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Fibrations of Categories

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Sammendrag. Det finnes en *kanonisk* modellstruktur på **Cat** som bruker ekvivalenser av kategorier, og vi forsøker å forstå fibrasjonene involvert. Første kapittel gir en detaljert utledning av at yonedaimbeddingen $r : \mathbf{C} \to \mathbf{Set}_{\mathbf{C}}$ er en *fri kokomplettering* av kategorien $\mathbf{C}$. Ut av dette springer et paradigme av "nerve og realisering"-adjunksjoner, som generaliserer en rekke klassiske konstruksjoner. I kapittel 2 motiveres først *diskrete fibrasjoner* via topologiske overdekningsrom, og disse fibrasjonene klassifiseres som funktorer $\mathbf{C} \to \mathbf{Set}$. Dette er intimt knyttet til den frie kokompletteringen og ulike linjer trekkes. Deretter går vi detaljert gjennom konstruksjonen av nevnt kanoniske modellstruktur. Til slutt gis en versjon av Quillen sitt *"small object argument"* for *presenterbare* kategorier.

Abstract. There is a *canonical* model structure on **Cat** whose weak equivalences are the categorical ones, and we attempt to develop an intrinsic understanding of the fibrations involved. In Chapter 1 we work out the category theory required to prove that the Yoneda embedding $r : \mathbf{C} \to \mathbf{Set}_{\mathbf{C}}$ is the *free cocompletion* of $\mathbf{C}$. This gives rise to a paradigm of "nerve-realization" adjunctions which subsume various classical constructions. Chapter 2 motivates *discrete fibrations* of categories through topological covering spaces, and the relation to Chapter 1 is made explicit by classifying discrete fibrations $\mathbf{E} \twoheadrightarrow \mathbf{C}$ as functors $\mathbf{C} \to \mathbf{Set}$. Observing this in the context of the free cocompletion proves fruitful. Finally, we explicitely construct the canonical model structure on **Cat** and we prove its uniqueness with respect to the set of weak equivalences. We conclude with an account of Quillen's *small object argument* for *presentable* categories.

Preface

Quillen introduced model structures in his seminal monograph *Homotopical Algebra* [Qui67] where he distilled out the properties of the category **Top** of topological spaces that are required to carry out arguments of homotopical nature. This has birthed a branch of "categorical" homotopy theory and made it worthwhile to search for model structures on the categories in which we work. Such structures consist of a selection of morphisms deemed *weak equivalences* and a selection of morphisms deemed *fibrations* (or equivalently, a selection of *cofibrations*), that interplay nicely. The modern axioms of a model structure are stronger than those initially posed by Quillen; today it is customary to work with Quillen's *closed* model categories in which the choice of either fibrations or cofibrations determines the other through *left-* and *right-lifting properties*.

A model structure on a category **C** associates a *homotopy category* to **C** by localizing **C** at the weak equivalences. The role of the fibrations (or cofibrations) is then seen as solely computational, and in general there are many possible choices of fibrations for a given set of weak equivalences. The weak equivalences in the canonical model structure on **Cat** are the equivalences of categories. A little-known fact is that this leaves no room for personal preference concerning the fibrations; the set of *isofibrations* (and *isocofibrations*) is the only valid choice. Perhaps less surprisingly, the same holds for the canonical model structure on **Set** where the weak equivalences are the bijections. Analogous model structures on higher categories have been constructed, but uniqueness of the set of fibrations (or cofibrations) with respect to the weak equivalences appears to be an open question. These higher analogues go by many names in the literature, e.g. "folk", "natural", "categorical", but also "canonical". Since the term "canonical" carries a connotation of being "uniquely determined", we suggest the idea of *defining* a canonical model structure as being such (Definition 2.30). In Section 2.2 these matters are discussed in more detail.

Discrete fibrations are the immediate generalization of topological covering maps to categories, and this can be seen by passing through the fundamental groupoids. This is done in Section 2.1. Classically, (connected) covering spaces are classified as (transitive) actions of the fundamental group. The categorical version classifies discrete fibrations $\mathbf{E} \twoheadrightarrow \mathbf{C}$ as functors $\mathbf{C} \to \mathbf{Set}$, and this suggests a geometric view of functors $\mathbf{C} \to \mathbf{Set}$ as "generalized covers" of the category **C**. While the first chapter develops "sterile" category theory, the author was guided and motivated by geometrical intuition while writing it. The end of Section 2.1 is an attempt at rendering some of the underlying ideas explicit.

Some authors (e.g. [Rie14a], [Hov07]) go as far as to require *functorial* factorizations of morphisms in their model categories. Quillen on the other hand obtained functorial factorizations for some of his model categories through his so-called *small object argument*.

While powerful, Quillen's small object argument is quite technical and may be a step out of the comfort zone for most homotopy theorists: It consists of a transfinite induction that doesn't "converge", but is halted at a sufficiently large cardinal. From a categorical perspective, the arbitrary stopping point feels very unnatural. Luckily, this was recently remedied in [Gar08] by Garner's *algebraic* small object argument.

The precise class of categories permitting the small object argument eludes the author, but a subset are the so-called *presentable*[1] categories—and we prove this is in Theorem 2.47. A feature of the small object argument is that it produces factorization systems on

[1] Some authors call these *locally presentable*, however we choose to follow [Lur09][A.1.1.2]

a category that satisfy the model structure axioms, and the model categories obtained in this way are *cofibrantly generated*. This gives a direct way to equip a presentable category with a cofibrantly generated model structure, these are called *combinatorial* model categories. These feature prominently in *Dugger's theorem*, which intuitively states that a combinatorial model category is the localization of a presentable $(\infty, 1)$-category (see e.g. [Lur09][A.3.7.6, A.3.7.8]).

Finally, some words about the contents of this thesis are in order. The author claims no originality concerning any of the results herein, though for some statements no reference has been found. However, due to its classical nature everything stated here is surely well-known to experts. Moreover, the author has made efforts to formulate and develop the theory for himself, and for this reason proofs are included even if they lack novelty.

Preliminaries

This text is intended to be read by other students at NTNU interested in topology and category theory. However, some familiarity with basic category theory is needed. Specifically, we assume familiarity with universal properties, (full, faithful, dense, (co)representable, categories of) functors, the Yoneda embedding, and basic simplicial methods. Adjunctions play an important role and are briefly introduced in Chapter 1.1, but the uninitiated reader will likely need to consult e.g. [Lan10][Ch.4] for more details. After a concept has been introduced, we will assume its dual as well and refer to the dual concept with a prefix "co-", e.g. limits and colimits.

Appreciation for most examples requires some knowledge of algebraic topology and category theory. If an example identifies a "new" construction as a known classical concept, the familiar reader may appreciate the "new" approach, whereas the unfamiliar reader may consider it to be a definition.

In Chapter 2.3 familiarity with cardinal arithmetic is needed for a rigorous understanding.

Notation and convention

Letters in boldface, e.g. $\mathbf{C}$, denote categories throughout, and $\mathbf{C}(-,-)$ is the associated hom-bifunctor. By $\mathbf{C}_0$ and $\mathbf{C}_1$ we mean the objects and morphisms of $\mathbf{C}$, respectively. In general $\mathbf{C}(a, b)$ is a "hom-class", or a "hom-set" whenever $\mathbf{C}$ is locally small (see "Size concerns" below). For short we will write $f^* := \mathbf{C}(f, C)$ and $f_* := \mathbf{C}(C, f)$ for the pre- and post-composition with f. Presheaf categories will be denoted by $\mathbf{Set}_{\mathbf{C}} = \mathbf{Set}^{\mathbf{C}^{op}}$ (only $\mathbf{Set}$-valued presheaves will be treated rigorously). For morphisms, a whole quiver of arrows $(\rightarrow, \twoheadrightarrow, \hookrightarrow, ...)$ will be defined and used; $L : \mathbf{C} \leftrightarrows \mathbf{D} : R$ will denote an adjunction where $L \vdash R$ (i.e. L is left adjoint to R), whereas $\leftrightarrow$ is reserved for isomorphisms. Natural transformations (resp. isomorphisms) will be denoted using double arrows $\Rightarrow$ (resp. $\cong$), except for natural transformations in the image of Δ (defined below), called *constant* natural transformations, for which we will also sometimes use single arrows—notably in diagrams.

Since $\mathbf{Cat}$ is cartesian closed, a bifunctor $B : \mathbf{C}\times\mathbf{D} \rightarrow \mathbf{E}$ corresponds to a functor $\mathbf{C} \rightarrow \mathbf{E}^{\mathbf{D}}$ which we will denote $B(-, =)$ to signify that the second parameter is *curried*. For example, the Yoneda embedding $r : \mathbf{C} \rightarrow \mathbf{Set}_{\mathbf{C}}$ may be written $\mathbf{C}(=, -)$. One application then removes one bar from each of the parameters, giving $r(c) = \mathbf{C}(-, c)$.

Some more categories deserve their own notation:

1. $\mathbf{1}, \mathbf{2}, \mathbf{3}...$ are the ordinal categories, so that $\mathbf{n}$ is the linear order on n objects.

2. Δ is the full subcategory of **Cat** spanned by the objects $\{\mathbf{n}\}_{n>0}$.

3. $\Delta : \mathbf{C} \to \mathbf{Set}_{\mathbf{C}}$ is also the *diagonal functor* sending an object $a \in \mathbf{C}_0$ to the *constant functor* $\Delta(a)$ defined as $\Delta(a)(f) = \mathrm{id}_a$ for all $f \in \mathbf{C}_1$ and $\Delta(a)(c) = a$ for $c \in \mathbf{C}_0$.

4. $\mathbf{I}$ is the category containing two objects 0 and 1 and an isomorphism $0 \leftrightarrow 1$.

Size concerns

A *small* category has a set of objects and morphisms. By **Cat** we mean (2)-category of small categories (and similarly for **Set**, **Top**, and **Ab**). A category $\mathbf{C}$ is *locally small* if all its hom-sets $\mathbf{C}(a, b)$ are sets. For example, **Set** is locally small. When discussing presheaf categories $\mathbf{Set}_{\mathbf{C}}$, the domain $\mathbf{C}$ will often be small. This ensures that $\mathbf{Set}_{\mathbf{C}}$ is locally small, since $\mathbf{Set}_{\mathbf{C}}(F, G) \subseteq \prod_{c \in \mathbf{C}_0} \mathbf{Set}(F(c), G(c))$ are sets.

In Chapter 2.3 more care is required and we follow [Lur09][1.2.15]: A category is κ-*small* for some cardinal κ when the cardinalities of $\mathbf{C}_0$ and $\mathbf{C}_1$ are smaller than κ. If κ is a regular cardinal, this ensures (as above) that $\mathbf{Set}_{\mathbf{C}}$ is *locally κ-small*, i.e. that $|\mathbf{C}(a, b)| < \kappa \ \forall a, b \in \mathbf{C}_0$. More details are given in the reference.

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Contents

1. Nerves and Realizations

We start by introducing colimits, hoping to convey the idea that these are general ways of constructing objects in a category. Especially important is their formulation in terms of an adjunction, which is how they feature in proofs by "abstract nonsense". Next up are limits (the dual of colimits) in **Set**, whose explicit descriptions will illuminate proofs in later chapters. Treating the **Set**-valued case first is useful, because it applies to any category through the Yoneda embedding. Ends are different formulations of limits, introduced to answer Question 1.48 (answered by Proposition 1.51) in a concise manner.

Comma categories are ubiquitous, and occur naturally as slice categories and categories of elements, both of which feature prominently throughout this text. When writing a presheaf as a canonical colimit of representable presheaves (see e.g. [Lan10][3.7]), the diagram category is the category of elements. Additionally, categories of elements give fibrations of categories which are studied in Chapter 2.

Finally, we argue that the presheaf category $\mathbf{Set}_{\mathbf{C}}$ is the *free cocompletion* of the category $\mathbf{C}$, and show that this gives rise to a paradigm of "nerve-realization" adjunctions that subsumes various familiar constructions. The approach taken here details that of [Dug99].

Everything stated here is classical, but some statements are perhaps not widely known or appreciated. All of [Lan10], [Rie14a], [Dug99], [Lur09] and the nLab have been put to use. Specific pages of the latter are cited in place.

1.1. Colimits and Adjunctions

We discuss colimits, developing intuition from the cases of **Set** and **Top**.

A colimit can be seen as a unique (or well-defined) way of assembling an object in a category from others. In order to formalize this, some definitions are needed.

Definition 1.1. Let $D : \mathbf{D} \to \mathbf{C}$ be a functor. We will call D a *diagram* to stress that it's domain $\mathbf{D}$ is small. A *cocone* under a diagram D is an object $c \in \mathbf{C}_0$ along with a natural transformation $\eta : D \Rightarrow \Delta c$.

Example 1.2. Let $\mathbf{D} := (b \leftarrow a \to c)$ be the category consisting of three objects $\mathbf{D}_0 = \{a, b, c\}$ and two morphisms $\mathbf{D}_1 := \{a \to b, a \to c\}$. A diagram $D : \mathbf{D} \to \mathbf{C}$ is called a *span (in* $\boldsymbol{C}$*)*. Dually, a diagram $D' : D^{op} \to \mathbf{C}$ is called a *cospan (in* $\boldsymbol{C}$*)*.

Any set D gives rise to a *discrete*[2] category $\mathbf{D}$ whose only morphisms are the identities. In this case a diagram $D : \mathbf{D} \to \mathbf{Set}$ is a collection of objects $(S_d)_{d \in \mathbf{D}_0}$, and a cocone under D is a set $S \in \mathbf{Set}_0$ along with functions $(S_d \to S)_{d \in \mathbf{D}_0}$ into S. Intuitively, we could think of S as "housing" the various S_d's. In particular, S could be the disjoint union $\coprod_{d \in \mathbf{D}} S_d$ along with inclusions $(S_d \subseteq S)_{d \in \mathbf{D}_0}$. More interesting situations occur when $\mathbf{D}$ isn't discrete:

Example 1.3. Let $\mathbf{D} := (i_0, i_1 : * \rightrightarrows I)$ be the subcategory of **Top** whose objects are the singleton $*$ and the interval I, and non-identity morphisms are the inclusions i_0, i_1 of the point $*$ into the interval I at either 0 or 1. A cocone under the inclusion $\mathbf{D} \hookrightarrow \mathbf{Top}$ is a space X along with two maps $x : * \to X$ and $p : I \to X$. The map x corresponds to

[2] A *discrete* category is one where the only morphisms are the identities.

a point $x(*)$ of X, and p is a path. Naturality of x and p means that $pi_0 = x(*) = pi_1$. This asserts that p is not just a path, but a loop based at $x(*)$ in X.

The set of loops in a topological space X is the set $\mathbf{Top}(S^1, X)$ of continuous functions from the circle S^1 into X. The previous example could then be stated as: Any cocone under $\mathbf{D} \hookrightarrow \mathbf{Top}$ (of Example 1.3) defines a (unique) loop $S^1 \to X$. Said differently, S^1 is the *colimit* of $\mathbf{D}$:

Definition 1.4. Let $D : \mathbf{D} \to \mathbf{C}$ be a diagram. When it exists, a *colimit of* D is an object $\mathrm{colim} D \in \mathbf{C}_0$ giving rise to a cocone $\eta : D \Rightarrow \Delta(\mathrm{colim} D)$ under D. Moreover, this cocone is universal, meaning that any other cocone $\gamma : D \Rightarrow \Delta C$ factors uniquely through a map $\hat{\gamma} : \mathrm{colim} D \to C$, i.e. $\gamma = \Delta(\hat{\gamma})\eta$.

When they exist, colimits are unique up to isomorphism. This justifies referring to *the* colimit of a diagram.

One way of intuiting Example 1.3 is to see the inclusions $i_0, i_1 : * \to I$ as points of I that should be glued together to form the colimit. "Gluing" is to form a quotient, and indeed $I/(0 \sim 1) \cong S^1$.

Example 1.5. Let $\mathbf{S}$ be the discrete category arising from a set S. The colimit of a diagram $S : \mathbf{S} \to \mathbf{Set}$ is the disjoint union $\coprod_{s \in S_0} S(s)$; the colimit of $A : \mathbf{S} \to \mathbf{Ab}$ is the direct products of abelian groups $\bigoplus_{s \in S_0} A(s)$, and the colimit of $G : \mathbf{S} \to \mathbf{Gp}$ is the free products of groups $\circledast_{s \in S_0} G(s)$.

Example 1.6. Let $\{U, V\}$ be an open cover of a topological space X. The colimit of the span $\mathbf{D} = \{* \leftarrow * \to *\} \mapsto \{U \supseteq U \cap V \subseteq V\}$ in $\mathbf{Top}$ is X, and we think of this as the result from gluing U and V together along their intersection inside X.

The colimit of a span is called a pushout:

Definition 1.7. Let $D : \mathbf{D} \to \mathbf{C}$ be a span admitting a colimit. The colimit $\mathrm{colim}_{\mathbf{D}} D$ is called the *pushout* of D, depicted as:

$$
\begin{tikzcd}
D(a) \arrow[r] \arrow[d] & D(c) \arrow[d, dashed, "\eta_c"] \\
D(b) \arrow[r, dashed, "\eta_b"] \arrow[ur, phantom, "\ulcorner"] & \mathrm{colim}_{\mathbf{D}} D
\end{tikzcd}
$$

The dashed arrows are the components of the cocone $\eta : D \Rightarrow \Delta(\mathrm{colim} D)$. The small angle "$\ulcorner$" signifies that this is a pushout square. Dualizing gives the notion of pullbacks and pullback squares (for which the angle "$\lrcorner$" is used).

Definition 1.8. Let $\mathbf{D}$ be a small category. A category $\mathbf{C}$ is ***D**-cocomplete* if all diagrams $D : \mathbf{D} \to \mathbf{C}$ admit colimits. When $\mathbf{C}$ is $\mathbf{D}$-cocomplete for all small categories $\mathbf{D}$, we say that $\mathbf{C}$ is *(small) cocomplete*.

Proposition 1.9. *Let $\boldsymbol{C}$ be a cocomplete category. Then for any small category $\boldsymbol{D}$, forming colimits of diagrams defines a functor* $\mathrm{colim} : \boldsymbol{C}^{\boldsymbol{D}} \to \boldsymbol{C}$.

The proof is a good exercice in working with universal properties.

The universal property of the colimit with respect to a diagram D is more concisely given as a natural isomorphism $\mathbf{C}^{\mathbf{D}}(D, \Delta -) \cong \mathbf{C}(\mathrm{colim}_{\mathbf{D}} D, -) : \mathbf{C} \to \mathbf{Set}$ of functors, which associates a cocone $D \Rightarrow \Delta c$ to the unique induced morphism $\mathrm{colim}_{\mathbf{D}} D \to c$. In particular, the colimit cocone $D \Rightarrow \Delta(\mathrm{colim}_{\mathbf{D}} D)$ is found as the preimage of $\mathrm{id}_{\mathrm{colim}_{\mathbf{D}} D}$ under this isomorphism.

When $\mathbf{C}$ is cocomplete, we also get naturality in $D \in \mathbf{C}^{\mathbf{D}}_0$:

$$\mathbf{C}(\mathrm{colim} -, -) \cong \mathbf{C}^{\mathbf{D}}(-, \Delta -) : \mathbf{D}^{op} \times \mathbf{C} \to \mathbf{Set}$$

If we denote such a natural isomorphism by Θ, this means that the following diagrams commute for all $f : c \to c' \in \mathbf{C}_1$ and $\eta : D \Rightarrow D' \in (\mathbf{C}^{\mathbf{D}})_1$:

$$\begin{tikzcd}
\mathbf{C}(\mathrm{colim} D', c) \arrow[r, "\Theta_{D',c}"] \arrow[d, "(\mathrm{colim}\eta)^* f_*"'] & \mathbf{C}^{\mathbf{D}}(D', \Delta c) \arrow[d, "\eta^*(\Delta f)_*"] \\
\mathbf{C}(\mathrm{colim} D, c') \arrow[r, "\Theta_{D,c'}"] & \mathbf{C}^{\mathbf{D}}(D, \Delta c')
\end{tikzcd}$$

This is an important example of an *adjunction*:

Definition 1.10. An *adjunction* between two categories $\mathbf{C}$ and $\mathbf{D}$ consists of two functors $L : \mathbf{C} \leftrightarrows \mathbf{D} : R$ and a natural isomorphism $\mathbf{D}(L-, -) \cong \mathbf{C}(-, R-) : \mathbf{C}^{op} \times \mathbf{D} \to \mathbf{Set}$. The functor L (resp. R) is then *left (resp. right) adjoint to R (resp. L)*, written $L \vdash R$.

One way of intuiting adjunctions is in terms of representability. Recall that a presheaf $P : \mathbf{C} \to \mathbf{Set}$ is *representable* if it is isomorphic to $\mathbf{C}(-, c)$ for some $c \in \mathbf{C}_0$. An adjunction $L : \mathbf{C} \leftrightarrows \mathbf{D} : R$ gives a functorial choice of representative $R(d)$ of the presheaf $\mathbf{D}(L-, d)$ for any $d \in \mathbf{D}_0$.

Adjunctions have many characterizations (see e.g. [Lan10][Ch.4]). In Example 1.3, the set of loops in a space X was identified as $\mathbf{Top}(S^1, X)$. This may also be stated as S^1 being the corepresenting object for the "set of loops"-functor. Identifying a functor as representable is useful; we will shortly recall the notion of a *continuous* functor, and representable functors are examples of such. Moreover, morphisms leaving representable functors are entirely determined by the Yoneda lemma: for all $c \in \mathbf{C}_0$ and $F : \mathbf{C} \to \mathbf{Set}$ we have $\mathbf{Set}^{\mathbf{C}}(\mathbf{C}(-, c), F) \cong F(c)$.

Similarly, we may look for a "set of adjunctions"-functor and seek a corepresenting object. In fact, such an object exists and is given by a 2-*category* we will call $\mathbf{Adj}$. The precise axioms of 2-categories are discussed in e.g. [Lan10][Ch.12]—for our present purposes it suffices to understand that a 2-*diagram in* ***Cat*** is a diagram that may also specify natural transformations between functors.

Definition 1.11. Let $\mathbf{Adj}$ be the (2−)category consisting of two objects a, b, two morphisms $L : a \to b$ and $R : b \to a$ along with two natural transformations $\eta : \mathrm{id}_a \Rightarrow RL$ and $\epsilon : LR \Rightarrow \mathrm{id}_b$. We call $\mathbf{Adj}$ the *free adjunction*, drawn as follows:

$$\begin{tikzcd}
a \arrow[dr, "L"] \arrow[dd, equal] \arrow[dd, phantom, shift left=4ex, "\overset{\eta}{\Rightarrow}"] & \\
 & b \arrow[dl, "R"] \arrow[dd, equal] \arrow[dd, phantom, shift right=4ex, "\overset{\epsilon}{\Rightarrow}"] \\
a \arrow[dr, "L"'] & \\
 & b
\end{tikzcd}$$

A 2-diagram D : **Adj** → **Cat** specifies exactly an adjunction between the two categories $D(a)$ and $D(b)$, this is quite immediate to see from the characterization of an adjunction in terms of the unit η and counit ϵ and the *triangular identities*:

$$\begin{tikzcd} L \arrow[r, "L(\eta)"] \arrow[dr, equal] & LRL \arrow[d, "\epsilon_{L-}"] \\ & L \end{tikzcd} \qquad \begin{tikzcd} RLR \arrow[d, "R(\epsilon)"'] & R \arrow[l, "\eta_{R-}"'] \arrow[dl, equal] \\ R & \end{tikzcd}$$

A special case arises when η and ϵ are isomorphisms. Then this gives an *adjoint equivalence* of categories, i.e. an equivalence of categories that also satisfies the triangular identities. A natural question to ask is what happens for other "orientations" of η and ϵ; let **Adj'** denote the category resulting from reversing the direction of η in **Adj**, so that $\eta : RL \Rightarrow \mathrm{id}_a$.

Question 1.12. What are 2-diagrams from $\mathbf{Adj}'$ into **Cat**?

After a definition, we will immediately answer this question.

Definition 1.13. An adjunction of contravariant functors is a *Galois correspondance*.

Given D : **Adj'** → **Cat**, if we perform $-^{op}$ on $D(a)$, the functors $D(R)$ and $D(L)$ become contravariant and η switches direction to give a Galois correspondance. The same would result from ϵ being flipped. The terminology originates from the classical correspondance between intermediate fields of an (infinite) Galois extension L/K and (closed) subgroups of the associated Galois group $\mathrm{Gal}(L/K)$. Here's another example:

Example 1.14. Let k be a field, and denote $\bar{k}$ its algebraic closure. The functor P : **Set** → **Cat** sends a set to its poset of subsets. For a family of polynomials $\{f_\alpha\}_{\alpha\in A} \in P(k[X_1, \ldots, X_n])$, denote the spanned ideal by $(f_\alpha)_{\alpha\in A}$.

