

**Master's thesis**

**NTNU**  
Norwegian University of Science and Technology  
Faculty of Information Technology and Electrical  
Engineering  
Department of Mathematical Sciences

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# Silting Subcategories and the Transitivity of Iterated Irreducible Silting Mutation

Master's thesis in Mathematical Sciences

Supervisor: Aslak Bakke Buan

June 2019



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## Preface

This thesis is the final product of my attendance in the 45-credits course *MA3911 Masteroppgave i matematiske fag* as part of the two-year Master's Degree Programme in Mathematical Sciences (MSMNFMA) at the Norwegian University of Science and Technology. It represents the accumulation of knowledge from over half a decade of mathematical studies, and at the time of writing, the zenith of my academic career.

A great thanks is extended to my supervisor, Prof. Aslak Bakke Buan for his mathematical insights and his stoicism when faced with the challenges of my pre-graduate mind.

Another great thanks is extended to my co-students, in particular to my fellow algebraists Erlend Due Børve, Didrik Fosse and Johan Lundin, for sharing in the joys and frustrations we have brought upon ourselves in this previous year.

Finally, the most enormous of thanks is reserved for my parents and my sister for allowing me to make my own mistakes and for bringing ice and cheers as to my attempt at forcing the brawl with this mathematical beast into a stumbling tango.

Also, thanks to me, as I am the one who actually did it.

*Audun Tamnes, June 2019*

## Abstract

In this thesis we expand the search for all tilting objects of a triangulated category to the more general silting objects. We show that mutation on these objects preserves the silting property, and that in the case of bounded derived categories of hereditary algebras, iterated irreducible mutation is indeed transitive. This is shown through the theory of exceptional sequences.

We give a first introduction to silting theory, and provide a nontrivial partial ordering on the collection of silting subcategories of any given triangulated category. A more detailed treatment is given to the silting theory in the setting of Krull-Schmidt triangulated categories. In particular, we show bijections between classes of silting objects of these and the silting objects in certain Verdier localizations.

The theory is supplemented throughout by examples from representation theory and Auslander-Reiten theory.

I denne tesen utvider vi vår søken etter tilteobjekter til de mer generelle silteobjektene. Vi viser at mutasjon på silteobjektene bevarer silteegenskapene, og at for bundne deriverte kategorier av hereditære algebraer, så er iterert irreduksibel mutasjon faktisk transitivt. Dette vises ved bruk av teori rundt eksepsjonelle sekvenser.

Det gis her en grunnleggende introduksjon til silteteori, og det introduseres en ikke-triviell partiell ordning på silteunderkategoriene til enhver gitt triangulert kategori. Ekstra plass blir tilsidesatt til å studere silteteori for Krull-Schmidt-triangulerte kategorier. Spesielt viser vi en bijeksjon mellom klasser av silteobjekt for slike kategorier og silteobjektene tilhørende en bestemt Verdierlokalisering av kategorien.

Teorien er supplementert med eksemper hentet fra representasjonsteori og Auslander-Reitenteori.

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# 1 Introduction and Preliminaries

## 1.1 Introduction

Rickard's Morita theorem [15] connects the study of derived equivalences to the study of *tilting* objects, as it asserts a derived equivalence of algebras if one appears as an endomorphism ring of a tilting object in the other. This motivates the search for a process which lets us find said tilting objects, and the natural candidate for such a process is that of *mutation*.

Mutation denotes a process defined on a class of structures in a category. As in biology, mutation indicates a small change in the fundamental building blocks as we pass from one instance to the next. In algebra, it changes the structure in a predetermined fashion such that the result is another instance of the same kind of structure. A question can be asked then of how much of the structure is actually preserved by the mutation, and what properties we may have lost. In [5] is defined mutation of quivers, quivers with potentials, cluster-tilting objects and tilting modules over 3-Calabi-Yau algebras. In this thesis, we consider the mutation of tilting objects in triangulated categories. In a perfect world, there would be an easily available mutation which allowed us to find all the tilting objects of a given triangulated category from a starting object. Attempts at finding a general mutation scheme for this purpose has not yet been entirely successful, as the tilting property has a tendency to get lost as part of the process.

To remedy this, we focus our attention on an article by *Aihara and Iyama* [1], and on their mutation of the more general *silting* objects and -subcategories of triangulated categories. Herein is provided a mutation which always works, i.e. the silting property is preserved by mutation, and the question which remains then is that of *transitivity* – that is, whether or not all silting objects are connected by a sequence of mutations. By introducing a partial ordering on the silting subcategories of a given triangulated category, the transitivity becomes a question of graph connectivity. *Silting reduction* then allows us to find connected components of said graphs by considering bijections of certain sets of silting subcategories to the silting subcategories of certain Verdier localizations. Finally, we present a positive result on the transitivity in the case of bounded derived categories of *piecewise hereditary algebras* by connecting the theory to that of *exceptional sequences*.

The categories of main interest to us are the bounded derived categories of hereditary path algebras. While an attempt has been made to provide general results, we often restrict to Krull-Schmidt triangulated categories, as they are sufficient generalizations of these.

Examples are amply provided, contextualizing the theory through the bounded derived categories of well-understood algebras. For these examples, we assume knowledge of AR-theory as taught by Happel [7].

Most main results, and the main bulk of notation in this thesis are as found in Aihara and Iyama [1]. Where the original article is advanced and to-the-point, a great effort has been put in by the author to make this thesis as detailed and approachable as possible. This includes the frequent introduction of lemmas to provide the necessary theory where [1] demands immediate expertise.



## 1.2 Preliminaries and Conventions

The specific assumptions on  $\mathcal{T}$  will be presented where needed. In the case of uncertainty,  $\mathcal{T}$  will usually denote a triangulated category [11] equipped with the shift functor, denoted by  $[1]$ . We assume that the triangulated categories have split idempotents: For any idempotent  $X \xrightarrow{f=f^2} X$ , there is an object  $Y$  and morphisms  $X \xrightarrow{\pi} Y$  and  $Y \xrightarrow{\iota} X$  such that  $\iota\pi = f$  and  $\pi\iota = 1_Y$ .

For general additive categories, when we say  $\mathcal{D}$  is a subcategory of  $\mathcal{C}$  it is always meant that it is *full* and *closed under isomorphism*.

For an additive category  $\mathcal{C}$  and a collection of objects  $\mathcal{X}$  in  $\mathcal{C}$ , denote by  $\text{smd } \mathcal{X}$  the smallest subcategory of  $\mathcal{C}$  which contains  $\mathcal{X}$  and is closed under taking summands. Furthermore, we denote by  $\text{add } \mathcal{X}$  the smallest subcategory of  $\mathcal{C}$  which contains  $\mathcal{X}$  and is closed under taking summands and finite coproduct.

When we say 'let  $\mathcal{C}$  be a category' or 'let  $\mathcal{C}$  be a subcategory of  $\mathcal{D}$ ', it is implied that  $\mathcal{C} = \text{add } \mathcal{C}$ . Under these conventions  $\text{smd } \mathcal{X}$  is not necessarily a subcategory of  $\mathcal{C}$ , but rather a collection of objects.

A *ring* is in this thesis always considered to be unital. For a ring  $R$ ,  $\text{mod } R$  denotes the category of finitely generated left  $R$ -modules. Furthermore,  $\mathcal{P}(\text{mod } R)$  denotes the subcategory of  $\text{mod } R$  of projective, finitely generated  $R$ -modules.

For  $\mathcal{A}, \mathcal{B} \subseteq \mathbf{Ob} \mathcal{T}$ , we denote by  $\mathcal{A} * \mathcal{B}$  the collection of objects  $X \in \mathcal{T}$  which exist as the central object of some triangle

$$A \longrightarrow X \longrightarrow B \longrightarrow X[1]$$

where  $A \in \mathcal{A}$  and  $B \in \mathcal{B}$ . The collection  $\mathcal{A} * \mathcal{B}$  is called the *extension* of  $\mathcal{A}$  and  $\mathcal{B}$  in  $\mathcal{T}$ .

Following are some basic properties of the extensions in a triangulated category.

**Remark 1.1.** *Let  $\mathcal{T}$  be a triangulated category and  $\mathcal{A}, \mathcal{B} \subseteq \mathbf{Ob} \mathcal{T}$  collections of objects containing 0. Then  $\mathcal{A}$  is contained in both  $\mathcal{A} * \mathcal{B}$  and  $\mathcal{B} * \mathcal{A}$ .*

*Proof.* Let  $A \in \mathcal{A}$ . We have the triangles  $A \longrightarrow A \longrightarrow 0 \longrightarrow A[1]$  and  $0 \longrightarrow A \longrightarrow A \longrightarrow 0$ . Thus as 0 is in both  $\mathcal{A}$  and  $\mathcal{B}$ , the result follows.  $\square$

In particular, this means  $\mathcal{X} \subseteq \mathcal{X} * \mathcal{X}$  and  $\mathcal{X} = \mathcal{X} * 0 = 0 * \mathcal{X}$ .

**Remark 1.2.** *Let  $\mathcal{A}, \mathcal{B}$  be collections of objects in  $\mathcal{T}$ , then*

$$(\mathcal{A} * \mathcal{B})[i] = \mathcal{A}[i] * \mathcal{B}[i].$$

*Proof.* Assume  $X \in \mathcal{A} * \mathcal{B}$ . There is a triangle

$$A \longrightarrow X \longrightarrow B \longrightarrow A[1],$$

and for any  $i \in \mathbb{Z}$  a triangle

$$A[i] \longrightarrow X[i] \longrightarrow B[i] \longrightarrow A[i+1].$$

Hence  $X[i] \in \mathcal{A}[i] * \mathcal{B}[i]$ , so  $(\mathcal{A} * \mathcal{B})[i] \subseteq \mathcal{A}[i] * \mathcal{B}[i]$ . Similarly, if  $Y \in \mathcal{A}[i] * \mathcal{B}[i]$ , there is a triangle

$$A[i] \longrightarrow Y \longrightarrow B[i] \longrightarrow A[i+1],$$

and also

$$A \longrightarrow Y[-i] \longrightarrow B \longrightarrow A[1].$$

Thus  $Y[-i] \in \mathcal{A} * \mathcal{B}$ , and so  $Y \in (\mathcal{A} * \mathcal{B})[i]$ , completing the inclusions.  $\square$

Having three collections of objects,  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$ , we can naturally define  $(\mathcal{A} * \mathcal{B}) * \mathcal{C}$  and  $\mathcal{A} * (\mathcal{B} * \mathcal{C})$ . As the next remark asserts, these two constructions are equal.

**Remark 1.3.** *Let  $\mathcal{A}, \mathcal{B}$  and  $\mathcal{C}$  be collections of objects in the triangulated category  $\mathcal{T}$ . Then*

$$(\mathcal{A} * \mathcal{B}) * \mathcal{C} = \mathcal{A} * (\mathcal{B} * \mathcal{C})$$

and we simply write  $\mathcal{A} * \mathcal{B} * \mathcal{C}$ .

*Proof.* Let  $X \in \mathcal{A} * (\mathcal{B} * \mathcal{C})$ , so there is a triangle

$$A \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} A[1]$$

with  $A \in \mathcal{A}$  and  $Y \in \mathcal{B} * \mathcal{C}$ . Thus there is also a triangle

$$B \xrightarrow{\varphi} Y \xrightarrow{\theta} C \xrightarrow{\sigma} B[1]$$

with  $B \in \mathcal{B}$  and  $C \in \mathcal{C}$ . Furthermore, we complete the morphism  $\theta g$  to a triangle

$$X \xrightarrow{\theta g} C \longrightarrow Z \longrightarrow X[1].$$

By rotating the first two triangles once, we get by the *octahedral axiom* the diagram

$$\begin{array}{ccccccc}
 X & \xrightarrow{g} & Y & \xrightarrow{h} & A[1] & \xrightarrow{-f[1]} & X[1] \\
 & \searrow & \downarrow \theta & & \downarrow & & \\
 & & C & \longrightarrow & Z & \longrightarrow & X[1] \\
 & \searrow \theta g & & & \downarrow & & \\
 & & & & B[1] & & \\
 & & & & \downarrow & & \\
 & & & & A[2] & & Y[1]
 \end{array}$$

The dotted vertical arrows indicate the triangle

$$A[1] \longrightarrow Z \longrightarrow B[1] \longrightarrow A[2]$$

meaning  $Z \in \mathcal{A}[1] * \mathcal{B}[1]$ , so by Remark 1.2  $Z[-1] \in \mathcal{A} * \mathcal{B}$ . By rotating this new triangle once, we arrive at

$$Z[-1] \longrightarrow X \longrightarrow C \longrightarrow Z$$

meaning  $X \in (\mathcal{A} * \mathcal{B}) * \mathcal{C}$ . The other inclusion is shown using an, arguably even simpler, dual argument.  $\square$

**Remark 1.4.** Let  $\mathcal{X}, \mathcal{Y}$  be collections of objects in  $\mathcal{T}$  which are closed under coproduct. Then  $\mathcal{X} * \mathcal{Y}$  is closed under coproduct.

*Proof.* For  $A, B \in \mathcal{X} * \mathcal{Y}$ , we have triangles

$$X \longrightarrow A \longrightarrow Y \longrightarrow X[1]$$

and

$$X' \longrightarrow B \longrightarrow Y' \longrightarrow X'[1].$$

Their term-wise coproduct is the triangle

$$X \oplus X' \longrightarrow A \oplus B \longrightarrow Y \oplus Y' \longrightarrow (X \oplus X')[1]$$

and so  $A \oplus B \in \mathcal{X} * \mathcal{Y}$  as well. □

A subcategory  $\mathcal{S}$  of  $\mathcal{T}$  is said to be *thick* [17] if it is a sub-triangulated category of  $\mathcal{T}$  and closed under taking direct summands. For  $\mathcal{S}$  a collection of objects, *thick*  $\mathcal{S}$  means the smallest sub-triangulated category of  $\mathcal{T}$  containing  $\mathcal{S}$  which is closed under taking direct summands. Similarly to the case with *smd* above, this may lead to some confusion as a sub-triangulated category  $\mathcal{S}$  of  $\mathcal{T}$  is not automatically assumed to be thick. In order to avoid such confusion, an attempt to clarify this will be made wherever necessary.

A category is said to be *skeletally small* if the collection of isomorphism classes of objects forms a set.

The *approximations* defined below will be used extensively throughout this thesis.

**Definition 1.5.** Let  $\mathcal{T}$  be any category,  $\mathcal{S}$  a subcategory of  $\mathcal{T}$  and  $T \in \mathcal{T}$ .

- (i) A morphism  $T \xrightarrow{f} S$  is a *left  $\mathcal{S}$ -approximation* of  $T$  if  $S$  is in  $\mathcal{S}$  and any  $T \xrightarrow{f'} S'$  with  $S'$  in  $\mathcal{S}$  factors through  $f$ . In other words,

$$\mathrm{Hom}_{\mathcal{T}}(S, S') \xrightarrow{- \circ f} \mathrm{Hom}_{\mathcal{T}}(T, S')$$

is surjective for all  $S'$  in  $\mathcal{S}$ .

If any object  $T \in \mathcal{T}$  has a left  $\mathcal{S}$ -approximation,  $\mathcal{S}$  is said to be *covariantly finite* in  $\mathcal{T}$ .

- (ii) A morphism  $S \xrightarrow{f} T$  is a *right  $\mathcal{S}$ -approximation* of  $T$  if  $S$  is in  $\mathcal{S}$ , and any  $S' \xrightarrow{f'} T$  with  $S'$  in  $\mathcal{S}$ , factors through  $f$ . In other words,

$$\mathrm{Hom}_{\mathcal{T}}(S', S) \xrightarrow{f \circ -} \mathrm{Hom}_{\mathcal{T}}(S', T)$$

is surjective for all  $S'$  in  $\mathcal{S}$ .

If any object  $T \in \mathcal{T}$  has a right  $\mathcal{S}$ -approximation,  $\mathcal{S}$  is said to be *contravariantly finite* in  $\mathcal{T}$ .

A subcategory  $\mathcal{S}$  of  $\mathcal{T}$  which is both contravariantly- and covariantly finite, is said to be *functorially finite* in  $\mathcal{T}$ .

Definition 1.5 (i) is visualized by the diagram

$$\begin{array}{ccc} T & \xrightarrow{f} & S \\ & \searrow f' & \downarrow g \\ & & S', \end{array}$$

signaling that for any morphism  $f'$  from  $T$  to an object  $S' \in \mathcal{S}$ , there is some  $S \xrightarrow{g} S'$  such that  $f' = gf$ . A dual diagram illustrates part (ii).

We note that *approximations are additive*.

**Lemma 1.6.** *Let  $\mathcal{T}$  be a triangulated category and  $\mathcal{S} \subseteq \mathcal{T}$  a subcategory.*

(i) *If  $T_1 \xrightarrow{f_1} S_1$  and  $T_2 \xrightarrow{f_2} S_2$  are left  $\mathcal{S}$ -approximations of  $T_1$  and  $T_2$ , then*

$$T_1 \oplus T_2 \xrightarrow{\begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}} S_1 \oplus S_2$$

*is a left  $\mathcal{S}$ -approximation of  $T_1 \oplus T_2$ .*

(ii) *If  $S_1 \xrightarrow{f_1} T_1$  and  $S_2 \xrightarrow{f_2} T_2$  are right  $\mathcal{S}$ -approximations of  $T_1$  and  $T_2$ , then*

$$S_1 \oplus S_2 \xrightarrow{\begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}} T_1 \oplus T_2$$

*is a right  $\mathcal{S}$ -approximation of  $T_1 \oplus T_2$ .*

*Proof.* Only part (i) is proved, as part (ii) is dual. let  $T_1 \oplus T_2 \xrightarrow{f''=(f_1'' \ f_2'')} S''$  be any morphism. Then there are morphisms  $S_i \xrightarrow{g_i} S''$  such that  $g_i f_i = f_i''$ , so by setting  $g := (g_1 \ g_2)$ , we have the commutative diagram

$$\begin{array}{ccc} T_1 \oplus T_2 & \xrightarrow{\begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}} & S_1 \oplus S_2 \\ & \searrow (f_1'' \ f_2'') & \downarrow (g_1 \ g_2) \\ & & S'' \end{array}$$

showing the top row is a left  $\mathcal{S}$ -approximation

□

The *image* of a functor  $\mathcal{C} \xrightarrow{F} \mathcal{D}$  is expressed by  $F\mathcal{C}$ . This will be used with both Hom- and shift functors. For a collection of objects  $\mathcal{X}$  in  $\mathcal{C}$ ,  $F\mathcal{X}$  will denote the collection of objects in  $\mathcal{D}$  obtained by applying  $F$  to all objects in  $\mathcal{X}$ .

Having established these basic notions, we combine them all to the following fact.

**Lemma 1.7.** *Let  $\mathcal{T}$  be a triangulated category and  $\mathcal{X}, \mathcal{Y}$  subcategories such that  $\text{Hom}_{\mathcal{T}}(\mathcal{X}, \mathcal{Y}) = 0$ . Then for any triangle*

$$X \xrightarrow{a} T \xrightarrow{b} Y \longrightarrow X[1]$$

*with  $X \in \mathcal{X}$  and  $Y \in \mathcal{Y}$ , we have that  $a$  is a right  $\mathcal{X}$ -approximation and  $b$  a left  $\mathcal{Y}$ -approximation of  $T$ .*

*Proof.* For some  $X' \in \mathcal{X}$ , apply  $\text{Hom}_{\mathcal{T}}(X', -)$  to the triangle to get the long exact sequence

$$\cdots \longrightarrow \text{Hom}_{\mathcal{T}}(X', X) \xrightarrow{a \circ -} \text{Hom}_{\mathcal{T}}(X', T) \longrightarrow \text{Hom}_{\mathcal{T}}(X', Y) \longrightarrow \cdots .$$

The right side vanishes by assumption, so the left morphism is surjective, and  $a$  is a right  $\mathcal{X}$ -approximation as asserted.

The second part is dual, and it is shown by applying  $\text{Hom}_{\mathcal{T}}(-, Y')$  to the triangle for any  $Y' \in \mathcal{Y}$ .  $\square$

We say that a morphism  $X \xrightarrow{f} Y$  is *right minimal* if any epimorphism  $X \xrightarrow{g} X$  such that  $fg = f$  is an automorphism. Dually,  $f$  is *left minimal* if any epimorphism  $Y \xrightarrow{h} Y$  such that  $hf = f$  is an automorphism.

Having several approximations of the same object, we see that the approximating objects are related as follows:

**Lemma 1.8.** *Let  $\mathcal{T}$  be a triangulated category,  $\mathcal{S}$  a subcategory. For some  $T \in \mathcal{T}$ , consider the triangles*

$$X \xrightarrow{f} T \xrightarrow{g} Y \longrightarrow S[1]$$

and

$$X' \xrightarrow{f'} T \xrightarrow{g'} Y' \longrightarrow S'[1].$$

- (i) *If  $f$  and  $f'$  are right  $\mathcal{S}$ -approximations of  $T$  and  $f'$  is right minimal, then  $X'$  is a direct summand of  $X$  and  $Y'$  is a direct summand of  $Y$ .*
- (ii) *If  $g$  and  $g'$  are left  $\mathcal{S}$ -approximations of  $T$  and  $g'$  is left minimal, then  $X'$  is a direct summand of  $X$  and  $Y'$  is a direct summand of  $Y$ .*

*Proof.* We prove part (i) only, as part (ii) is dual.

We obtain the solid parts of the commutative diagram

$$\begin{array}{ccccccc} X' & \xrightarrow{f'} & T & \xrightarrow{g'} & Y' & \longrightarrow & S'[1] \\ \downarrow h & & \parallel & & \downarrow \gamma & & \downarrow \\ X & \xrightarrow{f} & T & \xrightarrow{g} & Y & \longrightarrow & S[1] \\ \downarrow h' & & \parallel & & \downarrow \gamma' & & \downarrow \\ X' & \xrightarrow{f'} & T & \xrightarrow{g'} & Y' & \longrightarrow & S'[1]. \end{array}$$

As  $f$  and  $f'$  are right  $\mathcal{S}$ -approximations of  $T$ , they factor through each other by some  $X' \xrightarrow{h} X$  and  $X \xrightarrow{h'} X'$ . Then we can complete the diagram to a composition of triangle morphisms. As  $f'h'h = f$ ,  $h'h$  is an isomorphism, and so  $\gamma'\gamma$  is as well. This means that  $h$  and  $\gamma$  are split monomorphisms, which proves the assertion.  $\square$

We will often use the shift functor in conjunction with inequalities. As an example,  $X[> n]$  means any shift  $X[m]$  of  $X$  with  $m > n$ , while  $X[\leq n]$  means any shift with  $m \leq n$ . This is also combined with the notation above typically as  $\mathcal{M}[> n]$ , meaning  $M[m]$  for any object  $M \in \mathcal{M}$  and any  $m > 0$ .

Given a collection of objects  $\mathcal{X}$ , we sometimes want to consider the objects in  $\mathcal{T}$  whose Hom-sets with  $\mathcal{X}$  are 0.

**Definition 1.9.** For  $\mathcal{T}$  a category and  $\mathcal{X}$  a collection of objects in  $\mathcal{T}$ , denote by

- (i)  $\mathcal{X}^{\perp\tau} := \{T \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(\mathcal{X}, T) = 0\}$ , and
- (ii)  ${}^{\perp\tau}\mathcal{X} := \{T \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(T, \mathcal{X}) = 0\}$ ,

and call them the *right-* and *left orthogonal complements* of  $\mathcal{X}$ , respectively.

When there is no ambiguity concerning which is the ambient category  $\mathcal{T}$ , the notation is simplified to  $\mathcal{X}^{\perp}$  and  ${}^{\perp}\mathcal{X}$ . Immediately, we observe that any object  $X$  existing in both  $\mathcal{X}$  and an orthogonal complement is 0. This as  $\text{Hom}_{\mathcal{T}}(X, X) = 0$ . Furthermore, we observe the following property about the orthogonal complements

**Lemma 1.10.** *Let  $\mathcal{T}$  be a triangulated category and  $\mathcal{X}, \mathcal{Y}$  be collections of objects in  $\mathcal{T}$  such that  $\mathcal{Y} \subseteq \mathcal{X}$ . Then*

$${}^{\perp}\mathcal{X} \subseteq {}^{\perp}\mathcal{Y}.$$

*Proof.* Let  $T \in {}^{\perp}\mathcal{X}$ . Then  $\text{Hom}_{\mathcal{T}}(T, X) = 0$  for all  $X \in \mathcal{X}$ . By assumption any  $Y \in \mathcal{Y}$  is in  $\mathcal{X}$  as well, so  $\text{Hom}_{\mathcal{T}}(T, Y) = 0$ . Thus  $T \in {}^{\perp}\mathcal{Y}$ .  $\square$

Finally, ordered pairs of triangulated categories are given names based on properties between the two, and shared properties of the pair in relation to the triangulated category as a whole.

**Definition 1.11.** Let  $\mathcal{T}$  be a triangulated category and  $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{T}$  subcategories.

- (i) The ordered pair  $(\mathcal{X}, \mathcal{Y})$  is said to be a *torsion pair* in  $\mathcal{T}$  if  $\text{Hom}_{\mathcal{T}}(\mathcal{X}, \mathcal{Y}) = 0$  and  $\mathcal{T} = \mathcal{X} * \mathcal{Y}$ .
- (ii) The ordered pair  $(\mathcal{X}, \mathcal{Y})$  is a *t-structure* if  $(\mathcal{X}[1], \mathcal{Y})$  is a torsion pair and  $\mathcal{X}[1] \subseteq \mathcal{X}$ .
- (ii+) If in addition  $\mathcal{X}[1] = \mathcal{X}$ , we say it is a *stable t-structure*.
- (iii) The ordered pair  $(\mathcal{X}, \mathcal{Y})$  is a *co-t-structure* if  $(\mathcal{X}[-1], \mathcal{Y})$  is a torsion pair and  $\mathcal{X}[-1] \subseteq \mathcal{X}$ .

If  $(\mathcal{X}, \mathcal{Y})$  is a t-structure, we call  $\mathcal{X} \cap \mathcal{Y}$  its *heart*, and if it is a co-t-structure,  $\mathcal{X} \cap \mathcal{Y}$  is called its *coheart*.

Immediately, we see that the torsion pairs give rise to covariantly- and contravariantly finite subcategories.

**Corollary 1.12.** *Let  $\mathcal{T}$  be any triangulated category with a torsion pair  $(\mathcal{X}, \mathcal{Y})$ . Then  $\mathcal{X}$  is a contravariantly finite subcategory of  $\mathcal{T}$ , and  $\mathcal{Y}$  is a covariantly finite subcategory of  $\mathcal{T}$ . Left  $\mathcal{X}$ -approximations and right  $\mathcal{Y}$ -approximations of the objects in  $\mathcal{T}$  are given by the triangles which exist from  $\mathcal{T} = \mathcal{X} * \mathcal{Y}$ .*

*Proof.* Let  $T \in \mathcal{T}$ . We have a triangle

$$X \xrightarrow{u} T \xrightarrow{v} Y \longrightarrow X[1]$$

as per  $\mathcal{T} = \mathcal{X} * \mathcal{Y}$ . Then by Lemma 1.7,  $u$  is a right  $\mathcal{X}$ -approximation of  $T$ , and  $v$  is a left  $\mathcal{Y}$ -approximation of  $T$ .  $\square$

**Example 1.13.** The prototypical torsion pairs are those made from 'left complexes' and 'right complexes' in a derived category [13]. By cutting off the homologies at the right places, we easily see that the Hom-sets become 0, and what remains is to show the extension of the two is the whole category.

Let  $k$  be a field,  $\Lambda$  some  $k$ -algebra of finite global dimension, and

$$\mathcal{T} := D^b(\text{mod } \Lambda) \simeq K^b(\mathcal{P}(\text{mod } \Lambda))$$

We set  $\mathcal{X}$  to be the subcategory of complexes

$$\mathcal{X} := \{X \in \mathcal{T} \mid H^n(X) = 0 \text{ for all } n \geq 0\},$$

and set  $\mathcal{Y}$  to be

$$\mathcal{Y} := \{Y \in \mathcal{T} \mid H^n(Y) = 0 \text{ for all } n < 0\}.$$

For any  $T \in \mathcal{T}$ , it  $T$  is a complex

$$T = \dots \longrightarrow T^{-1} \xrightarrow{t^{-1}} T^0 \xrightarrow{t^0} T^1 \xrightarrow{t^1} \dots .$$

From this we define  $X$  and  $Y$  be

$$X = \dots \longrightarrow T^{-1} \xrightarrow{t^{-1}} \ker t^0 \longrightarrow 0 \longrightarrow \dots \in \mathcal{X}$$

and

$$Y = \dots \longrightarrow 0 \longrightarrow T^0 / \ker t^0 \xrightarrow{\bar{t}^0} T^1 \xrightarrow{t^1} \dots \in \mathcal{Y},$$

where  $T^0 / \ker t^0 \xrightarrow{\bar{t}^0} T^1$  is the morphism  $x + \ker t^0 \mapsto t^0(x)$ . Then we have

$$\begin{array}{ccccccc} Y[-1] & \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & T^0 / \ker t^0 & \xrightarrow{-\bar{t}^0} & \dots \\ \downarrow f & & & \downarrow & & \downarrow & & \downarrow & & \\ X & \dots & \longrightarrow & T^{-1} & \xrightarrow{t^{-1}} & \ker t^0 & \longrightarrow & 0 & \longrightarrow & \dots \\ \downarrow & & & \downarrow 1 & & \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & & \downarrow & & \\ \text{cone}(f) & \dots & \longrightarrow & T^{-1} & \xrightarrow{\begin{pmatrix} t^{-1} \\ 0 \end{pmatrix}} & \ker t^0 \oplus T^0 / \ker t^0 & \xrightarrow{\begin{pmatrix} 0 & \bar{t}^0 \end{pmatrix}} & T^1 & \xrightarrow{t^1} & \dots \end{array}$$

Since  $T^0 \cong \ker t^0 \oplus T^0 / \ker t^0$ , we get that  $\text{cone}(f) \cong T$ , and there is a triangle

$$X \longrightarrow T \longrightarrow Y \longrightarrow X[1], \text{ which means } (\mathcal{X}, \mathcal{Y}) \text{ is a torsion pair in } \mathcal{T}.$$

The pair  $(\mathcal{X}[-1], \mathcal{Y})$  does in addition form a t-structure on  $\mathcal{T}$ . Indeed,  $(\mathcal{X}[-1][1], \mathcal{Y}) = (\mathcal{X}, \mathcal{Y})$  is a torsion pair, and  $\mathcal{X}$  also satisfies  $\mathcal{X}[-1][1] = \mathcal{X} \subseteq \mathcal{X}[-1]$ . In this case, the *heart* of our t-structure,  $\mathcal{X}[-1] \cap \mathcal{Y}$ , is the category of stalk complexes concentrated in degree 0 – which again is equivalent to  $\text{mod } \Lambda$ . Such a t-structure is sometimes denoted by  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  or  $(D^{\leq 0}, D^{\geq 0})$ , the  $D$  referring to  $\mathcal{T}$  being a derived category.

## 2 Silting- and Tilting Subcategories of Triangulated Categories

The main structures to be studied in this thesis are the *silting subcategories* of triangulated categories. In this section we provide a definition and basic properties of silting subcategories as by Aihara and Iyama [1]. Also included is a short sidestep where we view the silting subcategories in connection to  $l$ -Calabi-Yau categories.

It is shown that the partial ordering on  $\text{silt } \mathcal{T}$  given by inclusion is a poor choice. We follow Aihara and Iyama in their generalization of the partial ordering on tilting objects by Riedtmann-Schofield [23] and Happel-Unger [10], as they introduce one based on the disappearance of Hom-sets. As part of proving our notion is actually a partial ordering, we introduce the subcategories  $\mathcal{T}_{\mathcal{M}}^{\leq 0}$  for silting subcategories  $\mathcal{M}$ . These will later be used to produce  $t$ -structures, and to help bridge the gap to abelian categories by introducing an analogue to projective resolutions.

The results provided in this section are mainly introductory. The main and final result of this section is however of interest in and of itself, as it asserts that the existence of one silting object restricts *all* silting subcategories to be the additive closures of objects.

### 2.1 Definitions, Properties, and -Examples

The basic definitions of *silting subcategories* and *silting objects* are the following. The generalization from tilting subcategories and objects is immediate from this.

**Definition 2.1.** Let  $\mathcal{T}$  be a triangulated category and  $\mathcal{M} \subseteq \mathcal{T}$  a subcategory.

- (i) we say  $\mathcal{M}$  is a *silting subcategory* of  $\mathcal{T}$  if  $\text{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{M}[> 0]) = 0$ , and  $\text{thick } \mathcal{M} = \mathcal{T}$ . The collection of all silting subcategories of  $\mathcal{T}$  is denoted by  $\text{silt } \mathcal{T}$ .
- (ii)  $\mathcal{M}$  is *tilting* if it is silting and  $\text{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{M}[< 0]) = 0$ .
- (iii) An object  $M \in \mathcal{T}$  is called a *silting object* if  $\text{add}\{M\}$  is a silting subcategory of  $\mathcal{T}$ . Similarly for tilting.

The property  $\text{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{M}[> 0]) = 0$  in (i) is often referred to as  $\mathcal{M}$  being *pre-silting*. Similarly, we can also say  $\mathcal{M}$  is *pre-tilting* if  $\text{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{M}[\neq 0]) = 0$ .

The first proposition of this section illustrates a close relationship between silting theory and representation theory.

**Proposition 2.2.** *Let  $A$  be a finite dimensional  $k$ -algebra and let  $\mathcal{T} = \text{K}^b(\mathcal{P}(\text{mod } A))$ . Then the stalk complex  $A$  is a tilting object in  $\mathcal{T}$ .*

*Proof.* Let  $\mathcal{M} := \text{add}\{A\} \subseteq \mathcal{T}$ , and  $M, M' \in \mathcal{M}$ . Then clearly, for any  $n \neq 0$ , we have  $\text{Hom}_{\mathcal{T}}(M, M'[n]) = 0$ , as these are stalk complexes of projectives which do not line up. Let  $X \in \mathcal{T}$ , so there are finitely generated projective  $A$ -modules  $P_1, \dots, P_n$  such that, up to shift

$$X = \dots \longrightarrow 0 \longrightarrow P_n \xrightarrow{p_n} \dots \longrightarrow P_1 \xrightarrow{p_1} P_0 \longrightarrow 0 \longrightarrow \dots .$$

As  $A$  is a finite dimensional  $k$ -algebra, the finitely generated projective  $A$ -modules are exactly the direct sums of direct summands of  $A$ , so the stalk complexes of the  $P_i$  are in  $\mathcal{M}$ . Consider the triangle  $P_1 \xrightarrow{p_1} P_0 \longrightarrow \text{cone}(p_1) \longrightarrow P_1[1]$ . As  $P_0$  and  $P_1[1]$  are in  $\text{thick } \mathcal{M}$ , so is  $\text{cone}(p_1)$ . By definition of the cone, it is

$$\dots \longrightarrow 0 \longrightarrow P_1 \xrightarrow{p_1} P_0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \dots .$$



It is clear from this that the morphism

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & P_2 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & & & \downarrow p_2 & & & & & & & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & P_1 & \xrightarrow{p_1} & P_0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

has the cone

$$\cdots \longrightarrow P_2 \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

which is also in thick  $\mathcal{M}$ , and that we will obtain  $X$  from  $n$  iterations of this process.

Thus  $\mathcal{T} = \text{thick } \mathcal{M}$ , and  $\mathcal{M}$  is tilting. □

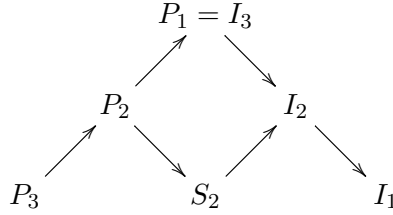
**Example 2.3.** Let  $k$  be a field and  $A_3$  the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3.$$

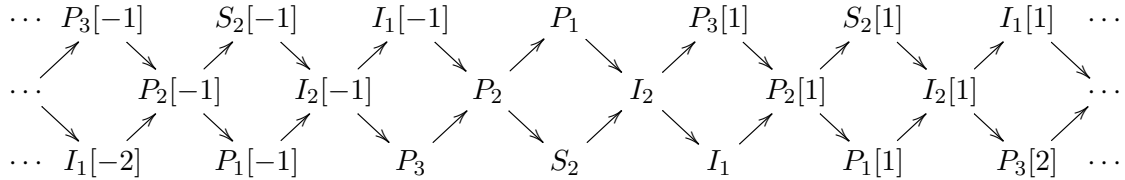
Let  $\Lambda$  denote the path algebra  $kA_3$ . The indecomposable objects in  $\text{mod } \Lambda$  are, up to isomorphism, given [2] by the representations

$$\begin{array}{ll} S_3 = P_3 = 0 \longrightarrow 0 \longrightarrow k, & I_2 = k \xrightarrow{1} k \longrightarrow 0, \\ P_2 = 0 \longrightarrow k \xrightarrow{1} k, & S_1 = I_1 = k \longrightarrow 0 \longrightarrow 0, \\ I_3 = P_1 = k \xrightarrow{1} k \xrightarrow{1} k, & S_2 = 0 \longrightarrow k \longrightarrow 0, \end{array}$$

and by the Krull-Schmidt theorem [2], all finitely generated (left)  $\Lambda$  modules are finite coproducts of these. From this, we obtain from Happel [7] the AR-quiver



in  $\text{mod } \Lambda$ . For the derived category  $\mathcal{T} := D^b(\text{mod } \Lambda) \simeq K^b(\mathcal{P}(\text{mod } \Lambda))$ , the indecomposable objects are exactly the stalks of indecomposable objects in  $\text{mod } \Lambda$  [7], and the objects in  $\mathcal{T}$  are finite coproducts of these indecomposables. Again, we obtain from Happel's construction the AR-quiver



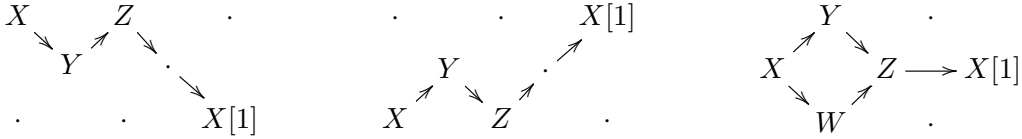
of  $\mathcal{T}$ .

For completeness, note that the arrows in the AR-quiver do not themselves represent morphisms. The *number* of arrows  $X \rightarrow Y$ , however, denotes the *dimension* of the subspace of  $\text{Hom}_{\mathcal{T}}(X, Y)$

consisting of *irreducible* morphisms from  $X$  to  $Y$ . Thus when we say that we 'read a triangle from the AR-quiver', we mean that it is known from AR-theory that there exists such a triangle.

In this particular instance, the Hom-spaces indicated by the arrows are also easily seen to be one-dimensional. Thus there is only one choice of morphism (up to multiplication by unit in  $k$ ) wherever there is an arrow in the AR-quiver, and we find triangles simply by taking cones of these.

