[r, "q_0"] & N \end{tikzcd}$$

(3) $\Rightarrow$ (1) : Let $f \in \operatorname{Hom}_\Lambda(P_{-1}, N)$, and since P_{-1} is projective, f lifts to some map $g : P_{-1} \to Q_0$ such that $f = q_0 g$.

$$\begin{tikzcd} & P_{-1} \arrow[r, "p_{-1}"] \arrow[d, "g"] \arrow[dr, "f"] & P_0 \arrow[r, "p_0"] & M \\ Q_{-1} \arrow[r, "q_{-1}"] & Q_0 \arrow[r, "q_0"] & N & \end{tikzcd}$$

Then g induces a map in $\operatorname{Hom}_{K^b(\operatorname{proj}(\Lambda))}(P, Q[1])$, which by assumption is null-homotopic, so there are maps $h_i : P_i \to Q_i$ for $i = -1, 0$ such that $g = h_0 p_{-1} - q_{-1} h_{-1}$. Then we have

$$f = q_0 g = q_0(h_0 p_{-1} - q_{-1} h_{-1}) = q_0 h_0 p_{-1}$$

so f factors through p_{-1}, making p_{-1}^* surjective. $\square$

Remark 2.42. An immediate consequence of lemma 2.41 is that a module M with minimal projective presentation $P_{-1} \to P_0$ is τ-rigid if and only if $P = (P_{-1} \to P)$ is presilting.

Lemma 2.43. *Let $M \in \mathrm{mod}(\Lambda)$ and $P_{-1} \xrightarrow{p_{-1}} P_0 \xrightarrow{p_0} M$ be a minimal projective presentation of M. Consider the 2-term complex P induced by the presentation. Then for any Q in $\mathrm{proj}(\Lambda)$, the following are equivalent.*

(1) $\mathrm{Hom}_\Lambda(Q, M) = 0$.

(2) $\mathrm{Hom}_{K^b(\mathrm{proj}(\Lambda))}(Q, P) = 0$.

Proof. (1) $\Rightarrow$ (2) : Let $f \in \mathrm{Hom}_{K^b(\mathrm{proj}(\Lambda))}(Q, P)$, then it is given by $f \in \mathrm{Hom}_\Lambda(Q, P_0)$. By assumption, $p_0 f = 0$, so f factors through P_{-1} as follows

$$\begin{tikzcd} & Q \arrow[dl, dashed] \arrow[d, "f"] & \\ P_{-1} \arrow[r, "p_{-1}"] & P_0 \arrow[r, "p_0"] & M \end{tikzcd}$$

so f is null-homotopic in $K^b(\mathrm{proj}(\Lambda))$.

(2) $\Rightarrow$ (1) : Let $f \in \mathrm{Hom}_\Lambda(Q, M)$, then it factors through p_0 via some map $g : Q \to P_0$ since Q is projective. By assumption, g is null-homotopic in $K^b(\mathrm{proj}(\Lambda))$, so there exists a map $h : Q \to P_{-1}$ such that $g = p_{-1}h$.

$$\begin{tikzcd} & Q \arrow[dl, dashed, "h"'] \arrow[d, dashed, "g"] \arrow[dr, "f"] & \\ P_{-1} \arrow[r, "p_{-1}"] & P_0 \arrow[r, "p_0"] & M \end{tikzcd}$$

Then $f = p_0 g = p_0 p_{-1} h = 0$. $\square$

Recall that a morphism $f : X \to Y$ of Λ-modules is **right minimal** if $fg = f$ implies that g is an automorphism for any $g \in \mathrm{End}_\Lambda(X)$. Furthermore, an epimorphism $g : P \to X$ with P being projective is a projective cover of X if and only if it is right minimal, see [ARS95, Proposition 4.1]

This next result from [ARS95, Proposition 2.2] will be used later. The proof is easy, but we exclude it because it's not part of our main focus.

Proposition 2.44. *Let $f : X \to Y$ be a morphism in $\mathrm{mod}(\Lambda)$. Then there is a decomposition $X = X_1 \oplus X_2$ such that $f|_{X_1}$ is right minimal and $f|_{X_2} = 0$.* $\square$

Next, we prove a lemma, which is similar to the one above, but it will allow us to draw some important conclusions about the 2-term complexes we're working with.

Lemma 2.45. *Let $f : X \to Y$ be a morphism in* $\mathrm{mod}(\Lambda)$*. Then f is right minimal if and only if $f|_{X'} \neq 0$ for all direct summands X' of X.*

Proof. Decompose X as $\bigoplus_{i\in I} X_i$ and suppose that $f|_{X_k} = 0$ for some $k \in I$. Then let $g \in \mathrm{End}_\Lambda(X)$, denote the component maps by $g_{ij} : X_i \to X_j$, and let

$$g = \begin{cases} 0 & \text{if } i \neq j \\ 0 & \text{if } i = j = k \\ 1 & \text{if } i = j \neq k \end{cases}$$

i.e. using matrix notation, g is the diagonal matrix with the identity morphism for the respective summands along the diagonal, and at index kk it is the zero-morphism. Then $fg = f$ but g is not an isomorphism, so f is not right minimal.

Conversely, suppose that $f|_{X_i} \neq 0$ for all $i \in I$. Let $g \in \mathrm{End}_\Lambda(X)$ be such that $fg = f$. Writing the equation using matrices, denoting the component maps of f and g by f_i and g_{ij} respectively, we get $\sum_{i\in I} f_i g_{ij} = f_j$ for all $j \in I$. Then we get $g_{ij} = 0$ for $i \neq j$ and g_{jj} is an isomorphism. Thus, g is an isomorphism, so f is right minimal □

For two Λ-modules M, N, the **radical** $\mathrm{rad}(M, N)$ consists of morphisms $f : M \to N$ such that there does not exist a decomposition $M = M_1 \oplus M_2$ and $N_1 \oplus N_2$ where M_2, N_2 are non-zero and $f|_{M_2}$ is an isomorphism. We prove now that every 2-term complex in $K^b(\mathrm{proj}(\Lambda))$ is actually isomorphic to a complex $P_{-1} \xrightarrow{d} P_0$ such that $d \in \mathrm{rad}(P_{-1}, P_0)$. It is important in proposition 2.47, because in particular it ensures that for a 2-term complex $P_{-1} \xrightarrow{d} P_0$, the map $P_0 \to \mathrm{Cok}(d)$ is a projective cover.

Lemma 2.46. *Let $P = (P_{-1} \xrightarrow{d} P_0)$ be a 2-term complex in $K^b(\mathrm{proj}(\Lambda))$. Then P is isomorphic to a 2-term complex $P' = (P'_{-1} \xrightarrow{d'} P'_0)$ such that $d' \in \mathrm{rad}(P'_{-1}, P'_0)$. Moreover, the map $P_0 \to \mathrm{Cok}(d)$ is right minimal up to isomorphism.*

Proof. Suppose that $d \notin \mathrm{rad}(P_{-1}, P_0)$, then there is a decomposition of P

$$P'_{-1} \oplus P''_{-1} \xrightarrow{\left(\begin{smallmatrix} d' & 0 \\ 0 & d'' \end{smallmatrix}\right)} P'_0 \oplus P''_0$$

where $d' = d|_{P'_{-1}}$ and $d'' = d|_{P''_{-1}}$ such that P''_{-1} and P''_0 are non-zero and d'' is an isomorphism. Then, it's clear from the following diagram that d'' is null-homotopic, and thus $P''_{-1} \xrightarrow{d''} P''_0$ is isomorphic to the zero-object in $K^b(\mathrm{proj}(\Lambda))$.

$$\begin{tikzcd} & P''_{-1} \arrow[r, "d''"] \arrow[dl, "1"'] \arrow[d, "d''"] & P''_0 \arrow[dl, "0"] \\ P''_{-1} \arrow[r, "d''"] & P''_0 & \end{tikzcd}$$

Then, $P \cong (P'_{-1} \xrightarrow{d'} P'_0)$. The same argument can be applied recursively to P until it's isomorphic to a 2-term complex with differential in the radical.

Now, let $P = (P_{-1} \xrightarrow{d} P_0)$ such that $d \in \mathrm{rad}(P_{-1}, P_0)$, and let $M = \mathrm{Cok}(d)$ with the canonical projection $\pi : P_0 \to M$. Suppose that there exists a direct summand X of P_0 such that $\pi|_X = 0$. Then there is an epimorphism $P_{-1} \to X$, which splits since X is projective, so X is a direct summand of P_{-1}. But then P_{-1} and P_0 share a common summand, so $d \notin \mathrm{rad}(P_{-1}, P_0)$. Therefore, $\pi|_X \neq 0$ for all direct summands of P_0, and π is right minimal (and a projective cover of M) by lemma 2.45. ☐

The following two results show an important correspondance between silting complexes and support τ-tilting complexes.

Proposition 2.47. *Let $P = (P_{-1} \xrightarrow{d} P_0)$ be a 2-term complex in $K^b(\mathrm{proj}(\Lambda))$ and let $M := \mathrm{Cok}(P)$.*

(1) If P is silting and d is right minimal, then M is τ-tilting.

(2) If P is silting, then M is support τ-tilting.

Proof. We prove (2) first because (1) follows directly.

(2) : By propotisiion 2.44, we have a decomposition $d = (d', 0) : P_{-1} = P'_{-1} \oplus P''_{-1} \to P_0$ such that d' is right minimal. Furthermore, the map $P_0 \to M$ is right minimal by lemma 2.46. Thus, the sequence

$$P'_{-1} \xrightarrow{d'} P_0 \longrightarrow M \longrightarrow 0$$

is then a minimal projective presentation of M. Since P is silting, and in particular pre-silting, M is τ-rigid by remark 2.42. Furthermore, since P''_{-1} is a summand of P_{-1} we have $\mathrm{Hom}_{K^b(\mathrm{proj}(\Lambda))}(P''_{-1}, P) = 0$. Then by lemma 2.43 we have $\mathrm{Hom}_\Lambda(P''_{-1}, M) = 0$. Because d' is a minimal projective presentation of M and taking projective presentations is additive, we have $|M| = |P'_{-1} \xrightarrow{d'} P_0|$ and therefore also

$$|M| + |P''_{-1}| = |P'_{-1} \xrightarrow{d'} P_0| + |P''_{-1}| = |P|$$

By proposition 2.40(2), we have

$$|M| + |P''_{-1}| = |P| = |\Lambda|$$

So (M, P''_{-1}) is a support τ-tilting pair, and then M is a support τ-tilting Λ-module by proposition 2.21.

(1) : This proof is the case where $P''_{-1} = 0$ in (2). ☐

Proposition 2.48. *Let $M \in \mathrm{mod}(\Lambda)$ and $P = (P_{-1} \xrightarrow{d} P_0)$ a minimal projective presentation of M, and let $Q \in \mathrm{proj}(\Lambda)$.*

(1) If M is τ-tilting, then $(P_{-1} \xrightarrow{d} P_0)$ is silting in $K^b(\mathrm{proj}(\Lambda))$.

(2) If (M, Q) is a support τ-tilting pair, then $(P_{-1} \oplus Q \xrightarrow{(d,0)} P_0)$ is silting in $K^b(\mathrm{proj}(\Lambda))$.

Proof. (2) : Let ${}_QP = (P_{-1} \oplus Q \xrightarrow{(d,0)} P_0)$. Since (M, Q) is a support τ-tilting pair, we have that M is τ-rigid and $\mathrm{Hom}_\Lambda(Q, M) = 0$. By remark 2.42 we have that P is presilting, and by lemma 2.43 that $\mathrm{Hom}_{K^b(\mathrm{proj}(\Lambda))}(Q, {}_Q P) = 0$. Therefore, ${}_QP$ is presilting. Because d is a minimal projective presentation of M, we have $|M| = |P|$. Furthermore, because (M, Q) is a support τ-tilting pair, we have $|M| + |Q| = |\Lambda|$, but then

$$|M| + |Q| = |P| + |Q| = |{}_QP| = |\Lambda|$$

so the complex ${}_QP = (P_{-1} \oplus Q \xrightarrow{(d,0)} P_0)$ is silting by proposition 2.40.

(1) : This proof is the case where $Q = 0$ in (2). $\square$

Denote by $2\mathrm{silt}\,\Lambda$ the set of isomorphism classes of basic 2-term silting complexes in $K^b(\mathrm{proj}(\Lambda))$. Now, we conclude this section with an important theorem from [AIR14, Theorem 3.2]. The work is all done, as the theorem follows directly from propositions 2.47 and 2.48.

Theorem 2.49. *There is a bijection*

$$2\mathrm{silt}\,\Lambda \longleftrightarrow \mathrm{s}\tau\text{-tilt}\,\Lambda$$

given by

$$2\mathrm{silt}\,\Lambda \ni P \mapsto H^0(P) \in \mathrm{s}\tau\text{-tilt}\,\Lambda$$

$$\mathrm{s}\tau\text{-tilt}\,\Lambda \ni (M, P) \mapsto (P_{-1} \oplus P \xrightarrow{(\,f\ 0\,)} P_0) \in 2\mathrm{silt}\,\Lambda$$

where $P_{-1} \xrightarrow{f} P_0$ is a minimal projective presentation of M.

Let's look at an example.

Example 2.50. Let k be an algebraically closed field, Γ the quiver

$$1 \longrightarrow 2 \longrightarrow 3$$

and $\Lambda = k\Gamma$ the path algebra over Γ. We have the AR-quiver of Γ

$$\begin{tikzcd}
& & P_1 \arrow[dr] & & \\
& P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ll, dashed, "\tau"'] \arrow[dr] & \\
P_3 \arrow[ur] & & S_2 \arrow[ll, dashed, "\tau"'] \arrow[ur] & & S_1 \arrow[ll, dashed, "\tau"']
\end{tikzcd}$$

Then $T = I_2 \oplus S_1$ is τ-tilting over $\Lambda/\langle e_3 \rangle$, so (T, P_3) is a support τ-tilting pair. We have a minimal projective presentation of T

$$P_3 \oplus P_2 \xrightarrow{f} P_1 \oplus P_1 \longrightarrow I_2 \oplus S_1 \longrightarrow 0$$

so by theorem 2.49, the complex $P = (P_3 \oplus P_2 \oplus P_3 \xrightarrow{(f\ 0)} P_1 \oplus P_1)$ is 2-silting. Indeed, it is clearly presilting and $|P| = 3 = |\Lambda|$.

Note that the choice of projective presentation is important, as the minimal one does not give a 2-silting complex. $\square$

3 Silting Modules

This section is dedicated to the concept of silting modules, first introduced in [HMV15]. Most of the results are from [HMV15, Section 3], but some additional results are included and some proofs are restructured. We specify where this is done.

3.1 Quasitilting modules, torsion classes and approximations

We begin by introducing the quasitilting modules. Their importance is due to the fact that they classify torsion classes which provide left approximation sequences. Recall proposition 2.22 that τ-rigid modules are support τ-tilting if and only if there is a particular left approximation sequence, which is part of the motivation for the classification of such torsion classes.

Lemma 3.1. *Let $T \in \mathrm{Mod}(A)$, then the following are equivalent.*

(1) $\mathrm{Pres}(T) = \mathrm{Gen}(T)$ *and T is* Ext*-projective in* $\mathrm{Gen}(T)$*.*

(2) $\mathrm{Gen}(T) = \mathrm{Sub}(\mathrm{Gen}(T)) \cap T^{\perp_1}$*, where* $\mathrm{Sub}(\mathrm{Gen}(T))$ *denotes the category of all submodules of modules in* $\mathrm{Gen}(T)$*.*

*A module T satisfying one of the equivalent conditions above is called **quasitilting**.*

Proof. (1) $\Rightarrow$ (2): Since T is Ext-projective in $\mathrm{Gen}(T)$, it is a torsion class by lemma 2.12. We will without further mention use that $\mathrm{Gen}(T)$ is closed under factors, extensions and coproducts for this part of the proof.

Clearly $\mathrm{Gen}(T) \subseteq \mathrm{Sub}(\mathrm{Gen}(T))$ and $\mathrm{Gen}(T) \subseteq T^{\perp_1}$ by assumption. So we must show that $\mathrm{Sub}(\mathrm{Gen}(T)) \cap T^{\perp_1} \subseteq \mathrm{Gen}(T)$.

Let $N \in \mathrm{Sub}(\mathrm{Gen}(T)) \cap T^{\perp_1}$ and let $M \in \mathrm{Gen}(T)$ such that there is a monomorphism $f : N \to M$. Then $C := \mathrm{Cok}(f) \in \mathrm{Gen}(T)$, and also $C \in \mathrm{Pres}(T)$ by assumption. Consider an $\mathrm{Add}(T)$-presentation of C:

$$T'' \xrightarrow{h} T' \xrightarrow{g} C \longrightarrow 0$$

Exactness at T' gives $\mathrm{Ker}(g) = \mathrm{Im}(h) \cong T''/\mathrm{Ker}(h)$ and since $T'' \in \mathrm{Gen}(T)$ we have $K := \mathrm{Ker}(g) \in \mathrm{Gen}(T)$. Now, apply $\mathrm{Hom}_A(T', -)$ to the exact sequence

$$0 \longrightarrow N \xrightarrow{f} M \xrightarrow{\pi} C \longrightarrow 0$$

to get the long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_A(T',N) \arrow[r] & \mathrm{Hom}_A(T',M) \arrow[r] & \mathrm{Hom}_A(T',C) \arrow[dll, rounded corners] \\ & \mathrm{Ext}^1_A(T',N) \arrow[r] & \dots & \end{tikzcd}$$

Since $T' \in \mathrm{Add}(T)$, there exists an A-module X such that $T' \oplus X \cong T^{(I)}$ for some set I, from which we get

$$\mathrm{Ext}^1_A(T', N) \bigoplus \mathrm{Ext}^1_A(X, N) \cong \mathrm{Ext}^1_A(T^{(I)}, N) \cong \prod_I \mathrm{Ext}^1_A(T, N) = 0.$$

Thus the map $\mathrm{Hom}_A(T', M) \to \mathrm{Hom}_A(T', C)$ in the long exact sequence above is surjective. Then there exists a map $b : T' \to M$ such that $g = \pi b$. Furthermore we have $gi = \pi bi = 0$, so $bi \in \mathrm{Ker}(\pi) = \mathrm{Im}(f)$, so there exists a map $a : K \to N$ such that $bi = fa$. We then have the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "i"] \arrow[d, dashed, "a"] & T' \arrow[r, "g"] \arrow[d, dashed, "b"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r, "f"] & M \arrow[r, "\pi"] & C \arrow[r] & 0
\end{tikzcd}$$

where at this point, all modules except N belong to $\mathrm{Gen}(T)$. We have $\mathrm{Cok}(a) \cong \mathrm{Cok}(b)$ by the Snake lemma. Since $\mathrm{Cok}(b) \in \mathrm{Gen}(T)$ we have $\mathrm{Cok}(a) \in \mathrm{Gen}(T)$. The canonical epimorphism $K \to \mathrm{Im}(a)$ gives us that $\mathrm{Im}(a) \in \mathrm{Gen}(T)$, so there is an exact sequence

$$0 \longrightarrow \mathrm{Im}(a) \longrightarrow N \longrightarrow \mathrm{Cok}(a) \longrightarrow 0$$

from which it follows that $N \in \mathrm{Gen}(T)$.

(2) $\Rightarrow$ (1) : We have by assumption that $\mathrm{Gen}(T) \subseteq T^{\perp_1}$, i.e. T is Ext-projective in $\mathrm{Gen}(T)$. Recall that $\mathrm{Pres}(T) \subseteq \mathrm{Gen}(T)$. Thus it is sufficient to show $\mathrm{Gen}(T) \subseteq \mathrm{Pres}(T)$.

Let $M \in \mathrm{Gen}(T)$ and consider the map $u : T^{(I)} \to M$ with $I = \mathrm{Hom}_A(T, M)$, which is universal in the sense that any map $T' \to M$ with $T' \in \mathrm{Add}(T)$ factors through u. Since M is generated by T, there is an epimorphism $f : T' \to M$ with $T' \in \mathrm{Add}(T)$, and f factors through the universal map, showing that u is surjective. Since $T^{(I)} \in \mathrm{Gen}(T)$, we have $\mathrm{Ext}^1_A(T, T^{(I)}) = 0$ by assumption. Now, apply $\mathrm{Hom}_A(T, -)$ to the short exact sequence

$$0 \longrightarrow \mathrm{Ker}(u) \xrightarrow{v} T^{(I)} \xrightarrow{u} M \longrightarrow 0$$

to get the long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(T, \mathrm{Ker}(u)) \arrow[r, "v_*"] & \mathrm{Hom}_A(T, T^{(I)}) \arrow[r, "u_*"] & \mathrm{Hom}_A(T, M) \arrow[dll, rounded corners] \\
& \mathrm{Ext}^1_A(T, \mathrm{Ker}(u)) \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

Since u_* is surjective by the arguments above, we have $\mathrm{Ker}(u) \in T^{\perp_1}$. We also have $\mathrm{Ker}(u) \in \mathrm{Sub}(\mathrm{Gen}(T))$, so $\mathrm{Ker}(u) \in \mathrm{Gen}(T)$. Then there is a map $w : T^{(J)} \to \mathrm{Ker}(u)$ and we have following $\mathrm{Add}(T)$-presentation of M which completes the proof

$$T^{(J)} \xrightarrow{vw} T^{(I)} \xrightarrow{u} M \longrightarrow 0$$

$\square$

The following two lemmas are useful, specifically for the proof of proposition 3.6.

Lemma 3.2. *Let T be an A-module and $\bar{A} = A/\operatorname{Ann}(T)$. Then $\operatorname{Ann}(T) = \operatorname{Ann}(\operatorname{Gen}(T))$ and $\operatorname{Gen}({}_A T) = \operatorname{Gen}({}_{\bar{A}} T)$.*

Proof. First we prove $\operatorname{Ann}(T) = \operatorname{Ann}(\operatorname{Gen}(T))$. Let $a \in \operatorname{Ann}(T)$ and $M \in \operatorname{Gen}(T)$. Then M is a factor of $T^{(I)}$ for some set I, and thus $aM = 0$. Conversely, note that $T \in \operatorname{Gen}(T)$, so if $a \in \operatorname{Ann}(\operatorname{Gen}(T))$ then also $a \in \operatorname{Ann}(T)$. The claim follows as T and any $M \in \operatorname{Gen}({}_A T)$ is trivially an $A/\operatorname{Ann}(T)$-module. □

Lemma 3.3. *Let T be an A-module, let $\bar{A} = A/\operatorname{Ann}(T)$ and suppose that $\operatorname{Gen}(T)$ is closed under extensions. Then $\operatorname{Ext}^1_A(T, \operatorname{Gen}(T)) = 0$ if and only if $\operatorname{Ext}^1_{\bar{A}}(T, \operatorname{Gen}(T)) = 0$.*

Proof. By lemma 3.2 we have $\operatorname{Gen}({}_A T) = \operatorname{Gen}({}_{\bar{A}} T)$, so we just write $\operatorname{Gen}(T)$. Suppose that $\operatorname{Ext}^1_A(T, \operatorname{Gen}(T)) = 0$, let $M \in \operatorname{Mod}(\bar{A})$ such that $M \in \operatorname{Gen}(T)$ and consider a short exact sequence in $\operatorname{Mod}(\bar{A})$.

$$0 \longrightarrow M \longrightarrow X \longrightarrow T \longrightarrow 0$$

Since $\operatorname{Gen}(T)$ is closed under extensions, also $X \in \operatorname{Gen}(T)$. Then the sequence can be taken in $\operatorname{Mod}(A)$, where it splits. Thus, it splits in $\operatorname{Mod}(\bar{A})$ as well, and then we have $\operatorname{Ext}^1_{\bar{A}}(T, \operatorname{Gen}(T)) = 0$.

Similarly one shows the other implication. □

A module is said to be **finendo** if it is finitely generated as a module over its endomorphism ring. The following proposition collects some characterizations of finendo modules which we will need. The first two results are from [CM93] and [CT95] respectively.

Proposition 3.4. *Let A be a unitary ring and Λ a finite dimensional k algebra over an algebraically closed field k. Then the following hold*

(1) A module $M \in \operatorname{Mod}(A)$ is finendo if and only if $\operatorname{Gen}(M)$ is closed under direct products.

(2) If a module $M \in \operatorname{Mod}(A)$ is tilting, then it is finendo..

Proof. (1) : See [CM93, Lemma on p.408].

(2) : See [CT95, Proposition 2.5]. □

Recall (definition 2.1 and proposition 2.22) that the existence of sequences with left approximations is a determining factor for a module to be tilting or support τ-tilting. The reason for considering quasitilting modules is because they classify torsion classes which admit left approximation sequences, and thus will turn up frequently in this section. For tilting modules, the left approximation is injective, but this is not always the case (for instance not for support τ-tilting modules). Recall that tilting modules are always faithful. We prove a lemma which says that for a module T and a left $\operatorname{Gen}(T)$-approximation of A, the approximation is injective if and only if T is faithful.

Lemma 3.5. *Let $T \in \mathrm{Mod}(A)$, and suppose there is morphism*

$$A \xrightarrow{\phi} T'$$

such that $T' \in \mathrm{Add}(T)$ and ϕ is a left $\mathrm{Gen}(T)$-approximation of A. Then $\mathrm{Ann}(T) = \mathrm{Ker}(\phi)$.