There are two contravariant functors:

$$\begin{aligned} I : P(\bar{k}^n) &\leftrightarrows P(k[X_1, \ldots, X_n]) : V \\ \{a_\beta\}_{\beta\in B} &\mapsto \{\ f \in k[X_1, \ldots, X_n] \mid f(a_\beta) = 0\ \forall\beta \in B\ \} \\ \{\ a \in \bar{k}^n \mid f_\alpha(a) = 0\ \forall\alpha \in A\ \} &\leftarrow\!\shortmid \{f_\alpha\}_{\alpha\in A} \end{aligned}$$

The functor I sends a subset of points of $\bar{k}^n$ to the set of polynomials vanishing upon it; V sends a familiy of polynomials to its zero locus.

Hilbert's Nullstellensatz states that I and V define a Galois correspondance. On one side we have $\alpha : \mathrm{id}_{P(k[X_1,\ldots,X_n])} \Rightarrow IV$ as the inclusion of a family $\{f_\alpha\}_{\alpha\in A} \subseteq \sqrt{(f_\alpha)_{\alpha\in A}}$ into the radical of it's spanned ideal. On the other side we have $\beta : \mathrm{id}_{P(\bar{k}^n)} \Rightarrow VI$ as the inclusion $S \subseteq \bar{S}$ of a set of points into its closure in the Zariski topology on $\bar{k}^n$.

The last part of this section is dedicated to functors preserving colimits:

Definition 1.15. A functor $L : \mathbf{B} \to \mathbf{C}$ is *cocontinuous* if for any diagram $D : \mathbf{D} \to \mathbf{B}$, a colimit cone $\delta : D \Rightarrow \Delta(\mathrm{colim}_{\mathbf{D}} D)$ produces a colimit cone $L_*(\delta) : LD \Rightarrow \Delta(\mathrm{colim}_{\mathbf{D}} LD)$ by applying L_*. In particular, $\mathrm{colim}_{\mathbf{D}} LD \cong L(\mathrm{colim}_{\mathbf{D}} D)$.

It is well-known that left adjoints are cocontinuous and right adjoints are continuous[3], and these facts will prove useful. Another important class of continuous functors are representable functors.

[3]These theorems go by "RAPL" (Right Adjoints Preserve Limits) and "LAPC", see e.g. [Rie14b].

The definition (1.15) of (co)continuity could be stated as "L_* preserves colimit cocones" as a functor $L_* : \mathbf{B}^{\mathbf{D}} \to \mathbf{C}^{\mathbf{D}}$. Since we in Proposition 1.9 formulated a colimit functor whose domain are such diagram categories, we could hope that cocontinuous functors also preserve general natural transformations of diagrams. In fact, they do:

Proposition 1.16. *Let $\boldsymbol{D}$ be a small category, and $\boldsymbol{B}$, $\boldsymbol{C}$ two $\boldsymbol{D}$-complete categories. For any cocontinuous functor $L : \boldsymbol{B} \to \boldsymbol{C}$, the following diagram commutes:*

$$\begin{tikzcd} \boldsymbol{B}^{\boldsymbol{D}} \arrow[r, "L_*"] \arrow[d, "\mathrm{colim}_{\boldsymbol{D}}"] & \boldsymbol{C}^{\boldsymbol{D}} \arrow[d, "\mathrm{colim}_{\boldsymbol{D}}"] \\ \boldsymbol{B} \arrow[r, "L"] & \boldsymbol{C} \end{tikzcd} \tag{1.17}$$

Proof. By definition of cocontinuity of L, the diagram commutes object-wise. Let $\alpha : D \Rightarrow D' \in \mathbf{B}^{\mathbf{D}}_1$, and denote $\delta : X \Rightarrow \Delta(\mathrm{colim}_{\mathbf{D}} D)$ and $\delta' : D' \Rightarrow \Delta(\mathrm{colim}_{\mathbf{D}} D')$ the colimit cocones. By definition of $\mathrm{colim}_{\mathbf{D}}$ on morphisms, $\mathrm{colim}_{\mathbf{D}}\alpha$ is the unique morphism making the diagram on the left commute:

$$\mathbf{B}^{\mathbf{D}} : \quad \begin{tikzcd} D \arrow[r, "\delta"] \arrow[d, "\alpha"] & \Delta(\mathrm{colim}_{\mathbf{D}} D) \arrow[d, "\Delta(\mathrm{colim}_{\mathbf{D}}\alpha)"] \\ D' \arrow[r, "\delta'"] & \Delta(\mathrm{colim}_{\mathbf{D}} D') \end{tikzcd} \quad \overset{L_*}{\longmapsto} \quad \begin{tikzcd} LD \arrow[r, "L_*(\delta)"] \arrow[d, "L_*(\alpha)"] & \mathrm{colim}_{\mathbf{D}}(LD) \arrow[d, "\Delta(L(\mathrm{colim}_{\mathbf{D}}\alpha))"] \\ LD' \arrow[r, "L_*(\delta')"] & \mathrm{colim}_{\mathbf{D}}(LD') \end{tikzcd} \quad : \mathbf{C}^{\mathbf{D}}$$

Applying L_* gives the diagram on the right, where we have used that $L_*\Delta = \Delta L$. Since $L(\mathrm{colim}_{\mathbf{D}}\alpha)$ makes the rightmost diagram commute, and this is the universal property of $\mathrm{colim}_{\mathbf{D}} L_*(\alpha)$, they are equal. We conclude that Diagram 1.17 commutes. □

In summary, a cocomplete category $\mathbf{C}$ carries an adjunction $\mathrm{colim}_{\mathbf{D}} \vdash \Delta$ associated with any small indexing category $\mathbf{D}$. It is a good exercice to obtain the dual notion of a *limit* as the right adjoint $\Delta \vdash \lim$ whenever $\mathbf{C}$ is complete.

1.2. Ends and Limits in **Set**

Computing (co)limits is difficult in a general category $\mathbf{C}$. However, if $\mathbf{C} = \mathbf{Set}$ the right perspective makes things easy. We start by computing limits in $\mathbf{Set}$ and then turn to how this may be leveraged for general $\mathbf{C}$ through the Yoneda embedding $r : \mathbf{C} \to \mathbf{Set}$. Recall that $* \in \mathbf{Set}_0$ denotes the singleton set, i.e. the terminal object.

Proposition 1.18. *Let $D : \boldsymbol{D} \to \boldsymbol{Set}$ be a diagram. Then the limit of D is the set*

$$\boldsymbol{Set}^{\boldsymbol{D}}(\Delta *, D) = \left\{ (x_d)_{d \in \boldsymbol{D}_0} \;\middle|\; D(\delta)(x_d) = x_{d'}, \delta{:}d{\to}d'{\in}\boldsymbol{D}_1 \right\} \subseteq \prod_{d \in \boldsymbol{D}_0} D(d) \tag{1.19}$$

In particular, $\boldsymbol{Set}$ is complete. Moreover, $\boldsymbol{Set}^{\boldsymbol{D}}(\Delta *, -) = \lim : \boldsymbol{Set}^{\boldsymbol{D}} \to \boldsymbol{Set}$ is the limit functor dual to colim *of Proposition 1.9.*

Proof. Denote $S := \mathbf{Set}^{\mathbf{D}}(\Delta *, D)$. Consider the map $\alpha \mapsto \alpha_d : S \to D(d)$ for all $d \in \mathbf{D}_0$. The property that $D(\delta)(x_d) = x_{d'}$ for any $x \in S$ and $\delta : d \to d' \in \mathbf{D}_1$ is exactly the naturality required for S to be a cone on D.

Let $\gamma : \Delta T \Rightarrow D$ be another cone. Then the map $t \mapsto (\gamma_d(t))_{d \in \mathbf{D}_0} : T \to S$ is the desired factorization. Uniqueness is immediately seen from the definitions.

We now turn to see that $\lim = \mathbf{Set}^{\mathbf{D}}(\Delta *, -)$. For any $\eta : D \Rightarrow D'$ in $\mathbf{Set}^{\mathbf{D}}$ and $d \in \mathbf{D}_0$, $\lim \eta : \lim D \to \lim D'$ is the unique morphism making the square on the right commute:

$$\begin{tikzcd}
\mathbf{Set}^{\mathbf{D}}(\Delta *, D) \arrow[r, equal] \arrow[d, "\eta_*"] & \lim_{\mathbf{D}} D \arrow[r, "-_d"] \arrow[d, "\lim \eta"] & D(d) \arrow[d, "\eta_d"] \\
\mathbf{Set}^{\mathbf{D}}(\Delta *, D') \arrow[r, equal] & \lim_{\mathbf{D}} D' \arrow[r, "-_d"] & D'(d)
\end{tikzcd}$$

This means that for all $x : \Delta * \Rightarrow D$, $(\lim \eta)_d = \eta_d(x_d) = (\eta_*(x))_d$. But this means $\lim \eta = \eta_*$, so $\lim = \mathbf{Set}^{\mathbf{D}}(\Delta *, -)$ on both objects and morphisms. □

In order to apply the above to an arbitrary category, we need two notions of how limits transfer through functors:

Definition 1.20. A functor $F : \mathbf{C} \to \mathbf{C}'$ *reflects* limits if for any diagram $D : \mathbf{D} \to \mathbf{C}$,

$$F(c) = \lim FD \in \mathbf{C}'_0 \implies c = \lim D \in \mathbf{C}_0$$

whenever $\lim FD$ exists. Reversing the implication gives the notion of a functor *preserving* limits. A functor that both reflects and preserves limits such that the limit exists in the domain whenever it does in the codomain, is said to *create* limits.

An important class of functors that reflect limits are fully faithful ones. Our proof of this uses the following lemma:

Lemma 1.21. *Let $F : \boldsymbol{C} \to \boldsymbol{B}$ be a fully faithful functor and $\boldsymbol{D}$ a category. Then the functor $F_* : \boldsymbol{C}^{\boldsymbol{D}} \to \boldsymbol{B}^{\boldsymbol{D}}$ defined by post-composition of F is also fully faithful.*

Proof. Let $D, D' : \mathbf{D} \to \mathbf{C}$ and $\eta, \eta' : D \Rightarrow D'$. If $F_*(\eta) = F_*(\eta')$ then $F(\eta_d) = F(\eta_{d'})$ $\forall d \in \mathbf{D}_0$. Since F is fully faithful, this gives $\eta_d = \eta'_{d'}$ $\forall d \in \mathbf{D}_0$. But this is exactly what $\eta = \eta'$ means, so F_* is faithful. Now let $\alpha : FD \Rightarrow FD'$. The preimage of each component of α defines a family $(F^{-1}(\alpha_d))_{d \in \mathbf{D}_0}$. To see that this defines a natural transformation $D \to D'$, let $f : d \to d' \in \mathbf{D}_1$. We then have

$$D(f)F^{-1}(\alpha_d) \overset{F}{\longmapsto} FD(f)\alpha_d = \alpha_{d'}FD'(f) \overset{F}{\longleftarrow\!\shortmid} F^{-1}(\alpha_{d'})D'(f)$$

Since F is faithful, the two preimages (on the sides) of the middle are equal. □

Proposition 1.22. *A fully faithful functor reflects limits.*

Proof. Let $F : \mathbf{C} \to \mathbf{B}$ be fully faithful, and $D : \mathbf{D} \to \mathbf{C}$ a diagram. Suppose that $F(c) = \lim FD$ for some $c \in \mathbf{C}_0$. We need to show that c is the limit of D. Let $x \in \mathbf{C}_0$:

$$\mathbf{C}(x, c) \overset{F \text{ f.f.}}{\cong} \mathbf{B}(F(x), F(c)) \cong \mathbf{B}^{\mathbf{D}}(F\Delta(x), FD) \overset{F_* \text{ f.f.}}{\cong} \mathbf{C}^{\mathbf{D}}(\Delta(x), D)$$

Note that we have used that $\Delta(F(x)) = F\Delta(x)$. □

Corollary 1.23. *Let $D : \boldsymbol{D} \to \boldsymbol{C}$ be a diagram. Whenever $\lim D$ exists in $\boldsymbol{C}$, we have:*

$$\boldsymbol{Set}^{\boldsymbol{D}}(\Delta *, \boldsymbol{C}(-, D)) \cong \boldsymbol{C}(-, \lim D) : \boldsymbol{C} \to \boldsymbol{Set}$$

Proof. The Yoneda embedding $r : \mathbf{C} \to \mathbf{Set}_{\mathbf{C}}$ is fully faithful. By Proposition 1.22, r reflects limits and $r(\lim D)$ is therefore the limit of $rD = \mathbf{C}(-, D)$ whenever $\lim D$ exists. On the other hand, limits in a presheaf category may be computed pointwise. Any $c \in \mathbf{C}_0$ defines a diagram $rD(c) = \mathbf{C}(c, D-) : \mathbf{D} \to \mathbf{Set}$ whose limit by Proposition 1.18 is $\mathbf{Set}^{\mathbf{D}}(\Delta *, \mathbf{C}(c, D-))$. In conclusion, $\mathbf{Set}^{\mathbf{D}}(\Delta *, \mathbf{C}(-, D)) \cong \mathbf{C}(-, \lim D)$. ☐

The construction (1.19) of limits in **Set** goes more generally as "small limits through finite products and equalizers" [Lan10][Ch.5.2]:

Construction 1.24. Let $D : \mathbf{D} \to \mathbf{C}$ be a diagram where $\mathbf{C}$ admits products indexed by the objects and arrows of $\mathbf{D}$, as well as (pairwise) equalizers. The limit of D is then the following equalizer:

$$\lim D \overset{u}{\dashrightarrow} \prod_{f:d\to d' \in \mathbf{D}_1} D(d) \underset{\prod_d \pi_d}{\overset{\prod_d D(f)\pi_d}{\rightrightarrows}} \prod_{d \in \mathbf{D}_0} D(d) \tag{1.25}$$

where $\pi_e : \prod_{f:d\to d'\in\mathbf{D}_1} D(d) \to D(e)$ is the projection. The equalizing property of $u : \lim D \to \prod_{f:d\to d'\in\mathbf{D}_1} D(d)$ unpacks to being a cone on D, and the universal property translates to being a universal such.

Comparing Equation 1.19 to 1.25, in the case of **Set** we identify in the $\lim D$ as a subset of $\prod_{d\in\mathbf{D}_0}$ containing collections satisfying exactly the relations of the equalizer 1.25, and u is in this case the set-inclusion.

Even if Construction 1.24 dualizes to make "small colimits by finite coproducts and coequalizers", there is a certain asymmetry to it: On one side D is being applied and the other side is simply the projection. Another diagram $D' : \mathbf{D}^{op} \to \mathbf{C}$ (contravariant this time) could be applied on the other side. The equalizer in this case is the *end* of the bifunctor $D' \times D : \mathbf{D}^{op} \times \mathbf{D} \to \mathbf{C}$,

$$\int_{d\in\mathbf{D}_0} D'(d)\times D(d) \overset{u}{\dashrightarrow} \prod_{\delta\in\mathbf{D}(d,d')} D'(d')\times D(d) \underset{\prod_d (D'(\delta)\times \mathrm{id}_{D(d)})\pi_\delta}{\overset{\prod_d (\mathrm{id}_{D'(d')}\times D(\delta))\pi_\delta}{\rightrightarrows}} \prod_{d\in\mathbf{D}_0} D'(d)\times D(d) \tag{1.26}$$

to be rigorously defined in 1.28. In this case, the property of the equalizer unpacks to the following universal diagram commuting for every $\delta : d \to d'$ in $\mathbf{D}$:

$$\begin{tikzcd} \int_{d\in\mathbf{D}_0} D'(d)\times D(d) \arrow[r] \arrow[d] & D'(d)\times D(d) \arrow[d, "\mathrm{id}\times D(\delta)"] \\ D'(d')\times D(d') \arrow[r, "D'\delta\times\mathrm{id}"] & D'(d)\times D(d') \end{tikzcd} \tag{1.27}$$

This is the assertion that $\int_{d\in\mathbf{D}_0} D(d)\times D'(d)$ is a *universal wedge* on $D'\times D$, and the maps leaving the wedge are components of a *dinatural transformation* from the constant bifunctor given by the end, to the bifunctor $D'\times D$.

Definition 1.28. Let $F, G : \mathbf{D}^{op} \times \mathbf{D} \to \mathbf{C}$ be two bifunctors, and c an object of $\mathbf{C}$.

1. A *dinatural transformation* $\eta : F \to G$ is a family $\{\eta_d : F(d,d) \to G(d,d)\}_{d \in \mathbf{D}_0}$ such that the following hexagon commutes for all $f : d \to d'$ in $\mathbf{D}_1$:

$$\begin{tikzcd}
 & F(d,d) \arrow[r, "\eta_d"] & G(d,d) \arrow[dr, "G(\mathrm{id},f)"] & \\
F(d',d) \arrow[ur, "F(f,\mathrm{id})"] \arrow[dr, "F(\mathrm{id},f)"'] & & & G(d,d') \\
 & F(d',d') \arrow[r, "\eta_{d'}"] & G(d',d') \arrow[ur, "G(f,\mathrm{id})"'] &
\end{tikzcd} \tag{1.29}$$

2. A dinatural transformation $\omega : \Delta(c) \to G$ is called a *wedge* on G; dually a dinatural transformation $\gamma : F \to \Delta(c)$ is called a *cowedge* on F. This means the following squares commute for all $f : d \to d'$ in $\mathbf{D}_1$:

$$\begin{tikzcd}
c \arrow[r, "\omega_d"] \arrow[d, "\omega_{d'}"] & G(d,d) \arrow[d, "G(\mathrm{id},f)"] \\
G(d',d') \arrow[r, "G(f,\mathrm{id})"] & G(d,d')
\end{tikzcd} \qquad
\begin{tikzcd}
F(d',d) \arrow[r, "F(f,\mathrm{id})"] \arrow[d, "F(\mathrm{id},f)"] & F(d,d) \arrow[d, "\gamma_d"] \\
F(d',d') \arrow[r, "\gamma_{d'}"] & c
\end{tikzcd}$$

3. The *end* of the bifunctor F is an object $\int_{d \in \mathbf{D}_0} F(d,d) \in \mathbf{C}_0$ (often $\int_{\mathbf{D}} F$ for short), along with a universal wedge $\delta : \Delta\left(\int_{\mathbf{D}} F\right) \to F$. This is the situation of diagram 1.27 above (replacing $D' \times D$ by F); universality means that any other wedge $\gamma : \Delta(c) \to F$ factors uniquely through a morphism $f : c \to \int_{\mathbf{D}} F$, i.e. $\gamma = \delta\Delta(f)$.

4. Dually, the *coend* of the bifunctor F is an object $\int^{\mathbf{D}} F$ of $\mathbf{C}$ along with a universal cowedge $F \to \Delta(\int^{\mathbf{D}} F)$.

Remark 1.30. Construction (1.26) shows that ends can be computed as limits. However, for any diagram $D : \mathbf{D} \to \mathbf{C}$ the limit can also be found as the end of the bifunctor $(d', d) \mapsto D(d)$ forgetting the contravariant parameter. As such, ends and limits are equally expressive and continuous functors preserve both.

A question immediately springs to mind:

Question 1.31. What is the end of $\mathbf{C}(-,-)$?

The answer follows immediately by proceeding as in 1.19. First consider singleton wedges; a singleton wedge $\eta : \Delta * \to \mathbf{C}(-,-)$ satisfies $f_*(\eta_{c'}) = f^*(\eta_c)$ i.e. $f\eta_{c'}(x) = \eta_c f(x)$, for any $f : c \to c'$ in $\mathbf{C}_1$: This is what it means for η to be in the *center* of $\mathbf{C}$, i.e. the endomorphism monoid $\mathbf{C}^{\mathbf{C}}(\mathrm{id}_{\mathbf{C}}, \mathrm{id}_{\mathbf{C}})$ of the identity functor on $\mathbf{C}$. The universal wedge is then the set of all such wedges, which is exactly the set (monoid) $\mathbf{C}^{\mathbf{C}}(\mathrm{id}_{\mathbf{C}}, \mathrm{id}_{\mathbf{C}})$.

The answer generalizes to the following useful example:

Example 1.32. Consider two functors $F, G : \mathbf{C} \to \mathbf{C}'$. The end of the bifunctor $\mathbf{C}'(F-, G-) : \mathbf{C}'^{op} \times \mathbf{C}' \to \mathbf{Set}$ is the set $\mathbf{C}'^{\mathbf{C}}(F, G)$ of natural transformations from F to G. Indeed, for any $\delta : c \to c'$ the desired diagram on the left commutes:

$$\begin{tikzcd}
\mathbf{C}'^{\mathbf{C}}(F,G) \arrow[r, "-_c"] \arrow[d, "-_{c'}"] & \mathbf{C}'(Fc, Gc) \arrow[d, "G\delta_*"] \\
\mathbf{C}'(Fc', Gc') \arrow[r, "F\delta^*"] & \mathbf{C}'(Fc, Gc')
\end{tikzcd} \qquad
\begin{tikzcd}
Fc \arrow[r, "\eta_c"] \arrow[d, "F\delta"] & Gc \arrow[d, "G\delta"] \\
Fc' \arrow[r, "\eta_{c'}"] & Gc'
\end{tikzcd}$$

The diagram on the right is the pointwise computation for any $\eta : F \Rightarrow G$, which commutes by naturality. Given any other wedge $\omega : \Delta W \to \mathbf{C}'(F-, G-)$, any point $w \in W$ defines a unique natural transformation $\omega_-(w) : F \Rightarrow G$, factoring ω through $\mathbf{C'^C}(F, G)$ as desired.

Ends will be central to the proofs appearing in the next sections. Their utility will not be to create new constructions, but in expressing familiar ones. By identifying a construction as an end, Remark 1.26 implies that this construction is preserved by continuous functors. An important class of continuous functors are the representable ones:

1.3. Comma categories

We introduce comma categories and a perspective on limits generalizing Prop. 1.9.

Definition 1.33. Consider a cospan of categories $\mathbf{A} \xrightarrow{F} \mathbf{C} \xleftarrow{G} \mathbf{B}$.

1. The *comma category* $(F{\downarrow}G)$ consists of objects:

$$(F{\downarrow}G)_0 := \{c : F(a) \to G(b) \mid a \in \mathbf{A}_0, b \in \mathbf{B}_0\} \subseteq \mathbf{C}_1$$

   A morphism $(F(a) \xrightarrow{c} G(b)) \to (F(a') \xrightarrow{c'} G(b'))$ is a couple $(f{:}a \to a', g{:}b \to b') \in \mathbf{A}_1 \times \mathbf{B}_1$ such that $d'F(f) = G(g)d$, i.e. a commutative square in $\mathbf{C}$:

$$\begin{tikzcd} F(a) \arrow[r, "F(f)"] \arrow[d, "c"] & F(a') \arrow[d, "c'"] \\ G(b) \arrow[r, "G(g)"] & G(b') \end{tikzcd}$$

2. For an object $c \in \mathbf{C}_0$, the category F *over* c (resp. G *under* c) is defined as $(F{\downarrow}c) := (F{\downarrow}\Delta c)$ (resp. $(c{\downarrow}G) := (\Delta c{\downarrow}G)$).

3. When $F = \mathrm{id}_{\mathbf{C}}$ (resp. $G = \mathrm{id}_{\mathbf{C}}$) we will simply write $\mathbf{C}/c$ (resp. $c/\mathbf{C}$) for $(F{\downarrow}c)$ (resp. $(c{\downarrow}G)$) and call it *the category over (resp. under)* c or simply refer to it as an *over (resp. under) category.* Morphisms are referred to as *morphisms over (resp. under)* c.

4. For any $a \in \mathbf{A}_0$, there is a *slice-functor* $a/F : (a/\boldsymbol{A}) \to (F(a)/\boldsymbol{C})$ by applying F to the objects and morphisms of $a/\mathbf{A}$.

Example 1.34. Many familiar constructions arise as over and under categories:

1. The over category $*/\mathbf{Top}$ consists of pointed topological spaces along with basepoint-preserving maps—in the litterature this is often denoted $\mathbf{Top}_*$. This category is the domain of the fundamental group functor $\pi_1 : */\mathbf{Top} \to \mathbf{Gp}$.

2. For a topological space X, the objects of the category $\mathbf{Top}/X$ are referred to as "spaces over X". Two important subcategories of $\mathbf{Top}/X$ are bundles over X, covers of X, and the (open) subsets of X.

3. For a commutative ring R, the category of commutative R-algebras may be identified as $R/\mathbf{CRing}$.

4. Let $\mathbf{C}$ be a category. The category $\mathbf{Cat}/\mathbf{C}$ is the category of diagrams in $\mathbf{C}$.

5. Let $\mathbf{C}$ be a category and $c \in \mathbf{C}_0$. In $c/\mathbf{C}$ (resp. $\mathbf{C}/c$), the object id_c is initial (resp. terminal). If c is initial (resp. terminal) in $\mathbf{C}$, then $\mathbf{C}$ is isomorphic to $c/\mathbf{C}$ (resp. $\mathbf{C}/c$). This characterizes categories with initial (resp. terminal) objects as exactly over (resp. under) categories.