The AR-triangles of  $K^b(\mathcal{P}(\text{mod } \Lambda))$  are read from this diagram by the subgraphs



as  $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$  and  $X \longrightarrow Y \oplus W \longrightarrow Z \longrightarrow X[1]$ , respectively. The right-most kind of triangle has morphisms chosen so that the composition along the top of the diamond shape is the negative of the composition along the bottom.

Consider the object  $M := P_1 \oplus P_2 \oplus P_3$ , and the category  $\mathcal{M} := \text{add}\{M\}$ . We see that any morphism from  $\mathcal{M}$  to  $\mathcal{M}[1]$  has to pass through  $I_2$ , and by the AR-triangle

$$P_1 \longrightarrow I_2 \longrightarrow P_3[1] \longrightarrow P_1[1]$$

and the commutativity of the diagram, we have that  $\text{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{M}[> 0]) = 0$ . Also,  $P_3 \in \text{thick } \mathcal{M}$ , and so we also see by the triangle above that  $I_2 \in \text{thick } \mathcal{M}$  as well. Then, by the triangles

$$P_2 \longrightarrow P_1 \oplus S_2 \longrightarrow I_2 \longrightarrow P_2[1]$$

and

$$I_2 \longrightarrow P_3[1] \oplus I_1 \longrightarrow P_2[1] \longrightarrow I_2[1]$$

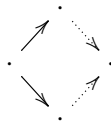
we get that  $S_2$  and  $I_1$  are in  $\text{thick } \mathcal{M}$  as well. It follows from  $\text{thick } \mathcal{M}$  being closed under shift that it is all of  $\mathcal{T}$ . Thus  $M$  is a silting object in  $\mathcal{T}$ . It is easily seen that  $M$  is in fact also a tilting object in  $\mathcal{T}$ . The object  $M' := P_1[1] \oplus P_2 \oplus P_3$  is similarly a silting object of  $\mathcal{T}$ , but is *not* tilting, as

$$0 \neq \text{Hom}_{\mathcal{T}}(P_2, P_1) \subseteq \text{Hom}_{\mathcal{T}}(\mathcal{M}', \mathcal{M}'[< 0]).$$

Similarly, we see that any object 'on a diagonal'



is a silting object in  $\mathcal{T}$ . There are several other silting objects in  $\mathcal{T}$ , such as the opposite diagonals, and the 'wedges'



**Example 2.4.** Let again  $k$  be a field. Denote now by  $\Lambda$  the quotient algebra

$$\Lambda = kA_3/\langle\beta\alpha\rangle$$

being the path algebra of  $A_3$  with relation. The indecomposable  $\Lambda$ -modules are given by the representations

$$\begin{aligned} S_3 = P_3 = 0 &\longrightarrow 0 \longrightarrow k, & I_1 = S_1 = k &\longrightarrow 0 \longrightarrow 0, \\ I_3 = P_2 = 0 &\longrightarrow k \xrightarrow{1} k, & S_2 = 0 &\longrightarrow k \longrightarrow 0, \\ I_2 = P_1 = k &\xrightarrow{1} k \longrightarrow 0, \end{aligned}$$

and the AR-quiver of  $\text{mod } \Lambda$  is

$$\begin{array}{ccccc} & & P_2 = I_3 & & I_2 = P_1 & & \\ & & \nearrow & & \nearrow & & \\ P_3 & & & & S_2 & & I_1. \\ & & \searrow & & \searrow & & \end{array}$$

The module  $I_1 = S_1$  has the projective resolution

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & P_3 & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & & & & & & & \downarrow & & & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & I_1 & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

and so  $\text{mod } \Lambda$  is not hereditary, but has global dimension 2. The stalk complexes of objects have endomorphism rings which are isomorphic to the endomorphism rings of the objects themselves. As the endomorphism rings of the indecomposable objects in  $\Lambda$  are one-dimensional vector spaces, it follows that the stalk complexes cannot have any proper direct summands. I.e. the stalk complexes of indecomposable  $\Lambda$ -modules are still indecomposable in  $K^b(\mathcal{P}(\text{mod } \Lambda))$ . Additionally, we similarly see that so are the shifts of the complex

$$M = \cdots \longrightarrow 0 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots .$$

It is straight forward to see that there is an isomorphism

$$kA_3/\langle\beta\alpha\rangle \cong \text{End}_{\text{mod } kA_3}(P_3 \oplus P_1 \oplus I_1).$$

Then by Rickard's Morita theorem for derived categories [15], there is a triangle equivalence

$$K^b(\mathcal{P}(\text{mod } \Lambda)) \simeq K^b(\mathcal{P}(\text{mod } kA_3)).$$

It follows that the complexes mentioned above are exactly the indecomposables of  $K^b(\mathcal{P}(\text{mod } \Lambda))$ . We then follow Happel's construction, and arrive at the AR-quiver

$$\begin{array}{ccccccccccccccc} \cdots & & P_3 & & I_1[-1] & & P_1 & & P_2[1] & & P_3[2] & & I_1[1] & & P_1[2] & & \cdots \\ \cdots & \nearrow & & \searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow & \searrow & \cdots \\ \cdots & & M[-1] & & S_2 & & M & & S_2[1] & & M[1] & & S_2[2] & & \cdots \\ \cdots & \nearrow & & \searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow & \searrow & \cdots \\ \cdots & & P_1[-1] & & P_2 & & P_3[1] & & I_1 & & P_1[1] & & P_2[2] & & P_3[3] & & \cdots \end{array}$$

Again, the subcategory  $\text{add}\{P_1 \oplus P_2 \oplus P_3\}$  is a silting subcategory, and also, so are the 'diagonals' and the 'wedges' as illustrated in Example 2.3.

We know the module categories of  $kA_3$  and  $kA_3/\langle\beta\alpha\rangle$  are non-equivalent – one being hereditary and the other not. As their derived categories are triangle equivalent, their silting objects appear as the same kinds configurations in the AR-quivers. Going forward, these derived categories will be the main source of examples. When appropriate, silting objects will simply be referred to as their configurations within the AR-quivers, and the AR-quiver simply by its general shape.

Our theory is heavily dependent on left- and right  $\mathcal{D}$ -approximations. This following lemma is used extensively throughout the theory, as it allows us to more easily locate approximations within triangles.

**Lemma 2.5.** *Let  $\mathcal{T}$  be a triangulated category,  $\mathcal{M} \in \text{silt } \mathcal{T}$ , and  $\mathcal{D}$  a subcategory of  $\mathcal{M}$ . Consider the triangle*

$$X \xrightarrow{f} D \xrightarrow{g} Y \xrightarrow{h} X[1],$$

where  $D \in \mathcal{D}$ .

- (i) *If  $X \in \mathcal{M}$ , then  $g$  is a right  $\mathcal{D}$ -approximation of  $Y$ .*
- (ii) *If  $Y \in \mathcal{M}$ , then  $f$  is a left  $\mathcal{D}$ -approximation of  $X$ .*

*Especially, if  $X \in \mathcal{M} \in \text{silt } \mathcal{T}$  and  $Y \in \mathcal{N} \in \text{silt } \mathcal{T}$ , and  $\mathcal{D}$  is a subcategory of both  $\mathcal{M}$  and  $\mathcal{N}$ , then both  $f$  and  $g$  are  $\mathcal{D}$ -approximations.*

*Proof.* If  $X \in \mathcal{M}$  then for any morphism  $D' \xrightarrow{g'} Y$  with  $D' \in \mathcal{D}$ ,  $hg' = 0$ , so  $g'$  factors through  $g$ . Similarly, if  $Y \in \mathcal{M}$ , then for any morphism  $X \xrightarrow{f'} D'$  with  $D' \in \mathcal{D}$ ,  $f'(-h[-1]) = 0$ , so  $f'$  factors through  $f$ .  $\square$

**Example 2.6.** Let  $k$  be a field and  $\Lambda$  the path algebra  $kA_3$ . As in Example 2.3, we consider the category  $\mathcal{T} := \mathbf{K}^b(\mathcal{P}(\text{mod } \Lambda))$ . Let  $\mathcal{M}'$  be the silting subcategory  $\text{add}\{P_1 \oplus P_2 \oplus S_2\}$ , and  $\mathcal{D}$  the subcategory  $\text{add}\{P_1 \oplus P_2\}$ .

From the AR-quiver, we read that there is a triangle

$$P_3 \longrightarrow P_2 \longrightarrow S_2 \longrightarrow P_3[1],$$

and we have  $P_2 \in \mathcal{D}$  and  $S_2 \in \mathcal{M}$ . There is only one morphism  $P_3 \rightarrow P_2$  up to a unit in  $k$ . By Lemma 2.5 (ii), it is then a left  $\mathcal{D}$ -approximation of  $P_3$ .

Similarly, if  $\mathcal{M}$  is the silting subcategory  $\text{add}\{P_1 \oplus P_2 \oplus P_3\}$  and  $\mathcal{D}$  is as before, we get that the morphism  $P_2 \rightarrow S_2$  is a right  $\mathcal{D}$ -approximation of  $S_2$ .

From this, we see that both  $\mathcal{M}$  and  $\mathcal{M}'$  are silting subcategories of  $\mathcal{T}$  having  $\mathcal{D}$  as a subcategory. The distinction between  $\mathcal{M}$  and  $\mathcal{M}'$  is the interchange of the indecomposable objects  $P_3$  and  $S_2$ . These objects are related by a triangle which has as its respective morphisms  $\mathcal{D}$ -approximations of  $P_3$  and  $S_2$ .

This example very much hints at what is to come in Section 4.1, where we generate new silting subcategories exactly by interchanging objects related by such triangles.

Aihara and Iyama provide the following interesting consequence of a triangulated category having silting subcategories. It asserts a certain 'closeness' property on the objects in  $\mathcal{T}$  meaning any two objects can be shifted sufficiently 'far away' from each other such that the morphisms between them vanish.

**Proposition 2.7.** *Let  $\mathcal{T}$  be a triangulated category with a silting subcategory. Then for any objects  $X$  and  $Y$  in  $\mathcal{T}$  there exists an  $n_{XY} \in \mathbb{Z}$  such that  $\mathrm{Hom}_{\mathcal{T}}(X, Y[n]) = 0$  for all  $n \geq n_{XY}$ .*

This property is sometimes denoted simply by  $\mathrm{Hom}_{\mathcal{T}}(X, Y[\geq n_{XY}]) = 0$  or  $\mathrm{Hom}_{\mathcal{T}}(X, Y[>> 0]) = 0$ . Note that this is equivalent to the similarly defined property  $\mathrm{Hom}_{\mathcal{T}}(X[<< 0], Y) = 0$ .

*Proof.* Let  $\mathcal{M} \in \mathrm{silt} \mathcal{T}$ , and consider the subcategory

$$\mathcal{U} := \{U \in \mathcal{T} \mid \forall M \in \mathcal{M} \exists n_{UM} \in \mathbb{Z} \text{ such that } \mathrm{Hom}_{\mathcal{T}}(U, M[\geq n_{UM}]) = 0\}$$

of  $\mathcal{T}$ . For clarity,  $n_{UM}$  is dependent on both  $U$  and  $M$ . This does not necessarily mean that for any  $U \in \mathcal{U}$  there is an  $n_U$  such that  $\mathrm{Hom}_{\mathcal{T}}(U, M[\geq n_U]) = 0$  for all  $M \in \mathcal{M}$ .

We first show that  $\mathcal{U}$  is all of  $\mathcal{T}$ . This by proving it forms a thick subcategory of  $\mathcal{T}$  containing  $\mathcal{M}$ . Using Neeman's definition [21] of a sub-triangulated category, this amounts to proving  $\mathcal{U}$  is an additive subcategory of  $\mathcal{T}$  which is closed under isomorphism, shift, extension and taking summand.

As it is a full subcategory, the Hom-sets and composition of morphisms of  $\mathcal{U}$  are as in  $\mathcal{T}$ . Thus the Hom-sets are abelian groups and composition is bilinear. Furthermore  $0 \in \mathcal{U}$ , and if  $n \in \mathbb{Z}$ ,  $M \in \mathcal{M}$  and  $U, U' \in \mathcal{U}$ :

$$\mathrm{Hom}_{\mathcal{T}}(U \oplus U', M[n]) \cong \mathrm{Hom}_{\mathcal{T}}(U, M[n]) \oplus \mathrm{Hom}_{\mathcal{T}}(U', M[n]), \quad (1)$$

each summand being 0 for large enough  $n$ . Thus  $U \oplus U' \in \mathcal{U}$ , and  $\mathcal{U}$  is an additive subcategory of  $\mathcal{T}$ . In addition, if  $U \oplus U'$  is in  $\mathcal{U}$ , each of the summands on the right side of (1) disappear whenever the left side disappears. Then both  $U$  and  $U'$  are in  $\mathcal{U}$ , and it is also closed under taking direct summands.

$\mathcal{U}$  is clearly closed under isomorphism, as  $\mathrm{Hom}_{\mathcal{T}}(U', M[n]) \cong \mathrm{Hom}_{\mathcal{T}}(U, M[n])$  for all  $U' \cong U$  in  $\mathcal{T}$  and all  $n \in \mathbb{Z}$ . For  $U \in \mathcal{U}$  and  $i \in \mathbb{Z}$ , we obtain

$$\mathrm{Hom}_{\mathcal{T}}(U[i], M[n+i]) \cong \mathrm{Hom}_{\mathcal{T}}(U, M[n]),$$

which means  $U[i] \in \mathcal{U}$ , and that  $\mathcal{U}$  is closed under shift. Let  $X \in \mathcal{U} * \mathcal{U}$  and  $M \in \mathcal{M}$ . There is then a triangle

$$U \longrightarrow X \longrightarrow U' \longrightarrow U[1],$$

in  $\mathcal{T}$  with  $U, U' \in \mathcal{U}$ . By definition of  $\mathcal{U}$ , there are  $n_{UM}, n_{U'M} \in \mathbb{Z}$  such that  $\mathrm{Hom}_{\mathcal{T}}(U, M[n]) = 0$  for any  $n \geq n_{UM}$  and  $\mathrm{Hom}_{\mathcal{T}}(U', M[n]) = 0$  for any  $n \geq n_{U'M}$ . Let  $n \geq \max(n_{UM}, n_{U'M})$  and apply  $\mathrm{Hom}_{\mathcal{T}}(-, M[n])$  to the triangle to obtain the long exact sequence

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{T}}(U', M[n]) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(X, M[n]) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(U, M[n]) \longrightarrow \cdots .$$

As both the left and the right terms vanish, the middle one does too by exactness. Thus for  $n_{XM} := \max\{n_{UM}, n_{U'M}\}$ , we have  $\mathrm{Hom}_{\mathcal{T}}(X, M[\geq n_{XM}]) = 0$ , so  $X \in \mathcal{U}$ . That is,  $\mathcal{U}$  is closed under extension in  $\mathcal{T}$ , which provides the final puzzle piece showing  $\mathcal{U}$  is a thick subcategory of  $\mathcal{T}$ . Since  $\mathcal{M}$  is silting,  $\mathcal{M} \subseteq \mathcal{U}$ . This means  $\mathcal{T} = \mathcal{U}$  by  $\mathcal{T} = \mathrm{thick} \mathcal{M}$  being the *smallest* thick

subcategory of  $\mathcal{T}$  containing  $\mathcal{M}$ , and it thus being a subcategory of all other thick subcategories of  $\mathcal{T}$  containing  $\mathcal{M}$ .

To complete the proof, we provide another, similarly defined subcategory of  $\mathcal{T}$ , and again prove that it is all of  $\mathcal{T}$ . For any  $X \in \mathcal{T}$ , we define the subcategory

$$\mathcal{V}_X := \{Y \in \mathcal{T} \mid \exists n_Y \in \mathbb{Z} \text{ such that } \text{Hom}_{\mathcal{T}}(X, Y[\geq n_Y]) = 0\}.$$

Again, the strategy involves showing this is a thick subcategory of  $\mathcal{T}$  containing  $\mathcal{M}$ . Seeing that  $\mathcal{V}_X$  contains  $\mathcal{M}$  is due to the previous part: As  $\mathcal{U} = \mathcal{T}$ ,  $X \in \mathcal{U}$ , and so  $\text{Hom}_{\mathcal{T}}(X, M[>> 0]) = 0$  for all  $M \in \mathcal{M}$ .

For  $Y \in \mathcal{V}_X$  and  $Y' \cong Y$  in  $\mathcal{T}$ ,

$$\text{Hom}_{\mathcal{T}}(X, Y[n]) \cong \text{Hom}_{\mathcal{T}}(X, Y'[n]).$$

so  $\mathcal{V}_X$  is closed under isomorphism in  $\mathcal{T}$ . Let  $n \geq n_Y$  so that  $\text{Hom}_{\mathcal{T}}(X, Y[n]) = 0$ . Then, for all  $i \in \mathbb{Z}$ ,  $\text{Hom}_{\mathcal{T}}(X, Y[i][n-i]) \cong \text{Hom}_{\mathcal{T}}(X, Y[n]) = 0$ , and so  $\mathcal{V}_X$  is closed under shift. Furthermore,

$$\text{Hom}_{\mathcal{T}}(X, (Y \oplus Y')[n]) \cong \text{Hom}_{\mathcal{T}}(X, Y[n]) \oplus \text{Hom}_{\mathcal{T}}(X, Y'[n]),$$

and by a similar argument as before,  $\mathcal{V}_X$  is closed under finite coproduct and taking direct summands. Let  $Z \in \mathcal{V}_X * \mathcal{V}_X$ . There is then a triangle

$$Y \xrightarrow{f} Z \xrightarrow{g} Y' \longrightarrow Y[1]$$

in  $\mathcal{T}$  with  $Y, Y' \in \mathcal{V}_X$ . We have that  $\text{Hom}_{\mathcal{T}}(X, Y[n]) = 0$  for all  $n$  greater than some  $n_Y$ , and that  $\text{Hom}_{\mathcal{T}}(X, Y'[n]) = 0$  for all  $n$  greater than some  $n_{Y'}$ . By applying  $\text{Hom}_{\mathcal{T}}(X, -)$  to the triangle, we get the long exact sequence

$$\cdots \longrightarrow \text{Hom}_{\mathcal{T}}(X, Y[n]) \longrightarrow \text{Hom}_{\mathcal{T}}(X, Z[n]) \longrightarrow \text{Hom}_{\mathcal{T}}(X, Y'[n]) \longrightarrow \cdots .$$

For  $n \geq \max(n_Y, n_{Y'})$ , both the left and the right terms disappear, and so the middle term disappears as well by exactness. That is, for  $n_Z := \max\{n_Y, n_{Y'}\}$ , we have  $\text{Hom}_{\mathcal{T}}(X, Z[\geq n_Z]) = 0$ , and so  $Z \in \mathcal{V}_X$ , so  $\mathcal{V}_X$  is closed under extensions. Again, we have a thick subcategory of  $\mathcal{T}$  containing  $\mathcal{M}$ , so  $\mathcal{V}_X = \mathcal{T}$  for any  $X \in \mathcal{T}$ , and the proposition has been proved.  $\square$

It is often desirable to have an additive category where the Hom-sets have more structure than just being additive groups. The definitions of  $k$ -linear and Hom-finite are standard.

**Definition 2.8.** Let  $\mathcal{T}$  be a category and  $k$  a field. We say that  $\mathcal{T}$  is  $k$ -linear if all Hom-sets are  $k$ -vector spaces and composition of morphisms is  $k$ -bilinear. Furthermore, we say that  $\mathcal{T}$  is Hom-finite over  $k$  if all these Hom-spaces are in addition finite dimensional over  $k$ .

The  $l$ -Calabi-Yau categories, in particular the 2-Calabi-Yau categories have played important roles in the development of the contemporary tilting theory [4, 5]. Before moving on, we put aside a page to Aihara and Iyama's note on how they are related to this this tilting definition.

**Definition 2.9.** Let  $\mathcal{T}$  be a  $k$ -linear, Hom-finite triangulated category, and let  $l \in \mathbb{Z}$ . We say that  $\mathcal{T}$  is  $l$ -Calabi-Yau if there is a natural isomorphism

$$\text{Hom}_{\mathcal{T}}(-, \sim) \cong \text{D Hom}_{\mathcal{T}}(\sim, -[l]),$$

where  $\text{D}$  is the dual  $\text{D} = \text{Hom}_k(-, k)$ .

The value of  $l$  in an  $l$ -Calabi-Yau category can provide useful information about the relationship between  $\mathcal{T}$  and the silting- and tilting subcategories of  $\mathcal{T}$ .

**Lemma 2.10.** *Let the triangulated category  $\mathcal{T}$  be a nonzero  $l$ -Calabi-Yau category for some  $l \in \mathbb{Z}$ . Then*

- (i) *If  $l = 0$ , every silting subcategory of  $\mathcal{T}$  is tilting.*
- (ii) *If  $l > 0$ ,  $\mathcal{T}$  has no silting subcategories.*
- (iii) *If  $l < 0$ ,  $\mathcal{T}$  has no tilting subcategories.*

Since any tilting subcategory is silting, the conclusion of Lemma 2.10 (i) is that  $\text{silt } \mathcal{T} = \text{tilt } \mathcal{T}$ . Likewise (ii) means  $\mathcal{T}$  has neither silting- nor tilting subcategories, and in the case of (iii) there are no tilting subcategories. Do note that part (iii) does *not* exclude the existence of silting subcategories of  $\mathcal{T}$ .

*Proof.* Assume  $\mathcal{T}$  is  $l$ -Calabi-Yau for some  $l$ . For objects  $A, B$ , in  $\mathcal{T}$ , we have

$$\text{Hom}_{\mathcal{T}}(A, B) \cong \text{Hom}_{\mathcal{T}}(A[l], B[l]).$$

Applying  $\text{D}(-)$  then yields

$$\text{D Hom}_{\mathcal{T}}(A, B) \cong \text{D Hom}_{\mathcal{T}}(A[l], B[l]) \cong \text{Hom}_{\mathcal{T}}(B, A[l]).$$

In particular, for  $X$  a nonzero object in  $\mathcal{T}$

$$\text{Hom}_{\mathcal{T}}(X, X[l]) \cong \text{D Hom}_{\mathcal{T}}(X, X) \not\cong 0.$$

- (i) Assume  $l = 0$ . Let  $\mathcal{M} \in \text{silt } \mathcal{T}$ ,  $X, Y \in \mathcal{M}$  and  $m < 0$  an integer.

$$\text{Hom}_{\mathcal{T}}(Y[m], X) \cong \text{Hom}_{\mathcal{T}}(Y, X[-m]) = 0,$$

i.e.

$$\text{Hom}_{\mathcal{T}}(X, Y[m]) \cong \text{D Hom}_{\mathcal{T}}(Y[m], X) = 0,$$

and so  $\text{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{M}[\neq 0]) = 0$ , and  $\mathcal{M} \in \text{tilt } \mathcal{T}$ , as proposed.

- (ii) Assume  $l > 0$ . Let  $0 \subsetneq \mathcal{M} \subsetneq \mathcal{T}$  be a subcategory. Then for any nonzero  $X$  in  $\mathcal{M}$ ,  $\text{Hom}_{\mathcal{T}}(X, X[l]) \cong \text{D Hom}_{\mathcal{T}}(X, X) \neq 0$ , and so  $\text{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{M}[> 0]) \neq 0$ , and  $\mathcal{M}$  is not silting.

- (iii) Assume  $l < 0$ , and let  $0 \subsetneq \mathcal{M} \subsetneq \mathcal{T}$  be a subcategory. For any nonzero  $X \in \mathcal{M}$ , we have  $\text{Hom}_{\mathcal{T}}(X, X[l]) \cong \text{D Hom}_{\mathcal{T}}(X, X) \neq 0$ . Then  $\text{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{M}[< 0]) \neq 0$ , and so  $\mathcal{M}$  is not tilting.  $\square$

**Example 2.11** (Non-example). By Example 2.3  $\mathcal{T} = \text{K}^b(\mathcal{P}(\text{mod } kA_3))$  has both tilting subcategories and silting subcategories which are not tilting. From Lemma 2.10, we then get that  $\mathcal{T}$  is *not*  $l$ -Calabi-Yau for any  $l \in \mathbb{Z}$ .

## 2.2 A Partial Ordering on the Silting Subcategories of $\mathcal{T}$

For the layman, the *inclusion* may present itself as a natural partial ordering on the silting subcategories of the triangulated category  $\mathcal{T}$ . In this section we show that this partial ordering is trivial. Instead, we follow Aihara and Iyama [1] in their generalization of the partial ordering on tilting modules as provided by Riedtmann-Schofield [23] and Happel-Unger [10].

This poset structure has the advantage that it is tightly intertwined with the theory of silting mutation of Section 4.

As part of the construction, we introduce the subcategories  $\mathcal{T}_{\mathcal{M}}^{\leq 0}$ , and provide some basic properties of these. In particular, we see in Proposition 2.25 how the partial ordering on  $\text{silt } \mathcal{T}$  is given by the inclusion of the corresponding categories  $\mathcal{T}_{\mathcal{M}}^{\leq 0}$ .

Finally, we show in Proposition 2.28 that the existence of a silting object in  $\mathcal{T}$  means that all silting subcategories of  $\mathcal{T}$  are given by silting objects.

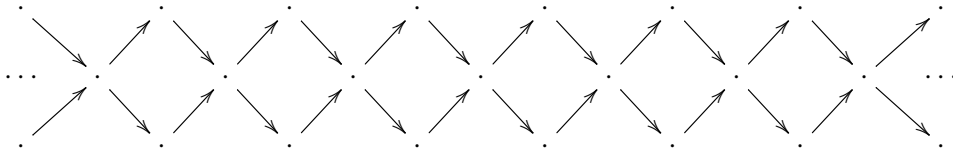
We start by introducing the notation  $\geq$  which will become our partial ordering.

**Definition 2.12.** Let  $\mathcal{T}$  be a triangulated category and  $\mathcal{M}$  and  $\mathcal{N}$  silting subcategories of  $\mathcal{T}$ . We say that  $\mathcal{M} \geq \mathcal{N}$  if

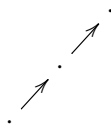
$$\text{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{N}[> 0]) = 0.$$

Immediately, it is clear that  $\mathcal{M} \geq \mathcal{M}$ . This relationship would hold also if we defined  $\geq$  on the set of pre-silting subcategories of  $\mathcal{T}$ .

**Example 2.13.** Assume  $\mathcal{T} = \text{K}^b(\mathcal{P}(\text{mod } \Lambda))$  as in Example 2.3 or Example 2.4. The AR-quiver has the underlying graph



Let  $M$  be a silting object of  $\mathcal{T}$  with three indecomposable summands appearing on a diagonal



and let  $N$  be a silting object all of whose indecomposable summands are either shared with  $M$  or to the right of  $M$ . Then  $\text{add}\{M\} \geq \text{add}\{N\}$ . Later, we will see that the silting subcategories of  $\mathcal{T}$  are exactly the additive closures of the silting objects in  $\mathcal{T}$ , and that these have exactly three indecomposable summands. The silting objects  $N$  of  $\mathcal{T}$  such that  $\text{add}\{M\} \geq \text{add}\{N\}$  are then exactly those for which the indecomposable summands are either shared with  $M$  or exist to the right of  $M$  in the AR-quiver.

The rest of this section is aimed at proving that this is a partial ordering on  $\text{silt } \mathcal{T}$ , and also to exploring the intermediate results used to get there. As we said in the introduction, several of the intermediate results used to get there are expressed in terms of the following subcategories of  $\mathcal{T}$ .



**Definition 2.14.** For  $\mathcal{T}$  a triangulated category and  $\mathcal{M} \in \text{silt } \mathcal{T}$ , define the subcategory  $\mathcal{T}_{\mathcal{M}}^{\leq 0} \subseteq \mathcal{T}$  as

$$\mathcal{T}_{\mathcal{M}}^{\leq 0} := \{X \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(\mathcal{M}, X[> 0]) = 0\}.$$

For ease of notation, define

$$\mathcal{T}_{\mathcal{M}}^{\leq l} := \mathcal{T}_{\mathcal{M}}^{\leq 0}[-l]$$

and

$$\mathcal{T}_{\mathcal{M}}^{\leq l} := \mathcal{T}_{\mathcal{M}}^{\leq l-1}$$

for any  $l \in \mathbb{Z}$ . Especially,  $\mathcal{T}_{\mathcal{M}}^{\leq 0} = \mathcal{T}_{\mathcal{M}}^{\leq 0}[1]$ .

The notation used here is reminiscent of the t-structure  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  from Example 1.13. As for now, this can be seen simply as a coincidence. In Section 3, we will see that under certain conditions,  $({}^{\perp}\mathcal{T}_{\mathcal{M}}^{\leq 0}, \mathcal{T}_{\mathcal{M}}^{\leq 0})$  will produce a co-t-structure of  $\mathcal{T}$ .

To more easily be able to follow Aihara and Iyama, we note the following basic properties of  $\mathcal{T}_{\mathcal{M}}^{\leq 0}$ .

**Remark 2.15.** Let  $\mathcal{T}$  be a triangulated category and  $\mathcal{M} \in \text{silt } \mathcal{T}$ .

- (i)  $\mathcal{T}_{\mathcal{M}}^{\leq 0}$  is closed under positive shift.
- (ii)  $\mathcal{T}_{\mathcal{M}}^{\leq 0}$  is closed under extensions.
- (iii)  $\mathcal{T}_{\mathcal{M}}^{\leq 0}$  is closed under taking direct summands.

*Proof.* The proofs are straight forward

- (i) For  $X \in \mathcal{T}_{\mathcal{M}}^{\leq 0}$  and  $n > 0$  an integer, we have that for any  $M \in \mathcal{M}$  and  $m > 0$ ,

$$\text{Hom}_{\mathcal{T}}(M, X[n][m]) \cong \text{Hom}_{\mathcal{T}}(M, X[n+m]) \subseteq \text{Hom}_{\mathcal{T}}(\mathcal{M}, X[> 0]) = 0.$$

That is,  $X[n] \in \mathcal{T}_{\mathcal{M}}^{\leq 0}$ .

- (ii) Let  $X \in \mathcal{T}_{\mathcal{M}}^{\leq 0} * \mathcal{T}_{\mathcal{M}}^{\leq 0}$ . There is then a triangle

$$A \longrightarrow X \longrightarrow B \longrightarrow A[1],$$

with  $A, B \in \mathcal{T}_{\mathcal{M}}^{\leq 0}$ . For any  $M \in \mathcal{M}$  apply  $\text{Hom}_{\mathcal{T}}(M, -)$  to the triangle to obtain the long exact sequence

$$\dots \longrightarrow \text{Hom}_{\mathcal{T}}(M, A[n]) \longrightarrow \text{Hom}_{\mathcal{T}}(M, X[n]) \longrightarrow \text{Hom}_{\mathcal{T}}(M, B[n]) \longrightarrow \dots$$

For  $n > 0$ , both the left and right terms vanish by definition of  $\mathcal{T}_{\mathcal{M}}^{\leq 0}$ , and then so does the middle term, by exactness. Thus  $\text{Hom}_{\mathcal{T}}(\mathcal{M}, X[> 0]) = 0$ , and so  $\mathcal{T}_{\mathcal{M}}^{\leq 0} * \mathcal{T}_{\mathcal{M}}^{\leq 0} \subseteq \mathcal{T}_{\mathcal{M}}^{\leq 0}$ . The other inclusion follows from Remark 1.1.

(iii) Assume  $X \oplus Y \in \mathcal{T}_{\mathcal{M}}^{\leq 0}$ . Then for any  $M \in \mathcal{M}$  and positive integer  $n$ ,

$$0 = \text{Hom}_{\mathcal{T}}(M, (X \oplus Y)[n]) \cong \text{Hom}_{\mathcal{T}}(M, X[n]) \oplus \text{Hom}_{\mathcal{T}}(M, Y[n]).$$

This means both  $\text{Hom}_{\mathcal{T}}(M, X[n])$  and  $\text{Hom}_{\mathcal{T}}(M, Y[n])$  vanish, and  $\mathcal{T}_{\mathcal{M}}^{\leq 0}$  is closed under taking direct summands.  $\square$

This next result makes it easier to prove Lemma 2.18. While it is stated in only two terms, we show that it easily generalizes to any finite number of terms.

**Lemma 2.16.** *Let  $\mathcal{T}$  be a triangulated category,  $\mathcal{M} \subseteq \mathcal{T}$  a pre-silting subcategory and  $m \geq n$  integers. Then*

$$\mathcal{M}[m] * \mathcal{M}[n] \subseteq \mathcal{M}[n] * \mathcal{M}[m].$$

*Proof.* Let  $m \geq n$ , and assume  $X \in \mathcal{M}[m] * \mathcal{M}[n]$ . Then there is a triangle

$$M[m] \longrightarrow X \longrightarrow M'[n] \xrightarrow{h} M[m+1]$$

with  $M, M' \in \mathcal{M}$ , and the morphism  $h = 0$ , so the triangle splits. By taking the coproduct of identity triangles, we obtain the solid parts of the diagram

$$\begin{array}{ccccccc} M'[n] & \longrightarrow & M'[n] \oplus M[m] & \longrightarrow & M[m] & \xrightarrow{0} & M'[n+1] \\ \downarrow 1 & & \downarrow \varphi & & \downarrow 1 & & \downarrow 1 \\ M'[n] & \longrightarrow & X & \longrightarrow & M[m] & \xrightarrow{0} & M'[n+1]. \end{array}$$

As the rightmost square commutes, this can be completed to a morphism of triangles by some  $\varphi$ , and as two of the three morphisms are identities, the  $\varphi$  is an isomorphism. This means that  $X \in \mathcal{M}[n] * \mathcal{M}[m]$ . I.e.  $\mathcal{M}[m] * \mathcal{M}[n] \subseteq \mathcal{M}[n] * \mathcal{M}[m]$ .  $\square$

By successive application of Lemma 2.16 and Remark 1.1, we get that if  $m \geq n \geq r$ ,

$$\begin{aligned} \mathcal{M}[r] * \mathcal{M}[m] * \mathcal{M}[n] &\subseteq \mathcal{M}[m] * \mathcal{M}[r] * \mathcal{M}[n] \\ &\subseteq \mathcal{M}[m] * \mathcal{M}[n] * \mathcal{M}[r] \\ &\subseteq \mathcal{M}[n] * \mathcal{M}[m] * \mathcal{M}[r]. \end{aligned}$$

Similarly, we get that for any sequence of integers  $n_1 \geq \dots \geq n_t$

$$\mathcal{M}[n_1] * \dots * \mathcal{M}[n_t] \subseteq \dots \subseteq \mathcal{M}[n_t] * \dots * \mathcal{M}[n_1].$$

**Example 2.17.** Let  $\mathcal{T}$  be as in Example 2.3,  $M$  the silting object  $P_1 \oplus P_2 \oplus P_3$ , and  $\mathcal{M} = \text{add}\{M\}$ . By the triangles  $P_3 \rightarrow P_2 \rightarrow S_2 \rightarrow P_3[1]$ ,  $P_1 \rightarrow I_2 \rightarrow P_3[1] \rightarrow P_1[1]$  and  $P_1 \rightarrow I_1 \rightarrow P_2[1] \rightarrow P_1[1]$  we get that

$$\mathcal{M} * \mathcal{M}[1] = \text{add}(\mathcal{M} \cup \mathcal{M}[1] \cup \{I_1, I_2, S_2\}).$$

I.e. both  $\mathcal{M}$  and  $\mathcal{M}[1]$ , and also all the objects "in between" the two in the AR-quiver. Also,  $\mathcal{M}[1] * \mathcal{M}$  is just  $\text{add}(\mathcal{M} \cup \mathcal{M}[1])$  as  $\mathcal{M}$  in this case is situated entirely to the left of  $\mathcal{M}[1]$  in the AR-quiver, and proceeding in triangles amount to movement to the right.

In addition, it is easily seen that for any  $n \geq 2$ , we have

$$\mathcal{M} * \mathcal{M}[n] = \mathcal{M}[n] * \mathcal{M} = \text{add}(\mathcal{M} \cup \mathcal{M}[n]).$$

Thus for this particular  $\mathcal{M}$ , there is a proper inclusion as by Lemma 2.16 for  $m = n + 1$ , and the lemma trivially holds for all other  $m$ .

Having subcategories of  $\mathcal{T}$ , we may want to attempt to describe  $\mathcal{T}$  in terms of these subcategories.

**Lemma 2.18.** *Let  $\mathcal{T}$  be a triangulated category.*

(i) *For any subcategory  $\mathcal{M} \subseteq \mathcal{T}$*

$$\text{thick } \mathcal{M} = \bigcup_{\substack{l > 0 \\ n_i \in \mathbb{Z}}} \text{smd}(\mathcal{M}[n_1] * \cdots * \mathcal{M}[n_l]).$$

(ii) *For any pre-silting subcategory  $\mathcal{M} \subseteq \mathcal{T}$*

$$\text{thick } \mathcal{M} = \bigcup_{l \geq 0} \text{smd}(\mathcal{M}[-l] * \cdots * \mathcal{M}[l]).$$

The expression  $\bigcup_{\substack{l > 0 \\ n_i \in \mathbb{Z}}} \text{smd}(\mathcal{M}[n_1] * \cdots * \mathcal{M}[n_l])$  may seem daunting at first. What Lemma 2.18 (i) says is that for any object  $X \in \text{thick } \mathcal{M}$  there is an  $l > 0$ , a set of  $l$  integers  $\{n_1, \dots, n_l\}$  and an object  $Y$  such that

$$X \oplus Y \in \mathcal{M}[n_1] * \cdots * \mathcal{M}[n_l].$$

This again, means that there are objects  $M_1, \dots, M_l \in \mathcal{M}$  such that  $X \oplus Y \in M_1[n_1] * \cdots * M_l[n_l]$ . Part (ii) is really the same result, only here pre-silting allows the  $n_i$  to appear in order from smallest to largest.

*Proof.* We prove both parts by showing inclusion both ways.

(i) For ease of notation, set

$$RHS := \bigcup_{\substack{l > 0 \\ n_i \in \mathbb{Z}}} \text{smd}(\mathcal{M}[n_1] * \cdots * \mathcal{M}[n_l]).$$

Let  $X \in RHS$ . Then for some object  $X'$ , some  $l > 0$ ,  $M_1, \dots, M_l \in \mathcal{M}$  and some set of integers  $n_1, \dots, n_l$

$$X \oplus X' \in M_1[n_1] * \cdots * M_l[n_l].$$

For  $N_i \in M_i[n_i] * \cdots * M_l[n_l]$  for  $1 < i < l$ , we have triangles

$$\begin{array}{ccc} M_i[n_i] & & \\ \downarrow & & \\ N_i & \longrightarrow & N_{i+1} \\ & & \downarrow \\ & & M_i[n_i + 1], \end{array}$$

and they all fit together into a diagram

$$\begin{array}{ccccccc}
M_1[n_1] & M_2[n_2] & \cdots & M_{l-1}[n_{l-1}] & M_l[n_l] & & (2) \\
\downarrow & \downarrow & & \downarrow & \Downarrow & & \\
X \oplus X' & \longrightarrow N_2 & \longrightarrow \cdots & \longrightarrow N_{l-1} & \longrightarrow N_l & \longrightarrow 0 & \\
& & \downarrow & \downarrow & \downarrow & \downarrow & \\
& & M_1[n_1 + 1] & \cdots & M_{l-2}[n_{l-2} + 1] & M_{l-1}[n_{l-1} + 1] & M_l[n_l + 1]
\end{array}$$

Now,  $N_l \cong M_l[n_l] \in \text{thick } \mathcal{M}$ , so  $N_{l-1} \in \text{thick } \mathcal{M}$ . Considering the second-to-last triangle, we similarly get that  $N_{l-2} \in \text{thick } \mathcal{M}$ . By iterating this process, we arrive at  $X \oplus X' \in \text{thick } \mathcal{M}$ . Since thick subcategories are closed under taking summands, it follows that  $X$  belongs to  $\text{thick } \mathcal{M}$ .