Proof. We first prove the inclusion $\mathrm{Ann}(T) \subseteq \mathrm{Ker}(\phi)$. Since $T' \in \mathrm{Add}(T)$ there exists an A-module X such that $T' \oplus X \cong T^{(I)}$ for some set I. Let $a \in \mathrm{Ann}(T)$, then we have

$$aT' \oplus aX \cong aT^{(I)} = 0$$

so $\mathrm{Ann}(T) \subseteq \mathrm{Ann}(T')$. Then we have $\phi(a) = a\phi(1) = 0$, i.e. $\mathrm{Ann}(T) \subseteq \mathrm{Ker}(\phi)$.

To prove the other inclusion, we first prove that

$$K := \bigcap_{f:A\to T} \mathrm{Ker}(f) = \mathrm{Ann}(T). \tag{3.1}$$

Let $a \in \mathrm{Ann}(T)$, then for all morphisms $f : A \to T$ we have $f(a) = af(1) = 0$, so $\mathrm{Ann}(T) \subseteq K$.

Conversely, let $a \in K$ and note that for every $t \in T$ there exists a unique morphism $f : A \to T$ such that $f(1) = t$. Then we have

$$f(a) = af(1) = at = 0 \quad \text{for all } t \in T$$

and thus $K \subseteq \mathrm{Ann}(T)$, so equation 3.1 holds.

Since ϕ is a left $\mathrm{Gen}(T)$-approximation of A, any morphism $f : A \to T$ factors through ϕ as indicated by the following commutative diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "f"'] & T' \arrow[dl, dashed, "f'"] \\ T & \end{tikzcd}$$

and thus $\mathrm{Ker}(\phi) \subseteq K$. We then conclude with

$$\mathrm{Ann}(T) \subseteq \mathrm{Ker}(\phi) \subseteq \mathrm{Ann}(T).$$

□

We continue with the following proposition, showing the interplay between finendo quasitilting modules and torsion classes with important properties.

Proposition 3.6. *The following are equivalent for an A-module T.*

(1) T is a finendo quasitilting module.

(2) T is a tilting $A/\operatorname{Ann}(T)$-module and $\operatorname{Gen}(T)$ is a torsion class containing all injective A-modules.

(3) T is Ext-projective in $\operatorname{Gen}(T)$ and there is an exact sequence

$$A \xrightarrow{\phi} T_0 \longrightarrow T_1 \longrightarrow 0$$

where $T_0, T_1 \in \operatorname{Add}(T)$ and ϕ is a left $\operatorname{Gen}(T)$-approximation.

Proof. Let $\bar{A} = A/\operatorname{Ann}(T)$. Note that by lemma 3.1 and lemma 2.12, $\operatorname{Gen}(T)$ is a torsion class in all three statements.

(1) $\Rightarrow$ (2): Since $\operatorname{Gen}({}_{\bar{A}}T) = \operatorname{Gen}({}_{A}T)$, the module T is finendo quasitilting over A if and only if it is finendo quasitilting over $\bar{A}$. Therefore, without loss of generality it is sufficient to show that any faithful finendo quasitilting A-module is a tilting A-module. We write $\operatorname{Gen}(T) = \operatorname{Gen}({}_{\bar{A}}T) = \operatorname{Gen}({}_{A}T)$. Because T is finendo, there is an exact sequence

$$\operatorname{End}_A(T)^n \longrightarrow T \longrightarrow 0$$

and because it is faithful we have that A is a subring of $\operatorname{End}_A(T)$. Let $R := \operatorname{End}_A(T)$. Now apply $\operatorname{Hom}_R(-,T)$ to the sequence above to get the following commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & {}_A A \arrow[r] \arrow[d] & {}_R R \arrow[d] \\ 0 \arrow[r] & {}_R R \arrow[r] & \operatorname{Hom}_R(R^{(n)},T) \end{tikzcd} \tag{3.2}$$

and since n is finite we have

$$\operatorname{Hom}_R(R^{(n)},T) \cong \bigoplus_n \operatorname{Hom}_R(R,T).$$

Let $\Phi : \operatorname{Hom}_R(R,T) \to T$ be defined by $\Phi(f) = f(1_R)$. The R-homomorphism Φ is an isomorphism, so then we have

$$\operatorname{Hom}_R(R^{(n)},T) \cong T^n$$

Therefore, by diagram (3.2) there is an injection $\phi : A \to T^{(n)}$. Now, let E be any injective A-module and $\psi : A^{(I)} \to E$ a surjection, then ψ factors through $\phi^{(I)} : A^{(I)} \to (T^{(n)})^{(I)}$ via a surjection as indicated by the following commutative diagram.

$$\begin{tikzcd} 0 \arrow[r] & A^{(I)} \arrow[r, "\phi^{(I)}"] \arrow[d, "\psi"] & (T^n)^{(I)} \arrow[dl, dashed] \\ & E & \end{tikzcd}$$

Then $E \in \mathrm{Gen}(T)$. By [Rot09, Prop. 3.38], which states that every A-module can be embedded as a submodule in an injective A-module, we have $\mathrm{Sub}(\mathrm{Gen}(T)) = \mathrm{Mod}(A)$ since $\mathrm{Gen}(T)$ contains all injective A-modules. Therefore, since T is quasitilting, we have

$$\mathrm{Gen}(T) = \mathrm{Sub}(\mathrm{Gen}(T)) \cap T^{\perp_1} = \mathrm{Mod}(A) \cap T^{\perp_1} = T^{\perp_1}$$

So T is tilting over A.

(2) $\Rightarrow$ (1) : Since T is tilting over $\bar{A}$, we have $\mathrm{Gen}(T) = T^{\perp_1}$, and thus $\mathrm{Pres}(T) = \mathrm{Gen}(T)$. Indeed, let $M \in \mathrm{Gen}(T)$ and $u : T^{(I)} \to M$ the universal surjective map for $I = \mathrm{Hom}_A(T, M)$. Then, by similar arguments as those used in the last part of the proof of lemma 3.1, it can be shown that $\mathrm{Ker}(u) \in \mathrm{Gen}(T)$ which induces an $\mathrm{Add}(T)$-presentation of M. By proposition 3.4(2) we have that T is finendo, and by lemma 3.3 we have that T is Ext-projective in $\mathrm{Gen}(T)$ as an A-module.

(2) $\Rightarrow$ (3) : Since T is tilting over $\bar{A}$ we have a short exact sequence

$$0 \longrightarrow \bar{A} \xrightarrow{\phi} T_0 \longrightarrow T_1 \longrightarrow 0$$

where $T_0, T_1 \in \mathrm{Add}(T)$ and ϕ is a left $\mathrm{Gen}(T)$-approximation of $\bar{A}$. Compose ϕ with the projection $\pi : A \to \bar{A}$ to get $\psi := \phi\pi : A \to T_0$. We show that ψ is a left $\mathrm{Gen}(T)$-approximation in $\mathrm{Mod}(A)$. Let $M \in \mathrm{Gen}(T)$ and $f \in \mathrm{Hom}_A(A, M)$, then since $\mathrm{Ann}(T) = \mathrm{Ann}(\mathrm{Gen}(T))$ by lemma 3.2, there exists a map $f' : \bar{A} \to M$ with $f'(a + \mathrm{Ann}(T)) = f(a)$. Then, $f'\pi = f$. Furthermore, $M \in \mathrm{Gen}(T)$ also as a $\bar{A}$-module. So the map f' factors through ϕ via some map $f'' : T_0 \to M$, i.e. $f''\phi = f'$, as indicated by the following diagram

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[d, "f"] & \bar{A} \arrow[r, "\phi"] \arrow[dl, dashed, "f'"'] & T_0 \arrow[dll, dashed, "f''"] \\ M & & \end{tikzcd}$$

Then $f = f''\phi\pi = f''\psi$, so ψ is a left $\mathrm{Gen}(T)$-approximation A. Finally, T is Ext-projective in $\mathrm{Gen}(T)$ by lemma 3.3.

(3) $\Rightarrow$ (2) : $\mathrm{Gen}(T)$ is a torsion class and contained in $\mathrm{Ker}(\mathrm{Ext}^1_{\bar{A}}(T, -))$ by lemma 3.3. By assumption, there is an exact sequence

$$A \xrightarrow{\phi} T_0 \longrightarrow T_1 \longrightarrow 0$$

By lemma 3.5 we have $\mathrm{Ann}(T) = \mathrm{Ker}(\phi)$, so we get a short exact sequence

$$0 \longrightarrow \bar{A} \xrightarrow{\psi} T_0 \longrightarrow T_1 \longrightarrow 0$$

Finally, let $X \in \mathrm{Ker}(\mathrm{Ext}^1_{\bar{A}}(T, -))$ and apply $\mathrm{Hom}_{\bar{A}}(-, X)$ to the short exact sequence above to get the following long exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\bar{A}}(T_1, X) \longrightarrow \mathrm{Hom}_{\bar{A}}(T_0, X) \xrightarrow{\psi^*} \mathrm{Hom}_{\bar{A}}(\bar{A}, X) \longrightarrow 0 \longrightarrow \dots$$

There is a surjective map $g : \bar{A}^{(I)} \to X$, and since ψ^* is surjective it factors through some map $g' : T_0^{(I)} \to X$ which must also be surjective. So then $X \in \operatorname{Gen}(T)$, and

$$\operatorname{Gen}(T) = \operatorname{Ker}(\operatorname{Ext}^1_{\bar{A}}(T, -)) = {}_{\bar{A}}T^{\perp_1}$$

In other words, T is tilting over $\bar{A} = A/\operatorname{Ann}(T)$. □

There exists an equivalence relation on the class of quasitilting modules, which is given by the following lemma. For our purposes, the existence of such a relation is important because it gives an equivalence relation on silting modules in section 3.2.

Lemma 3.7. *Let T be a quasitilting A-module. Then $\operatorname{Add}(T)$ is the class of Ext-projective modules in $\operatorname{Gen}(T)$.*

Proof. If T is Ext-projective in $\operatorname{Gen}(T)$, then so are the modules in $\operatorname{Add}(T)$. Conversely, if M is Ext-projective in $\operatorname{Gen}(T) = \operatorname{Pres}(T)$, then there is a surjection $f : T' \to M$ where $\operatorname{Ker}(f) \in \operatorname{Gen}(T)$. By applying $\operatorname{Hom}_A(-, \operatorname{Ker}(f))$ to the short exact sequence induced by f one sees that it splits, and thus $M \in \operatorname{Add}(T)$. □

So, for a quasitilting module T, its associated torsion class, i.e. $\operatorname{Gen}(T)$, completely determines the class $\operatorname{Add}(T)$. In particular, if T_1, T_2 are two quasitilting modules, then we have $\operatorname{Add}(T_1) = \operatorname{Add}(T_2)$ if and only if $\operatorname{Gen}(T_1) = \operatorname{Gen}(T_2)$. Therefore, we will say that the two quasitilting modules are **equivalent** if $\operatorname{Add}(T_1) = \operatorname{Add}(T_2)$.

Theorem 3.8. *The following are equivalent for a torsion class $\mathcal{T}$ in $\operatorname{Mod}(A)$.*

(1) For every A-module M, there is a sequence

$$M \xrightarrow{\phi} B \longrightarrow C \longrightarrow 0$$

such that ϕ is a left $\mathcal{T}$-approximation of M and C is Ext-projective in $\mathcal{T}$.

(2) There is a finendo quasitilting A-module T such that $\operatorname{Gen}(T) = \mathcal{T}$.

In particular, there is a bijection between equivalence classes of finendo quasitilting A-modules and torsion classes $\mathcal{T}$ in $\operatorname{Mod}(A)$ such that every A-module has a left $\mathcal{T}$-approximation with Ext-projective cokernel.

Proof. (1)⇒ (2): Set $M = A$ with the given exact sequence

$$A \xrightarrow{\phi} B \longrightarrow C \longrightarrow 0$$

and let $T = B \oplus C$. Now, $B \in \mathcal{T}$ by assumption, and $C \in \mathcal{T}$ because it is a factor of B, so $T \in \mathcal{T}$ and thus $\operatorname{Gen}(T) \subseteq \mathcal{T}$. Conversely, let $X \in \mathcal{T}$, then any surjection $f : A^{(I)} \to X$

factors through the left $\mathcal{T}$-approximation $\phi^{(I)}$ via a surjection $B^{(I)} \to X$ as indicated by the following commutative diagram.

$$\begin{tikzcd}
A^{(I)} \arrow[r, "\phi^{(I)}"] \arrow[d, "f"'] & B^{(I)} \arrow[r] \arrow[dl, dashed] & C^{(I)} \\
X & &
\end{tikzcd}$$

Thus $X \in \operatorname{Gen}(T)$ since $B^{(I)} \in \operatorname{Add}(T)$, and so $\mathcal{T} = \operatorname{Gen}(T)$.

Due to proposition 3.6 it remains to show that T is Ext-projective in $\operatorname{Gen}(T)$, but this property holds for C by assumption, so it suffices to show it for B.

By lemma 3.5 we have $\operatorname{Ker}(\phi) = \operatorname{Ann}(T)$. Let $\bar{A} = A/\operatorname{Ann}(T)$ as before, then there is a short exact sequence in $\operatorname{Mod}(\bar{A})$

$$0 \longrightarrow \bar{A} \xrightarrow{\bar{\phi}} B \longrightarrow C \longrightarrow 0$$

We have by assumption that $\operatorname{Gen}(T) \subseteq \operatorname{Ker}(\operatorname{Ext}^1_A(C,-))$. But $\operatorname{Gen}({}_AT) = \operatorname{Gen}({}_{\bar{A}}T)$ by lemma 3.2, and since $C \in \operatorname{Gen}(T)$, it is an $\bar{A}$-module, and then we have

$$\operatorname{Gen}(T) \subseteq \operatorname{Ker}(\operatorname{Ext}^1_{\bar{A}}(C,-))$$

by lemma 3.3. Now, let $X \in \operatorname{Gen}(T)$, and apply $\operatorname{Hom}_{\bar{A}}(-,X)$ to the short exact sequence above to get the long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_{\bar{A}}(C,X) \arrow[r] & \operatorname{Hom}_{\bar{A}}(B,X) \arrow[r] & \operatorname{Hom}_{\bar{A}}(\bar{A},X) \arrow[dll, rounded corners] & \\
& \underbrace{\operatorname{Ext}^1_{\bar{A}}(C,X)}_{=0} \arrow[r] & \operatorname{Ext}^1_{\bar{A}}(B,X) \arrow[r] & \underbrace{\operatorname{Ext}^1_{\bar{A}}(\bar{A},X)}_{=0} \arrow[r] & \dots
\end{tikzcd}$$

The last term is 0 since $\bar{A}_{\bar{A}}$ is projective in $\operatorname{Mod}(\bar{A})$. Then we have $\operatorname{Ext}^1_{\bar{A}}(B,X) = 0$ for all $X \in \operatorname{Gen}(T)$, so we have $\operatorname{Gen}(T) \subseteq \operatorname{Ker}(\operatorname{Ext}^1_{\bar{A}}(B,-))$.

Finally, let $X \in \operatorname{Gen}(T)$ again, and suppose there is a short exact sequence in $\operatorname{Mod}(A)$.

$$0 \longrightarrow X \longrightarrow N \longrightarrow B \longrightarrow 0$$

Since $\operatorname{Gen}(T) = \mathcal{T}$ is closed under extensions we have $N \in \operatorname{Gen}(T)$. Furthermore, since $\operatorname{Gen}({}_AT) = \operatorname{Gen}({}_{\bar{A}}T)$ by lemma 3.2, the sequence is short exact in $\operatorname{Mod}(\bar{A})$. We have established that $\operatorname{Ext}^1_{\bar{A}}(B, \operatorname{Gen}(T)) = 0$, so the sequence splits in $\operatorname{Mod}(\bar{A})$. Thus it splits over $\operatorname{Mod}(A)$ as well, and so $\operatorname{Ext}^1_A(B, \operatorname{Gen}(T)) = 0$ as desired.

(2) $\Rightarrow$ (1): By Proposition 3.6 there is exact sequence

$$A \xrightarrow{\phi} T_0 \longrightarrow T_1 \longrightarrow 0$$

where $T_0, T_1 \in \mathrm{Add}(T)$ and ϕ is a left $\mathrm{Gen}(T)$-approximation of A. Let $M \in \mathrm{Mod}(A)$ and $\pi : A^{(I)} \to M$ a surjection. Consider the pushout diagram

$$\begin{tikzcd}
A^{(I)} \arrow[r, "\phi^{(I)}"] \arrow[d, "\pi"] & T_0^{(I)} \arrow[r] \arrow[d, "\sigma"] & T_1^{(I)} \arrow[r] \arrow[d, "\wr"] & 0 \\
M \arrow[r, "\phi'"] & B \arrow[r] & \mathrm{Cok}(\phi') \arrow[r] & 0
\end{tikzcd}$$

Since π is surjective, so is σ and therefore $B \in \mathrm{Gen}(T)$. Let $X \in \mathrm{Gen}(T)$ and consider a morphism $f : M \to X$. The composition $f\pi$ factors through the $\mathrm{Gen}(T)$-approximation $\phi^{(I)}$, but then the pushout property yields a morphism $\theta : B \to X$ such that $f = \theta\phi'$, that is ϕ' is in fact a left $\mathrm{Gen}(T)$-approximation of M.

$$\begin{tikzcd}
A^{(I)} \arrow[r, "\phi^{(I)}"] \arrow[d, "\pi"] & T_0^{(I)} \arrow[d, "\sigma"] \arrow[ddr, bend left, "f'"] & \\
M \arrow[r, "\phi'"] \arrow[drr, bend right, "f"'] & B \arrow[dr, dashed, "\theta"] & \\
& & X
\end{tikzcd}$$

Clearly $C := \mathrm{Cok}(\phi')$ is Ext-projective in $\mathrm{Gen}(T)$ as it is isomorphic to $T_1^{(I)}$. So the desired sequence is

$$M \xrightarrow{\phi'} B \longrightarrow C \longrightarrow 0$$

□

We finish this section with proving that the statements (1) and (3) in lemma 2.41 are also equivalent statements in $\mathrm{Mod}(A)$. This will be used later, and even though the proof is very similar, we include it for completion.

Lemma 3.9. *Let $M, N \in \mathrm{Mod}(A)$ and let $P_{-1} \xrightarrow{p_{-1}} P_0 \xrightarrow{p_0} M \to 0$ and $Q_{-1} \xrightarrow{q_{-1}} Q_0 \xrightarrow{q_0} N \to 0$ be projective presentations of M and N respectively. Let P and Q denote the 2-term complexes of the projective presentations of M and N respectively. Then the following are equivalent.*

(1) $\mathrm{Hom}_A(P_0, N) \xrightarrow{p_{-1}^*} \mathrm{Hom}_A(P_{-1}, N)$ *is surjective.*

(2) $\mathrm{Hom}_{D(A)}(P, Q[1]) = 0$.

Proof. (1) $\Rightarrow$ (2): Let $f \in \mathrm{Hom}_{D(A)}(P, Q[1])$, then it is determined by some morphism $f \in \mathrm{Hom}_A(P_{-1}, Q_0)$. By assumption, the composition $q_0 f$ factors through p_{-1} via some morphism $f' \in \mathrm{Hom}_A(P_0, N)$. But since P_0 is projective and q_0 is surjective, f' lifts to a morphism $h_0 \in \mathrm{Hom}_A(P_0, Q_0)$ such that $f' = q_0 h_0$, as indicated by the following diagram

$$\begin{tikzcd}
 & P_{-1} \arrow[r, "p_{-1}"] \arrow[d, "f"] & P_0 \arrow[dl, "h_0"] \arrow[dl, bend left, "f'", to path={-- (\tikztotarget)}, phantom] \arrow[ddl, bend left, "f'"] \\
Q_{-1} \arrow[r, "q_{-1}"] & Q_0 \arrow[d, "q_0"] & \\
 & N &
\end{tikzcd}$$

We then have

$$q_0 f = f' p_{-1} = q_0 h_0 p_{-1} \quad \Rightarrow \quad q_0(f - h_0 p_{-1}) = 0$$

and so $(f - h_0 p_{-1})$ factors through q_{-1} via some morphism $h_{-1} \in \mathrm{Hom}_A(P_{-1}, Q_{-1})$, yielding

$$f = q_{-1} h_{-1} + h_0 p_{-1}$$

(2) $\Rightarrow$ (1): Let $f \in \mathrm{Hom}_A(P_{-1}, N)$, and since $q_0 \in \mathrm{Hom}_A(Q_0, N)$ is surjective, f lifts to a morphism $f' \in \mathrm{Hom}_A(P_{-1}, Q_0)$ such that $f = q_0 f'$. But f' induces a null-homotopic morphism by assumption, so there are morphisms $h_i \in \mathrm{Hom}_A(P_i, Q_i)$ for $i = -1, 0$ such that

$$f' = h_0 p_{-1} + q_{-1} h_{-1}.$$

Then we conclude with

$$f = q_0 f' = q_0(h_0 p_{-1} + q_{-1} h_{-1}) = q_0 h_0 p_{-1}$$

i.e. p_{-1}^* is surjective. $\square$

3.2 Silting modules

We will now turn to the concept of (partial) silting modules. First, we define an important class of A-modules and prove some of its properties. For a morphism $\sigma \in \mathrm{Proj}(A)$, consider the class of A-modules

$$\mathcal{D}_\sigma := \{X \in \mathrm{Mod}(A) \mid \mathrm{Hom}_A(\sigma, X) \text{ is surjective}\}$$

Lemma 3.10. *Let $\sigma : P_{-1} \to P_0$ be a morphism in $\mathrm{Proj}(A)$ with $\mathrm{Cok}(\sigma) = T$.*

(1) The class $\mathcal{D}_\sigma$ is closed under factors, extensions, and direct products.

(2) The class $\mathcal{D}_\sigma$ is contained in $T^{\perp_1}$.

(3) An A-module X belongs to $\mathcal{D}_\sigma$ if and only if for some (resp. all) projective presentation(s) ω of X the condition $\mathrm{Hom}_{D(A)}(\sigma, \omega[1]) = 0$ is satisfied.

Proof. (1) :

Proving closure under epimorphic images. Suppose $f : X \to Y$ is a surjection with $X \in \mathcal{D}_\sigma$, and let $\phi : P_{-1} \to Y$ be any morphism. Because P_{-1} is projective and f is surjective, there is a morphism $\psi : P_{-1} \to X$ such that $\phi = f\psi$.

$$\begin{tikzcd}
& P_{-1} \arrow[r, "\sigma"] \arrow[d, "\phi"] \arrow[dl, dashed, "\psi"'] & P_0 \\
X \arrow[r, "f"] & Y &
\end{tikzcd}$$

But $X \in \mathcal{D}_\sigma$, so there is a morphism $\psi' : P_0 \to X$ such that $\psi = \psi'\sigma$.

$$\begin{tikzcd}
& P_{-1} \arrow[r, "\sigma"] \arrow[d, "\phi"] \arrow[dl, "\psi"'] & P_0 \arrow[dll, dashed, bend right, "\psi'"'] \\
X \arrow[r, "f"] & Y &
\end{tikzcd}$$

Then we conclude that $\phi = f\psi = f\psi'\sigma$, so $Y \in \mathcal{D}_\sigma$.

Proving closure under extensions. Let $X, Z \in \mathcal{D}_\sigma$ and suppose there is a short exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

Further suppose there is a morphism $p : P_{-1} \to Y$, then $gp : P_{-1} \to Z$ factors through σ since $Z \in \mathcal{D}_\sigma$, as shown in the following diagram with commutative square.

$$\begin{tikzcd}
& & P_{-1} \arrow[r, "\sigma"] \arrow[d, "p"] & P_0 \arrow[d, dashed, "p'"] & \\
0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0
\end{tikzcd}$$

The projectivity of P_0 gives a morphism $h_0 : P_0 \to Y$ such that $p' = gh_0$. Then $gp = gh_0\sigma$, and consequently $g(p - h_0\sigma) = 0$. Therefore, there is a map $h_{-1} : P_{-1} \to X$ such that $fh_{-1} = (p - h_0\sigma)$. Then, because $X \in \mathcal{D}_\sigma$, there is a morphism $q : P_0 \to X$ such that $h_{-1} = q\sigma$.

$$\begin{tikzcd}
& & P_{-1} \arrow[r, "\sigma"] \arrow[d, "p"] \arrow[dl, "h_{-1}"'] & P_0 \arrow[d, "p'"] \arrow[dl, "h_0"'] \arrow[dll, bend right, "q"'] & \\
0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0
\end{tikzcd}$$

Then we conclude that

$$p = fh_{-1} + h_0\sigma = fq\sigma + h_0\sigma = (fq + h_0)\sigma$$

so $Y \in \mathcal{D}_\sigma$.

Proving closure under direct products. Suppose $X_i \in \mathcal{D}_\sigma$ for all i in some set I. Let $f_i : P_{-1} \to X_i$ be morphisms for each i which factor through σ as $f_i = f_i'\sigma$. Consider the product morphisms

$$\Pi_I f_i : P_{-1} \to \Pi_I X_i \quad \text{and} \quad \Pi_I f_i' : P_0 \to \Pi_i X_i$$

Writing them in matrix form, we have

$$(\Pi_i f_i')\sigma = \begin{pmatrix} f_1'\sigma & 0 & 0 & \dots & 0 \\ 0 & f_2'\sigma & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix} = \begin{pmatrix} f_1 & 0 & 0 & \dots & 0 \\ 0 & f_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix} = \Pi_i f_i$$

That is $\Pi_I X_i \in \mathcal{D}_\sigma$.