Remark 1.35. Equality on morphisms in comma categories is inherited from the domain categories $\mathbf{A}$ and $\mathbf{B}$ (of Def. 1.33). It would be a mistake to equate two commuting squares by only considering them in $\mathbf{C}$. To illustrate this, consider a functor $F : \mathbf{A} \to \mathbf{C}$. The following two parallel morphisms in $(F{\downarrow}c)$ are equal if and only if $f = g$ in $\mathbf{A}_1$:

$$\begin{tikzcd} F(d) \arrow[rr, "F(f)"] \arrow[dr, "\phi"'] & & F(d') \arrow[dl, "\psi"] \\ & c & \end{tikzcd} \quad \underset{(f \neq g)}{\neq} \quad \begin{tikzcd} F(d) \arrow[rr, "F(g)"] \arrow[dr, "\phi"'] & & F(d') \arrow[dl, "\psi"] \\ & c & \end{tikzcd}$$

Even if $F(f) = F(g)$, which makes these triangles indistinguishable in $\mathbf{C}$.

The following is a first step in generalizing (the dual of) Proposition 1.9 to the category $\mathbf{Cat}/\mathbf{C}$ of all diagrams in a category $\mathbf{C}$. See the subsequent remark (1.37) for a sketch of the full-fledged generalization.

Proposition 1.36 ([Lan10][Ex.5b, p.115])**.** *If $\mathbf{C}$ is a complete category, there is a contravariant limit functor* $\lim : \mathbf{Cat}/\mathbf{C} \to \mathbf{C}$ *sending any diagram to its limit in $\mathbf{C}$.*

Proof. An object of $\mathbf{Cat}/\mathbf{C}$ is a diagram $D : \mathbf{D} \to \mathbf{C}$. Completeness of $\mathbf{C}$ gives a limit $\lim D \in \mathbf{C}_0$ for any such diagram D. If $D' : \mathbf{D'} \to \mathbf{C}$ is another diagram, and $F : \mathbf{D} \to \mathbf{D'}$ is a morphism from D to D' over $\mathbf{C}$, then the limit cone $\delta' : \Delta(\lim D') \Rightarrow D'$ restricts to a cone $\delta'_{F-} : \Delta(\lim D') \Rightarrow D'F = D$ by precomposing with F. Universality of $\lim D$ gives a unique morphism $\lim F : \lim D' \to \lim D$, making $\lim : \mathbf{Cat}/\mathbf{C} \to \mathbf{C}$ well-defined on objects and morphisms—it remains to check functoriality.

$$\begin{tikzcd} \mathbf{D} \arrow[d, "F"'] \arrow[dr, "D"] & \\ \mathbf{D'} \arrow[r, "D'"] \arrow[d, "F'"'] & \mathbf{C} \\ \mathbf{D''} \arrow[ur, "D''"'] & \end{tikzcd} \qquad \begin{tikzcd} \Delta(\lim D') \arrow[r, Rightarrow, "\delta'"] & D' \\ \Delta(\lim D'') \arrow[u, dashed, "\Delta(\lim F')"] \arrow[ur, Rightarrow, "\delta''_{F'-}"'] & \end{tikzcd} \quad \overset{F^*}{\mapsto} \quad \begin{tikzcd} \Delta(\lim D) \arrow[dr, Rightarrow, "\delta"] & \\ \Delta(\lim D') \arrow[u, dashed, "\Delta(\lim F)"] \arrow[r, Rightarrow, "\delta'_{F-}"] & D'F = D \\ \Delta(\lim D'') \arrow[u, "\Delta(\lim F')"] \arrow[ur, Rightarrow, "\delta''_{F'F-}"'] & \end{tikzcd}$$

Consider two composable morphisms F and F' as on the left above. In the middle, $\lim F'$ is induced by the universal property of the limit cone $\delta' : \lim D' \Rightarrow D'$. The middle diagram transfers through F^* into the bottom triangle of the diagram on the right. Since everything commutes on the right, the composition $\Delta(\lim F')\Delta(\lim F) = \Delta(\lim F \lim F')$ is the unique morphism in the image of Δ making the outer diagram commute, hence $\lim F' \lim F = \lim F'F$. □

Dualizing the proposition (1.36) gives a *covariant* functor $\mathrm{colim} : \mathbf{Cat}/\mathbf{C} \to \mathbf{C}$ whenever $\mathbf{C}$ is a cocomplete category.

Remark 1.37. The reason the limit functor of the previous proposition (1.36) isn't a proper generalization of the limit functor of (the dual of) Proposition 1.9 is that the morphisms involved in the latter (i.e. natural transformations of diagrams) are not present in the over category $\mathbf{Cat}/\mathbf{C}$. The remedy is to define a *weaker* version of our over categories, where a morphism $F : D \to D'$ over $\mathbf{C}$ (drawn below) is allowed to commute *up to a natural transformation* $\eta : D \to D'F$.

$$\begin{tikzcd} \mathbf{D} \arrow[d, "F"'] \arrow[dr, "D"] & \\ \mathbf{D}' \arrow[r, "D'"'] \arrow[ur, phantom, "\Uparrow\eta"] & \mathbf{C} \end{tikzcd}$$

A natural transformation of diagrams $\alpha \in \mathbf{C}^{\mathbf{D}}_1$ is then given by the above morphism when $F = \mathrm{id}_{\mathbf{D}}$ and $\eta = \alpha$. The generalization for colimits requires the direction of η to be reversed, since $\mathrm{colim} : \mathbf{Cat}/\mathbf{C} \to \mathbf{C}$ is then covariant.

The generality of the above will not be strictly needed for our purposes, but it is the formal justification for why we may speak of *the* limit and colimit functors while applying them to morphisms of both $\mathbf{C}^{\mathbf{D}}_1$ and $(\mathbf{Cat}/\mathbf{C})_1$.

Having now changed (or, in view of the preivous remark (1.37), expanded) the domain of our limit functor, we are confronted with the question of whether a continuous functor $R : \mathbf{B} \to \mathbf{C}$ between complete categories respects $\lim : \mathbf{Cat}/\mathbf{B} \to \mathbf{B}$ on morphisms as well as objects. Luckily it does:

Proposition 1.38. *Let $\boldsymbol{B}$ and $\boldsymbol{C}$ be complete categories. If $R : \boldsymbol{B} \to \boldsymbol{C}$ is a continuous functor, the following diagram commutes:*

$$\begin{tikzcd} \boldsymbol{Cat}/\boldsymbol{B} \arrow[r, "\boldsymbol{Cat}/R"] \arrow[d, "\lim"] & \boldsymbol{Cat}/\boldsymbol{C} \arrow[d, "\lim"] \\ \boldsymbol{B} \arrow[r, "R"] & \boldsymbol{C} \end{tikzcd}$$

Proof. By definition of continuity of R, the diagram commutes object-wise. Consider a morphism $F : D \to D'$ in $\mathbf{Cat}/\mathbf{B}_1$ between two diagrams $D : \mathbf{D} \to \mathbf{B}$ and $D' : \mathbf{D'} \to \mathbf{B}$. Write δ and δ' for the respective limit cones of D and D'. On the left below $(\mathbf{Cat}/R)(F)$ is drawn, whereas on the right we see the diagram inducing $\lim F$ by universality of δ.

$$\begin{tikzcd} \mathbf{D} \arrow[d, "F"] \arrow[dr, "RD"] & & & \Delta(\lim RD) \arrow[r, Rightarrow, "R_*(\delta)"] & RD \arrow[dd, equal] & & \Delta(\lim D) \arrow[r, Rightarrow, "\delta"] & D \arrow[dd, equal] \\ & \mathbf{C} & {} \arrow[r, hook, dashed, "\lim"] & {} & & {} & {} \arrow[l, mapsto, dashed, "R_*"'] & \\ \mathbf{D'} \arrow[ur, "RD'"'] & & & \Delta(\lim RD') \arrow[uu, dashed] \arrow[r, Rightarrow, "R_*(\delta'_{F-})"] & RD & & \Delta(\lim D') \arrow[uu, dashed, "\Delta(\lim F)"] \arrow[r, Rightarrow, "\delta'_{F-}"] & D'F \end{tikzcd}$$

The middle diagram is both the one inducing $\lim(\mathbf{Cat}/\mathbf{C})(R)$, and the image of the rightmost diagram. This means $\lim(\mathbf{Cat}/\mathbf{C})(R) = R(\lim F)$, as desired. □

For a fixed diagram category $\mathbf{D}$ and a $\mathbf{D}$-complete category $\mathbf{C}$, the limit functor $\lim : \mathbf{C}^{\mathbf{D}} \to \mathbf{C}$ is right adjoint to the diagonal $\Delta : \mathbf{C} \to \mathbf{C}^{\mathbf{D}}$. How does this generalize?

Question 1.39. When $\mathbf{C}$ is complete, does $\lim : (\mathbf{Cat}/\mathbf{C})^{op} \to \mathbf{C}$ admit a left adjoint?

The answer to this question is likely hidden somewhere in the literature. It is a natural question to ask at this point, but in order to give a well motivated answer, we shall return to it in Proposition 2.25 at the end of Section 2.1.

One may check that forming comma categories defines a functor $(-\downarrow -) : \mathbf{Cat}/\mathbf{C} \times \mathbf{C}/\mathbf{Cat} \to \mathbf{Cat}$, though we will not need this construction. However, we will need the following related fact:

Proposition 1.40. *Let $\boldsymbol{C}$ be a small category. There is an* over *(or* slice*) functor $\boldsymbol{C}/- : \boldsymbol{C} \to \boldsymbol{Cat}$ associating any object c in $\boldsymbol{C}_0$ to its over category $\boldsymbol{C}/c$.*

Proof. Let $f : c \to c'$ be a morphism in $\mathbf{C}$. We define a functor $\mathbf{C}/f : \mathbf{C}/c \to \mathbf{C}/c'$ by postcomposition on objects and "doing nothing" on morphisms:

$$\mathbf{C}/c : \quad \begin{tikzcd} a \arrow[d, "u"'] \arrow[dr, "v"] & \\ b \arrow[r, "w"'] & c \end{tikzcd} \quad \overset{\mathbf{C}/f}{\dashmapsto} \quad \begin{tikzcd} a \arrow[d, "u"'] \arrow[dr, dotted, "v"] \arrow[drr, "fv"] & & \\ b \arrow[r, dotted, "w"'] \arrow[rr, bend right, "fw"'] & c \arrow[r, dotted, "f"] & c' \end{tikzcd} \quad : \mathbf{C}/c'$$

This is functorial since $\mathbf{C}/f$ acts as the identity on $u \in \mathbf{C}_1$. □

1.4. Categories of Elements

An important occurrance of comma categories are so-called categories of elements:

Definition 1.41. Let $\mathbf{C}$ be a small category, and $P : \mathbf{C} \to \mathbf{Set}$ a presheaf. The *category of elements of* P is the under category $(*\downarrow P)$. An object $x : * \to P(c)$ will rather be denoted $x \in P(c)$, and a morphism $P(f) : P(c) \to P(c')$ from $x \in P(c)$ to $x' \in P(c')$ will be written $P(f)(x) = x'$. There is a forgetful functor $U_P : (*\downarrow P)^{op} \to \mathbf{C}$ defined by:

$$(*\downarrow P) : \quad \begin{tikzcd} x \in P(c) \arrow[d, "P(f)"] \\ x' \in P(c') \end{tikzcd} \quad \overset{U_P}{\longmapsto} \quad \begin{tikzcd} c \\ c' \arrow[u, "f"'] \end{tikzcd} \quad : \mathbf{C}$$

Proposition 1.42. *Let $\boldsymbol{C}$ be a small category. Forming categories of elements gives rise to a faithful functor $(*\downarrow -) : \boldsymbol{Set}_{\boldsymbol{C}} \to \boldsymbol{Cat}$.*

Proof. Let $\eta : P \Rightarrow P'$ be a natural transformation of presheaves over $\mathbf{C}$. The functor $(*\downarrow \eta)$ sends an object $x \in P(c)$ to $\eta_c(x) \in P'(c)$, and a morphism $x' = P(f)(x)$ with $f : c' \to c \in \mathbf{C}_1$ is sent to $\eta_{c'}(x') = P'(f)(\eta_c(x))$ in $(*\downarrow P')$. This is well-defined since naturality of η makes the following commute:

$$\begin{tikzcd} x \in P(c) \arrow[r, mapsto, "\eta_c"] \arrow[d, "P(f)"] & \eta_c(x) \in P'(c) \arrow[d, "P'(f)"] \\ x' \in P(c') \arrow[r, mapsto, "\eta_{c'}"] & \eta_{c'}(x') \in P'(c') \end{tikzcd}$$

As such, $(*\downarrow\eta)$ is functorial by stacking diagrams such as the above and applying naturality. Functoriality of $(*\downarrow-)$ is also clear; natural transformations are composed component-wise, and $(*\downarrow\eta)$ is just η acting component-wise.

Claim: $(*\downarrow-)$ is faithful. Let $\eta, \eta' : P \Rightarrow P'$. If $\eta \neq \eta'$, then $\eta_c \neq \eta'_c$ for some $c \in \mathbf{C}_0$. As these are morphisms in **Set**, this means there is an $x \in P(c)$ such that $\eta_c(x) \neq \eta'_c(x)$. But this means exactly $(*\downarrow\eta)(x \in P(c)) \neq (*\downarrow\eta')(x \in P(c))$, so $(*\downarrow\eta) \neq (*\downarrow\eta')$. □

Remark 1.43. The failure of $(*\downarrow-)$ to be full stems from the fact for a natural transformation $\eta : P \Rightarrow P'$, the induced functor $(*\downarrow\eta) : (*\downarrow P) \to (*\downarrow P')$ sends an object $x \in P(c)$ to $\eta_c(x) \in P'(c)$ whilst respecting the object c. By this we mean that $(*\downarrow\eta)$ gives rise to a morphism from U_P to $U_{P'}$ in $\mathbf{Cat}/\mathbf{C}$, since

$$U_P(x \in P(c)) = c = U_{P'}(\eta_c(x) \in P'(c))$$

A general functor $F : (*\downarrow P) \to (*\downarrow P')$ does not satisfy this constraint.

In Theorem 2.18 a refined picture is given: The image of $(*\downarrow-) : \mathbf{Set}_\mathbf{C} \to \mathbf{Cat}$ is identified as the category of "covers" of $\mathbf{C}$, and $U_P : (*\downarrow P) \to \mathbf{C}$ is then seen as a "categorical covering space" over $\mathbf{C}$.

Categories of elements arise prominently as the diagram categories in the following:

Theorem 1.44. *Let $\boldsymbol{C}$ be a small category. Any presheaf $P : \boldsymbol{C} \to \boldsymbol{Set}$ is a colimit of representable presheaves in a canonical way:*

$$P = \operatorname*{colim}_{x \in P(c)} \left[(*\downarrow P)^{op} \xrightarrow{U_P} \boldsymbol{C} \xrightarrow{r} \boldsymbol{Set}_{\boldsymbol{C}} \right] \tag{1.45}$$

Proof. See e.g. [Lan10][Ch. 3.7]. □

The above colimit (1.45) can be written in many ways; e.g. $P = \operatorname{colim}_{x\in P(c)} \mathbf{C}(-, c)$ or even $P = \operatorname{colim}_{\mathbf{C}(-,c)\Rightarrow P} \mathbf{C}(-, c)$. In the latter case, the diagram category is $(r\downarrow P)$, i.e. the category of representable presheaves over P. Using the Yoneda lemma, one immediately sees that $(r\downarrow P) \cong (*\downarrow P)^{op}$. The colimit cocone $p : rU_P \Rightarrow \Delta P$ is indexed by $(*\downarrow P)$, and the components are given by $\hat{x} : \mathbf{C}(-, c) \Rightarrow P$ for $x \in P(c)$, where $x \mapsto \hat{x} : F(c) \leftrightarrow \mathbf{Set}^\mathbf{C}(\mathbf{C}(-, c), P)$ is the Yoneda lemma.

We will need a slight improvement on Theorem 1.44 which allows us to also obtain the morphisms of $\mathbf{Set}_\mathbf{C}$ through colimits:

Corollary 1.46. *Let $\boldsymbol{C}$ be a small category, and $\eta : P \Rightarrow P'$ a natural transformation of presheaves over $\boldsymbol{C}$. Then* colim : $\boldsymbol{Cat}/\boldsymbol{Set}_{\boldsymbol{C}} \to \boldsymbol{Set}_{\boldsymbol{C}}$ *(dual to Prop. 1.36) gives:*

$$\operatorname{colim}(*\downarrow\eta) = \eta$$

Proof. Write the canonical cocones of P and P' as $p : rU_P \Rightarrow \Delta P$ and $p' : rU_{P'} \Rightarrow \Delta P'$. By definition, $\operatorname{colim}(*\downarrow\eta)$ is induced as the unique morphism factoring the restriction $p'_{(*\downarrow\eta)-} : rU_P \Rightarrow \Delta P'$ through the colimit cocone p.

Claim: $p'_{(*\downarrow\eta)-} = \Delta(\eta)p$. For $x \in P(c)$ denote $\hat{x} : \mathbf{C}(-, c) \Rightarrow P$ the natural transformation given by the Yoneda lemma. Note that by definition, $\hat{x}_c(\mathrm{id}_c) = x$, and in this notation $p_{x\in P(c)} = \hat{x}$. Consequently, for any $x \in P(c)$:

$$(\Delta(\eta)p)_{x\in P(c)} = \eta\hat{x} = \widehat{\eta_c(x)} = p'_{(*\downarrow\eta)(x\in P(c))}$$

Showing that η satisfies the universal property of $\operatorname{colim}(*\downarrow\eta)$. □

Remark 1.47. Theorem 1.44 and Corollary 1.46 assert the fact that objects and morphisms of $\mathbf{Set_C}$ are obtained as colimits of diagrams passing through $\mathbf{C}$. The bigger picture is that $\mathbf{Set_C}$ is the *free cocompletion* of $\mathbf{C}$. This will be proven in Theorem 1.55 of the next section (1.5).

Upon pondering Theorem 1.44 and the functor $(*{\downarrow}-)$, the author arrived at the following question[4]:

Question 1.48. For a presheaf $P : \mathbf{C} \to \mathbf{Set}$, do we have that $(*{\downarrow}P) = \operatorname{colim}_{x \in P(c)}(*{\downarrow}\mathbf{C}(-,c))$?

That is, do we get any category of elements the colimit of "representable" categories of elements? We now address this question, whose answer is given by Corollary 1.52.

Our first step will be to see that the category of elements of a representable presheaf $\mathbf{C}(-,c)$ is the over category $\mathbf{C}/c$.

Lemma 1.49. *Let $\boldsymbol{C}$ be a small category and $c \in \boldsymbol{C}_0$. Then $(*{\downarrow}\boldsymbol{C}(-,c)) \cong \boldsymbol{C}/c$.*

Proof. An object of $(*{\downarrow}\mathbf{C}(-,c))$ is a morphism $f \in \mathbf{C}(a,c)$ for some $a \in \mathbf{C}_0$, which is exactly an object $f : a \to c$ in $\mathbf{C}/c$. Likewise, a morphism $f = h^*(g)$ as on the left below, corresponds exactly to the triangle over c on the right:

$$(*{\downarrow}\mathbf{C}(-,c)) : \quad \begin{tikzcd} & \mathbf{C}(a,c) \\ * \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & \mathbf{C}(a',c) \arrow[uu, "h^*"'] \end{tikzcd} \qquad\qquad \begin{tikzcd} a \arrow[dr, "f"] \arrow[dd, "h"'] & \\ & c \\ a' \arrow[ur, "g"'] & \end{tikzcd} \quad : \mathbf{C}/c$$

$\square$

In view of the lemma (1.49), if $(*{\downarrow}-)$ preserves colimits, the category of elements of any presheaf would be obtained as a colimit of over categories by applying $(*{\downarrow}-)$ to the colimit 1.45. One way of proving cocontinuity is to construct a right adjoint. To do so, we will require the following reformulation:

Proposition 1.50. *Let $\boldsymbol{C}$ be a small category, and denote $d : \boldsymbol{Set} \to \boldsymbol{Cat}$ the functor sending a set D to its discrete category $\boldsymbol{D} = d(D)$. The category of elements of a presheaf $P : \boldsymbol{C}^{op} \to \boldsymbol{Set}$ may be computed as the coend $\int^{c \in \boldsymbol{C}_0} dP(c) \times \boldsymbol{C}/c$.*

The proof relies on the fact that $\mathbf{Cat}$ is *cartesian closed*, meaning that $\mathbf{C} \times - \dashv -^{\mathbf{C}}$ is an adjunction in $\mathbf{Cat}$. The counit is the evaluation $\mathrm{ev} : \mathbf{C} \times -^{\mathbf{C}} \Rightarrow \mathrm{id}_{\mathbf{Cat}}$ for all categories $\mathbf{C}$, which will be useful in the following.

Proof. The goal is to see $(*{\downarrow}P)$ as a universal cowedge on $dP \times \mathbf{C}/- : \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{Cat}$. The presheaf P defines a family of functors $\hat{P}_c : (\mathbf{C}/c)^{op} \to P(c)/P$ for any $c \in \mathbf{C}_0$ by applying P to any triangle over c (the directions flip because P is contravariant):

$$\mathbf{C}/c : \quad \begin{tikzcd} a \arrow[dr] \arrow[dd] & \\ & c \\ b \arrow[ur] & \end{tikzcd} \quad \overset{\hat{P}}{\longmapsto} \quad \begin{tikzcd} & P(a) \\ P(c) \arrow[ur] \arrow[dr] & \\ & P(b) \arrow[uu] \end{tikzcd} \quad : P(c)/P$$

[4]The answer is likely obvious to experts, though the author hasn't found it in the litterature.

Moreover, evaluation defines a family of functors $\hat{\mathrm{ev}}_c : d(P(c)) \times P(c)/P \to (*{\downarrow}P)$. The diagramattical description of this is simplest by writing an element $x \in P(c)$ as a function $x : * \to P(c)$. "Evaluation at x" is then restriction along x:

$$d(P(c))\times P(c)/P : \quad \left(x \in d(P(c)),\ \begin{tikzcd} & P(a) \\ P(c) \arrow[ur, "P(f)"] \arrow[dr, "P(g)"'] & \\ & P(b) \arrow[uu, "P(h)"] \end{tikzcd} \right) \overset{\hat{\mathrm{ev}}_c}{\longmapsto} \begin{tikzcd} & & P(a) \\ * \arrow[r, dotted, "x"] \arrow[urr, bend left, "P(f)(x)"] \arrow[drr, bend right, "P(g)(x)"'] & P(c) \arrow[ur, dotted, "P(f)"] \arrow[dr, dotted, "P(g)"'] & \\ & & P(b) \arrow[uu, "P(h)"'] \end{tikzcd} \quad : (*{\downarrow}P)$$

Claim: the family $\{\hat{ev}_c(id_{d(P(c))}\times\hat{P}_c)\}_{c\in \boldsymbol{C}_0}$ *defines a cowedge* $dP\times\boldsymbol{C}/- \to \Delta(*{\downarrow}P)$. We want the following square to commute for all $f : c \to c' \in \mathbf{C}_1$:

$$\begin{tikzcd} dP(c')\times\mathbf{C}/c \arrow[r, "dP(f)\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\mathbf{C}/f"] \arrow[dr, phantom, "?"] & dP(c)\times\mathbf{C}/c \arrow[d, "\hat{\mathrm{ev}}_c(\mathrm{id}\times\hat{P}_c)"] \\ dP(c')\times\mathbf{C}/c' \arrow[r, "\hat{\mathrm{ev}}_{c'}(\mathrm{id}\times\hat{P}_{c'})"] & (*{\downarrow}P) \end{tikzcd}$$

On objects, the equation underlying the diagram is $P(fg)(x) \overset{?}{=} P(g)(P(f)(x))$ for all $x \in dP(c')$ and $g \in (\mathbf{C}/c)_0$, which holds true by functoriality of P. Verifying that the square commutes for morphisms in $\mathbf{C}/c$ follows immediately from expanding the definitions and applying functoriality. We needn't consider non-identity morphisms in $dP(c')$ since it is a discrete category.

Claim: The cowedge $\epsilon := \hat{ev}_-(id\times\hat{P}_-) : dP\times\boldsymbol{C}/- \to \Delta(*{\downarrow}P)$ *is universal.* Consider another cowedge $\delta : dP\times\mathbf{C}/- \to \mathbf{D}$. We now define a functor $F : (*{\downarrow}P) \to \mathbf{D}$ such that $\Delta(F)\epsilon = \delta$, and argue that F is uniquely determined.

Let $f : c' \to c \in \mathbf{C}_1$, and consider a morphism $(x'{=}P(f)(x))$ in $(*{\downarrow}P)_1$. Define $F(x'{=}P(f)(x)) := \delta_c(x,f)$ on morphisms. This induces a definition on objects by $F(x{\in}P(c)) = \delta_c(x,\mathrm{id}_c)$. In order for this to give a well-defined functor, parallel morphisms must be sent to parallel morphisms. This is indeed the case, since if $(x'{=}P(f)(x))$ and $(x'{=}P(g)(x))$ are two parallel morphisms in $(*{\downarrow}P)$, then $\delta_c(x,f)$ and $\delta_c(x,g)$ are parallel in $\mathbf{D}$ since δ_c is a functor.