To show the other inclusion, we show  $RHS$  is a thick subcategory containing  $\mathcal{M}$ . Then as  $\text{thick } \mathcal{M}$  is the smallest such category, the inclusion holds.

Let  $A \oplus B \in RHS$ . Then for some  $X' \in \mathcal{T}$  and some  $n_1, \dots, n_l \in \mathbb{Z}$

$$A \oplus B \oplus X' \in \mathcal{M}[n_1] * \cdots * \mathcal{M}[n_l].$$

That is,  $A \in RHS$ , and  $RHS$  is closed under taking direct summands.

If  $X \oplus X' \in \mathcal{M}[n_1] * \cdots * \mathcal{M}[n_l]$ , and  $Y \oplus Y' \in \mathcal{M}[m_1] * \cdots * \mathcal{M}[m_r]$ , then by Remarks 1.1 and 1.4, we have that

$$X \oplus Y \oplus X' \oplus Y' \in \mathcal{M}[n_1] * \cdots * \mathcal{M}[n_l] * \mathcal{M}[m_1] * \cdots * \mathcal{M}[m_r].$$

This means that  $X \oplus Y \in RHS$ , and hence  $RHS$  is closed under finite coproducts.

Consider a triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1] \quad (3)$$

in  $\mathcal{T}$  with  $X, Z \in RHS$ . Say  $X \oplus X' \in \mathcal{M}[n_1] * \cdots * \mathcal{M}[n_l]$  and  $Z \oplus Z' \in \mathcal{M}[m_1] * \cdots * \mathcal{M}[m_r]$ . By adding (3) to the identity triangles

$$X' \longrightarrow X' \longrightarrow 0 \longrightarrow X'[1] \quad \text{and} \quad 0 \longrightarrow Z' \longrightarrow Z' \longrightarrow 0,$$

we get the triangle

$$X \oplus X' \longrightarrow Y \oplus X' \oplus Z' \longrightarrow Z \oplus Z' \longrightarrow (X \oplus X')[1].$$

This means that

$$Y \oplus X' \oplus Z' \in \mathcal{M}[n_1] * \cdots * \mathcal{M}[n_l] * \mathcal{M}[m_1] * \cdots * \mathcal{M}[m_r],$$

and so  $Y \in RHS$ , and it is closed under extensions.

By Remark 1.2  $X \in \mathcal{M}[n_1] * \cdots * \mathcal{M}[n_l]$  implies  $X[i] \in \mathcal{M}[n_1 + i] * \cdots * \mathcal{M}[n_l + i]$ , so  $RHS$  is closed under shift. Then  $RHS$  is a thick subcategory of  $\mathcal{T}$  containing  $\mathcal{M}$ , and so  $\text{thick } \mathcal{M} \subseteq RHS$ . This shows the other inclusion.

(ii) Let the subcategory  $\mathcal{M} \subseteq \mathcal{T}$  be pre-silting. We set

$$RHS' := \bigcup_{l \geq 0} \text{smd}(\mathcal{M}[-l] * \cdots * \mathcal{M}[l])$$

and show that  $RHS' = RHS = \text{thick } \mathcal{M}$  from (i). The inclusion  $RHS' \subseteq RHS$  is obvious.

Let  $X \in RHS$ , so  $X \oplus X' \in \mathcal{M}[n_1] * \cdots * \mathcal{M}[n_l]$ , and let  $\{n'_1, \dots, n'_l\} = \{n_1, \dots, n_l\}$  with  $n'_1 \leq \cdots \leq n'_l$ . Then by Lemma 2.16

$$X \oplus X' \in \mathcal{M}[n'_1] * \cdots * \mathcal{M}[n'_l]$$

In addition, by Remark 1.1, we can "fill in the blanks" to get

$$X \oplus X' \in \mathcal{M}[-s] * \mathcal{M}[-s+1] * \cdots * \mathcal{M}[s-1] * \mathcal{M}[s],$$

i.e.

$$X \in \text{smd}(\mathcal{M}[-s] * \mathcal{M}[-s+1] * \cdots * \mathcal{M}[s-1] * \mathcal{M}[s]) \subseteq RHS',$$

where  $s = \max\{|n'_1|, |n'_l|\}$ . □

Lemma 2.18 is still a bit unwieldy. We will iterate on this result twice: Proposition 2.22 iterates on this by letting  $\mathcal{M}$  be silting, and in Section 3 Proposition 3.15 iterates this again, allowing us to remove the  $\text{smd}$ -parts.

As an attempt to relieve ourselves of unnecessary instances of diagrams such as (2), we show the following lemma.

**Lemma 2.19.** *Let  $\mathcal{T}$  be a triangulated category and  $\mathcal{M}$  a pre-silting subcategory of silt  $\mathcal{T}$ . Then for any integers  $i < j$  and any  $l, l' \geq 0$ ,*

$$\text{Hom}_{\mathcal{T}}(\mathcal{M}[i-l'] * \cdots * \mathcal{M}[i], \mathcal{M}[j] * \cdots * \mathcal{M}[j+l]) = 0.$$

*Proof.* Without loss of generality, let  $X = X_i \in M_{i-l'}[i-l'] * \cdots * M_i[i]$  and  $Y = Y_j \in M_j[j] * \cdots * M_{j+l}[j+l]$ . For  $j \leq s < j+l$ , there are triangles

$$M_s[s] \longrightarrow Y_s \longrightarrow Y_{s+1} \longrightarrow M_s[s+1] \tag{4}$$

where  $Y_s \in M_s[s] * \cdots * M_{j+l}[j+l]$ . Similarly, for  $i-l' < t \leq i$ , there are triangles

$$M_t[t-1] \longrightarrow X_{t-1} \longrightarrow X_t \longrightarrow M_t[t] \tag{5}$$

where  $X_t \in M_{i-l'}[i-l'] * \cdots * M_t[t]$ .

For each  $t$ , apply  $\text{Hom}_{\mathcal{T}}(M_t[t], -)$  to the triangles (4) to get long exact sequences

$$\cdots \longrightarrow \text{Hom}_{\mathcal{T}}(M_t[t], M_s[s]) \longrightarrow \text{Hom}_{\mathcal{T}}(M_t[t], Y_s) \longrightarrow \text{Hom}_{\mathcal{T}}(M_t[t], Y_{s+1}) \longrightarrow \cdots$$

for  $i-l' \leq t \leq i$  and  $j \leq s < j+l$ .

The left terms vanish by  $t < s$  and  $\mathcal{M}$  being silting. Furthermore, if  $s = j+l-1$ , we get that the right term is  $\text{Hom}_{\mathcal{T}}(M_t[t], M_{j+1}[j+1]) = 0$ . Then by exactness,  $\text{Hom}_{\mathcal{T}}(M_t[t], Y_{j+l-1}) = 0$  as well. By repeatedly using the exactness of the sequences, we get that  $\text{Hom}_{\mathcal{T}}(M_t[t], Y_s) = 0$  for all

$i - l' \leq t \leq i$  and  $j \leq s < j + l$ . In particular, as  $X_{i-l'} = M_{i-l'}[i - l']$ , we get  $\text{Hom}_{\mathcal{T}}(X_{i-l'}, Y_s) = 0$  for all  $s$ .

Now, for each  $s$ , apply  $\text{Hom}_{\mathcal{T}}(-, Y_s)$  to the triangles (5) to get long exact sequences

$$\cdots \longrightarrow \text{Hom}_{\mathcal{T}}(M_t[t], Y_s) \longrightarrow \text{Hom}_{\mathcal{T}}(X_t, Y_s) \longrightarrow \text{Hom}_{\mathcal{T}}(X_{t-1}, Y_s) \longrightarrow \cdots$$

for  $i - l' \leq t \leq i$  and  $j \leq s \leq j + l$ . From the previous part, we see that the left terms all vanish. Also, for  $t = i - l' + 1$ , the right term vanishes as well, as noted above. Then, by exactness,  $\text{Hom}_{\mathcal{T}}(X_{i-l'-1}, Y_s) = 0$ , and by repeatedly using the exactness of the sequences, we get that  $\text{Hom}_{\mathcal{T}}(X_t, Y_s) = 0$  for all  $t$  and  $s$ .

This all accumulates in the conclusion that

$$\text{Hom}_{\mathcal{T}}(X_i, Y_j) = \text{Hom}_{\mathcal{T}}(X, Y) = 0,$$

which proves the lemma.  $\square$

For absolute generality of Lemma 2.19, we get the following corollary by a simple application of Lemma 2.16.

**Corollary 2.20.** *Let  $\mathcal{T}$  be a triangulated category and  $\mathcal{M} \in \text{silt } \mathcal{T}$ . If  $\{n_1, \dots, n_r\}$  and  $\{m_1, \dots, m_s\}$  are sets of integers such that  $n_i < m_j$  for all  $i$  and  $j$ , then*

$$\text{Hom}_{\mathcal{T}}(\mathcal{M}[n_1] * \cdots * \mathcal{M}[n_r], \mathcal{M}[m_1] * \cdots * \mathcal{M}[m_s]) = 0.$$

Now we present a nice technical tool which helps us determine if an object in an extension  $\mathcal{M} * \mathcal{N}$  is just an object of  $\mathcal{M}$  or  $\mathcal{N}$ . It will be used already to prove Proposition 2.22 below.

**Lemma 2.21.** *Let  $\mathcal{T}$  be a triangulated category,  $\mathcal{M}$  and  $\mathcal{N}$  subcategories of  $\mathcal{T}$  and  $X$  be an object in  $\text{smd}(\mathcal{M} * \mathcal{N})$ .*

(i) *If  $\text{Hom}_{\mathcal{M}}(\mathcal{M}, X) = 0$ , then  $X \in \mathcal{N}$ .*

(ii) *If  $\text{Hom}_{\mathcal{M}}(X, \mathcal{N}) = 0$ , then  $X \in \mathcal{M}$ .*

*Proof.* Let  $X \in \text{smd}(\mathcal{M} * \mathcal{N})$ , and assume  $\text{Hom}_{\mathcal{T}}(\mathcal{M}, X) = 0$ . Then, for some  $M \in \mathcal{M}$ ,  $N \in \mathcal{N}$  and  $Y \in \mathcal{T}$ , we have a triangle

$$M \xrightarrow{\begin{pmatrix} a=0 \\ b \end{pmatrix}} X \oplus Y \xrightarrow{\begin{pmatrix} c & d \end{pmatrix}} N \longrightarrow M[1].$$

Consider the two triangles

$$0 \longrightarrow X \xrightarrow{1} X \longrightarrow 0,$$

and, for some  $Z$ ,

$$M \xrightarrow{b} Y \xrightarrow{\gamma} Z \longrightarrow M[1].$$

obtained by completing  $b$  to a triangle. Take the sum of these two triangles, and we get the solid part of the diagram

$$\begin{array}{ccccccc} M & \xrightarrow{\begin{pmatrix} 0 \\ b \end{pmatrix}} & X \oplus Y & \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix}} & X \oplus Z & \longrightarrow & M[1] \\ \downarrow 1 & & \downarrow 1 & & \downarrow \text{dotted} & & \downarrow 1 \\ M & \xrightarrow{\begin{pmatrix} 0 \\ b \end{pmatrix}} & X \oplus Y & \xrightarrow{\begin{pmatrix} c & d \end{pmatrix}} & N & \longrightarrow & M[1]. \end{array}$$

We see that the left square commutes and complete it to a morphism of triangles by some  $X \oplus Z \rightarrow N$ . As two of the vertical morphisms are identities, the third is an isomorphism, so also  $X \oplus Z \cong N \in \mathcal{N}$ . We have that  $\mathcal{N}$  is closed under direct summands, so  $X \in \mathcal{N}$ , which proves (i). Part (ii) is proved analogously.  $\square$

We continue the effort to describe  $\mathcal{T}$  in terms of a silting subcategory as started by Lemma 2.18. By  $\mathcal{M}$  being a silting subcategory of  $\mathcal{T}$ , we immediately get part (i) from Lemma 2.18 (ii) by thick  $\mathcal{M} = \mathcal{T}$ .

**Proposition 2.22.** *Let  $\mathcal{T}$  be a triangulated category and  $\mathcal{M}$  a silting subcategory of  $\mathcal{T}$ . Then*

(i)

$$\mathcal{T} = \bigcup_{l \geq 0} \text{smd}(\mathcal{M}[-l] * \cdots * \mathcal{M}[l]),$$

and

(ii)

$$\mathcal{T}_{\mathcal{M}}^{\leq 0} = \bigcup_{l \geq 0} \text{smd}(\mathcal{M} * \cdots * \mathcal{M}[l]).$$

*Proof.* For part (ii), let  $X \in \mathcal{T}_{\mathcal{M}}^{\leq 0}$ . Then, as  $\mathcal{T}_{\mathcal{M}}^{\leq 0} \subseteq \mathcal{T}$ , it follows by part (i) that  $X \in \text{smd}(\mathcal{M}[-l] * \cdots * \mathcal{M}[l])$  for some  $l \geq 0$ . By definition of  $\mathcal{T}_{\mathcal{M}}^{\leq 0}$ , we have  $\text{Hom}_{\mathcal{T}}(\mathcal{M}, X[> 0]) = 0$ , and so equivalently  $\text{Hom}_{\mathcal{T}}(\mathcal{M}[< 0], X) = 0$ . In particular,  $\text{Hom}_{\mathcal{T}}(\mathcal{M}[-l], X) = 0$ . Lemma 2.21 (i) then gives us that

$$X \in \mathcal{M}[-l+1] * \cdots * \mathcal{M}[l] \subseteq \text{smd}(\mathcal{M}[-l+1] * \cdots * \mathcal{M}[l]).$$

We arrive at  $X \in \mathcal{M} * \cdots * \mathcal{M}[l]$  by

$$\text{Hom}_{\mathcal{T}}(\mathcal{M}[-l+1], X) = \cdots = \text{Hom}_{\mathcal{T}}(\mathcal{M}[-1], X) = 0$$

and successively applying Lemma 2.21  $l$  times.

For the other inclusion, let  $X \in \bigcup_{l \geq 0} \text{smd}(\mathcal{M} * \cdots * \mathcal{M}[l])$ . Then there is an  $Y$ , and  $n \geq 0$  and  $M_0, \dots, M_l$  such that

$$X \oplus Y \in M_0 * \cdots * M_l[l].$$

For any  $M \in \mathcal{M}$  and  $n > 0$ , we have

$$\text{Hom}_{\mathcal{T}}(M, X[n]) \oplus \text{Hom}_{\mathcal{T}}(M, Y[n]) \cong \text{Hom}_{\mathcal{T}}(M, (X \oplus Y)[n]).$$

By Remark 1.2, the right side is contained in  $\text{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{M}[n] * \cdots * \mathcal{M}[l+n])$ , which is 0 by Lemma 2.19. Thus both direct summands on the left side are 0, and so  $X \in \mathcal{T}_{\mathcal{M}}^{\leq 0}$ .  $\square$

The following property of silting subcategories is an important one: In it, Aihara and Iyama assert that the different silting subcategories of  $\mathcal{T}$  cannot be nested within each other. That is,  $\mathcal{T}$  will never have a silting subcategory properly contained in another silting subcategory.

**Theorem 2.23.** *Let  $\mathcal{T}$  be a triangulated category and  $\mathcal{M}, \mathcal{N} \in \text{silt } \mathcal{T}$ . If  $\mathcal{M} \subseteq \mathcal{N}$ , then  $\mathcal{M} = \mathcal{N}$ .*

*Proof.* The theorem is proven by showing the inclusion  $\mathcal{N} \subseteq \mathcal{M}$ . To this end, pick an object  $X \in \mathcal{N}$ . By  $\mathcal{N}$  silting and  $\mathcal{M} \subseteq \mathcal{N}$ , we get that

$$\mathrm{Hom}_{\mathcal{T}}(\mathcal{M}, X[> 0]) \subseteq \mathrm{Hom}_{\mathcal{M}}(\mathcal{N}, \mathcal{N}[> 0]) = 0,$$

and so  $X \in \mathcal{T}_{\mathcal{M}}^{\leq 0}$ . By Proposition 2.22, this means there is an  $l \geq 0$  and  $M_0, \dots, M_l \in \mathcal{M}$  such that  $X \in \mathrm{smd}(M_0 * \dots * M_l[l])$ .

Let  $Y = Y_1 \in M_1[1] * \dots * M_l[l]$ , so for  $Y_i \in M_i[i] * \dots * M_l[l]$ , we have triangles

$$M_i[i] \longrightarrow Y_i \longrightarrow Y_{i+1} \longrightarrow M_i[i+1]$$

for  $1 \leq i \leq l-1$ . Apply  $\mathrm{Hom}_{\mathcal{T}}(X, -)$  to the triangles to obtain long exact sequences

$$\dots \longrightarrow \mathrm{Hom}_{\mathcal{T}}(X, M_i[i]) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(X, Y_i) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(X, Y_{i+1}) \longrightarrow \dots$$

For all these  $i$ , the left term vanishes as  $\mathrm{Hom}_{\mathcal{T}}(X, M_i[i]) \subseteq \mathrm{Hom}_{\mathcal{T}}(\mathcal{N}, \mathcal{N}[> 0]) = 0$ . Also, for  $i = l-1$ , we have  $Y_{i+1} = Y_l = M_l[l]$ , so for the same reason as above,  $\mathrm{Hom}_{\mathcal{T}}(X, Y_l) = 0$ . Then by exactness, the middle term,  $\mathrm{Hom}_{\mathcal{T}}(X, Y_{l-1})$  vanishes as well. By repeated application of the exactness, it then follows that  $\mathrm{Hom}_{\mathcal{T}}(X, Y_i) = 0$  for all  $i$ . In particular  $\mathrm{Hom}_{\mathcal{T}}(X, Y) = 0$ , and so  $X \in \mathcal{M}$  by Lemma 2.21 (ii).  $\square$

The following result is one of two we need to show  $\geq$  is actually a partial ordering on silt  $\mathcal{T}$ . Notice how once we show  $({}^{\perp}\mathcal{M}_{\mathcal{M}}^{\leq 0}, \mathcal{T}_{\mathcal{M}}^{\leq 0})$  is a co-t-structure, this provides us with the coheart.

**Proposition 2.24.** *If  $\mathcal{M} \in \mathrm{silt} \mathcal{T}$ , then  $\mathcal{T}_{\mathcal{M}}^{\leq 0} \cap {}^{\perp}(\mathcal{T}_{\mathcal{M}}^{\leq 0}) = \mathcal{M}$ .*

*Proof.* For ease of notation, set  $\mathcal{N} := \mathcal{T}_{\mathcal{M}}^{\leq 0} \cap {}^{\perp}(\mathcal{T}_{\mathcal{M}}^{\leq 0})$ . As  $\mathrm{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{M}[> 0]) = 0$ ,  $\mathcal{M} \subseteq \mathcal{T}_{\mathcal{M}}^{\leq 0}$ . Also, let  $X \in \mathcal{T}_{\mathcal{M}}^{\leq 0}$ , so  $X = Y[1]$  for some  $Y \in \mathcal{T}_{\mathcal{M}}^{\leq 0}$ . We have  $\mathrm{Hom}_{\mathcal{T}}(\mathcal{M}, Y[> 0]) = 0$ , and so  $\mathrm{Hom}_{\mathcal{T}}(M, X) = \mathrm{Hom}_{\mathcal{T}}(M, Y[1]) = 0$ . As this holds for all such  $M$  and  $X$ ,

$$\mathrm{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{T}_{\mathcal{M}}^{\leq 0}) = 0,$$

and so  $\mathcal{M} \subseteq {}^{\perp}(\mathcal{T}_{\mathcal{M}}^{\leq 0})$ . Then  $\mathcal{M} \subseteq \mathcal{T}_{\mathcal{M}}^{\leq 0} \cap {}^{\perp}(\mathcal{T}_{\mathcal{M}}^{\leq 0}) = \mathcal{N}$ .

We show that  $\mathcal{N}$  is a silting subcategory of  $\mathcal{T}$ , and conclude from this that  $\mathcal{M} = \mathcal{N}$  by Theorem 2.23. By definition,  $\mathcal{N} \subseteq \mathcal{T}_{\mathcal{M}}^{\leq 0}$  and  $\mathcal{N} \subseteq {}^{\perp}(\mathcal{T}_{\mathcal{M}}^{\leq 0})$ , so

$$\mathrm{Hom}_{\mathcal{T}}(\mathcal{N}, \mathcal{N}[> 0]) \subseteq \mathrm{Hom}_{\mathcal{T}}({}^{\perp}(\mathcal{T}_{\mathcal{M}}^{\leq 0}), \mathcal{T}_{\mathcal{M}}^{\leq 0}[> 0]).$$

Let  $X \in {}^{\perp}(\mathcal{T}_{\mathcal{M}}^{\leq 0})$ , so  $\mathrm{Hom}_{\mathcal{T}}(X, \mathcal{T}_{\mathcal{M}}^{\leq 0}[1]) = 0$ . Furthermore, let  $Y \in \mathcal{T}_{\mathcal{M}}^{\leq 0}$  and  $n > 0$  an integer.  $\mathcal{T}_{\mathcal{M}}^{\leq 0}$  is closed under positive shift, and so  $Y[n-1] \in \mathcal{T}_{\mathcal{M}}^{\leq 0}$ . Then

$$\mathrm{Hom}_{\mathcal{T}}(X, Y[n]) = \mathrm{Hom}_{\mathcal{T}}(X, Y[n-1][1]) \subseteq \mathrm{Hom}_{\mathcal{T}}(X, \mathcal{T}_{\mathcal{M}}^{\leq 0}) = 0,$$

and as this holds for all such  $X, Y, n$ ,

$$\mathrm{Hom}_{\mathcal{T}}(\mathcal{N}, \mathcal{N}[> 0]) = 0.$$

By definition, thick  $\mathcal{N}$  contains  $\mathcal{N}$ , and then also  $\mathcal{M}$ . As thick  $\mathcal{M} = \mathcal{T}$  is the *smallest* thick subcategory of  $\mathcal{T}$  containing  $\mathcal{M}$ ,

$$\mathcal{T} \subseteq \mathrm{thick} \mathcal{N} \subseteq \mathcal{T}.$$

This proves thick  $\mathcal{N} = \mathcal{T}$ , and  $\mathcal{N}$  is a silting subcategory of  $\mathcal{T}$  containing the silting subcategory  $\mathcal{M}$ .  $\square$



As with Proposition 2.24, Proposition 2.25 is needed to show the  $\leq$  being an actual partial ordering on silt  $\mathcal{T}$ . It suggests an identification of the to-be-partial ordering with the partial ordering given by inclusion on the  $\{\mathcal{T}_{\mathcal{M}}^{\leq 0} \mid \mathcal{M} \in \text{silt } \mathcal{T}\}$ .

**Proposition 2.25.** *For  $\mathcal{M}, \mathcal{N}$  in silt  $\mathcal{T}$  the two are related as  $\mathcal{M} \geq \mathcal{N}$  if and only if  $\mathcal{T}_{\mathcal{M}}^{\leq 0} \supseteq \mathcal{T}_{\mathcal{N}}^{\leq 0}$ .*

*Proof.* Assume  $\mathcal{T}_{\mathcal{M}}^{\leq 0} \supseteq \mathcal{T}_{\mathcal{N}}^{\leq 0}$ . Since  $\mathcal{N} \subseteq \mathcal{T}_{\mathcal{N}}^{\leq 0} \subseteq \mathcal{T}_{\mathcal{M}}^{\leq 0}$ , we have that  $\text{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{N}[> 0]) = 0$ , which is exactly what we want for  $\mathcal{M} \geq \mathcal{N}$ .

For the converse, assume  $\mathcal{M} \geq \mathcal{N}$ . Then  $\text{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{N}[> 0]) = 0$ , so  $\mathcal{N} \subseteq \mathcal{T}_{\mathcal{M}}^{\leq 0}$ . By Remark 2.15 (i) and (ii)  $\mathcal{T}_{\mathcal{M}}^{\leq 0}$  is closed under extensions and positive shift, so

$$\mathcal{N} * \cdots * \mathcal{N}[l] \subseteq \mathcal{T}_{\mathcal{M}}^{\leq 0}$$

for any integer  $l \geq 0$ . Also by Remark 2.15 (iii),  $\mathcal{T}_{\mathcal{M}}^{\leq 0}$  is closed under taking summand, and so

$$\text{smd}(\mathcal{N} * \cdots * \mathcal{N}[l]) \subseteq \mathcal{T}_{\mathcal{M}}^{\leq 0}$$

for all  $l \geq 0$ . That is

$$\mathcal{T}_{\mathcal{N}}^{\leq 0} = \bigcup_{l \geq 0} \text{smd}(\mathcal{N} * \cdots * \mathcal{N}[l]) \subseteq \mathcal{T}_{\mathcal{M}}^{\leq 0},$$

which completes the proof. □

Finally, we reach the conclusion of this section, which asserts that what we introduced as a partial ordering on the silting subcategories of  $\mathcal{T}$  has the property we claim it has.

**Theorem 2.26.** *Let  $\mathcal{T}$  be a triangulated category. Then the relation  $\geq$  on silt  $\mathcal{T}$  as given in Definition 2.12 is a partial ordering.*

*Proof.* We need to show that the ordering is reflexive, antisymmetric and transitive.

Assume  $\mathcal{M} \in \text{silt } \mathcal{T}$ . Then as  $\text{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{M}[> 0]) = 0$ ,  $\mathcal{M} \geq \mathcal{M}$ , which shows  $\geq$  is reflexive.

Assume further that  $\mathcal{N} \in \text{silt } \mathcal{T}$  such that  $\mathcal{M} \geq \mathcal{N}$  and  $\mathcal{N} \geq \mathcal{M}$ . Then by Proposition 2.25,  $\mathcal{T}_{\mathcal{M}}^{\leq 0} = \mathcal{T}_{\mathcal{N}}^{\leq 0}$ . Then also

$$\mathcal{T}_{\mathcal{M}}^{\leq 0} = \mathcal{T}_{\mathcal{M}}^{\leq 0}[1] = \mathcal{T}_{\mathcal{N}}^{\leq 0}[1] = \mathcal{T}_{\mathcal{N}}^{\leq 0},$$

and naturally,

$${}^{\perp}(\mathcal{T}_{\mathcal{M}}^{\leq 0}) = {}^{\perp}(\mathcal{T}_{\mathcal{N}}^{\leq 0}).$$

It follows from Proposition 2.24 that

$$\mathcal{M} = \mathcal{T}_{\mathcal{M}}^{\leq 0} \cap {}^{\perp}(\mathcal{T}_{\mathcal{M}}^{\leq 0}) = \mathcal{T}_{\mathcal{N}}^{\leq 0} \cap {}^{\perp}(\mathcal{T}_{\mathcal{N}}^{\leq 0}) = \mathcal{N},$$

showing  $\geq$  is antisymmetric.

To finish it off, assume  $\mathcal{M}, \mathcal{L}, \mathcal{N} \in \text{silt } \mathcal{T}$  with  $\mathcal{M} \geq \mathcal{L} \geq \mathcal{N}$ . By Proposition 2.25 we then have that  $\mathcal{T}_{\mathcal{M}}^{\leq 0} \supseteq \mathcal{T}_{\mathcal{L}}^{\leq 0} \supseteq \mathcal{T}_{\mathcal{N}}^{\leq 0}$ , which, again by Proposition 2.25 gives us that  $\mathcal{M} \geq \mathcal{N}$ , showing  $\geq$  is transitive. □

When we later introduce *silting mutation*, we will see that employing *left mutation* will allow us to 'move down' with respect to this partial ordering. Likewise, using *right mutation* we can will 'move up'. This will be expressed precisely.

As Aihara and Iyama notes, the partial ordering relates links in proper descending (or ascending) chains to each other by the following:

**Proposition 2.27.** *Let  $\mathcal{T}$  be a triangulated category and assume  $\mathcal{M}, \mathcal{L}$  and  $\mathcal{N}$  are silting subcategories of  $\mathcal{T}$  such that  $\mathcal{M} \geq \mathcal{L} \geq \mathcal{N}$ . Then  $\mathcal{M} \cap \mathcal{N} \subseteq \mathcal{L}$ .*

*Proof.* Under these assumptions, Proposition 2.25 gives  $\mathcal{T}_{\mathcal{M}}^{\leq 0} \supseteq \mathcal{T}_{\mathcal{L}}^{\leq 0} \supseteq \mathcal{T}_{\mathcal{N}}^{\leq 0}$ , and immediately also that  $\mathcal{T}_{\mathcal{M}}^{\leq 0} \supseteq \mathcal{T}_{\mathcal{L}}^{\leq 0} \supseteq \mathcal{T}_{\mathcal{N}}^{\leq 0}$ . Furthermore, by Lemma 1.10, we get that

$${}^{\perp}(\mathcal{T}_{\mathcal{M}}^{\leq 0}) \subseteq {}^{\perp}(\mathcal{T}_{\mathcal{L}}^{\leq 0}) \subseteq {}^{\perp}(\mathcal{T}_{\mathcal{N}}^{\leq 0}).$$

Then, we get

$$\mathcal{M} \cap \mathcal{N} \subseteq {}^{\perp}(\mathcal{T}_{\mathcal{M}}^{\leq 0}) \cap \mathcal{T}_{\mathcal{N}}^{\leq 0} \subseteq {}^{\perp}(\mathcal{T}_{\mathcal{L}}^{\leq 0}) \cap \mathcal{T}_{\mathcal{L}}^{\leq 0} = \mathcal{L},$$

by Proposition 2.24, which is what we wanted to show.  $\square$

Finally, we conclude Aihara and Iyama by applying the results of this section to show how one silting object means all silting subcategories are given by objects.

While the proposition is easy to state and remember, it also allows us to distance ourselves from thinking about silting in terms of *subcategories*, and approaches a setting where silting is studied as a property of *objects*. In particular, it shows how all silting subcategories of  $\mathbb{K}^b(\mathcal{P}(\text{mod } kA_3))$  are given by silting objects, i.e. configurations of the AR-quivers.

**Proposition 2.28.** *Let  $\mathcal{T}$  be a triangulated category. If there is a silting object  $M \in \mathcal{T}$ , then for any  $\mathcal{N} \in \text{silt } \mathcal{T}$ , there is an object  $N$  such that  $\mathcal{N} = \text{add}\{N\}$ .*

*Proof.* Let  $M \in \mathcal{T}$  be a silting object, and let  $\mathcal{N}$  be a silting subcategory of  $\mathcal{T}$ . Then by Lemma 2.18 (ii)

$$M \in \mathcal{T} = \text{thick } \mathcal{N} = \bigcup_{l \geq 0} \text{smd}(\mathcal{N}[-l] * \cdots * \mathcal{N}[l]).$$

i.e. there are objects  $N_{-l}, \dots, N_l$  in  $\mathcal{N}$  such that  $M \in \text{smd}(N_{-l}[-l] * \cdots * N_l[l])$ . Pick any such set of objects, and define

$$\mathcal{N}' := \text{add}\{N_{-l} \oplus \cdots \oplus N_l\} \subseteq \mathcal{N}.$$

First, observe that  $\text{Hom}_{\mathcal{T}}(\mathcal{N}', \mathcal{N}'[> 0]) = 0$  as  $\mathcal{N}' \subseteq \mathcal{N}$ . Next, note that each  $N_i \in \mathcal{N}' \subseteq \text{thick } \mathcal{N}'$ . Since  $\text{thick } \mathcal{N}'$  is closed under shift, extensions and direct summands, we get

$$M \in \text{smd}(N_{-l}[-l] * \cdots * N_l[l]) \subseteq \text{thick } \mathcal{N}'.$$

As  $\text{thick } \mathcal{N}'$  is closed under isomorphisms, direct summands and finite coproducts, it then follows that  $\text{add}\{M\} \subseteq \text{thick } \mathcal{N}'$ . This means that  $\mathcal{T} = \text{thick}(\text{add}\{M\}) \subseteq \text{thick } \mathcal{N}' \subseteq \mathcal{T}$ , and  $\text{thick } \mathcal{N}' = \mathcal{T}$ .

This shows  $\mathcal{N}'$  is silting and contained in  $\mathcal{N}$ . By Theorem 2.23,  $\mathcal{N}' = \mathcal{N}$ , and we have proven the result by using  $N := N_{-l} \oplus \cdots \oplus N_l$ .  $\square$

### 3 Krull-Schmidt triangulated categories

The categories of main interest to us are the easy-to-understand path algebras of finite quivers without cycles, i.e. the hereditary path algebras [2]. For  $\Lambda$  such an algebra, any finitely generated left  $\Lambda$ -module has a finite decomposition into a coproduct of left  $\Lambda$ -modules, all of whose endomorphism rings are local [2]. Such a decomposition is also unique up to isomorphism of its components.

As shown in Proposition 3.3, this is also true for  $K^b(\mathcal{P}(\text{mod } \Lambda))$ , and we fittingly label these categories as *Krull-Schmidt categories*.

In this section we study the triangulated categories which inhabit such a Krull-Schmidt property. The expressions in Proposition 2.22 reach the simpler forms in Proposition 3.15, which allows us to easier understand the category in terms of its sifting subcategories.

We begin by properly defining the central concept:

**Definition 3.1.** We say that an additive category  $\mathcal{T}$  is a *Krull-Schmidt category* if

- (i)  $\mathcal{T}$  is Hom-finite over some field  $k$ .
- (ii) Any object  $T \in \mathcal{T}$  has a unique (up to isomorphism) finite decomposition

$$T = \bigoplus_{i < \infty} X_i^{d_i},$$

where the  $X_i$  are mutually non-isomorphic objects with local endomorphism ring. The  $d_i \in \mathbb{N}$  for all  $i$  and denotes the number of copies of  $X_i$  in  $T$ . In the case where all the  $d_i = 1$ , we say that  $T$  is a *basic* object of  $\mathcal{T}$ .

Let  $X \in \mathcal{T}$  be a nonzero object such that  $\text{End}_{\mathcal{T}}(X)$  is local. Assume a decomposition  $X \cong X_1 \oplus X_2$ , and without loss of generality, let  $X_1 \neq 0$ . Consider the morphism

$$X_1 \oplus X_2 \xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} X_1 \oplus X_2.$$

This is clearly not an isomorphism, and as any element in a local ring is either an isomorphism or nilpotent, it is nilpotent: For some  $n \in \mathbb{N}$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This is only possible when  $X_2 = 0$ , and so  $X$  is indecomposable. Furthermore, as any indecomposable object only has one decomposition, all indecomposable objects in a Krull-Schmidt category have local endomorphism rings. We will often refer to an object in such a category simply by its decomposition into indecomposables.

Now let  $X \xrightarrow{f} Y$  be a split monomorphism between nonzero, indecomposable objects. Then for some  $Y \xrightarrow{f'} X$ ,  $f'f = 1_X$ . As  $ff' \in \text{End}_{\mathcal{T}}(Y)$ , which is a local ring, it is either an isomorphism or nilpotent. If it is nilpotent, there is an  $n \geq 1$  such that

$$(ff')^n = f(ff')^{n-1}f' = ff' = 0.$$

Then  $f' = f'(ff') = 0$ , and so  $1_X = 0$ , which is impossible. Thus it is an isomorphism, and so  $f$  is a split epimorphism, and so it is an isomorphism. Similarly we see that any split epimorphism between nonzero indecomposable objects are isomorphisms. In the case where  $\mathcal{T}$  is in addition triangulated,

any monomorphism (epimorphism) is split, and then any monomorphism (epimorphism) between indecomposable objects is an isomorphism.

The next result is often taught in graduate level university courses in terms of modules over Artin algebras. We provide here a more general variant for use in Krull-Schmidt categories, as it significantly reduces the amount of effort needed in some later proofs.

**Proposition 3.2.** *Let  $\mathcal{T}$  be a Krull-Schmidt triangulated category and  $X \in \mathcal{T}$ . We then have the equivalence of categories*

$$\text{add}\{X\} \xrightarrow{\text{Hom}_{\mathcal{T}}(X, -)} \mathcal{P}(\text{mod } \Gamma),$$

where  $\Gamma := \text{End}_{\mathcal{T}}(X)^{\text{op}}$ .

*Proof.* First, we note that as  $\mathcal{T}$  is Hom-finite over  $k$ , the algebra  $\Gamma$  is a finite dimensional  $k$ -algebra, and the finitely generated projective  $\Gamma$ -modules are, up to isomorphism, the finite direct sums of direct summands of  $\Gamma$ . I.e  $\mathcal{P}(\text{mod } \Gamma) = \text{add}\{\Gamma\}$ . For any  $Y \in \text{add}\{\Gamma\}$ , there is an  $n \geq 1$  and an  $Z$  such that  $Y \oplus Z \cong X^n$ . Then, as  $\Gamma$ -modules,

$$\text{Hom}_{\mathcal{T}}(X, Y) \oplus \text{Hom}_{\mathcal{T}}(X, Z) \cong \text{Hom}_{\mathcal{T}}(X, X^n) \cong \text{Hom}_{\mathcal{T}}(X, X)^n = \Gamma^n,$$

so  $\text{Hom}_{\mathcal{T}}(X, Y) \in \mathcal{P}(\text{mod } \Gamma)$ .

To complete the proof, we show that  $\text{Hom}_{\mathcal{T}}(X, -)$  restricted to  $\text{add}\{X\}$  is full, faithful and dense. We do this by showing

$$\text{Hom}_{\mathcal{T}}(A, B) \xrightarrow{\text{Hom}_{\mathcal{T}}(X, -)} \text{Hom}_{\text{mod } \Gamma}(\text{Hom}_{\mathcal{T}}(X, A), \text{Hom}_{\mathcal{T}}(X, B)) \quad (6)$$

is an isomorphism for all  $A, B \in \text{add}\{X\}$ , and by showing  $\text{Hom}_{\mathcal{T}}(X, -)$  is dense in  $\text{mod } \Gamma$ .

Note that the morphism  $\text{Hom}_{\mathcal{T}}(X, -)$  in (6) is the one which maps any  $A \xrightarrow{f} B$  to  $f \circ -$ .

We show the isomorphism (6) in three steps:

(i)  $A = X$ : Assume  $X \xrightarrow{f} B$  is such that  $f \circ - = 0$ . I.e. for all  $X \xrightarrow{g} X$ , the composition  $fg = 0$ . Then especially  $f1_X = f = 0$ , so the map is a monomorphism.