(2) : Consider the factorisation of $\sigma = i\pi$ through its image, with $\pi : P_{-1} \to \mathrm{Im}(\sigma)$ and $i : \mathrm{Im}(\sigma) \to P_0$, which induces a short exact sequence

$$0 \longrightarrow \mathrm{Im}(\sigma) \xrightarrow{i} P_0 \longrightarrow T \longrightarrow 0$$

Let $X \in \mathcal{D}_\sigma$, then applying $\mathrm{Hom}_A(-, X)$ to the short exact sequence above we get a long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_A(T,X) \arrow[r] & \mathrm{Hom}_A(P_0,X) \arrow[r, "i^*"] & \mathrm{Hom}_A(\mathrm{Im}(\sigma),X) \arrow[dll, rounded corners] \\ & \mathrm{Ext}^1_A(T,X) \arrow[r] & 0 \arrow[r] & \dots \end{tikzcd}$$

We will prove that i^* is surjective, which completes the proof. Suppose there is a morphism $f : \mathrm{Im}(\sigma) \to X$. Because $X \in \mathcal{D}_\sigma$, the composition $f\pi$ factors through σ via some morphism $g : P_0 \to X$, as indicated in the following commutative diagram.

$$\begin{tikzcd} P_{-1} \arrow[r, "\sigma"] \arrow[d, "\pi"] & P_0 \arrow[d, dashed, "g"] \\ \mathrm{Im}(\sigma) \arrow[r, "f"] & X \end{tikzcd}$$

Then we have

$$f\pi = g\sigma = gi\pi$$

but since π is an epimorphism we get $f = gi$ which proves that i^* is surjective.

(3): Suppose that $X \in \mathcal{D}_\sigma$. Then for all projective presentations ω of X we have $\mathrm{Hom}_{D(A)}(\sigma, \omega[1]) = 0$ by lemma 3.9. Conversely, if there exists a projective presentation ω of X such that $\mathrm{Hom}_{D(A)}(\sigma, \omega[1]) = 0$, then by lemma 3.9 we have $X \in \mathcal{D}_\sigma$. $\square$

We are now ready to define silting modules.

Definition 3.11. We call an A-module T

- **partial silting** if there is a projective presentation σ of T such that

  (S1) $\mathcal{D}_\sigma$ is a torsion class.

  (S2) $T \in \mathcal{D}_\sigma$.

- **silting** if there is a projective presentation σ of T such that $\mathrm{Gen}(T) = \mathcal{D}_\sigma$.

We say that T is (partial) silting with respect to σ, and the class $\mathrm{Gen}(T)$ is then called a **silting class**.

Remark 3.12. (1) Because $\mathcal{D}_\sigma$ is always closed under factors and extensions by lemma 3.10(1), it is then a torsion class if and only if it is closed under coproducts by remark 2.6.

(2) Note that the definitions in 3.11 depend on the choice of σ, as not all projective presentations will satisfy the conditions (S1) and (S2). We will give some examples demonstrating this later.

Following up on remark 3.12(1), the class $\mathcal{D}_\sigma$ for $\sigma : P_{-1} \to P_0$ is closed under coproducts if $\mathrm{Hom}_A(P_i, -)$ preserves coproducts for $i = -1, 0$. This property on P_i is known under several names, **small** and **dually slender** being the most common ones. For an A-modules M and any family of A-modules $\{N_i\}_{i\in I}$, there is a canonical injection $\Phi : \bigoplus_{i\in I} \mathrm{Hom}_A(M, N_i) \to \mathrm{Hom}_A(M, \bigoplus_{i\in I} N_i)$. Then M is small if and only if Φ is a surjection. It is useful to know when modules are small so that one has more control over when $\mathcal{D}_\sigma$ is a torsion class. We prove a lemma which provides some cases of when modules are small.

Lemma 3.13. *Let A be a ring, then the following holds*

(1) The module ${}_AA$ is small.

(2) Any finitely generated A-module is small.

(3) Any finitely presented A-module is small.

Proof. (1) : Let $\{M_i\}_{i\in I}$ be an infinite family of A-modules, and let $f \in \mathrm{Hom}_A({}_AA, \bigoplus_{i\in I} M_i)$. Then f is an infinite family of maps $(f_i)_{i\in I}$ where $f_i : {}_AA \to M_i$. Clearly, $f_i(1) = 0$ if and only if $f_i = 0$. Suppose that $f_i \neq 0$ for all $i \in I$, so then $f(1) = \bigoplus_{i\in I} f_i(1) \in \bigoplus_{i\in I} M_i$ has infinitely many non-zero components. This contradicts the structure of the coproduct $\bigoplus_{i\in I} M_i$, so then $(f_i)_{i\in I}$ has in fact only finitely many non-zero components. The map Φ above is then surjective, and ${}_AA$ is small.

(2) : Let M be an A-module, and suppose it is not small. We show that then M cannot be finitely generated. Let $\{X_i\}_{i\in I}$ be an infinite family of A-modules and let $\pi_j : \bigoplus_{i\in I} X_i \to X_j$

be the canonical projection. Then since M is not small, the map $\Phi : \bigoplus_{i\in I} \operatorname{Hom}_A(M, X_i) \to \operatorname{Hom}_A(M, \bigoplus_{i\in I} X_i)$ is not surjective, so there is some map $f \in \operatorname{Hom}_A(M, \bigoplus_{i\in I} X_i)$ such that $\pi_j f \neq 0$ for infinitely many j. Without loss of generality, suppose $I = \mathbb{N}$, and let $M_n = \{m \in M \mid \pi_i f(m) = 0 \,\forall i \geq n\}$. Then $M = \bigcup_{i\in\mathbb{N}} M_n$, but clearly $M \neq M_n$ for all $n \in \mathbb{N}$. In particular, M cannot be finitely generated.

(3) : Any finitely presented module is finitely generated, so it follows from (2). □

Remark 3.14. If Λ is a finite dimensional k-algebra over an algebraically closed field k as in sections 2.2 and 2.3, then a minimal projective presentation σ of a module $T \in \operatorname{mod}(\Lambda)$ is a map in $\operatorname{proj}(\Lambda)$, and thus always yields a torsion class $\mathcal{D}_\sigma$.

We saw in lemma 2.12 that partial tilting modules give rise to torsion pairs, and the following lemma shows that partial silting modules share that property. In particular, the torsion class is given by $\operatorname{Gen}(T)$ in both cases.

Corollary 3.15. *If $T \in \operatorname{Mod}(A)$ is partial silting, then $\operatorname{Gen}(T) \subseteq \mathcal{D}_\sigma \subseteq T^{\perp_1}$ and $(\operatorname{Gen}(T), T^{\perp_0})$ is a torsion pair. Furthermore, every silting module is partial silting.*

Proof. Let $T \in \operatorname{Mod}(A)$ be partial silting, then since $\mathcal{D}_\sigma$ is a torsion class, $T^{(I)} \in \mathcal{D}_\sigma$ for all sets I. Then, also all factors $T^{(I)}/K \in \mathcal{D}_\sigma$, so $\operatorname{Gen}(T) \subseteq \mathcal{D}_\sigma$. By lemma 3.10 we have $\operatorname{Gen}(T) \subseteq T^{\perp_1}$ and by lemma 2.12, the pair $(\operatorname{Gen}(T), T^{\perp_0})$ is a torsion pair.

Now, let T be silting. As above, combining the lemmas 3.10 and 2.12, we get that T is partial silting. □

The definition of silting modules clearly resembles the definition $\operatorname{Gen}(T) = T^{\perp_1}$ for tilting modules. We prove a proposition relating tilting, silting and quasitilting modules.

Proposition 3.16. *The following hold in* $\operatorname{Mod}(A)$.

(1) All tilting A-modules are silting A-modules.

(2) All silting A-modules are finendo quasitilting A-modules.

Proof. (1): Let $T \in \operatorname{Mod}(A)$ be tilting and suppose that $\operatorname{p.d.}(T) = 1$. Take a projective presentation σ of T, which is injective.

$$0 \longrightarrow P_{-1} \xrightarrow{\sigma} P_0 \longrightarrow T \longrightarrow 0$$

Let $X \in \operatorname{Mod}(A)$ and apply $\operatorname{Hom}_A(-, X)$ to the short exact sequence above to get a long exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(T, X) \longrightarrow \operatorname{Hom}_A(P_0, X) \xrightarrow{\sigma^*} \operatorname{Hom}_A(P_{-1}, X) \longrightarrow \operatorname{Ext}^1_A(T, X) \longrightarrow \dots$$

By assumption we have $\mathrm{Gen}(T) = T^{\perp_1}$. Suppose that $X \in \mathcal{D}_\sigma$, then σ^* is surjective and so $X \in T^{\perp_1} = \mathrm{Gen}(T)$. Conversely, suppose that $X \in \mathrm{Gen}(T) = T^{\perp_1}$, then σ^* is surjective and $X \in \mathcal{D}_\sigma$. Thus, $\mathrm{Gen}(T) = \mathcal{D}_\sigma$ and T is silting.

If $\mathrm{p.d.}(T) = 0$, then T is projective and thus $T^{\perp_1} = \mathrm{Mod}(A)$. Furthermore, the projective presentation of T is $0 \to T$, and clearly we have $X \in \mathcal{D}_0$ for every $X \in \mathrm{Mod}(A)$.

(2): Let $T \in \mathrm{Mod}(A)$ be silting with respect to a projective presentation σ. Then $\mathrm{Gen}(T) = \mathcal{D}_\sigma$ is closed under direct products and $\mathcal{D}_\sigma \subseteq T^{\perp_1}$ by lemma 3.10, so T is Ext-projective in $\mathrm{Gen}(T)$, and also T is finendo by proposition 3.4(1). It then remains to show that $\mathrm{Pres}(T) = \mathrm{Gen}(T)$ by lemma 3.1(1). Recall that $\mathrm{Pres}(T) \subseteq \mathrm{Gen}(T)$, so we need to show $\mathrm{Gen}(T) \subseteq \mathrm{Pres}(T)$.

Let $M \in \mathrm{Gen}(T)$, let $I = \mathrm{Hom}_A(T, M)$ and take the universal morphism $u : T^{(I)} \to M$. Let $K := \mathrm{Ker}(u)$ and $k : K \to T^{(I)}$ be the canonical inclusion. We will show that

$$K \in \mathcal{D}_\sigma = \mathrm{Gen}(T)$$

which then yields an $\mathrm{Add}(T)$-presentation of M.

Suppose there is a morphism $f : P_{-1} \to K$ and consider the composition kf. Since $T^{(I)} \in \mathcal{D}_\sigma$, there is a morphism $g : P_0 \to T^{(I)}$ such that $kf = g\sigma$. Furthermore, since $ug\sigma = ukf = 0$, the morphism ug factors through $T = \mathrm{Cok}(\sigma)$ via some morphism $h : T \to M$, i.e. the following diagram commutes.

$$\begin{tikzcd}
 & P_{-1} \arrow[r, "\sigma"] \arrow[d, "f"] & P_0 \arrow[r, "\pi"] \arrow[d, "g"] & T \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & K \arrow[r, "k"] & T^{(I)} \arrow[r, "u"] & M \arrow[r] & 0
\end{tikzcd}$$

By the universal property of u, there exists a morphism $h' : T \to T^{(I)}$ such that $h = uh'$. Then we have

$$ug = h\pi = uh'\pi \quad \Rightarrow \quad u(g - h'\pi) = 0$$

So there is a morphism $g' : P_0 \to K$ such that

$$g - h'\pi = kg'$$

and then we have

$$kf = g\sigma = (kg' + h'\pi)\sigma = kg'\sigma$$

Since k is injective, we have $f = g'\sigma$, and so $K \in \mathcal{D}_\sigma = \mathrm{Gen}(T)$. Then there is a surjection $v : T^{(J)} \to K$, which induces an $\mathrm{Add}(T)$-presentation of M

$$T^{(J)} \xrightarrow{kv} T^{(I)} \xrightarrow{u} M \longrightarrow 0$$

$\square$

Recall that quasitilting modules are equivalent if they have the same additive closure. Consequently, as all silting modules are finendo quasitilting, we say that two silting modules T_1 and T_2 are **equivalent** if $\mathrm{Add}(T_1) = \mathrm{Add}(T_2)$.

As was shown in the proposition above, tilting modules are examples of silting modules. The following proposition provides further connections between tilting and silting modules.

Proposition 3.17.

(1) An A-module T is (partial) tilting if and only if T is (partial) silting with respect to an injective projective presentation σ.

(2) The following are equivalent for an A-module T.

(i) T is tilting.

(ii) T is faithful silting.

(iii) T is faithful finendo quasitilting.

Proof. (1) : The 'only if' direction is clear since all (partial) tilting modules are (partial) silting by proposition 3.16. We show the 'if' direction for a partial silting module, as the proof for silting modules follows the same line of arguments. So, let T be a partial silting A-module with respect to an injective projective presentation $\sigma : P_{-1} \to P_0$. Then, $\mathcal{D}_\sigma$ is a torsion class and $T \in \mathcal{D}_\sigma$. It is then sufficient to show that $\mathcal{D}_\sigma = T^{\perp_1}$. So take any $X \in \mathrm{Mod}(A)$, and apply $\mathrm{Hom}_A(-, X)$ to the short exact sequence

$$0 \longrightarrow P_{-1} \xrightarrow{\sigma} P_0 \longrightarrow T \longrightarrow 0$$

to get a long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_A(T,X) \arrow[r] & \mathrm{Hom}_A(P_0,X) \arrow[r, "\sigma^*"] & \mathrm{Hom}_A(P_{-1},X) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}] \\ & \mathrm{Ext}^1_A(T,X) \arrow[r] & 0 \arrow[r] & \dots \end{tikzcd}$$

Suppose that $X \in T^{\perp_1}$, then σ^* is surjective and $X \in \mathcal{D}_\sigma$. Conversely, if $X \in \mathcal{D}_\sigma$, then σ^* is surjective and $X \in T^{\perp_1}$.

(2) : $(i) \Rightarrow (ii)$: All tilting modules are silting by proposition 3.16 and faithful by lemma 2.3.

$(ii) \Rightarrow (iii)$: All silting modules are finendo quasitilting by proposition 3.16, the claim follows.

$(iii) \Rightarrow (i)$: By proposition 3.6, we have in particular that T is a tilting module over $A/\mathrm{Ann}(T)$. But $\mathrm{Ann}(T) = 0$ since T is faithful, so T is a tilting module over A. $\square$

For an A-module T, the definition for silting $\mathrm{Gen}(T) = \mathcal{D}_\sigma$ resembles the definition for tilting $\mathrm{Gen}(T) = T^{\perp_1}$, as was mentioned earlier. There is also a strong resemblence between the axioms ($S1$) and ($S2$) for partial silting modules and axioms ($PT1$) and ($PT2$) for partial tilting modules, and also axioms ($T1$) and ($T2$) for tilting modules. The following proposition provides a third axiom ($S3$) for silting modules, which is analog to ($T3$) for tilting modules.

Proposition 3.18. *Let $T \in \mathrm{Mod}(A)$ with a projective presentation σ. Then the following are equivalent.*

(1) T is a silting module with respect to σ.

(2) T is a partial silting module with respect to σ, and

(S3) there is an exact sequence

$$A \xrightarrow{\phi} T_0 \longrightarrow T_1 \longrightarrow 0$$

where $T_0, T_1 \in \mathrm{Add}(T)$ and ϕ is a left $\mathcal{D}_\sigma$-approximation.

Proof. (1) $\Rightarrow$ (2): By corollary 3.15 T is partial silting. Furthermore, by propostion 3.16, T is finendo quasitilting and then by proposition 3.6, there is an exact sequence

$$A \xrightarrow{\phi} T_0 \longrightarrow T_1 \longrightarrow 0$$

where $T_0, T_1 \in \mathrm{Add}(T)$ and ϕ is a left $\mathrm{Gen}(T)$-approximation of A. The claim follows since $\mathrm{Gen}(T) = \mathcal{D}_\sigma$.

(2) $\Rightarrow$ (1): Let $M \in \mathrm{Gen}(T)$. Then there is a surjection $f : T^{(I)} \to M$ for some set I, and so $M \cong T^{(I)}/\mathrm{Ker}(f)$. Since $\mathcal{D}_\sigma$ is a torsion class and $T^{(I)} \in \mathcal{D}_\sigma$ we have $M \in \mathcal{D}_\sigma$.

Conversely, let $M \in \mathcal{D}_\sigma$ and consider any surjection $f : A^{(I)} \to M$. Then f factors through the left $\mathcal{D}_\sigma$-approximation $\phi^{(I)}$, say as $f = f'\phi^{(I)}$, where $f' : T_0^{(I)} \to M$ must be surjective. Thus $M \in \mathrm{Gen}(T)$. □

We now include a small lemma which will be used in the proof of theorem 3.20.

Lemma 3.19.

(1) Let $(\theta_i)_{i \in I}$ be a family of maps in $\mathrm{Proj}(A)$ and $\theta = \bigoplus_{i \in I} \theta_i$. Then $\mathcal{D}_\theta = \bigcap_{i \in I} \mathcal{D}_{\theta_i}$.

(2) Let $\alpha : Q_{-1} \to Q_0$ and $\beta : Q_{-1} \to Q'_0$ be maps in $\mathrm{Proj}(A)$, and

$$\begin{aligned} (\alpha, \beta) : Q_{-1} &\to Q_0 \oplus Q'_0 \\ q &\mapsto (\alpha(q), \beta(q)) \end{aligned}$$

Then $\mathcal{D}_\alpha \subseteq \mathcal{D}_{(\alpha,\beta)}$.

Proof. For the sake of simplicity, we write $\bigoplus_I$ and $\bigcap_I$ instead of $\bigoplus_{i\in I}$ and $\bigcap_{i\in I}$ respectively.

(1) : Set $\theta_i : P_i \to Q_i$ and $\theta : P = \bigoplus_I P_i \to \bigoplus_I Q_i = Q$. Let $X \in \mathcal{D}_\theta$ and take a map $f : P \to X$. Then f factors through θ via some map $f' : Q \to X$. Now, f is really the direct sum $f = \bigoplus_I f_i$ of component maps $f_i : P_i \to X$, and similarly for f'. So,

$$\bigoplus_I f_i = \bigoplus_I f'_i\theta_i$$

Therefore, each f_i factors through θ_i, and so $X \in \mathcal{D}_{\theta_i}$ for all $i \in I$, and then $X \in \bigcap_I \mathcal{D}_{\theta_i}$.

Conversely, let $X \in \bigcap_I \mathcal{D}_{\theta_i}$. So for any map $f_i : P_i \to X$ there is a map $f'_i : Q_i \to X$ such that $f_i = f'_i\theta_i$, i.e. for all $i \in I$ there is a commutative diagram

$$\begin{tikzcd} P_i \arrow[r, "\theta_i"] \arrow[d, "f_i"'] & Q_i \arrow[dl, "f'_i"] \\ X & \end{tikzcd}$$

Taking the direct sum indexed by I of the top rows induces a commutative diagram which shows that $X \in \mathcal{D}_\theta$.

(2) : Let $X \in \mathcal{D}_\alpha$ and take a map $f : Q_{-1} \to X$. Then there is a map $f' : Q_0 \to X$ such that $f = f'\alpha$. Consider the following diagram

$$\begin{tikzcd} Q_{-1} \arrow[r, "\binom{\alpha}{\beta}"] \arrow[d, "f"] & Q_0 \oplus Q'_0 \arrow[dl, "{(f',0)}"] \\ X & \end{tikzcd}$$

Clearly it commutes, so $X \in \mathcal{D}_{(\alpha,\beta)}$. □

The following theorem generalizes Bongartz completion to silting modules.

Theorem 3.20. *Every partial silting A-module T with respect to a projective presentation σ is a direct summand of a silting A-module $\overline{T} = T \oplus M$ with $\mathrm{Gen}(\overline{T}) = \mathcal{D}_\sigma = \mathrm{Gen}(T)$.*

Proof. Let $T \in \mathrm{Mod}(A)$ be partial silting w.r.t a projective presentation $\sigma : P_{-1} \to P_0$. First we construct an approximation sequence of A. Consider the universal map $\psi : P_{-1}^{(I)} \to A$ with $I = \mathrm{Hom}_A(P_{-1}, A)$, and take the pushout to get the following commutative diagram.

$$\begin{tikzcd} P_{-1}^{(I)} \arrow[r, "\sigma^{(I)}"] \arrow[d, "\psi"] & P_0^{(I)} \arrow[r] \arrow[d, "\theta"] & T^{(I)} \arrow[r] \arrow[d, equal] & 0 \\ A \arrow[r, "\phi"] & M \arrow[r, "\pi"] & T^{(I)} \arrow[r] & 0 \end{tikzcd} \tag{3.3}$$

If $M \in \mathcal{D}_\sigma$, then follows directly from the pushout property that ϕ is a left $\mathcal{D}_\sigma$-approximation of A. So we will show that every map $g : P_{-1} \to M$ factors through σ, thereby showing that

$M \in \mathcal{D}_\sigma$. Since $T^{(I)} \in \mathcal{D}_\sigma$, the composition πg factors through σ, so we have the following commutative diagram.

$$\begin{tikzcd}
 & P_{-1} \arrow[r, "\sigma"] \arrow[d, "g"] & P_0 \arrow[d, dashed, "g'"] & \\
A \arrow[r, "\phi"] & M \arrow[r, "\pi"] & T^{(I)} \arrow[r] & 0
\end{tikzcd}$$

Furthermore, because P_0 is projective, there is a map $h_0 : P_0 \to M$ such that $g' = \pi h_0$. Then by the commutativity of the square, $\pi(g - h_0\sigma) = 0$, so $(g - h_0\sigma)$ factors through ϕ, say as $g - h_0\sigma = \phi h_{-1}$, as illustrated in the following diagram

$$\begin{tikzcd}
 & P_{-1} \arrow[r, "\sigma"] \arrow[d, "g"] \arrow[dl, dashed, "h_{-1}"'] & P_0 \arrow[d, "g'"] \arrow[dl, dashed, "h_0"'] & \\
A \arrow[r, "\phi"] & M \arrow[r, "\pi"] & T^{(I)} \arrow[r] & 0
\end{tikzcd}$$

Now, the universal property of ψ and the commutative square in diagram (3.3) gives two component maps $\tilde{\psi} : P_{-1} \to A$ and $\tilde{\theta} : P_0 \to M$ such that

$$\phi\tilde{\psi} = \tilde{\theta}\sigma = g - g''\sigma$$

Then g factors through σ, and thus $M \in \mathcal{D}_\sigma$.

The claim now is that $\overline{T} := T \oplus M$ is silting with $\mathrm{Gen}(\overline{T}) = \mathcal{D}_\sigma$. The left square of diagram 3.3 is a pushout square, so there is a projective presentation of M

$$P_{-1}^{(I)} \xrightarrow{\begin{pmatrix} \psi \\ \sigma^{(I)} \end{pmatrix}} A \oplus P_0^{(I)} \xrightarrow{(\phi,\theta)} M \longrightarrow 0$$

and by setting $\gamma = \sigma \oplus \begin{pmatrix} \psi \\ \sigma^{(I)} \end{pmatrix}$ we get a projective presentation of $\overline{T}$

$$P_{-1} \oplus P_{-1}^{(I)} \xrightarrow{\gamma} P_0 \oplus A \oplus P_0^{(I)} \xrightarrow{(c,\phi,\theta)} \overline{T} \longrightarrow 0$$

where c is the cokernel map $c : P_0 \to T$.

By lemma 3.19(1), we have

$$\mathcal{D}_\gamma = \mathcal{D}_\sigma \cap \mathcal{D}_{(\psi,\sigma^{(I)})^T} \quad \text{and} \quad \mathcal{D}_{\sigma^{(I)}} = \bigcap_I \mathcal{D}_\sigma = \mathcal{D}_\sigma$$

By lemma 3.19(2) we have

$$\mathcal{D}_{\sigma^{(I)}} = \mathcal{D}_\sigma \subseteq \mathcal{D}_{(\psi,\sigma^{(I)})^T}$$

But then

$$\mathcal{D}_\sigma \cap \mathcal{D}_{(\psi,\sigma^{(I)})^T} = \mathcal{D}_\sigma$$

and so $\mathcal{D}_\gamma = \mathcal{D}_\sigma$.

Since $M, T \in \mathcal{D}_\sigma = \mathcal{D}_\gamma$ we have $\overline{T} \in \mathcal{D}_\gamma$. Therefore, $\overline{T}$ is partial silting. Furthermore, the sequence

$$A \xrightarrow{\phi} M \xrightarrow{\pi} T^{(I)} \longrightarrow 0$$

is exact with $M, T^{(I)} \in \mathrm{Add}(\overline{T})$, and ϕ is a left $\mathcal{D}_\gamma$-approximation of A. Finally, $\overline{T}$ is then silting by proposition 3.18. $\square$

We now prove that silting modules actually generalize support τ-tilting modules. That is, silting modules over a finite dimensional k-algebra Λ for an algebraically closed field k coincide with support τ-tilting modules.

Theorem 3.21. *Let Λ be a finite dimensional k-algebra over some algebraically closed field k, and let $T \in \mathrm{mod}(\Lambda)$ with minimal projective presentation $\sigma : P_{-1} \to P_0$. Then the following hold.*

(1) T is partial silting if and only if it is τ-rigid.