We will appeal to δ's dinaturality in order to show functoriality of F. Consider two composable morphisms $f : c' \to c$ and $f' : c'' \to c'$ in $\mathbf{C}$, giving rise to two composable morphisms $(x'{=}P(f)(x))$ and $(x''{=}P(f')(x'))$ in $(*{\downarrow}P)$, for any $x \in P(c)$. Then F is functorial if and only if the following holds:

$$F(x''{=}P(f')(x'))F(x'{=}P(f)(x)) \overset{\text{def.}}{=} \delta_{c'}(x',f')\delta_c(x,f) \overset{(?)}{=} \delta_c(x,ff') \overset{\text{def.}}{=} F(x''{=}P(f'f)(x))$$

The following diagram commutes by dinaturality of δ.

$$\begin{tikzcd} dP(c)\times\mathbf{C}/c' \arrow[r, "dP(f)\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\mathbf{C}/f"] & dP(c')\times\mathbf{C}/c' \arrow[d, "\delta_{c'}"] \\ dP(c)\times\mathbf{C}/c \arrow[r, "\delta_c"] & \mathbf{D} \end{tikzcd}$$

Choosing $(x,f') \in (dP(c)\times\mathbf{C}/c')_0$ shows that the equation (?) above holds.

It is clear that $\Delta(F)\epsilon = \delta$ from the definition. To see universality of F, suppose $G : (*\downarrow P) \to \mathbf{D}$ is another functor such that $\Delta(G)\epsilon = \delta$. Remark that a general morphism $(x' = P(f{:}c' \to c)(x))$ in $(*\downarrow P)_1$ is in the image of ϵ_c by the following:

$$
\begin{tikzcd}[row sep=small]
 & P(c') \\
* \arrow[ur, "x'"] \arrow[dr, "x"'] & \\
 & P(c) \arrow[uu, "P(f)"']
\end{tikzcd}
\xleftarrow{\ \mathrm{ev}_c\ }\!\!\shortmid
\left(
\begin{tikzcd}[row sep=small]
 & P(c') \\
x, P(c) \arrow[ur, "P(f)"] \arrow[dr, equal] & \\
 & P(c) \arrow[uu, "P(f)"']
\end{tikzcd}
\right)
\xleftarrow{\ \mathrm{id}\times\hat{P}_c\ }\!\!\shortmid
\left(
\begin{tikzcd}[row sep=small]
 & c \\
x, c \arrow[ur, equal] & \\
 & c' \arrow[uu, "f"'] \arrow[ul, "f"]
\end{tikzcd}
\right)
$$

Denote the preimage on the right by $(x, f{=}f{\circ}\mathrm{id}_c)$. We then have:

$$G(x'{=}P(f)(x)) = G(\epsilon_c(x, f{=}f{\circ}\mathrm{id}_c)) = \delta_c(x, f{\circ}\mathrm{id}_c) = F(x'{=}P(f)(x))$$

giving $G = F$ on morphisms, which implies equality on objects as well. $\square$

The hard work of identifying $(*\downarrow -)$ as a coend pays off immediately in our first proof by "abstract nonsense":

Proposition 1.51.[5] *Let $\boldsymbol{C}$ be a small category. There is an adjunction*

$$(*\downarrow -) : \boldsymbol{Set}_{\boldsymbol{C}} \leftrightarrows \boldsymbol{Cat} : \boldsymbol{Cat}(\boldsymbol{C}/{=}, -)_0$$

Before the proof, recall that the inclusion $d : \mathbf{Set} \to \mathbf{Cat}$ is left adjoint to the functor $-_0$ sending a category $\mathbf{C}$ to its set of objects $\mathbf{C}_0$.

Proof. For any presheaf $P : \mathbf{C} \to \mathbf{Set}$ and small category $\mathbf{X}$, (co)end calculus gives:

$$
\begin{aligned}
\mathbf{Cat}((*\downarrow P), \mathbf{X}) &\cong \mathbf{Cat}\Big(\int^{c\in\mathbf{C}_0} dP(c) \times \mathbf{C}/c, \mathbf{X}\Big) && \text{(Proposition 1.50)} \\
&\cong \int_{c\in\mathbf{C}_0} \mathbf{Cat}(dP(c) \times \mathbf{C}/c, \mathbf{X}) && \text{(Remark 1.30, } \mathbf{Cat}(-,\mathbf{X}) \text{ continuous)} \\
&\cong \int_{c\in\mathbf{C}_0} \mathbf{Cat}(dP(c), \mathbf{Cat}(\mathbf{C}/c, \mathbf{X})) && \text{(}\mathbf{Cat} \text{ is cartesian closed)} \\
&\cong \int_{c\in\mathbf{C}_0} \mathbf{Set}(P(c), \mathbf{Cat}(\mathbf{C}/c, \mathbf{X})_0) && (d \dashv -_0) \\
&\cong \mathbf{Set}_{\mathbf{C}}(P, \mathbf{Cat}(\mathbf{C}/-, \mathbf{X})_0) && \text{(Example 1.32)}\square
\end{aligned}
$$

Identifying $(*\downarrow -)$ as a left adjoint grants us the desired cocontinuity. We conclude:

Corollary 1.52. *Any category of elements is a canonical colimit of over categories.*

Proof. Let $P : \mathbf{C} \to \mathbf{Set}$ be a presheaf, written canonically as $P = \mathrm{colim}_{x\in P(c)}\mathbf{C}(-, c)$. Cocontinuity of $(*\downarrow -)$ gives $(*\downarrow P) = \mathrm{colim}_{x\in P(c)}(*\downarrow\mathbf{C}(-, c)) = \mathrm{colim}_{x\in P(c)}\mathbf{C}/c$. $\square$

This settles Question 1.48 in the positive.

It is not hard to see that any small category $\mathbf{C}$ arises as the category of elements of the terminal object $\Delta *$ in $\mathbf{Set}_{\mathbf{C}}$; the forgetful functor $U_{\Delta *} : (*\downarrow\Delta *) \to \mathbf{C}$ is fully faithful and bijective on objects. The following proposition is the analogous statement to Theorem 1.44 giving any category as a canonical colimit of its slices, a curiosity the author hasn't seen elsewhere.

[5] This is Thm. 3.2 of nLab's Category of elements, added in rev. 26 by an anonymous contributor.

Proposition 1.53. *Any small category* $\boldsymbol{C}$ *is the colimit of its slices* $\boldsymbol{C}/- : \boldsymbol{C} \to \boldsymbol{Cat}$.

In the proof a morphism in a slice $\mathbf{C}/c$, as on the left in diagram 1.54 below, will be denoted by $\phi = \phi' \circ \psi$ and referred to as a "triangle over c". Explicitely, ϕ is the domain and ϕ' the codomain of the morphism ψ in $\mathbf{C}/c$. The placement of the $\circ$ carries meaning, reflecting the equality of triangles (morphisms) over c. For example, $(gf = g \circ f) \neq (gf = \mathrm{id} \circ gf)$ in general—the actual sides of the triangle matter.

Proof. To see that $\mathbf{C}$ is a cone on $\mathbf{C}/-$, consider the projections $U_c : \mathbf{C}/c \to \mathbf{C}$.

$$\begin{tikzcd}
\mathbf{C}/c : & c' \arrow[d, "\psi"'] \arrow[dr, "\phi"] & & & c' \arrow[d, "\psi"] & \\
& {} & c & \stackrel{U_c}{\longmapsto} & {} & : \mathbf{C} \\
& c'' \arrow[ur, "\phi'"'] & & & c'' &
\end{tikzcd} \tag{1.54}$$

The collection $(U_c)_{c \in \mathbf{C}}$ defines a natural tranformation $\mathbf{C}/c \Rightarrow \Delta\mathbf{C}$: For any morphism $f : c \to c'$ of $\mathbf{C}$, the slice functor sends f to its post-composition $\mathbf{C}/f = f^* : \mathbf{C}/c \to \mathbf{C}/c'$. Post-composition doesn't change the domain of a morphism, so $U_c = U_{c'} f^*$ on objects. Now let $\phi = \phi' \circ \psi$ be a triangle over c as on the left in the diagram above. Then f^* sends this to the triangle $f\phi = f\phi' \circ \psi$ over c', which projects to $U_{c'}(f\phi = f\phi' \circ \psi) = \psi = U_c(\phi = \phi' \circ \psi)$. Consequently, for any morphism $f : c \to c'$ of $\mathbf{C}$ we have that $U_c = U_{c'} f^*$ as functors. In other words, $U_- : \mathbf{C}/- \Rightarrow \Delta\mathbf{C}$ is a cocone.

It remains to show universality of the cocone U_-. Let $\alpha : \mathbf{C}/- \Rightarrow \Delta\mathbf{D}$ be another cocone, for which we seek to define a factorization $\hat{\alpha} : \mathbf{C} \to \mathbf{D}$. A concise approach is to define $\hat{\alpha}$ on morphisms, and verify well-definedness on objects afterwards. To do this, the fact that id_c is terminal in $\mathbf{C}/c$ will be useful. That is, any morphism $\phi : b \to c$ of $\mathbf{C}$ defines an object of $\mathbf{C}/c$, and $\phi = \mathrm{id}_c \circ \phi$ is the (unique) terminal triangle to id_c. Posing $\hat{\alpha}(\phi : b \to c) := \alpha_c(\phi = \mathrm{id}_c \circ \phi)$ is therefore well-defined as a function (a map of sets); we need to verify that this is well-defined on objects and respects composition.

The definition of $\hat{\alpha}$ on morphisms induces one on objects by looking at the identities. Explicitely, $\hat{\alpha}(c)$ must be the (co)domain of $\hat{\alpha}(\mathrm{id}_c) = \alpha_c(\mathrm{id}_c = \mathrm{id}_c \circ \mathrm{id}_c)$, i.e. $\alpha_c(\mathrm{id}_c)$. In order for this to be well-defined as the object-function of a functor, we need that any morphism $\phi : b \to c$ in $\mathbf{C}$ is sent to $\hat{\alpha}(\phi) : \alpha_b(\mathrm{id}_b) \to \alpha_c(\mathrm{id}_c)$. The case of the codomain is immediate from the definition,

$$\hat{\alpha}(\phi) = \alpha_c(\phi = \mathrm{id}_c \circ \phi) : \alpha_c(\phi) \to \alpha_c(\mathrm{id}_c)$$

As for the domain, notice that $\phi = \phi^*(\mathrm{id}_b)$ in $\mathbf{C}/c$. Naturality of α then applies to give $\alpha_c(\phi) = \alpha_c(\phi^*(\mathrm{id}_b)) = \alpha_b(\mathrm{id}_b)$. We conclude that $\hat{\alpha}$ is well-defined on objects.

To prove functoriality, consider a composition $a \xrightarrow{f} b \xrightarrow{g} c$ in $\mathbf{C}$.

$$\begin{tikzcd}
\mathbf{C}/b : & a \arrow[r, "f"] \arrow[dr, "f"] & b \arrow[d, equal] & & a \arrow[r, "f"] \arrow[dr, "gf"] & b \arrow[r, "g"] \arrow[d, "g"] & c \arrow[dl, equal] & : \mathbf{C}/c \\
& & b & \stackrel{g^*}{\dashrightarrow} & & c & &
\end{tikzcd}$$

On the left above is the terminal triangle $f = \mathrm{id}_b \circ f$. On the right, the terminal triangle $gf = \mathrm{id} \circ gf$ is written as the composition of $g^*(f = \mathrm{id}_b \circ f) = (gf = g \circ f)$ and the terminal triangle $g = \mathrm{id}_c \circ g$. Naturality of α implies that the image of a triangle is invariant under post-composition. Therefore:

$$\begin{aligned}
\hat{\alpha}(gf) &= \alpha_c(gf = \mathrm{id}_c \circ gf) && \text{(definition of } \hat{\alpha}\text{)} \\
&= \alpha_c\left((g = \mathrm{id}_c \circ g)g^*(f = \mathrm{id}_b \circ f)\right) && \text{(composition in } \mathbf{C}/c\text{)} \\
&= \alpha_c(g = \mathrm{id}_c \circ g)\alpha_c\left(g^*(f = \mathrm{id}_b \circ f)\right) && \text{(functoriality of } \alpha_c\text{)} \\
&= \alpha_c(g = \mathrm{id}_c \circ g)\alpha_b(f = \mathrm{id}_b \circ f) && \text{(naturality of } \alpha\text{)} \\
&= \hat{\alpha}(g)\hat{\alpha}(f)
\end{aligned}$$

□

1.5. The Free Cocompletion

Being familiar now with the category of elements of a presheaf $P : \mathbf{C} \to \mathbf{Set}$, and how to recover P as a colimit indexed by $(*{\downarrow}P)$, it is perhaps intuitive that any functor $F : \mathbf{C} \to \mathbf{D}$ lets us "redirect" this colimit into $\mathbf{D}$ as $FU_P : (*{\downarrow}P)^{op} \to \mathbf{D}$, and that if $\mathbf{D}$ is cocomplete this is actually sufficient to induce a functor $R : \mathbf{Set}_{\mathbf{C}} \to \mathbf{D}$ satisfying:

$$\begin{tikzcd}
 & \mathbf{Set}_{\mathbf{C}} \arrow[d, dashed, "R"] \\
\mathbf{C} \arrow[ur, "r"] \arrow[r, "F"'] & \mathbf{D}
\end{tikzcd}$$

The next proposition confirms this intuition, and in fact gives even more: the induced functor is a left adjoint.

Theorem 1.55. *Let $\mathbf{C}$ and $\mathbf{D}$ be small categories with $\mathbf{D}$ cocomplete, and consider a functor $F : \mathbf{C} \to \mathbf{D}$. The Yoneda embedding $r{:}\mathbf{C} \to \mathbf{Set}_{\mathbf{C}}$ is the* free cocompletion:

1. *There exists a unique cocontinuous functor $R : \mathbf{Set}_{\mathbf{C}} \to \mathbf{D}$ such that $Rr = F$.*
2. *Let $N := \mathbf{D}(F =, -)$. Then $R \dashv N$.*

Proof. 1) Consider a functor $F : \mathbf{C} \to \mathbf{D}$ with $\mathbf{D}$ cocomplete. A presheaf $P \in \mathbf{Set}_{\mathbf{C}}$ is the colimit of the diagram $rU_P : (*{\downarrow}P)^{op} \to \mathbf{Set}_{\mathbf{C}}$, where U_P is the forgetful functor seen on the left below. On objects, define $R : \mathbf{Set}_{\mathbf{C}} \to \mathbf{D}$ by $P \mapsto \mathrm{colim}_{x \in P(c)} F(c)$.

$$\begin{tikzcd}
 & & \mathbf{Set}_{\mathbf{C}} \arrow[d, "R"] \\
(*{\downarrow}P)^{op} \arrow[r, "U_P"] & \mathbf{C} \arrow[ur, "r"] \arrow[dr, "F"'] & {} \\
 & & \mathbf{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
(*{\downarrow}P)^{op} \arrow[dd, "(*{\downarrow}\eta)^{op}"'] \arrow[dr, "U_P"] & & \\
 & \mathbf{C} \arrow[r, "F"] & \mathbf{D} \\
(*{\downarrow}P')^{op} \arrow[ur, "U_{P'}"'] & &
\end{tikzcd}$$

For a natural transformation $\eta : P \Rightarrow P'$, the functor $(*{\downarrow}\eta)^{op}$ is part of a triangle in the over category $\mathbf{Cat}/\mathbf{D}$, seen on the right. The dual of Proposition 1.36 gives a colimit functor which we may use to define $R(\eta) := \mathrm{colim}(*{\downarrow}\eta)$ on morphisms. This is clearly functorial, since both colim and $(*{\downarrow}-)$ are.

That R is cocontinuous will follow from being a left adjoint, which is proven in (2). Supposing cocontinuity of R for a moment, arguing unicity is easy: if $G : \mathbf{Set}_{\mathbf{C}} \to \mathbf{D}$ is another cocontinuous functor factoring F, then for any $\eta : P \Rightarrow P'$,

$$\begin{aligned}
G(\eta) &= G(\operatorname{colim}(*\downarrow\eta)) && \text{(Corollary 1.46)}\\
&= \operatorname{colim}\left[(\mathbf{Cat}/G)(*\downarrow\eta) : GrU_P \to GrU_{P'}\right] && \text{(Prop. 1.38, } G \text{ cocontinuous)}\\
&= \operatorname{colim}[(*\downarrow\eta) : FU_P \to FU_{P'}] && (Gr = F)\\
&= R(\eta)
\end{aligned}$$

Equality of R and G on morphisms implies equality on objects.

2) We verify that $N := \mathbf{D}(F{=}, -) : \mathbf{D} \to \mathbf{Set}_{\mathbf{C}}$ is indeed right adjoint to R. Let $P \in \mathbf{Set}_{\mathbf{C}}$ and $d \in \mathbf{D}$. We then have:

$$\begin{aligned}
\mathbf{D}(R(P), d) &= \mathbf{D}(\operatorname*{colim}_{x \in P(c)}\left[RrU_P(x)\right], d) && \text{(def. of } R(P))\\
&\cong \lim_{x \in P(c)} \mathbf{D}(Rr(c), d) && \text{(continuity of } \mathbf{D}(F-, d)\text{, def. of } U_P)\\
&\cong \lim_{x \in P(c)} \mathbf{D}(F(c), d) && (Rr = F)\\
&\cong \lim_{x \in P(c)} \mathrm{N}(d)(c) && \text{(def. of N)}\\
&\cong \lim_{x \in P(c)} \mathbf{Set}_{\mathbf{C}}(r(c), \mathrm{N}(d)) && \text{(Yoneda lemma)}\\
&\cong \mathbf{Set}_{\mathbf{C}}(P, \mathrm{N}(d)) && \text{(continuity of } \mathbf{Set}_{\mathbf{C}}(-, \mathrm{N}(d)))
\end{aligned}$$

Where we have also used that $P = \operatorname{colim}_{x \in P(c)} r(c)$ on the last line. □

The functors R and N of the theorem (1.55) are called *realization* and *nerve* functors; together they make up a "realization-nerve"-adjunction. The following examples motivate the choice of terminology.

Example 1.56. The category **Top** is cocomplete (Construction 1.24). Consider the functor $\mathbf{n} \mapsto \blacktriangle^n : \Delta \to \mathbf{Top}$ mapping the poset $\mathbf{n}$ to the geometric n-simplex $\blacktriangle^n := \{ p \in \mathbb{R}^n \mid p_i \geqslant 0, \sum_{i=1}^n p_i \leqslant 0 \}$. The induced adjunction consists of the classical *geometric realization* $|-|$: $\mathbf{Set}_\Delta \to \mathbf{Top}$ of a simplicial set. The right adjoint is therefore $X \mapsto \mathbf{Top}(\blacktriangle^-, X)$, which is the classical *singular* functor.

Example 1.57. The category **Cat** is cocomplete (Construction 1.24), and the inclusion $\Delta \subseteq \mathbf{Cat}$ induces an adjunction $h \dashv \mathrm{N}$, where the left adjoint $h : \mathbf{Set}_\Delta \to \mathbf{Cat}$ is the functor associating a simplicial set to its *homotopy category*. For a *Kan* simplicial set S, this category can be explicitely described as:

$$\begin{aligned}
&\text{(objects)} && h(S)_0 := S_0\\
&\text{(morphisms)} && h(S)(x, y) := \{ f \in S_1 \mid d_1(f) = x \text{ and } d_0(f) = y \}/\simeq\\
& && \qquad \text{where } d_1(s) \sim d_0(s) d_2(s) \quad \forall s \in S_2\\
& && \qquad \text{and } \simeq \text{ is the equivalence relation spanned by } \sim.
\end{aligned}$$

However, for a general simplicial set the relations are difficult to write down explicitely. A simplicial set $S = N(\mathbf{C})$ arising as the nerve of some category $\mathbf{C}$ is in particular Kan. By definition $S_2 = \mathbf{Cat}(\Delta^2, \mathbf{C})$ is the set of commuting triangles in $\mathbf{C}$.

$$\begin{tikzcd} & s(1) \arrow[dr, "d_1(s)"] & \\ s(0) \arrow[ur, "d_2(f)"] \arrow[rr, "d_1(f)"'] & & s(2) \end{tikzcd}$$

We then have that the 2-simplex $s : \Delta^2 \to \mathbf{C}$ pictured above exists in S if and only if $d_1(s) = d_0(s)d_2(s)$ as morphisms in $\mathbf{C}_1$. This means that $hN(\mathbf{C}) \cong \mathbf{C}$, i.e. the counit of the adjunction $h \dashv N$ is an isomorphism.

Remark 1.58. In the literature, the functor h of Example 1.57 is also written τ_1 (or t_1) and called the *fundamental functor* or the 1-*truncation*. Our terminology here mirrors that of [Rie14a] and [Lur09]. In the latter, Lurie defines a *localization functor* as a left adjoint to a fully faithful functor. Since the nerve $N : \mathbf{Cat} \to \mathbf{Set}_\Delta$ is fully faithful, h gives an example of such.

Identifying $(*\downarrow -)$ as a left adjoint is much easier given Theorem 1.55:

Example 1.59. Let $\mathbf{C}$ be a category. The slice functor $\mathbf{C}/- : \mathbf{C} \to \mathbf{Cat}$ sending an object $c \in \mathbf{C}_0$ to its over category $\mathbf{C}/c$ must factor through $\mathbf{Set}_\mathbf{C}$ since $\mathbf{Cat}$ is cocomplete. In Lemma 1.49, the category of elements of a representable presheaf was identified as the category over the representing object, giving that $\mathbf{C}/- \cong (*\downarrow -)r$. Since $(*\downarrow -)$ is cocontinuous, this implies that the induced adjunction is $(*\downarrow -) \vdash \mathbf{Cat}(\mathbf{C}/=, -)_0$ of Proposition 1.51.

Example 1.60 (The Dold-Kan correspondance)**.** Consider the category $\mathbf{Ch}_\mathbb{N}(\mathbf{Ab})$ of bounded chain complexes of abelian groups, denote $\mathbb{Z} : \mathbf{Set} \leftrightarrows \mathbf{Ab} : U$ the free-forgetful adjunction. The functor $N_\Delta \mathbb{Z}r : \Delta \to \mathbf{Ch}_\mathbb{N}(\mathbf{Ab})$ sends an object $\mathbf{n}$ to the *Moore complex* of Δ^n, constructed from the linearization

$$C\mathbb{Z}_*(\Delta^n) := (\cdots \to \mathbb{Z}^{\oplus \Delta^n_k} \xrightarrow{\sum d^k_i} \mathbb{Z}^{\oplus \Delta^n_{k-1}} \to \cdots \to \mathbb{Z}^{\oplus \Delta^n_0})$$

by "normalizing": $N_\Delta(C\mathbb{Z}_*(\Delta^n))_k := \bigcap_{i=0}^{k-1} \ker d^k_i$ for $0 \leqslant k \leqslant n$ and differential (the restriction of) d^k_k. Now, $\mathbf{Ch}_\mathbb{N}(\mathbf{Ab})$ has coproducts computed pointwise, and the coequalizer of $\alpha, \beta : C_* \rightrightarrows C'_*$ is just the cokernel of $\alpha - \beta$. This means $\mathbf{Ch}_\mathbb{N}(\mathbf{Ab})$ is cocomplete (Construction 1.24), and we obtain an adjunction $L : \mathbf{Set}_\Delta \leftrightarrows \mathbf{Ch}_\mathbb{N}(\mathbf{Ab}) : \Gamma$.

Remark 1.61. The category of presheaves being the free cocompletion generalizes to *enriched* categories, where $\mathbf{Set}$ is replaced by a suitably nice[6] *symmetric monoidal* category, e.g. $\mathbf{Ab}$ and $\mathbf{Cat}$.

In view of the previous remark, Example 1.59 is begging to be enriched:

Example 1.62. The fact that $\mathbf{Cat}$ is a 2-category is the reason for the postfix "$_0$" in the right adjoint $\mathbf{Cat}(\mathbf{C}/=, -)_0$ of Example 1.59. By considering the whole functor categories (not just their objects) we obtain a a functor $\mathbf{Cat}(\mathbf{C}/=, -) : \mathbf{Cat} \to \mathbf{Cat}_\Delta$ into the category of simplicial objects in $\mathbf{Cat}$. The left adjoint is the enriched category of elements, called the *Grothendieck construction* $\int^\Delta : \mathbf{Cat}_\Delta \to \mathbf{Cat}$ sends a $\mathbf{Cat}$-valued presheaf to weak version of the under category $(*\downarrow P)$ (see Remark 1.37).

[6]Mimicking the properties of **Set**, specifically we need a complete, cocomplete, symmetric monoidal closed category.

2. Model Categories

This chapter aims to motivate and introduce canonical model categories, specifically the most "categorical" model structure on **Cat**. We will start by defining a *cover of groupoids* by translating the notion of a topological covering space to the fundamental groupoids. The immediate generalization of this gives *discrete fibrations of categories* $\boldsymbol{E} \twoheadrightarrow \boldsymbol{C}$, which we will see are intimately related to functors $\mathbf{C} \to \mathbf{Set}$. This observatoin will shine a new light on the contents of the previous chapter.