To show it is a split epimorphism, let  $\text{Hom}_{\mathcal{T}}(X, X) \xrightarrow{h} \text{Hom}_{\mathcal{T}}(X, B)$ . We map  $h$  to the morphism  $X \xrightarrow{f := h(1_X)} B$ , and we show that  $f \circ - = h$ . This is clear since  $h$  is a  $\Gamma$ -homomorphism by assumption: For any  $X \xrightarrow{g} X$ , we can consider  $g$  as an element in the left  $\Gamma$ -module  $\Gamma$ . Thus  $fg = h(1_X)g = h(1_X g) = h(g)$ , so the map is a split epimorphism. Thus (6) is an isomorphism for  $A = X$ .

(ii)  $A = X^n$ : To reduce the size of the diagrams, we define  $e_X(-) := \text{Hom}_{\mathcal{T}}(X, -)$  Due to Hom-functors being additive, we get the diagram

$$\begin{array}{ccc} \text{Hom}_{\mathcal{T}}(X^n, B) & \xrightarrow{\alpha} & \text{Hom}_{\text{mod } \Gamma}(e_X(X^n), e_X(B)) \\ \text{iso } \downarrow \gamma & & \text{iso } \downarrow \delta \\ & & \text{Hom}_{\text{mod } \Gamma}(e_X(X)^n, e_X(B)) \\ & & \text{iso } \downarrow \rho \\ \text{Hom}_{\mathcal{T}}(X, B)^n & \xrightarrow{\beta} & \text{Hom}_{\text{mod } \Gamma}(e_X(X), e_X(B))^n \end{array}$$

with

$$f \xrightarrow{\gamma} (f_1 \cdots f_n) \xrightarrow{\beta} (f_1 \circ - \cdots f_n \circ -)$$

and

$$f \xrightarrow{\alpha} f \circ - \xrightarrow{\delta} (f_1 \cdots f_n) \circ - \xrightarrow{\rho} (f_1 \circ - \cdots f_n \circ -).$$

I.e. the diagram commutes, and as each component of  $\beta$  is an isomorphism from (i), the morphisms  $\beta$ , and then also  $\alpha$  are isomorphisms too.

(iii)  $A \oplus A' = X^n$ : We obtain a similar diagram as in (ii):

$$\begin{array}{ccc} \text{Hom}_{\mathcal{T}}(A \oplus A', B) & \xrightarrow[\text{iso}]{\alpha} & \text{Hom}_{\text{mod } \Gamma}(e_X(A \oplus A'), e_X(B)) \\ \downarrow \text{iso } \gamma & & \downarrow \text{iso } \delta \\ \text{Hom}_{\mathcal{T}}(A, B) & & \text{Hom}_{\text{mod } \Gamma}(e_X(A) \oplus e_X(A'), e_X(B)) \\ \oplus & \xrightarrow{\sigma = \begin{pmatrix} a & 0 \\ 0 & a' \end{pmatrix}} & \downarrow \text{iso } \rho \\ \text{Hom}_{\mathcal{T}}(A', B) & & \text{Hom}_{\text{mod } \Gamma}(e_X(A), e_X(B)) \\ & & \oplus \\ & & \text{Hom}_{\text{mod } \Gamma}(e_X(A'), e_X(B)) \end{array}.$$

This diagram commutes for the same reasons as the one in (ii), so  $\sigma$  is an isomorphism as well, and finally

$$\text{Hom}_{\mathcal{T}}(A, B) \xrightarrow{a = \text{Hom}_{\mathcal{T}}(X, -)} \text{Hom}_{\text{mod } \Gamma}(\text{Hom}_{\mathcal{T}}(X, A), \text{Hom}_{\mathcal{T}}(X, B))$$

is an isomorphism.

Thus the functor  $\text{Hom}_{\mathcal{T}}(X, -)$  is *fully faithful*.

To show it is dense, we let  $P \in \mathcal{P}(\text{mod } \Gamma) = \text{add}\{\Gamma\}$ , so for some  $Q$  and some  $n \geq 1$ , we have  $P \oplus Q \cong \Gamma^n$ . We consider then the idempotent

$$P \oplus Q \xrightarrow{f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} P \oplus Q$$

in  $\text{Hom}_{\text{mod } \Gamma}(\Gamma^n, \Gamma^n)$ . It has the kernel  $\ker(f) = P$ . From the first part, we have the isomorphism

$$\text{Hom}_{\mathcal{T}}(X^n, X^n) \xrightarrow{e_X = \text{Hom}_{\mathcal{T}}(X, -)} \text{Hom}_{\text{mod } \Gamma}(\Gamma^n, \Gamma^n),$$

so there is some idempotent  $X^n \xrightarrow{u} X^n$  such that  $e_X(u) = f$ . As idempotents in  $\mathcal{T}$  split, we have the commutative diagram

$$\begin{array}{ccc} X^n & \xrightarrow{u} & X^n \\ & \searrow \pi & \nearrow \iota \\ & & Y \end{array}$$

where  $\pi \iota = 1_Y$ . That is,  $\iota$  is a split monomorphism, and  $Y$  is a direct summand of  $X^n$ . This also means that we have the exact sequence

$$0 \longrightarrow Y \xrightarrow{\iota} X^n \xrightarrow{u} X^n$$

in  $\mathcal{T}$ , which yields the exact sequence

$$0 \longrightarrow e_X(Y) \longrightarrow \Gamma^n \xrightarrow{f} \Gamma^n$$

in  $\text{mod } \Gamma$ . Thus  $e_X(Y) \cong P$  as both are the kernel of  $f$ .  $\square$

We are now able to see that our favorite bounded homotopy categories of projective modules are Krull-Schmidt, and thus that the Krull-Schmidt triangulated categories are a meaningful abstract substitution. It is assumed known that these categories are triangulated, see e.g. [26].

**Proposition 3.3.** *Let  $k$  be a field and  $\Lambda$  a finite-dimensional algebra over  $k$ . Then  $\mathcal{T} := \text{K}^b(\mathcal{P}(\text{mod } \Lambda))$  is a Krull-Schmidt triangulated category.*

*Proof.* Let  $A \in \mathcal{T}$ . By Proposition 3.2 we have the equivalence

$$\text{add}\{A\} \xrightarrow{\text{Hom}_{\mathcal{T}}(A, -)} \mathcal{P}(\text{mod } \Gamma),$$

where  $\Gamma := \text{End}_{\mathcal{T}}(A)^{\text{op}}$ . The endomorphism ring of  $A$  consists of homotopy classes of chain morphisms  $A \rightarrow A$ . Each degree of such a chain morphism is itself an element in an endomorphism ring of some object in  $\text{mod } \Lambda$ , which is a finite-dimensional  $k$ -vector space. As  $A$  only has a finite number of degrees where it is nonzero, it follows that the chain morphisms  $A \rightarrow A$  form a finite-dimensional  $k$ -vector space. Furthermore, the homotopy classes of chain morphisms  $A \rightarrow A$  form a finite dimensional  $k$ -vector space. Thus we have that  $\Gamma$  is a finite dimensional  $k$ -vector space, and that  $\text{mod } \Gamma$  is a Krull-Schmidt category.

Then  $P := \text{Hom}_{\mathcal{T}}(A, A)$  has a unique (up to isomorphism) decomposition into indecomposable  $\Gamma$ -modules

$$P = P_1 \oplus \cdots \oplus P_n.$$

As each  $P_i$  is in  $\mathcal{P}(\text{mod } \Gamma)$  and  $\text{Hom}_{\mathcal{T}}(A, -)$  is dense, each  $P_i \cong \text{Hom}_{\mathcal{T}}(A, A_i)$  for some  $A_i \in \text{add}\{A\}$ . Thus

$$\text{Hom}_{\mathcal{T}}(A, A) \cong \bigoplus_{i=1}^n \text{Hom}_{\mathcal{T}}(A, A_i) \cong \text{Hom}_{\mathcal{T}}(A, \bigoplus_{i=1}^n A_i),$$

and by  $\text{Hom}_{\mathcal{T}}(A, -)$  being an equivalence,  $A \cong \bigoplus_{i=1}^n A_i$ . Equivalences preserve decomposability, so each  $A_i$  is indecomposable.

Assume  $A$  has two decompositions into indecomposable objects

$$A \cong \bigoplus_{i=1}^n A_i \cong \bigoplus_{i=1}^{n'} A'_i$$

where for some  $j$ ,  $A_j \not\cong A'_i$  for all  $i$ . Then  $P$  has two decompositions

$$P \cong \bigoplus_{i=1}^n \text{Hom}_{\mathcal{T}}(A, A_i) \cong \bigoplus_{i=1}^{n'} \text{Hom}_{\mathcal{T}}(A, A'_i),$$

so  $n = n'$ . Also, by  $\text{Hom}_{\mathcal{T}}(A, -)$  being an equivalence,  $\text{Hom}_{\mathcal{T}}(A, A_j) \not\cong \text{Hom}_{\mathcal{T}}(A, A'_i)$  for all  $i$ . This is a contradiction, so the decomposition of  $A$  is unique up to isomorphism.

Let  $X$  be indecomposable in  $K^b(\mathcal{P}(\text{mod } \Lambda))$ . Similar to above, there is now an equivalence

$$\text{add}\{X\} \xrightarrow{\text{Hom}_{\mathcal{T}}(X, -)} \mathcal{P}(\text{mod } \Gamma)$$

with  $\Gamma = \text{End}_{\mathcal{T}}(X, X)$ . As  $X$  is indecomposable, it is sent to an the indecomposable object  $\text{Hom}_{\mathcal{T}}(X, X)$  by the equivalence. We saw that this a finite dimensional  $k$ -vector space above, and as it is indecomposable, it must be one-dimensional. I.e.  $\text{Hom}_{\mathcal{T}}(X, X) \cong k$ , which is especially a local ring.

Thus the indecomposable objects in  $\mathcal{T}$  have local rings, and  $\mathcal{T}$  is Krull-Schmidt.  $\square$

Krull-Schmidt categories all have the following nice property.

**Proposition 3.4.** *Let  $\mathcal{T}$  be a Krull-Schmidt category and  $M \in \mathcal{T}$ . Then  $\text{add}\{M\}$  is functorially finite in  $\mathcal{T}$ .*

The proof is a straight forward basis argument, but the bookkeeping and indexing still makes it rather ugly.

*Proof.* We show  $\text{add}\{M\}$  is contravariantly finite. The proof of it being covariantly finite is dual.

Let  $T \in \mathcal{T}$ . As  $\mathcal{T}$  is Krull-Schmidt, there is a decomposition  $M = \bigoplus_{i=1}^n M_i$ , where the  $M_i$  are indecomposable. Also,  $\text{Hom}_{\mathcal{T}}(M_i, T)$  is a finite dimensional  $k$ -vector space for each  $i$ , and so we choose for each  $i$  a basis  $\{f_{i1}, \dots, f_{it_i}\}$  for  $\text{Hom}_{\mathcal{T}}(M_i, T)$ . Define the object  $M'_i := \bigoplus_{j=1}^{t_i} M_i$ , and let  $f'_i$  be the morphism

$$M'_i = \bigoplus_{j=1}^{t_i} M_i \xrightarrow{f'_i := (f_{i1} \dots f_{it_i})} T.$$

Do the same again to define the object  $M'$  and the morphism  $M' \xrightarrow{f'} T$  by

$$M' := \bigoplus_{i=1}^n M'_i \xrightarrow{f' := (f'_1 \dots f'_n)} T.$$

We show this is a right  $\text{add}\{M\}$ -approximation of  $T$ .

To this end, let  $M'' \xrightarrow{f''} T$  be any morphism with  $M'' \in \text{add}\{M\}$ . By definition of  $\text{add}\{M\}$ , the object  $M''$  is a finite direct sum of summands of  $M$ . I.e. there are  $d_1, \dots, d_n$  such that  $M'' = \bigoplus_{i=1}^n M_i^{d_i}$ , and the morphism  $f''$  is then given by

$$M'' = \bigoplus_{i=1}^n M_i^{d_i} \xrightarrow{f'' := (f''_1 \dots f''_n)} T,$$

with  $M_i^{d_i} \xrightarrow{f''_i} T$  for  $1 \leq i \leq n$ .

For each  $i$ ,  $f''_i$  is then given by  $f''_i = (f''_{i1} \dots f''_{id_i})$ , where  $M_i \xrightarrow{f''_{ij}} T$  for each  $1 \leq j \leq d_i$ . As  $f''_{ij} \in \text{Hom}_{\mathcal{T}}(M_i, T)$ , there are  $c_{ij}^1, \dots, c_{ij}^{t_i}$  for each  $j$  such that  $f''_{ij} = c_{ij}^1 f_{i1} + \dots + c_{ij}^{t_i} f_{it_i}$ . Define

$$g_i := \begin{pmatrix} c_{i1}^1 & \dots & c_{id_i}^1 \\ \vdots & & \vdots \\ c_{i1}^{t_i} & \dots & c_{id_i}^{t_i} \end{pmatrix}.$$

Then  $f_i'' = f_i' g_i$ , and it is easy to check that the diagram

$$\begin{array}{ccc} & M' & \xrightarrow{f'} T \\ \left( \begin{array}{c} g_1 \\ \vdots \\ g_n \end{array} \right) \uparrow & & \nearrow f'' \\ & M'' & \end{array}$$

commutes. This shows  $f'$  is a right  $\text{add}\{M\}$ -approximation of  $T$ , and that  $\text{add}\{M\}$  is contravariantly finite in  $\mathcal{T}$ .  $\square$

This result allows us to easily generate functorially finite subcategories of silting subcategories by simply picking an object  $X$  in the silting subcategory and considering  $\text{add}\{X\}$ . In particular, for our example category  $\text{K}^b(\mathcal{P}(\text{mod } kA_3))$ , we see that choosing any summand of a silting object yields a functorially finite subcategory of the corresponding silting subcategory. Such subcategories are important when we in Section 4.1 generate new silting subcategories from old ones by *mutation*.

The following remark is an easy observation when working in Krull-Schmidt categories. It often reduces the amount of work needed when working with silting categories with additive generators.

**Remark 3.5.** *Let  $\mathcal{T}$  be a Krull-Schmidt triangulated category and  $X \in \mathcal{T}$ . Then  $\mathcal{M} := \text{add}\{X\}$  is a silting subcategory of  $\mathcal{T}$  if and only if  $\text{Hom}_{\mathcal{T}}(X, X[> 0]) = 0$  and  $\text{thick}\{X\} = \mathcal{T}$ .*

*Proof.* Let  $X \in \mathcal{T}$  be such that  $\text{Hom}_{\mathcal{T}}(X, X[> 0]) = 0$  and  $\text{thick}\{X\} = \mathcal{T}$ . By  $\mathcal{T}$  being Krull-Schmidt, there is a decomposition

$$X = X_1 \oplus \cdots \oplus X_m$$

with  $X_i$  indecomposable. Then  $\text{Hom}_{\mathcal{T}}(X_i, X_j[> 0]) = 0$  for all  $i$  and  $j$ . Furthermore, any  $Y \in \mathcal{M}$  is of the form  $Y = X_1^{d_1} \oplus \cdots \oplus X_m^{d_m}$ , for some  $d_i \geq 0$ . Then for any  $Y, Y' \in \mathcal{M}$  and  $n > 0$ ,

$$\text{Hom}_{\mathcal{T}}(Y, Y'[n]) = \bigoplus_{i,j=1}^m \text{Hom}_{\mathcal{T}}(X_i, X_j[n])^{d_i d_j} = 0,$$

showing  $\text{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{M}[> 0]) = 0$ . In addition,  $X \in \text{thick}(\mathcal{M})$ , so  $\text{thick}(\mathcal{M}) = \mathcal{T}$ , and  $\mathcal{M}$  is silting.

For the other direction, assume  $\mathcal{M}$  is silting. then clearly  $\text{Hom}_{\mathcal{T}}(X, X[> 0]) = 0$ , and as  $\text{thick}\{X\}$  contains all summands of  $X$  and all finite coproducts of such summands,  $\mathcal{M} \subseteq \text{thick}\{X\}$ . This shows  $\mathcal{T} = \text{thick}(\mathcal{M}) \subseteq \text{thick}\{X\}$ , and so both directions hold.  $\square$

Note that in this Krull-Schmidt case, the subcategories  $\mathcal{M}$  of  $\mathcal{T}$  containing an additive generator  $M$  also contains a basic additive generator  $M'$ . This basic generator is the coproduct of the mutually non-isomorphic summands of  $M$  called the *basic version* of  $M$ .

**Proposition 3.6.** *Let  $\mathcal{T}$  be a Krull-Schmidt triangulated category with a silting object. Then there is a bijection between the silting subcategories of  $\mathcal{T}$  and the basic silting objects in  $\mathcal{T}$ .*

*Proof.* As by Proposition 2.28,  $\mathcal{T}$  having a silting object means any silting subcategory is  $\text{add}\{M\}$  for some object  $M$ . Then the basic version  $M'$  of  $M$  generates the same category, and we have our bijection.  $\square$



If  $M \xrightarrow{f} N$  is a right  $\mathcal{M}$ -approximation of  $N$ ,  $M' \in \mathcal{M}$ , and  $M' \xrightarrow{f'} N$  is some morphism, then  $M \oplus M' \xrightarrow{(f \ f')} N$  is a right  $\mathcal{M}$ -approximation of  $N$  as well. Indeed, let  $M'' \in \mathcal{M}$  and  $M'' \xrightarrow{f''} N$  be any morphism. Then there exists some  $M'' \xrightarrow{h} M$  such that  $f'' = fh$ . Then clearly is easily seen that the diagram

$$\begin{array}{ccc} M \oplus M' & \xrightarrow{(f \ f')} & N \\ \begin{pmatrix} h \\ 0 \end{pmatrix} \uparrow & \nearrow f'' & \\ M'' & & \end{array}$$

commutes as well. Conversely, we show in Lemma 3.7 that for any  $\mathcal{M}$ -approximation, there is a decomposition such that one of the components is a minimal approximation. This lemma, and by extension Proposition 3.2, are the main reasons for why we in this thesis assume the Krull-Schmidt categories to be Hom-finite.

**Lemma 3.7.** *Let  $\mathcal{T}$  be a Hom-finite category over some field  $k$ . For any morphism  $M \xrightarrow{f} N$  in  $\mathcal{T}$ , there are decompositions*

(i)

$$M \cong X \oplus Y \xrightarrow{f=(f_X \ f_Y)} N$$

where  $X \xrightarrow{f_X} N$  is a right minimal morphism and  $f_Y = 0$ .

(ii)

$$M \xrightarrow{f=\begin{pmatrix} f_X \\ f_Y \end{pmatrix}} X \oplus Y \cong N$$

where  $M \xrightarrow{f_X} X$  is a left minimal morphism and  $f_Y = 0$ .

*Proof.* We prove part (i). The a proof of part (ii) is dual.

Define  $A := M \oplus N$ . From Proposition 3.2 we have the equivalence

$$\text{add}\{A\} \xrightarrow{\text{Hom}_{\mathcal{T}}(A, -)} \mathcal{P}(\text{mod } \Gamma)$$

where  $\Gamma := \text{End}_{\mathcal{T}}(A)^{\text{op}}$ . By applying this equivalence to the morphism  $M \xrightarrow{f} N$ , we then obtain

$$M_P := \text{Hom}_{\mathcal{T}}(A, M) \xrightarrow{f_P := \text{Hom}_{\mathcal{T}}(A, f)} \text{Hom}_{\mathcal{T}}(A, N) =: N_P,$$

i.e. the morphism  $M_P \xrightarrow{f_P} N_P$  in  $\text{mod } \Gamma$ . In this case, we know [3] there is a decomposition

$$M_P \cong M'_P \oplus M''_P \xrightarrow{f_P=(f'_P \ f''_P)} N_P$$

where  $M'_P \xrightarrow{f'_P} N_P$  is right minimal and  $f''_P = 0$ .

Since  $\text{Hom}_{\mathcal{T}}(A, -)$  is dense, there are  $M'$  and  $M''$  in  $\text{add}\{A\}$  such that  $\text{Hom}_{\mathcal{T}}(A, M') \cong M'_P$  and  $\text{Hom}_{\mathcal{T}}(A, M'') \cong M''_P$ . It follows that

$$\text{Hom}_{\mathcal{T}}(A, M' \oplus M'') \cong M'_P \oplus M''_P \cong M_P \cong \text{Hom}_{\mathcal{T}}(A, M),$$

and as  $\text{Hom}_{\mathcal{T}}(A, -)$  is an equivalence,  $M \cong M' \oplus M''$ . Also, as  $\text{Hom}_{\mathcal{T}}(A, -)$  is full, there are morphisms  $M' \xrightarrow{f'} N$  and  $M'' \xrightarrow{f''} N$  such that  $\text{Hom}_{\mathcal{T}}(A, f') = f'_P$  and  $\text{Hom}_{\mathcal{T}}(A, f'') = f''_P$ . As

$$\begin{aligned} f_P &= (f'_P \ f''_P) \\ &= (\text{Hom}_{\mathcal{T}}(A, f') \ \text{Hom}_{\mathcal{T}}(A, f'')) \\ &= \text{Hom}_{\mathcal{T}}(A, (f' \ f'')) \end{aligned}$$

we have  $f = (f' \ f'')$ . As  $\text{Hom}_{\mathcal{T}}(A, -)$  is faithful,  $f'' = 0$ . To see that  $f'$  is right minimal, let  $M' \xrightarrow{g} M'$  be such that  $f'g = f'$ . Then by applying  $\text{Hom}_{\mathcal{T}}(A, -)$ , we see that  $\text{Hom}_{\mathcal{T}}(A, g)$  is an isomorphism by the minimality of  $f'_P$ . Again, by  $\text{Hom}_{\mathcal{T}}(A, -)$  being an equivalence, it follows that  $g$  is an isomorphism, and that  $f'$  is right minimal.  $\square$

By applying Lemma 3.7, we immediately get the following equivalent definitions of right- and left minimal morphisms

**Corollary 3.8.** *Let  $k$  be a field and  $\mathcal{T}$  an additive category such that for all  $A, B \in \mathcal{T}$ ,  $\text{Hom}_{\mathcal{T}}(A, B)$  is a finite dimensional vector space over  $k$ . Let  $X \xrightarrow{f} Y$  be a morphism in  $\mathcal{T}$ .*

(I) *The following are equivalent:*

(i)  *$f$  is right minimal.*

(ii) *For any decomposition*

$$X = X' \oplus X'' \xrightarrow{f=(f' \ f'')} Y$$

*with  $X' \neq 0$ ,  $f' \neq 0$ .*

(II) *The following are equivalent:*

(iii)  *$f$  is left minimal.*

(iv) *For any decomposition*

$$X \xrightarrow{f=\begin{pmatrix} f' \\ f'' \end{pmatrix}} Y' \oplus Y'' = Y$$

*with  $Y' \neq 0$ ,  $f' \neq 0$ .*

*Proof.* We prove (I). Let  $f$  be right minimal. To arrive at a contradiction, assume there is a decomposition

$$X = X' \oplus X'' \xrightarrow{f=(f' \ f'')} Y,$$

with  $X'$  and  $X''$  nonzero but  $f' = 0$ . Then the diagram

$$\begin{array}{ccc} X' \oplus X'' & \xrightarrow{f} & Y \\ h=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \downarrow & & \downarrow 1 \\ X' \oplus X'' & \xrightarrow{f} & Y \end{array}$$

commutes. This is a contradiction, as  $h$  is not an isomorphism. Thus no such decomposition exists.

Assume (ii). By Lemma 3.7, there is a decomposition of  $X$  and  $f$  as

$$X = X' \oplus X'' \xrightarrow{f=(f' \ f'')} Y,$$

where  $f'$  is right minimal and  $f'' = 0$ . By our assumption, we then have that  $X'' = 0$ , and so  $X = X'$  and  $f = f'$ , which is right minimal.

The proof of (II) is dual.  $\square$

It is now easy to show that sums of minimal morphisms are themselves minimal. More precisely, we have the following result:

**Corollary 3.9.** *Let  $\mathcal{T}$  be a Krull-Schmidt triangulated category, and let  $X \xrightarrow{f} Y$  and  $X' \xrightarrow{f'} Y'$  be morphisms.*

(i) *If  $f$  and  $f'$  are left minimal, then  $X \oplus X' \xrightarrow{f \oplus f'} Y \oplus Y'$  is left minimal.*

(ii) *If  $f$  and  $f'$  are right minimal, then  $X \oplus X' \xrightarrow{f \oplus f'} Y \oplus Y'$  is right minimal.*

*Proof.* A proof is provided for part (ii). As  $\mathcal{T}$  is Krull-Schmidt, there are unique (up to isomorphism) decompositions

$$X = \bigoplus_{i=1}^n X_i \xrightarrow{f=(f_1 \cdots f_n)} Y,$$

and

$$X' = \bigoplus_{i=n+1}^{n+m} X_i \xrightarrow{f'=(f_{n+1} \cdots f_{n+m})} Y',$$

where the  $X_i$  are indecomposable. By Corollary 3.8 (I), the  $f_i$  are all nonzero. The direct sum of these is

$$\bigoplus_{i=1}^{n+m} X_i \xrightarrow{\begin{pmatrix} f_1 & \cdots & f_n & 0 & \cdots & 0 \\ 0 & \cdots & 0 & f_{n+1} & \cdots & f_{n+m} \end{pmatrix} = f \oplus f'} Y \oplus Y',$$

and the restriction of this morphism to an indecomposable summand is either  $\begin{pmatrix} f_i \\ 0 \end{pmatrix} \neq 0$  or  $\begin{pmatrix} 0 \\ f_i \end{pmatrix} \neq 0$ . Again, by Corollary 3.8, this gives that  $f \oplus f'$  is a right minimal morphism. The proof for part (i) is dual.  $\square$

From Kelly [16], we have the following notions of an *ideal* of a (pre)-additive category

**Definition 3.10.** For a pre-additive category  $\mathcal{A}$ , we define an *ideal*  $\mathcal{I}$  in  $\mathcal{A}$  as follows:

(i)  $\mathbf{Ob} \mathcal{I} = \mathbf{Ob} \mathcal{A}$ .

(ii) For any objects  $X, Y \in \mathcal{I}$ ,  $\mathrm{Hom}_{\mathcal{I}}(X, Y) \subseteq \mathrm{Hom}_{\mathcal{A}}(X, Y)$  is a subgroup with the property that for any  $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ ,  $g \in \mathrm{Hom}_{\mathcal{I}}(B, C)$  and  $h \in \mathrm{Hom}_{\mathcal{A}}(C, D)$

$$A \xrightarrow{f \in \mathcal{A}} B \xrightarrow{g \in \mathcal{I}} C \xrightarrow{h \in \mathcal{A}} D$$

then  $gf \in \mathrm{Hom}_{\mathcal{I}}(A, C)$  and  $hg \in \mathrm{Hom}_{\mathcal{I}}(B, D)$ .

We say that  $\mathcal{I}$  is a *maximal ideal* in  $\mathcal{A}$  if it is a proper ideal in  $\mathcal{A}$  and it is not properly contained in any proper ideal of  $\mathcal{A}$ . The *Jacobson radical* of  $\mathcal{A}$ ,  $J_{\mathcal{A}}$  is defined to be the intersection of all the maximal ideals of  $\mathcal{A}$ .

Also from [16], the Jacobson radical  $J_{\mathcal{A}}$  has the following properties.

**Proposition 3.11.** *Let  $\mathcal{A}$  be an additive category.*

- (i) *A morphism  $X \xrightarrow{f} Y$  between indecomposable objects in  $\mathcal{T}$  is in the Jacobson radical if and only if it is not an isomorphism.*
- (ii) *A morphism  $\bigoplus_{i=1}^n X_i \xrightarrow{f} \bigoplus_{j=1}^m Y_j$  with  $X_i, Y_j$  indecomposable is in the Jacobson radical if and only if all its components  $X_i \xrightarrow{f_{ij}} Y_j$  are not isomorphisms.*
- (iii) *A morphism  $r$  is in the Jacobson radical if and only if  $1 - r$  is an isomorphism.*

**Lemma 3.12.** *Let  $\mathcal{T}$  be a Krull-Schmidt triangulated category, and assume there is a triangle*

$$X \xrightarrow{g} Y \xrightarrow{f} Z \longrightarrow X[1]$$

*in  $\mathcal{T}$  with  $f$  a right minimal morphism. Then  $g$  is in  $J_{\mathcal{T}}$ .*

*Proof.* Assume  $g$  is not in  $J_{\mathcal{T}}$ . Then there is an indecomposable summand  $X' \neq 0$  of  $X$  which is isomorphic to a summand  $Y'$  of  $Y$  under  $g$ . In other words, by letting the vertical morphisms be the inclusions into the coproducts, there is an isomorphism  $g'$  such that the diagram

$$\begin{array}{ccccc} X' & \xrightarrow{g'} & Y' & & \\ \downarrow \iota_X & & \downarrow \iota_Y & & \\ X & \xrightarrow{g} & Y & \xrightarrow{f} & Z \longrightarrow X[1]. \end{array}$$

commutes. Since  $f$  is right minimal, we have by Corollary 3.8 that its restriction to any nonzero summand is nonzero. I.e.  $f\iota_Y \neq 0$ , and as  $g'$  is an isomorphism, neither is  $f\iota_Y g'$ . But since  $f$  and  $g$  are successive morphisms in a triangle,  $f\iota_Y g' = fg\iota_X = 0$ . Thus we have arrived at a contradiction, showing that  $g$  is in  $J_{\mathcal{T}}$ .  $\square$

The following lemma is an important tool used in later proofs. The parts of it correspond to Propositions 2.1, 2.3 and the dual to 2.3 by Iyama and Yoshino [14], respectively. Again, we honor tradition and refer to parts (ii) and (iii) collectively as 'Wakamatsu's Lemma'.

**Lemma 3.13.** *Let  $\mathcal{T}$  be a Krull-Schmidt triangulated category, and  $\mathcal{M}$  a subcategory of  $\mathcal{T}$ .*

- (i) *If  $\mathcal{N} \subseteq \mathcal{T}$  is a subcategory such that  $\text{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{N}) = 0$ . Then  $\mathcal{M} * \mathcal{N}$  is closed under taking direct summands:  $\text{smd}(\mathcal{M} * \mathcal{N}) = \mathcal{M} * \mathcal{N}$ .*
- (ii) *If  $\mathcal{M}$  is contravariantly finite in  $\mathcal{T}$  and  $\mathcal{M} * \mathcal{M} \subseteq \mathcal{M}$ , then  $(\mathcal{M}, \mathcal{M}^{\perp \tau})$  is a torsion pair.*
- (iii) *If  $\mathcal{M}$  is covariantly finite in  $\mathcal{T}$  and  $\mathcal{M} * \mathcal{M} \subseteq \mathcal{M}$ , then  $({}^{\perp \tau} \mathcal{M}, \mathcal{M})$  is a torsion pair.*

*Proof.* We show parts (i) and (ii), as (iii) is just dual to (ii).

- (i) Let  $X_1 \oplus X_2 \in \mathcal{M} * \mathcal{N}$ . There is a triangle

$$M \xrightarrow{\begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix}} X_1 \oplus X_2 \xrightarrow{\begin{pmatrix} b'_1 & b'_2 \end{pmatrix}} N \longrightarrow M[1]$$

with  $M \in \mathcal{M}$  and  $N \in \mathcal{N}$ . Each of the  $a'_i$  are easily seen to be right  $\mathcal{M}$ -approximations of  $X_i$ , respectively, and so by Lemma 3.7, these decompose as

$$M = M_i \oplus M''_i \xrightarrow{a'_i = (a_i \ 0)} X_i$$

with  $M_i \xrightarrow{a_i} X_i$  right minimal  $\mathcal{M}$ -approximations of the  $X_i$ . We complete both these to triangles

$$M_i \xrightarrow{a_i} X_i \longrightarrow U_i \longrightarrow M_i[1]$$

and add them to obtain

$$M_1 \oplus M_2 \xrightarrow{\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}} X_1 \oplus X_2 \longrightarrow U_1 \oplus U_2 \longrightarrow M_1[1] \oplus M_2[1].$$

Here, the left morphism is a right minimal  $\mathcal{M}$ -approximation of  $X_1 \oplus X_2$  by Corollary 3.9, so by Lemma 1.8  $M_1 \oplus M_2$  is a direct summand of  $M$ , and more importantly,  $U_1 \oplus U_2$  is a direct summand of  $N$ . As  $\mathcal{N}$  is closed under direct summands,  $U_i \in \mathcal{N}$ , and so  $X_i \in \mathcal{M} * \mathcal{N}$ .

(ii) Clearly, by definition, we have that  $\text{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{M}^\perp) = 0$ . What remains then is to show that  $\mathcal{T} = \mathcal{M} * \mathcal{M}^\perp$ .

Let  $T \in \mathcal{T}$ . As  $\mathcal{M}$  is contravariantly finite in  $\mathcal{T}$  we have by Lemma 3.7 that there is a triangle

$$M \xrightarrow{a} T \xrightarrow{b} N \xrightarrow{c} M[1]$$

with  $a$  a right minimal  $\mathcal{M}$ -approximation of  $T$ . Then by Lemma 3.12,  $c \in J_{\mathcal{T}}$ . To finish the proof we need only see that  $N \in \mathcal{M}^\perp$ .

Let  $M' \xrightarrow{f} N$  be any morphism with  $M' \in \mathcal{M}$ , and complete it to a triangle

$$M' \xrightarrow{f} N \xrightarrow{g} L \xrightarrow{h} M'[1].$$

By the octahedral axiom, we then have the diagram

$$\begin{array}{ccccccc} T & \xrightarrow{b} & N & \xrightarrow{c} & M[1] & \xrightarrow{-a[1]} & T[1] \\ & \searrow & \downarrow g & & \downarrow \text{dotted} & & \\ & & L & \xrightarrow{\quad} & M''[1] & \xrightarrow{-d[1]} & T[1] \\ & \searrow gb & & & \downarrow \text{dotted} & & \\ & & & & M'[1] & \xrightarrow{-f[1]} & N[1] \\ & & & & \downarrow \text{dotted} & & \\ & & & & M[2] & & \end{array}$$

The vertical triangle gives us  $M'' \in \mathcal{M} * \mathcal{M} = \mathcal{M}$ , and a morphism  $M'' \xrightarrow{d} T$ . We refit the triangles

into the solid parts of the diagram

$$\begin{array}{ccccccc}
& & M'' & & M' & & \\
& & \downarrow d & & \downarrow f & & \\
M & \xrightarrow{a} & T & \xrightarrow{b} & N & \xrightarrow{c} & M[1] \\
& & \downarrow gb & & \downarrow g & & \\
& & L & \xrightarrow{g'} & L & & \\
& & \downarrow & & \downarrow h & & \\
& & M''[1] & & M'[1] & & 
\end{array}$$

As  $a$  is a right  $\mathcal{M}$ -approximation of  $T$ ,  $d$  factors through  $a$ , and so  $bd = 0$ . Then for some  $g'$  we get  $b = g'gb$ , and so  $(1_N - g'g)b = 0$ . This again means that  $1_N - g'g$  factors through  $c$ , and that  $1_N - g'g \in J_{\mathcal{T}}$ . Then  $g'g$  is an isomorphism by Proposition 3.11, and so  $g$  is a split monomorphism. By [11] this means  $f = 0$ , so  $N \in \mathcal{X}^{\perp}$ .  $\square$

This next result is a fundamental property of torsion pairs, and is used to make Proposition 3.15 easier to prove.

**Lemma 3.14.** *Let  $\mathcal{T}$  be any triangulated category. If  $(\mathcal{X}, \mathcal{Y})$  and  $(\mathcal{X}', \mathcal{Y})$  are torsion pairs in  $\mathcal{T}$ , then  $\mathcal{X}' = \mathcal{X}$ . Likewise, if  $(\mathcal{X}, \mathcal{Y})$  and  $(\mathcal{X}, \mathcal{Y}')$  are torsion pairs, then  $\mathcal{Y} = \mathcal{Y}'$ .*

*Proof.* Let  $X \in \mathcal{X}$ , so  $X \in \mathcal{T} = \mathcal{X}' * \mathcal{Y} \subseteq \text{smd}(\mathcal{X}' * \mathcal{Y})$ . Since  $\text{Hom}_{\mathcal{T}}(X, \mathcal{Y}) = 0$ , we have by Lemma 2.21 that  $X \in \mathcal{X}'$ . Then by a symmetric argument, we arrive at  $\mathcal{X} = \mathcal{X}'$ . This line of reasoning is easily adjusted to show the second statement.  $\square$

As alluded to earlier, in the setting of Krull-Schmidt triangulated categories, Proposition 2.22 reaches a simpler form. Proposition 3.15 (i) and (ii) update Proposition 2.22 to this new setting, while part (iii) shows the left orthogonal complement of  $\mathcal{T}_{\mathcal{M}}^{\leq 0}$  is the 'negative' part of  $\mathcal{T}$  as given by (i). Finally, we see how the  $\mathcal{T}_{\mathcal{M}}^{\leq 0}$  are used to make torsion pairs and co-t-structures.

**Proposition 3.15.** *Let  $\mathcal{T}$  be a Krull-Schmidt triangulated category, and  $\mathcal{M} \in \text{silt } \mathcal{T}$ . Then*

(i)

$$\mathcal{T} = \bigcup_{l \geq 0} \mathcal{M}[-l] * \cdots * \mathcal{M}[l],$$

(ii)

$$\mathcal{T}_{\mathcal{M}}^{\leq 0} = \bigcup_{l \geq 0} \mathcal{M} * \cdots * \mathcal{M}[l],$$

and

(iii)

$${}^{\perp}(\mathcal{T}_{\mathcal{M}}^{\leq 0}) = \bigcup_{l > 0} \mathcal{M}[-l] * \cdots * \mathcal{M}[-1].$$

Furthermore,  $({}^{\perp}(\mathcal{T}_{\mathcal{M}}^{\leq 0}), \mathcal{T}_{\mathcal{M}}^{\leq 0})$  is a torsion pair of  $\mathcal{T}$ , and  $({}^{\perp}(\mathcal{T}_{\mathcal{M}}^{\leq 0}), \mathcal{T}_{\mathcal{M}}^{\leq 0})$  is a co-t-structure with coheart  $\mathcal{M}$ .

*Proof.* By Lemma 2.19

$$\mathrm{Hom}_{\mathcal{T}}(\mathcal{M}[-l], \mathcal{M}[-l+1] * \cdots * \mathcal{M}[l]) = 0$$

for all  $l \geq 0$ . Then by Lemma 3.13 (i)  $\mathrm{smd}(\mathcal{M}[-l] * \cdots * \mathcal{M}[l]) = \mathcal{M}[-l] * \cdots * \mathcal{M}[l]$ . By Proposition 2.22 (i), we then get part (i).

Part (ii) is proved similarly by Proposition 2.22 (ii) and observing that by Lemma 2.19

$$\mathrm{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{M}[1] * \cdots * \mathcal{M}[l]) = 0$$

for any  $l \geq 0$ .