(2) T is silting if and only if it is support τ-tilting.

(3) T is (finendo) quasitilting if and only if it is support τ-tilting.

Proof. (1): Suppose that T is partial silting. Then $\mathrm{Gen}(T) \subseteq T^{\perp_1}$ by corollary 3.15, and it follows by theorem 2.37(2) that T is τ-rigid.

Conversely, suppose that T is τ-rigid. Then $\mathrm{Hom}_\Lambda(\sigma, T)$ is surjective by theorem 2.37(1), i.e. $T \in \mathcal{D}_\sigma$. Furthermore, by the discussion following lemma 3.13 we have that $\mathcal{D}_\sigma$ is a torsion class, thus T is partial silting.

(2): Suppose that T is silting, then it is τ-rigid by (1). Furthermore, by proposition 3.18 there is an exact sequence (which can be taken in $\mathrm{mod}(\Lambda)$)

$$\Lambda \xrightarrow{\phi} T_0 \longrightarrow T_1 \longrightarrow 0$$

where $T_0, T_1 \in \mathrm{add}(T)$ and ϕ is a left $\mathrm{gen}(T)$-approximation of Λ. Since $\mathrm{add}(T) \subseteq \mathrm{gen}(T)$ we have that T is support τ-tilting by proposition 2.22.

The converse follows immediately from theorem 2.37(3).

(3): Clearly all finitely generated Λ-modules are finendo. Then by (2), it suffices to prove that T is silting if and only if T is quasitilting. If T is silting, then it is quasitilting by proposition 3.16. Suppose then that T is quasitilting, then by proposition 3.6 there is an exact sequence

$$\Lambda \xrightarrow{\phi} T_0 \longrightarrow T_1 \longrightarrow 0$$

where $T_0, T_1 \in \mathrm{Add}(T)$ and ϕ is a left $\mathrm{Gen}(T)$-approximation of Λ and T is Ext-projective in $\mathrm{Gen}(T)$. Then T satisfies condition $(S3)$ in proposition 3.18. Furthermore, since $\mathrm{Gen}(T) \subseteq T^{\perp_1}$, theorem 2.37(2) gives us that T is τ-rigid. But then we have that T is partial silting by (1), and since it satisfies condition $(S3)$ in proposition 3.18 we conclude that T is silting. $\square$

3.3 Examples

Because all tilting modules are silting, and all silting Λ-modules over a finite dimensional k-algebra Λ are support τ-tilting, there is no shortage of examples of silting modules. We first give an example showing explicitly the link between silting modules and support τ-tilting modules.

Example 3.22. Let k be an algebraically closed field and consider the quiver

$$\Gamma: \qquad 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

Let $\Lambda = k\Gamma$. There are 6 indecomposable modules over Λ, and we have the AR-quiver

$$\begin{tikzcd}
 & & P_1 \arrow[dr] & & \\
 & P_2 \arrow[ur] \arrow[dr] & & I_2 \arrow[ll, dashed, "\tau"'] \arrow[dr] & \\
P_3 \arrow[ur] & & S_2 \arrow[ll, dashed, "\tau"'] \arrow[ur] & & S_1 \arrow[ll, dashed, "\tau"']
\end{tikzcd}$$

We adapt example 2.50 to silting modules, so let $T = I_2 \oplus S_1$. We have

$$\mathrm{Gen}(T) = \mathrm{Add}(\{I_2, S_1\})$$

and T admits the following projective presentation

$$P_3 \oplus P_2 \xrightarrow{\sigma} P_1 \oplus P_1 \longrightarrow I_2 \oplus S_1 \longrightarrow 0$$

but $\mathrm{Gen}(T) \subsetneq \mathcal{D}_\sigma$. For instance $P_1 \in \mathcal{D}_\sigma$ but $P_1 \notin \mathrm{Gen}(T)$. It's clear from the AR-quiver that $P_3^{\perp_0} = \mathrm{Add}(\{S_2, I_2, S_1\})$ and $S_2 \notin \mathcal{D}_\sigma$. Then we have

$$\mathcal{D}_\sigma \cap P_3^{\perp_0} = \mathrm{Gen}(T).$$

Then letting $\gamma = (\sigma, 0)$ be the following projective presentation of T

$$P_3 \oplus P_2 \oplus P_3 \xrightarrow{(\sigma,0)} P_1 \oplus P_1 \longrightarrow I_2 \oplus S_1 \longrightarrow 0$$

we have $\mathcal{D}_\gamma = \mathrm{Gen}(T)$, and so T is silting with respect to γ. Note that T is support τ-tilting by example 2.50. Furthermore, γ is precisely the unique 2-term silting complex corresponding to T by theorem 2.49. The link between 2-term silting complexes and silting modules will be described in detail in section 5.2.

The need to extend σ to γ is an example of the choice of projective presentation mentioned in remark 3.12. $\square$

Next we give an example of a silting module which is neither finitely presented nor tilting.

Example 3.23. Consider the Krönecker quiver Γ with countably many arrows

$$\Gamma: \qquad 1 \underset{\vdots}{\rightrightarrows} 2$$

Let k be an algebraically closed field and let $A = k\Gamma$. We have the two indecomposable projective modules $P_i = Ae_i$ for $i = 1, 2$

$$P_1: \qquad k \underset{\vdots}{\rightrightarrows} k^{(\mathbb{N})} \qquad\qquad P_2: \qquad 0 \underset{\vdots}{\rightrightarrows} k$$

and the two indecomposable injective modules

$$I_1: \qquad k \underset{\vdots}{\rightrightarrows} 0 \qquad\qquad I_2: \qquad k \underset{\vdots}{\rightrightarrows} k^{(\mathbb{N})}$$

Let $T = P_1/\operatorname{soc}(P_1)$ which is the simple module corresponding to vertex 1. Clearly, $\operatorname{Gen}(T) = \operatorname{Add}(T)$ and $\operatorname{Gen}(T) \subseteq P_2^{\perp_0}$. In fact, $\operatorname{Gen}(T)$ consists precisely of all semisimple injective A-modules. The inclusion is in fact an equality. Indeed, let M be the A-module given by some representation

$$M: \qquad U \underset{\vdots}{\rightrightarrows} V$$

and suppose that $M \in P_2^{\perp_0}$. Then $\operatorname{Hom}_k(k, V) = 0$ and so $V = 0$, and thus $M \in \operatorname{Gen}(T)$. Furthermore, since $\operatorname{Gen}(T)$ consists of injective A-modules, we have

$$P_2^{\perp_0} = \operatorname{Gen}(T) \subseteq T^{\perp_1}.$$

However, $I_2 \notin \operatorname{Gen}(T)$ but clearly $I_2 \in T^{\perp_1}$, so the inclusion above is proper, and therefore T is not tilting.

The A-module T admits the following (infinite) presentation.

$$0 \longrightarrow P_2^{(\mathbb{N})} \xrightarrow{\sigma} P_1 \longrightarrow T \longrightarrow 0 \tag{3.4}$$

Now, applying $\operatorname{Hom}_A(-, X)$ to the sequence above for some A-module X, one easily checks that $\mathcal{D}_\sigma = T^{\perp_1}$. We adjust sequence 3.4 slightly to a new projective presentation as such

$$0 \longrightarrow P_2^{(\mathbb{N})} \oplus P_2 \xrightarrow{\sigma \oplus 0} P_1 \longrightarrow T \longrightarrow 0$$

and let $\gamma = \sigma \oplus 0$. Then

$$\mathcal{D}_\gamma = \mathcal{D}_\sigma \cap P_2^{\perp_0} = T^{\perp_1} \cap P_2^{\perp_0} = P_2^{\perp_0} = \operatorname{Gen}(T).$$

Then, T is silting with respect to γ, but it is not tilting (nor finitely presented). The exact same construction can be applied to the path algebra of the quiver

$$1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} 2$$

to show that $P_1/\operatorname{soc}(P_1)$ is silting but not tilting. Of course, it admits a finite presentation in this case.

Finally, note the similarity between the construction of the silting class $\mathcal{D}_\gamma$ and the construction of 2-term silting complexes from minimal projective presentations of support τ-tilting Λ-modules in example 2.50. $\square$

4 Tilting in the morphism category

In this section we present the theory from [MŠ18], which connects the theory of silting modules with tilting in the morphism category. In particular, we prove that (partial) silting A-modules correspond bijectively to (partial) tilting objects in the morphism category $\mathrm{Mor}(A)$. The results are mostly from [MŠ18], but we provide some additional prerequisite results regarding the morphism category.

For any unitary ring A, we define the morphism category $\mathrm{Mor}(A)$ where the objects are A-homomorphisms $f : M \to N$, and morphisms are given by commutative diagrams

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "h_1"] & N \arrow[d, "h_2"] \\ M' \arrow[r, "g"] & N' \end{tikzcd}$$

We denote the object $f : M \to N$ in $\mathrm{Mor}(A)$ as Z_f. The morphism given by the diagram above is then written as $h : Z_f \to Z_g$, and for any given morphism h we always write h_1, h_2 for the two component A-homomorphisms. The zero-object $Z_0 \in \mathrm{Mor}(A)$ is $0 \to 0$. We show what kernels, cokernels, monomorphisms and epimorphisms look like in $\mathrm{Mor}(A)$.

Lemma 4.1. *Let $Z_f = f : M \to N$ and $Z_g = g : M' \to N'$ be objects in $\mathrm{Mor}(A)$ and $h = (h_1, h_2) : Z_f \to Z_g$ a morphism in $\mathrm{Mor}(A)$.*

(1) The object $\mathrm{Ker}(h)$ is given by $\mathrm{Ker}(h_1) \to \mathrm{Ker}(h_2)$

(2) The object $\mathrm{Cok}(h)$ is given by $\mathrm{Cok}(h_1) \to \mathrm{Cok}(h_2)$

(3) The map h is a monomorphism in $\mathrm{Mor}(A)$ if and only if h_1, h_2 are monomorphisms in $\mathrm{Mod}(A)$.

(4) The map h is an epimorphism in $\mathrm{Mor}(A)$ if and only if h_1, h_2 are epimorphisms in $\mathrm{Mod}(A)$.

Proof. (1) : Let $\mathrm{Ker}(h) = X \to Y$, and let $Z_i = B \to C$ an object in $\mathrm{Mor}(A)$ with a morphism $p = (p_1, p_2) : Z_i \to Z_f$ such that $hp = (h_1p_1, h_2p_2) = 0$. Then p factors through $\mathrm{Ker}(h)$, in particular p_1 factors through $\mathrm{Ker}(h_1)$ and p_2 factors through $\mathrm{Ker}(h_2)$, thus $X = \mathrm{Ker}(h_1)$ and $Y = \mathrm{Ker}(h_2)$, as illustrated in the following commutative diagram.

$$\begin{tikzcd} B \arrow[r] \arrow[dr, "p_1"'] \arrow[rr, dashed, bend left] & C \arrow[dr, "p_2"] \arrow[rr, dashed, bend left] & X \arrow[r] \arrow[dl] & Y \arrow[dl] \\ & M \arrow[r, "f"] \arrow[d, "h_1"] & N \arrow[d, "h_2"] & \\ & M' \arrow[r, "g"] & N' & \end{tikzcd}$$

(2) : The proof is similar to the proof of (1).

(3) : Suppose that both h_1, h_2 are monomorphisms, then $\mathrm{Ker}(h) = 0 \to 0$ by (1), so h is a monomorphism.

Conversely, suppose that h_1 is a monomorphism and h_2 is not. Then it follows that there is a non-zero object $0 \to \mathrm{Ker}(h_2)$ in $\mathrm{Mor}(A)$ and a map $Z_{(0 \to \mathrm{Ker}(h_2))} \to Z_f$ such that composition with h is zero. Thus, h cannot be a monomorphism. Similar arguments show that if h_1 is not a monomorphism then h cannot be a monomorphism.

(4) : The proof is similar to the proof of (3). $\square$

Objects in $\mathrm{Mor}(A)$ can be thought of as chain complexes of A-modules concentrated in degrees $-1, 0$. The morphism category also goes by the name **arrow category**, and is then denoted by $C^{\to}(A)$ or $\mathrm{Arr}(A)$.

Denote by $T_2(A)$ the ring of lower triangular matrices over A. The following proposition shows that $\mathrm{Mor}(A)$ is an abelian category.

Proposition 4.2. *The category* $\mathrm{Mor}(A)$ *is equivalent to the category* $\mathrm{Mod}(T_2(A))$.

Proof. Define a functor $F : \mathrm{Mor}(A) \to \mathrm{Mod}(T_2(A))$ as follows. The functor sends an object $Z_f = M \xrightarrow{f} N$ in $\mathrm{Mor}(A)$ to the abelian group $M \oplus N$, where for $m \in M$ and $n \in N$, the $T_2(A)$-action is defined as

$$\begin{pmatrix} a_1 & 0 \\ a_2 & a_3 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} a_1 m \\ a_2 f(m) + a_3 n \end{pmatrix}$$

The functor acts on morphisms of objects in $\mathrm{Mor}(A)$ as follows

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "h_1"] & N \arrow[d, "h_2"] & \mapsto & M \oplus N \arrow[d, "h_1 \oplus h_2"] \\ M' \arrow[r, "f'"] & N' & & M' \oplus N' \end{tikzcd} \tag{4.1}$$

It is straightforward to check that F is indeed a functor. We will show that it is an equivalence.

F is faithful: Fix two objects $Z_f = f : M \to N$ and $Z_{f'} = f' : M' \to N'$ and consider diagram 4.1. Clearly, if $h_1 \oplus h_2 = 0$ we have $h_1 = h_2 = 0$, so

$$F : \mathrm{Hom}_{\mathrm{Mor}(A)}(Z_f, Z_{f'}) \to \mathrm{Hom}_{T_2(A)}(M \oplus N, M' \oplus N')$$

is injective.

F is full: Again, fix $Z_f = M \xrightarrow{f} N$ and $Z_{f'} = M' \xrightarrow{f'} N'$ in $\mathrm{Mor}(A)$, and let

$$h : M \oplus N \to M' \oplus N'$$

be a map in $\mathrm{Mod}(T_2(A))$. We write $h((m,n)^T) = (m', n')^T$. First we show that there are induced maps $M \to M'$ and $N \to N'$. Let $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $e_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then

$$h(e_1(m,n)^T) = h((m,0)^T) = (m', 0)^T$$
$$h(e_2(m,n)^T) = h((0,n)^T) = (0, n')^T$$

and so we have the induced maps $h_1 : M \to M'$ with $h_1(m) = m'$ and $h_2 : N \to N'$ with $h_2(n) = n'$. Let $a \in A$, then we have

$$\begin{aligned} h \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} &= h \begin{pmatrix} 0 \\ af(m) \end{pmatrix} = \begin{pmatrix} 0 \\ ah_2 f(m) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} h \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \begin{pmatrix} h_1(m) \\ h_2(n) \end{pmatrix} = \begin{pmatrix} 0 \\ af'h_1(m) \end{pmatrix}. \end{aligned}$$

In particular, for $a = 1_A$ we have $h_2 f(m) = f'h_1(m)$, so the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} \\ M' & \xrightarrow{f'} & N' \end{array}$$

which implies that F is full.

F is dense: Let $M \oplus N$ be a $T_2(A)$-module. Then there must be some map $f : M \to N$ such that

$$\begin{pmatrix} a_1 & 0 \\ a_2 & a_3 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} a_1 m \\ a_2 f(m) + a_3 n \end{pmatrix}$$

The object $Z_f = M \xrightarrow{f} N$ is then sent to $M \oplus N$ via F. □

The following subcategories of $\mathrm{Mor}(A)$ will be of particular interest to us.

$$\mathcal{L} := \mathrm{Mor}(\mathrm{proj}(A)) = \{Z_\sigma \mid \sigma \in \mathrm{proj}(A)\}$$

$$\mathcal{BL} := \mathrm{Mor}(\mathrm{Proj}(A)) = \{Z_\sigma \mid \sigma \in \mathrm{Proj}(A)\}$$

They are clearly both additive and full subcategories of $\mathrm{Mor}(A)$. We now define exact categories in the sense of Quillen, for which we adopt the definitions due to Happel, [Hap88, Chapter I.2].

Definition 4.3. Let $\mathcal{B}$ be an additive category, embeddded as a full and extension-closed subcategory of an abelian category $\mathcal{A}$. Let $\mathcal{S}$ be the set of short exact sequences in $\mathcal{A}$ with terms in $\mathcal{B}$, then $(\mathcal{B}, \mathcal{S})$ is called an **exact category**. Let the following be a sequence in $\mathcal{S}$.

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

then f is called a **proper monomorphism** and g an **proper epimorphism**.

There is also an axiomatic approach to defining exact categories, where the axioms mimic some key properties of short exact sequences, see [Qui73]. We also need the following definition of projective and injective objects in an exact category.

Definition 4.4. Let $(\mathcal{B}, \mathcal{S})$ be an exact category as defined above. Then

(1) An object P in $\mathcal{B}$ is said to be $\mathcal{S}$-projective if for all proper epimorphisms $g : Y \to Z$ and morphisms $p : P \to Z$, there exists a morphism $p' : P \to Y$ such that $p = gp'$.

(2) An object I in $\mathcal{B}$ is said to be $\mathcal{S}$-injective if for all proper monomorphisms $f : X \to Y$ and morphisms $i : X \to I$, there exists a morphism $i' : Y \to I$ such that $i = i'f$.

For the sake of simplicity, when no confusion can arise we will just use the terms projective and injective when working with exact categories.

Lemma 4.5. *The categories $\mathcal{L}$ and $\mathcal{BL}$ are closed under extensions. Consequently, they are exact categories.*

Proof. We only prove that $\mathcal{BL}$ is closed under extensions as the proof for $\mathcal{L}$ follows the same arguments.

Let $Z_f, Z_h \in \mathcal{BL}$ and suppose there is a short exact sequence in $\operatorname{Mor}(A)$.

$$0 \longrightarrow Z_f \xrightarrow{\alpha} Z_g \xrightarrow{\beta} Z_h \longrightarrow 0$$

Since monomorphisms and epimorphisms in $\operatorname{Mor}(A)$ are given by degree-wise monomorphisms and epimorphisms in $\operatorname{Mod}(A)$ by lemma 4.1, the short exact sequence above is given by a commutative diagram with exact rows as follows

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r, "\alpha_1"] \arrow[d, "f"] & P' \arrow[r, "\beta_1"] \arrow[d, "g"] & P'' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & Q \arrow[r, "\alpha_2"] & Q' \arrow[r, "\beta_2"] & Q'' \arrow[r] & 0
\end{tikzcd}$$

Since $P'', Q'' \in \operatorname{Proj}(A)$, the two rows split, therefore $P', Q' \in \operatorname{Proj}(A)$ and $Z_g \in \mathcal{BL}$. $\square$

As a consequence of lemma 4.5, the exact sequences in $\mathcal{L}$ and $\mathcal{BL}$ (as exact categories) are given by degree-wise short exact sequences in $\operatorname{Mod}(A)$.

Recall that an abelian category $\mathcal{A}$ is called **hereditary** if $\operatorname{g.d.}(\mathcal{A}) \leq 1$ (global dimension at most 1), or equivalently if the functor $\operatorname{Ext}^2_{\mathcal{A}}$ vanishes. The category $\mathcal{A}$ is said to have **enough projectives** if for any object X there exists a projective object P and an epimorphism $P \to X$. Dually, $\mathcal{A}$ is said to have **enough injectives** if for any object X there exists an injective object I and a monomorphism $X \to I$.

Lemma 4.6. *The categories $\mathcal{L}$ and $\mathcal{BL}$ are hereditary, and they have enough projectives and enough injectives. In particular, for an object $P \in \operatorname{Proj}(A)$ (respectively $\operatorname{proj}(A)$) the object $Z_{1_P} \in \operatorname{Mor}(A)$ is projective-injective in $\mathcal{BL}$ (respectively $\mathcal{L}$).*

Proof. First we prove that $\mathcal{BL}$ has enough projectives and injectives. The proof for $\mathcal{L}$ follows the same arguments.

Let the following be a short exact sequence in $\mathrm{Mor}(A)$ with terms in $\mathcal{BL}$.

$$0 \longrightarrow Z_f \xrightarrow{\alpha} Z_g \xrightarrow{\beta} Z_h \longrightarrow 0$$

given by the diagram

$$\begin{tikzcd}
0 \arrow[r] & P' \arrow[r, "\alpha_1"] \arrow[d, "f"] & P'' \arrow[r, "\beta_1"] \arrow[d, "g"] & P''' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & Q' \arrow[r, "\alpha_2"] & Q'' \arrow[r, "\beta_2"] & Q''' \arrow[r] & 0
\end{tikzcd} \tag{4.2}$$

Now take any P in $\mathrm{Proj}(A)$, consider the object $Z_{(0\to P)}$ and suppose there is a morphism $\phi : Z_{(0\to P)} \to Z_h$. The morphism $\phi_1 : 0 \to P'''$ factors trivially through P'', and the morphism $\phi_2 : P \to Q'''$ factors through Q'' because P is projective in $\mathrm{Mod}(A)$ and β_2 is an epimorphism. Thus, $Z_{(0\to P)}$ is projective in $\mathcal{BL}$ as it is projective with respect to proper epimorphisms (in fact, it is projective even in $\mathrm{Mor}(A)$). A similar argument shows that Z_{1_P} is projective in $\mathcal{BL}$ (and in $\mathrm{Mor}(A)$).

Now, for the same P and exact sequence as above, consider the object $Z_{(P\to 0)}$ and suppose there is a morphism $\psi : Z_f \to Z_{(P\to 0)}$. The exact rows in diagram 4.2 split since P''' and Q''' are projective A-modules, so the morphism $\psi_1 : P' \to P$ factors through P'' via the summand P'. The morphism $\psi_2 : Q' \to 0$ factors trivially through Q''. Thus, $Z_{(P\to 0)}$ is injective in $\mathcal{BL}$ as it is injective with respect to proper monomorphisms (note that it is not always injective in $\mathrm{Mor}(A)$, as the proof relies on the fact that the exact sequences split degree-wise). A similar argument shows that Z_{1_P} is injective in $\mathcal{BL}$ (not always in $\mathrm{Mor}(A)$)

Then for any object $Z_f = f : P \to Q$ in $\mathcal{BL}$, there exists a projective object $Z \in \mathcal{BL}$ and an epimorphism $Z \to Z_f$ given by the following diagram.

$$\begin{tikzcd}
P \arrow[r, "1"] \arrow[d, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] & P \arrow[d, "f"] \\
P \oplus Q \arrow[r, "(\, f \ 1\,)"] & Q
\end{tikzcd}$$

So $\mathcal{BL}$ has enough projectives. Similarly, one sees that $\mathcal{BL}$ has enough injectives.

Now we show that $\mathcal{BL}$ is hereditary. The proof for $\mathcal{L}$ follows the same arguments. Take some object $Z_\sigma = \sigma : P \to Q$ in $\mathcal{BL}$, and consider the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & P \arrow[r, "1"] \arrow[d, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"] & P \arrow[r] \arrow[d, "\sigma"] & 0 \\
0 \arrow[r] & P \arrow[r, "\left(\begin{smallmatrix}1\\-\sigma\end{smallmatrix}\right)"] & P \oplus Q \arrow[r, "(\, \sigma \ 1\,)"] & Q \arrow[r] & 0
\end{tikzcd} \tag{4.3}$$

The diagram induces a projective resolution of Z_σ of length 1

$$0 \longrightarrow P_1(Z_\sigma) \longrightarrow P_0(Z_\sigma) \longrightarrow Z_\sigma \longrightarrow 0$$

Then we have $\mathrm{p.d.}(\mathcal{BL}) \leq 1$, and similarly one can show that $\mathrm{i.d.}(\mathcal{BL}) \leq 1$. Then $\mathrm{Ext}^n_{\mathrm{Mor}(A)}$ vanishes on objects of $\mathcal{BL}$ for all $n \geq 2$, so $\mathcal{BL}$ is hereditary. $\square$

Extensions of objects in $\mathcal{BL}$ or $\mathcal{L}$ are fairly easy to deal with as we have shown in the two last lemmas, but extensions in $\mathrm{Mor}(A)$ are significantly harder to work with. The following lemma provides a very useful isomorphism concerning extension groups when the left-most term is in $\mathcal{BL}$.

Lemma 4.7. *For all objects Z_σ in $\mathcal{BL}$ and Z_g in $\mathrm{Mor}(A)$, we have the isomorphism*

$$\mathrm{Ext}^1_{\mathrm{Mor}(A)}(Z_\sigma, Z_g) \cong \mathrm{Hom}_{K^b(A)}(Z_\sigma, Z_g[1])$$

Proof. Let $Z_\sigma \in \mathcal{BL}$ be given by $\sigma : P \to Q$. By the proof of lemma 4.6, a projective resolution of Z_σ

$$0 \longrightarrow P_1(Z_\sigma) \longrightarrow P_0(Z_\sigma) \longrightarrow Z_\sigma \longrightarrow 0 \tag{4.4}$$

in $\mathcal{BL}$ is given by a commutative diagram of A-modules.