2.0.1. Notation

Let $\mathbf{B}$ be a groupoid. For an object $b \in \mathbf{B}_0$, we will denote $\mathbf{B}(b)$ the automorphism group $\mathbf{B}(b, b)$. Other authors use e.g. $\mathrm{Aut}_{\mathbf{B}}(b)$ or $\pi_1(\mathbf{B}, b)$. The full subgroupoid spanned by an object b will be written $\mathbf{b}$, it can also be thought of as the one-object groupoid given by the group $\mathbf{C}(c)$. More generally, if G is a group we will write $\mathbf{G}$ when we think of G as a one-object groupoid.

2.1. Covers of Groupoids

We start by recalling the definiton of a cover of topological spaces:

Definition 2.1. A continuous surjection $p : X \to Y$ of topological spaces is a *covering map* if for every $y \in Y$ there exists an open neighbourhood V of y along with a homeomorphism $\Phi : p^{-1}(V) \leftrightarrow \coprod_{x \in p^{-1}(y)} V$ making the following commute:

$$\begin{tikzcd} p^{-1}(V) \arrow[rr, leftrightarrow, "\Phi"] \arrow[dr, "p"'] & & \coprod_{x \in p^{-1}(y)} V \arrow[dl, "\coprod \mathrm{id}_V"] \\ & V & \end{tikzcd}$$

When $p : X \to Y$ is a covering map, X is a *covering space of* Y.

When studying covering spaces most authors (e.g. [May99][Ch.3]) prefer to work in a convenient subcategory of **Top**, usually comprised of connected and locally path-connected spaces. In this case we have the following fundamental lifting property:

Proposition 2.2. *Let $p : X \to Y$ be a cover of connected, locally path-connected topological spaces. Any path $f : I \to Y$ has a unique lift $f_x : I \to X$ to any point $x \in p^{-1}(f(0))$ such that $pf_x = f$. Moreover, the lifts respect homotopy classes of paths.*

Proof. See e.g. [May99][Ch.3.2]. □

Recall that the *fundamental groupoid functor* $\pi_1 : \boldsymbol{Top} \to \boldsymbol{Cat}$ carries a (connected, locally path-connected) topological space X to its *fundamental groupoid* $\pi_1 X$ whose objects are the points of X and a morphism between two points $x_0, x_1 \in X$ is a homotopy class of paths $f : I \to X$ starting at x_0 and ending at x_1.

Proposition 2.2 translates well to the fundamental groupoids: The cover $p : X \to Y$ gives a functor $\pi_1 p : \pi_1 X \to \pi_1 Y$ that is surjective on objects, and any morphism $f : y_0 \to y_1$ in $\pi_1 Y$ induces a unique lift $f_{x_0} : x_0 \to x_1$ for any $x_0 \in \pi_1 X$ such that $\pi_1 p(x_0) = y_0$. This could be differently stated as $\pi_1 p$ inducing a bijection

$$(x_0/\pi_1 p)_0 : (x_0/\pi_1 X)_0 \leftrightarrow (p(x_0)/\pi_1 Y)_0$$

for any $x_0 \in X$. This leads us to the following:

Definition 2.3. Let $P : \mathbf{E} \to \mathbf{B}$ be a functor between two groupoids.

1. The groupoid $\mathbf{B}$ (or $\mathbf{E}$) is *connected* if none of its hom-sets are empty.
2. If $\mathbf{B}$ and $\mathbf{E}$ are connected, then P is a *cover of groupoids* if there is a bijection

$$e/P : (e/\mathbf{E})_0 \leftrightarrow (P(e)/\mathbf{B})_0$$

   for all $e \in \mathbf{E}_0$, referred to as the *slice-map induced by P at e*. In this case we denote $P : \mathbf{E} \twoheadrightarrow \mathbf{B}$ using the arrow "$\twoheadrightarrow$".

3. When P is a cover, *the fiber over $b \in \boldsymbol{B}_0$* is the set $\mathbf{E}_b := \{e \in \mathbf{E}_0 \mid P(e) = b\}$.
4. The *category of covers over $\boldsymbol{B}$* is the full subcategory $\mathbf{Cov_B} \subseteq (\mathbf{Gpd}{\downarrow}\mathbf{B})$ spanned by covers of groupoids.

Note that when $P : \mathbf{E} \twoheadrightarrow \mathbf{B}$ is a cover, the cardinality of the fibers $\mathbf{E}_b$ is invariant of $b \in \mathbf{B}_0$. We call a cover *finite* if all it's fibers $\mathbf{E}_b$ are finite for $b \in \mathbf{B}_0$.

For groupoids, the "pointwise" bijection induced by covering maps is actually an isomorphism of categories:

Proposition 2.4. *For any groupoid cover $P : \boldsymbol{E} \twoheadrightarrow \boldsymbol{B}$, the slice-maps are isomorphisms:*

$$e/P : e/\boldsymbol{E} \leftrightarrow P(e)/\boldsymbol{B} \quad \forall e \in \boldsymbol{E}_0$$

Proof. Let $e \in \mathbf{E}_0$, then any morphism on the right has a unique preimage on the left:

$$\begin{tikzcd} & e' \arrow[dd, dashed, "\hat{g}\hat{f}^{-1}"] & & & b' \arrow[dd] \\ e \arrow[ur, "\hat{f}"] \arrow[dr, "\hat{g}"'] & & \arrow[r, mapsto, "e/P"] & P(e) \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & e'' & & & b'' \end{tikzcd}$$

Functoriality follows immediately from the unicity of the above construction. $\square$

Definition 2.5. Let $\mathbf{B}$ be a groupoid. A *$\boldsymbol{B}$-action* is a functor $\mathbf{B} \to \mathbf{Set}$, and $\mathbf{Set}^{\mathbf{B}}$ is the category of $\mathbf{B}$-actions (also called $\mathbf{B}$-sets). The action of $\mathbf{B}$ is transitive (resp. finite) if $\mathbf{B}(b)$ is transitive (resp. finite) as a group action for any $b \in \mathbf{B}_0$.

By considering a group G as a single-object groupoid, one recovers the notion of G-set as a functor $\mathbf{G} \to \mathbf{Set}$. The previous definition is the generalization to groupoids.

Remark 2.6. Given a $\mathbf{B}$-action $A : \mathbf{B} \to \mathbf{Set}$, the restriction of A to any object $b \in \mathbf{B}_0$ gives a $\mathbf{B}(b)$-set $A(b)$. The terminology "$\mathbf{B}$-set", while useful, is therefore a bit misleading; the action of a proper groupoid $\mathbf{B}$ involves many sets, not a single one as in the classical case. Indeed, a $\mathbf{B}$-set is a collection of $\mathbf{B}(b)$-sets for $b \in \mathbf{B}_0$, along with morphisms between these.

Covers of groupoids support the following fundamental lifting property.

Theorem 2.7. *Let $\boldsymbol{X}$ be a connected groupoid, and $P : \boldsymbol{E} \twoheadrightarrow \boldsymbol{B}$ a cover. In the diagram below, a lift $\hat{Q}$ exists if and only if there are $b \in \boldsymbol{B}$ and $x \in \boldsymbol{X}_b$, $e \in \boldsymbol{E}_b$ satisfying $Q(\boldsymbol{X}(x)) \subseteq P(\boldsymbol{E}(e)) \subseteq \boldsymbol{B}(b)$. In this case, $\hat{Q}$ is uniquely determined by the choice of image of x in $\boldsymbol{E}_b$.*

$$\begin{tikzcd} & \boldsymbol{E} \arrow[d, two heads, "P"] \\ \boldsymbol{X} \arrow[r, "Q"'] \arrow[ur, dashed, "\hat{Q}"] & \boldsymbol{B} \end{tikzcd}$$

Proof. $\Rightarrow$) Any lift $\hat{Q}$ and $x \in \mathbf{X}$ give $\hat{Q}(\mathbf{X}(x)) \subseteq \mathbf{E}(\hat{Q}(x))$, and therefore $Q(\mathbf{X}(x)) = P\hat{Q}(\mathbf{X}(x)) \subset P(\mathbf{E}(e))$, with $e = \hat{Q}(x)$.

$\Leftarrow$) Pick a basepoint $x \in \mathbf{X}$ and let $e \in \mathbf{E}_{Q(x)}$ be such that $Q(\mathbf{X}(x)) \subseteq P(\mathbf{E}(e))$. Consider the slice $x/\mathbf{X}$. Connectivity of $\mathbf{X}$ means that any point x' of $\mathbf{X}$ is obtained as the codomain of a path $X_{x'} : x \to x' \in x/\mathbf{X}$. First we define $\hat{Q}(x') := \operatorname{cod} \widehat{Q(X_{x'})}$ where $\widehat{Q(X_{x'})}$ is the lift into $e/\mathbf{E}$. This independent of choice of path $X_{x'}$ since if $X'_{x'} : x \to x'$ is another path, then $X_{x'}^{-1}X'_{x'} : x \to x$ is a loop at x and:

$$\begin{tikzcd}[column sep=small] & & x' \arrow[dd, "X_{x'}^{-1}"] & & & & Q(x') \arrow[dd, "Q(X_{x'})^{-1}"] & & & & \hat{Q}(x') \arrow[dd, "\widehat{Q(X_{x'})}^{-1}"] \\ x \arrow[urr, "X'_{x'}"] \arrow[drr, "X_{x'}^{-1}X'_{x'}"'] & & & \arrow[r, mapsto, dashed, "Q"] & Q(x) \arrow[urr, "Q(X'_{x'})"] \arrow[drr, "Q(X_{x'})^{-1}Q(X'_{x'})"'] & & & \arrow[r, mapsto, dashed, "(e/P)^{-1}"] & e \arrow[urr, "\widehat{Q(X'_{x'})}"] \arrow[drr, "\widehat{Q(X_{x'})}^{-1}\widehat{Q(X'_{x'})}"'] & & \\ & & x & & & & Q(x) & & & & e \end{tikzcd}$$

In general, the rightmost triangle needn't have the bottom-right vertex as e, but this is what $Q(\mathbf{X}(x)) \subseteq P(\mathbf{E}(e))$ buys us. As such, $\hat{Q}$ becomes well-defined on objects. Now, for any $f : x \to x' \in x/\mathbf{X}$ let $\hat{Q}(f) := \widehat{Q(f)}$ which is clearly compatible with $\hat{Q}$ on objects. This induces a total defintion of $\hat{Q}$, since for any morphism $g : x' \to x''$ in $\mathbf{X}$ there is a unique lift of $Q(g)$ into $\hat{Q}(x')/\mathbf{E}$. Functoriality is immediately seen through the isomorphism $Q(x')/\mathbf{B} \cong \hat{Q}(x')/\mathbf{E}$. □

Definition 2.8. A cover of groupoids $Q : \mathbf{X} \twoheadrightarrow \mathbf{B}$ is *universal* if $Q(\mathbf{X}(x))$ is the trivial subgroup $\{\mathrm{id}_{Q(x)}\} \subseteq \mathbf{B}(Q(x))$ for all $x \in \mathbf{X}_0$.

Corollary 2.9. *Let $Q : \boldsymbol{X} \twoheadrightarrow \boldsymbol{B}$ be a universal cover. For any other cover $P : \boldsymbol{E} \twoheadrightarrow \boldsymbol{B}$,*

$$\boldsymbol{Cov_B}(Q, P) = \boldsymbol{E}_b \quad \forall b \in \boldsymbol{B} \tag{2.10}$$

Proof. Let $b \in \mathbf{B}_0$, and $x \in \mathbf{X}_b$. By definition of being a universal cover $Q(\mathbf{X}(x))$ is trivial for all $x \in \mathbf{X}_0$. This means that for any $e \in \mathbf{E}_b$, we automatically have $Q(\mathbf{X}(x)) \subseteq P(\mathbf{E}(e))$, satisfying the hypotheses of Theorem 2.7. Therefore any map in $\mathbf{Cov_B}(Q, P)$ corresponds uniquely to a choice of basepoint $e \in \mathbf{E}_b$. □

Corollary 2.9 is actually a shadow of the Yoneda lemma for groupoids, something the next theorem (2.18) will render precise. It will be most interesting for us to state and prove Theorem 2.18 for categories; the groupoid case then follows immediately. In order to do this, we require a notion of a "cover of categories":

Definition 2.11. Let $\mathbf{C}, \mathbf{C}'$ be categories. A functor $F : \mathbf{C} \to \mathbf{C}'$ is a *discrete fibration* if the *slice-maps* $c/F : (c/\mathbf{C})_0 \to (F(c)/\mathbf{C}')_0$ are bijections for all $c \in \mathbf{C}_0$. The full subcategory of $\mathbf{Cat}/\mathbf{C}'$ spanned by discrete fibrations is denoted $\mathbf{Fib_{C'}}$, and we denote the objects using the arrow "$\twoheadrightarrow$".

Remark 2.12. Covers of groupoids are discrete fibrations of categories. The latter is a significantly weaker notion since there is no appeal to "connectedness" of the domain and codomain categories.

Example 2.13. Let $\mathbf{C}$ be a category. Two familiar constructions give discrete fibrations:

1. Let $c \in \mathbf{C}_0$. There is a forgetful functor $P_c : c/\mathbf{C} \to \mathbf{C}$ defined by:

$$c/\mathbf{C} : \quad \begin{tikzcd} & c' \arrow[d, "f"] \\ c \arrow[ur] \arrow[dr] & c'' \end{tikzcd} \quad \xmapsto{P_c} \quad \begin{tikzcd} c' \arrow[d, "f"] \\ c'' \end{tikzcd} \quad : \mathbf{C}$$

   To see that P_c is a discrete fibration, consider an object $g : c \to c' \in (c/\mathbf{C})_0$. Then $P_c(g) = c'$, and we need to find a unique lift of any $h : c' \to c'' \in (c'/\mathbf{C})_0$ into $(g/(c/\mathbf{C}))_0$. An object of the latter is just a morphism leaving g in $c/\mathbf{C}$. But h is exactly that, and is the only such satisfying:

$$(g/(c/\mathbf{C}))_0 : \quad \begin{tikzcd} & c' \arrow[d, "h"] \\ c \arrow[ur, "g"] \arrow[dr, "hg"'] & c'' \end{tikzcd} \quad \xmapsto{g/P_c} \quad \begin{tikzcd} c' \arrow[d, "h"] \\ c'' \end{tikzcd} \quad : (c'/\mathbf{C})_0$$

   Consequently, g/P_c is a bijection, as desired.

2. Consider a functor $P : \mathbf{C} \to \mathbf{Set}$. There is a forgetful functor $U_P : (*{\downarrow}P) \to \mathbf{C}$ sending a morphism $(x' = P(f)(x)) \in (*{\downarrow}P)_1$ to $f \in \mathbf{C}_1$. This is a discrete fibration: for any object $y \in P(c)$ of $(*{\downarrow}P)$, any object $g \in (c/\mathbf{C})_0$ has a unique preimage in $(y/(*{\downarrow}P))_0$ given by:

$$(y/(*{\downarrow}P))_0 : \quad \begin{tikzcd} y \in P(c) \arrow[d, "P(g)"] \\ P(f)(y) \in P(c') \end{tikzcd} \quad \xmapsto{y/U_P} \quad \begin{tikzcd} c \arrow[d, "g"] \\ c' \end{tikzcd} \quad : (c/\mathbf{C})_0$$

   This is just expressing the fact that any morphism $g : c \to c'$ gives a function $P(g)$ and an element $y \in P(c)$ has a unique image $P(g)(y)$ through $P(g)$. The forgetful functor $U_P : (*{\downarrow}P) \twoheadrightarrow \mathbf{C}$ is called a *Grothendieck fibration*.

Remark 2.14. The attentive reader may have noticed that in the previous chapter we mostly encountered over categories, whereas Definition 2.11 and Example 2.13 employ under categories. For groupoids the choice is irrelevant since $b/\mathbf{B} \cong \mathbf{B}/b$ for any groupoid $\mathbf{B}$ and $b \in \mathbf{B}_0$. However for categories the choice matters, and the under categories were chosen because they are the categorical model of paths starting at a point x_0 in a topological space X, which is where we began this chapter.

Remark 2.15. In Example 2.13[2] we may instead wish to consider a presheaf $P \in \mathbf{Set}_{\mathbf{C}}$. In this case we get a contravariant forgetful functor $U_P : (*{\downarrow}P)^{op} \to \mathbf{C}$, and we see that for an object $x \in P(c)$ the slice-map x/U_P must map into the (objects of the) over category $\mathbf{C}/c$ since P flips the direction of arrows. This extra complication is another reason we prefer contravariant functors when developing this section.

Example 2.13 implicitely defines two functors $P_- : \mathbf{C} \to \mathbf{Fib_C}$ and $U_- : \mathbf{Set^C} \to \mathbf{Fib_C}$ which will be needed. We only verify that U_- is well-defined since both are very similiar.

Lemma 2.16. *Let $\boldsymbol{C}$ be a small category. The association $P \mapsto U_P : \boldsymbol{Set^B} \to \boldsymbol{Fib_C}$ of a functor to its Grothendieck fibration defines a functor denoted U_-.*

Proof. In Example 2.13 we argued that U_P is indeed a discrete fibration for $P \in \mathbf{Set^C}$, so U_P is well-defined on objects. We need to define U_- on morphisms:

Let $P, P' : \mathbf{C} \to \mathbf{Set}$ be two functors and let $\eta : P \Rightarrow P' \in \mathbf{Set^C}$. We define $U_\eta := (*{\downarrow}\eta)$. Functoriality is then given by functoriality of $(*{\downarrow}-)$; we need only verify that this defines a morphism in $\mathbf{Fib_C}$, i.e. that $U_\eta = (*{\downarrow}\eta)$ makes the following commute:

$$\begin{tikzcd}
(*{\downarrow}P) \arrow[d, "(*{\downarrow}\eta)"'] \arrow[dr, "U_P"] & \\
(*{\downarrow}P') \arrow[r, "U_{P'}"'] & \mathbf{C}
\end{tikzcd}$$

On objects it is clear that $U_P = (*{\downarrow}\eta)U_{P'}$ since $U_P(x \in P(c)) = U_{P'}(\eta_c(x) \in P'(c))$. Now consider $f : c \to c' \in \mathbf{C}_1$ giving rise to a morphism $(x' = P(f)(x)) \in (*{\downarrow}P)_1$.

$$\begin{tikzcd}
c \arrow[d, "f"] & x \in P(c) \arrow[d, "P(f)"] \arrow[l, mapsto, "U_P"'] \arrow[r, mapsto, "(*{\downarrow}\eta)"] & \eta_c(x) \in P'(c) \arrow[d, "P'(f)"] \arrow[r, mapsto, "U_{P'}"] & c \arrow[d, "f"] \\
c' & x' \in P(c') & \eta_{c'}(x') \in P'(c') & c'
\end{tikzcd}$$

Above we see that $P(f)$ as a morphism in $(*{\downarrow}P)$ maps to $f \in \mathbf{C}_1$ under both U_P and $U_{P'}(*{\downarrow}P)$. In conclusion, $(*{\downarrow}\eta) : U_P \to U_{P'}$ is a morphism in $\mathbf{Fib_C}$. $\square$

In the previous chapter, we determined a right adjoint to $(*{\downarrow}-) : \mathbf{Set_C} \to \mathbf{Cat}$ in Proposition 1.51. Dualizing gives the adjunction $(*{\downarrow}-) \vdash \mathbf{Cat}(=/\mathbf{C}, -)_0$ for $(*{\downarrow}-) : \mathbf{Set^C} \to \mathbf{C}$, i.e. the covariant case. The previous lemma refines our understanding of the category of elements: It is a discrete fibration $U_P : (*{\downarrow}P) \twoheadrightarrow \mathbf{C}$. There is an obvious forgetful functor $\mathbf{Fib_C} \to \mathbf{C}$ and the current picture is then as follows:

$$\begin{tikzcd}
\mathbf{Set^C} \arrow[rr, shift left, "(*{\downarrow}-)"] \arrow[dr, "U_-"'] & & \mathbf{Cat} \arrow[ll, shift left, "\mathbf{Cat}(=/\mathbf{C},-)_0"] \\
& \mathbf{Fib_C} \arrow[ur] &
\end{tikzcd}$$

The composition $\mathbf{Cat}(=/\mathbf{C}, -)_0F : \mathbf{Fib_C} \to \mathbf{Set^C}$ features in the upcoming Theorem 2.18 and has a nice description:

Lemma 2.17. *Let $\boldsymbol{C}$ be a small category. There is a* fiber functor $(-)_= : \boldsymbol{Fib_C} \to \boldsymbol{Set^C}$.

Proof. Let $F : \mathbf{E} \twoheadrightarrow \mathbf{C}$ be a discrete fibration. The functor $F_- : \mathbf{C} \to \mathbf{Set}$ is defined as sending an object $c \in \mathbf{C}_0$ to the fiber $F_c = \mathbf{E}_c \subseteq \mathbf{C}_0$. Any object $f : c \to c'$ in $(c/\mathbf{C})_0$ has a unique lift f_e into $(e/\mathbf{E})_0$ for any $e \in F_b$. Define $F_f(e)$ to be the codomain of f_e. Using unicity of lifts, it is not hard to see that this is functorial, and therefore that F_- is a functor.

Now let $G : \mathbf{D} \twoheadrightarrow \mathbf{C}$ be another discrete fibration, and $H : \mathbf{E} \to \mathbf{D}$ a morphism of the fibrations F, G drawn on the left. Let $f : b \to b' \in \mathbf{C}_1$.

$$\begin{tikzcd} \mathbf{E} \arrow[rr, "H"] \arrow[dr, "F"'] & & \mathbf{D} \arrow[dl, "G"] \\ & \mathbf{C} & \end{tikzcd} \qquad \begin{tikzcd} F_b \arrow[r, "H_b"] \arrow[d, "F_f"] & G_b \arrow[d, "G_f"] \\ F_{b'} \arrow[r, "H_{b'}"] & G_{b'} \end{tikzcd}$$

We need to define a natural transformation $H_- : F_- \Rightarrow G_-$. Luckily $H_b(c) := H(c) : F_b \to G_b$ is well-defined for all $b \in \mathbf{C}_0$, since if $F(c) = b$ then $G(H(c)) = b$, so $H(c) \in G_b$.

Naturality amounts to ensuring that the codomain of the lift $f_{H_b(c)}$ of f into $H_b(c)/\mathbf{D}$ is equal to $H_{b'}(c')$, where $f_c : c \to c'$ is the lift into $c/\mathbf{C}$. But this must be the case, since $f = F(f_c) = GH(f_c)$, so $f_{H_b(c)} = H(f_c) : H_b(c) \to H_b(c')$ by unicity of lifts. □

When $\mathbf{B}$ is a groupoid, this means that any cover $P : \mathbf{E} \twoheadrightarrow \mathbf{B}$ defines an action $P_- : \mathbf{B} \to \mathbf{Set}$ of $\mathbf{B}$ on the fibers.

Theorem 2.18. *Let $\boldsymbol{C}$ be a small category. We have an equivalence of categories:*

$$U_- : \boldsymbol{Set}^{\boldsymbol{C}} \simeq \boldsymbol{Fib}_{\boldsymbol{C}} : (-)_=$$

A short proof of the corresponding theorem for presheaves is found in [LR19][2.1.2].

Proof. We start by constructing $\epsilon : \mathrm{id}_{\mathbf{Set}^{\mathbf{C}}} \cong (U_-)_=$. Let $F : \mathbf{C} \to \mathbf{Set}$. By definition, $(U_F)_c = U_F^{-1}(c) = F(c)$ for all $c \in \mathbf{C}_0$. As such, we define $\epsilon_c = \mathrm{id}_{F(c)}$ and check naturality: Let $f : c \to c' \in \mathbf{C}_1$ and $x \in F(c)$. The definition of $(U_F)_f$ says that $(U_F)_f(x)$ is the codomain of the unique lift of $f \in \mathbf{C}_1$ to the fiber $(x \in F(c)) \in (*\downarrow F)_0$. But the unique lift is the morphism $F(f) \in (*\downarrow F)_1$ on the left below, whose codomain is $F(f)(x)$. In other words, $(F_F)_f(x) = F(f)(x)$ and the diagram on the right commutes:

$$\begin{tikzcd} x \in F(c) \arrow[d, "F(f)"] \\ F(f)(x) \in F(c') \end{tikzcd} \qquad \begin{tikzcd} (U_F)_c \arrow[r, equal, "\epsilon_c"] \arrow[d, "U_f"] & F(c) \arrow[d, "F(f)"] \\ (U_F)_{c'} \arrow[r, equal, "\epsilon_{c'}"] & F(c') \end{tikzcd}$$

In other words, $\epsilon : \mathrm{id}_{\mathbf{Set}^{\mathbf{C}}} \cong (U_-)_=$ is a natural isomorphism.