For part (iii), let

$$RHS := \bigcup_{l > 0} \mathcal{M}[-l] * \cdots * \mathcal{M}[-1].$$

Let  $X$  be any object in  $RHS$ , so  $X \in \mathcal{M}[-l'] * \cdots * \mathcal{M}[-1]$  for some  $l' > 0$ . By part (ii), any  $Y \in \mathcal{T}_{\mathcal{M}}^{\leq 0}$  is in  $\mathcal{M} * \cdots * \mathcal{M}[l]$  for some  $l \geq 0$ . Then we have by Lemma 2.19 that  $\mathrm{Hom}_{\mathcal{T}}(X, Y) = 0$ , so we have that

$$\mathrm{Hom}_{\mathcal{T}}(RHS, \mathcal{T}_{\mathcal{M}}^{\leq 0}) = 0.$$

Let  $T \in \mathcal{T}$ . By part (i) we have that  $T \in \mathcal{M}[-l] * \cdots * \mathcal{M}[l]$  for some  $l \geq 0$ . That is, by Remark 1.3

$$X \in (\mathcal{M}[-l] * \cdots * \mathcal{M}[-1]) * (\mathcal{M} * \cdots * \mathcal{M}[l]) \subseteq RHS * \mathcal{T}_{\mathcal{M}}^{\leq 0},$$

showing  $\mathcal{T} = RHS * \mathcal{T}_{\mathcal{M}}^{\leq 0}$ .

Then  $(RHS, \mathcal{T}_{\mathcal{M}}^{\leq 0})$  is a torsion pair in  $\mathcal{T}$ . By Corollary 1.12,  $\mathcal{T}_{\mathcal{M}}^{\leq 0}$  is a covariantly finite subcategory of  $\mathcal{T}$ . Also,  $\mathcal{T}_{\mathcal{M}}^{\leq 0}$  is closed under extensions, and it follows from Lemma 3.13 (iii) that  $({}^{\perp}(\mathcal{T}_{\mathcal{M}}^{\leq 0}), \mathcal{T}_{\mathcal{M}}^{\leq 0})$  is also a torsion pair in  $\mathcal{T}$ .

We then get by Lemma 3.14 that  $RHS = {}^{\perp}(\mathcal{T}_{\mathcal{M}}^{\leq 0})$ , which shows part (iii).

It is easy to check from first principles that  ${}^{\perp}(\mathcal{T}_{\mathcal{M}}^{\leq 0})[-1] = {}^{\perp}(\mathcal{T}_{\mathcal{M}}^{\leq 0})$ , so  $({}^{\perp}(\mathcal{T}_{\mathcal{M}}^{\leq 0})[-1], \mathcal{T}_{\mathcal{M}}^{\leq 0})$  is a torsion pair in  $\mathcal{T}$ . Similarly, we see that  ${}^{\perp}(\mathcal{T}_{\mathcal{M}}^{\leq 0})[-1] \subseteq {}^{\perp}(\mathcal{T}_{\mathcal{M}}^{\leq 0})$ , so by Definition 1.11 we have that  $({}^{\perp}(\mathcal{T}_{\mathcal{M}}^{\leq 0}), \mathcal{T}_{\mathcal{M}}^{\leq 0})$  is a co-t-structure in  $\mathcal{T}$ .

By Proposition 2.24,  $\mathcal{T}_{\mathcal{M}}^{\leq 0} \cap {}^{\perp}(\mathcal{T}_{\mathcal{M}}^{\leq 0}) = \mathcal{M}$ , and this co-t-structure has coheart  $\mathcal{M}$ .  $\square$

The following proposition concerns the kinds of triangles which appear in Example 3.17. We show that there are restrictions which can be made to these triangles, which then allows for an updated definition of the  $\mathcal{M}$ -resolution of an object in  $\mathcal{T}$ .

**Proposition 3.16.** *Let  $\mathcal{T}$  be a Krull-Schmidt triangulated category. For any silting subcategory  $\mathcal{M}$  of  $\mathcal{T}$ , and any  $N_0 \in \mathcal{T}_{\mathcal{M}}^{\leq 0}$ , there is, for some  $l \geq 0$  a sequence of triangles*

$$\begin{array}{ccccccc} N_1 & \xrightarrow{g_1} & M_0 & \xrightarrow{f_0} & N_0 & \longrightarrow & N_1[1], \\ & & & & & & \dots \\ & & & & & & \dots \\ N_l & \xrightarrow{g_l} & M_{l-1} & \xrightarrow{f_{l-1}} & N_{l-1} & \longrightarrow & N_l[1], \\ & & & & & & \dots \\ 0 & \xrightarrow{g_{l+1}} & M_l & \xrightarrow{f_l} & N_l & \longrightarrow & 0, \end{array}$$

such that  $f_i$  is a minimal right  $\mathcal{M}$ -approximation of  $N_i$  and  $g_{i+1}$  belongs to  $\mathcal{J}_{\mathcal{T}}$  for each  $0 \leq i \leq l$ .

*Proof.* As  $N_0 \in \mathcal{T}_{\mathcal{M}}^{\leq 0}$ , we have by Proposition 3.15 that  $N_0 \in \mathcal{M} * \cdots * \mathcal{M}[l]$  for some  $l \geq 0$ . Then by the rotation axiom there is a triangle

$$N'_1 \xrightarrow{u} M'_0 \xrightarrow{f'_0} N_0 \xrightarrow{v} N'_1[1] \quad (7)$$

where  $M'_0 \in \mathcal{M}$  and  $N'_1[1] \in \mathcal{M}[1] * \cdots * \mathcal{M}[l]$ , and  $f'_0$  is a right  $\mathcal{M}$ -approximation of  $N_0$  by Lemma 1.7 and Lemma 2.19.

Thus by Lemma 3.7 we can write  $M'_0 = M_0 \oplus M''_0 \xrightarrow{f=(f_0 \ 0)} N_0$ , where  $f_0$  is a minimal right  $\mathcal{M}$ -approximation of  $N_0$ , and we have a triangle

$$N_1 \xrightarrow{g_1} M_0 \xrightarrow{f_0} N_0 \longrightarrow N_1[1], \quad (8)$$

where by Lemma 3.12,  $g_1$  is in the Jacobson radical. By Lemma 1.8, we also have that  $N_1$  is a direct summand of  $N'_1$ , so  $N_1 \in \mathcal{M}[1] * \cdots * \mathcal{M}[l]$  by Lemmas 3.13 (i) and 2.19.

Then, make a triangle (7) for  $N_1$ , and repeat the exact same arguments as done for the previous one to obtain the next triangle of the form (8). Then do the same again for  $N_2$ , and so on, until we reach  $N_l \in \mathcal{M}$ , providing us with the final triangle where  $f_l$  is an isomorphism.  $\square$

**Example 3.17.** In abelian categories  $\mathcal{A}$  with enough projectives, we consider the projective resolutions  $p(X)$  of objects  $X$ . These are complexes

$$\cdots \longrightarrow P^{-1} \xrightarrow{p^{-1}} P^0 \xrightarrow{p^0} X \longrightarrow 0 \longrightarrow \cdots,$$

which are exact everywhere, except at  $X$ , where  $\text{cok } p^{-1} = X$ . The notion of being a projective object  $P$  is however dependent on the existence of an exact structure on  $\mathcal{A}$ , as the property is given by  $\text{Hom}_{\mathcal{T}}(P, -)$  being exact. Thus projectivity and projective resolutions lose much of its merit in this setting.

By using the theory we introduce in this thesis we can mimic much of the construction of projective resolutions. This is not meant as a straight up alternative, but rather as a way to help understand this new theory in terms of something familiar.

Recall that the construction of the projective resolution is as follows: For  $X \in \mathcal{A}$  take a projective cover  $P^0 \xrightarrow{p^0} X$ , and the kernel  $\ker p^0 \xrightarrow{\iota^0} P^0$  of this morphism. Then take a projective cover  $P^{-1} \xrightarrow{\pi^{-1}} \ker p^0$ , and compose the two to the morphism  $p^{-1} := \iota^0 \pi^{-1}$ . Continue this by repeating these steps, using  $p^{-1}$  in the place of  $p^0$ , and so on. The following diagram illustrates this.

$$\begin{array}{ccccccc} \cdots & \xrightarrow{p^{-3}} & P^{-2} & \xrightarrow{p^{-2}} & P^{-1} & \xrightarrow{p^{-1}} & P^0 \xrightarrow{p^0} X \\ & \searrow \pi^{-3} & \nearrow \iota^{-2} & \searrow \pi^{-2} & \nearrow \iota^{-1} & \searrow \pi^{-1} & \nearrow \iota^0 \\ & & \ker p^{-2} & & \ker p^{-1} & & \ker p^0 \end{array}$$

Note that  $\ker p^i = \ker \pi^i$  as the  $\iota^{i+1}$  are inclusions.

We mimic this algorithmic construction by using triangles in place of short exact sequences, i.e. substitute the kernels by the corresponding weak kernels (cylinders) from the triangles.



Let  $\mathcal{T}$  be a Krull-Schmidt triangulated category and  $\mathcal{M} \in \text{silt } \mathcal{T}$ . For any  $X \in \mathcal{T}$  there are by Proposition 3.15 (i) an  $l \geq 0$  and  $M_{-l}, \dots, M_l \in \mathcal{M}$  such that  $X \in M_{-l}[-l] * \dots * M_l[l]$ . From this we produce the diagram

$$\begin{array}{ccccccc}
& & & 0 & \cdots & & N_{-l+2}[1] \\
& & & \nearrow & & & \nearrow \\
& & & N_l & \cdots & & N_{-l+1} \\
& & f_l \nearrow & & g_l \searrow & & f_{-l+1} \nearrow & & g_{-l+1} \searrow \\
& & M_l[-l] & \cdots & M_{l-1}[-l] & \cdots & M_{-l+1}[-l] & \cdots & M_{-l}[-l] \\
& & \cdots & & \cdots & & \cdots & & \cdots \\
& & \nearrow & & \searrow & & \nearrow & & \searrow \\
0 & & g_{l+1} & & & & g_{-l+2} & & f_{-l} \\
& & & & & & N_{-l+2} & & X \\
& & & & & & \searrow & & \searrow \\
& & & & & & N_{-l+3}[1] & & N_{-l+1}[1]
\end{array}$$

where the diagonal parts are triangles with  $N_{-l+i} \in M_{-l+i}[-l] * \dots * M_l[l-i]$ . Similarly to how the projective resolution produces a complex of projective objects, this yields a complex

$$\cdots \longrightarrow 0 \longrightarrow M_l[-l] \xrightarrow{g_l f_l} \cdots \xrightarrow{g_{-l+1} f_{-l+1}} M_{-l}[-l] \xrightarrow{f_{-l}} X \longrightarrow 0 \longrightarrow \cdots$$

with objects  $M_i[-l] \in \mathcal{M}[-l]$ . We call this an  $\mathcal{M}[-l]$ -resolution of  $X$ .

By Proposition 3.15 (ii), we can do even better by restricting  $X$  to be in  $\mathcal{T}_{\mathcal{M}}^{\leq 0}$ : Then for some  $l \geq 0$  there are  $M_0, \dots, M_l \in \mathcal{M}$  such that  $X \in M_0 * \dots * M_l[l]$ , and we get the diagram

$$\begin{array}{ccccccc}
& & & 0 & \cdots & & N_4[1] & & & & N_2[1] & & (9) \\
& & & \nearrow & & & \nearrow & & & & \nearrow & & \\
& & & N_l & \cdots & & N_3 & & & & N_1 & & \\
& & f_l \nearrow & & g_l \searrow & & g_3 \searrow & & & & f_1 \nearrow & & g_1 \searrow \\
& & M_l & \cdots & M_{l-1} & \cdots & M_2 & \cdots & M_1 & \cdots & M_0 & & \\
& & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \\
& & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow \\
0 & & g_{l+1} & & f_{l-1} & & g_{l-1} & & f_2 & & g_2 & & f_0 \\
& & & & & & N_{l-1} & & N_2 & & N_2 & & X \\
& & & & & & \searrow & & \searrow & & \searrow & & \searrow \\
& & & & & & N_l[1] & & \cdots & & N_3[1] & & N_1[1],
\end{array}$$

where the diagonal parts are triangles with  $N_i \in M_i * \dots * M_l[l-i]$ . As before, this yields a complex

$$\cdots \longrightarrow M_l \xrightarrow{g_l f_l} \cdots \xrightarrow{g_2 f_2} M_1 \xrightarrow{g_1 f_1} M_0 \longrightarrow X \longrightarrow 0 \longrightarrow \cdots,$$

and time the objects  $M_i$  are in  $\mathcal{M}$ . The similarities to projective resolutions are evident: Instead of projective covers, we take  $f_i$ , which are right minimal  $\mathcal{M}$ -approximations by Lemmas 1.7 and 2.19, and instead of the kernel of said covers, we take the *weak kernel* of  $f_i$  by choosing the morphism  $g_{i+1}$ .

This complex is now an  $\mathcal{M}$ -resolution of  $X$ , and we can define the *length* of it to be the largest index  $i$  of a nonzero  $M_i$ . Continuing down this path of analogous concepts, we define the  $\mathcal{M}$ -dimension of  $X$  to be the smallest  $l \geq 0$  such that there exists an  $\mathcal{M}$ -resolution of  $X$  of length  $l$ . Now it is clear that  $\mathcal{T}_{\mathcal{M}}^{\leq 0}$  is exactly the subcategory of  $\mathcal{T}$  of objects of *finite  $\mathcal{M}$ -dimension*, where

$$\mathcal{M} * \dots * \mathcal{M}[l]$$

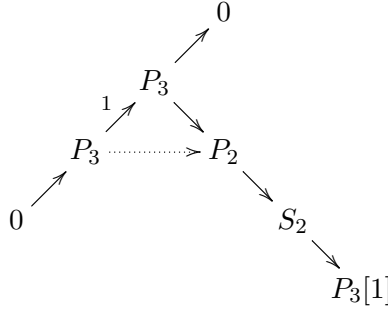
is the collection of objects in of  $\mathcal{M}$ -dimension less than or equal to  $l$ .

Building on this, Proposition 3.16 asserts that for any  $X \in \mathcal{T}_{\mathcal{M}}^{\leq 0}$ , we have a diagram (9) where the  $f_i$  are right minimal  $\mathcal{M}$ -approximations and  $g_{i+1}$  are in  $J_{\mathcal{T}}$ . We then say that the complex

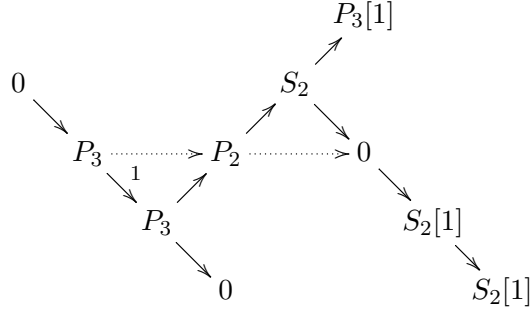
$$\dots \longrightarrow 0 \longrightarrow M_l \xrightarrow{g_l f_l} \dots \xrightarrow{g_2 f_2} M_1 \xrightarrow{g_1 f_1} M_0 \longrightarrow X \longrightarrow 0 \longrightarrow \dots$$

is a *minimal  $\mathcal{M}$ -resolution of  $X$* . Note that as  $g_{i+1} \in J_{\mathcal{T}}$ , so is  $g_{i+1} f_i$ . Thus in a minimal  $\mathcal{M}$ -resolution, all the differentials are non-isomorphisms on the summands of the  $M_i$ .

Now a concrete example: In the case where  $\mathcal{T} = \text{K}^b(\mathcal{P}(\text{mod } kA_3))$  and  $\mathcal{M} = \text{add}\{P_1 \oplus P_2 \oplus P_3\}$ , we get for the objects  $S_2$  and  $S_2[1]$  the diagrams



and



and the corresponding  $\mathcal{M}$ -resolutions

$$\dots \longrightarrow 0 \longrightarrow P_3 \longrightarrow P_2 \longrightarrow S_2 \longrightarrow 0 \longrightarrow \dots$$

and

$$\dots \longrightarrow 0 \longrightarrow P_3 \longrightarrow P_2 \longrightarrow 0 \longrightarrow S_2[1] \longrightarrow 0 \longrightarrow \dots$$

respectively.

This showcases the distinction between projective resolutions and  $\mathcal{M}$ -resolutions. A projective resolution ends once  $\ker p^i = 0$ , whereas this needs not be the case for  $\mathcal{M}$ -resolutions, allowing for *more* resolutions in  $\mathcal{T}$  than in  $\text{mod } kA_3$ .

Having two objects such that we can apply Proposition 3.16, we can read from their corresponding sets of triangles the following property.

**Lemma 3.18.** *Let  $\mathcal{T}$  be a Krull-Schmidt triangulated category,  $\mathcal{M}$  a silting subcategory of  $\mathcal{T}$  and  $N_0, N'_0 \in \mathcal{T}_{\mathcal{M}}^{\leq 0}$ . For  $N_0$  and  $N'_0$ , get the respective triangles*

$$\begin{array}{ccc}
N_1 \xrightarrow{g_1} M_0 \xrightarrow{f_0} N_0 \longrightarrow N_1[1], & N'_1 \xrightarrow{g'_1} M'_0 \xrightarrow{f'_0} N'_0 \longrightarrow N'_1[1], \\
\cdots & \cdots \\
N_l \xrightarrow{g_l} M_{l-1} \xrightarrow{f_{l-1}} N_{l-1} \longrightarrow N_l[1], & N'_{l'} \xrightarrow{g'_{l'}} M'_{l'-1} \xrightarrow{f'_{l'-1}} N'_{l'-1} \longrightarrow N'_{l'}[1], \\
0 \xrightarrow{g_{l+1}} M_l \xrightarrow{f_l} N_l \longrightarrow 0. & 0 \xrightarrow{g'_{l'+1}} M'_{l'} \xrightarrow{f'_{l'}} N'_{l'} \longrightarrow 0.
\end{array}$$

from Proposition 3.16.

If  $\text{Hom}_{\mathcal{T}}(N_0, N'_0[l]) = 0$ , then  $\text{add}\{M_l\} \cap \text{add}\{M'_0\} = 0$ .

*Proof.* The objects  $X \in \text{add}\{M_l\} \cap \text{add}\{M'_0\}$  are exactly the ones which are finite coproducts of indecomposable, simultaneous summands of  $M_l$  and  $M'_0$ .

From Proposition 3.11 we know  $a \in \text{Hom}_{\mathcal{T}}(M_l, M'_0)$  belongs to the Jacobson radical if and only if each component of  $a$  is a non-isomorphism. If  $M_l$  and  $M'_0$  share an isomorphic summand, there exists a morphism  $M_l \rightarrow M'_0$  with a component which is an isomorphism. We then show  $\text{add}\{M_l\} \cap \text{add}\{M'_0\} = 0$  by proving any  $a \in \text{Hom}_{\mathcal{T}}(M_l, M'_0)$  is in  $J_{\mathcal{T}}$ .

Assume first  $l = 0$ . Then from  $N_0$  we have the triangle  $0 \longrightarrow M_0 \xrightarrow{f_0} N_0 \longrightarrow 0$ , with  $f_0$  an isomorphism. Apply  $\text{Hom}_{\mathcal{T}}(M_0, -)$  to the triangle to get the long exact sequence

$$\cdots \longrightarrow \text{Hom}_{\mathcal{T}}(M_0, N'_1) \xrightarrow{g'_1 \circ -} \text{Hom}_{\mathcal{T}}(M_0, M'_0) \xrightarrow{f'_0 \circ -} \text{Hom}_{\mathcal{T}}(M_0, N'_0) \longrightarrow \cdots .$$

The right term vanishes by assumption as  $M_0 \cong N_0$  and  $\text{Hom}_{\mathcal{T}}(N_0, N'_0) = 0$ . That is,  $- \circ g'_1$  is surjective, and all morphisms  $M_0 \rightarrow M'_0$  factor through  $g'_1$ . As  $g'_1$  is in the Jacobson radical, any morphism  $M_0 \rightarrow M'_0$  is as well.

Now, assume  $l > 0$ , and apply  $\text{Hom}_{\mathcal{T}}(-, N'_0)$  to the triangles for  $N_0$ . From the first triangle, we get the sequence

$$\cdots \longrightarrow \text{Hom}_{\mathcal{T}}(M_0[-l+1], N'_0) \longrightarrow \text{Hom}_{\mathcal{T}}(N_1[-l+1], N'_0) \longrightarrow \text{Hom}_{\mathcal{T}}(N_0[-l], N'_0) \longrightarrow \cdots ,$$

where the right side vanishes by  $\text{Hom}_{\mathcal{T}}(N_0, N'_0[l]) = 0$  and the left side by

$$\text{Hom}_{\mathcal{T}}(M_0[-l+1], N'_0) \subseteq \text{Hom}_{\mathcal{T}}(\mathcal{M}[-l+1], \mathcal{M} * \cdots * \mathcal{M}[l'])$$

which is 0 by Lemma 2.19. It follows by exactness that the middle term vanishes as well. If there is more than one triangle, we get the sequence

$$\cdots \longrightarrow \text{Hom}_{\mathcal{T}}(M_1[-l+2], N'_0) \longrightarrow \text{Hom}_{\mathcal{T}}(N_2[-l+2], N'_0) \longrightarrow \text{Hom}_{\mathcal{T}}(N_1[-l+1], N'_0) \longrightarrow \cdots ,$$

from the second one. Here the right side vanishes by the previous triangle, and the left side again by Lemma 2.19. So by exactness, the middle term vanishes as well. Continue this for all the triangles, and we get

$$\text{Hom}_{\mathcal{T}}(N_{l-1}[-1], N'_0) \cong \text{Hom}_{\mathcal{T}}(N_{l-2}[-2], N'_0) \cong \cdots \cong \text{Hom}_{\mathcal{T}}(N_0[-l], N'_0) = 0.$$

As  $M_l \cong N_l$ , we rotate the second-to-last triangle from  $N_0$  to get

$$N_{l-1}[-1] \longrightarrow M_l \xrightarrow{g_l} M_{l-1} \xrightarrow{f_{l-1}} N_{l-1}.$$

Let  $a \in \text{Hom}_{\mathcal{T}}(M_l, M'_0)$ , and consider the solid part of the following diagram.

$$\begin{array}{ccccccc} N_{l-1}[-1] & \xrightarrow{\alpha} & M_l & \xrightarrow{g_l} & M_{l-1} & \xrightarrow{f_{l-1}} & N_{l-1} \\ \downarrow & \swarrow \alpha' & \downarrow a & \swarrow h & \downarrow b & & \downarrow \\ N'_1 & \xrightarrow{g'_1} & M'_0 & \xrightarrow{f'_0} & N'_0 & \longrightarrow & N'_1[1]. \end{array}$$

Since  $g_l$  is a left  $\mathcal{M}$ -approximation of  $M_l$  there is a morphism  $M_{l-1} \xrightarrow{h} M'_0$  such that  $a = hg_l$ . Then  $f'_0 a \alpha = f'_0 h g_l \alpha = 0$ , so there is a morphism  $N_{l-1}[-1] \rightarrow N'_1$  making the left square commute. The morphism  $M_{l-1} \xrightarrow{b} N'_0$  then exists by axiom. As  $f'_0$  is a right  $\mathcal{M}$ -approximation of  $N'_0$ , there is a morphism  $M_{l-1} \xrightarrow{b'} M'_0$  such that  $b = f'_0 b'$ . Now, as  $f'_0(a - b'g_l) = 0$ , we have a morphism  $M_l \xrightarrow{a'} N'_1$  such that  $a - b'g_l = g'_1 a'$ . Both  $g_l$  and  $g'_1$  are in  $\mathcal{J}_{\mathcal{T}}$  and so is  $a = g'_1 a' + b'g_l$ , and the proof is complete.  $\square$

The easiest situation to be in regarding silting is when there exists an indecomposable silting object  $M \in \mathcal{T}$ . Given such a situation, Aihara and Iyama provides the following result, asserting that the silting subcategories of  $\mathcal{T}$  are exactly those which are additively generated by a shift of  $M$ . Later, we will show that all silting objects in a Krull-Schmidt category have the same number of indecomposable summands. This result then shows a special case where all silting objects have *one* indecomposable summand.

The most trivial example of such a triangulated category is  $D^b(\text{mod } kA_1)$  – the derived category of the path algebra which arrives from the quiver with only one vertex and no arrows.

**Lemma 3.19.** *Let  $\mathcal{T}$  be a Krull-Schmidt triangulated category. If  $M \in \mathcal{T}$  is an indecomposable silting object, i.e.  $\mathcal{M} := \text{add}\{M\} \in \text{silt } \mathcal{T}$ , then  $\text{silt } \mathcal{T} = \{\text{add}\{M[i]\} \mid i \in \mathbb{Z}\}$*

*Proof.* Let  $\mathcal{N} \in \text{silt } \mathcal{T}$ . By Proposition 2.28 there is an  $N' \in \mathcal{T}$  such that  $\mathcal{N} = \text{add}\{N'\}$ . Assume without loss of generality that  $N'$  is basic.

By Proposition 2.7 we have that  $\text{Hom}_{\mathcal{T}}(M, N'[\gg 0]) = 0$ . If  $\text{Hom}_{\mathcal{T}}(M, N'[k]) = 0$  for all  $k \in \mathbb{Z}$ , then  $N' \in \mathcal{T}_{\mathcal{M}}^{\leq 0}$  and by Proposition 3.15 (ii) there is some  $l' \geq 0$  such that  $N' \in \mathcal{M} * \cdots * \mathcal{M}[l']$ .

As  $\text{Hom}_{\mathcal{T}}(M, N'[k]) = 0$  for all  $k$ , we also have that  $N'[-l' - 1] \in \mathcal{T}_{\mathcal{M}}^{\leq 0}$ . By Remark 1.2 and Proposition 3.15 (iii)

$$N'[-l' - 1] \in \mathcal{M}[-l' - 1] * \cdots * \mathcal{M}[-1] \subseteq {}^{\perp}(\mathcal{T}_{\mathcal{M}}^{\leq 0}).$$

This means

$$\text{Hom}_{\mathcal{T}}(N', N') \cong \text{Hom}_{\mathcal{T}}(N'[-l' - 1], N'[-l' - 1]) = 0,$$

and  $N' = 0$ . This is impossible as  $\text{thick}\{N'\} = \mathcal{T} \neq \text{thick}\{0\}$  by Remark 3.5.

We then let  $k$  be the largest integer such that  $\text{Hom}_{\mathcal{T}}(M, N'[k]) \neq 0$ , and set  $N := N'[k]$ . Clearly,  $N \in \mathcal{T}_{\mathcal{M}}^{\leq 0}$ , and so we have triangles

$$\begin{array}{ccccccc} N_1 & \xrightarrow{g_1} & M_0 & \xrightarrow{f_0} & N & \longrightarrow & N_1[1], \\ & & & & & & \\ \dots & & & & & & \\ N_l & \xrightarrow{g_l} & M_{l-1} & \xrightarrow{f_{l-1}} & N_{l-1} & \longrightarrow & N_l[1], \\ & & & & & & \\ 0 & \xrightarrow{g_{l+1}} & M_l & \xrightarrow{f_l} & N_l & \longrightarrow & 0. \end{array}$$

for  $N$  as provided by Proposition 3.16. In order to arrive at a contradiction, assume  $M_0 = 0$ . Then we have the triangle

$$N_1 \longrightarrow 0 \longrightarrow N \longrightarrow N_1[1],$$

and so  $N \cong N_1[1] \in \mathcal{M}[1] * \dots * \mathcal{M}[l]$ . Then  $\text{Hom}_{\mathcal{T}}(M, N) \cong \text{Hom}_{\mathcal{T}}(M, N_1) = 0$  by Lemma 2.19, which contradicts the construction of  $N$ . We conclude that  $M_0 \neq 0$ . Going forward, we then allow ourselves to assume that the  $l \geq 0$  is chosen minimally, so that  $M_l \neq 0$ .

Assume for now that  $l > 0$ . Then as  $\mathcal{N}$  is silting,

$$\text{Hom}_{\mathcal{T}}(N, N[l]) \cong \text{Hom}_{\mathcal{T}}(N'[k], N'[k+l]) \cong \text{Hom}_{\mathcal{T}}(N', N'[l]) = 0,$$

and by Lemma 3.18,  $\text{add}\{M_l\} \cap \text{add}\{M_0\} = 0$ . As  $M$  is indecomposable and  $M_l, M_0 \in \text{add}\{M\}$  are nonzero, they both share the summand  $M \neq 0$ . These are contradictions, and we conclude that  $l = 0$ . The first (and only) triangle is then

$$0 \longrightarrow M_0 \longrightarrow N \longrightarrow 0.$$

This gives  $N \cong M_0 \in \text{add}(M)$ . As  $N'$  is basic  $N = N'[k]$  is as well. I.e  $N \cong M$ , so we have  $N' \cong M[-k]$ . This means that  $\text{add}\{N'\} = \text{add}\{M[-k]\}$ , which proves the lemma.  $\square$

The upcoming Corollary 3.25 provides an interesting fact about the silting objects of a Krull-Schmidt triangulated category in that they will have the same number of indecomposable summands. We build towards this result as done by Aihara and Iyama by considering the *Grothendieck group* of the Krull-Schmidt triangulated category in question. To begin the journey towards this corollary, we define the Grothendieck group of a triangulated category by simply exchanging the short exact sequences in the traditional definition [2] by triangles.

**Definition 3.20.** For a triangulated category  $\mathcal{T}$ , let  $F(\mathcal{T})$  be the free abelian group where the elements are isomorphism classes  $[T]$  of objects in  $\mathcal{T}$ . Define the subgroup  $R(\mathcal{T})$  as

$$R(\mathcal{T}) := \left\langle \{[Y] - [X] - [Z] \mid \exists \text{ triangle } X \longrightarrow Y \longrightarrow Z \longrightarrow X[1] \text{ in } \mathcal{T}\} \right\rangle$$

The *Grothendieck group*  $K_0(\mathcal{T})$  is the group

$$K_0(\mathcal{T}) = F(\mathcal{T})/R(\mathcal{T}).$$

Following the definition, the Grothendieck group of  $\mathcal{T}$ , identifies the isomorphism class  $[Y]$  of a center object in a triangle in  $\mathcal{T}$  by the sum  $[X] + [Z]$  of the isomorphism classes of the left and right objects.

Let  $[T] \in K_0(\mathcal{T})$ . Then, as  $\mathcal{T}$  is Krull-Schmidt, there is a decomposition  $T \cong \bigoplus_{i=1}^n T_i$ , where the  $T_i$  are indecomposable in  $\mathcal{T}$ . There is a triangle

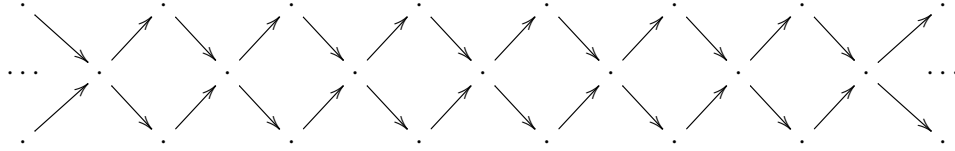
$$T_1 \longrightarrow T \longrightarrow \bigoplus_{i=2}^n T_i \longrightarrow T_1[1],$$

giving  $[T] = [T_1] + [\bigoplus_{i=2}^n T_i]$ . This is clearly extendable so that

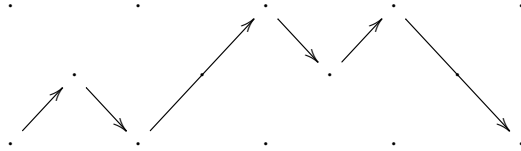
$$[T] = \sum_{i=1}^n [T_i].$$

Thus any element in  $K_0(\mathcal{T})$  is generated by the  $\{[X] \mid X \in \mathcal{T} \text{ indecomposable}\}$ . Theorem 3.24 takes it one step further, showing  $K_0(\mathcal{T})$  is generated by the indecomposable objects of any sifting subcategory of  $\mathcal{T}$ .

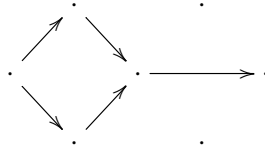
**Example 3.21.** Let  $\Lambda$  be  $kA_3$  so that the AR-quiver has the shape



We know the triangles in  $D^b(\text{mod } \Lambda)$  from triplets of successive arrows in the shapes



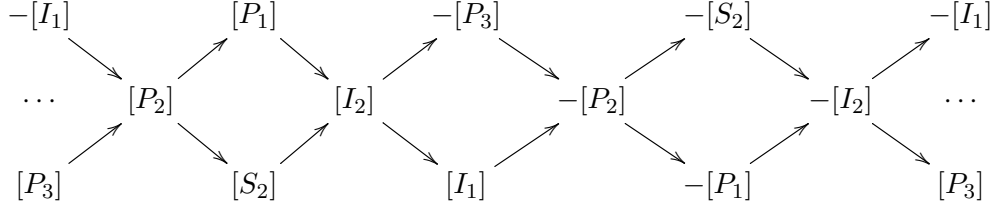
and from



Thus we can easily see the structure of the Grothendieck group. Any central object is identified by the sum of the two objects diagonally above it, and also the two diagonally below it. Additionally, it can be identified as the sum of the top and bottom objects to the left and the object directly to its right, and the other way around. Any object on the top row is identified with the sum of the object diagonally down to the left and the object two spots diagonally down to the right, and vice versa. Similarly any object on the bottom row is identified by the sum of the object diagonally up to the left and the one two spots diagonally up to the right, and vice versa. Finally the sum of the objects at the top and the bottom of a square is the sum of the objects to the left and right of the square.

In particular, we have  $[P_2] = [P_3] + [S_2]$  and also  $[S_2] = [P_2] + [P_3[1]]$ . This gives  $[P_3] + [P_2] + [P_3[1]] = [P_2]$ , so  $[P_3[1]] = -P_3$ . As this holds in general, the Grothendieck group of

$\mathcal{T}$  can be represented in its entirety by



We could come up with a similar story when  $\Lambda$  is  $kA_3/\langle\beta\alpha\rangle$  as well by simply interchanging the actual objects in the triangles to be those from the corresponding AR-quiver.

For the next two results, we will need a certain map. It is very much the map which in a clever way sends an object in  $\mathcal{T}$  to a corresponding element in  $K_0(\mathcal{T})$ .

**Definition 3.22.** Let  $\mathcal{T}$  be a Krull-Schmidt triangulated category, and  $\mathcal{M} \in \text{silt } \mathcal{T}$ . We define the map

$$\mathbf{Ob}\mathcal{T} \xrightarrow{\gamma} \mathbb{Z}^{\text{ind } \mathcal{M}}.$$

by its properties:

- (i) It bijectively maps the objects of  $\mathcal{M}$  to the elements in  $\mathbb{Z}^{\text{ind } \mathcal{M}}$  with nonnegative entries. It does this by counting up the indecomposable summands of  $M$ , and adding 1 to the corresponding copy,  $\mathbb{Z}^{M_i}$ , each time  $M_i$  is counted. The inverse is the obvious one.
- (ii) For any  $N \in \mathcal{T}_{\mathcal{M}}^{\leq 0}$ , take  $l \geq 0$  and triangles as provided by Proposition 3.16, and define

$$\gamma(N) := \sum_{i=0}^l (-1)^i \gamma(M_i).$$

- (iii) Finally, for general  $T \in \mathcal{T}$ , we have by Proposition 3.15 (i) that  $T \in \mathcal{M}[-k] * \cdots * \mathcal{M}[k]$  for some  $l \geq 0$ . Then by Remark 1.2,  $T[k] \in \mathcal{M} * \cdots * \mathcal{M}[2k]$ , which is in  $\mathcal{T}_{\mathcal{M}}^{\leq 0}$  by Proposition 3.15 (ii).

We set

$$\gamma(T) := (-1)^k \gamma(T[k]).$$

We prove the following important property of  $\gamma$ .

**Lemma 3.23.** For  $\mathcal{T}$  a Krull-Schmidt triangulated category with  $\mathcal{M} \in \text{silt } \mathcal{T}$ , let  $\mathbf{Ob}\mathcal{T} \xrightarrow{\gamma} \mathbb{Z}^{\text{ind } \mathcal{M}}$  be as in Definition 3.22. If  $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$  is a distinguished triangle in  $\mathcal{T}$ , then  $\gamma(Y) = \gamma(X) + \gamma(Z)$ . Furthermore, by defining  $\gamma([T]) = \gamma(T)$ , we get a homomorphism  $K_0(\mathcal{T}) \xrightarrow{\gamma} \mathbb{Z}^{\text{ind } \mathcal{M}}$ .

*Proof.* As  $X, Y, Z \in \mathcal{T}$ , there are  $l_X, l_Y, l_Z \geq 0$  such that

$$X \in \mathcal{M}[-l_X] * \cdots * \mathcal{M}[l_X],$$

$$Y \in \mathcal{M}[-l_Y] * \cdots * \mathcal{M}[l_Y],$$

and

$$Z \in \mathcal{M}[-l_Z] * \cdots * \mathcal{M}[l_Z].$$

Then for  $L := \max\{l_X, l_Y, l_Z\}$

$$X[L] \in \mathcal{M} * \cdots * \mathcal{M}[l_X + L] \subseteq \mathcal{T}_{\mathcal{M}}^{\leq 0},$$

$$Y[L] \in \mathcal{M} * \cdots * \mathcal{M}[l_Y + L] \subseteq \mathcal{T}_{\mathcal{M}}^{\leq 0},$$

and

$$Z[L] \in \mathcal{M} * \cdots * \mathcal{M}[l_Z + L] \subseteq \mathcal{T}_{\mathcal{M}}^{\leq 0}.$$

So if we show  $\gamma(Y[L]) = \gamma(X[L]) + \gamma(Z[L])$ , then by Definition 3.22 (iii), we would have

$$\gamma(Y) = \gamma(X) + \gamma(Z). \quad (10)$$

To this end, we rename the objects such that  $X, Y$  and  $Z$  are in  $\mathcal{T}_{\mathcal{M}}^{\leq 0}$ .

To prove the Lemma, we want to use induction on  $l \geq 0$ . For a fixed  $l$ , consider the following statements:

(i)<sub>l</sub> (10) holds if  $X, Y, Z \in \mathcal{M} * \cdots * \mathcal{M}[l]$ .

(ii)<sub>l</sub> (10) holds if  $X, Y \in \mathcal{M} * \cdots * \mathcal{M}[l]$  and  $Z \in \mathcal{M} * \cdots * \mathcal{M}[l + 1]$ .

(iii)<sub>l</sub> (10) holds if  $X \in \mathcal{M} * \cdots * \mathcal{M}[l]$  and  $Y, Z \in \mathcal{M} * \cdots * \mathcal{M}[l + 1]$ .

For (i)<sub>0</sub>, we have  $X, Y, Z \in \mathcal{M}$ , so the triangle splits, and  $Y \cong X \oplus Z$ . Then clearly, by Definition 3.22 (i), (10) holds. To complete the proof, we show

$$(i)_l \Rightarrow (ii)_l \Rightarrow (iii)_l \Rightarrow (i)_{l+1},$$

and the result will follow from induction.