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & P \arrow[r, "1"] \arrow[d, "\binom{1}{0}"] & P \arrow[r] \arrow[d, "\sigma"] & 0 \\
0 \arrow[r] & P \arrow[r, "\binom{1}{-\sigma}"] & P \oplus Q \arrow[r, "(\sigma\ 1)"] & Q \arrow[r] & 0
\end{tikzcd}$$

Let the object $Z_g \in \mathrm{Mor}(A)$ be given by $g : M \to N$, and apply $\mathrm{Hom}_{\mathrm{Mor}(A)}(-, Z_g)$ to sequence 4.4 to get the long exact sequence.

$$\begin{tikzcd}
\dots \arrow[r] & \mathrm{Hom}_{\mathrm{Mor}(A)}(P_0(Z_\sigma), Z_g) \arrow[r] & \mathrm{Hom}_{\mathrm{Mor}(A)}(P_1(Z_\sigma), Z_g) \arrow[dll, rounded corners] \\
\mathrm{Ext}^1_{\mathrm{Mor}(A)}(Z_\sigma, Z_g) \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

So every element in $\mathrm{Ext}^1_{\mathrm{Mor}(A)}(Z_\sigma, Z_g)$ is represented by some morphism
$h \in \mathrm{Hom}_{\mathrm{Mor}(A)}(P_1(Z_\sigma), Z_g)$ given by the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & P \arrow[d, "h"] \\
M \arrow[r, "g"] & N
\end{tikzcd}$$

Now consider Z_σ and Z_g as 2-term complexes in $K^b(A)$, then an element in
$\mathrm{Hom}_{K^b(A)}(Z_\sigma, Z_g[1])$ is given by the diagram

$$\begin{tikzcd}
 & P \arrow[r, "\sigma"] \arrow[d, "h"] & Q \\
M \arrow[r, "-g"] & N &
\end{tikzcd}$$

So, elements of $\mathrm{Ext}^1_{\mathrm{Mor}(A)}(Z_\sigma, Z_g)$ and $\mathrm{Hom}_{K^b(A)}(Z_\sigma, Z_g[1])$ both rely on finding a morphism $h \in \mathrm{Hom}_A(P, N)$.

Now we show that elements of these two sets are 0 under the same conditions. By the exact sequence above, an element of $\mathrm{Ext}^1_{\mathrm{Mor}(A)}(Z_\sigma, Z_g)$ is 0 if and only if the morphisms $P_1(Z_g) \to Z_g$ factor through the inclusion $P_1(Z_\sigma) \to P_0(Z_\sigma)$. This is true if and only if there exists morphisms $P \xrightarrow{s} M$ and $Q \xrightarrow{t} N$ such that the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & P \arrow[r, "s"] \arrow[d, "\binom{1}{0}"'] & M \arrow[d, "g"] \\
P \arrow[r, "\binom{1}{-\sigma}"] \arrow[rr, bend right, "h"'] & P \oplus Q \arrow[r, "(gs,t)"] & N
\end{tikzcd}$$

Then $h = gs - t\sigma$, so the chain complex morphism above is null-homotopic. □

The lemma is useful in the sense that working with $\mathrm{Hom}_{K^b(A)}(Z_\sigma, -)$ is far easier than working with $\mathrm{Ext}^1_{\mathrm{Mor}(A)}(Z_\sigma, -)$. The class $\Sigma^{\perp_1}$ for some set Σ of objects in $\mathcal{BL}$ is needed in order to consider tilting objects in $\mathrm{Mor}(A)$. The following lemma gives an explicit construction of $\Sigma^{\perp_1}$, and moreover it provides a way of translating information between $\mathrm{Mor}(A)$ and $\mathrm{Mod}(A)$. The importance of such a link is self-evident. We now extend the class $\mathcal{D}_\sigma$ from earlier sections slightly. For a set of objects Σ in $\mathcal{BL}$, consider the full subcategory $\mathcal{D}_\Sigma$ of $\mathrm{Mod}(A)$

$$\mathcal{D}_\Sigma = \{X \in \mathrm{Mod}(A) \mid \mathrm{Hom}_A(\sigma, X) \text{ is surjective for all } \sigma \in \Sigma\}$$

It follows from lemma 3.10 that $\mathcal{D}_\Sigma$ is always closed under extensions and factors.

Lemma 4.8. *Let Σ be a set of objects in $\mathcal{BL}$. Then the full subcategory $\Sigma^{\perp_1}$ of $\mathrm{Mor}(A)$ is given by*

$$\Sigma^{\perp_1} = \{Z_g \in \mathrm{Mor}(A) \mid \mathrm{Cok}(g) \in \mathcal{D}_\Sigma\}.$$

Furthermore, $\Sigma^{\perp_1}$ is closed under extensions and factors.

Proof. Let $Z_g \in \mathrm{Mor}(A)$, such that $Z_g \in \Sigma^{\perp_1}$. Then we have

$$\mathrm{Hom}_{K^b(A)}(Z_\sigma, Z_g[1]) = 0$$

for all $Z_\sigma \in \Sigma$ by lemma 4.7. Let $Z_\sigma = \sigma : (P_{-1} \to P_0) \in \Sigma$, then the morphism

$$\mathrm{Hom}_A(P_0, \mathrm{Cok}(g)) \to \mathrm{Hom}_A(P_{-1}, \mathrm{Cok}(g))$$

is surjective by lemma 3.9, and thus $\mathrm{Cok}(g) \in \mathcal{D}_\Sigma$. Similarly, if $\mathrm{Cok}(g) \in \mathcal{D}_\Sigma$ then

$$\mathrm{Hom}_{K^b(A)}(Z_\sigma, Z_g[1]) = 0$$

for all $Z_\sigma \in \Sigma$ by lemma 3.9, and thus $Z_g \in \Sigma^{\perp_1}$ by lemma 4.7.

Let $Z_f, Z_g \in \Sigma^{\perp_1}$, and suppose there is a short exact sequence

$$0 \longrightarrow Z_g \longrightarrow Z_h \longrightarrow Z_f \longrightarrow 0$$

Let $Z_\sigma \in \Sigma$ and apply $\mathrm{Hom}_{\mathrm{Mor}(A)}(Z_\sigma, -)$ to the short exact sequence to get a long exact sequence

$$\begin{tikzcd} \dots \arrow[r] & \underbrace{\mathrm{Ext}^1_{\mathrm{Mor}(A)}(Z_\sigma, Z_g)}_{=0} \arrow[r] & \mathrm{Ext}^1_{\mathrm{Mor}(A)}(Z_\sigma, Z_h) \arrow[dll, rounded corners] \\ & \underbrace{\mathrm{Ext}^1_{\mathrm{Mor}(A)}(Z_\sigma, Z_f)}_{=0} \arrow[r] & \dots \end{tikzcd}$$

from which it follows that $Z_h \in \Sigma^{\perp_1}$, and $\Sigma^{\perp_1}$ is therefore closed under extensions.

Now, let Z_f and $Z_{f'}$ be given by $f : M \to N$ and $f' : M' \to N'$ respectively such that $Z_{f'}$ is a factor of Z_f, i.e. the following diagram is commutative and the vertical maps are surjective

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "h_1"] & N \arrow[d, "h_2"] \\ M' \arrow[r, "f'"] & N' \end{tikzcd}$$

Suppose that $Z_f \in \Sigma^{\perp_1}$, we show that then $Z_{f'} \in \Sigma^{\perp_1}$. Take some object $Z_\sigma \in \Sigma$ given by $\sigma : P \to Q$, then by lemma 4.7 an object in $\mathrm{Ext}^1_{\mathrm{Mor}(A)}(Z_\sigma, Z_{f'})$ is given by the following diagram in $K^b(A)$

$$\begin{tikzcd} & P \arrow[r, "\sigma"] \arrow[d, "g"] & Q \\ M' \arrow[r, "-f'"] & N' & \end{tikzcd}$$

Since P is projective and $h_2 : N \to N'$ is surjective, there is a map $t : P \to N$ such that $g = h_2 t$. But, the map t is null-homotopic since $Z_f \in \Sigma^{\perp_1}$, i.e. there are maps $d_{-1} : P \to M$ and $d_0 : Q \to N$ such that $t = d_0\sigma - f d_{-1}$. Then we have

$$g = h_2 t = h_2 d_0 \sigma - h_2 f d_{-1} = h_2 d_0 \sigma - f' h_1 d_{-1}$$

which shows that g is null-homotopic, i.e. $Z_{f'} \in \Sigma^{\perp_1}$. □

Remark 4.9. Since $\Sigma^{\perp_1}$ is closed under extensions and factors, it is a torsion class if and only if it is closed under coproducts in $\mathrm{Mor}(A)$.

We include a small but important lemma which shows that for an object $Z_\sigma \in \mathcal{BL}$, the class $Z_\sigma^{\perp_1}$ is a torsion class if and only if $\mathcal{D}_\sigma$ is a torsion class.

Lemma 4.10. *Let $Z_\sigma \in \mathcal{BL}$. Then $Z_\sigma^{\perp_1}$ is a torsion class if and only if $\mathcal{D}_\sigma$ is a torsion clas.*

Proof. The class $Z_\sigma^{\perp_1}$ is closed under extensions and epimorphic images in $\mathrm{Mor}(A)$ by lemma 4.8. Similarly, the class $\mathcal{D}_\sigma$ is closed under extensions and epimorphic images in $\mathrm{Mod}(A)$ by lemma 3.10. Therefore, we only need to show that $Z_\sigma^{\perp_1}$ is closed under coproducts in $\mathrm{Mor}(A)$ if and only if $\mathcal{D}_\sigma$ is closed under coproducts in $\mathrm{Mod}(A)$.

Suppose that $Z_\sigma^{\perp_1}$ is closed under coproducts in $\mathrm{Mor}(A)$, and let $\{T_i\}_{i\in I}$ be a family of A-modules such that $T_i \in \mathcal{D}_\sigma$ for all $i \in I$. Let $f_i : P_i \to Q_i$ be a projective presentation of T_i for all $i \in I$. Then by lemma 4.8 we have $Z_{f_i} \in Z_\sigma^{\perp_1}$ for all $i \in I$. By assumption we then have $\bigoplus_{i\in I} Z_{f_i} \in Z_\sigma^{\perp_1}$, and so by lemma 4.8

$$\mathrm{Cok}(\bigoplus_{i\in I} f_i) = \bigoplus_{i\in I} T_i \in \mathcal{D}_\sigma.$$

Thus, $\mathcal{D}_\sigma$ is closed under coproducts in $\mathrm{Mod}(A)$.

Suppose now that $\mathcal{D}_\sigma$ is closed under coproducts in $\mathrm{Mod}(A)$, and let $\{Z_{g_i}\}_{i\in I}$ be a family of objects in $\mathrm{Mor}(A)$ such that $Z_{g_i} \in Z_\sigma^{\perp_1}$ for all $i \in I$. Then by lemma 4.8 we have $\mathrm{Cok}(g_i) \in \mathcal{D}_\sigma$ for all $i \in I$, and by assumption we also have

$$\bigoplus_{i\in I} \mathrm{Cok}(g_i) = \mathrm{Cok}(\bigoplus_{i\in I} g_i) \in \mathcal{D}_\sigma$$

which by lemma 4.8 again implies that $\bigoplus_{i\in I} Z_{g_i} \in Z_\sigma^{\perp_1}$. □

The next lemma shows that (partial) silting A-modules correspond bijectively to (partial) tilting objects in $\mathrm{Mor}(A)$.

Lemma 4.11. *Let T be an A-module with projective presentation $\sigma : P_{-1} \to P_0$. Then the following hold.*

(1) T is partial silting with respect to σ if and only if Z_σ is partial tilting in $\mathrm{Mor}(A)$.

(2) T is silting with respect to σ if and only if $Z_\sigma \oplus Z_{1_A}$ is tilting in $\mathrm{Mor}(A)$.

Proof. (1) : First, by lemma 4.8, $T \in \mathcal{D}_\sigma$ if and only if $Z_\sigma \in Z_\sigma^{\perp_1}$. Furthermore, the class $Z_\sigma^{\perp_1}$ is a torsion class in $\mathrm{Mor}(A)$ if and only if $\mathcal{D}_\sigma$ is a torsion class in $\mathrm{Mod}(A)$ by lemma 4.10. This proves (1).

(2) : Suppose that $Z := Z_\sigma \oplus Z_{1_A}$ is tilting in $\mathrm{Mor}(A)$, so $\mathrm{Gen}(Z) = Z^{\perp_1} = Z_\sigma^{\perp_1}$. First note that for any morphism $f : M \to N$ in $\mathrm{Mod}(A)$, we have

$$\mathrm{Hom}_{\mathrm{Mor}(A)}(Z_{1_A}, Z_f) \cong \mathrm{Hom}_A(A, M) \cong M.$$

In particular, we have $\mathrm{Hom}_{\mathrm{Mor}(A)}(Z_{1_A}, Z_{(0\to N)}) = 0$, and thus we have $Z_{(0\to N)} \in \mathrm{Gen}(Z)$ if and only if $Z_{(0\to N)} \in \mathrm{Gen}(Z_\sigma)$. It's easy to see that $Z_{(0\to N)} \in \mathrm{Gen}(Z_\sigma)$ if and only if $N \in \mathrm{Gen}(T)$. By lemma 4.8, $Z_{(0\to N)} \in Z_\sigma^{\perp_1}$ if and only if $N \in \mathcal{D}_\sigma$. Thus, we have

$$N \in \mathrm{Gen}(T) \iff Z_{(0\to N)} \in \mathrm{Gen}(Z) \iff Z_{(0\to N)} \in Z_\sigma^{\perp_1} \iff N \in \mathcal{D}_\sigma$$

so T is silting with respect to σ.

Conversely, suppose that T is silting. In particular, it is partial silting by corollary 3.15, and so Z_σ is partial tilting in $\mathrm{Mor}(A)$ by (1). By lemma 4.6, the object Z_{1_A} is projective-injective in $\mathrm{Mor}(A)$, and so by lemma 2.11 the object $Z := Z_\sigma \oplus Z_{1_A}$ is again partial tilting. Therefore, we know that $\mathrm{Gen}(Z) \subseteq Z^{\perp_1}$, so we only need to show the other inclusion.

Let $f : M \to N$ be a morphism in $\mathrm{Mod}(A)$ such that $Z_f \in Z^{\perp_1}$. Then we have $\mathrm{Cok}(f) \in \mathcal{D}_\sigma = \mathrm{Gen}(T)$ by lemma 4.8. Then there is a surjection $p : T^{(I)} \to \mathrm{Cok}(f)$, and since $P_0^{(I)}$ is projective, p lifts to a map $\tilde{p} : Z_\sigma^{(I)} \to Z_f$ given by the following commutative diagram.

$$\begin{tikzcd}
P_{-1}^{(I)} \arrow[r, "\sigma^{(I)}"] \arrow[d, dashed, "p''"] & P_0^{(I)} \arrow[r, "\pi_\sigma^{(I)}"] \arrow[d, dashed, "p'"] & T^{(I)} \arrow[r] \arrow[d, "p"] & 0 \\
M \arrow[r, "f"] & N \arrow[r, "\pi_f"] & \mathrm{Cok}(f) \arrow[r] & 0
\end{tikzcd} \tag{4.5}$$

Next, there is a surjection $q : A^{(J)} \to M$, which extends to a map $\tilde{q} : Z_{1_A}^{(J)} \to Z_f$ given by the following commutative diagram.

$$\begin{tikzcd}
A^{(J)} \arrow[r, "1"] \arrow[d, "q"] & A^{(J)} \arrow[d, "fq"] \\
M \arrow[r, "f"] & N
\end{tikzcd}$$

Then we have a map $(\tilde{p}, \tilde{q}) : Z_\sigma^{(I)} \oplus Z_{1_A}^{(J)} \to Z_f$ given by the commutative diagram

$$\begin{tikzcd}
P_{-1}^{(I)} \oplus A^{(J)} \arrow[r, "\begin{pmatrix} \sigma^{(I)} & 0 \\ 0 & 1 \end{pmatrix}"] \arrow[d, "(p''\text{,}q)"] & P_0^{(I)} \oplus A^{(J)} \arrow[d, "(p'\text{,}fq)"] \\
M \arrow[r, "f"] & N
\end{tikzcd}$$

We will prove that $(\tilde{p}, \tilde{q})$ is surjective, i.e. that both (p'', q) and (p', fq) are surjective. Since q is surjective by construction, the map (p'', q) is surjective. To prove that (p', fq) is surjective, we do a diagram chase in diagram 4.5.

Let $n \in N$ such that $n \notin \mathrm{Im}(f)$. Then $0 \neq \pi_f(n) \in \mathrm{Cok}(f)$, and since p is surjective there is some $t \in T^{(I)}$ such that $p(t) = \pi_f(n)$. Furthermore, the map $\pi_\sigma^{(I)}$ is surjective, so there is some $\alpha \in P_0^{(I)}$ such that

$$\pi_\sigma^{(I)}(\alpha) = t \quad \text{and} \quad p\pi_\sigma^{(I)}(\alpha) = \pi_f(n)$$

Since the diagram commutes, we have

$$\pi_f p'(\alpha) = p\pi_\sigma^{(I)}(\alpha) = \pi_f(n) \quad \Rightarrow \quad \pi_f(p'(\alpha) - n) = 0.$$

So $(p'(\alpha) - n) \in \operatorname{Ker}(\pi_f) = \operatorname{Im}(f)$, i.e. there is some $m \in M$ such that $f(m) = p'(\alpha) - n$. But then

$$N = \operatorname{Im}(p', f)$$

and since $q : A^{(J)} \to M$ is surjective, we also have

$$N = \operatorname{Im}(p', fq).$$

Then (p', fq) is surjective which makes the map $(\tilde{p}, \tilde{q})$ surjective, proving that $Z_f \in \operatorname{Gen}(Z)$. $\square$

5 Silting Complexes

5.1 Silting complexes, t-structures and co-t-structures

Now we turn to the more general notion of silting complexes over $K^b(\mathrm{Proj}(A))$, compared to the silting complexes in $K^b(\mathrm{proj}(\Lambda))$ from section 2.3. The structure and results of this section closely follow that of section 4.2 in [HMV15]. We also include an important theorem from [Wei13] and use it to split up the proof of one of the main results. The section culminates in the proof of theorem 5.21, showing that there are bijections between equivalence classes of silting complexes in $D(A)$, silting t-structures in $D(A)$, intermediate co-t-structures in $D(A)$ with the aisle being closed under coproducts and triples of subcategories of $D(A)$ such that the middle term appears in both a co-t-structure and an intermediate t-structure.

The first part of this section shows how silting complexes in $K^b(\mathrm{Proj}(A))$ concentrated in n degrees are in bijection with certain t-structures in $D(A)$, specifically proven in lemma 5.16. The concept of t-structures and co-t-structures is central in this section, so we begin by giving their definitions and presenting some motivational examples.

Definition 5.1. Let D be a triangulated category.

A **t-structure** (respectively a **co-t-structure**) in D is a pair of full subcategories $(\mathcal{V}^{\leq 0}, \mathcal{V}^{\geq 0})$ (respectively $(\mathcal{U}_{\geq 0}, \mathcal{U}_{\leq 0})$) such that

(1) $\mathrm{Hom}_D(\mathcal{V}^{\leq 0}, \mathcal{V}^{\geq 0}[-1]) = 0$ (respectively, $\mathrm{Hom}_D(\mathcal{U}_{\geq 0}, \mathcal{U}_{\leq 0}[1]) = 0$)

(2) $\mathcal{V}^{\leq 0}[1] \subseteq \mathcal{V}^{\leq 0}$ (respectively, $\mathcal{U}_{\geq 0}[-1] \subseteq \mathcal{U}_{\geq 0}$)

(3) For every Y in D there is a triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

such that $X \in \mathcal{V}^{\leq 0}$ and $Z \in \mathcal{V}^{\geq 0}[-1]$ (respectively, $X \in \mathcal{U}_{\geq 0}$ and $Z \in \mathcal{U}_{\leq 0}[1]$)

We use the notations

$$\begin{array}{ll} \mathcal{V}^{\leq n} := \mathcal{V}^{\leq 0}[-n] & \mathcal{U}_{\geq n} := \mathcal{U}_{\geq 0}[-n] \\ \mathcal{V}^{\geq n} := \mathcal{V}^{\geq 0}[-n] & \mathcal{U}_{\leq n} := \mathcal{U}_{\leq 0}[-n] \end{array}$$

The category $\mathcal{V}^{\leq 0}$ is called the **aisle**, the category $\mathcal{V}^{\geq 0}$ the **co-aisle** and the intersection $\mathcal{V}^{\leq 0} \cap \mathcal{V}^{\geq 0}$ the **heart** of the t-structure. The notation is motivated by example 5.3.

Remark 5.2. Note that for a t-structure $(\mathcal{V}^{\leq 0}, \mathcal{V}^{\geq 0})$ (respectively for a co-t-structure $(\mathcal{U}_{\geq 0}, \mathcal{U}_{\leq 0})$), we have $\mathcal{V}^{\geq 0} = (\mathcal{V}^{\leq 0})^{\perp_0}[1]$ (respectively $\mathcal{U}_{\leq 0} = (\mathcal{U}_{\geq 0})^{\perp_0}[-1]$). That is, the aisles completely determine the (co-)t-structures. It then follows that $\mathcal{V}^{\geq 0}[-1] \subseteq \mathcal{V}^{\geq 0}$ and $\mathcal{U}_{\leq 0}[1] \subseteq \mathcal{U}_{\leq 0}$.

We give some examples to illustrate how (co-) t-structures arise in in different categories.

Example 5.3. Let $D^{\leq 0}$ and $D^{\geq 0}$ denote the subcategories of $D(A)$ consisting of complexes which only have homologies lying in degrees $n \leq 0$ and $n \geq 0$ respectively. The pair $(D^{\leq 0}, D^{\geq 0})$ form a t-structure in $D(A)$ called the **standard t-structure**. Its associated truncation functors are the **smart** or **soft** truncation functors, denoted by $\tau^{\leq n}$ and $\tau^{\geq n}$ for all $n \in \mathbb{Z}$. For a complex $X = (X_i, d_i)_{i \in \mathbb{Z}}$ in $D(A)$, $\tau^{\leq n} X$ and $\tau^{\geq n+1} X$ are given by

$$\tau^{\leq n} X = \dots \xrightarrow{d_{n-2}} X_{n-1} \xrightarrow{d_{n-1}} \operatorname{Ker}(d_n) \longrightarrow 0 \longrightarrow \dots$$

$$\tau^{\geq n+1} X = \dots \longrightarrow 0 \longrightarrow X_n / \operatorname{Ker}(d_n) \xrightarrow{d_n} X_{n+1} \xrightarrow{d_{n+1}} \dots$$

where $\tau^{\geq n+1} X = X / \tau^{\leq n} X$ for every $n \in \mathbb{Z}$. Then, for $X \in D(A)$ there exists a triangle

$$\tau^{\leq 0} X \longrightarrow X \longrightarrow \tau^{\geq 1} X \longrightarrow (\tau^{\leq 0} X)[1]$$

It is easily verifiable that the pair $(D^{\leq 0}, D^{\geq 0})$ satisfies axioms (1), (2) for t-structures. Axiom (3) follows from the existence of triangles like the one above.

The truncation functors are called smart because they preserve homology in the degree they truncate, i.e. $\tau^{\leq n} X$ has homology in degrees $\leq n$ and $\tau^{\geq n+1} X$ has homology in degrees $\geq n+1$. There are other definitions of the smart truncations, but note that since they preserve homology, they are naturally isomorphic in $D(A)$.

Example 5.4. Let $K_p(A)$ denote the triangulated subcategory of $K(A)$ consisting of homotopically projective complexes, that is all complexes $X \in K(A)$ such that $\operatorname{Hom}_{K(A)}(X, Y) = 0$ for all exact complexes Y, see [Kel98]. Let $K_{\geq 0}$ and $K_{\leq 0}$ denote the subcategories of $K_p(A)$ consisting of complexes whose components are zero for all $n < 0$ and $n > 0$ respectively. The pair $(K_{\geq 0}, K_{\leq 0})$ form a co-t-structure in $K_p(A)$ called the **standard co-t-structure**. For a complex $X = (X_i, d_i)$ in $K_p(A)$, the triangle satisfying axiom (3) is obtained by the so-called **stupid truncation** of X, where each X_i is replaced by zero outside the chosen degree. While smart truncations preserve homology at the degree they truncate, stupid truncations to not.

Example 5.5. It was shown in [HRS96, Theorem 2.1] that torsion pairs in $\operatorname{Mod}(A)$ induce t-structures in $D(A)$. We present the description of such t-structures now without proof.

Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\operatorname{Mod}(A)$, and consider the following subcategories of $D(A)$.

$$D^{\leq 0}_{\mathcal{T}} := \{X \in D(A) \mid H^0(X) \in \mathcal{T},\ H^i(X) = 0\ \forall i > 0\}$$
$$D^{\geq 0}_{\mathcal{F}} := \{X \in D(A) \mid H^{-1}(X) \in \mathcal{F},\ H^i(X) = 0\ \forall i < -1\}$$

Then $(D^{\leq 0}_{\mathcal{T}}, D^{\geq 0}_{\mathcal{F}})$ is a t-structure in $D(A)$.

This final example will be of particular interest as it relates directly to silting complexes.

Example 5.6. It was shown in [AJS03, Proposition 3.2] that for any X in $D(A)$, the smallest triangulated subcategory of $D(A)$ containing X which is closed under coproducts, extensions and positive shifts is an aisle in a t-structure. Denote this subcategory by aisle(X), then the pair $(\text{aisle}(X), X^{\perp_{<0}})$ is a t-structure in $D(A)$.