Now we construct $\eta : \mathrm{id}_{\mathbf{Fib}_{\mathbf{C}}} \cong U_{(-)_=}$. For $F : \mathbf{E} \twoheadrightarrow \mathbf{C}$, we need the following to commute:

$$\begin{tikzcd} \mathbf{E} \arrow[rr, leftrightarrow, "\eta_F"] \arrow[dr, "F"'] & & (*\downarrow F_-) \arrow[dl, "U_{F_-}"] \\ & \mathbf{C} & \end{tikzcd} \tag{2.19}$$

Define $\eta_F(e) := (e \in F_{F(e)})$ on objects, this is well-defined since e is tautologically in the fiber of $F(e) \in \mathbf{C}_0$. For a morphism $f : e \to e'$, applying F gives $F(f) : F(e) \to F(e')$ and the unique lift $F_{F(f)}$ of $F(f)$ to e is again f. In other words, the following defines a fully faithful functor:

$$\mathbf{E}: \quad \begin{tikzcd} e \arrow[d, "f"] \\ e' \end{tikzcd} \quad \overset{\eta_F}{\longmapsto} \quad \begin{tikzcd} e \in F_{F(e)} \arrow[d, "F_{F(f)}"] \\ e' \in F_{F(e')} \end{tikzcd} \quad : (*{\downarrow}F_{-})$$

Moreover, η_F is surjective on objects since any $(x \in F_c) \in (*{\downarrow}F_{-})_0$ is the image of $x \in \mathbf{E}_0$. It is clear that the definition of η_F respects Diagram 2.19. Consequently, η_F is an isomorphism of discrete fibrations over $\mathbf{C}$.

Checking naturality of η is a bit tedious but straightforward. □

Corollary 2.20. *Let $\boldsymbol{B}$ be a connected groupoid. Under Theorem 2.18:*

1. *Universal covers of $\boldsymbol{B}$ correspond to representable functors $\boldsymbol{B} \to \boldsymbol{Set}$.*
2. *Connected covers correspond to transitive $\boldsymbol{B}$-actions.*
3. *Finite covers correspond to finite $\boldsymbol{B}$-actions.*

Proof. We only prove (1), as (3) isn't too difficult to see and (2) can be found in [Bro06][10.4.2(c)]. The category of elements of a representable functor $\mathbf{B}(b, -)$ is the slice $b/\mathbf{B}$ (dual of Prop. 1.49), which is connected whenever $\mathbf{B}$ is. Moreover, the automorphism groups are trivial since for any $f : b \to b' \in \mathbf{B}_1$ the identity $\mathrm{id}_{c'}$ is the only morphism making the following commute:

$$b/\mathbf{B}: \quad \begin{tikzcd} & b' \arrow[dd, equal, "\mathrm{id}_{b'}"] \\ b \arrow[ur, "f"] \arrow[dr, "f"'] & \\ & b' \end{tikzcd}$$

The cover $U_b : b/\mathbf{B} \to \mathbf{B}$ is therefore universal. By Corollary 2.9, the isomorphism class of representable functors must correspond to the isomorphism class of universal covers. □

In [May99][Ch.3], May prefers to construct universal covers in the topological case and leaves the details of the groupoid case to the reader. The construction through representable functors given here is presumably not the one May had in mind; but supposing Theorem 2.18 we find the construction to be quite direct with few details to verify—though it may have drawbacks that escape the present author.

Remark 2.21. Revisiting Corollary 2.9 we now see the relation to the Yoneda lemma: A universal cover $Q : \mathbf{X} \twoheadrightarrow \mathbf{B}$ corresponds to a representable functor Q_{-}, and the fiber $\mathbf{E}_b$ is exactly the value of the functor $\mathbf{E}_{-} : \mathbf{B} \to \mathbf{Set}$ on the object $b \in \mathbf{B}_0$.

By considering a group G as one-object groupoid $\mathbf{G}$, Corollary 2.20 associates a universal cover to the single representable functor $\mathbf{G}(*, -) : \mathbf{G} \to \mathbf{Set}$.

Example 2.22. The additive group of the integers defines a one-point groupoid which we will denote $\mathbb{Z}$ in this example. By Corollary 2.20, the universal covering space of $\mathbb{Z}$ is the slice $(*/\mathbb{Z})$ whose objects are the integers, and there is a single morphism $n \to m \in (*/\mathbb{Z})$ given by $m - n$. Any such morphism factors as a sequence of 1's, which is just the trivial statement $m - n = \sum_{i=1}^{m-n} 1$. We may draw this as:

$$(*/\mathbb{Z}) = (\cdots \xrightarrow{1} -1 \xrightarrow{1} 0 \xrightarrow{1} 1 \xrightarrow{1} \cdots) \twoheadrightarrow \mathbb{Z}$$

This can be seen as a categorical analogue of the infamous topological result that the real line $\mathbb{R}$ is the universal covering space of the circle S^1.

Example 2.23. Consider the cyclic groups C_2 and C_3. The universal covers of $\mathbf{C_2}$ and $\mathbf{C_3}$ corresponds to their respective slices which we may draw as follows:

$$\begin{tikzcd}
(*/\mathbf{C_2}) & 0 \arrow[r, bend left, "1"] & 1 \arrow[l, bend left, "1"] \\
(\mathbf{C_2}) & * \arrow[loop right, "1"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& 1 \arrow[dl, "2"] \arrow[dr, "1"] & & \\
0 \arrow[ur, bend left, "1"] \arrow[rr, "2"] & & 2 \arrow[ul, bend right, "2"'] \arrow[ll, bend left, "1"] & (*/\mathbf{C_3}) \\
& * \arrow[loop left, "2"] \arrow[loop right, "1"] & & (\mathbf{C_3})
\end{tikzcd}$$

Specifically, the objects of $\mathbf{C_2}$ are the elements of C_2; and for any two objects $x, y \in C_2$ there is exactly one morphism $y - x : x \to y$ in $\mathbf{C_2}$. Similarly for $\mathbf{C_3}$.

Another interesting corollary of Theorem 2.18 is the following:

Corollary 2.24. *Let $\boldsymbol{B}$ be a connected groupoid. For all $b \in \boldsymbol{B}_0$, the automorphism group of b is naturally isomorphic to the automorphism group of the universal cover given by the slice under b, i.e. $\boldsymbol{Cov_B}(b/\boldsymbol{B}) \cong \boldsymbol{B}(b)$.*

Proof. For any $b \in \mathbf{B}_0$, the Yoneda embedding and Theorem 2.18 combine to give:

$$\mathbf{B}(b) \cong \mathbf{Set}^{\mathbf{B}}(\mathbf{B}(b, -)) \cong \mathbf{Cov_B}(b/\mathbf{B}) \qquad \square$$

We conclude this section with a new perspective on the contents previous chapter. If the reader is allergic to philosophy, they should skip directly ahead to Proposition 2.25.

In Proposition 1.53 we reconstructed a category $\mathbf{C}$ as the colimit of its slice functor $\mathbf{C} = \mathrm{colim}_{\mathbf{C}}\,[\mathbf{C}/- : \mathbf{C} \to \mathbf{Cat}]$, and the colimit cocone was the collection of discrete fibrations $(P_c : \mathbf{C}/c \to \mathbf{C})_{c \in \mathbf{C}_0}$. We may now see this more geometrically as reconstructing $\mathbf{C}$ from the "universal covers $\mathbf{C}/c$" at each "point" $c \in \mathbf{C}_0$.

The free cocompletion (Theorem 1.55) relates to Theorem 2.18 as well. As in the previous chapter we consider presheaves again, but we will reuse the notation of Theorem 2.18—a short proof for preshaves is found in [LR19][2.1.2].

With our newfound insight that a slice $\mathbf{C}/c$ is not simply a category, but a discrete fibration $P_c : \mathbf{C}/c \to \mathbf{C}$, the free cocompletion (Theorem 1.55) applied to the diagram on the left

$$\begin{tikzcd}
U_- : \mathbf{Set_C} \arrow[rr, leftrightarrow] & & \mathbf{Fib_C} : (-)_= \\
& \mathbf{C} \arrow[ul, "r"] \arrow[ur, "P_-"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Set_C} \arrow[r, shift left, "(*\downarrow -)"] \arrow[dr, shift left, Rightarrow] & \mathbf{Cat} \arrow[l, shift left] \arrow[d, dashed, shift right] \\
\mathbf{C} \arrow[u, "r"] \arrow[r, "\mathbf{C}/-"'] & \mathbf{Fib_C} \arrow[ul, shift left, Rightarrow] \arrow[u, shift right, "\mathcal{U}"']
\end{tikzcd}$$

gives us that $U_- : \mathbf{Set}_{\mathbf{C}} \simeq \mathbf{Fib}_{\mathbf{C}} : (-)_=$ is actually an *adjoint* equivalence, drawn as the center arrows in the diagram to the right. The topmost adjunction in the right diagram is Proposition 1.51's $(*\downarrow -) \dashv \mathbf{Cat}(\mathbf{C}/{=}, -)_0$, whereas $\mathcal{U} : \mathbf{Fib}_{\mathbf{C}} \to \mathbf{Cat}$ is the obvious forgetful functor.

Adjoint equivalences are symmatric in the sense that $(-)_= \vdash U_-$ as well. By composing left adjoints in the diagram on the right, we find a left adjoint to $\mathcal{U}$ given by $\mathcal{L} := U_- \circ \mathbf{Cat}(\mathbf{C}/=, -)_0 : \mathbf{Cat} \to \mathbf{Fib}_{\mathbf{C}}$.

Enlarging the codomain of $\mathbf{C}/-$ to be all of $\mathbf{Cat}/\mathbf{C}$ (and not just the full subcategory $\mathbf{Fib}_{\mathbf{C}}$) gives in general an adjunction that is not an equivalence through the free cocompletion. Denote this adjunction $U_- : \mathbf{Set}_{\mathbf{C}} \leftrightarrows \mathbf{Cat}/\mathbf{C} : \mathcal{F}$ (reusing then the symbol U_-). Since $\mathcal{F}$ is then not in general also a left adjoint, it is unclear if $\mathcal{L} : \mathbf{Cat} \leftrightarrows \mathbf{Fib}_{\mathbf{C}} : \mathcal{U}$ "expands" to some adjunction $\hat{\mathcal{L}} : \mathbf{Cat} \leftrightarrows \mathbf{Cat}/\mathbf{C} : \mathcal{U}$.

We now finally return to answer Question 1.39. Leaving aside size concerns for a moment, consider the following three diagrams:

$$\begin{tikzcd} \mathbf{Set}_{\mathbf{C}} & \mathbf{Cat}/\mathbf{Set}_{\mathbf{C}} \arrow[l, "\mathrm{colim}"'] \\ \mathbf{C} \arrow[u, "r"] & \mathbf{Cat}/\mathbf{C} \arrow[l, "\mathrm{colim}"] \arrow[u, "\mathbf{Cat}/r"'] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Set}_{\mathbf{C}} \arrow[dr, "U_-"'] & \mathbf{Cat}/\mathbf{Set}_{\mathbf{C}} \arrow[l, "\mathrm{colim}"'] \\ & \mathbf{Cat}/\mathbf{C} \arrow[u, "\mathbf{Cat}/r"'] \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Set}_{\mathbf{C}} \arrow[dr, "U_-"] & \\ \mathbf{C} \arrow[u, "r"] \arrow[r, "\mathbf{C}/-"] & \mathbf{Cat}/\mathbf{C} \end{tikzcd}$$

The left diagram commutes if and only if $\mathbf{C}$ is cocomplete (so $\mathrm{colim} : \mathbf{Cat}/\mathbf{C} \to \mathbf{C}$ is well-defined) and the Yoneda embedding r is cocontinuous (dual of Proposition 1.38). This is far from the case in general, but seeing this diagram alongside the others will inspire an idea.

The middle diagram commutes starting from the top left corner, and this is in fact Theorem 1.44 manifested in a diagram: U_- sends a presheaf P to its category of elements as a cover $U_P : (*\downarrow P)^{op} \to \mathbf{C}$, which is then sent by $\mathbf{Cat}/r$ to the natural diagram $rU_P : (*\downarrow P)^{op} \to \mathbf{Set}_{\mathbf{C}}$ admitting P as a colimit.

The right diagram commutes and is the one inducing $U_- \vdash \mathcal{F}$ just mentioned. Juxtaposing the middle and right diagrams over the left diagram proposes $\mathbf{C}/-$ as a candidate for a right adjoint to colim:

Proposition 2.25. *Let $\mathbf{C}$ be a cocomplete category. There is an adjunction:*

$$\mathrm{colim} : \mathbf{Cat}/\mathbf{C} \leftrightarrows \mathbf{C} : \mathbf{C}/-$$

Proof. We will construct a natural isomorphism Θ:

$$\Theta : \mathbf{C}(\mathrm{colim}-, -) \cong \mathbf{Cat}/\mathbf{C}(-, \mathbf{C}/-) : \mathbf{Cat}/\mathbf{C} \times \mathbf{C} \to \mathbf{Set}$$

Let $D : \mathbf{D} \to \mathbf{C}$ and $c \in \mathbf{C}_0$. Denote by $\delta : D \Rightarrow \Delta(\mathrm{colim}_{\mathbf{D}} D)$ the colimit cocone of D. To a morphism $f : \mathrm{colim}_{\mathbf{D}} D \to c \in \mathbf{C}_1$ we associate a morphism $\Theta(f) \in (\mathbf{Cat}/\mathbf{C})_1$:

$$\begin{tikzcd} \mathbf{D} \arrow[rr, "\Theta(f)"] \arrow[dr, "D"'] & & \mathbf{C}/c \arrow[dl, "P_c"] \\ & \mathbf{C} & \end{tikzcd}$$

This is defined by $\Theta(f)(d) := f\delta_d : D(d) \to c$ on objects and any morphism $e : d \to d' \in \mathbf{D}_1$ is sent to $\Theta(f)(e) = D(e)$ as a morphism over c. This is well-defined by naturality of δ:

$$\begin{tikzcd} D(d) \arrow[rr, "D(e)"] \arrow[dr, "f\delta_d"'] & & D(d') \arrow[dl, "f\delta_{d'}"] \\ & c & \end{tikzcd}$$

Moreover, it is clear that $\Theta(f)$ is a well-defined as a functor by functoriality of D, and also as a morphism over $\mathbf{C}$.

Claim: Θ is injective. If $\Theta(f) = \Theta(g)$, then $f\delta_d = g\delta_d$ for all $d \in \mathbf{D}_0$, giving $\Delta(f)\delta = \Delta(g)\delta$ as natural transformations $D \Rightarrow \Delta c$. But such natural transformations are in bijective correspondance with arrows $\mathrm{colim}_{\mathbf{D}} D \to c$, so $f = g$.

Claim: Θ is surjective. Any morphism $F : D \to P_c$ over $\mathbf{C}$ gives rise to a natural transformation $(F(d))_{d\in\mathbf{D}_0} : D \to \Delta c$. To see that this is natural, remark that since P_c is faithful, $F(e) = D(e)$ for any morphism $e \in \mathbf{D}_1$. The preimage of F under Θ induced by universal property of $\mathrm{colim}_{\mathbf{D}} D$:

$$\begin{tikzcd} D \arrow[rr, Rightarrow, "\delta"] \arrow[dr, Rightarrow, "(F(d))_{d\in\mathbf{D}_0}"'] & & \Delta(\mathrm{colim}_{\mathbf{D}} D) \arrow[dl, dashed, "\Theta^{-1}(F)"] \\ & \Delta c & \end{tikzcd}$$

Verifying naturality of Θ is straightforward. $\square$

Dualizing Proposition 2.25 settles Question 1.39 in the positive.

2.2. Canonical Model Structures

A model structure equips a category with a notion of *weak equivalence*, a version of homotopy equivalence abstracted from classical topology. Localizing a model category at the weak equivalences yields a *homotopy category* where maps are defined "up-to-homotopy". Moreover, a model structure carries sets of "nice surjections" (fibrations) and "nice injections" (cofibrations) that allow lifting maps in certain diagrams. After a precise definition of model structures, we turn to a notion of *canonicity* and two examples.

Notation. Objects of the functor category $\mathbf{C}^{\mathbf{2}}$ will be written as arrows $f : c \to c' \in \mathbf{C}_1$. A morphism $f \Rightarrow g \in (\mathbf{C}^{\mathbf{2}})_1$ is a commutative diagram:

$$\begin{tikzcd} c \arrow[r, "a"] \arrow[d, "f"] & d \arrow[d, "g"] \\ c' \arrow[r, "a'"] & d' \end{tikzcd}$$

The above morphism is denoted $(a, a') : f \Rightarrow g$. If we write $u : f \Rightarrow g$, then this implicitely defines $u = (u^1, u^2) : f \Rightarrow g$. We may also denote morphisms in $\mathbf{C}^{\mathbf{2}}$ with single arrows, to avoid confusing them with natural transformations.

Definition 2.26. Let $\mathbf{C}$ be a category and $a, b \in \mathbf{C}_0$. If $ri = \mathrm{id}_a$,

$$\mathrm{id}_a : a \xrightarrow{i} b \xrightarrow{r} a$$

then we call i a *section* of r and r a *retraction* of i. When such a diagram exists, a is a *section* of b and b is a retraction of a.

Definition 2.27. A *model structure* on a category $\mathbf{C}$ consists of three sets of maps: the weak equivalences $\mathcal{W}$, the fibrations $\mathcal{F}$, and the cofibrations $\mathcal{C}$, all of which contain the identities and are closed under composition. Maps which are both a fibration (resp. cofibration) and a weak equivalence are called *acyclic* fibrations (resp. acyclic cofibrations). These are subject to the following axioms:

(i) *"Finite limits & colimits"*: $\mathbf{C}$ is finitely complete and finitely cocomplete.

(ii) *"3-for-2"*: Let $f, g \in \mathbf{C}$. If two of f, g, gf are weak equivalences, so is the third.

(iii) *"Retracts preserve"*: If a map f is a retract in $\mathbf{C^2}$ of another map g (as in the diagram on the left below), then f is a weak equivalence, fibration, or cofibration whenever g is.

$$\begin{tikzcd} A \arrow[r, "b"] \arrow[d, "f"] \arrow[rr, bend left, equal] & B \arrow[r, "a"] \arrow[d, "g"] & A \arrow[d, "f"] \\ A' \arrow[r, "b'"] \arrow[rr, bend left, equal] & B' \arrow[r, "a'"] & A' \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d, hook, "i"] & C \arrow[d, two heads, "p"] \\ B \arrow[r] \arrow[ur, dashed, "h"] & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, dashed, hook, "\sim"] \arrow[d, dashed, hook] \arrow[dr, "f"] & A' \arrow[d, dashed, two heads] \\ B' \arrow[r, dashed, two heads, "\sim"] & B \end{tikzcd}$$

(iv) *"Lifting"*: In the middle diagram above, p is a fibration and i is a cofibration. If either of them is acyclic, then the lift h exists.

(v) *"2 factorizations"*: Any map $f \in \mathbf{C}$ can be factored $f = pi$ with p a fibration and i a cofibration and either p of i can be chosen acyclic (as in the diagram on the right above).

When the category $\mathbf{C}$ is equipped with a model structure, $\mathbf{C}$ becomes a *model category*. In this case we denote the fibrations using the arrow "$\twoheadrightarrow$", cofibrations using "$\hookrightarrow$" and weak equivalences by "$\xrightarrow{\sim}$".

The above axioms are those of [DS95]. Quillen first introduced model categories in [Qui67] with greater generality; the axioms above correspond to Quillen's "closed" model categories in which either set of fibrations or cofibrations determines the other:

Proposition 2.28. *The fibrations $\mathcal{F}$ are exactly the maps with the* right-lifting property *with respect to acyclic cofibrations, i.e. $\mathcal{F} = rlp(\mathcal{W} \cap \mathcal{C})$. Dually, $\mathcal{C} = llp(\mathcal{W} \cap \mathcal{F})$, and furthermore $\mathcal{W} \cap \mathcal{F} = rlp(\mathcal{C})$ and $\mathcal{W} \cap \mathcal{C} = llp(\mathcal{F})$.*

Proof. Proving $\mathcal{F} = \mathrm{rlp}(\mathcal{W} \cap \mathcal{C})$ suffices—the other statements follow by duality or analogous arguments.

Let $f : A \to B$ have the right-lifting property with respect to acyclic cofibrations. Factor $f = pi$ with i acyclic. The lift h exists in the diagram on the left, exhibiting f as a retract of the fibration p on the right:

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, hook, "i"] & A \arrow[d, "f"] \\ A' \arrow[r, two heads, "p"] \arrow[ur, dashed, "h"] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"] \arrow[rr, bend left, equal] & A' \arrow[r, "h"] \arrow[d, two heads, "p"] & A \arrow[d, "f"] \\ B \arrow[r, equal] & B \arrow[r, equal] & B \end{tikzcd}$$

□

Proposition 2.28 justifies forgetting about either fibrations or cofibrations, since we may retrieve one or the other. We may therefore specify a model structure as simply $(\mathcal{W}, \mathcal{F})$. In general there are many choices of fibrations for a fixed set of weak equivalences, however these choices don't affect the notion of homotopy in the category, and one may chose the fibrations which are most convenient for a given problem. Before discussing matters further, a lemma:

Proposition 2.29. *Let $\boldsymbol{C}$ be a model category. Then pullbacks preserve fibrations and pushouts preserve cofibrations.*

Proof. The statement for fibrations is proved, as the proof for cofibrations is similar.

Consider the pullback square on the left where p is a fibration and f is our candidate:

$$\begin{tikzcd} E \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, two heads, "p"] \\ A' \arrow[r] & B \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r] \arrow[d, hook, "\wr"] & E \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, two heads, "p"] \\ C' \arrow[r] \arrow[ur, dashed, "h"] \arrow[urr, dashed, "h'"'] & A' \arrow[r] & B \end{tikzcd}$$

The rectangle on the right is a lifting problem with f on the left, glued with the pullback square on the right. The outer rectangle admits a lift h' since p is a fibration, and this induces the desired h by virtue of being a pullback square. $\square$

We now turn to *canonical* model structures. This term is not well-established in the literature, but it is used for the "categorical" model structures on the category of (higher) categories. A little-known fact is that these are unique (in the sense of the following definition) in the 0- and 1-categorical cases of **Set** and **Cat**, respectively. This will be proved after the precise definition:

Definition 2.30. A model structure $(\mathcal{W}, \mathcal{F})$ on a category $\mathbf{C}$ is *canonical* if $\mathcal{F}$ is the only set of fibrations such that this is a model structure; i.e. if $(\mathcal{W}, \mathcal{F}')$ is also a model structure on $\mathbf{C}$, then $\mathcal{F}' = \mathcal{F}$.

Example 2.31 (The canonical model structure on **Set**)**.** Taking $\mathcal{W}$ as bijections and all maps fibrant and cofibrant defines a canonical model structure on **Set**.

"Finite limits & colimits": **Set** is both complete and cocomplete by Construction 1.24 and its dual.

"3-for-2": Bijections are closed under composition.

"Retracts preserve": Let $(a, a') : g \to f$ be a retraction with right-inverse (b, b'). There is nothing to verify if g is a fibration or cofibration, so let g be a bijection. Then $ag^{-1}b'$ is right- and left-inverse to f:

$$f(ag^{-1}b') = (a'g)(g^{-1}b') = \mathrm{id} \qquad (ag^{-1}b')f = (ag^{-1})(gb) = \mathrm{id}$$

"Lifting": If either side in a commutative square is a bijection, lifting is trivial.

"2 factorizations": Given a map $f : S \to T$, the two factorizations are $\mathrm{id} \circ f$ (cofibration, then weak equivalence and fibration) and $f \circ \mathrm{id}$ (weak equivalence and cofibration, then fibration).

Finally, the model structure is canonical since all maps have both the left- and right-lifting property with respect to bijections.

The category **Cat** supports a variety of model structures. The most "categorical" one, in which the weak equivalences are equivalences of categories, is the one we will consider here. It will turn out that this model structure is also canonical, and the fibrations are so-called *isofibrations*:

Definition 2.32. Let $F : \mathbf{E} \to \mathbf{C}$ be a functor between small categories and $c \in \mathbf{C}_0$.

1. The full category of $c/\mathbf{C}$ spanned by isomorphisms is called the *iso-over category*, denoted $c \mathbin{\rotatebox[origin=c]{-45}{$\wr$}} \mathbf{C}$.

2. The *iso-comma category* $(F \wr \mathbf{C})$ is the full subcategory of $(F{\downarrow}\mathbf{C})$ spanned by isomorphisms.

3. If the *(iso-)slice-maps* $e \mathbin{\rotatebox[origin=c]{-45}{$\wr$}} F : (e \mathbin{\rotatebox[origin=c]{-45}{$\wr$}} \boldsymbol{E})_0 \to (F(e) \mathbin{\rotatebox[origin=c]{-45}{$\wr$}} \boldsymbol{C})_0$ are surjections for all $e \in \mathbf{E}_0$ then F is an *isofibration.*

4. If F is injective on objects, we will call F an *isocofibration.*

Remark 2.33. Any discrete fibration is an isofibration; as is often the case in category theory, restrictions are relaxed when notions are generalized. For groupoids, isofibrations and discrete fibrations coincide. Moreover, isofibrations regain the symmetry of the groupoidal situation that was lost for discrete fibrations, in that $c \mathbin{\rotatebox[origin=c]{-45}{$\wr$}} \mathbf{C} \cong \mathbf{C} \mathbin{\rotatebox[origin=c]{-45}{$\wr$}} c$ (defining the iso-under category dually) and could have used iso-under categories in the definition.