Let  $l \geq 0$ , and assume (i)<sub>l</sub>. Take a triangle  $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$  with  $X, Y \in \mathcal{M} * \cdots * \mathcal{M}[l]$  and  $Z \in \mathcal{M} * \cdots * \mathcal{M}[l + 1]$ . There are  $M_0, \dots, M_{l+1} \in \mathcal{M}$  such that  $Z \in M_0 * \cdots * M_{l+1}[l + 1]$ . Then there is a triangle

$$Z_1 \longrightarrow M_0 \longrightarrow Z \longrightarrow Z_1[1],$$

with  $Z_1 \in M_1 * \cdots * M_{l+1}[l]$ . By Definition 3.22 (ii), we have

$$\gamma(Z_1) = \sum_{i=0}^l (-1)^i \gamma(M_{i+1}) = \sum_{i=1}^{l+1} (-1)^{i-1} \gamma(M_i),$$

and also

$$\gamma(M_0) - \gamma(Z_1) = (-1)^0 \gamma(M_0) + \sum_{i=1}^{l+1} (-1)^i \gamma(M_i) = \sum_{i=0}^{l+1} (-1)^i \gamma(M_i) = \gamma(Z).$$

By the octahedral axiom, we have, for some  $W$ , the diagram

$$\begin{array}{ccccccc} Y[-1] & \longrightarrow & Z[-1] & \longrightarrow & X & \longrightarrow & Y \\ & & \searrow & & \downarrow & & \\ & & & & Z_1 & \longrightarrow & W & \longrightarrow & Y \\ & & & & \downarrow & & \downarrow & & \\ & & & & & & M_0 & & \\ & & & & & & \downarrow & & \\ & & & & & & X[1] & \longrightarrow & Z, \end{array}$$





for some  $M'_0 \in \mathcal{M} * \cdots * \mathcal{M}[l]$  and  $M'_{-1} \in \mathcal{M}[-l+1] * \cdots * \mathcal{M}$ . Furthermore, split  $[M'_0]$  as

$$[M'_0] = [M_0] - [M'_1],$$

where  $M_0 \in \mathcal{M}$  and  $M'_1 \in \mathcal{M} * \cdots * \mathcal{M}[l-1]$ . Do the same for  $[M'_i]$ ,  $1 \leq i < l$  to get

$$[M'_0] = [M_0] - [M_1] + \cdots + (-1)^l [M_l].$$

Similarly, we can split  $[M'_{-1}]$  as

$$[M'_{-1}] = [M_{-1}] - [M'_{-2}],$$

where  $M_{-1} \in \mathcal{M}$  and  $M'_{-2} \in \mathcal{M}[-l+2] * \cdots * \mathcal{M}$ . Do the same for  $[N'_i]$ ,  $-l < i \leq -2$ , and we have

$$[T] = \sum_{i=-l}^l (-1)^i [M_i]$$

where each  $M_i \in \mathcal{M}$ . By the triangle

$$M'_i \longrightarrow \bigoplus_{j=1}^n M'_j \longrightarrow \bigoplus_{\substack{j=1 \\ j \neq i}}^n M'_j \longrightarrow M'_i[1],$$

the summands  $[M_i]$  with  $M_i$  in  $\mathcal{M}$  are generated by the  $[M']$  with  $M'$  indecomposable in  $\mathcal{M}$ . Then so is  $[T]$ , and  $K_0(\mathcal{T})$  is generated by  $\text{ind } \mathcal{M}$ . To prove the theorem it remains to show the generators are linearly independent.

To this end, consider the map  $\gamma$  from Definition 3.22. By Lemma 3.23,  $\gamma$  defines a homomorphism  $K_0(\mathcal{T}) \rightarrow \mathbb{Z}^{\text{ind } \mathcal{M}}$ . Now since  $\gamma(\text{ind } \mathcal{M})$  is a basis for  $\mathbb{Z}^{\text{ind } \mathcal{M}}$ ,  $\text{ind } \mathcal{M}$  is a linearly independent set in  $K_0(\mathcal{T})$ .  $\square$

As we alluded, the theorem is now applied to show that the number of indecomposable summands does not vary across silting objects. It is important for many practical examples, as it helps identify the silting subcategories by excluding the silting objects with the wrong number of summands. This is particularly useful when we are looking at derived categories of path algebras, where the subcategories are in bijection with the basic silting objects.

**Corollary 3.25.** *Let  $\mathcal{T}$  be a Krull-Schmidt triangulated category, and let  $M, N \in \mathcal{T}$  be basic silting objects in  $\mathcal{T}$ . Then  $M$  and  $N$  have the same number of non-isomorphic indecomposable summands.*

*Proof.* Since the Grothendieck group is a free abelian group with bases  $\text{ind}(\text{add}\{M\})$  and  $\text{ind}(\text{add}\{N\})$ , the two have the same cardinality. This means the number of isomorphism classes of indecomposable summands of  $M$  is the same as the number of isomorphism classes of indecomposable summands of  $N$  – as asserted.  $\square$

**Example 3.26.** For  $\mathcal{T} = D^b(\text{mod } kA_3)$  we now know all the silting objects have exactly three indecomposable summands – by the fact that  $P_1 \oplus P_2 \oplus P_3$  is such an object. It should be noted that this does *not* mean that all the basic objects having three indecomposable summands are silting objects. For example we see that  $P_3 \oplus P_2 \oplus S_2$  is not silting, as there is a nonzero morphism  $S_2 \rightarrow P_3[1]$ .

## 4 Obtaining New Silting Subcategories From Old Ones

As is tradition in algebra, we borrow the notion of mutation from another field of science, in this case biology, and creatively mold its meaning into what we want it to be.

To us, mutation denotes a process of obtaining from a structure of some kind a new structure of the same kind. In [5] is discussed mutation of quivers and cluster-tilting objects. In Section 4.1 of this thesis we consider Aihara and Iyama's approach as in [1] in its discussion of the mutation of silting-, and to some degree tilting subcategories. While the process of mutation on silting subcategories will be considered in general, our main focus will be on the class of Krull-Schmidt triangulated categories as defined in Section 3.

Section 4.2 introduces *silting reduction*. Here we consider the *Verider localization* of a triangulated category, and connect certain silting subcategories of  $\mathcal{T}$  to the silting subcategories certain Verdier localizations  $\mathcal{T}/\mathcal{S}$  bijectively. The final result here shows how mutation commutes with this bijection, thus allowing us to understand mutation in  $\text{silt } \mathcal{T}$  through mutation in  $\text{silt } \mathcal{T}/\mathcal{S}$ .

### 4.1 Mutation of Silting Subcategories

In this section we introduce a kind of mutation of a subcategory  $\mathcal{M}$  of  $\mathcal{T}$  with respect to to a covariant- or contravariant subcategory  $\mathcal{D}$  of  $\mathcal{M}$ . When performed on a silting subcategory, it will be named *silting mutation*, and we will see that the end result will itself be a silting subcategory of  $\mathcal{M}$ . Silting mutation is thus a method of obtaining new silting subcategories from old ones. We give conditions under which this holds for tilting subcategories as well.

We also look at the relationship between silting mutation and the partial ordering on  $\text{silt } \mathcal{T}$  introduced in Section 2.2.

We begin by defining the mutation:

**Definition 4.1.** Let  $\mathcal{T}$  be a triangulated category. For  $\mathcal{M} \in \text{silt } \mathcal{T}$  and a covariantly finite subcategory  $\mathcal{D} \subseteq \mathcal{M}$ , we define a subcategory  $\mu^+(\mathcal{M}; \mathcal{D}) \subseteq \mathcal{T}$ , called the *left mutation of  $\mathcal{T}$  with respect to  $\mathcal{D}$* , by the following process: Take any object  $M \in \mathcal{M}$ , and a left  $\mathcal{D}$ -approximation  $M \xrightarrow{f} D$  of  $M$ . Complete the approximation to a triangle

$$M \xrightarrow{f} D \longrightarrow N_M \longrightarrow M[1].$$

Then we define

$$\mu^+(\mathcal{M}; \mathcal{D}) := \text{add}(\mathcal{D} \cup \{N_M \mid M \in \mathcal{M}\}).$$

Dually, if  $\mathcal{D}$  is contravariantly finite, we define the *right mutation* by

$$\mu^-(\mathcal{M}; \mathcal{D}) := \text{add}(\mathcal{D} \cup \{N'_M \mid M \in \mathcal{M}\})$$

where  $N'_M$  is an object which appears in

$$N'_M \longrightarrow D \xrightarrow{g} M \longrightarrow N'_M[1],$$

which completes a right  $\mathcal{D}$ -approximation  $D \xrightarrow{g} M$  to a triangle.

**Example 4.2.** Let  $\mathcal{T} := \text{K}^b(\mathcal{P}(\text{mod } kA_3))$  and  $\mathcal{M}$  be the silting subcategory  $\text{add}\{P_1 \oplus P_2 \oplus P_3\}$ . By Propositions 3.3 and 3.4  $\text{add}\{P_1 \oplus P_2\}$ ,  $\text{add}\{P_1 \oplus P_3\}$  and  $\text{add}\{P_2 \oplus P_3\}$  are all functorially finite in  $\mathcal{T}$ . Then in particular, they are functorially finite subcategories of  $\mathcal{M}$ .

We find  $\mu^+(\mathcal{M}; \mathcal{D})$  for  $\mathcal{D} = \text{add}\{P_1 \oplus P_2\}$ . By definition, we have

$$\mu^+(\mathcal{M}; \mathcal{D}) = \text{add}(\{P_1 \oplus P_2\} \cup \{N_M \mid M \in \mathcal{M}\}).$$

A general object in  $\mathcal{M}$  is of the form  $M = P_1^{n_1} \oplus P_2^{n_2} \oplus P_3^{n_3}$  for some  $n_i \geq 0$ . As  $P_1$  and  $P_2$  are in  $\mathcal{M}$ , the identities are approximations of these. A left  $\mathcal{D}$ -approximation of  $P_3$  is given by the morphism  $P_3 \rightarrow P_2$ , which has at its 0'th degree the map  $f$ , given by

$$\begin{array}{ccccc} 0 & \longrightarrow & 0 & \longrightarrow & k \\ \downarrow & & \downarrow & & \downarrow 1 \\ 0 & \longrightarrow & k & \xrightarrow{1} & k. \end{array}$$

This is exactly the morphism indicated by the arrow  $P_3 \rightarrow P_2$  in the AR-quiver in Example 2.3, and so the right side of the triangle is  $S_2$ . As approximations are additive, we obtain a triangle

$$P_1^{n_1} \oplus P_2^{n_2} \oplus P_3^{n_3} \longrightarrow P_1^{n_1} \oplus P_2^{n_2} \oplus P_2^{n_3} \longrightarrow 0 \oplus 0 \oplus S_2^{n_3} \longrightarrow (P_1^{n_1} \oplus P_2^{n_2} \oplus P_3^{n_3})[1]$$

with the leftmost morphism a left  $\mathcal{D}$ -approximation of  $M$ . Thus  $N_M \cong S_2^{n_3}$ , and  $\mu^+(\mathcal{M}; \mathcal{D}) = \text{add}\{P_1 \oplus P_2 \oplus S_2\}$ .

This mutation can be illustrated by

$$\begin{array}{c} \cdot \\ \nearrow P_1 \\ P_2 \\ \searrow \\ P_3 \cdots \cdots \rightarrow S_2. \end{array}$$

This is easily adapted to see that we similarly get left mutations

$$\mu^+(\mathcal{M}; \text{add}\{P_2 \oplus P_3\}) = \text{add}\{P_2 \oplus P_3 \oplus P_1[1]\}$$

and

$$\mu^+(\mathcal{M}; \text{add}\{P_1 \oplus P_3\}) = \text{add}\{P_1 \oplus P_3 \oplus I_2\},$$

which are illustrated by the diagrams

$$\begin{array}{ccc} \cdot & P_1 & \cdot & \cdot \\ & \nearrow P_2 & \cdot & \cdot \\ P_3 & & \cdot & \cdot \\ & & \cdot & \cdot \\ & & \cdot & P_1[1] \end{array} \qquad \begin{array}{ccc} \cdot & & \cdot \\ & \nearrow P_1 & \cdot \\ P_3 & \nearrow P_2 & \cdot \\ & \cdot & \cdot \\ & \cdot & \cdot \\ & \cdot & I_1. \end{array}$$

We can also find the right mutations of  $\mathcal{M}$  with respect to these subcategories, and this is done similarly by finding left approximations and consulting the AR-quiver to get the corresponding triangles.

In addition, we note that the mutations here are themselves silting subcategories. Also, the subcategory which we mutated with respect to is still a (functorially finite) subcategory after the mutation. By Lemma 1.7, we then get that left mutation with respect to the same subcategory will bring us right back to  $\mathcal{M}$ , as the approximations are given by the same triangles as before.

These are not just coincidences, but direct consequences of Theorem 4.3 and Proposition 4.7.

In the next two theorems we present an advantage Aihara and Iyama shows the silting subcategories have over their tilting counterparts. The mutation of a silting subcategory will always produce a silting subcategory. As the tilting subcategories are also silting, their mutations will thus be silting too. Theorem 4.5 shows a special condition under which the mutation of a tilting subcategory will in fact be tilting. By Proposition 4.7 we see that as long as the mutation of  $\mathcal{M}$  is nontrivial, it produces a silting subcategory different from  $\mathcal{M}$ .

**Theorem 4.3.** *Let  $\mathcal{T}$  be a triangulated category,  $\mathcal{M} \in \text{silt } \mathcal{T}$ , and  $\mathcal{D}$  a covariantly (resp. contravariantly) finite subcategory of  $\mathcal{M}$ . Then  $\mu^+(\mathcal{M}; \mathcal{D})$  (resp.  $\mu^-(\mathcal{M}; \mathcal{D})$ ) is a silting subcategory of  $\mathcal{T}$ .*

For the time being,  $\mu^+(\mathcal{M}; \mathcal{D})$  is constructed by making explicit choices for the left  $\mathcal{D}$ -approximations of the  $M \in \mathcal{M}$ . This will later turn out to be unnecessary, as any such choice will generate the same category.

A proof of Theorem 4.3 is provided for the case of left mutation. The right case is dual.

*Proof.* In order to prove  $\mu^+(\mathcal{M}; \mathcal{D})$  is silting, we show that

- (i)  $\text{thick}(\mu^+(\mathcal{M}; \mathcal{D})) = \mathcal{T}$ , and
- (ii)  $\text{Hom}_{\mathcal{T}}(\mu^+(\mathcal{M}; \mathcal{D}), \mu^+(\mathcal{M}; \mathcal{D})[> 0]) = 0$ .

For any  $M \in \mathcal{M}$  there is a triangle

$$M \xrightarrow{f} D \longrightarrow N_M \longrightarrow M[1],$$

and by rotation, a triangle

$$N_M[-1] \longrightarrow M \xrightarrow{f} D \longrightarrow N_M,$$

with  $f$  a left  $\mathcal{D}$ -approximation, and  $D, N_M \in \mu^+(\mathcal{M}; \mathcal{D})$ . As  $\text{thick}$  implies closed under shift,  $N_M[-1] \in \text{thick}(\mu^+(\mathcal{M}; \mathcal{D}))$ , and as it is also closed under extension,  $M \in \text{thick}(\mu^+(\mathcal{M}; \mathcal{D}))$ . Thus  $\text{thick}(\mu^+(\mathcal{M}; \mathcal{D}))$  is a thick subcategory of  $\mathcal{T}$  containing  $\mathcal{M}$ , so

$$\mathcal{T} = \text{thick } \mathcal{M} \subseteq \text{thick}(\mu^+(\mathcal{M}; \mathcal{D})) \subseteq \mathcal{T},$$

showing (i).

Recall that

$$\mu^+(\mathcal{M}; \mathcal{D}) = \text{add}(\mathcal{D} \cup \{N_M \mid M \in \mathcal{M}\}),$$

where the  $N_M$  are objects appearing as the right sides of triangles

$$M \xrightarrow{f} D \longrightarrow N_M \longrightarrow M[1], \tag{11}$$

completing a left  $\mathcal{D}$ -approximation  $M \xrightarrow{f} D$  of  $M$ . To show (ii), then means to show the following four points:

- (I)  $\text{Hom}_{\mathcal{T}}(\mathcal{D}, \mathcal{D}[> 0]) = 0$ .
- (II)  $\text{Hom}_{\mathcal{T}}(\mathcal{D}, N_M[> 0]) = 0$  for any  $M \in \mathcal{M}$ .

(III)  $\text{Hom}_{\mathcal{T}}(N_M, \mathcal{D}[> 0]) = 0$  for any  $M \in \mathcal{M}$ .

(IV)  $\text{Hom}_{\mathcal{T}}(N_M, N_{M'}[> 0]) = 0$  for any  $M, M' \in \mathcal{M}$ .

Part (I) is seen straight from  $\mathcal{D} \subseteq \mathcal{M} \in \text{silt } \mathcal{T}$ .

For any  $M \in \mathcal{M}$ , take some  $D' \in \mathcal{D}$  and apply  $\text{Hom}_{\mathcal{T}}(D', -)$  to triangle (11) to get the long exact sequence

$$\cdots \longrightarrow \text{Hom}_{\mathcal{T}}(D', D[n]) \longrightarrow \text{Hom}_{\mathcal{T}}(D', N_M[n]) \longrightarrow \text{Hom}_{\mathcal{T}}(D', M[n+1]) \longrightarrow \cdots .$$

For  $n > 0$  the left and right sides are 0 by  $D', D, M \in \mathcal{M}$  and  $n, n+1 > 0$ . Hence the middle term is also 0. This shows (II).

Next, apply  $\text{Hom}_{\mathcal{T}}(-, D')$  to (11) to get a long exact sequence

$$\cdots \longrightarrow \text{Hom}_{\mathcal{T}}(M[1], D'[n]) \longrightarrow \text{Hom}_{\mathcal{T}}(N_M, D'[n]) \longrightarrow \text{Hom}_{\mathcal{T}}(D, D'[n]) \longrightarrow \cdots ,$$

and note that the right side is 0 for all  $n > 0$ . If  $n > 1$ , we get

$$\text{Hom}_{\mathcal{T}}(M[1], D'[n]) \cong \text{Hom}_{\mathcal{T}}(M, D'[n-1]) = 0,$$

and by exactness,  $\text{Hom}_{\mathcal{T}}(N_M, D'[n]) = 0$ . The case  $n = 1$  requires an additional step: Since  $f$  is a left  $\mathcal{D}$ -approximation of  $M$ ,

$$\text{Hom}_{\mathcal{T}}(D, D') \xrightarrow{-\circ f} \text{Hom}_{\mathcal{T}}(M, D')$$

is surjective. Then, if we apply  $\text{Hom}_{\mathcal{T}}(-, D')$  to the triangle to get the exact sequence

$$\cdots \longrightarrow \text{Hom}_{\mathcal{T}}(D, D') \xrightarrow{-\circ f} \text{Hom}_{\mathcal{T}}(M, D') \longrightarrow \text{Hom}_{\mathcal{T}}(N_M[-1], D') \longrightarrow \text{Hom}_{\mathcal{T}}(D[-1], D') \longrightarrow \cdots ,$$

where the left morphism is surjective and the right term  $\text{Hom}_{\mathcal{T}}(D[-1], D') = 0$ . It follows by exactness that

$$\text{Hom}_{\mathcal{T}}(N_M, D'[1]) \cong \text{Hom}_{\mathcal{T}}(N_M[-1], D') = 0,$$

which shows (III).

Next, for some  $M' \in \mathcal{M}$  apply  $\text{Hom}_{\mathcal{T}}(-, M')$  to the triangle (11) corresponding to  $M$  to get the long exact sequence

$$\cdots \longrightarrow \text{Hom}_{\mathcal{T}}(M[1], M'[n]) \longrightarrow \text{Hom}_{\mathcal{T}}(N_M, M'[n]) \longrightarrow \text{Hom}_{\mathcal{T}}(D, M'[n]) \longrightarrow \cdots . \quad (12)$$

For  $n > 1$  the left and right sides are 0, so by exactness  $\text{Hom}_{\mathcal{T}}(N_M, \mathcal{M}[> 1]) = 0$ . Then, apply  $\text{Hom}_{\mathcal{T}}(N_M, -)$  to the triangle (11) corresponding to  $M' \in \mathcal{M}$  to get the long exact sequence

$$\cdots \longrightarrow \text{Hom}_{\mathcal{T}}(N_M, D[n]) \longrightarrow \text{Hom}_{\mathcal{T}}(N_M, N_{M'}[n]) \longrightarrow \text{Hom}_{\mathcal{T}}(N_M, M'[n+1]) \longrightarrow \cdots .$$

For  $n > 0$  the left and right sides vanish by (III) and (12), giving  $\text{Hom}_{\mathcal{T}}(N_M, N_{M'}[> 0]) = 0$ . This shows (IV), which again shows (ii), which completes the proof.  $\square$

By this next corollary, we see that our choice of approximations do not matter for the construction of  $\mu^{\pm}(\mathcal{M}; \mathcal{D})$ . Having this, we can allow ourselves to focus the examples completely on the AR-quiver of our hereditary algebra without worrying that the approximations read from this would be inferior candidates.

**Corollary 4.4.** *Let  $\mathcal{T}$  be a triangulated category,  $\mathcal{M} \in \text{silt } \mathcal{T}$ , and  $\mathcal{D}$  a covariantly (resp. contravariantly) finite subcategory of  $\mathcal{M}$ . Then the construction of  $\mu^+(\mathcal{M}; \mathcal{D})$  (resp.  $\mu^-(\mathcal{M}; \mathcal{D})$ ) is well-defined. I.e. it is independent of the choice of left (resp. right)  $\mathcal{D}$ -approximations of the objects in  $\mathcal{M}$ .*

*Proof.* Let  $\mathcal{D}$  be a covariantly finite subcategory of  $\mathcal{M} \in \text{silt } \mathcal{T}$ . For any  $M \in \mathcal{M}$ , make a choice of left  $\mathcal{D}$ -approximation  $f$  of  $M$ , and construct

$$\mu^+(\mathcal{M}; \mathcal{D}) = \text{add}(\mathcal{D} \cup \{N_M \mid M \in \mathcal{M}\})$$

from this. Similarly, make another (possibly different) choice of left  $\mathcal{D}$ -approximation  $f'$  of  $M$ , and construct

$$\mu^+(\mathcal{M}; \mathcal{D})' = \text{add}(\mathcal{D} \cup \{N'_M \mid M \in \mathcal{M}\})$$

from this. Then for any  $M \in \mathcal{M}$ , we have the diagram

$$\begin{array}{ccccccc} M & \xrightarrow{f'} & D' & \xrightarrow{g'} & N'_M & \longrightarrow & M[1] \\ \downarrow 1 & & \downarrow \delta & & & & \downarrow 1 \\ M & \xrightarrow{f} & D & \xrightarrow{g} & N_M & \longrightarrow & M[1], \end{array}$$

with the rows being the triangles used in the constructions above. Now,  $f$  factors through  $f'$  by some  $\delta$ , so the left square commutes. Then by an axiom equivalent to the octahedral axiom [12], there is a triangle

$$D' \longrightarrow D \oplus N'_M \longrightarrow N_M \longrightarrow D'[1].$$

The rightmost morphism is 0 as  $N_M$  and  $D$  are in  $\mu^+(\mathcal{M}; \mathcal{D})$ , so the triangle splits, and  $D \oplus N'_M \cong D' \oplus N_M$ . This means  $N'_M \in \mu^+(\mathcal{M}; \mathcal{D})$ , and so  $\mu^+(\mathcal{M}; \mathcal{D}) \subseteq \mu^+(\mathcal{M}; \mathcal{D})'$ . These are silting subcategories of  $\mathcal{T}$ , and it follows that they must be equal.  $\square$

Since any tilting subcategory is in particular silting, a mutation of such a tilting subcategory will by Theorem 4.3 again be silting. As alluded to in the introduction, the mutation of a tilting subcategory need not itself be tilting, which can be used as an argument to why silting is an *improvement* upon tilting. Even in the reasonably nice  $K^b(\mathcal{P}(\text{mod } kA_3))$ , we saw in Example 4.2 that the tilting subcategory  $\text{add}\{P_1 \oplus P_2 \oplus P_3\}$  can be right mutated into  $\text{add}\{P_2 \oplus P_3 \oplus P_1[1]\}$ . This is easy to check that this is only silting and not tilting.

There are, however, conditions under which mutation on a tilting subcategory always works.

**Theorem 4.5.** *Let  $\mathcal{T}$  be a triangulated category, and  $\mathcal{M}$  a tilting subcategory of  $\mathcal{T}$ :*

(i) *If  $\mathcal{D} \subseteq \mathcal{M}$  is a covariantly finite subcategory, then the following are equivalent:*

(I)  $\mu^+(\mathcal{M}; \mathcal{D})$  *is tilting.*

(II) *For all  $M \in \mathcal{M}$ , there is a left  $\mathcal{D}$ -approximation  $M \xrightarrow{f} D$  such that  $\text{Hom}_{\mathcal{T}}(D', f)$  is injective for all  $D' \in \mathcal{D}$ .*

(ii) *If  $\mathcal{D} \subseteq \mathcal{M}$  is a contravariantly finite subcategory, then the following are equivalent:*

(III)  $\mu^-(\mathcal{M}; \mathcal{D})$  *is tilting.*

(IV) For all  $M \in \mathcal{M}$ , there is a right  $\mathcal{D}$ -approximation  $D \xrightarrow{g} M$  such that  $\text{Hom}_{\mathcal{T}}(g, D')$  is injective for all  $D' \in \mathcal{D}$ .

We call these kinds of mutation *tilting mutation*. Only statement (i) is proven here, as (ii) is done in a dual manner.

*Proof.* It is already known from Theorem 4.3 that  $\mu^+(\mathcal{M}; \mathcal{D})$  is silting. To see it is tilting, it then suffices to check that  $\text{Hom}_{\mathcal{T}}(\mu^+(\mathcal{M}; \mathcal{D}), \mu^+(\mathcal{M}; \mathcal{D})[< 0]) = 0$ . In other words, we need to show that (II) is equivalent to

- (a)  $\text{Hom}_{\mathcal{T}}(\mathcal{D}, \mathcal{D}[< 0]) = 0$ . (This holds in general from  $\mathcal{M}$  being tilting.)
- (b)  $\text{Hom}_{\mathcal{T}}(D', N_M[< 0]) = 0$  for any  $D' \in \mathcal{D}$  and  $M \in \mathcal{M}$ .
- (c)  $\text{Hom}_{\mathcal{T}}(N_{M'}, N_M[< 0]) = 0$  for all  $M', M \in \mathcal{M}$ .
- (d)  $\text{Hom}_{\mathcal{T}}(N_M, D'[< 0]) = 0$  for all  $M \in \mathcal{M}$ ,  $D' \in \mathcal{D}$ .

Assume (I). For any  $M \in \mathcal{M}$ , any corresponding triangle as in (11) and any  $D' \in \mathcal{D}$ , apply  $\text{Hom}_{\mathcal{T}}(D', -)$  to the triangle and get the long exact sequence

$$\cdots \longrightarrow \text{Hom}_{\mathcal{T}}(D', N_M[-1]) \longrightarrow \text{Hom}_{\mathcal{T}}(D', M) \xrightarrow{f \circ -} \text{Hom}_{\mathcal{T}}(D', D) \longrightarrow \cdots .$$

As  $\text{Hom}_{\mathcal{T}}(D', N_M[-1]) = 0$  by (b), we have by exactness that  $f \circ -$  is injective. This shows (I)  $\Rightarrow$  (II).

We then assume (II), pick any  $M \in \mathcal{M}$ , and use the  $f$  as guaranteed by (II) to get a triangle

$$M \xrightarrow{f} D \xrightarrow{g} M_N \longrightarrow M[1]$$

as in (11). For any  $D' \in \mathcal{D}$ , we apply  $\text{Hom}_{\mathcal{T}}(D', -[n])$  to the triangle to get the long exact sequence

$$\longrightarrow \text{Hom}_{\mathcal{T}}(D', D[n]) \longrightarrow \text{Hom}_{\mathcal{T}}(D', N_M[n]) \longrightarrow \text{Hom}_{\mathcal{T}}(D', M[n+1]) \xrightarrow{f[n+1] \circ -} \text{Hom}_{\mathcal{T}}(D', D[n+1]) \longrightarrow .$$

Since  $\mathcal{M}$  is tilting, we have  $\text{Hom}_{\mathcal{T}}(D', D[n]) = 0$  for all  $n < 0$ . If in addition  $n < -1$  we have  $\text{Hom}_{\mathcal{T}}(D', M[n+1]) = 0$  for the same reason, and by exactness,  $\text{Hom}_{\mathcal{T}}(D', N_M[n]) = 0$ . If  $n = -1$ , i.e.  $n+1 = 0$ , and so  $f[n+1] \circ - = f \circ -$  is injective by our choice of  $f$ . This shows  $\text{Hom}_{\mathcal{T}}(D', N_M[n]) = 0$  by exactness, which gives us (II)  $\Rightarrow$  (b).

For any  $M \in \mathcal{M}$ , take triangle as in (11). For  $D' \in \mathcal{D}$ , apply  $\text{Hom}_{\mathcal{T}}(-, D')$  to it to get the long exact sequence

$$\cdots \longrightarrow \text{Hom}_{\mathcal{T}}(M[1], D'[n]) \longrightarrow \text{Hom}_{\mathcal{T}}(N_M, D'[n]) \longrightarrow \text{Hom}_{\mathcal{T}}(D, D'[n]) \longrightarrow \cdots .$$

For  $n < 0$ , the left term vanishes as it is isomorphic to  $\text{Hom}_{\mathcal{T}}(M, D[n-1])$  and  $\mathcal{D} \subseteq \mathcal{M}$  – which is tilting. The same goes for the right term. Thus  $\text{Hom}_{\mathcal{T}}(N_M, D'[n]) = 0$  by exactness, showing (II)  $\Rightarrow$  (d).

For some  $M' \in \mathcal{M}$  apply  $\text{Hom}_{\mathcal{T}}(M', -)$  to a triangle as in (11) to get the long exact sequence

$$\cdots \longrightarrow \text{Hom}_{\mathcal{T}}(M', D[n]) \longrightarrow \text{Hom}_{\mathcal{T}}(M', N_M[n]) \longrightarrow \text{Hom}_{\mathcal{T}}(M', M[n+1]) \longrightarrow \cdots . \quad (13)$$

Again, both the left and right terms vanish for  $n < -1$ , so the middle term vanishes as well.



Now we take a triangle as in (11) for the object  $M'$ , and for some  $M \in \mathcal{M}$  we apply  $\mathrm{Hom}_{\mathcal{T}}(-, N_M)$  to it to get the long exact sequence

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{T}}(M'[1], N_M[n]) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(N_{M'}, N_M[n]) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(D', N_M[n]) \longrightarrow \cdots .$$

For  $n < 0$ , the left term is isomorphic to  $\mathrm{Hom}_{\mathcal{T}}(M', N_M[n-1])$ , which vanishes by (13). The right term vanishes by (b), and it follows from exactness that the middle term vanishes as well. Thus (II)  $\Rightarrow$  (c). This completes our list, and finally provides us with the logical equivalence (I)  $\Leftrightarrow$  (II).  $\square$

We introduced the partial ordering on silt  $\mathcal{T}$ , claiming that it would come in handy when studying silting mutation. Following Theorem 4.3, one may wonder if  $\mathcal{M}$  and  $\mu^{\pm}(\mathcal{M}; \mathcal{D})$  are related by the partial ordering, and if so, how they are related. The answer to this starts with Proposition 4.7, wherein we show that any nontrivial mutation of a silting subcategory is related to the original in a well-ordered manner.

Another important consequence of Proposition 4.7 is that mutation is reversible: Having left (right) mutated  $\mathcal{M}$  with respect to  $\mathcal{D}$ , we can right (left) mutate the result – again with respect to  $\mathcal{D}$ , to get back to  $\mathcal{M}$ . This relies on the following proposition.

**Proposition 4.6.** *Let  $\mathcal{T}$  be a triangulated category, and let  $\mathcal{M} \in \mathrm{silt} \mathcal{T}$ .*

- (i) *If  $\mathcal{D}$  is a covariantly finite subcategory of  $\mathcal{M}$ , then it is a contravariantly finite subcategory of  $\mu^+(\mathcal{M}; \mathcal{D})$ .*
- (ii) *If  $\mathcal{D}'$  is a contravariantly finite subcategory of  $\mathcal{M}$  then it is a covariantly finite subcategory of  $\mu^-(\mathcal{M}; \mathcal{D}')$ .*

*Proof.* As per usual, only  $\mathcal{D}$  being contravariantly finite in  $\mu^+(\mathcal{M}; \mathcal{D})$  is proven. The second part is dual.

By definition, any object  $X \in \mu^+(\mathcal{M}; \mathcal{D})$  is of the form  $X = \bigoplus_{i=1}^n X_i$ , where  $X_i \in \mathcal{D}$ , or  $X_i \oplus X'_i \cong N_M$  for some  $X'_i \in \mu^+(\mathcal{M}; \mathcal{D})$  and some  $M \in \mathcal{M}$ . If  $X_i \in \mathcal{D}$ , the identity is a right  $\mathcal{D}$ -approximation, and  $N_{X_i} = 0$ . If  $X_i \oplus X'_i \cong N_M$ , there is an  $M \in \mathcal{M}$  and a triangle

$$M \xrightarrow{f} D \xrightarrow{g = \begin{pmatrix} g_{X_i} \\ g_{X'_i} \end{pmatrix}} X_i \oplus X'_i \longrightarrow M[1]$$

with  $f$  a left  $\mathcal{D}$ -approximation. We get that  $g$  is a right  $\mathcal{D}$ -approximation by Lemma 2.5, and easily see that  $g_{X_i}$  and  $g_{X'_i}$  are left  $\mathcal{D}$ -approximations as well.

Thus any direct summand of  $X$  has a right  $\mathcal{D}$ -approximation, and the direct sum of these approximations is clearly a left  $\mathcal{D}$ -approximation of  $X$ . As this holds for any  $X \in \mu^+(\mathcal{M}; \mathcal{D})$ , we get that  $\mathcal{D}$  is contravariantly finite in  $\mu^+(\mathcal{M}; \mathcal{D})$ .  $\square$

**Proposition 4.7.** *Let  $\mathcal{T}$  be a triangulated category, and  $\mathcal{M}$  a silting subcategory of  $\mathcal{T}$ .*

- (i) *If  $\mathcal{D}$  is a covariantly finite subcategory of  $\mathcal{M}$ , then*

$$\mu^-(\mu^+(\mathcal{M}; \mathcal{D}); \mathcal{D}) = \mathcal{M}$$

and

$$\mathcal{M} \geq \mu^+(\mathcal{M}; \mathcal{D}),$$

with equality if and only if  $\mathcal{D} = \mathcal{M}$ .

(ii) If  $\mathcal{D}$  is a contravariantly finite subcategory of  $\mathcal{M}$ , then

$$\mu^+(\mu^-(\mathcal{M}; \mathcal{D}); \mathcal{D}) = \mathcal{M}$$

and

$$\mu^-(\mathcal{M}; \mathcal{D}) \geq \mathcal{M},$$

with equality if and only if  $\mathcal{D} = \mathcal{M}$ .

*Proof.* As before, only the first part will be proven, as the second part is dual. For  $M \in \mathcal{M}$ , there is a triangle

$$M \xrightarrow{f} D \xrightarrow{g} N_M \longrightarrow M[1] \quad (14)$$

with  $f$  and  $g$   $\mathcal{D}$ -approximations, and  $N_M \in \mu^+(\mathcal{M}; \mathcal{D})$ . We have that

$$\mu^-(\mu^+(\mathcal{M}; \mathcal{D}); \mathcal{D}) = \text{add}(\mathcal{D} \cup \{N'_X \mid X \in \mu^+(\mathcal{M}; \mathcal{D})\})$$

where  $N'_X$  is given by a triangle

$$N'_X \xrightarrow{f'} D' \xrightarrow{g'} X \longrightarrow N'_X[1]$$

completing a right  $\mathcal{D}$ -approximation  $g'$ . The triangle (14) gives  $M \cong N'_{N_M}$ , and  $\mathcal{M} \subseteq \mu^-(\mu^+(\mathcal{M}; \mathcal{D}); \mathcal{D})$ . These are silting subcategories of  $\mathcal{T}$ . As one is contained within the other, they must be the same.

Next, for any  $M \in \mathcal{M}$ , complete a left  $\mathcal{D}$ -approximation to a triangle as above, and apply  $\text{Hom}_{\mathcal{T}}(M', -)$  to it for any  $M' \in \mathcal{M}$  to get the long exact sequence

$$\cdots \longrightarrow \text{Hom}_{\mathcal{T}}(M', D[n]) \longrightarrow \text{Hom}_{\mathcal{T}}(M', N_M[n]) \longrightarrow \text{Hom}_{\mathcal{T}}(M', M[n+1]) \longrightarrow \cdots .$$

For  $n > 0$ , the left and right terms vanish by  $\mathcal{M}$  being silting and  $\mathcal{D} \subseteq \mathcal{M}$ . It then follows by exactness that the middle term vanishes too. Let  $M \in \mathcal{M}$ ,  $X \in \mu^+(\mathcal{M}; \mathcal{D})$  and  $n > 0$ . Then for some  $m \geq 0$ , we have  $X = \bigoplus_{i=1}^m X_i$ , where either  $X_i \in \mathcal{D}$  or  $X_i \oplus Y_i = N_{M'}$  for some  $Y_i$  and some  $M' \in \mathcal{M}$ . Additionally, both  $\text{Hom}_{\mathcal{T}}(M, -)$  and shift are additive functors, so

$$\text{Hom}_{\mathcal{T}}(M, X[n]) \cong \bigoplus_{i=1}^m \text{Hom}_{\mathcal{T}}(M, X_i[n]).$$

Whenever  $X_i \in \mathcal{D}$ , we get that  $\text{Hom}_{\mathcal{T}}(M, X_i[n]) = 0$ , and if  $X_i \oplus Y_i = N_{M'}$ , then

$$\text{Hom}_{\mathcal{T}}(M, X_i[n]) \oplus \text{Hom}_{\mathcal{T}}(M, Y_i[n]) \cong \text{Hom}_{\mathcal{T}}(M, N_{M'}[n]) = 0,$$

so  $\text{Hom}_{\mathcal{T}}(M, X_i[n]) = 0$ . This shows  $\text{Hom}_{\mathcal{T}}(\mathcal{M}, \mu^+(\mathcal{M}; \mathcal{D})[> 0]) = 0$ , and that  $\mathcal{M} \geq \mu^+(\mathcal{M}; \mathcal{D})$ .