In [DBB83, Theorem 1.3.6] it was shown that the heart of a t-structure in a triangulated category D is always an abelian subcategory of D. The following example illustrates how in some cases, starting with a module category, forming the standard t-structure in the derived category and then taking the heart, one recovers the module category.

Example 5.7. Let k be an algebraically closed field, and $\Lambda = k\Gamma$ the path algebra over the quiver Γ

$$1 \longrightarrow 2$$

There are three isomorphism classes of indecomposable left Λ-modules in $\text{mod}(\Lambda)$, given by the representations

$$P_2 = 0 \longrightarrow k \qquad\qquad P_1 = k \longrightarrow k \qquad\qquad S_1 = k \longrightarrow 0$$

Taking their stalk complexes in $D^b(\Lambda)$, they induce isomorphism classes M_1, M_2, M_3, respectively, of indecomposable objects in $D^b(\Lambda)$. We get further isomorphism classes recursively via the shift functor, letting $M_i[1] = M_{i+3}$ and $M_i[-1] = M_{i-3}$. Then we have the AR-quiver of $D^b(\Lambda)$

$$\begin{tikzcd}
 & M_0 \arrow[dr, dashed, "\leq 0"] & & M_2 \arrow[dr] & & M_4 \arrow[dr] & \\
\cdots \arrow[ur] & & M_1 \arrow[ur] & & M_3 \arrow[ur, dashed, "\geq 0"] & & \cdots
\end{tikzcd}$$

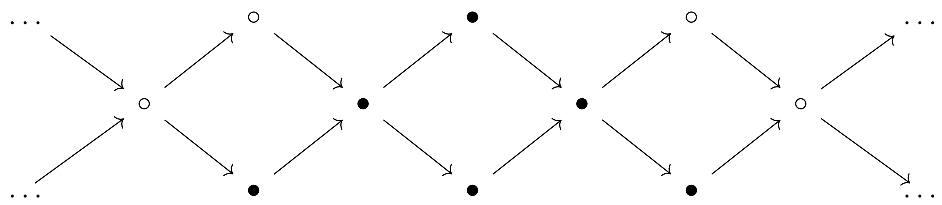
Take the standard t-structure $(D^{\leq 0}, D^{\geq 0})$ in $D^b(\Lambda)$. In the quiver above, all objects to the left of the dashed line marked ≥ 0 lie in $D^{\geq 0}$ and all objects to the right of the dashed line marked ≤ 0 lie in $D^{\leq 0}$. Clearly, the heart $D^{\leq 0} \cap D^{\geq 0} = \text{add}(M_1 \oplus M_2 \oplus M_3) \cong \text{mod}(\Lambda)$. $\square$

Example 5.8. Another example, which is slightly more interesting, is to follow the same construction as in the previous example, but for the following quiver Γ instead.

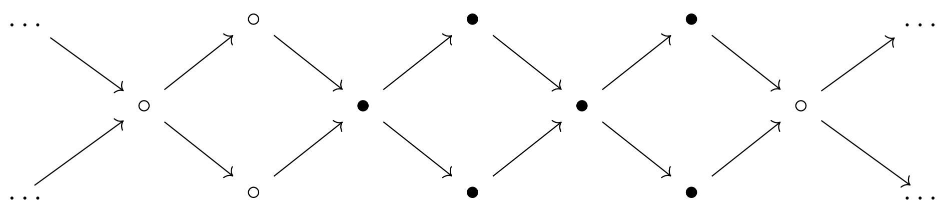
$$1 \longrightarrow 2 \longrightarrow 3$$

Let k be an algebraically closed field and $\Lambda = k\Gamma$ the path algebra of Γ. See for instance example 3.22 for the AR-quiver of Λ in $\text{mod}(\Lambda)$. As in the the previous example, the stalk complexes of the indecomposable modules in $\text{mod}(\Lambda)$ induce isomorphism classes of indecomposable objects in $D^b(\Lambda)$.

Form the standard t-structure $(D^{\leq 0}, D^{\geq 0})$ in $D^b(\Lambda)$. The following diagram is the AR-quiver of $D^b(\Lambda)$, but with dots representing the objects. The black dots represent the objects which are in the heart $D^{\leq 0} \cap D^{\geq 0}$, and one easily sees that $D^{\leq 0} \cap D^{\geq 0} \cong \mathrm{mod}(\Lambda)$.



One can manually adjust the t-structure to a new t-structure $(N^{\leq 0}, N^{\geq 0})$ so that the heart $N^{\leq 0} \cap N^{\geq 0}$ is given in the following AR-quiver.



Then $N^{\leq 0} \cap N^{\geq 0} \cong \mathrm{mod}(\Lambda')$ where Λ' is the path algebra over the quiver Γ'

$$1 \longrightarrow 2 \longleftarrow 3$$

$\square$

The following definition of silting complexes is from [Wei13, Definition 3.1], in the reference called **big semi tilting**.

Definition 5.9. A complex σ in $K^b(\mathrm{Proj}(A))$ is called **presilting** if it satisfies

(1) $\mathrm{Hom}_{D(A)}(\sigma, \sigma^{(I)}[i]) = 0$ for all sets I and all $i > 0$.

and it is called **silting** if it also satisfies

(2) The smallest triangulated subcategory of $D(A)$ which contains $\mathrm{Add}(\sigma)$ is $K^b(\mathrm{Proj}(A))$.

A complex of projective A-modules concentrated between degress $-n+1$ and 0 will be called n-term. An n-term complex will be called n-presilting, respectively n-silting, if it is presilting, or silting. We say that a complex X in $D(A)$ *generates* $D(A)$ if whenever $\mathrm{Hom}_{D(A)}(X[i], Y) = 0$ for all $i \in \mathbb{Z}$, then $Y = 0$.

Remark 5.10. If $\mathcal{X}$ is a class of objects all of which generate $D(A)$, then $\mathcal{X}$ is called a class of generators. It follows from [AJS00, Proposition 4.5] and [NS09, Lemma 2.2(1)] that an object X in $D(A)$ generates $D(A)$ if and only if the smallest triangulated subcategory of $D(A)$ containing X which is closed under coproducts is $D(A)$.

We include two results from [Kel98] which will prove useful when working with $D(A)$. The first is [Kel98, Theorem on p.158], which gives an equivalence of categories, which we will not prove since for our purpose, we only need that the equivalence exists.

Theorem 5.11. *Let $K_p(A)$ be the triangulated subcategory of $K(A)$ from example 5.4, then there is an equivalence of categories*

$$K_p(A) \cong D(A).$$

$\square$

The second is [Kel98, Proposition on p. 158], which we will prove as the proof gives insight when used in the proof of proposition 5.14.

Proposition 5.12. *Let U be a triangulated subcategory of $D(A)$. Then U equals $D(A)$ if and only if U contains A and is closed under coproducts.*

Proof. By theorem 5.11 we have $D(A) \cong K_p(A)$, so we prove the corresponding statement for $K_p(A)$.

Let U be the smallest triangulated subcategory of $K_p(A)$ which contains A and is closed under coproducts. Note that U contains every free A-module F since $A \in U$. Furthermore, by taking cones, U also contains every finite complex of free A-modules. We show that U also contains all projective A-modules.

Let $P \in \mathrm{Mod}(A)$ be projective, then there exists an A-module Q such that $P \oplus Q \cong F$ for some free A-module F. Let

$$e := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} : P \oplus Q \to P \oplus Q$$

where $\mathrm{Im}(e) = P$. Then there is a free resolution of P, denoted by F_P, as follows

$$\dots \xrightarrow{1-e} P \oplus Q \xrightarrow{e} P \oplus Q \xrightarrow{1-e} P \oplus Q \longrightarrow 0.$$

Consider the morphism $f : F_P \to F_P$ constructed as the composition

$$F_P \longrightarrow P \longrightarrow F_P$$

where $f = e$ in degree 0, and zero elsewhere. We will show that $1 - f$ is null-homotopic, which implies that P as a stalk complex is isomorphic in $K(A)$ to its free resolution F_P. We have the following commutative diagram

$$\begin{tikzcd}
F_P \arrow[d, "1-f"] & \dots \arrow[r, "1-e"] & P \oplus Q \arrow[r, "e"] \arrow[d, "1"] & P \oplus Q \arrow[r, "1-e"] \arrow[d, "1"] \arrow[dl, "1"'] & P \oplus Q \arrow[r] \arrow[d, "1-e"] \arrow[dl, "1"'] & 0 \\
F_P & \dots \arrow[r, "1-e"] & P \oplus Q \arrow[r, "e"] & P \oplus Q \arrow[r, "1-e"] & P \oplus Q \arrow[r] & 0
\end{tikzcd}$$

i.e. $1-f$ is null-homotopic. So $P \cong F_P$ in $K(A)$. Now, let F_{-n} denote the complex concentrated in degrees $0, -1, \ldots, -n$ with a free A-modules F in each degree and idempotent differentials as above. We now invoke Milnor's triangle

$$\bigoplus_{n\in\mathbb{N}} F_{-n} \longrightarrow \bigoplus_{m\in\mathbb{N}} F_{-m} \longrightarrow \varinjlim F_{-m} \longrightarrow \big(\bigoplus_{n\in\mathbb{N}} F_{-n}\big)[1]$$

where $\varinjlim F_{-m}$ is an infinite complex right bounded at 0, with a free A-module F in each degree and idempotent differentials. Each $F_{-n} \in U$ by the arguments in the beginning of the proof, and then also coproducts of F_{-n} are in U. Then, since the first two and last terms in Milnor's triangle are in U, so is the term $\varinjlim F_{-m}$. But such limits are precisely the free resolutions of projective A-modules, which we have shown are isomorphic in $K(A)$. Therefore, U contains all projective A-modules. Then it follows that U equals $K_p(A)$ since it is the smallest triangulated subcategory containing all projectives. □

We also need the following, which is a combination of results from [Wei13]. The proofs requires a different approach to silting as well as several preliminary results, so we only state the results.

Proposition 5.13. *Let $\sigma \in K^b(\mathrm{Proj}(A))$ and $\sigma \in D^{\leq 0}$, then the following hold.*

(1) If σ is presilting, then it is silting if and only if $\sigma^{\perp_{>0}} \subseteq D^{\leq 0}$.

(2) If σ is n-silting, then $D^{\leq -n} \subseteq \sigma^{\perp_{>0}} \subseteq D^{\leq 0}$, and $\sigma^{\perp_{>0}}$ is closed under coproducts.

Proof. (1): See [Wei13, Proposition 3.12].
(2): See [Wei13, Lemma 4.1] and [Wei13, Proposition 4.2]. □

We can now prove the following proposition, which is the first step towards a correspondence between n-silting complexes in $D(A)$ and t-structures in $D(A)$.

Proposition 5.14. *Let σ be an n-term complex in $K^b(\mathrm{Proj}(A))$. The following are equivalent.*

(1) σ is n-silting.

(2) σ is a presilting generator of $D(A)$ and $\sigma^{\perp_{>0}} \cap D^{\leq 0}$ is closed under coproducts in $D(A)$.

(3) $\mathrm{aisle}(\sigma) = \sigma^{\perp_{>0}}$.

(4) σ is presilting and $\sigma^{\perp_{>0}} \subseteq D^{\leq 0}$.

Proof. (1) $\Rightarrow$ (2) : By proposition 5.13(2) we have that $\sigma^{\perp_{>0}}$ is closed under coproducts in $D(A)$. Since σ is silting by assumption, then the smallest triangulated subcategory of $D(A)$ containing $\mathrm{Add}(\sigma)$ is $K^b(\mathrm{Proj(A)})$, which contains A as a stalk complex. Therefore, by proposition 5.12 the smallest triangulated subcategory of $D(A)$ which contains σ and is closed under coproducts is $D(A)$. Therefore by remark 5.10 we have that σ generates $D(A)$.

(2) $\Rightarrow$ (3) : It is easy to see that the subcategory $\sigma^{\perp_{>0}} \cap D^{\leq 0}$ is closed under positive shifts and extensions. By assumption, it is also closed under coproducts, and since σ is n-presilting, we have $\sigma \in \sigma^{\perp_{>0}} \cap D^{\leq 0}$. By example 5.6, the smallest such category is aisle(σ), therefore aisle(σ) $\subseteq \sigma^{\perp_{>0}} \cap D^{\leq 0}$. For any Y in $\sigma^{\perp_{>0}}$, there is a triangle associated with the t-structure (aisle(σ)$, \sigma^{\perp_{<0}}$)

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

where $X \in$ aisle(σ) and $Z \in \sigma^{\perp_{<0}}[-1] = \sigma^{\perp_{\leq 0}}$. Because aisle($\sigma$) $\subseteq \sigma^{\perp_{>0}} \cap D^{\leq 0}$, we have $X \in \sigma^{\perp_{>0}}$, and $Y \in \sigma^{\perp_{>0}}$ by assumption. Then it follows by the triangle that $Z \in \sigma^{\perp_{>0}}$. But then $Z \in \sigma^{\perp_{>0}} \cap \sigma^{\perp_{\leq 0}}$, and so $Z = 0$ since σ is a generator of $D(A)$. Then it follows that $X \cong Y$ and aisle(σ) $= \sigma^{\perp_{>0}}$.

(3) $\Rightarrow$ (4) : Since $\sigma \in$ aisle(σ) $= \sigma^{\perp_{>0}}$, it is n-presilting. We show that aisle(σ) $\subseteq D^{\leq 0}$. Consider the class $\mathcal{X}$ of objects coming from isomorphisms, coproducts and positive shifts starting with σ. These objects must all belong to aisle(σ) and $D^{\leq 0}$ since $\sigma \in D^{\leq 0}$. Let $X, Z \in \mathcal{X} \subseteq D^{\leq 0}$ and let the following be a triangle.

$$X \longrightarrow Y \longrightarrow Z \xrightarrow{f} X[1] \longrightarrow Y[1]$$

Now, $Z \in D^{\leq 0}$ and $X[1] \in D^{\leq -1}$. We have $Y[1] \cong \mathrm{Cone}(f)$, and writing out the complexes

$$\begin{array}{ccccccccccc}
Z = & \dots & \longrightarrow & Z_{-2} & \longrightarrow & Z_{-1} & \longrightarrow & Z_0 & \longrightarrow 0 & \longrightarrow & \dots \\
\downarrow & & & \downarrow{\scriptstyle f_{-2}} & & \downarrow{\scriptstyle f_{-1}} & & \downarrow & \downarrow & & \\
X[1] = & \dots & \longrightarrow & X_{-1} & \longrightarrow & X_0 & \longrightarrow & 0 & \longrightarrow 0 & \longrightarrow & \dots \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow & \downarrow & & \\
Y[1] = & \dots & \longrightarrow & X_{-1} \oplus Z_{-1} & \longrightarrow & X_0 \oplus Z_0 & \longrightarrow & 0 & \longrightarrow 0 & \longrightarrow & \dots
\end{array}$$

we have $Y[1] \in D^{\leq -1}$ and $Y \in D^{\leq 0}$. Thus, all objects coming from isomorphisms, coproducts, positive shifts and extensions starting with σ are contained in $D^{\leq 0}$. That is,

$$\mathrm{aisle}(\sigma) = \sigma^{\perp_{>0}} \subseteq D^{\leq 0}$$

and we are done.

(4) $\Rightarrow$ (1) : It follows directly from proposition 5.13(1). $\square$

Proposition 5.14 gives a particular bijection between t-structures and n-silting complexes. We aim to prove a more general bijection, for which we need the following definition.

Definition 5.15. (1) A t-structure $(\mathcal{V}^{\leq 0}, \mathcal{V}^{\geq 0})$ (respectively a co-t-structure $(\mathcal{U}_{\geq 0}, \mathcal{U}_{\leq 0})$) in $D(A)$ is said to be **intermediate** if there are $a, b \in \mathbb{Z}$, $a \leq b$, such that $D^{\leq a} \subseteq \mathcal{V}^{\leq 0} \subseteq D^{\leq b}$ (respectively $K_{\leq a} \subseteq \mathcal{U}_{\leq 0} \subseteq K_{\leq b}$)

(2) A t-structure $(\mathcal{V}^{\leq 0}, \mathcal{V}^{\geq 0})$ is said to be **silting** if it is intermediate and there is a generator σ of $D(A)$ such that $\mathcal{V}^{\leq 0} \cap {}^{\perp_0}(\mathcal{V}^{\leq 0}[1]) = \mathrm{Add}(\sigma)$. It is called **n-silting** if it also satisfies $D^{\leq -n+1} \subseteq \mathcal{V}^{\leq 0} \subseteq D^{\leq 0}$.

We now show that n-silting t-structures are precisely those which correspond to n-silting complexes. Later, we will show that equivalent silting complexes correspond to the same silting t-structure.

Lemma 5.16. *A t-structure $(\mathcal{V}^{\leq 0}, \mathcal{V}^{\geq 0})$ is n-silting with $\mathcal{V}^{\leq 0} \cap {}^{\perp_0}(\mathcal{V}^{\leq 0}[1]) = \mathrm{Add}(\sigma)$ if and only if σ is an n-silting complex and $\mathcal{V}^{\leq 0} = \sigma^{\perp_{>0}}$.*

Proof. Suppose that $(\mathcal{V}^{\leq 0}, \mathcal{V}^{\geq 0})$ is an n-silting t-structure with $\mathcal{V}^{\leq 0} \cap {}^{\perp_0}(\mathcal{V}^{\leq 0}[1]) = \mathrm{Add}(\sigma)$. Since $\sigma \in \mathrm{Add}(\sigma)$ we have

$$\mathrm{Hom}_{D(A)}(\sigma, \sigma[i]) = 0 \quad \text{for all } i > 0$$

and so σ is presilting. We will show that $\sigma^{\perp_{>0}} = \mathcal{V}^{\leq 0}$. Since $\mathcal{V}^{\leq 0} \subseteq \mathrm{Add}(\sigma)$, then $\sigma^{\perp_{>0}}$ will be closed under coproducts in $D(A)$. Then σ will be a silting complex by Proposition 5.14.

First we show $\mathcal{V}^{\leq 0} \subseteq \sigma^{\perp_{>0}}$. Let $X \in \mathcal{V}^{\leq 0}$, and since $\sigma \in \mathrm{Add}(\sigma)$, we have

$$\mathrm{Hom}_{D(A)}(\sigma, X[1]) = 0.$$

Recall that $\mathcal{V}^{\leq 0}[1] \subseteq \mathcal{V}^{\leq 0}$, so clearly

$$\mathrm{Hom}_{D(A)}(\sigma, X[i]) = 0 \quad \text{for all } i > 0$$

i.e. $\mathcal{V}^{\leq 0} \subseteq \sigma^{\perp_{>0}}$.

Next we show $\mathcal{V}^{\leq 0} \supseteq \sigma^{\perp_{>0}}$. Let $X \in \sigma^{\perp_{>0}}$ and consider the triangle associated with the t-structure

$$Y \longrightarrow X \longrightarrow Z \longrightarrow Y[1]$$

where $Y \in \mathcal{V}^{\leq 0}$ and $Z \in \mathcal{V}^{\geq 0}[-1] = \mathcal{V}^{\geq 1}$. Since $\sigma \in {}^{\perp_0}(\mathcal{V}^{\leq 0}[1])$ we have

$$\mathrm{Hom}_{D(A)}(\sigma, Y[i]) = 0 \quad \text{for all } i > 0$$

and by our assumption on X we have

$$\mathrm{Hom}_{D(A)}(\sigma, X[i]) = 0 \quad \text{for all } i > 0.$$

Therefore, by the triangle above we also have

$$\mathrm{Hom}_{D(A)}(\sigma, Z[i]) = 0 \quad \text{for all } i > 0.$$

Next, since $\sigma \in \mathcal{V}^{\leq 0}$, it's clear that

$$\mathrm{Hom}_{D(A)}(\sigma, Z[i]) = 0 \quad \text{for all } i \leq 0$$

and thus

$$\mathrm{Hom}_{D(A)}(\sigma, Z[i]) = 0 \quad \text{for all } i \in \mathbb{Z}.$$

By assumption, σ is a generator of $D(A)$, and therefore $Z = 0$. Thus $X \cong Y$, i.e. $X \in \mathcal{V}^{\leq 0}$.

Because $\mathcal{V}^{\leq 0} = \sigma^{\perp_{>0}} \subseteq \mathrm{Add}(\sigma)$, which is closed under coproducts in $D(A)$, then $\sigma^{\perp_{>0}}$ is closed under coproducts, and so σ is a silting complex in $D(A)$ by proposition 5.14.

It remains to show that σ is an n-term complex. The t-structure $(\mathcal{V}^{\leq 0}, \mathcal{V}^{\geq 0})$ is n-silting by assumption, so $\sigma \in \mathcal{V}^{\leq 0} \subseteq D^{\leq 0}$. Let $\sigma \in K^b(\mathrm{Proj}(A))$ be of the form $(P_i, d_i)_{i\in\mathbb{Z}}$ and $P_i = 0$ for all $i > 0$. Since $(\mathcal{V}^{\leq 0}, \mathcal{V}^{\geq 0})$ is n-silting, $D^{\leq -n+1} \subseteq \mathcal{V}^{\leq 0}$ and $\sigma \in {}^{\perp_0}(\mathcal{V}^{\leq 0}[1])$, so therefore

$$\mathrm{Hom}_{D(A)}(\sigma, X) = 0 \quad \text{for all } X \in D^{\leq -n}.$$

Consider the co-t-structure $(K_{\geq 0}, K_{\leq 0})$ in $K_p(A)$ and fit σ in a triangle given by stupid truncations

$$X \longrightarrow \sigma \xrightarrow{\;u\;} Y \longrightarrow X[1]$$

where $X \in K_{\geq -n+1} \cap K_{\leq 0}$ and $Y \in K_{\leq -n} = D^{\leq -n}$. Then u is zero, so the triangle splits and σ is a summand of X, i.e. σ is n-term.

Now we prove the other implication in the lemma. Let σ be n-silting. We will show that $(\sigma^{\perp_{>0}}, \sigma^{\perp_{<0}})$ is an n-silting t-structure satisfying the required properties of the lemma. Clearly, $D^{\leq -n+1} \subseteq \sigma^{\perp_{>0}}$, and from Proposition 5.14 we have $\sigma^{\perp_{>0}} \subseteq D^{\leq 0}$ and that σ generates $D(A)$. Clearly $\mathrm{Add}(\sigma) \subseteq \sigma^{\perp_{>0}} \cap {}^{\perp_0}(\sigma^{\perp_{>0}}[1])$ since $\sigma \in \sigma^{\perp_{>0}} \cap {}^{\perp_0}(\sigma^{\perp_{>0}}[1])$. To show the other inclusion, let $X \in \sigma^{\perp_{>0}} \cap {}^{\perp_0}(\sigma^{\perp_{>0}}[1])$. Let $I = \mathrm{Hom}_{D(A)}(\sigma, X)$, then there is a universal morphism $u : \sigma^{(I)} \to X$ which induces a triangle by (TR1)

$$K \longrightarrow \sigma^{(I)} \xrightarrow{\;u\;} X \xrightarrow{\;v\;} K[1]$$

Apply $\mathrm{Hom}_{D(A)}(\sigma, -)$ to the triangle to get a long exact

$$\cdots \longrightarrow \mathrm{Hom}_{D(A)}(\sigma, K) \longrightarrow \mathrm{Hom}_{D(A)}(\sigma, \sigma^{(I)}) \xrightarrow{\;u_*\;} \mathrm{Hom}_{D(A)}(\sigma, X) \longrightarrow \mathrm{Hom}_{D(A)}(\sigma, K[1]) \longrightarrow \cdots$$

and observe that the morphism u_* is surjective because of the universal property of u. Therefore, $\mathrm{Hom}_{D(A)}(\sigma, K[1]) = 0$. Furthermore, we have

$$\mathrm{Hom}_{D(A)}(\sigma, \sigma^{(I)}[i+1]) = 0 = \mathrm{Hom}_{D(A)}(\sigma, X[i]) \quad \text{for all } i > 0$$

so we have $\mathrm{Hom}_{D(A)}(\sigma, K[i]) = 0$ for all $i > 0$. By assumption, $X \in {}^{\perp_0}(\sigma^{\perp_{>0}}[1])$, and we have shown that $K \in \sigma^{\perp_{>0}}$, so then the morphism v in the triangle is zero. Then the triangle splits and $X \in \mathrm{Add}(\sigma)$. Then $(\sigma^{\perp_{>0}}, \sigma^{\perp_{<0}})$ is an n-silting t-structure. $\square$

It is clear from the lemma that two silting complexes σ and ω satisfy $\mathrm{Add}(\sigma) = \mathrm{Add}(\omega)$ if and only if $\sigma^{\perp_{>0}} = \omega^{\perp_{>0}}$. Thus, we define two silting complexes σ and ω to be equivalent if $\mathrm{Add}(\sigma) = \mathrm{Add}(\omega)$.

We are now almost ready to give the proof of theorem 5.21, one of the main results in [HMV15]. It is however quite involved, so we will first prove a few lemmas to make proof easier to digest. The following definitions are from [Wei13].

Definition 5.17. Let $\mathcal{C} \subseteq \mathcal{D}$ be a triangulated subcategory. We say that

(1) $\mathcal{C}$ is **specially covariantly finite** in $\mathcal{D}$ if for every $D \in \mathcal{D}$, there exists a triangle

$$D \longrightarrow C \longrightarrow B \longrightarrow D[1] \tag{5.1}$$

for some $C \in \mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{D}}(B, C'[1]) = 0$ for all $C' \in \mathcal{C}$.