The relaxation that the slice-maps are merely surjective means isofibrations lack uniqueness of lifts compared to discrete fibrations.

Example 2.34 (The canonical model structure on **Cat**).[7] Let $\mathcal{W}$ be equivalences of categories, $\mathcal{F}$ isofibrations and $\mathcal{C}$ isocofibrations. We claim that $(\mathcal{W}, \mathcal{F}, \mathcal{C})$ defines a model structure on **Cat**:

Finite limits & colimits: **Cat** is both complete and cocomplete by Construction 1.24 and its dual.

"3-for-2": Equivalences of categories are closed under composition.

"Retracts preserve": Let $(a, a') : G \to F$ be a retraction in $\mathbf{Cat}^{\mathbf{2}}$ with right-inverse (b, b'), drawn on the left:

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "a"] \arrow[d, "F"] & \mathbf{D} \arrow[r, "a"] \arrow[d, "G"] & \mathbf{C} \arrow[d, "F"] \\
\mathbf{C}' \arrow[r, "b'"] & \mathbf{D}' \arrow[r, "a'"] & \mathbf{C}'
\end{tikzcd}
\qquad
\begin{tikzcd}
c \mathbin{\rotatebox[origin=c]{-45}{$\wr$}} \mathbf{C} \arrow[r] \arrow[d] & b(c) \mathbin{\rotatebox[origin=c]{-45}{$\wr$}} \mathbf{D} \arrow[r] \arrow[d] & c \mathbin{\rotatebox[origin=c]{-45}{$\wr$}} \mathbf{C} \arrow[d] \\
F(c) \mathbin{\rotatebox[origin=c]{-45}{$\wr$}} \mathbf{C}' \arrow[r] & Gb(c) \mathbin{\rotatebox[origin=c]{-45}{$\wr$}} \mathbf{D}' \arrow[r] & F(c) \mathbin{\rotatebox[origin=c]{-45}{$\wr$}} \mathbf{C}'
\end{tikzcd}$$

To start off, suppose G is an isofibration and $\phi : F(c) \leftrightarrow c' \in (F(c) \mathbin{\rotatebox[origin=c]{-45}{$\wr$}} C')_0$. To show that F is an isofibration, we need to find a lift in $c \mathbin{\rotatebox[origin=c]{-45}{$\wr$}} \mathbf{C}$ of ϕ. This will be done through a chase in the diagram above on the right. Since G is an isofibration, the middle map on the right is surjective on objects and we obtain a preimage $\hat{\phi} : b(c) \leftrightarrow d \in (b(c) \mathbin{\rotatebox[origin=c]{-45}{$\wr$}} \mathbf{D})_0$ of $b'(\phi)$. Applying a gives the desired lift $a(\hat{\phi}) : c \to a(c)$.

Next, consider the case of G being an isocofibration. Then $Gb = b'F$ is an isocofibration as well, but then F must be as well by left-cancellation of monomorphisms (injections) in **Set**.

[7]This construction details nLab's Canonical model structure on Cat, parts written by Todd Trimble in revision 17.

Lastly, suppose G is a weak equivalence and denote by $\beta : \mathrm{id}_B \cong G^{-1}G$ one of the natural isomorphisms. Then $aG^{-1}b'$ is an inverse equivalence to F:

$$(aG^{-1}b')F = (aG^{-1})(Gb') \overset{a(\beta_b)}{\cong} ab = \mathrm{id}_{\mathbf{C}}$$

The other side of the equivalence can be equivalently verified.

"Lifting": First, consider the case of an acyclic cofibration I and a fibration P. The goal is then to define a lifting $H : \mathbf{B} \to C$:

$$\begin{tikzcd}
\mathbf{A} \arrow[r, "F"] \arrow[d, hook, "I", "\wr"'] & \mathbf{C} \arrow[d, two heads, "P"] \\
\mathbf{B} \arrow[r, "G"] \arrow[ur, dashed, "H"] & \mathbf{D}
\end{tikzcd}$$

Weak equivalences are fully faithful dense functors, so for any object $b \in B_0$ we may pick an isomorphism $\phi_b : I(a_b) \cong b$ (using the axiom of choice) such that the codomain is in the image of I (by density), and with $\phi_{I(a)} = \mathrm{id}_{I(a)}$ for $a \in A_0$. Applying G, we obtain an isomorphism $G(\phi_b)$ in $\mathbf{D}$ which lifts to $\mathbf{C}$ since P is an isofibration. Denote this lift by $\psi_b : F(a_b) \leftrightarrow c_b$. Now define $H(b) := c_b$, and

$$H(f : b \to b') := \psi_{b'} F I^{-1}(\phi_{b'}^{-1} f \phi_b)\psi_b^{-1}$$

where $I^{-1} : \mathbf{B}(I(a_b), I(a_{b'})) \leftrightarrow \mathbf{A}(a_b, a_{b'})$ is given by full faithfulness of I. This is well-defined on objects since I is an isocofibration, so that the correspondance $a \mapsto c_{I(a)}$ is well-defined. Functoriality of H is straightforward to check.

Then $HI = F$, since for any $a \in \mathbf{A}_0$, $HI(a) = F(a)$ since in this case $\psi_{I(a)}$ is the identity. Moreover, for any morphism $f : a \to a'$ in $\mathbf{A}_1$ we have by definition:

$$HI(f) = \mathrm{id}_{I(a')} F I^{-1}(\mathrm{id}_{I(a')} I(f) \mathrm{id}_{I(a)})\mathrm{id}_{I(a)} = F(f)$$

Last we need to verify that $PH = G$. This holds on objects since for any $b \in \mathbf{B}_0$, $H(b) = c_b$ is sent to $G(b)$ by P. Now let $g : b \to b'$ be any morphism in $\mathbf{B}_1$. Then

$$PH(g) = p(\psi_{b'} F I^{-1}(\phi_{b'}^{-1} g \phi_b)\psi_b^{-1}) = G(\phi_{b'})G(\phi_{b'}^{-1} g \phi_b)G(\phi_b^{-1}) = G(g)$$

where we have used that e.g. $P(\psi_b) = G(\phi_b)$. In conclusion, $H : \mathbf{B} \to \mathbf{C}$ is the desired lift.

When I is only an isocofibration and P is an acyclic isofibration, the argument is similar. The fundamental insight is that the property of being an acyclic isofibration means that P surjective on objects: Since a weak equivalence is dense, any object $d \in \mathbf{D}_0$ is the codomain of an isomorphism $P(c) \leftrightarrow d$ for some $c \in \mathbf{C}_0$. The existance of a lift gives that d is in the image of P as well.

"2 factorizations": Let $F : \mathbf{C} \to \mathbf{D}$ be a functor. We start by factoring F as an acyclic isocofibration followed by an isofibration as on the left where F'' is the obvious forgetful functor:

$$\begin{tikzcd}
\mathbf{C} \arrow[dr, hook, "F'", "\sim"'] \arrow[dd, "F"'] & \\
& (F \wr \mathbf{D}) \arrow[dl, two heads, "F''"] \\
\mathbf{D} &
\end{tikzcd}
\qquad
\begin{tikzcd}
F(c) \arrow[r, "F(f)"] \arrow[d, equal] & F(c') \arrow[d, equal] \\
F(c) \arrow[r, "F(f)"] & F(c')
\end{tikzcd}$$

The functor F' gives the morphism in $(F \wr \mathbf{C})(\mathrm{id}_{F(c)}, \mathrm{id}_{F(c')})$ on the right above for a morphism $f : c \to c' \in \mathbf{C}_1$. The objects are drawn vertically, so in particular $F'(c) = \mathrm{id}_{F(c)}$. This is clearly a faithful isocofibration, but it is also full: A general morphism by replacing the bottom "$F(f)$" above by a general isomorphism $\phi : F(c) \to F(c') \in \mathbf{D}_1$. But since the diagram commutes, $\phi = F(f)$. Moreover F' is dense, because for any object $\psi : F(c) \to d$ in $(F \wr \mathbf{D})_0$ we have an isomorphism:

$$\begin{tikzcd}
F(c) \arrow[r, "\mathrm{id}_{F(c)}"] \arrow[d, "\psi"] & F(c) \arrow[d, "\psi"] \\
d \arrow[r, "\mathrm{id}_d"] & d
\end{tikzcd}$$

In conclusion F' is an acyclic isocofibration.

Now we argue that F'' is an isofibration: Let $\psi : F(c) \to d$ be an arbitrary object of $F \mathbin{\rotatebox[origin=c]{-45}{$\wr$}} \mathbf{D}$, which is sent to d by F''. We need to give a lift to $(\phi \mathbin{\rotatebox[origin=c]{-45}{$\wr$}} (F \wr \mathbf{D}))_0$ for any isomorphism $\phi : d \to d'$, i.e. object of $d \mathbin{\rotatebox[origin=c]{-45}{$\wr$}} \mathbf{D}$. The lift is defined as:

$$\begin{tikzcd}
F(c) \arrow[r, "\phi\psi"] \arrow[d, "\psi"] & d' \arrow[d, equal] \\
d \arrow[r, "\phi"] & d'
\end{tikzcd}$$

which is indeed a lift, since it sent to ϕ by F''. In conclusion, F'' is an isofibration. Writing out the definitions one sees that $F = F''F'$, giving the desired factorization.

The interested reader may consult the given reference for the other factorization.

Proposition 2.35.[8] *The model structure of Example 2.34 on* $\mathbf{Cat}$ *is canonical.*

Before the proof, recall that the forgetful functor $A \mapsto A_0 : \mathbf{Set} \to \mathbf{Cat}$ is left adjoint to the $K : \mathbf{Set} \to \mathbf{Cat}$ sending a set S to the category $K(S)$ where all objects are connected by a single isomorphism. An example is the *free isomorphism category* $\boldsymbol{I} := K(\boldsymbol{2}_0)$ (or the "categorical interval") consisting of two objects and an isomorphism.

Proof. Consider a model structure $(\mathcal{W}, \mathcal{F}, \mathcal{C})$ on $\mathbf{Cat}$ with weak equivalences being equivalences of categories. We will see that the cofibrations are exactly the isocofibrations $\mathcal{C}_c$ and conclude by Proposition 2.28. The proof is divided in steps.

Step 1: The unique morphism $U : \boldsymbol{0} \to \boldsymbol{1}$ *is a cofibration.* Factor it as $U = PI$ with $P : \mathbf{K} \to \mathbf{1}$ an acyclic fibration and $I : \mathbf{0} \to \mathbf{K}$ a fibration. Picking any object $k \in \mathbf{K}_0$ induces a section of P which we denote P^{-1}. Consequently, P is a retraction in $\mathbf{Cat}$. We may exhibit U as a retract of I as in the diagram on the left, which commutes since $U : \mathbf{0} \to \mathbf{1}$ is the unique morphism between these categories.

$$\begin{tikzcd}
\mathbf{0} \arrow[r, equal] \arrow[d, "U"] & \mathbf{0} \arrow[r, equal] \arrow[d, hook, "I"] & \mathbf{0} \arrow[d, "U"] \\
\mathbf{1} \arrow[r, "P^{-1}"] & \mathbf{K} \arrow[r, "P"] & \mathbf{1}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{0} \arrow[r] \arrow[d, hook, "U"'] & \mathbf{C} \arrow[d, two heads, "P", "\wr"'] \\
\mathbf{1} \arrow[r, "D"] \arrow[ur, dashed, "H"] & \mathbf{D}
\end{tikzcd}$$

[8] This construction details nLab's Canonical model structure on Cat, parts written by Todd Trimble in revision 17.

Step 2: The isocofibrations are cofibrations, i.e. $\mathcal{C}_c \subseteq \mathcal{C}$. Let $P : \mathbf{C} \to \mathbf{D}$ be an acyclic fibration. If P isn't surjective on objects, we may pick a $d \in \mathbf{D}_0$ that isn't hit by P. Since U is a cofibration, the lift H should exist in the diagram above on the right—but this is absurd, so P must be surjective on objects. The upshot is that the acyclic fibrations are necessarily acyclic isofibrations, and we deduce that isocofibrations are cofibrations: $\mathcal{W} \cap \mathcal{F} \subseteq \mathcal{W} \cap \mathcal{F}_c \overset{\text{llp}}{\Longrightarrow} \mathcal{C}_c \subseteq \mathcal{C}$.

Step 3: If there is a cofibration identifying two objects, then the unique morphism $\boldsymbol{I} \to \boldsymbol{1}$ *is a cofibration.* Suppose there is a cofibration $I : \mathbf{A} \to \mathbf{B}$ that isn't injective on objects, so that $I(a) = b = I(a')$ for two $a, a' \in \mathbf{A}_0$. Let $i : I_0 \to A_0$ send $0, 1 \in \mathbf{I}_0$ to $a, a' \in \mathbf{A}_0$ respectively. Pick any retraction $r : \mathbf{A}_0 \to \mathbf{I}_0$ of i, i.e. so that $r(a) = 0$ and $r(a') = 1$.

Under the adjunction $-_0 \dashv K$, the map $r \in \mathbf{Set}(\mathbf{A}_0, \mathbf{I}_0)$ corresponds to $R : \mathbf{A} \to \mathbf{I} = K(\mathbf{I}_0)$, and R notably doesn't identify a and a', i.e. $R(a) \neq R(a')$ because R equals r on objects. On the left below we form the pushout of I and R, producing a cofibration $\mathbf{I} \hookrightarrow \mathbf{U}$ by Proposition 2.29. Moreover, the natural map $\mathbf{U} \hookrightarrow K(\mathbf{U}_0)$ given by the unit of the adjunction $-_0 \vdash K$ is also a cofibration since it is injective on objects, thus an isocofibration which by Step 2 is a cofibration. The composition J below is then also a cofibration:

$$
\begin{tikzcd}
\mathbf{A} \arrow[r, "R"] \arrow[d, hook, "I"] & \mathbf{I} \arrow[d, hook] \arrow[dr, "J"] & \\
\mathbf{B} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \mathbf{U} \arrow[r, hook] & K(\mathbf{U}_0)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{I} \arrow[r, equal] \arrow[d] & \mathbf{I} \arrow[r, equal] \arrow[d, hook, "J"] & \mathbf{I} \arrow[d] \\
\mathbf{1} \arrow[r, "U"] & K(\mathbf{U}_0) \arrow[r] & \mathbf{1}
\end{tikzcd}
$$

Since I identifies a and a' while R does not, the pushout of I must identify $R(a)$ and $R(a')$, and J as well. Denote $U : \mathbf{1} \to K(\mathbf{U}_0)$ the unique morphism hitting $JR(a) = JR(a')$ in $K(\mathbf{U}_0)_0$. Then the squares above on the right commute, exhibiting $\mathbf{I} \hookrightarrow \mathbf{1}$ as a retract of the cofibration J.

Step 4: If $\boldsymbol{I} \to \boldsymbol{1}$ *is a cofibration, any isomorphism in a category is an identity.* Remark that $\mathbf{I} \to \mathbf{1}$ is an equivalence of categories, hence an acyclic cofibration under the hypothesis. Consider any isomorphism ϕ in a category $\mathbf{A}$. Factor the unique morphism $U_{\mathbf{A}} : \mathbf{A} \to \mathbf{1}$ as $U : \mathbf{A} \overset{I}{\hookrightarrow} \bar{\mathbf{A}} \twoheadrightarrow \mathbf{1}$ (doesn't matter which one is acyclic), then $I(\phi)$ is in particular also an isomorphism. The existance of the lift in the following diagram asserts that ϕ is in fact an identity:

$$
\begin{tikzcd}
\mathbf{I} \arrow[r, "I(\phi)"] \arrow[d, hook, "\wr"] & \bar{\mathbf{A}} \arrow[d, two heads] \\
\mathbf{1} \arrow[r, equal] \arrow[ur, dashed] & \mathbf{1}
\end{tikzcd}
$$

Conclusion: If there is a cofibration that isn't an isocofibration, Step 3 and Step 4 give that all isomorphisms in categories are identities. This is of course absurd; consider any non-trivial group as a single object category (groupoid). Therefore, $\mathcal{C} = \mathcal{C}_c$ and consequently $\mathcal{W} \cap \mathcal{C} = \mathcal{W} \cap \mathcal{C}_c \overset{\text{rlp}}{\Longrightarrow} \mathcal{F}_c = \mathcal{F}$. □

In the literature, the previous canonical model structures (Examples 2.31 and 2.34) go by many names: "folk", "categorical", and "natural" are all used alongside "canonical". These terms also refer to model structures on (models of) the category $n\mathbf{Cat}$ of *n-categories*. For higher categories there are several notions of equivalence of varying *strictness*, but one could hope for canonicity in the sense of Def. 2.30 for any, or some, such choices.

Constructing the analogous model structures on n**Cat** is more involved: In [Lac04] and [Lac10] Lack constructs appropriate model structures for (models of) 2- and 3-categories respectively. These are colloquially referred to as "canonical" on e.g. the nLab—however canonicity in the precise sense of Def. 2.30 appears to remain unknown. A promising approach is taken by [LMW10], where a model structure on ω**Cat** is constructed, which restricts along the inclusions of **Set**, **Cat**, 2**Cat** into ω**Cat** to give the ones seen here, as well as Lack's model structure on 2-categories.

In short, the author is left with the follwing question: *Are the "natural" model structures on n**Cat** canonical in the sense of Def. 2.30—and if so, why?*

2.3. The Small Object Argument

The factorizations of morphisms in Examples 2.31 and 2.34 are in fact *functorial*, and shortly define what that means. This is a stronger property that facilitates working with a model category. Some authors (e.g. [Rie14a], [Hov07]) go as far as to require the factorization of morphisms to be functorial when posing their model structure axioms. However, as we will see, functorial factorizations are often obtainable through Quillen's so-called *small object argument.*

The approach taken here follows both [Lur09][A.1.2.5] and [Rie14a]. The former proves the small object argument for *presentable*[9] categories, and we do the work required to define these. For the rest of this section, let κ be a regular cardinal.

Defining functorial factorizations in a category $\mathbf{C}$ is nicest done under a "simplicial" perspective on $\mathbf{C}$. By this we mean that a morphism $f : c \to c' \in \mathbf{C}_1$ defines a unique 1-simplex $\sigma_f \in N(\mathbf{C})_1 = \mathbf{C}^\mathbf{2}$. Similarly, a pair of composable morphisms f, g defines a unique 2-simplex $\sigma_{f,g} \in N(\mathbf{C})_2 = \mathbf{C}^\mathbf{2}$, drawn as:

$$\begin{tikzcd} & c' \arrow[dr, "g"] \arrow[d, phantom, "\sigma_{f,g}"] & \\ c \arrow[ur, "f"] \arrow[rr, "gf"'] & {} & c'' \end{tikzcd}$$

From the 2-simplex $\sigma_{f,g}$ we recover f and g using the simplicial maps: $f = d_2(\sigma_{gf})$ and $g = d_0(\sigma_{gf})$. We may even compose: $gf = d_1(\sigma_{f,g})$.

Definition 2.36. Let $\mathbf{C}$ be a category. A *functorial factorization* on $\mathbf{C}$ is a functor $T : \mathbf{C}^\mathbf{2} \to \mathbf{C}^\mathbf{3}$ such that $d_1 T(f) = f$ for any morphism f of $\mathbf{C}$.

Functorial factorizations will be created using transfinite compositions of morphisms. For this we introduce some notation:

Notation. For an ordinal α, we will also denote by α the underlying poset of α, thought of as a category. The morphisms of α are denoted $\beta < \beta'$. If $d : \alpha \to \mathbf{C}$ is a diagram, we will write $d_\beta := d(\beta)$ on objects, and also denote the colimit of d (when it exists) by $\mathrm{colim}_{\beta<\alpha} d_\beta$. Moreover, for any $\alpha' < \alpha$ we may restrict d to the full subcategory of α spanned by α'. The colimit in this case will be written $\mathrm{colim}_{\beta<\alpha'} d_\beta$.

[9]Some authors refer to these as *locally presentable*, but we follow [Lur09] by writing simply presentable.

Definition 2.37. Let α be an ordinal. An *α-composite* in a category $\mathbf{C}$ is a diagram $d : \alpha \to \mathbf{C}$ such that the colimit $d_\alpha := \operatorname{colim}_{\beta<\alpha} d_\beta$ exists, and for any limit ordinal $\beta < \alpha$ we have $d_\beta = \operatorname{colim}_{\gamma<\beta} d_\gamma$. When the colimit of d exists, we will denote it $d_\alpha := \operatorname{colim}_{\beta<\alpha} d_\beta$ and refer to the unique induced morphism $d_0 \to d_\alpha$ as an *α-composition.*

Note that for an α-composite $d : \alpha \to \mathbf{C}$, we always have $d_\beta = \operatorname{colim}_{\gamma<\beta} d_\gamma$ for a non-limit (i.e. successor) ordinal β. The condition on limit ordinals thus asserts a certain "continuity" of the sequence of colimits in $\mathbf{C}$. α-composites are important examples of *filtered colimits*:

Definition 2.38. A *κ-filtered* category $\mathbf{F}$ admits a cocone for any κ-small diagram $D : \mathbf{D} \to \mathbf{F}$. A cocone f of two objects $f_0, f_1 \in \mathbf{F}_0$ will be called an *upper bound* of f_0 and f_1, displayed as:

$$\begin{tikzcd} f_0 \arrow[dr] & \\ & f \\ f_1 \arrow[ur] & \end{tikzcd}$$

When $\mathbf{F}$ is κ-filtered, we call the colimit of a diagram $F : \mathbf{F} \to \mathbf{C}$ a *κ-filtered colimit.*

The "small objects" referred to in this sections heading are objects that interact nicely with filtered colimits:

Definition 2.39. Let $\mathbf{C}$ be a cocomplete category. An object $c \in \mathbf{C}_0$ is κ-compact if for any ordinal $\alpha > \kappa$, the functor corepresented by c is cocontinuous, i.e.:

$$\operatorname*{colim}_{\beta<\alpha} \mathbf{C}(c, d_\beta) = \mathbf{C}(c, d_\alpha)$$

for all $d : \alpha \to \mathbf{C}$. We say an object is *small* when such a regular cardinal κ exists.

The small object argument will be proved for the following class of categories:

Definition 2.40. A locally small, cocomplete category $\mathbf{C}$ is *presentable* if there is a set $S \subset \mathbf{C}_0$ of small objects generating $\mathbf{C}$ through colimits, i.e. such that for any object $c \in \mathbf{C}_0$ there exists a diagram $D : \mathbf{D} \to \mathbf{C}$ taking values in S such that $c = \operatorname{colim}_{\mathbf{D}} D$.

Our first goal is to prove that a κ-small colimit of κ-compact objects is also κ-compact. As a corollary, all objects of a presentable category are small. This is asserted by Lurie between parentheses in his definition of presentable categories[Lur09][A.1.1.2]. The result will follow as a corollary of Proposition 2.43 saying that filtered colimits and limits of appropriate sizes commute. Before the proof, a remark:

Remark 2.41. The dual of Proposition 1.18 identifies a colimit in **Set** as the quotient of a disjoint union. In the case of Definition (2.39), we see that

$$\operatorname*{colim}_{\beta<\alpha} \mathbf{C}(c, d_\beta) = \Big(\coprod_{\beta<\alpha} \mathbf{C}(c, d_\beta)\Big)/\sim \tag{2.42}$$

where for $\beta < \beta'$, $(f : c \to d_\beta) \sim (f' : c \to d_{\beta'})$ if and only if $f' = d(\beta < \beta')f$. Denote the equivalence associated class by $\bar{f}$.

An object $c \in \mathbf{C}_0$ being small then means that any arrow $c \to d_\alpha$ from c into a α-transfinite composition, defines a unique equivalence class in 2.42, and therefore factors through some d_β with $\beta < \alpha$:

$$\begin{tikzcd}
 & c \arrow[d, dashed, "\exists"] \arrow[rrrr, "\forall"] & & & & \\
\cdots \arrow[r] & d_\beta \arrow[r, "d(\beta<\beta')"] & d_{\beta'} \arrow[r] & \cdots \arrow[r] & d_\alpha
\end{tikzcd}$$

The following proposition is proved for finite limits and finitely-filtered colimits in [Lan10][Ch.9.2]. Here we give a more explicit generalized proof.