Clearly, if  $\mathcal{D} = \mathcal{M}$ , then  $\mu^+(\mathcal{M}; \mathcal{D}) = \mathcal{M}$  as all the  $\mathcal{D}$ -approximations are identities and the  $N_M$  are just zero. Furthermore, if  $\mathcal{D} \subsetneq \mathcal{M}$ , take  $M \in \mathcal{M} \setminus \mathcal{D}$ , and a triangle

$$M \xrightarrow{f} D \xrightarrow{g} N_M \xrightarrow{h} M[1]$$

as before. If  $f$  is a split monomorphism,  $D \cong M \oplus N_M$ , and so  $M \in \mathcal{D}$ , which we assumed it wasn't. Thus  $f$  is not a split monomorphism, and so  $h \neq 0$  [11]. In other words,  $\text{Hom}_{\mathcal{T}}(N_M, M[1]) \neq 0$ , and so  $N_M$  is not in  $\mathcal{M}$ , proving the mutation is a silting subcategory different from  $\mathcal{M}$ .  $\square$



where  $N_X$  is obtained from a triangle

$$X \xrightarrow{f} M \xrightarrow{g} N_X \longrightarrow X[1]$$

completing a left  $\mathcal{M}_X$ -approximation  $X \xrightarrow{f} M$ . Since  $\mathcal{M}_X \subsetneq \mathcal{M}$ , we have by Proposition 4.7 (i) that  $\mathcal{M} > \mathcal{L}$ . To show  $\mathcal{L} \geq \mathcal{N}$  we need to show  $\text{Hom}_{\mathcal{T}}(L, \mathcal{N}[> 0]) = 0$  for all  $L \in \mathcal{L}$ . As we have seen before,  $\text{Hom}_{\mathcal{T}}(L', \mathcal{N}[> 0]) = 0$  is clearly true for all summands  $L'$  of  $L$  which lie in  $\mathcal{M}_X$ . It then only remains to show that  $\text{Hom}_{\mathcal{T}}(N_X, \mathcal{N}[> 0]) = 0$  so that  $\text{Hom}_{\mathcal{T}}(L', \mathcal{N}[> 0]) = 0$  also for any summand of  $N_X$ .

To this end, let  $N \in \mathcal{N}$  and apply  $\text{Hom}_{\mathcal{T}}(-, N)$  to the triangle above to obtain the long exact sequence

$$\cdots \longrightarrow \text{Hom}_{\mathcal{T}}(X, N[n]) \longrightarrow \text{Hom}_{\mathcal{T}}(N_X[-1], N[n]) \longrightarrow \text{Hom}_{\mathcal{T}}(M[-1], N[n]) \longrightarrow \cdots$$

For  $n > 0$  the left term vanishes as it is in  $\text{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{N}[> 0])$ , and the right term as it is isomorphic to  $\text{Hom}_{\mathcal{T}}(M, N[n+1]) \subseteq \text{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{N}[> 0])$ . Then by exactness we have  $\text{Hom}_{\mathcal{T}}(N_X, \mathcal{N}[> 1]) = 0$ , and we are left with checking  $\text{Hom}_{\mathcal{T}}(N_X, \mathcal{N}[1]) = 0$ . Again, we employ the usual strategy and apply  $\text{Hom}_{\mathcal{T}}(-, N)$  to our triangle for some  $N \in \mathcal{N}$  to obtain the long exact sequence

$$\cdots \longrightarrow \text{Hom}_{\mathcal{T}}(M, N) \xrightarrow{-\circ f} \text{Hom}_{\mathcal{T}}(X, N) \longrightarrow \text{Hom}_{\mathcal{T}}(N_X[-1], N) \longrightarrow \text{Hom}_{\mathcal{T}}(M[-1], N) \longrightarrow \cdots$$

Again, the rightmost term is 0. We show the leftmost morphism  $-\circ f$  is surjective, and then conclude from exactness that  $\text{Hom}_{\mathcal{T}}(N_X, \mathcal{N}[1]) = 0$ . As  $N \in \mathcal{T}_{\mathcal{M}}^{\leq 0}$ , we obtain  $l' \geq 0$  and  $l'$  corresponding triangles

$$\begin{array}{ccccccc} N'_1 & \xrightarrow{g'_1} & M'_0 & \xrightarrow{f'_0} & N & \longrightarrow & N'_1[1], \\ N'_2 & \xrightarrow{g'_2} & M'_1 & \xrightarrow{f'_1} & N'_1 & \longrightarrow & N'_2[1], \\ \dots & & & & & & \\ 0 & \longrightarrow & M'_{l'} & \xrightarrow{f'_{l'}} & N'_{l'} & \longrightarrow & 0. \end{array}$$

with  $M'_i \in \mathcal{M}$  and  $N'_i \in M'_i * \cdots * M'_{l'}[l' - i]$  as by Proposition 3.16. For any morphism  $X \xrightarrow{a} N$ , consider the solid parts of the diagram

$$\begin{array}{ccccccc} N_X[-1] & \longrightarrow & X & \xrightarrow{f} & M & \xrightarrow{g} & N_X \\ & & \swarrow b & \downarrow c & \searrow a & & \\ N'_1 & \xrightarrow{g'_1} & M'_0 & \xrightarrow{f'_0} & N & \longrightarrow & N'_1[1]. \end{array}$$

Since  $f'_0$  is a right  $\mathcal{M}$ -approximation of  $N$ ,  $a$  factors through  $f'_0$  by some  $X \xrightarrow{b} M'_0$ . Since both  $N_0$  and  $N'_0 := N$  are in  $\mathcal{N}$ , it follows by Lemma 3.18 that  $\text{add}\{M_l\} \cap \text{add}\{M'_0\} = 0$ . We picked  $X$  as a summand of  $M_l$ , so  $X \in \text{add}\{M_l\}$  and so it cannot be in  $\text{add}\{M'_0\}$ . This means that  $M'_0 \in \mathcal{M}$  but does not have the summand  $X$ , so by definition of  $\mathcal{M}_X$ , we have  $M'_0 \in \mathcal{M}_X$ .

Since  $f$  is chosen to be a left  $\mathcal{M}_X$ -approximation of  $X$ ,  $b$  factors through  $f$  by some morphism  $M \xrightarrow{c} M'_0$ . That is,  $a = f'_0 b = f'_0 c f$ , which means that  $-\circ f$  is surjective. Thus  $\text{Hom}_{\mathcal{T}}(N_X, \mathcal{N}[> 0]) = 0$ , and we can conclude that  $\mathcal{L} \geq \mathcal{N}$ .  $\square$

For these Krull-Schmidt categories, irreducible silting mutation corresponds to *minimal movement* within the partial ordering structure on  $\text{silt } \mathcal{T}$ . That is, by applying left (right) irreducible mutation to  $\mathcal{M}$ , we get a silting subcategory  $\mathcal{M} > \mathcal{N}$  ( $\mathcal{N} > \mathcal{M}$ ) where no silting subcategory  $\mathcal{L}$  is properly placed in between the two as  $\mathcal{M} > \mathcal{L} > \mathcal{N}$  ( $\mathcal{N} > \mathcal{L} > \mathcal{M}$ ).

This is shown in Theorem 4.12. To streamline the proof, we first prove Lemma 4.11. This again is proved by use of Lemma 4.10, which says that minimal approximations appear together in triangles.

**Lemma 4.10.** *Let  $\mathcal{T}$  be a Krull-Schmidt triangulated category,  $\mathcal{M} \in \text{silt } \mathcal{T}$ , and  $X \in \mathcal{M}$  an object which has no summands in  $\mathcal{D}$ .*

(i) *If  $\mathcal{D}$  is a covariantly finite subcategory of  $\mathcal{M}$ , then for any triangle*

$$X \xrightarrow{f} D \xrightarrow{g} N_X \longrightarrow X[1],$$

where  $X \xrightarrow{f} D$  is a left minimal  $\mathcal{D}$ -approximation of  $X$ , we get that  $D \xrightarrow{g} N_X$  is a right minimal  $\mathcal{D}$ -approximation of  $N_X$ .

(ii) *If  $\mathcal{D}$  is a contravariantly finite subcategory of  $\mathcal{M}$ , then for any triangle*

$$N'_X \xrightarrow{f} D \xrightarrow{g} Y \longrightarrow X[1]$$

where  $D \xrightarrow{g} Y$  is a right minimal  $\mathcal{D}$ -approximation of  $Y$ , we get that  $N'_X \xrightarrow{f} D$  is a left minimal  $\mathcal{D}$ -approximation of  $N'_X$ .

*Proof.* We prove part (i), as part (ii) is dual. As  $f$  is a left  $\mathcal{D}$ -approximation of  $X$ ,  $g$  is a right  $\mathcal{D}$ -approximation of  $N_X$  by Lemma 2.5 (i). By Lemma 3.7, we can decompose  $D$  and  $g$  as

$$D = D' \oplus D'' \xrightarrow{g=(g' \ 0)} N_X,$$

where  $g'$  is right minimal. Complete  $g'$  to a triangle  $X' \xrightarrow{f'} D' \xrightarrow{g'} N_X \longrightarrow X'[1]$ , and we have the solid part of the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & D' \oplus D'' & \xrightarrow{(g' \ 0)} & N_X & \longrightarrow & X[1] \\ \vdots \downarrow & & \downarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & & \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & & \downarrow \vdots \\ X' \oplus D'' & \xrightarrow{\begin{pmatrix} f' & 0 \\ 0 & 1 \end{pmatrix}} & D' \oplus D'' & \xrightarrow{\begin{pmatrix} g' & 0 \\ 0 & 0 \end{pmatrix}} & N_X \oplus 0 & \longrightarrow & (X' \oplus D'')[1]. \end{array}$$

It can be completed to a morphism of triangles, and by two of the vertical morphisms being isomorphisms, so is the third one. This means  $D'' = 0$ , so  $D = D'$  and  $g = g'$ , which is right minimal.  $\square$

As we claim that mutation is an 'interchange of indecomposable summands' of silting objects, we claim that when interchanging something indecomposable, what we gain is again indecomposable. Lemma 4.11 shows that this is in fact the case.

**Lemma 4.11.** *Let  $\mathcal{T}$  be a Krull-Schmidt triangulated category,  $\mathcal{M} \in \text{silt } \mathcal{T}$ , and  $X \in \mathcal{M}$  indecomposable.*

(i) *If  $\mathcal{D}$  is a covariantly finite subcategory of  $\mathcal{M}$ , then for any triangle*

$$X \xrightarrow{f} D \xrightarrow{g} N_X \longrightarrow X[1],$$

*where  $X \xrightarrow{f} D$  is a left minimal  $\mathcal{D}$ -approximation of  $X$ , we have that  $N_X$  is indecomposable as well.*

(ii) *If  $\mathcal{D}$  is a contravariantly finite subcategory of  $\mathcal{M}$ , then for any triangle*

$$N'_X \xrightarrow{f} D \xrightarrow{g} X \longrightarrow X[1],$$

*where  $D \xrightarrow{g} X$  is a right minimal  $\mathcal{D}$ -approximation of  $X$ , we have that  $N'_X$  is indecomposable as well.*

*Proof.* A proof of part (i) is provided.

To arrive at a contradiction, we assume  $N_X$  is not indecomposable. As  $\mathcal{T}$  is Krull-Schmidt, we have a decomposition  $N_X = \bigoplus_{i=1}^n Y_i$  where the  $Y_i$  are (nonzero) indecomposable objects, and since our mutations are additive categories, the  $Y_i$  are all in  $\mu^+(\mathcal{M}; \mathcal{D})$ . By Proposition 4.6,  $\mathcal{D}$  is contravariantly finite in  $\mu^+(\mathcal{M}; \mathcal{D})$ , so we take right minimal approximations  $D_i \xrightarrow{g_i} Y_i$  of the  $Y_i$ . we complete these to triangles

$$X_i \xrightarrow{f_i} D_i \xrightarrow{g_i} Y_i \longrightarrow X_i[1],$$

and add them together to get the solid part of the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & D & \xrightarrow{g} & N_X & \longrightarrow & X[1] \\ \uparrow \text{---} \downarrow & & \uparrow \text{---} \downarrow & & \uparrow \text{---} \downarrow & & \uparrow \text{---} \downarrow \\ \bigoplus_{i=1}^n X_i & \xrightarrow{\bigoplus_{i=1}^n f_i} & \bigoplus_{i=1}^n D_i & \xrightarrow{\bigoplus_{i=1}^n g_i} & \bigoplus_{i=1}^n Y_i & \longrightarrow & \bigoplus_{i=1}^n X_i[1]. \end{array}$$

By Lemma 4.10 (i),  $g$  is a right minimal  $\mathcal{D}$ -approximation of  $N_X$ , and by Corollary 3.9 (ii)  $\bigoplus_{i=1}^n g_i$  is also a right minimal  $\mathcal{D}$ -approximation of  $N_X$ . It then follows by Lemma 1.8 that  $D \cong \bigoplus_{i=1}^n D_i$  and  $X \cong \bigoplus_{i=1}^n X_i$ .

By Lemma 4.10, the  $f_i$  are left minimal approximations of the  $X_i$ , and by Corollary 3.9 (i), and Lemma 1.6 we have that  $\bigoplus_{i=1}^n f_i$  is a left minimal  $\mathcal{D}$ -approximations as well.

Assume that there is a  $j$  such that  $Y_j \in \mathcal{M}_X$ . The minimal right approximation  $g_j$  is then the identity, which means that  $X_j = 0$  and  $f_j = 0$ . But by Lemma 3.8 (II) this is a contradiction, as now  $\bigoplus_{i=1}^n f_i$  is not left minimal. This means  $Y$  has no summands in  $\mathcal{M}_X$ , and so all the  $X_i \neq 0$ , which means  $X$  decomposes nontrivially. This is again a contradiction, and we conclude that  $N_X$  is indecomposable.  $\square$

We can now formally state and prove the claim that irreducible mutation is reversible, and that it amounts to 'minimal movement' within the partial ordering on  $\text{silt } \mathcal{T}$ .

**Theorem 4.12.** *Let  $\mathcal{T}$  be a Krull-Schmidt triangulated category such that for any  $\mathcal{X} \in \text{silt } \mathcal{T}$  and any indecomposable object  $X \in \mathcal{X}$ , the category  $\mathcal{X}_X$  is functorially finite in  $\mathcal{X}$ . If  $\mathcal{M}, \mathcal{N} \in \text{silt } \mathcal{T}$ , then the following are equivalent:*

- (i)  $\mathcal{N}$  is an irreducible left mutation of  $\mathcal{M}$ .
- (ii)  $\mathcal{M}$  is an irreducible right mutation of  $\mathcal{N}$ .
- (iii)  $\mathcal{M} > \mathcal{N}$ , and if  $\mathcal{L} \in \text{silt } \mathcal{T}$  with  $\mathcal{M} \geq \mathcal{L} \geq \mathcal{N}$  then  $\mathcal{M} = \mathcal{L}$  or  $\mathcal{L} = \mathcal{N}$ .

*Proof.* The theorem is proved by showing (i)  $\Leftrightarrow$  (ii) and (i)  $\Leftrightarrow$  (iii).

Assume (i), so  $\mathcal{N}$  is an irreducible left mutation of  $\mathcal{M}$ . That is,  $\mathcal{N} = \mu^+(\mathcal{M}; \mathcal{M}_X)$  for some indecomposable object  $X \in \mathcal{M}$ . By definition,  $\mathcal{N} = \text{add}(\mathcal{M}_X \cup \{N_X\})$ , where  $N_X$  is obtained from some triangle

$$X \xrightarrow{f} D \xrightarrow{g} N_X \xrightarrow{h} X[1]$$

completing a left  $\mathcal{M}_X$ -approximation  $X \xrightarrow{f} D$ . Since  $\mu^+(\mathcal{M}; \mathcal{M}_X)$  does not depend on the specific approximation, we can assume  $f$  is left minimal by Lemma 3.7. By Proposition 4.7 (i),  $\mathcal{M} = \mu^-(\mathcal{N}; \mathcal{M}_X)$ . Thus  $\mathcal{M}$  is a right mutation of  $\mathcal{N}$ . By Lemma 4.11,  $N_X$  is indecomposable, and so  $\mathcal{M}_X = \mathcal{N}_Y$  for some indecomposable object  $Y \in \mathcal{N}$ . In other words;  $\mathcal{M}$  is a right minimal approximation of  $\mathcal{N}$ . Showing (ii)  $\Rightarrow$  (i) is dual.

To show (i)  $\Leftrightarrow$  (iii), first assume (iii). As  $\mathcal{M} > \mathcal{N}$ , we have by Proposition 4.9 that there exists an irreducible left mutation  $\mathcal{L}$  of  $\mathcal{M}$  such that  $\mathcal{M} > \mathcal{L} \geq \mathcal{N}$ . By our assumption, we then have  $\mathcal{L} = \mathcal{N}$ , showing (i).

Assume (i), so  $\mathcal{N} = \mu^+(\mathcal{M}; \mathcal{M}_X)$  for an indecomposable  $X \in \mathcal{M}$ . Assume, in order to arrive at a contradiction, that there exists a silting subcategory  $\mathcal{L}$  of  $\mathcal{T}$  such that  $\mathcal{M} > \mathcal{L} > \mathcal{N}$ . Again, by Proposition 4.9, there exists an irreducible left mutation  $\mathcal{K}$  of  $\mathcal{M}$  such that  $\mathcal{M} > \mathcal{K} \geq \mathcal{L} > \mathcal{N}$ , and so, by applying Proposition 2.27, we get that  $\mathcal{M} \cap \mathcal{N} \subseteq \mathcal{K}$ . We have that

$$\mathcal{M} = \text{add}(\mathcal{M}_X \cup \{X\}),$$

and

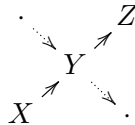
$$\mathcal{N} = \text{add}(\mathcal{M}_X \cup \{N_X\}),$$

meaning  $\mathcal{M} \cap \mathcal{N} = \mathcal{M}_X \subseteq \mathcal{K}$ . Then both  $\mathcal{N}$  and  $\mathcal{K}$  are irreducible left mutations of  $\mathcal{M}$  made by substituting the same indecomposable object  $X$ , and so  $\mathcal{K} = \mathcal{N}$ . This is a contradiction to  $\mathcal{K} \geq \mathcal{L} > \mathcal{N}$ , and so no such  $\mathcal{L}$  exists.  $\square$

The theory is contextualized by an example:

**Example 4.13.** As per usual, let  $\mathcal{T}$  be  $\text{K}^b(\mathcal{P}(\text{mod } \Lambda))$ , where  $\Lambda$  is the path algebra  $kA_3$  or  $kA_3/\langle \beta\alpha \rangle$ .

Let  $M$  be the silting object  $X \oplus Y \oplus Z$ , where the  $X, Y, Z$  are on a diagonal



of the AR-quiver of  $\mathcal{T}$ .

From Proposition 3.4, we know that for any summand  $N$  of  $M$ ,  $\text{add}\{N\}$  is functorially finite in  $\mathcal{T}$ . In particular,  $\text{add}\{N\}$  is covariantly finite in  $\text{add}\{M\}$ .

Choosing  $N := Y \oplus Z$ , we get the left  $\text{add}\{N\}$ -approximation  $X \rightarrow Y$  from the corresponding arrow in the AR-quiver. Similarly,  $X \rightarrow Y$  is also a right  $\text{add}\{X \oplus Z\}$ -approximation of  $Y$ , and  $Y \rightarrow Z$  is both a left  $\text{add}\{X \oplus Z\}$ -approximation of  $Y$  and a right  $\text{add}\{X \oplus Y\}$ -approximation of  $Z$  by Lemma 2.5.

For ease of notation, all summands in this example are denoted by  $X, Y$  and  $Z$ , even though they change as by the mutation performed on the silting objects.

Left mutation of  $M$  with respect to  $X, Y$  and  $Z$  produces the respective silting objects  $M_X, M_Y$  and  $M_Z$ , of which the summands are given by the diagrams

$$M_X : \begin{array}{ccc} \cdot & & Z \\ & \nearrow Y & \\ & & \cdot \\ \cdot & \xrightarrow{\quad} & X, \end{array} \quad M_Y : \begin{array}{ccc} \cdot & & Z \\ & \nearrow & \searrow \\ & \cdot & \cdot \\ X & \xrightarrow{\quad} & Y, \end{array} \quad M_Z : \begin{array}{ccc} \cdot & & \cdot \\ & \nearrow Y & \searrow \\ & \cdot & \cdot \\ X & \xrightarrow{\quad} & Z. \end{array}$$

respectively. The dotted arrows indicate which summands are interchanged. These new objects correspond to the mutations  $\mu^+(\text{add}\{M\}; \text{add}\{M\}_X)$ ,  $\mu^+(\text{add}\{M\}; \text{add}\{M\}_Y)$  and  $\mu^+(\text{add}\{M\}; \text{add}\{M\}_Z)$ .

By Proposition 4.7,  $\text{add}\{M\} > \mathcal{N}$ , where  $\mathcal{N}$  is any of these mutations. This coincides with what was seen before: The silting objects  $N$  of  $\mathcal{T}$  such that  $\text{add}\{M\} \geq \text{add}\{N\}$  are exactly those whose summands are shared with  $M$  or are to the right of  $M$  in the AR-quiver. We also know from Theorem 4.12 that these cannot be irreducibly mutated into each other, as this would violate the fact that each was obtained through irreducible left mutation of  $\text{add}\{M\}$ .

Also by Theorem 4.12 and Proposition 4.7, these are right mutated back to  $\text{add}\{M\}$  by mutating with the same summand of  $M$ . We also see this by 4.10, as we interchange the summand by going back through the triangle and arriving again at  $X$ .

Futhermore, as  $\text{add}\{M_X\}$  is again a silting subcategory, it can itself be left mutated to yield the silting objects  $M_{X,X}, M_{X,Y}$  and  $M_{X,Z}$ , given by

$$M_{X,X} : \begin{array}{ccc} \cdot & & Z \\ & \nearrow Y & \\ & & \cdot \\ \cdot & \xrightarrow{\quad} & X \\ & \cdot & \cdot \\ & \cdot & \cdot \end{array}, \quad M_{X,Y} : \begin{array}{ccc} \cdot & & Z \\ & \nearrow & \searrow \\ & \cdot & \cdot \\ \cdot & \xrightarrow{\quad} & Y \\ & \cdot & \cdot \\ & \cdot & X \end{array}, \quad M_{X,Z} : \begin{array}{ccc} \cdot & & \cdot \\ & \nearrow Y & \searrow \\ & \cdot & \cdot \\ \cdot & \xrightarrow{\quad} & Z \\ & \cdot & \cdot \\ & \cdot & \cdot \end{array}.$$

Similarly for  $M_Y$  and  $M_Z$ , we get

$$M_{Y,X} : \begin{array}{ccc} \cdot & & Z \\ & \nearrow & \searrow \\ & \cdot & \cdot \\ \cdot & \xrightarrow{\quad} & X \\ & \cdot & \cdot \\ & \cdot & Y \end{array}, \quad M_{Y,Y} : \begin{array}{ccc} \cdot & & Z \\ & \nearrow & \searrow \\ & \cdot & \cdot \\ X & \xrightarrow{\quad} & Y \\ & \cdot & \cdot \\ & \cdot & \cdot \end{array}, \quad M_{Y,Z} : \begin{array}{ccc} \cdot & & \cdot \\ & \nearrow & \searrow \\ & \cdot & \cdot \\ \cdot & \xrightarrow{\quad} & Z \\ & \cdot & \cdot \\ X & \cdot & Y \end{array},$$

and

$$M_{Z,X} : \begin{array}{ccc} \cdot & & \cdot \\ & \nearrow Y & \\ & \cdot & \cdot \\ \cdot & \xrightarrow{\quad} & X \\ & \cdot & \cdot \\ & \cdot & Z \end{array}, \quad M_{Z,Y} : \begin{array}{ccc} \cdot & & \cdot \\ & \nearrow & \searrow \\ & \cdot & \cdot \\ \cdot & \xrightarrow{\quad} & Y \\ & \cdot & \cdot \\ X & \cdot & Z \end{array}, \quad M_{Z,Z} : \begin{array}{ccc} \cdot & & \cdot \\ & \nearrow Y & \searrow \\ & \cdot & \cdot \\ \cdot & \xrightarrow{\quad} & Z \\ & \cdot & \cdot \\ X & \cdot & \cdot \end{array}.$$



When starting at a given silting object, distinct irreducible left mutations amount to interchanging different summands, and so the – in this case three, different mutations all give distinct results. That is, there is exactly one irreducible left mutation for each indecomposable object in  $\mathcal{M}$ , and similarly for right mutation.

Note that mutation of  $M_{X,Y}$  to  $M_{X,Y,X}$  is given by

$$M_{X,Y,X} : \begin{array}{ccc} \cdot & Z & \cdot \\ & \searrow & \\ \cdot & & Y \\ & & \searrow \\ \cdot & \cdots \cdots \cdots & X, \end{array}$$

which is the same set of summands as in  $M_{Y,X}$ . This shows that there may be more than one sequence of mutations sharing a start- and endpoint, and these sequences may even be of different length.

## 4.2 Silting Reduction and Verdier Localization of $\mathcal{T}$

Having introduced both silting mutation theory and Krull-Schmidt triangulated categories, we start nudging the focus towards the question of *transitivity* of Section 5. In this section we will consider the *Verdier localization* of a triangulated Krull-Schmidt category  $\mathcal{T}$  with respect to a thick subcategory  $\mathcal{S}$ . The goal is very much the same as in Aihara and Iyama [1] to show that under certain conditions on  $\mathcal{S}$ , the silting subcategories of this localization are in bijection with certain classes of silting subcategories of  $\mathcal{T}$ .

In Theorem 5.9 of Section 5.1 we apply this to see that these classes of silting subcategories are indeed transitive under iterated irreducible silting mutation.

The Verdier localization is a generalized variant of the localization employed when obtaining a derived category from a homotopy category [26].

**Definition 4.14.** Let  $\mathcal{T}$  be a triangulated category, and let  $\mathcal{S} \subseteq \mathcal{T}$  a thick subcategory. We say that a morphism  $X \xrightarrow{f} Y$  in  $\mathcal{T}$  is a *quasi-isomorphism* with respect to  $\mathcal{S}$  if for any triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1]$$

$Z \in \mathcal{S}$ . Define the *Verdier localization* [20],  $\mathcal{T}/\mathcal{S}$  of  $\mathcal{T}$  with respect to  $\mathcal{S}$  as follows:

- (i)  $\mathbf{Ob} \mathcal{T}/\mathcal{S} := \mathbf{Ob} \mathcal{T}$ .
- (ii) For  $X, Y \in \mathcal{T}/\mathcal{S}$ , the morphisms in  $\mathrm{Hom}_{\mathcal{T}/\mathcal{S}}(X, Y)$  are given by *roof* diagrams

$$\begin{array}{ccc} & U & \\ q \swarrow & & \searrow f \\ X & & Y \end{array}$$

for  $U \xrightarrow{f} Y$  a morphism and  $U \xrightarrow{q} X$  a quasi-isomorphism in  $\mathcal{T}$ .

The *canonical functor*, or *localization functor*  $\mathcal{T} \xrightarrow{L} \mathcal{T}/\mathcal{S}$  is the one which takes objects to themselves and morphisms  $X \xrightarrow{f} Y$  to the corresponding roof

$$\begin{array}{ccc} & X & \\ 1 \swarrow & & \searrow f \\ X & & Y \end{array}$$

For ease of notation, we will denote  $\mathcal{T}/\mathcal{S}$  simply by  $\mathcal{U}$ .

Consider the subcategory

$$\mathcal{S}^{\perp\tau} = \{T \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(\mathcal{S}, T) = 0\} \subseteq \mathcal{T}.$$

If we assume  $\mathcal{S} \subseteq \mathcal{T}$  is a thick and contravariantly finite subcategory, then by Lemma 3.13 (ii), we have a torsion pair  $(\mathcal{S}, \mathcal{S}^{\perp\tau})$  in  $\mathcal{T}$ . As  $\mathcal{S}[1] = \mathcal{S}$ , we get by Definition 1.11 that it additionally forms a stable t-structure of  $\mathcal{T}$ . In particular, this means that  $\mathcal{T} = \mathcal{S} * \mathcal{S}^{\perp}$ , i.e. for any  $T \in \mathcal{T}$  there is a triangle

$$S \xrightarrow{a} T \xrightarrow{b} U \xrightarrow{c} S[1] \quad (15)$$

with  $S \in \mathcal{S}$  and  $U \in \mathcal{S}^{\perp}$ .

From [25] and [20], we know that the Verdier localization has the following important properties.

**Proposition 4.15.** *Let  $\mathcal{T}$  be a triangulated category and  $\mathcal{S} \subseteq \mathcal{T}$  a thick subcategory. For  $\mathcal{U}$  the Verdier localization of  $\mathcal{T}$  with respect to  $\mathcal{S}$ , we have:*

- (i)  $\mathcal{U}$  is a triangulated category, where the triangles are the images of triangles in  $\mathcal{T}$ .
- (ii) The canonical functor  $L$  maps quasi-isomorphisms in  $\mathcal{T}$  to isomorphisms in  $\mathcal{U}$ .
- (iii) If  $\mathcal{S}$  is contravariantly finite in  $\mathcal{T}$ , the restriction

$$L|_{\mathcal{S}^{\perp}} : \mathcal{S}^{\perp} \xrightarrow{\sim} \mathcal{U}$$

of  $L$  to  $\mathcal{S}^{\perp\tau}$  is an equivalence.

Straight away, we get the following result.

**Proposition 4.16.** *Let  $\mathcal{T}$  be a Krull-Schmidt triangulated category, and  $\mathcal{S}$  a thick subcategory of  $\mathcal{T}$ . For any  $T \in \mathcal{T}$  and a triangle*

$$S \xrightarrow{a} T \xrightarrow{b} U \xrightarrow{c} S[1],$$

with  $S \in \mathcal{S}$  and  $U \in \mathcal{S}^{\perp\tau}$ ,  $a$  is a right minimal  $\mathcal{S}$ -approximation, and  $b$  is a left minimal  $\mathcal{S}^{\perp\tau}$ -approximation of  $T$ .

*Proof.* That  $a$  and  $b$  are approximations is by Lemma 1.7. It remains to check the minimality. By Lemma 3.7, there is a decomposition  $S = S' \oplus S'' \xrightarrow{a=(a' \ 0)} T$ , where  $a'$  is a right minimal  $\mathcal{S}$ -approximation of  $T$ . Complete  $a'$  to a triangle  $S' \xrightarrow{a'} T \xrightarrow{b'} U' \xrightarrow{c'} S'[1]$ , and we have the solid part of the commutative diagram

$$\begin{array}{ccccccc} S & \xrightarrow{a} & T & \xrightarrow{b} & U & \xrightarrow{c} & S[1] \\ \downarrow 1 & & \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & & \downarrow \varphi & & \downarrow 1 \\ S' \oplus S'' & \xrightarrow{\begin{pmatrix} a' & 0 \\ 0 & 0 \end{pmatrix}} & T \oplus 0 & \xrightarrow{\begin{pmatrix} b' & 0 \\ 0 & 0 \end{pmatrix}} & U' \oplus S''[1] & \xrightarrow{\begin{pmatrix} c' & 0 \\ 0 & 1 \end{pmatrix}} & S'[1] \oplus S''[1]. \end{array}$$

As the left square commutes, the diagram is completed to a morphism of triangles by some  $\varphi$ , which is an isomorphism as the two leftmost vertical morphisms are isomorphisms. This means  $S''[1] \in \mathcal{S}^{\perp\tau}$ . As  $\mathcal{S}$  is closed under shift, we also have  $S''[1] \in \mathcal{S}$ . Then

$$\mathrm{Hom}_{\mathcal{T}}(S'', S'') \cong \mathrm{Hom}_{\mathcal{T}}(S''[1], S''[1]) = 0,$$

which means  $S = S'$  and so  $a = a'$ . A proof showing  $b$  is left minimal is dual.  $\square$

As the functor  $L$  is dependent on the subcategory  $\mathcal{S}$ , it has the following properties in regards to  $\mathcal{S}$  and the triangles which appear from the torsion pair  $(\mathcal{S}, \mathcal{S}^{\perp})$ .

**Lemma 4.17.** *Let  $\mathcal{T}$  be a Krull-Schmidt triangulated category, and  $\mathcal{S}$  a thick and contravariantly finite subcategory. For  $\mathcal{U} = \mathcal{T}/\mathcal{S}$  the Verdier localization and  $\mathcal{T} \xrightarrow{L} \mathcal{U}$  the canonical functor, we have that:*

(i) *For any  $T \in \mathcal{T}$  and a triangle*

$$S \xrightarrow{a} T \xrightarrow{b} U \xrightarrow{c} S[1]$$

*with  $S \in \mathcal{S}$  and  $U \in \mathcal{S}^{\perp}$ , then  $T \cong U$  in  $\mathcal{U}$ .*

(ii)  $L\mathcal{S} = 0$ .

(iii) *Let  $T \xrightarrow{f} T'$  be any morphism in  $\mathcal{T}$ . Then  $Lf$  in  $\mathcal{U}$  is given by  $g$  in a morphism of triangles*

$$\begin{array}{ccccccc} S & \longrightarrow & T & \longrightarrow & U & \longrightarrow & S[1] \\ \downarrow & & \downarrow f & & \downarrow g & & \downarrow \\ S' & \longrightarrow & T' & \longrightarrow & U' & \longrightarrow & S'[1] \end{array}$$

*where the rows are triangles with  $S, S' \in \mathcal{S}$  and  $U, U' \in \mathcal{S}^{\perp}$ .*

*Proof.* From Proposition 4.16  $T \xrightarrow{b} U$  is a left minimal  $\mathcal{S}^{\perp}$ -approximation of  $T$ . As  $S \in \mathcal{S}$ , which is closed under shift,  $S[1] \in \mathcal{T}$ , and so  $T \xrightarrow{b} U$  is a quasi-isomorphism, and by Proposition 4.15 (ii),  $T \xrightarrow{Lf} U$  is an isomorphism in  $\mathcal{U}$ . This shows part (i).

For part (ii), note that for any  $S \in \mathcal{S}$ , (15) takes the form

$$S \xrightarrow{1} S \longrightarrow 0 \longrightarrow S[1].$$

It follows from (i) that  $S \cong 0$  in  $\mathcal{U}$ .

Now let  $T \xrightarrow{f} T'$  be any morphism in  $\mathcal{T}$ . By (15), we then have the solid part of the diagram

$$\begin{array}{ccccccc} S & \xrightarrow{a} & T & \xrightarrow{b} & U & \longrightarrow & S[1] \\ \vdots & & \downarrow f & & \downarrow g & & \vdots \\ S' & \xrightarrow{a'} & T' & \xrightarrow{b'} & U' & \longrightarrow & S'[1], \end{array} \tag{16}$$

Now  $b'fa \in \text{Hom}_{\mathcal{T}}(\mathcal{S}, \mathcal{S}^\perp) = 0$   $fa$  factors through  $a'$ , and we can complete the diagram to a morphism of triangles by the dotted vertical maps as indicated. Applying  $L$  to the diagram yields the morphism of triangles

$$\begin{array}{ccccccc} S & \xrightarrow{La} & T & \xrightarrow{Lb} & U & \longrightarrow & S[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S' & \xrightarrow{La'} & T' & \xrightarrow{Lb'} & U' & \longrightarrow & S'[1] \end{array}$$

in  $\mathcal{U}$ .

Again,  $Lb$  and  $Lb'$  are isomorphisms by part (i), so we have an isomorphism

$$\text{Hom}_{\mathcal{U}}(T, T') \xrightarrow{Lb' \circ \circ (Lb)^{-1}} \text{Hom}_{\mathcal{U}}(U, U').$$

In particular, it maps  $Lf$  isomorphically to  $Lg$ . Furthermore, by Proposition 4.15 (iii),  $L$  is an isomorphism on the Hom-sets

$$\text{Hom}_{\mathcal{T}}(U, U') \xrightarrow{L} \text{Hom}_{\mathcal{U}}(U, U'),$$

and so  $g$  is mapped isomorphically to  $Lg$ . Then

$$\text{Hom}_{\mathcal{T}}(U, U') \cong \text{Hom}_{\mathcal{U}}(U, U') \cong \text{Hom}_{\mathcal{U}}(T, T')$$

and for all intents and purposes,  $g$ ,  $Lg$  and  $Lf$  are the same morphism. By abuse of notation,  $Lf$  is thus given by  $U \xrightarrow{g} U'$  as in (16).  $\square$

Before introducing the main result of the section, Theorem 4.19, we show the following lemma.

**Lemma 4.18.** *Let  $\mathcal{T}$  be a Krull-Schmidt triangulated category and  $\mathcal{M} \in \text{silt } \mathcal{T}$ . Assume  $\mathcal{S}$  is a thick and contravariantly finite subcategory of  $\mathcal{T}$ , and that  $\mathcal{D} \in \text{silt } \mathcal{S}$  is such that  $\mathcal{D} \subseteq \mathcal{M}$ .*

*If  $M \in \mathcal{M}$ , then in a triangle*

$$S \xrightarrow{a} M \xrightarrow{b} U \longrightarrow S[1]$$

*with  $S \in \mathcal{S}$  and  $U \in \mathcal{S}^\perp$  the object  $S$  is in  $\mathcal{S}_{\mathcal{D}}^{\leq 0}$ .*

Recall that

$$\mathcal{S}_{\mathcal{D}}^{\leq 0} = \{X \in \mathcal{S} \mid \text{Hom}_{\mathcal{S}}(\mathcal{D}, X[> 0]) = 0\},$$

and

$${}^{\perp \mathcal{S}} \mathcal{S}_{\mathcal{D}}^{\leq 0} = \{Y \in \mathcal{S} \mid \text{Hom}_{\mathcal{T}}(Y, \mathcal{S}_{\mathcal{D}}^{\leq 0}) = 0\}.$$

*Proof.* Let  $M \in \mathcal{M}$ , and consider the triangle

$$S \xrightarrow{a} M \xrightarrow{b} U \longrightarrow S[1]$$

with  $S \in \mathcal{S}$  and  $U \in \mathcal{S}^\perp$ . By Proposition 3.15

$$\left( {}^{\perp \mathcal{S}}(\mathcal{S}_{\mathcal{D}}^{\leq 0}), \mathcal{S}_{\mathcal{D}}^{\leq 0} \right)$$

is a torsion pair in  $\mathcal{S}$ . In particular  ${}^{\perp_S} \mathcal{S}_{\mathcal{D}}^{\leq 0} * \mathcal{S}_{\mathcal{D}}^{\leq 0} = \mathcal{S}$ , there is a triangle

$$S'' \xrightarrow{c} S \xrightarrow{d} S' \longrightarrow S''[1]$$

with  $S' \in \mathcal{S}_{\mathcal{D}}^{\leq 0}$ , and  $S'' \in {}^{\perp_S} \mathcal{S}_{\mathcal{D}}^{\leq 0}$ .

By Proposition 3.15 (iii) there is an  $l > 0$  such that  $S'' \in \mathcal{D}[-l] * \cdots * \mathcal{D}[-1]$ . As  $\mathcal{D} \subseteq \mathcal{M}$ , we then have

$$\mathrm{Hom}_{\mathcal{T}}(S'', M) \subseteq \mathrm{Hom}_{\mathcal{T}}(\mathcal{M}[-l] * \cdots * \mathcal{M}[-1], \mathcal{M})$$

which is 0 by Lemma 2.19.