(2) $\mathcal{C}$ is **coresolving** if it is closed under extensions and $\mathcal{C}[1] \subseteq C$.

(3) $\mathcal{D}$ has **finite $\mathcal{C}$-coresolutions** if for every $D \in \mathcal{D}$, there exists a collection of objects $\{C_i\}$ in $\mathcal{C}$ for $0 \leq i \leq n$ and a finite sequence of triangles

$$\begin{gathered} X_0 \to C_0 \to X_1 \to X_0[1] \\ X_1 \to C_1 \to X_2 \to X_1[1] \\ \cdots \\ X_{n-1} \to C_{n-1} \to C_n \to X_{n-1}[1] \end{gathered}$$

The following theorem from [Wei13] is important. It gives a bijection between equivalence classes of silting complexes in $D^{\leq 0}$ and subcategories of $D^{\leq 0}$ with certain properties, which turns out to be precisely the subcategories corresponding to intermediate co-t-structures in $D^-(A)$ with the aisle being closed under coproducts in $D(A)$. We only state the theorem, see [Wei13, Theorem 5.3] for the proof, but the properties given in the theorem allows us to prove the two important lemmas which follow.

Theorem 5.18. *There is a bijection between equivalence classes of silting complexes in $D^{\leq 0}$ and subcategories $\mathcal{U} \subseteq D^{\leq 0}$ which are coresolving, specially covariantly finite in $D^{\leq 0}$, closed under coproducts in $D(A)$ such that $D^-(A)$ has finite $\mathcal{U}$-coresolutions. The bijection is given by $\sigma \mapsto \sigma^{\perp_{>0}}$.* □

The aim of the following two lemmas is to show that silting complexes in $D^{\leq 0}$ correspond bijectively to intermediate co-t-structures $(\mathcal{U}_{\geq 0}, \mathcal{U}_{\leq 0})$ in $D^-(A)$ with $\mathcal{U}_{\leq 0}$ being closed under coproducts in $D(A)$. We do this in two steps, first we show that the subcategories $\mathcal{U}$ in theorem 5.18 being coresolving and specially covariantly finite in $D^-(A)$ is equivalent to $({}^{\perp_0}(\mathcal{U}[1]), \mathcal{U})$ being a co-t-structure in $D^-(A)$. Next, we show that additionally, $D^-(A)$ having finite $\mathcal{U}$-coresolutions is equivalent to $({}^{\perp_0}(\mathcal{U}[1]), \mathcal{U})$ being intermediate.

Lemma 5.19. *Let $\mathcal{U} \subseteq D^-(A)$ be a subcategory. Then $\mathcal{U}$ is coresolving and specially covariantly finite in $D^-(A)$ if and only if $({}^{\perp_0}(\mathcal{U}[1]), \mathcal{U})$ is a co-t-structure in $D^-(A)$.*

Proof. Suppose that $\mathcal{U}$ is coresolving and specially covariantly finite in $D^-(A)$. We verify the co-t-structure axioms for $({}^{\perp_0}(\mathcal{U}[1]),\mathcal{U})$ in $D^-(A)$.

Axiom 1: $\mathrm{Hom}_{D^-(A)}({}^{\perp_0}(\mathcal{U}[1]),\mathcal{U}[1]) = 0$ by definition.

Axiom 2: Let $X \in {}^{\perp_0}(\mathcal{U}[1])$. Since $\mathcal{U}$ is coresolving, we have $\mathcal{U}[2] \subseteq \mathcal{U}[1]$ and therefore $\mathrm{Hom}_{D^-(A)}(X[-1],\mathcal{U}[1]) = \mathrm{Hom}_{D^-(A)}(X,\mathcal{U}[2]) = 0$. Thus, ${}^{\perp_0}(\mathcal{U}[1])$ is closed under $[-1]$.

Axiom 3: Let $X \in D^-(A)$. Since $\mathcal{U}$ is specially covariantly finite in $D^-(A)$ there exists a triangle

$$X[-1] \longrightarrow Y \longrightarrow Z \longrightarrow X$$

where $Y \in \mathcal{U}$ and $\mathrm{Hom}_{D^-(A)}(Z,U[1]) = 0$ for all $U \in \mathcal{U}$. So $Z \in {}^{\perp_0}(\mathcal{U}[1])$, and because $\mathcal{U}$ is coresolving, $Y[1] \in \mathcal{U}$. Then the following is a co-t-structure triangle

$$Z \longrightarrow X \longrightarrow Y[1] \longrightarrow Z[1]$$

Conversely, suppose that $({}^{\perp_0}(\mathcal{U}[1]),\mathcal{U})$ is a co-t-structure in $D^-(A)$.

Proving $\mathcal{U}$ is coresolving: By remark 5.2, we have $\mathcal{U}[1] \subseteq \mathcal{U}$ and $\mathcal{U} = ({}^{\perp_0}(\mathcal{U}[1]))^{\perp_0}[-1]$. Consider a triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

where $X, Z \in \mathcal{U}$. Let $W \in {}^{\perp_0}(\mathcal{U}[1])$ and apply $\mathrm{Hom}_{D^-(A)}(W[-1],-)$ to the triangle. One sees that $\mathrm{Hom}_{D^-(A)}(W[-1],Y) = 0$, so

$$Y \in ({}^{\perp_0}(\mathcal{U}[1]))^{\perp_0}[-1] = \mathcal{U}$$

Thus, $\mathcal{U}$ is coresolving.

Proving $\mathcal{U}$ is specially covariantly finite in $D^-(A)$: Let $X \in D^-(A)$, then there is a co-t-structure triangle

$$Y \longrightarrow X[1] \longrightarrow Z \longrightarrow Y[1]$$

where $Y \in {}^{\perp_0}(\mathcal{U}[1])$ and $Z \in \mathcal{U}[1]$. Then we also have a triangle

$$X \longrightarrow Z[-1] \longrightarrow Y \longrightarrow X[1]$$

where $Z[-1] \in \mathcal{U}$ and $Y \in {}^{\perp_0}(\mathcal{U}[1])$, satisfying property 5.17(1). □

Lemma 5.20. *There is a bijection between equivalence classes of silting complexes in $D^{\leq 0}$ and intermediate co-t-structures $(\mathcal{U}_{\geq 0},\mathcal{U}_{\leq 0})$ in $D^-(A)$ with $\mathcal{U}_{\leq 0}$ closed under coproducts in $D(A)$, given by the bijection $\sigma \mapsto \sigma^{\perp_{>0}}$ from theorem 5.18.*

Proof. Suppose that $\sigma \in D^{\leq 0}$ is silting and $\mathcal{U} \subseteq D^-(A)$ the corresponding subcategory according to theorem 5.18. By lemma 5.19, the equivalence classes of silting complexes in $D(A)$ correspond bijectively to co-t-structures $({}^{\perp_0}(\mathcal{U}[1]),\mathcal{U})$ in $D^-(A)$. By theorem 5.18, $\mathcal{U}$

is closed under coproducts in $D(A)$. It remains to show that the co-t-structures arising in this way from silting complexes are intermediate. To do that, we prove that $D^-(A)$ having finite $\mathcal{U}$-coresolutions is equivalent to $({}^{\perp_0}(\mathcal{U}[1]), \mathcal{U})$ being intermediate.

First, since σ belongs to both $K^b(\mathrm{Proj}(A))$ and $D^{\leq 0}$, it is concentrated between degrees $-n+1$ and 0 for some $n \in \mathbb{N}$, i.e. it is n-silting. Then by proposition 5.14 we have that $\sigma^{\perp_{>0}} \subseteq D^{\leq 0}$. It's obvious that $D^{\leq -n+1} \subseteq \sigma^{\perp_{>0}}$. The subcategory $\mathcal{U} \subseteq D^-(A)$ is determined by the assignment $\sigma \mapsto \sigma^{\perp_{>0}}$. So then $D^{\leq -n+1} \subseteq \mathcal{U} \subseteq D^{\leq 0}$ and the co-t-structure $({}^{\perp_0}(\mathcal{U}[1]), \mathcal{U})$ is intermediate.

Conversely, suppose that $(\mathcal{U}_{\geq 0}, \mathcal{U}_{\leq 0})$ is a co-t-structure in $D^-(A)$ with $\mathcal{U}_{\leq 0}$ closed under coproducts such that $K_{\leq -n} \subseteq \mathcal{U}_{\leq 0} \subseteq K_{\leq 0}$ for some n. We then show that $D^-(A)$ has finite $\mathcal{U}_{\leq 0}$-coresolutions, which finishes the proof.

Let $X \in D^-(A)$ with $H^i(X) = 0$ for all $i > k$ for some integer k. Before constructing a finite $\mathcal{U}_{\leq 0}$-coresolution of X, we show that X can actually be taken in $D^{\leq 0}$. That is, we will show that starting with $X \in D^-(A)$, we can construct a sequence of triangles yielding an object $Y \in D^{\leq 0}$, and we then show that any $\mathcal{U}_{\leq 0}$-coresolutions of Y must be finite.

If $k = 0$ then $X \in D^{\leq 0}$, so suppose that $k > 0$ and consider the triangle coming from the co-t-structure $(\mathcal{U}_{\geq 0}, \mathcal{U}_{\leq 0})$

$$X \longrightarrow U_0 \longrightarrow C_0 \longrightarrow X[1]$$

where $U_0 \in \mathcal{U}_{\leq 0}$ and $C_0 \in \mathcal{U}_{\geq 0}$. Taking homology of the triangle and using that $\mathcal{U}_{\leq 0} \subseteq K_{\leq 0} = D^{\leq 0}$, we have

$$\cdots \longrightarrow H^k(X) \longrightarrow \underbrace{H^k(U_0)}_{=0} \longrightarrow H^k(C_0) \longrightarrow \underbrace{H^{k+1}(X)}_{=0} \longrightarrow \cdots$$

and thus $H^i(C_0) = 0$ for all $i > k-1$. We can apply the same process to C_0, yielding an object C_1 such that $H^i(C_1) = 0$ for all $i > k-2$. Therefore, we can construct a finite sequence of co-t-structure triangles

$$\begin{gathered} X \to U_0 \to C_0 \to X[1] \\ C_0 \to U_1 \to C_1 \to C_0[1] \\ \cdots \\ C_{k-2} \to U_{k-1} \to C_{k-1} \to C_{k-2}[1] \end{gathered}$$

where $C_{k-1} \in D^{\leq 0}$. So, without loss of generality, assume we start with $X \in D^{\leq 0}$ and construct a sequence of co-t-structure triangles

$$\begin{gathered} X \to U_0 \to C_0 \to X[1] \\ C_0 \to U_1 \to C_1 \to C_0[1] \\ \cdots \\ C_{n-1} \to U_n \to C_n \to C_{n-1}[1] \end{gathered}$$

where $U_i \in \mathcal{U}_{\leq 0}$ and $C_i \in \mathcal{U}_{\geq 0}$ for all $0 \leq i \leq n$, where n is the natural number such that $K_{\leq -n} \subseteq \mathcal{U}_{\leq 0}$.

Applying the functor $\mathrm{Hom}_{D(A)}(C_n, -)$ to all the triangles yield n long exact sequences, and because $\mathrm{Hom}_{D(A)}(\mathcal{U}_{\geq 0}, \mathcal{U}_{\leq 0}[1]) = 0$, many of the terms become zero and we get a sequence of isomorphisms. The following "diagram" shows the isomorphisms, where the vertical sequences are the long exact sequences coming from all the triangles, and the top and bottom "rows" are all zero. It is a bit hard to digest, but we include it so that the reader does not have to write out the sequences by hand. For the sake of simplicity, we write $\mathrm{Hom}_{D(A)}(X, Y) = (X, Y)$ in the "diagram".

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] \\
(C_n, U_0[n]) \arrow[d] & (C_n, U_1[n-1]) \arrow[d] & \cdots & (C_n, U_{n-2}[2]) \arrow[d] & (C_n, U_{n-1}[1]) \arrow[d] \\
(C_n, C_0[n]) \arrow[d, "\cong"] \arrow[dr, equal] & (C_n, C_1[n-1]) \arrow[d, "\cong"] \arrow[dr, equal] & \cdots \arrow[dr, equal] & (C_n, C_{n-2}[2]) \arrow[d, "\cong"] \arrow[dr, equal] & (C_n, C_{n-1}[1]) \arrow[d, "\cong"] \\
(C_n, X[n+1]) \arrow[d] & (C_n, C_0[n]) \arrow[d] & \cdots & (C_n, C_{n-3}[3]) \arrow[d] & (C_n, C_{n-2}[2]) \arrow[d] \\
(C_n, U_0[n+1]) \arrow[d] & (C_n, U_1[n]) \arrow[d] & \cdots & (C_n, U_{n-2}[3]) \arrow[d] & (C_n, U_{n-1}[2]) \arrow[d] \\
\vdots & \vdots & & \vdots & \vdots
\end{tikzcd}$$

From the "diagram" above, we get

$$\mathrm{Hom}_{D(A)}(C_n, C_{n-1}[1]) \cong \mathrm{Hom}_{D(A)}(C_n, C_{n-i}[i]) \quad \text{for all } 1 \leq i \leq n$$

and in particular we get the isomorphism

$$\mathrm{Hom}_{D(A)}(C_n, C_{n-1}[1]) \cong \mathrm{Hom}_{D(A)}(C_n, X[n+1])$$

Now, since $X \in D^{\leq 0}$ we get $X[n+1] \in D^{\leq -n-1} = D^{\leq -n}[1]$. We have by assumption that $D^{\leq -n} = K_{\leq -n} \subseteq \mathcal{U}_{\leq 0}$, so then $D^{\leq -n}[1] \subseteq \mathcal{U}_{\leq 0}[1]$. Moreover, $C_n \in \mathcal{U}_{\geq 0}$ and

$$\mathrm{Hom}_{D(A)}(\mathcal{U}_{\geq 0}, \mathcal{U}_{\leq 0}[1]) = 0$$

so we have

$$\mathrm{Hom}_{D(A)}(C_n, X[n+1]) = 0 = \mathrm{Hom}_{D(A)}(C_n, C_{n-1}[1])$$

Thus the triangle

$$C_{n-1} \longrightarrow U_n \longrightarrow C_n \xrightarrow{0} C_{n-1}[1]$$

splits, so $C_{n-1} \in \mathcal{U}_{\leq 0}$. Then since

$$\operatorname{Hom}_{D(A)}(C_n[-1], C_{n-1}) = \operatorname{Hom}_{D(A)}(C_n, C_{n-1}[1]) = 0$$

the solid part of the following diagram commutes, so there exists a map $U_n \to C_{n-1}$ indicated by the dashed arrow such that the whole diagram commutes

$$\begin{tikzcd} C_n[-1] \arrow[r] \arrow[dr, bend left, "0"'] & C_{n-1} \arrow[r] \arrow[d, "1"] & U_n \arrow[r] \arrow[dl, dashed] & C_n \\ & C_{n-1} & & \end{tikzcd}$$

and $C_{n-1} \cong U_n$. So $C_n = 0$ and therefore the sequence constructed above stops, i.e. the $\mathcal{U}_{\leq 0}$-coresolution of X is finite. □

We are now ready to prove the main theorem.

Theorem 5.21. *There exists bijections between*

(1) equivalence classes of silting complexes in $D(A)$.

(2) silting t-structures in $D(A)$.

(3) intermediate co-t-structures $(\mathcal{U}_{\geq 0}, \mathcal{U}_{\leq 0})$ in $D(A)$ with $\mathcal{U}_{\leq 0}$ being closed under coproducts in $D(A)$.

(4) triples $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of subcategories of $D(A)$ such that $(\mathcal{A}, \mathcal{B})$ is a co-t-structure and $(\mathcal{B}, \mathcal{C})$ is an intermediate t-structure.

Proof. We claim that the following assignments are bijections

Bijection	Assignment
(1) $\Rightarrow$ (2)	$\Psi : \sigma \mapsto (\sigma^{\perp_{>0}}, \sigma^{\perp_{<0}})$
(2) $\Rightarrow$ (1)	$\Theta : (\mathcal{V}^{\leq 0}, \mathcal{V}^{\geq 0}) \mapsto \sigma$ with $\operatorname{Add}(\sigma) = \mathcal{V}^{\leq 0} \cap {}^{\perp_0}(\mathcal{V}^{\leq 0}[1])$
(1) $\Rightarrow$ (3)	$\Phi : \sigma \mapsto ({}^{\perp_0}(\sigma^{\perp_{>0}}[1]), \sigma^{\perp_{>0}})$
(1) $\Rightarrow$ (4)	$\Omega : \sigma \mapsto ({}^{\perp_0}(\sigma^{\perp_{>0}}[1]), \sigma^{\perp_{>0}}, \sigma^{\perp_{<0}})$

Because two silting complexes σ and ω are equivalent if and only if $\sigma^{\perp_{>0}} = \omega^{\perp_{>0}}$, the assignments Ψ, Φ and Ω do not depend on the choice of representatives of the equivalence class. We will assume without loss of generality that silting complexes σ in $D(A)$ are concentrated in degrees less than or equal to 0, or that $\sigma^{\perp_{>0}}$ is contained in $D^{\leq 0} = K_{\leq 0}$. By lemma 5.16 we have that Ψ and Θ are mutual inverses. Clearly, if Ψ and Φ are bijections, then so is Ω as well. Thus, we only need to show that Φ is a bijective.

By lemma 5.20, equivalence classes of silting complexes σ are in bijection with intermediate co-t-structures $(\mathcal{U}_{\geq 0}, \mathcal{U}_{\leq 0})$ in $D^-(A)$ with $\mathcal{U}_{\leq 0}$ closed under coproducts in $D(A)$. We will prove that such co-t-structures in $D^-(A)$ are in bijection with the corresponding co-t-structures in $D(A)$. Specifically, we will prove that for such co-t-structures $(\mathcal{U}_{\geq 0}, \mathcal{U}_{\leq 0})$

in $D^-(A)$, the pair $({}^{\perp_0}(\mathcal{U}_{\leq 0}[1]), \mathcal{U}_{\leq 0})$ is an intermediate co-t-structure in $D(A)$. Note that by the proof of lemma 5.20, we retain the property $\mathcal{U}_{\leq 0} \subseteq K_{\leq 0}$.

The pair $({}^{\perp_0}(\mathcal{U}_{\leq 0}[1]), \mathcal{U}_{\leq 0})$ clearly satisfies axioms (1) and (2) in definition 5.1 for being a co-t-structure in $D(A)$, so we only have to show that it satisfies axiom (3).

Now we use the equivalence of categories $D(A) \cong K_p(A)$ from theorem 5.11, and the standard co-t-structure $(K_{\geq 0}, K_{\leq 0})$ in $K_p(A)$ from example 5.4.

Let $X \in D(A) \cong K_p(A)$, and using stupid truncations there is a triangle

$$Y \longrightarrow X \xrightarrow{\phi} Z \longrightarrow Y[1] \tag{5.2}$$

where $Y \in K_{\geq 1}$ and $Z \in K_{\leq 0}$. So then $Z \in D^-(A)$, and by the co-t-structure $(\mathcal{U}_{\geq 0}, \mathcal{U}_{\leq 0})$ there is a triangle

$$C[-1] \longrightarrow Z \xrightarrow{\theta} U \longrightarrow C \tag{5.3}$$

where $U \in \mathcal{U}_{\leq 0}$ and $C \in \mathcal{U}_{\geq 0} \subset {}^{\perp_0}(\mathcal{U}_{\leq 0}[1])$. The composition $X \xrightarrow{\theta\phi} U$ induces a triangle

$$X \xrightarrow{\theta\phi} U \longrightarrow \mathrm{Cone}(\theta\phi) \longrightarrow X[1] \tag{5.4}$$

By the octahedral axiom (T4)

$$\begin{tikzcd}
X \arrow[r, "\phi"] \arrow[d, equal] & Z \arrow[r] \arrow[d, "\theta"] & Y[1] \arrow[r] \arrow[d, dashed] & X[1] \arrow[d, equal] \\
X \arrow[r, "\theta\phi"] & U \arrow[r] \arrow[d] & \mathrm{Cone}(\theta\phi) \arrow[r] \arrow[d, dashed] & X[1] \arrow[d, "\phi[1]"] \\
 & C \arrow[r, equal] \arrow[d] & C \arrow[r] \arrow[d, dashed] & Z[1] \\
 & Z[1] \arrow[r] & Y[2] &
\end{tikzcd}$$

the three triangles 5.2, 5.3, 5.4 give rise to a fourth triangle

$$Y[1] \longrightarrow \mathrm{Cone}(\theta\phi) \longrightarrow C \longrightarrow Y[2]$$

Now, $Y[1] \in K_{\geq 0}$ by construction of triangle 5.2 and $\mathcal{U}_{\leq 0}[1] \subseteq K_{\leq -1}$ by assumption. So $Y[1] \in {}^{\perp_0}K_{\leq -1}$ and in particular $Y[1] \in {}^{\perp_0}(\mathcal{U}_{\leq 0}[1])$. Since $C \in {}^{\perp_0}(\mathcal{U}_{\leq 0}[1])$ by construction of triangle 5.3, we also get $\mathrm{Cone}(\theta\phi) \in {}^{\perp_0}(\mathcal{U}_{\leq 0}[1])$. Thus, in the triangle

$$\mathrm{Cone}(\theta\phi)[-1] \longrightarrow X \xrightarrow{\theta\phi} U \longrightarrow \mathrm{Cone}(\theta\phi)$$

we have $\mathrm{Cone}(\theta\phi) \in {}^{\perp_0}(\mathcal{U}_{\leq 0}[1])$ and $U \in \mathcal{U}_{\leq 0}$. So $({}^{\perp_0}(\mathcal{U}_{\leq 0}[1]), \mathcal{U}_{\leq 0})$ satisfies axiom (3) in definition 5.1.

Lastly, note that passing from the intermediate co-t-structure $(\mathcal{U}_{\geq 0}, \mathcal{U}_{\leq 0})$ in $D^-(A)$ to the intermediate co-t-structure $({}^{\perp_0}(\mathcal{U}_{\leq 0}[1]), \mathcal{U}_{\leq 0})$ in $D(A)$ is an injective assignment. The inverse is given by taking the intersection with $D^-(A)$. $\square$

5.2 Silting modules and 2-term silting complexes

The structure and results of this section closely follow that of section 4.2 in [HMV15]. The aim is to develop a correlation between silting modules and silting complexes, which also makes it possible to relate (co-)t-structures to silting modules via the results in section 5.1. We will show that taking homology of a 2-term silting complex induces a bijection between equivalence classes of silting modules and equivalence classes of 2-term silting complexes.

By proposition 5.14, the subcategory $\sigma^{\perp_0} \subseteq D(A)$ for a n-term complex σ is important for establishing if σ silting or not. Similarly, for an A-module T with projective presentation σ, the class $\mathcal{D}_\sigma$ from section 3.2 is important for establishing if T is silting or not. This next lemma provides a useful way of translating information between σ and $\mathcal{D}_\sigma$ for an 2-term complex σ.

Lemma 5.22. *The following hold for any 2-term complex $\sigma : P_{-1} \to P_0$ in $K^b(\mathrm{Proj}(A))$ with $T = H^0(\sigma)$.*

(1) Let $X \in D^{\leq 0}$. Then $X \in \sigma^{\perp_{>0}}$ if and only if $H^0(X) \in \mathcal{D}_\sigma$.
Furthermore, $\mathcal{D}_\sigma = \sigma^{\perp_{>0}} \cap \mathrm{Mod}(A)$.

(2) Let $X \in D^{\geq 0}$. Then $X \in \sigma^{\perp_{\leq 0}}$ if and only if $H^0(X) \in T^{\perp_0}$.
Furthermore, $T^{\perp_0} = \sigma^{\perp_{\leq 0}} \cap \mathrm{Mod}(A)$.

Proof. (1): Let $X = (X_i, d_i)_{i \in \mathbb{Z}} \in D^{\leq 0}$, and suppose then without loss of generality that $X_i = 0$ for all $i > 0$. Suppose that $X \in \sigma^{\perp_{>0}}$. Any map $h : P_{-1} \to H^0(X)$ lifts to some map $\tilde{h} : P_{-1} \to X_0$ via $\pi : X_0 \to H^0(X)$ as P_{-1} is projective. Then $\tilde{h}$ induces a map in $\mathrm{Hom}_{K(A)}(\sigma, X[1])$ which by assumption is null-homotopic as indicated by the following diagram.

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] \arrow[d] & P_{-1} \arrow[r, "\sigma"] \arrow[d, "\tilde{h}"] \arrow[dl, "s_{-1}"'] & P_0 \arrow[r] \arrow[d] \arrow[dl, "s_0"'] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & X_{-1} \arrow[r, "d_{-1}"] & X_0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

Then we have

$$h = \pi \tilde{h} = \pi(s_0 \sigma + d_{-1} s_{-1}) = \pi s_0 \sigma$$

so h factors through σ and $H^0(X) \in \mathcal{D}_\sigma$.