Proposition 2.43. *In* ***Set****,* κ*-filtered colimits commute with* κ*-small limits:*

$$\operatorname*{colim}_{f \in \boldsymbol{F}} \lim_{l \in \boldsymbol{L}} D(f,l) = \lim_{l \in \boldsymbol{L}} \operatorname*{colim}_{f \in \boldsymbol{F}} D(f,l)$$

for all diagrams $D : \boldsymbol{F} \times \boldsymbol{L} \to \boldsymbol{Set}$ *with* $|\boldsymbol{L}_0| < \kappa$ *and* $\boldsymbol{F}$ *a* κ*-filtered category.*

The proof relies heavily on the equivalence relation on

$$\operatorname*{colim}_{f \in \mathbf{F}} \lim_{l \in \mathbf{L}} D(f,l) = (\coprod_{f \in \mathbf{F}} \mathbf{Set}^{\mathbf{L}}(\Delta *, D(f,-)))/\sim$$

Here $(\eta : \Delta * \Rightarrow D(f_\eta,-)) \sim (\eta' : \Delta * \Rightarrow D(f_{\eta'},-))$ if and only if there exists an upper bound f of $f_\eta, f_{\eta'}$ (on the left) inducing the commuting square on the right:

$$\begin{tikzcd}
f_\eta \arrow[dr, "\phi_\eta"] & & & & \mathbf{Set}^{\mathbf{L}}(\Delta *, D(f_\eta,-)) \arrow[dr, "D(\phi_\eta,-)_*"] & \\
 & f \arrow[r, mapsto, dashed, "\lim_{l \in \mathbf{L}} D(-,l)"] & * \arrow[urr, "\eta"] \arrow[drr, "\eta'"'] & & & \mathbf{Set}^{\mathbf{L}}(\Delta *, D(f,-)) \\
f_{\eta'} \arrow[ur, "\phi_{\eta'}"'] & & & & \mathbf{Set}^{\mathbf{L}}(\Delta *, D(f_{\eta'},-)) \arrow[ur, "D(\phi_{\eta'},-)_*"'] &
\end{tikzcd} \tag{2.44}$$

We denote the equivalence class by a bar, so that $\bar{\eta} = \bar{\eta'}$ above.

Proof. Let $D : \mathbf{F} \times \mathbf{L} \to \mathbf{Set}$ be a diagram where $\mathbf{F}$ is κ-filtered and $\mathbf{L}$ is κ-small. We have the following morphisms:

$$\begin{aligned}
D(-,-) &\Rightarrow \Delta(\operatorname*{colim}_{f \in \mathbf{F}} D(f,-)) && \text{(colimit)} \\
\lim_{l \in \mathbf{L}} D(-,l) &\Rightarrow \Delta(\lim_{l \in \mathbf{L}} \operatorname*{colim}_{f \in \mathbf{F}} D(f,l)) && \text{(post-compose with lim)} \\
\Phi : \operatorname*{colim}_{f \in \mathbf{F}} \lim_{l \in \mathbf{L}} D(f,l) &\to \lim_{l \in \mathbf{L}} \operatorname*{colim}_{f \in \mathbf{F}} D(f,l) && \text{(colimit)}
\end{aligned}$$

Consider an arbitrary element $\alpha : \Delta * \Rightarrow D(f_\alpha,-) \in \lim_{l \in \mathbf{L}_0} D(-,l)$. Then α defines an equivalence class $\overline{\alpha}$ that we may apply Φ to. Inspecting the construction of Φ reveals that $\Phi(\overline{\alpha}) := (\overline{\alpha_l})_{l \in \mathbf{L}_0}$ independent of choice of representative. Moreover, $\Phi(\overline{\alpha})$ is an element of

$$\lim_{l \in \mathbf{L}} \operatorname*{colim}_{f \in \mathbf{F}} D(f,l) = \mathbf{Set}^{\mathbf{L}}(\Delta *, [\coprod_{f \in \mathbf{F}} D(f,-)]/\sim)$$

where two natural transformations $\beta, \gamma : \Delta * \Rightarrow \operatorname{colim} D(f, -)$ coincide if the pointwise equivalence classes $\beta_l = \gamma_l$ do for all $l \in \mathbf{L}$. This is witnessed by a common representative $r_l \in D(f_l, l)$ for some $f_l \in \mathbf{F}_0$.

We start by proving injectivity of Φ: Suppose $(\overline{\alpha_l}) = (\overline{\beta_l})$ for $\alpha : \Delta * \Rightarrow D(f_\alpha, -)$ and $\beta : \Delta \Rightarrow D(f_\beta, -)$. This means $(\alpha_l \in D(f_\alpha, l)) \sim (\beta_l \in D(f_\beta, l))\ \forall l \in \mathbf{L}_0$. Diagram 2.44 portrays this as:

$$\begin{tikzcd}
f_\alpha \arrow[dr, "\phi_{\alpha_l}"] & & & & D(f_\alpha, l)) \arrow[dr, "D(\phi_{\alpha_l}, l)"] & \\
& f_l \arrow[r, mapsto, dashed, "D(-,l)"] & * \arrow[urr, "\alpha_l"] \arrow[drr, "\beta_l"'] & & & D(f_l, l)) \\
f_\beta \arrow[ur, "\phi_{\beta_l}"] & & & & D(f_\beta, l)) \arrow[ur, "D(\phi_{\beta_l}, l)"] &
\end{tikzcd}$$

Since $\mathbf{F}$ is κ-filtered and $\mathbf{L}$ is κ-small, an upper bound f' of $\{f_l\}_{l \in \mathbf{L}}$ exists; denote $\psi_l : f_l \to f'$ the induced maps. An upper bound f of the following diagram exists:

$$\begin{tikzcd}
f_\alpha \arrow[dr, Rrightarrow, "(\psi_l \phi_{\alpha_l})_{l \in \mathbf{L}}"] & & \\
& f' \arrow[r, dashed, "\psi"] & f \\
f_\beta \arrow[ur, Rrightarrow, "(\psi_l \phi_{\beta_l})_{l \in \mathbf{L}}"'] & &
\end{tikzcd}$$

Denote ϕ_α, ϕ_β the unique induced arrows from f_α and f_β into f respectively, i.e. the compositions in the diagram above. Then $D(\phi_\alpha, l)(\alpha_l) = D(\phi_\beta, l)(\beta_l)\ \forall l \in \mathbf{L}_0$ since both ϕ_α and ϕ_β factor through f_l. But this means $D(\phi_\alpha, -)_*(\alpha) = D(\phi_\beta, -)_*(\beta)$, which is exactly:

$$\begin{tikzcd}
f_\alpha \arrow[dr, "\phi_\alpha"] & & & & \mathbf{Set}^{\mathbf{L}}(\Delta *, D(f_\alpha, -)) \arrow[dr, "D(\phi_\alpha, -)_*"] & \\
& f \arrow[r, mapsto, dashed, "\mathbf{Set}^{\mathbf{L}}(\Delta *, D(-,-))"] & * \arrow[urr, "\alpha"] \arrow[drr, "\beta"'] & & & \mathbf{Set}^{\mathbf{L}}(\Delta *, D(f, -)) \\
f_\beta \arrow[ur, "\phi_\beta"] & & & & \mathbf{Set}^{\mathbf{L}}(\Delta *, D(f_\beta, -)) \arrow[ur, "D(\phi_\beta, -)_*"] &
\end{tikzcd}$$

The diagram asserts that $\alpha \sim \beta$, as desired for injectivity of Φ.

To show surjectivity, let $\gamma \in \lim_{\mathbf{L}} \operatorname{colim}_{\mathbf{F}} D(f, l)$, and pick representatives $c_l \in D(f_l, l)$ of the equivalence classes γ_l. Naturality of γ means that for any $L : l \to l' \in \mathbf{L}$, we have $(*)\ D(f_l, L)(c_l) \sim c_{l'}$. Pick for any $L \in \mathbf{L}_1$ an upper bound $\phi_l^L : f_l \to f_L \leftarrow f_{l'} : \phi_{l'}^L$ of $(*)$ inducing a commutative diagram like 2.44. Again, since $\mathbf{F}$ is κ-filtered we may pick an upper bound f' of $\{f_L\}_{l \in \mathbf{L}_1}$ with maps $\psi_L : f_L \to f'$, and find an upper bound f for the following diagram consisting of all triangles:

$$f_l \xrightarrow{\psi_L \phi_l^L} f' \xleftarrow{\psi_L \phi_{l'}^L} f_{l'} \qquad \forall L : l \to l' \in \mathbf{L}_1$$

Denote $\psi : f' \to f$ and $\phi_l := \phi \psi_L \phi_l^L : f_l \to f$ the induced maps (independent of L). Then $(D(\phi_l, l)(c_l))_{l \in \mathbf{L}_0} : \Delta * \Rightarrow D(f, -)$ is a natural transformation: For any $L : l \to l'$,

$$\begin{tikzcd}
f_l \arrow[dr, "\phi_l^L"'] \arrow[drr, bend left, "\phi_l"] & & & & D(f_l, l') \arrow[d] \arrow[dr, bend left, "D(\phi_l, l')"] & \\
& f_L \arrow[r, "\psi\psi_L"] & f \arrow[r, mapsto, dashed, "D(-,l')"] & * \arrow[ur, "D(f_l, L)(c_l)"] \arrow[dr, "c_{l'}"'] & D(f_L, l') \arrow[r, "D(\psi\psi_L, l')"] & D(f, l') \\
f_{l'} \arrow[ur, "\phi_{l'}^L"] \arrow[urr, bend right, "\phi_{l'}"'] & & & & D(f_{l'}, l') \arrow[u] \arrow[ur, bend right, "D(\phi_{l'}, l')"'] &
\end{tikzcd}$$

The right diagram commuting means exactly that $(D(\phi_l, l)(c_l))_{l\in\mathbf{L}_0}$ forms a natural tranformation. Since $\overline{D(\phi_l, l)(c_l)} \sim c_l$, these are both representatives of γ_l. It is then clear that $\Phi(\overline{(D(\phi_l, l)(c_l))_l}) = \gamma$, meaning Φ is surjective. ☐

Corollary 2.45. *Let $\boldsymbol{C}$ be cocomplete and $c : \boldsymbol{D} \to \boldsymbol{C}$ a κ-small diagram such that $c_d := c(d)$ is κ-compact for all $d \in \boldsymbol{D}_0$. Then $c := \operatorname{colim}_{d\in\boldsymbol{D}_0} c_d$ is also κ-compact.*

Proof. Let $y : \alpha \to \mathbf{C}$ be an α-composite for an ordinal $\alpha > \kappa$. Then:

$$\begin{aligned}
\mathbf{C}(c, y_\alpha) &= \lim_{d\in\mathbf{D}_0} \mathbf{C}(c_d, y_\alpha) && (\mathbf{C}(-, y_\alpha) \text{ continuous})\\
&\cong \lim_{d\in\mathbf{D}_0} \operatorname*{colim}_{\beta<\alpha} \mathbf{C}(c_d, y_\beta) && (c_d \text{ compact})\\
&\cong \operatorname*{colim}_{\beta<\alpha} \lim_{d\in\mathbf{D}_0} \mathbf{C}(c_d, y_\beta) && (\text{Proposition } 2.43)\\
&= \operatorname*{colim}_{\beta<\alpha} \mathbf{C}(c, y_\beta) && (\mathbf{C}(-, y_\beta) \text{ continuous } \forall \beta < \alpha)
\end{aligned}$$ ☐

Definition 2.46. Let $\mathbf{C}$ be a cocomplete category. A set $S \subseteq \mathbf{C}_1$ is *weakly saturated* if it contains all isomorphisms, and is closed under transfinite composition, pushouts and retracts.

For a more detailed definition, see e.g. [Lur09][A.1.2.2]

A general subset $A \subseteq \mathbf{C}_1$ generates a weakly saturated set which we denote $\bar{A}$, given as the intersection of all weakly saturated sets containing A.

Theorem 2.47 ("Small object argument")**.** *Let $\boldsymbol{C}$ be a presentable category. Any set of morphisms $A \subseteq \boldsymbol{C}_1$ induces a functorial factorization $T : \boldsymbol{C}^{\boldsymbol{2}} \to \boldsymbol{C}^{\boldsymbol{3}}$ into $(\bar{A}, rlp(A))$.*

The proof will be faciliated by some remarks and two lemmas. First of all, a "lifting problem" between two morphisms $a, f \in \mathbf{C}_1$ is an element (u_1, u_2) of $\mathbf{C}^\mathbf{2}(a, f)$ displayed on the left:

$$\begin{tikzcd}
c \arrow[r, "u_1"] \arrow[d, "a"'] & y \arrow[d, "f"] \\
d \arrow[r, "u_2"'] \arrow[ur, dashed, "h"] & z
\end{tikzcd}
\qquad
\begin{tikzcd}
 & y' \arrow[d, "h'"] \arrow[dd, bend left, "f''"] \\
c \arrow[ur, "f'"] \arrow[d, "a"'] & y \arrow[d, "f"] \\
d \arrow[r, "u_2"'] \arrow[ur, dashed, "h"] & z
\end{tikzcd}$$

A solution to the lifting problem is an h as on the left above. One class of squares that admit such lifts are the ones where y arises as the pushout of a by some f' such that $u_1 = h'f'$, and f as the unique arrow out of this pushout induced by u_2 and f'' above. Then u_2 above will factor through h, computed as the pushout of f' by a.

When c is small, u_1 will factor through an f' whenever y is given by a transfinite composition. The small object argument consists in constructing such a transfinite composition which also gives rise to the desired pushout squares at each level. We start with two lemmas:

Lemma 2.48. *If $\boldsymbol{C}$ is α-cocomplete for an ordinal α, then so is $\boldsymbol{C}^{\boldsymbol{2}}$.*

Proof. This is a consequence of the well-known fact that colimits in functor categories may be computed pointwise. ☐

Working with diagrams in the functor category $\mathbf{C^2}$ gets unwieldy without good notation. We will write an α-composite in $\mathbf{C^2}$ as $(f_- : X_- \to Y_-) : \alpha \to \mathbf{C^2}$ to be understood as:

$$
\begin{tikzcd}
X_0 \arrow[r, "x_0"] \arrow[d, "f_0"] & X_1 \arrow[r] \arrow[d, "f_1"] & \cdots \arrow[r] \arrow[d] & X_\alpha \arrow[d, "\operatorname*{colim}_{\beta<\alpha} f_\beta = f_\alpha"] \\
Y_0 \arrow[r, "y_0"] & Y_1 \arrow[r] & \cdots \arrow[r] & Y_\alpha
\end{tikzcd}
\tag{2.49}
$$

By Lemma 2.48, this also defines two α-composites $X_-, Y_- : \alpha \to \mathbf{C}$ and a natural transformation $f_- : X_- \Rightarrow Y_-$. Such that $f_\alpha = \mathrm{colim}_{\beta<\alpha} f_\beta$.

Lemma 2.50. *Let $C, D \in \mathbf{C}_0$ be κ-compact objects in a cocomplete category $\mathbf{C}$. Then any morphism $a : C \to D$ is κ-compact in $\mathbf{C^2}$.*

Proof. Consider an α-composite $(f_- : X_- \to Y_-) : \alpha \to \mathbf{C^2}$ for $\alpha > \kappa$ and a morphism $a : C \to D \in \mathbf{C}_1$. We have an injection of $\mathrm{colim}_{\beta<\alpha}\mathbf{C^2}(a, f_\beta)$ into $\mathbf{C^2}(a, f_\alpha)$ that we now explain:

$$
\operatorname*{colim}_{\beta<\alpha} \mathbf{C^2}(a, f_\beta) = [\coprod_{\beta<\alpha} \mathbf{C^2}(a, f_\beta)]/\sim \quad \subseteq \mathbf{C^2}(a, f_\alpha) \tag{2.51}
$$

The injection is given by post-composition with the universal arrow $(i_\beta, j_\beta) : f_\beta \Rightarrow f_\alpha$. This is independent of choice of representative: If $(u_{\beta'}, v_{\beta'}) : a \Rightarrow f_{\beta'}$ is equivalent to f_β, then (supposing $\beta \leqslant \beta'$):

$$
\begin{tikzcd}
C \arrow[r, "u_\beta"] \arrow[d, "a"] \arrow[rr, bend right, "u_{\beta'}", start anchor=south east, end anchor=west, shift right=8] & X_\beta \arrow[rr, "i_\beta"] \arrow[dr, "x_{\beta<\beta'}"] \arrow[dd, "f_\beta"] & & X_\alpha \arrow[dd, "f_\alpha"] \\
& & X_{\beta'} \arrow[ur, "i_{\beta'}"'] \arrow[dd, "f_{\beta'}"] & \\
D \arrow[r, "v_\beta"] \arrow[rr, bend right, "v_{\beta'}", shift right=8] & Y_\beta \arrow[rr, "j_\beta"] \arrow[dr, "y_{\beta<\beta'}"'] & & Y_\alpha \\
& & Y_{\beta'} \arrow[ur, "j_{\beta'}"'] &
\end{tikzcd}
$$

If $\beta = \beta'$ then $f_\beta = f_{\beta'}$ since the only endomorphism of β in α is id_β.

Now we argue surjectivity of (2.51). Let $(u_\alpha, v_\alpha) : a \to f_\alpha$. Since C and D are small, we obtain factorizations u_β and v_β (for big enough β, but smaller than α):

$$
\begin{tikzcd}
u_\alpha : C \arrow[r, "u_\beta"] \arrow[d, "a"] \arrow[dr, phantom, "?"] & X_\beta \arrow[r, "i_\beta"] \arrow[d, "f_\beta"] & X_\alpha \arrow[d, "f_\alpha"] \\
v_\alpha : D \arrow[r, "v_\beta"] & Y_\beta \arrow[r, "j_\beta"] & Y_\alpha
\end{tikzcd}
$$

The question is whether the left square commutes, i.e. if (u_β, v_β) is the desired factorization of (u_α, v_α) in $\mathbf{C^2}$. Now, C is small and Y_α is a comlimit in $\mathbf{C}$. Since both $f_\beta u_\beta$ and $v_\beta a$ post-composed with j_β produces $f_\alpha u_\alpha C \to Y_\alpha$, they are both representatives of the same equivalence class under the isomorphism $\mathrm{colim}_{\beta<\alpha}\mathbf{C}(C, Y_\beta) \cong \mathbf{C}(C, Y_\alpha)$ given by smallness of C. But there is at most one such representative of any equivalence class at any β, so $f_\beta u_\beta = v_\beta a$. Hence the left square commutes and $(i_\beta, j_\beta)(u_\beta, v_\beta) = (u_\alpha, v_\alpha)$, meaning Equation 2.51 is also a surjection. $\square$

Proof of Theorem 2.47. Let $A \subseteq \mathbf{C}_1$ be a set of morphisms, and denote κ a regular cardinal such the domans and codomains of morphisms in A are all κ-compact. For short, we will denote $u := (u^1, u^2)$ for a morphism in $\mathbf{C^2}$. Let $f : X \to Z$ be a morphism in $\mathbf{C}$ that we seek to factorize into $(\bar{A}, \mathrm{rlp}(A))$.

Define $X_0 := X$, $f_0 := f_0$, and for a successor ordinal $\beta+1 \leqslant \kappa$ define $f_{\beta+1} : X_{\beta+1} \to Z$ through the pushout on the left:

$$
\begin{tikzcd}
\coprod\limits_{a\in A, u\in \mathbf{C^2}(a,f_\beta)} \operatorname{cod} a \arrow[r, "\coprod u^1"] \arrow[d, "\coprod a"] & X_\beta \arrow[d, "x_\beta"] \arrow[dr, "f_\beta"] & \\
\coprod\limits_{a\in A, u\in \mathbf{C^2}(a,f_\beta)} \operatorname{dom} a \arrow[r, "y_\beta"] \arrow[rr, bend right, "\coprod u^2"'] & X_{\beta+1} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, dashed, "f_{\beta+1}"] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "x_0"] \arrow[dr, "f"'] & X_1 \arrow[r] \arrow[d, "f_1"] & \cdots \arrow[dl] \arrow[r, "(i_\beta)_{\beta<\kappa}"] & X_\kappa \arrow[dll, "f_\kappa"] \\
& Z & &
\end{tikzcd}
\tag{2.52}
$$

Let $(f_\alpha : X_\alpha \to Z) := \mathrm{colim}_{\beta<\alpha}(f_\beta : X_\beta \to Z)$ be the colimit in $\mathbf{C^2}$ for any limit ordinal $\alpha < \kappa$. This defines a κ-composite whose colimit is $f_\kappa : X_\kappa \to Z$, defining the κ-composition $\hat{f} : X_0 \to X_\kappa$. By construction $f_\kappa i_0 = f$ is a factorization as seen on the right above.

Claim: $i_0 \in \bar{A}$. First we verify that $\coprod_{a\in A} a$ above is a transfinite composition. Since A is a set and $\mathbf{C}$ is locally small, the union $\bigcup_{a\in A} \mathbf{C^2}(a, f_\beta)$ is also a set. We may order this set and for sake of simplicity index it by κ, producing $(a_k)_{k<\kappa}$. Define $z : \kappa \to \mathbf{C}$ on successor ordinals as follows:

$$\cdots z_k := \coprod_{k'<k} \operatorname{cod} a_{k'} \to \coprod_{k'<k+1} \operatorname{dom} a_{k'} := z_{k+1} \cdots$$

Where the morphism displayed is induced by $(a_{k'})_{k'<k+1}$ and identities. For a limit ordinal $K < \kappa$, let $z_K := \mathrm{colim}_{k<K} z_k$ in order to make z a κ-composite. Then we obtain $\coprod_{a\in A} a$ as the induced κ-composition $z_0 \to z_\kappa$.

Since $\bar{A}$ is closed under pushouts, each x_β pictured above (left part of 2.52) is in $\bar{A}$. But this means that the transfinite composition $i_0 : X_0 \to X_\kappa$ is in $\bar{A}$ as well.

Claim: $f_\kappa \in rlp(A)$. Consider a lifting problem $u : a \Rightarrow f_\kappa$ pictured as the outer square on the right below. Since a is small, this factors through an $u_\beta : a \Rightarrow f_\beta$ on the left (where $i^a_\beta : \operatorname{dom} a \hookrightarrow \coprod_{a\in A, u\mathbf{C^2}(a,f_\beta)} \operatorname{dom} a$):

$$
\begin{tikzcd}
\operatorname{cod} a \arrow[rr, "u^1_\beta"] \arrow[dd, "a"'] & & X_\beta \arrow[dl, "x_\beta"] \arrow[dd, "f_\beta"] \\
& X_{\beta+1} \arrow[dr, "f_{\beta+1}"] & \\
\operatorname{dom} a \arrow[ur, "y_\beta i^a_\beta"] \arrow[rr, "u^2 = u^2_\beta"'] & & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
\operatorname{cod} a \arrow[rrr, "u^1"] \arrow[dr, "u^1_\beta"] \arrow[dd, "a"'] & & & X_\kappa \arrow[dd, "f_\kappa"] \\
& X_\beta \arrow[r, "x_\beta"] & X_{\beta+1} \arrow[ur, "i_{\beta+1}"] \arrow[dr, "f_{\beta+1}"] & \\
\operatorname{dom} a \arrow[urr, "y_\beta i^a_\beta"] \arrow[rrr, "u^2"] & & & Z
\end{tikzcd}
$$

The important thing to notice is that the bijection $\mathrm{colim}_{\beta<\kappa}\mathbf{C^2}(a, f_\beta) \cong \mathbf{C^2}(a, f_\kappa)$ leaves the bottom component u^2 untouched, since all the $(i_\beta, j_\beta) : f_\beta \Rightarrow f_\kappa$ are just the identity on the "bottom" ($j_\beta = \mathrm{id}_Z$). The diagram on the left here is obtained from the left of Diagram 2.52 by including a into the coproduct. On the right, we obtain the desired lift as $i_{\beta+1} y_\beta i^a_\beta : \operatorname{dom} a \to X_\kappa$.

Claim: We have a functor $T : \boldsymbol{C^2} \to \boldsymbol{C^3}$. On objects $T(f) = \sigma_{\hat{f}, f_\kappa}$ in the notation introduced before Def. 2.36. For a morphism $u : f \Rightarrow g$ in $\mathbf{C^1}$, we must find a $v : X_\kappa \to Y_\kappa$ making the following commute:

$$\begin{tikzcd}
X \arrow[rrr, "u^1"] \arrow[dr, "\hat{f}"] \arrow[dd, "f"'] & & & Y \arrow[dl, "\hat{g}"'] \arrow[dd, "g"] \\
\sigma_{\hat{f}, f_\kappa} & X_\kappa \arrow[r, dashed, "v"] \arrow[dl, "f_\kappa"] & Y_\kappa \arrow[dr, "g_\kappa"'] & \sigma_{\hat{g}, g_\kappa} \\
Z \arrow[rrr, "u^2"] & & & Z'
\end{tikzcd}$$

We will do this in the simple case of when $\hat{f} \in A$. The general case of $\bar{A}$ follows by extending this argument through the relevant constructions of Definition 2.46.

Finding v can be formulated as the lifting problem $(\hat{g}u^1, u^2 f_\kappa) : \hat{f} \Rightarrow g_\kappa$ in $\mathbf{C^2}$. But this does indeed have a solution when $\hat{f} \in A$ since g_κ was proven to be in $\mathrm{rlp}(A)$. Using universality of X_κ as a colimit, it is straightforward to check functoriality of T. $\square$

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