Conversely, let $H^0(X) \in \mathcal{D}_\sigma$, and take a map $f \in \mathrm{Hom}_{K(A)}(\sigma, X[1])$. The composition of $f_{-1} : P_{-1} \to X_0$ with $\pi : X_0 \to H^0(X)$ factors through σ via some map $h : P_0 \to X_0$. Then, since P_0 is projective, h lifts via π to some map $\tilde{h} : P_0 \to X_0$, so we have the following diagram

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] \arrow[d] & P_{-1} \arrow[r, "\sigma"] \arrow[d, "f_{-1}"] & P_0 \arrow[r] \arrow[d] \arrow[dl, dashed, "\tilde{h}"'] \arrow[ddl, dashed, "h"] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & X_{-1} \arrow[r, "d_{-1}"] & X_0 \arrow[r] \arrow[d, "\pi"] & 0 \arrow[r] & 0 \arrow[r] & \dots \\
& & H^0(X) & & &
\end{tikzcd}$$

Then we have
$$\pi(f_{-1} - \tilde{h}\sigma) = \pi f_{-1} - \pi\tilde{h}\sigma = \pi f_{-1} - h\sigma = 0$$
so $(f_{-1} - \tilde{h}\sigma)$ factors through X_{-1} via some map $g : P_{-1} \to X_{-1}$. Thus $f_{-1} = d_{-1}g + \tilde{h}\sigma$, and so f is null-homotopic, implying $X \in \sigma^{\perp_{>0}}$.

The arguments for showing $\mathcal{D}_\sigma = \sigma^{\perp_{>0}} \cap \mathrm{Mod}(A)$ are similar to the ones above.

(2): Let $X = (X_i, d_i)_{i\in\mathbb{Z}} \in D^{\geq 0}$ and suppose without loss of generality that $X_i = 0$ for all $i < 0$. Suppose that $X \in \sigma^{\perp_{\leq 0}}$. Since $X \in D^{\geq 0}$, we have a t-structure triangle of the form
$$\tau^{\leq 0}X \longrightarrow X \longrightarrow \tau^{\geq 1}X \longrightarrow (\tau^{\leq 0}X)[1].$$
In $D(A)$ the stalk complex of $H^0(X)$ is isomorphic to $\tau^{\leq 0}(X)$, yielding the follow triangle
$$(\tau^{\geq 1}X)[-1] \longrightarrow H^0(X) \longrightarrow X \longrightarrow \tau^{\geq 1}X. \tag{5.5}$$
It is clear that
$$\mathrm{Hom}_{D(A)}(\sigma, (\tau^{\geq 1}X)[-1]) = 0 = \mathrm{Hom}_{D(A)}(\sigma, X)$$
so then we also have $\mathrm{Hom}_{D(A)}(\sigma, H^0(X)) = 0$. This implies that $\mathrm{Hom}_A(T, H^0(X)) = 0$.

Conversely, suppose that $H^0(X) \in T^{\perp_0}$. Consider again the triangle (5.5), and apply $\mathrm{Hom}_{D(A)}(\sigma, -)$ to it. Any map $f \in \mathrm{Hom}_{D(A)}(\sigma, H^0(X))$ factors through $\mathrm{Cok}(\sigma) = T$, but then $f = 0$. And clearly, $\mathrm{Hom}_{D(A)}(\sigma, H^0(X)[i]) = 0$ for all $i < 0$.

Any map $g \in \mathrm{Hom}_{D(A)}(\sigma, (\tau^{\geq 1}X)[-1])$ factors through $X[-1]$ by the following diagram
$$\begin{tikzcd}
 & \sigma \arrow[dl, dashed] \arrow[d, "g"] \arrow[dr, bend left, "=0"] & & & \\
X[-1] \arrow[r] & (\tau^{\geq 1}X)[-1] \arrow[r] & H^0(X) \arrow[r] & X \arrow[r] & \tau^{\geq 1}X
\end{tikzcd}$$
but clearly $\mathrm{Hom}_{D(A)}(\sigma, X[-1]) = 0$, so then $\mathrm{Hom}_{D(A)}(\sigma, (\tau^{\geq 1}X)[-1]) = 0$.

Now, because $X_i = 0$ for all $i < 0$ by assumption, the complex $\tau^{\geq 1}X$ is isomorphic to
$$(\dots \longrightarrow 0 \longrightarrow 0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow 0 \longrightarrow \dots)$$
in $D(A)$, and therefore $\mathrm{Hom}_{D(A)}(\sigma, \tau^{\geq 1}X) = 0$. Now we conclude that
$$\mathrm{Hom}_{D(A)}(\sigma, X[i]) = 0 \quad \text{for all } i \leq 0$$
The arguments for showing that $T^{\perp_0} = \sigma^{\perp_{\leq 0}} \cap \mathrm{Mod}(A)$ are similar to the ones above. □

Recall remark 3.12(1), that $\mathcal{D}_\sigma$ for σ a morphism between projective A-modules, is a torsion class if and only if it is closed under coproducts in $\mathrm{Mod}(A)$. Lemma 5.23 gives us that $\mathcal{D}_\sigma$ is a torsion class if and only if $\sigma^{\perp_{>0}} \cap D^{\leq 0}$ is closed under coproducts in $D(A)$, which by proposition 5.14 bridges the gap between presilting/silting A-modules and presilting/silting complexes in $K^b(\mathrm{Proj}(A))$.

Lemma 5.23. *Let σ be a 2-term complex in $K^b(\mathrm{Proj}(A))$. Then $\sigma^{\perp_{>0}} \cap D^{\leq 0}$ is closed under coproducts in $D(A)$ if and only if $\mathcal{D}_\sigma$ is closed under coproducts in* $\mathrm{Mod}(A)$*.*

Proof. Suppose $\sigma^{\perp_{>0}} \cap D^{\leq 0}$ is closed under coproducts in $D(A)$, and take a family of objects $\{Z_i\}_{i\in I}$ in $\mathcal{D}_\sigma$. Consider the stalk complexes of the Z_i's. Clearly, each $Z_i \in D^{\leq 0}$, and consider then $f \in \mathrm{Hom}_{D(A)}(\sigma, Z_i[1])$. By the definition of $\mathcal{D}_\sigma$, the map f is null-homotopic, as can be seen in the following diagram

$$\begin{tikzcd}
\dots \arrow[r] & 0 \arrow[r] \arrow[d] & P^{-1} \arrow[r, "\sigma"] \arrow[d, "f"] & P^0 \arrow[r] \arrow[d] \arrow[dl, dashed, "f'"'] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & Z_i \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

Since $Z_i[1] \in D^{\leq -1} \subseteq \sigma^{\perp_{>0}} \cap D^{\leq 0}$ for all $i \in I$, we have that each $Z_i \in \sigma^{\perp_{>0}} \cap D^{\leq 0}$. Then $\bigoplus_{i\in I} Z_i \in \mathcal{D}_\sigma$ by our assumption.

Conversely, suppose that $\mathcal{D}_\sigma$ is closed under coproducts in $\mathrm{Mod}(A)$, and take a family of objects $\{X_i\}_{i\in I}$ in $\sigma^{\perp_{>0}} \cap D^{\leq 0}$. First, for any X_i consider the triangle

$$(\tau^{\leq -1} X_i)[1] \longrightarrow X_i[1] \longrightarrow H^0(X_i)[1] \longrightarrow (\tau^{\leq -1} X_i)[2]$$

and apply $\mathrm{Hom}_{D(A)}(\sigma, -)$ to it. Clearly $\mathrm{Hom}_{D(A)}(\sigma, (\tau^{\leq -1}X_i)[2]) = 0$ and by assumption $\mathrm{Hom}_{D(A)}(\sigma, X_i[1]) = 0$. Then $\mathrm{Hom}_{D(A)}(\sigma, H^0(X_i)[1]) = 0$, i.e. $H^0(X_i) \in \mathcal{D}_\sigma$ for all $i \in I$. Because H^0 commute with coproducts, we have that

$$H^0\big(\bigoplus_I X_i\big) = \bigoplus_I H^0(X_i) \in \mathcal{D}_\sigma$$

by assumption. Then by lemma 5.22(1), we have that $\bigoplus_I X_i \in \sigma^{\perp_{>0}} \cap D^{\leq 0}$. □

Lemma 5.24. *Let $\sigma \in K^b(\mathrm{Proj}(A))$ be a 2-term complex and $H^0(\sigma) = T$. Then T is partial silting with respect to σ if and only if σ is presilting and $\sigma^{\perp_{>0}} \cap D^{\leq 0}$ is closed for coproducts in $D(A)$.*

Proof. Suppose that $T \in \mathrm{Mod}(A)$ is partial tilting with respect to σ. Then by lemma 5.23 we have that $\sigma^{\perp_{>0}} \cap D^{\leq 0}$ is closed under coproducts. Since $T \in \mathcal{D}_\sigma$ it follows by lemma 3.10(3) that $\sigma \in \sigma^{\perp_{>0}}$.

Suppose that σ is presilting and that $\sigma^{\perp_{>0}} \cap D^{\leq 0}$ is closed under coproducts. Then by lemma 5.23 we have that $\mathcal{D}_\sigma$ is closed under coproducts as well. Since $\sigma \in \sigma^{\perp_{>0}}$ it follows by lemma 3.10(3) $T \in \mathcal{D}_\sigma$. □

Now we have all the tools to prove another important result, in particular that silting A-modules correspond bijectively to 2-silting complexes in $K^b(\mathrm{Proj}(A))$ by taking homology.

Theorem 5.25. *Let σ be a 2-term complex in $K^b(\mathrm{Proj}(A))$ and $H^0(\sigma) = T$. Then the following are equivalent.*

(1) σ is 2-silting.

(2) σ is a presilting generator of $D(A)$.

(3) T is a silting A-module with respect to σ.

(4) $(\mathcal{D}_\sigma, T^{\perp_0})$ is a torsion pair in $\mathrm{Mod}(A)$.

Proof. (1) $\Rightarrow$ (2) : Follows directly from proposition 5.14.

(2) $\Rightarrow$ (1) : By proposition 5.14, if $\sigma^{\perp_{>0}} \subseteq D^{\leq 0}$, then σ is 2-silting. So let $X \in \sigma^{\perp_{>0}}$, then there is a triangle associated with the standard t-structure $(D^{\leq 0}, D^{\geq 0})$

$$\tau^{\leq 0}X \longrightarrow X \longrightarrow \tau^{\geq 1}X \longrightarrow (\tau^{\leq 0}X)[1]$$

First note that $\mathrm{Hom}_{D(A)}(\sigma[i], \tau^{\geq 1}X) = 0$ for all $i \geq 0$. Applying $\mathrm{Hom}_{D(A)}(\sigma[i], -)$ to the triangle with $i < 0$ we get a long exact sequence

$$\dots \longrightarrow \mathrm{Hom}_{D(A)}(\sigma[i], X) \longrightarrow \mathrm{Hom}_{D(A)}(\sigma[i], \tau^{\geq 1}X) \longrightarrow \mathrm{Hom}_{D(A)}(\sigma[i], (\tau^{\leq 0}X)[1]) \longrightarrow \dots$$

By our assumption on X, we have

$$\mathrm{Hom}_{D(A)}(\sigma[i], X) = \mathrm{Hom}_{D(A)}(\sigma, X[-i]) = 0 \quad \text{for all } i < 0$$

and since $(\tau^{\leq 0}X)[1-i] \in D^{\leq -2}$ for $i < 0$, we get that

$$\mathrm{Hom}_{D(A)}(\sigma[i], (\tau^{\leq 0}X)[1]) = \mathrm{Hom}_{D(A)}(\sigma, (\tau^{\leq 0}X)[1-i]) = 0.$$

Then $\mathrm{Hom}_{D(A)}(\sigma[i], \tau^{\geq 1}X) = 0$ for all $i \in \mathbb{Z}$. Because σ generates $D(A)$, we conclude that $\tau^{\geq 1}X = 0$, which implies that $X \in D^{\leq 0}$.

(1) $\Rightarrow$ (3) : Combining proposition 5.14(2) and lemma 5.24, we get that $\mathcal{D}_\sigma$ is closed under coproducts, i.e. it is a torsion class. Furthermore, by lemma 3.10(3) we have that $T \in \mathcal{D}_\sigma$, so T is partial silting with respect to σ. B corollary 3.15 we have $\mathrm{Gen}(T) \subseteq \mathcal{D}_\sigma$.

We show that the inclusion above is an equality. Let $M \in \mathcal{D}_\sigma$ and take the universal map $u : \sigma^{(I)} \to M$ where $I = \mathrm{Hom}_{D(A)}(\sigma, M)$. We will show that $H^0(u) : T^{(I)} \to M$ is surjective, i.e. that $\mathcal{D}_\sigma \subseteq \mathrm{Gen}(T)$. There is a triangle

$$\sigma^{(I)} \xrightarrow{u} M \xrightarrow{v} C \xrightarrow{w} \sigma^{(I)}[1] \tag{5.6}$$

where $C = \mathrm{Cone}(u)$. Taking homology of the triangle we get an exact sequence

$$\dots \longrightarrow T^{(I)} \xrightarrow{H^0(u)} M \xrightarrow{H^0(v)} H^0(C) \longrightarrow 0 \longrightarrow \dots \tag{5.7}$$

There is a surjection $M \to H^0(C)$, so $H^0(C) \in \mathcal{D}_\sigma$ since $\mathcal{D}_\sigma$ is a torsion class. By the last part of lemma 5.22(1) we have that $H^0(C) \in \sigma^{\perp_{>0}}$ and $C \in D^{\leq 0}$. There is a triangle associated with the standard t-structure on $D(A)$

$$\tau^{\leq -1}C \longrightarrow C \longrightarrow H^0(C) \longrightarrow (\tau^{\leq -1}C)[1] \tag{5.8}$$

Take a map $f \in \mathrm{Hom}_{D(A)}(\sigma, C)$, then since σ is presilting, the composition $wf = 0$ in triangle (5.6), so f factors through v via some map $g : \sigma \to M$. By the universal property of u, the map g factors through u via some map $g' : \sigma \to \sigma^{(I)}$ such that $f = vug'$, illustrated by the following commutative diagram

$$\begin{tikzcd}
 & & \sigma \arrow[d, "f"] \arrow[dll, "g'"'] & \\
\sigma^{(I)} \arrow[r, "u"] & M \arrow[r, "v"] & C \arrow[r, "w"] & \sigma^{(I)}[1]
\end{tikzcd}$$

Then $f = 0$ and so $\mathrm{Hom}_{D(A)}(\sigma, C) = 0$. Furthermore since $(\tau^{\leq -1}C)[1] \in D^{\leq -2}$ we have

$$\mathrm{Hom}_{D(A)}(\sigma, (\tau^{\leq -1}C)[1]) = 0.$$

By triangle 5.8 we then have

$$\mathrm{Hom}_{D(A)}(\sigma, H^0(C)) = 0.$$

Then, since $H^0(C) \in \mathcal{D}_\sigma$, we have

$$\mathrm{Hom}_{D(A)}(\sigma, H^0(C)[1]) = 0$$

and then since σ is 2-term we get

$$\mathrm{Hom}_{D(A)}(\sigma, H^0(C)[i]) = 0 \quad \text{for all } i \in \mathbb{Z}$$

Because σ generates $D(A)$, we get $H^0(C) = 0$. Finally, from the exact sequence 5.7 we get that $H^0(u) : T^{(I)} \to M$ is surjective, i.e. $\mathrm{Gen}(T) = \mathcal{D}_\sigma$. So T is silting with respect to σ.

(3) $\Rightarrow$ (4) : This follows immediately from corollary 3.15.

(4) $\Rightarrow$ (1) : Assume that $(\mathcal{D}_\sigma, T^{\perp_0})$ is a torsion pair. We have $\mathrm{Hom}_{D(A)}(T, T^{\perp_0}) = 0$, and therefore $T \in \mathcal{D}_\sigma$ by the comments following definition 2.4. So T is partial silting with respect to σ. Then by lemma 5.24, σ is presilting and $\sigma^{\perp_{>0}} \cap D^{\leq 0}$ is closed for coproducts in $D(A)$. By proposition 5.14, it remains to show that σ is a generator for $D(A)$. To that end, let $X \in D(A)$ such that

$$\mathrm{Hom}_{D(A)}(\sigma, X[i]) = 0 \quad \text{for all} i \in \mathbb{Z}$$

Because σ is 2-term, this is equivalent to $\mathrm{Hom}_{D(A)}(\sigma, \tau^{\leq 0}(X[i])) = 0$ for all $i \in \mathbb{Z}$. Then clearly, $\mathrm{Hom}_{D(A)}(\sigma, H^0(X[i])) = 0$ for all $i \in \mathbb{Z}$, which implies $H^i(X) \in T^{\perp_0}$ for all $i \in \mathbb{Z}$. There is a triangle associated with the standard t-structure in $D(A)$.

$$H^0(X[i+1])[-1] \longrightarrow \tau^{\leq -1}(X[i+1]) \longrightarrow \tau^{\leq 0}(X[i+1]) \longrightarrow H^0(X[i+1])$$

Apply $\mathrm{Hom}_{D(A)}(\sigma, -)$ to the triangle to get a long exact sequence

$$\begin{tikzcd} \dots \arrow[r] & \mathrm{Hom}_{D(A)}(\sigma, H^0(X[i+1])[-1]) \arrow[r] & \mathrm{Hom}_{D(A)}(\sigma, \tau^{\leq -1}(X[i+1])) \arrow[dll, hook] \\ \mathrm{Hom}_{D(A)}(\sigma, \tau^{\leq 0}(X[i+1])) \arrow[r] & \mathrm{Hom}_{D(A)}(\sigma, H^0(X[i+1])) \arrow[r] & \dots \end{tikzcd}$$

Now by our assumption on X we have

$$\mathrm{Hom}_{D(A)}(\sigma, \tau^{\leq 0}(X[i+1])) = 0$$

and

$$\mathrm{Hom}_{D(A)}(\sigma, H^0(X[i+1])[-1]) = 0.$$

Then

$$\mathrm{Hom}_{D(A)}(\sigma, \tau^{\leq -1}(X[i+1])) = 0 = \mathrm{Hom}_{D(A)}(\sigma, \tau^{\leq 0}(X[i])[1]).$$

Thus, $\tau^{\leq 0}(X[i]) \in \sigma^{\perp_{>0}}$ for all $i \in \mathbb{Z}$ since σ is 2-term. By lemma 5.22(1) we have

$$H^i(X) = H^0(\tau^{\leq 0}(X[i])) \in \mathcal{D}_\sigma \quad \text{for all } i \in \mathbb{Z}$$

So $H^i(X) \in \mathcal{D}_\sigma \cap T^{\perp_0}$ for all $i \in \mathbb{Z}$, but then $H^i(X) = 0$ for all $i \in \mathbb{Z}$ because $(\mathcal{D}_\sigma, T^{\perp_0})$ is a torsion pair. Then X is isomorphic to the zero complex in $D(A)$, and so σ generates $D(A)$. □

Corollary 5.26. *Let σ be a 2-term complex in $K^b(\mathrm{Proj}(A))$ and $H^0(\sigma) = T$. If any of the equivalent statements of theorem 5.25 is satisfied, then*

$$\sigma^{\perp_{>0}} = D^{\leq 0}_{\mathcal{D}_\sigma} = \{X \in D(A) \mid H^0(X) \in \mathcal{D}_\sigma,\ H^i(X) = 0\, \forall i > 0\}.$$

Proof. Recall from example 5.5 that $(D^{\leq 0}_{\mathcal{D}_\sigma}, D^{\geq 0}_{T^{\perp_0}})$ is a t-structure in $D(A)$ if $(\mathcal{D}_\sigma, T^{\perp_0})$ is a torsion pair in $\mathrm{Mod}(A)$. By proposition 5.14 we have $\mathrm{aisle}(\sigma) = \sigma^{\perp_{>0}} \subseteq D^{\leq 0}$ since σ is 2-silting. Let $X \in \mathrm{aisle}(\sigma)$, then $H^0(X) \in \mathcal{D}_\sigma$ by lemma 5.22(1) and $H^i(X) = 0$ for all $i > 0$ because $X \in D^{\leq 0}$. So $\mathrm{aisle}(\sigma) = \sigma^{\perp_{>0}} \subseteq D^{\leq 0}_{\mathcal{D}_\sigma}$. By remark 5.2, the aisle of a t-structure determines the entire t-structure, so in our case $D^{\geq 0}_{T^{\perp_0}} = (D^{\leq 0}_{\mathcal{D}_\sigma})^{\perp_0}[1]$. We will therefore show that $\mathrm{aisle}(\sigma)^{\perp_0} \subseteq D^{\geq 1}_{T^{\perp_0}}$ to complete the proof. By the t-structure $(\mathrm{aisle}(\sigma), \sigma^{\perp_{>0}})$ we get $\mathrm{aisle}(\sigma)^{\perp_0} = \sigma^{\perp_{<0}}[-1] = \sigma^{\perp_{\leq 0}}$. Now let $X \in \mathrm{aisle}(\sigma)^{\perp_0}$ and consider the triangle

$$(\tau^{\geq 0}X)[i-1] \longrightarrow (\tau^{\leq -1}X)[i] \longrightarrow X[i] \longrightarrow (\tau^{\geq 0}X)[i]$$

Since $\sigma \in D^{\leq 0}$ we have

$$\mathrm{Hom}_{D(A)}(\sigma, (\tau^{\geq 0}X)[i-1]) = 0 \quad \text{for all } i \leq 0.$$

By our assumption on X we have

$$\mathrm{Hom}_{D(A)}(\sigma, X[i]) = 0 \quad \text{for all } i \leq 0.$$

Then by the triangle we have

$$\mathrm{Hom}_{D(A)}(\sigma, (\tau^{\leq -1}X)[i]) = 0 \quad \text{for all } i \leq 0.$$

and since $(\tau^{\leq -1}X) \in D^{\leq -1}$ we have

$$\mathrm{Hom}_{D(A)}(\sigma, (\tau^{\leq -1}X)[i]) = 0 \quad \text{for all } i > 0.$$

But then $\tau^{\leq -1}X = 0$ as σ generates $D(A)$. Therefore, $X \in D^{\geq 0}$ and by lemma 5.22(2) we have that $H^0(X) \in T^{\perp_0}$. Then

$$\mathrm{aisle}(\sigma)^{\perp_0} \subseteq D^{\geq 1}_{T^{\perp_0}} = \{X \in D(A) \mid H^0(X) \in T^{\perp_0},\ H^i(X) = 0\, \forall i < 0\}.$$

and thus

$$\sigma^{\perp_{>0}} = D^{\leq 0}_{\mathcal{D}_\sigma} = \{X \in D(A) \mid H^0(X) \in \mathcal{D}_\sigma,\ H^i(X) = 0\, \forall i > 0\}.$$

□

Corollary 5.27. *Let σ and ω be 2-silting complexes in $K^b(\mathrm{Proj}(A))$, with $T_0 = H^0(\sigma)$ and $T_1 = H^0(\omega)$ the corresponding silting modules. Then σ and ω are equivalent if and only if T_0 and T_1 are equivalent.*

Proof. Suppose that T_0 and T_1 are equivalent, i.e. $\mathrm{Add}(T_0) = \mathrm{Add}(T_1)$. Then clearly $\mathrm{Gen}(T_0) = \mathcal{D}_\sigma = \mathcal{D}_\omega = \mathrm{Gen}(T_1)$. By corollary 5.26 we have $\sigma^{\perp_{>0}} = \omega^{\perp_{>0}}$, so σ and ω are equivalent.

Conversely, suppose that σ and ω are equivalent, i.e. $\mathrm{Add}(\sigma) = \mathrm{Add}(\omega)$. Since H^0 commutes with coproducts, we have $\mathrm{Add}(T_0) = \mathrm{Add}(T_1)$. □

Theorem 5.21 can be modified to the case of 2-silting complexes which then also accomodates silting modules.

Theorem 5.28. *There exists bijections between*

(1) Equivalence classes of 2-silting complexes;

(2) Equivalence classes of silting A-modules;

(3) 2-silting t-structures in $D(A)$;

(4) co-t-structures $(\mathcal{U}_{\geq 0}, \mathcal{U}_{\leq 0})$ in $D(A)$ such that $K_{\leq -1} \subseteq \mathcal{U}_{\leq 0} \subseteq K_{\leq 0}$ and $\mathcal{U}_{\leq 0}$ is closed under coproducts in $D(A)$.

Proof. We claim that the following assignments are bijections

Bijection	Assignment
(1) $\to$ (2)	$H^0 : \sigma \mapsto H^0(\sigma)$
(1) $\to$ (3)	$\Psi : \sigma \mapsto (\sigma^{\perp_{>0}}, \sigma^{\perp_{<0}})$
(1) $\to$ (4)	$\Phi : \sigma \mapsto ({}^{\perp_0}(\sigma^{\perp_{>0}}[1]), \sigma^{\perp_{>0}})$

(1) $\to$ (2) : By corollary 5.27, H^0 is both well-defined and injective. By theorem 5.25, if $T = H^0(\sigma)$ is a silting module with respect to σ, then σ is a 2-silting complex. So H^0 is surjective as well.

(1) $\to$ (3) : The bijection Ψ in theorem 5.21 follows from lemma 5.16, relating n-silting t-structures to n-silting complexes. Therefore, Ψ induces a bijection between equivalence classes of 2-silting t-structures and 2-silting complexes.

(1) $\to$ (4) : First, by the proof of lemma 5.20, the co-t-structure

$$({}^{\perp_0}(\sigma^{\perp_{>0}}[1]), \sigma^{\perp_{>0}}) = \Phi(\sigma)$$

in theorem 5.21 satisfies $K_{\leq -n} \subseteq \sigma^{\perp_{>0}} \subseteq K_{\leq 0}$, and $\sigma^{\perp_{>0}}$ is closed under coproducts in $D(A)$. Restricting Φ to 2-silting complexes, we have our bijection. $\square$

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