Then  $ac = 0$  so there is some  $S' \xrightarrow{e} M$  such that  $a = ed$ .

$$\begin{array}{ccccccc} S'' & \xrightarrow{c} & S & \begin{array}{c} \xleftarrow{f} \\ \xrightarrow{d} \end{array} & S' & \longrightarrow & S''[1] \\ & & \downarrow a & & \swarrow e & & \\ S & \xrightarrow{a} & M & \xrightarrow{b} & U & \longrightarrow & S[1] \end{array}$$

Similarly,  $be \in \mathrm{Hom}_{\mathcal{T}}(\mathcal{S}, \mathcal{S}^{\perp}) = 0$ , so there is a morphism  $S' \xrightarrow{f} S$  such that  $e = af$ . Combining these, we get  $a = afd$ , so  $a(1_S - fd) = 0$ .

$$\begin{array}{ccccc} & & S & & \\ & \swarrow & \downarrow 1_S - fd & & \\ U[-1] & \longrightarrow & S & \xrightarrow{a} & M \xrightarrow{b} U \end{array}$$

It follows that  $1_S - fd$  factors through  $U[-1]$ . Since  $\mathcal{S}$  is closed under shift, and

$$\mathrm{Hom}_{\mathcal{T}}(S[1], U) \subseteq \mathrm{Hom}_{\mathcal{T}}(\mathcal{S}, \mathcal{S}^{\perp}) = 0,$$

we get that  $1_S - fd$  factors through 0. That is,  $fd = 1_S$ , and  $d$  is a split monomorphism, i.e. the triangle

$$S \xrightarrow{d} S' \longrightarrow S''[1] \xrightarrow{-c[1]} S[1]$$

splits. We conclude from this that  $S \oplus S''[1] \cong S' \in \mathcal{S}_{\mathcal{D}}^{\leq 0}$ , and so  $S \in \mathcal{S}_{\mathcal{D}}^{\leq 0}$ .  $\square$

Having Lemma 4.18, we can now state and prove the main result.

**Theorem 4.19.** *Let  $\mathcal{T}$  be a Krull-Schmidt triangulated category, and let  $\mathcal{S}$  be a thick subcategory of  $\mathcal{T}$ . Let  $\mathcal{U}$  be the Verdier localization  $\mathcal{T}/\mathcal{S}$ , and  $\mathcal{T} \xrightarrow{L} \mathcal{U}$  the canonical functor.*

(i) *If  $\mathcal{S}$  is a contravariantly finite subcategory of  $\mathcal{T}$ , then for any  $\mathcal{D} \in \mathrm{silt} \mathcal{S}$ , the map*

$$\{\mathcal{M} \in \mathrm{silt} \mathcal{T} \mid \mathcal{D} \subseteq \mathcal{M}\} \rightarrow \mathrm{silt} \mathcal{U}$$

*given by  $\mathcal{M} \mapsto L\mathcal{M}$  is injective.*

(ii) *If  $\mathcal{S}$  is a functorially finite subcategory of  $\mathcal{T}$ , then the map in (i) is bijective.*

The consequence of Theorem 4.19 (ii) is immediate, as  $\mathrm{silt} \mathcal{S}^{\perp}$  in part (ii) represents all the silting subcategories of  $\mathcal{M}$  which contain any silting subcategory of  $\mathcal{S}$ .

*Proof.* Part (i) is shown by proving  $L\mathcal{M}$  is a silting subcategory of  $\mathcal{U}$ , and then that  $\mathcal{M} \mapsto L\mathcal{M}$  is injective. Then we prove (ii) by showing the map is also surjective.

(i) Let  $X \in \text{thick } \mathcal{M} = \mathcal{T}$ . Then by Proposition 3.15 there is an  $l > 0$  such that  $X \in \mathcal{M}[-l] * \cdots * \mathcal{M}[l]$ . I.e. there are  $M_i \in \mathcal{M}$  and  $M'_i \in \mathcal{M}[i] * \cdots * \mathcal{M}[l]$  and triangles

$$\begin{array}{ccc} & M_i[i] & \\ & \downarrow & \\ & M'_i & \longrightarrow M'_{i+1} \\ & & \downarrow \\ & & M_i[i+1] \end{array}$$

which fit together in the diagram

$$\begin{array}{ccccccccc} M_{-l}[-l] & M_{-l+1}[-l+1] & \cdots & M_{l-1}[l-1] & M_l[l] & & & & \\ \downarrow & \downarrow & & \downarrow & \parallel & & & & \\ X & \longrightarrow M'_2 & \longrightarrow \cdots & \longrightarrow M'_{l-1} & \longrightarrow M'_l & \longrightarrow 0 & & & \\ & \downarrow & & \downarrow & \downarrow & & & & \\ & M_{-l}[-l+1] & \cdots & M_{l-2}[l-1] & M_{l-1}[l] & M_l[l+1] & & & \end{array}$$

in  $\mathcal{T}$ . We now apply  $L$  to the whole diagram to get a new diagram

$$\begin{array}{ccccccccc} LM_{-l}[-l] & LM_{-l+1}[-l+1] & \cdots & LM_{l-1}[l-1] & LM_l[l] & & & & \\ \downarrow & \downarrow & & \downarrow & \parallel & & & & \\ LX & \longrightarrow LM'_2 & \longrightarrow \cdots & \longrightarrow LM'_{l-1} & \longrightarrow LM'_l & \longrightarrow 0 & & & \\ & \downarrow & & \downarrow & \downarrow & & & & \\ & LM_{-l}[-l+1] & \cdots & LM_{l-2}[l-1] & LM_{l-1}[l] & LM_l[l+1] & & & \end{array}$$

of triangles in  $\mathcal{U}$ . Starting with the rightmost triangle, we have  $LM_{l-1}[l-1]$  and  $LM_l[l]$  in  $\text{thick } LM$ , so the middle term,  $LM'_l$  is as well. We then do the same for the second-to-last triangle, the third-to-last, and so on. After  $2l - 1$  steps we conclude  $LX \in \text{thick } LM$ , which means that  $L \text{thick } \mathcal{M} \subseteq \text{thick } LM$ . We then conclude from  $\mathcal{M} \in \text{silt } \mathcal{T}$  that

$$\mathcal{U} = L\mathcal{T} = L \text{thick } \mathcal{M} \subseteq \text{thick } LM,$$

and we are halfway done showing  $LM \in \text{silt } \mathcal{U}$ . What remains is to show that  $\text{Hom}_{\mathcal{U}}(LM, LM[> 0]) = 0$ . To this end we let  $M$  and  $M'$  be any objects in  $\mathcal{M}$ . As before,  $(\mathcal{S}, \mathcal{S}^\perp)$  is a torsion pair, and we have the triangles

$$S \xrightarrow{a} M \xrightarrow{b} U \longrightarrow S[1] \tag{17}$$

and

$$S' \xrightarrow{a'} M' \xrightarrow{b'} U' \longrightarrow S'[1]$$

with  $S, S' \in \mathcal{S}$  and  $U, U' \in \mathcal{S}^\perp$ . By applying  $\text{Hom}_{\mathcal{T}}(M', -)$  to (17) we get the long exact sequence

$$\cdots \longrightarrow \text{Hom}_{\mathcal{T}}(M', M[n]) \longrightarrow \text{Hom}_{\mathcal{T}}(M', U[n]) \longrightarrow \text{Hom}_{\mathcal{T}}(M', S[n+1]) \longrightarrow \cdots,$$

and for  $n > 0$ , the left term vanishes as  $\mathcal{M}$  is silting. By Lemma 4.18 we know that  $S \in \mathcal{S}_{\mathcal{D}}^{\leq 0}$ , and as  $\mathcal{D} \subseteq \mathcal{M}$ , we have

$$\mathcal{S}_{\mathcal{D}}^{\leq 0} = \bigcup_{l \geq 0} \mathcal{D} * \cdots * \mathcal{D}[l] \subseteq \bigcup_{l \geq 0} \mathcal{M} * \cdots * \mathcal{M}[l] = \mathcal{T}_{\mathcal{M}}^{\leq 0}$$

so

$$\mathrm{Hom}_{\mathcal{T}}(M', S[n+1]) \subseteq \mathrm{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{T}_{\mathcal{M}}^{\leq 0}[> 0]) = 0.$$

Hence both the left and right terms in the exact sequence vanish, and it follows by exactness that also  $\mathrm{Hom}_{\mathcal{T}}(M', U[n]) = 0$ .

For the next step, we apply  $\mathrm{Hom}_{\mathcal{T}}(-, U')$  to (17) and obtain the long exact sequence

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{T}}(S[1], U'[n]) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(U, U'[n]) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(M, U'[n]) \longrightarrow \cdots$$

Consider any part of the sequence where  $n > 0$ . Since  $\mathcal{S}$  is a thick subcategory of  $\mathcal{T}$ , it is closed under shift, and

$$\mathrm{Hom}_{\mathcal{T}}(S[1], U'[n]) \cong \mathrm{Hom}_{\mathcal{T}}(S[1-n], U') \subseteq \mathrm{Hom}_{\mathcal{T}}(\mathcal{S}, \mathcal{S}^{\perp}) = 0.$$

Then both the left and right terms of the exact sequence disappear, and it follows by exactness that the middle term vanishes as well. Since  $M \cong U$  and  $M' \cong U'$  in  $\mathcal{U}$ , we get that

$$\mathrm{Hom}_{\mathcal{U}}(M, M'[n]) \cong \mathrm{Hom}_{\mathcal{U}}(U, U'[n]) = 0,$$

for all  $M, M' \in L\mathcal{M}$  and  $n > 0$ , proving  $L\mathcal{M}$  is a silting subcategory of  $\mathcal{U}$ .

Next we show that this map is injective. Assume  $\mathcal{M}, \mathcal{N} \in \mathrm{silt} \mathcal{T}$  with  $\mathcal{D} \subseteq \mathcal{M}, \mathcal{N}$  and that  $L\mathcal{M} = L\mathcal{N}$ . Let  $M \in \mathcal{M}$  and  $N \in \mathcal{N}$ , and we obtain triangles

$$S_M \xrightarrow{a_M} M \xrightarrow{b_M} U_M \longrightarrow S_M[1] \quad \text{and} \quad S_N \xrightarrow{a_N} N \xrightarrow{b_N} U_N \longrightarrow S_N[1]$$

for  $M$  and  $N$  as by (17). For some  $n > 0$  and some morphism  $M \xrightarrow{f} N[n]$ , consider the solid part of the diagram

$$\begin{array}{ccccccc} S_M & \longrightarrow & M & \longrightarrow & U_M & \longrightarrow & S_M[1] \\ \downarrow & \nearrow \varphi & \downarrow f & & \downarrow 0 & & \downarrow \\ S_N[n] & \xrightarrow{a_N[n]} & N[n] & \xrightarrow{b_N[n]} & U_N[n] & \longrightarrow & S_N[n+1] \end{array}$$

As

$$\mathrm{Hom}_{\mathcal{T}}(S_M, U_N[n]) \cong \mathrm{Hom}_{\mathcal{T}}(S_M[-n], U_N) \subseteq \mathrm{Hom}_{\mathcal{T}}(\mathcal{S}, \mathcal{S}^{\perp}) = 0,$$

there is a morphism  $S_M \rightarrow S_N[n]$  making the left square commute. Then there is also a morphism  $U_M \rightarrow U_N[n]$  completing the diagram to a morphism of triangles. This morphism is 0 as  $L\mathcal{M} = L\mathcal{N}$  is silting, and so  $\mathrm{Hom}_{\mathcal{T}}(U_M, U_N[> 0]) = 0$ . It follows that  $b_N[n]f = 0$ , and  $f$  factors through  $a_N[n]$  by some  $M \xrightarrow{\varphi} S_N[n]$ . Now, as  $S_N \in \mathcal{S}_{\mathcal{D}}^{\leq 0}$  by Lemma 4.18,  $\mathrm{Hom}_{\mathcal{T}}(M, S_N[n])$  is also contained in

$$\mathrm{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{T}_{\mathcal{M}}^{\leq 0}[> 0]) = 0$$

as  $\mathcal{D} \subseteq \mathcal{M}$ . Thus  $\varphi = 0$ , so  $f = 0$ . That is,  $\mathrm{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{N}[> 0]) = 0$ , and so  $\mathcal{M} \geq \mathcal{N}$ .

By a completely symmetric argument, we also have  $\mathcal{N} \geq \mathcal{M}$ , showing the two are the same by the antisymmetric property of  $\geq$ . That is,  $\mathcal{M} = \mathcal{N}$ , and the map is injective.

(ii) To show  $\mathcal{M} \mapsto L\mathcal{M}$  is surjective on silt  $\mathcal{U}$ , assume  $\mathcal{S}$  is in addition a covariantly finite subcategory of  $\mathcal{T}$ . As seen before,  $({}^{\perp s}(\mathcal{S}_{\mathcal{D}}^{\leq 0}), \mathcal{S}_{\mathcal{D}}^{\leq 0})$  is a torsion pair in  $\mathcal{S}$  by Proposition 3.15 (ii). Then, by Remark 1.2 and  $\mathcal{S}$  being closed under shift,

$${}^{\perp s}(\mathcal{S}_{\mathcal{D}}^{\leq 0}) * \mathcal{S}_{\mathcal{D}}^{\leq 0} = {}^{\perp s}(\mathcal{S}_{\mathcal{D}}^{\leq 0})[1] * \mathcal{S}_{\mathcal{D}}^{\leq 0}[1] = ({}^{\perp s}(\mathcal{S}_{\mathcal{D}}^{\leq 0}) * \mathcal{S}_{\mathcal{D}}^{\leq 0})[1] = \mathcal{S}[1] = \mathcal{S},$$

and so  $({}^{\perp s}(\mathcal{S}_{\mathcal{D}}^{\leq 0}), \mathcal{S}_{\mathcal{D}}^{\leq 0})$  is also a torsion pair in  $\mathcal{S}$ . In particular, it follows from Corollary 1.12 that  $\mathcal{S}_{\mathcal{D}}^{\leq 0}$  is covariantly finite in  $\mathcal{S}$ . Combining this with the fact that  $\mathcal{S}$  is covariantly finite in  $\mathcal{T}$ , we get that  $\mathcal{S}_{\mathcal{D}}^{\leq 0}$  is covariantly finite in  $\mathcal{T}$ . By Lemma 3.13 (iii), we then have that  $({}^{\perp \tau}(\mathcal{S}_{\mathcal{D}}^{\leq 0}), \mathcal{S}_{\mathcal{D}}^{\leq 0})$  is a torsion pair in  $\mathcal{T}$ .

Fix a silting category  $\mathcal{N}' \in \text{silt } \mathcal{U}$ . As  $\mathcal{S}^{\perp} \xrightarrow{L} \mathcal{U}$  is an equivalence,  $\mathcal{N}' = L\mathcal{N}$  for some  $\mathcal{N} \in \text{silt } \mathcal{S}^{\perp}$ . For any  $N \in \mathcal{N}$ , there is a triangle

$$S_N \xrightarrow{\alpha} M_N \xrightarrow{\beta} N \xrightarrow{\gamma} S_N[1], \quad (18)$$

with  $M_N \in {}^{\perp \tau}(\mathcal{S}_{\mathcal{D}}^{\leq 0})$  and  $S_N[1] \in \mathcal{S}_{\mathcal{D}}^{\leq 0}$  by the torsion pair above. From  $\mathcal{S} \subseteq \mathcal{T}$  we have that  ${}^{\perp s}(\mathcal{S}_{\mathcal{D}}^{\leq 0}) \subseteq {}^{\perp \tau}(\mathcal{S}_{\mathcal{D}}^{\leq 0})$ , and then by Proposition 2.24,

$$\mathcal{D} = \mathcal{S}_{\mathcal{D}}^{\leq 0} \cap {}^{\perp s}(\mathcal{S}_{\mathcal{D}}^{\leq 0}) \subseteq \mathcal{S}_{\mathcal{D}}^{\leq 0} \cap {}^{\perp \tau}(\mathcal{S}_{\mathcal{D}}^{\leq 0}).$$

As any object in  $\mathcal{S}_{\mathcal{D}}^{\leq 0} \cap {}^{\perp \tau}(\mathcal{S}_{\mathcal{D}}^{\leq 0})$  is in particular in  $\mathcal{S}_{\mathcal{D}}^{\leq 0} \subseteq \mathcal{S}$ , the inclusion  $\supseteq$  also holds, and it follows that  $\mathcal{D} = \mathcal{S}_{\mathcal{D}}^{\leq 0} \cap {}^{\perp \tau}(\mathcal{S}_{\mathcal{D}}^{\leq 0})$ .

Note that  $\beta$  is a right  $({}^{\perp}(\mathcal{S}_{\mathcal{D}}^{\leq 0}))$ -approximation of  $N$ , as by Corollary 1.12. Then, by Lemma 3.7, there exists a decomposition  $M_N = X \oplus Y \xrightarrow{\beta=(\beta_X \ 0)} N$ , where  $\beta_X$  is a right minimal  $({}^{\perp}(\mathcal{S}_{\mathcal{D}}^{\leq 0}))$ -approximation of  $N$ . Complete  $\beta_X$  to a triangle  $Z_N \xrightarrow{\alpha_X} X \xrightarrow{\beta_X} N \longrightarrow Z_N[1]$ , and add it to the identity triangle for  $Y$ . This yields the solid part of the diagram

$$\begin{array}{ccccccc} S_N & \xrightarrow{\alpha} & M_N & \xrightarrow{\beta} & N & \xrightarrow{\gamma} & S_N[1] \\ \downarrow \text{dotted} & & \downarrow 1 & & \downarrow \binom{1}{0} & & \downarrow \text{dotted} \\ Z_N \oplus Y & \xrightarrow{\binom{\alpha_X \ 0}{0 \ 1}} & X \oplus Y & \xrightarrow{\binom{\beta_X \ 0}{0 \ 0}} & N \oplus 0 & \longrightarrow & (Z_N \oplus Y)[1]. \end{array}$$

The central square commutes, so it is completed to a morphism of triangles by the outermost vertical morphism. As the two central vertical morphisms are isomorphisms, then so is the last one. This means  $Y$  is a direct summand in  $\mathcal{S}_{\mathcal{D}}^{\leq 0} \cap {}^{\perp \tau}(\mathcal{S}_{\mathcal{D}}^{\leq 0}) = \mathcal{D}$ . By Lemma 1.8  $X$  is independent of the right minimal  $({}^{\perp \tau}(\mathcal{S}_{\mathcal{D}}^{\leq 0}))$ -approximation of  $N$ . It follows that  $M_N$  is uniquely given by  $X$  and a summand in  $\mathcal{D}$ .

The next step now is to show that the category

$$\mathcal{M} := \text{add}(\mathcal{D} \cup \{M_N \mid N \in \mathcal{N}\})$$

is in silt  $\mathcal{T}$ . Note that by our previous endeavor, this construction is well-defined. We begin by showing  $\text{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{M}[> 0]) = 0$  – by checking for each combination of objects from  $\mathcal{M}$ .

It is easily seen that  $\mathcal{D}[> 0] \subseteq \mathcal{S}_{\mathcal{D}}^{\leq 0}$ , and so  $\text{Hom}_{\mathcal{T}}(M_N, \mathcal{D}[> 0]) = 0$ . We choose some  $N \in \mathcal{N}$ , some  $D' \in \mathcal{D}$ , and we apply  $\text{Hom}_{\mathcal{T}}(D', -)$  to the triangle (18) for  $N$ . This yields the long exact sequence

$$\cdots \longrightarrow \text{Hom}_{\mathcal{T}}(D', S_N[n]) \longrightarrow \text{Hom}_{\mathcal{T}}(D', M_N[n]) \longrightarrow \text{Hom}_{\mathcal{T}}(D', N[n]) \longrightarrow \cdots$$



For  $n > 0$ , the left side disappears as  $S_N \in \mathcal{S}_{\mathcal{D}}^{\leq 0}$  by Lemma 4.18, and the right side as  $N \in \mathcal{N} \subseteq \mathcal{S}^{\perp}$ , so  $\mathrm{Hom}_{\mathcal{T}}(D'[-n], N) \subseteq \mathrm{Hom}_{\mathcal{T}}(\mathcal{S}, \mathcal{S}^{\perp}) = 0$ . Then the middle term also vanishes by exactness. Similarly, for any  $N' \in \mathcal{N}$ , apply  $\mathrm{Hom}_{\mathcal{T}}(-, N')$  to (18), and get the exact sequence

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{T}}(N, N'[n]) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(M_N, N'[n]) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(S_N, N'[n]) \longrightarrow \cdots . \quad (19)$$

For  $n > 0$ , the left term disappears as  $\mathcal{N}$  is silting, and the right side again as  $\mathrm{Hom}_{\mathcal{T}}(S_N[-n], N') \subseteq \mathrm{Hom}_{\mathcal{T}}(\mathcal{S}, \mathcal{S}^{\perp}) = 0$ . Then the middle term disappears as well, again by exactness. For any  $N' \in \mathcal{N}$ , make a triangle

$$S_{N'} \longrightarrow M_{N'} \longrightarrow N' \longrightarrow S_{N'}[1]$$

with  $M_{N'} \in {}^{\perp}(\mathcal{S}_{\mathcal{D}}^{\leq 0})$  and  $S_{N'}[1] \in \mathcal{S}_{\mathcal{D}}^{\leq 0}$  as we did for  $N$ . Then, apply  $\mathrm{Hom}_{\mathcal{T}}(M_{N'}, -)$  to the triangle for  $N$  to get the long exact sequence

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{T}}(M_{N'}, S_N[n]) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(M_{N'}, M_N[n]) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(M_{N'}, N[n]) \longrightarrow \cdots .$$

For  $n > 0$ , the left term disappears by definition of  ${}^{\perp}(\mathcal{S}_{\mathcal{D}}^{\leq 0})$ , and the right term by (19). Thus  $\mathrm{Hom}_{\mathcal{T}}(M_{N'}, M_N[> 0]) = 0$ . This completes the argument, and  $\mathcal{M} \subseteq \mathcal{T}$  is pre-silting.

The object  $S_N$  is in

$$\mathcal{S}_{\mathcal{D}}^{\leq 0} = \bigcup_{l \geq 0} \mathcal{D} * \cdots * \mathcal{D}[l] \subseteq \bigcup_{l \geq 0} \mathcal{M} * \cdots * \mathcal{M}[l]$$

and by Remark 1.1, we have

$$\bigcup_{l \geq 0} \mathcal{M} * \cdots * \mathcal{M}[l] \subseteq \bigcup_{l \geq 0} \mathcal{M}[-l] * \cdots * \mathcal{M}[l].$$

Furthermore, as  $\mathrm{Hom}_{\mathcal{T}}(\mathcal{M}[-l], \mathcal{M}[-l+1] * \cdots * \mathcal{M}[l]) = 0$  by Lemma 2.19, we get from Lemma 3.13 (i) that

$$\bigcup_{l \geq 0} \mathcal{M}[-l] * \cdots * \mathcal{M}[l] = \bigcup_{l \geq 0} \mathrm{smd}(\mathcal{M}[-l] * \cdots * \mathcal{M}[l])$$

which equals thick  $\mathcal{M}$  by Lemma 2.18 (ii). The conclusion of this is that  $S_N$ , and then also  $S_N[1] \in \mathrm{thick} \mathcal{M}$ . As  $M_N$  is in thick  $\mathcal{M}$  by default, we see from triangle (18) that  $N \in \mathrm{thick} \mathcal{M}$ . Then

$$\mathcal{S}^{\perp} = \mathrm{thick} \mathcal{N} \subseteq \mathrm{thick} \mathcal{M},$$

and, as  $\mathcal{D} \subseteq \mathcal{M}$ ,

$$\mathcal{S} = \mathrm{thick} \mathcal{D} \subseteq \mathrm{thick} \mathcal{M}.$$

Then we see from  $(\mathcal{S}, \mathcal{S}^{\perp})$  being a torsion pair in  $\mathcal{T}$  that

$$\mathcal{T} = \mathcal{S} * \mathcal{S}^{\perp} \subseteq \mathrm{thick} \mathcal{M},$$

which means that  $\mathrm{thick} \mathcal{M} = \mathcal{T}$ , and finally that  $\mathcal{M}$  is a silting subcategory of  $\mathcal{T}$ .

To complete the proof, we need to show that  $LM \simeq \mathcal{N}$ . We do this by showing  $LM = LN$ , and use that  $LN \simeq \mathcal{N}$ . First, let  $N \in \mathcal{N}$ , and consider again the triangle (18). As  $L$  is a triangle functor, this yields the triangle

$$LS_N \longrightarrow LM_N \longrightarrow LN \longrightarrow LS_N[1]$$

in  $\mathcal{U}$ .  $LS_N$  and  $LS_N[1]$  are 0 as  $S_N, S_N[1] \in \mathcal{S}$ . I.e. we have

$$0 \longrightarrow LM_N \longrightarrow LN \longrightarrow 0,$$

and so  $LN \cong LM_N \in LM$  and  $LN \subseteq LM$ .

For the other inclusion, let  $X \in LM$ . Then  $X = LM$  for some  $M \in \mathcal{M}$ . By definition of  $\mathcal{M}$ , we have  $M = \bigoplus_{i=1}^n M_i$ , where  $M_i \in \mathcal{D}$ , or for some  $M'_i$  and some  $N_i \in \mathcal{N}$ , we have  $M_i \oplus M'_i = M_{N_i}$ . In the latter case, there is a triangle

$$S_{N_i} \longrightarrow M_i \oplus M'_i \longrightarrow N_i \longrightarrow S_{N_i}[1],$$

and by applying  $L$ , a triangle

$$0 \longrightarrow LM_i \oplus LM'_i \longrightarrow LN_i \longrightarrow 0,$$

which means  $LM_i$  is a summand of  $LN_i \in LN$ . Then  $LM$  is a summand of  $\bigoplus_{i=1}^m LN_i \in LN$ , and so it follows that  $LM \in LN$ , and that  $LM = LN$ .  $\square$

**Example 4.20.** Let  $\mathcal{T}$  be  $D^b(\text{mod } kA_3)$  and  $\mathcal{S} := \text{thick}\{P_3 \oplus P_2\}$ . The AR-quiver for  $\mathcal{T}$  is as in Example 2.3 given by

$$\begin{array}{cccccccccccc}
\cdots & P_3[-1] & & S_2[-1] & & I_1[-1] & & P_1 & & P_3[1] & & S_2[1] & & I_1[1] & \cdots \\
& \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\cdots & & P_2[-1] & & I_2[-1] & & P_2 & & I_2 & & P_2[1] & & I_2[1] & \cdots \\
& \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
\cdots & I_1[-2] & & P_1[-1] & & P_3 & & S_2 & & I_1 & & P_1[1] & & P_3[2] & \cdots
\end{array}$$

It is straight forward to check now that  $\mathcal{S} = \text{add}\{(P_3 \oplus P_2 \oplus S_2)[i] \mid i \in \mathbb{Z}\}$ , and furthermore that  $\mathcal{S}^\perp = \text{add}\{I_1[i] \mid i \in \mathbb{Z}\}$ . The set of silting subcategories of  $\mathcal{S}^\perp$  is then  $\{\mathcal{N}_i := \text{add}\{I_1[i]\} \mid i \in \mathbb{Z}\}$ .

Let  $\mathcal{D} \subseteq \mathcal{S}$  be the subcategory  $\text{add}\{P_2 \oplus P_3\}$ . Clearly,  $\text{thick } \mathcal{D} = \mathcal{S}$ , and we also see from the AR-quiver that  $\text{Hom}_{\mathcal{T}}(\mathcal{D}, \mathcal{D}[> 0]) = 0$ , so  $\mathcal{D} \in \text{silt } \mathcal{S}$ . The silting subcategories of  $\mathcal{T}$  containing  $\mathcal{D}$  are, as we have seen from previous examples, the

$$\{\mathcal{M}_i := \text{add}\{P_3 \oplus P_2 \oplus P_1[i]\} \mid i \geq 0\}$$

and the

$$\{\mathcal{N}_j := \text{add}\{P_3 \oplus P_2 \oplus I_1[j]\} \mid j < 0\}.$$

We now apply the canonical functor  $L$  to these. As  $LS = 0$ , we obtain from any  $\mathcal{N}_j$ , the category  $LN_j \simeq \text{add}\{LI_1[j]\}$  for the corresponding  $j$ . Since  $I_1[j] \in \mathcal{S}^\perp$ , this corresponds to  $\text{add}\{I_1[j]\} \in \text{silt } \mathcal{S}^\perp$  where  $j < 0$ . For  $\mathcal{M}_i$ , we similarly get  $LM_i \simeq \text{add}\{LP_1[i]\}$  for the corresponding  $i \geq 0$ .

There is a triangle  $P_1 \xrightarrow{f} I_1 \rightarrow P_2[1] \rightarrow P[1]$ , and as  $P_2[1] \in \mathcal{S}$ , the morphism  $f$  is a quasi-isomorphism. Thus  $LP_1 \cong LI_1$  in  $\mathcal{T}/\mathcal{S}$ , and  $LM_i$  corresponds to  $\text{add}\{I_1[i]\}$ , where  $i \geq 0$ .

Thus we have the bijection.

This example also illustrates a relationship between the indecomposable objects in the  $\mathcal{M}_i$  and  $\mathcal{N}_j$  which are not in  $\mathcal{D}$  and the indecomposable objects in  $LM_i$  and  $LN_j$ , respectively. This next proposition shows that this is no coincidence.

**Proposition 4.21.** *Let  $\mathcal{T}$  be a Krull-Schmidt triangulated category and  $\mathcal{M} \in \text{silt } \mathcal{T}$ . Assume  $\mathcal{S}$  is a thick and contravariantly finite subcategory of  $\mathcal{T}$ , and that  $\mathcal{D} \in \text{silt } \mathcal{S}$  is such that  $\mathcal{D} \subseteq \mathcal{M}$ . Define  $\mathcal{U}$  as the Verdier localization  $\mathcal{T}/\mathcal{S}$ , and let  $\mathcal{T} \xrightarrow{L} \mathcal{U}$  be the canonical functor. Consider the ideal  $[\mathcal{D}]$  of  $\mathcal{M}$  in which the morphisms are those that factor through objects in  $\mathcal{D}$ . Then the functor  $\mathcal{M} \xrightarrow{L} L\mathcal{M}$  induces an equivalence*

$$\mathcal{M}/[\mathcal{D}] \xrightarrow{L} L\mathcal{M}$$

from the quotient category  $\mathcal{M}/[\mathcal{D}]$  to  $L\mathcal{M}$ . In particular,  $L$  induces a bijection between  $\text{ind } \mathcal{M} \setminus \text{ind } \mathcal{D}$  and  $\text{ind } L\mathcal{M}$ .

*Proof.* The quotient category  $\mathcal{M}/[\mathcal{D}]$  has as objects the objects of  $\mathcal{M}$ , and a morphism  $M \rightarrow M'$  in  $\mathcal{M}/[\mathcal{D}]$  is a coset  $\bar{f} := f + \text{Hom}_{[\mathcal{D}]}(M, M')$ . For  $M \xrightarrow{\bar{f}} M'$  and  $M' \xrightarrow{\bar{g}} M''$  morphisms in  $\mathcal{M}/[\mathcal{D}]$ , the composition  $\bar{g}\bar{f}$  is given by

$$gf + \text{Hom}_{[\mathcal{D}]}(M, M'').$$

With this in mind, define  $\mathcal{M}/[\mathcal{D}] \xrightarrow{L} L\mathcal{M}$  by

$$\left( M \xrightarrow{f + \text{Hom}_{[\mathcal{D}]}(M, M')} M' \right) \mapsto \left( M \xrightarrow{Lf} M' \right).$$

Then  $\bar{f} = \bar{g}$  means  $f - g$  factors through  $\mathcal{D}$ , and so

$$L\bar{f} - L\bar{g} = Lf - Lg = L(f - g) = 0,$$

and the induced functor is then well-defined.

To show this induced functor is an equivalence, we show it is fully faithful and dense. As the objects of  $\mathcal{M}/[\mathcal{D}]$  are exactly the objects of  $\mathcal{M}$ , the induced functor is clearly dense.

For any morphism  $M \xrightarrow{f} M'$  in  $\mathcal{M}$ , we have seen that  $M \xrightarrow{Lf} M'$  is given, up to equivalence, by  $U \xrightarrow{g} U'$  in the diagram

$$\begin{array}{ccccccc} S & \xrightarrow{a} & M & \xrightarrow{b} & U & \xrightarrow{c} & S[1] \\ \downarrow & & \downarrow f & & \downarrow g & & \downarrow \\ S' & \xrightarrow{a'} & M' & \xrightarrow{b'} & U' & \xrightarrow{c'} & S'[1], \end{array}$$

where the rows are triangles as by (17). Assume  $g = 0$ . Then  $b'f = 0$ , and  $f$  factors through  $a'$  by some  $M \xrightarrow{\varphi} S'$ . By Lemma 4.18,  $S' \in \mathcal{S}_{\mathcal{D}}^{\leq 0}$ , so by Proposition 3.15 (ii) there is some  $l \geq 0$  such that  $S' \in \mathcal{D} * \dots * \mathcal{D}[l]$ . This implies the triangle

$$D \xrightarrow{\alpha} S' \xrightarrow{\beta} D' \longrightarrow D[1],$$

with  $D \in \mathcal{D}$  and  $D' \in \mathcal{D}[1] * \dots * \mathcal{D}[l]$ . Apply  $\text{Hom}_{\mathcal{T}}(M, -)$  to it to get the long exact sequence

$$\dots \longrightarrow \text{Hom}_{\mathcal{T}}(M, D) \xrightarrow{\alpha \circ -} \text{Hom}_{\mathcal{T}}(M, S') \xrightarrow{\beta \circ -} \text{Hom}_{\mathcal{T}}(M, D') \longrightarrow \dots$$

Since  $\text{Hom}_{\mathcal{T}}(M, D') \subseteq \text{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{M}[1] * \dots * \mathcal{M}[l])$ , which is 0 by Lemma 2.19, the left morphism is surjective. Thus  $\varphi$  factors through  $D \in \mathcal{D}$ , and it follows that so does  $f$ . This means  $f$  is 0 in  $\mathcal{M}/[\mathcal{D}]$ , so the induced  $L$  is faithful.

To show fullness, let  $U \xrightarrow{g} U'$  be some morphism. Again, we have the diagram

$$\begin{array}{ccccccc} S & \xrightarrow{a} & M & \xrightarrow{b} & U & \xrightarrow{c} & S[1] \\ & & \vdots \downarrow f & & \downarrow g & & \\ S' & \xrightarrow{a'} & M' & \xrightarrow{b'} & U' & \xrightarrow{c'} & S'[1], \end{array}$$

where the rows are triangles as by (17). As before, we see by Lemma 4.18, and also by Remark 1.1 that  $S'[1] \in \mathcal{D}[1]*\dots*\mathcal{D}[l+1]$  for some  $l \geq 0$ . Then  $\text{Hom}_{\mathcal{T}}(M, S'[1]) \subseteq \text{Hom}_{\mathcal{T}}(\mathcal{M}, \mathcal{M}[1]*\dots*\mathcal{M}[l+1]) = 0$  by Lemma 2.19, and there is a morphism  $M \xrightarrow{f} M'$  making the center square commute. This means there is an  $M \xrightarrow{f} M'$  such that  $g = Lf$ , and that  $L$  is full, and thus an equivalence.

In particular, it gives a bijection of the indecomposable objects in  $\mathcal{M}/[\mathcal{D}]$  and the indecomposable objects in  $L\mathcal{M}$ . The construction of  $\mathcal{M}/[\mathcal{D}]$  can be viewed as just setting all objects in  $\mathcal{D}$  to be 0. Thus the (nonzero) indecomposable objects in  $\mathcal{M}/[\mathcal{D}]$  are the indecomposable objects in  $\mathcal{M}$  which do not belong to  $\mathcal{D}$ , and the bijection

$$\text{ind } \mathcal{M} \setminus \text{ind } \mathcal{D} \leftrightarrow \text{ind } L\mathcal{M}$$

follows. □

Before moving on to the final parts of the thesis, we'll see that given nice conditions, the functor  $L$  commutes with mutation. Recall that for  $\mathcal{X}$  a collection of objects in the silting subcategory  $\mathcal{M} \in \text{silt } \mathcal{T}$ , we defined  $\mathcal{M}_{\mathcal{X}}$  as

$$\mathcal{M}_{\mathcal{X}} = \text{add}(\text{ind } \mathcal{M} \setminus \text{ind } \mathcal{X}).$$

The lemma is as follows:

**Lemma 4.22.** *Let  $\mathcal{T}$  be a Krull-Schmidt triangulated category with a silting object, and  $\mathcal{M} \in \text{silt } \mathcal{T}$ . Assume  $\mathcal{S}$  is a thick and contravariantly finite subcategory of  $\mathcal{T}$ , and  $\mathcal{D} \in \text{silt } \mathcal{S}$  such that  $\mathcal{D} \subseteq \mathcal{M}$ . Let  $\mathcal{T} \xrightarrow{L} \mathcal{U} := \mathcal{T}/\mathcal{S}$  be the canonical functor.*

(i) *For any covariantly finite subcategory  $\mathcal{X}$  of  $\mathcal{M}$  such that  $\mathcal{X} \cap \mathcal{D} = 0$ ,*

$$L\mu^+(\mathcal{M}; \mathcal{M}_{\mathcal{X}}) = \mu^+(L\mathcal{M}; L\mathcal{M}_{L\mathcal{X}}).$$

(ii) *For any contravariantly finite subcategory  $\mathcal{X}$  of  $\mathcal{M}$  such that  $\mathcal{X} \cap \mathcal{D} = 0$ ,*

$$L\mu^-(\mathcal{M}; \mathcal{M}_{\mathcal{X}}) = \mu^-(L\mathcal{M}; L\mathcal{M}_{L\mathcal{X}}).$$

*Proof.* As  $\mathcal{T}$  has a silting object, it follows from Proposition 2.28 that all silting subcategories are of the form  $\text{add}\{Z\}$  for some (basic) object  $Z$ .

Let  $\mathcal{M} = \text{add}\{M\}$  be in  $\text{silt } \mathcal{T}$  for a basic object  $M$ . Then  $\mathcal{D} = \text{add}\{D\}$  and  $\mathcal{X} = \text{add}\{X\}$  for summands  $D$  and  $X$  of  $M$ . As  $\mathcal{D} \cap \mathcal{X} = 0$ , we also have that for some  $Y$ ,

$$M = D \oplus X \oplus Y,$$

and  $\mathcal{M} = \text{add}\{D \oplus X \oplus Y\}$ . It follows that  $\mathcal{M}_{\mathcal{X}} = \text{add}\{D \oplus Y\}$  and, as  $LS = 0$ , we have  $L(\mathcal{M}_{\mathcal{X}}) = \text{add}\{LY\}$ . Also,  $L\mathcal{M} = \text{add}\{LX \oplus LY\}$ , and  $L\mathcal{X} = \text{add}\{LX\}$ , so  $(L\mathcal{M})_{L\mathcal{X}} = \text{add}\{LY\}$  as well.

By definition, we have

$$\begin{aligned} L\mu^+(\mathcal{M}; \mathcal{M}_{\mathcal{X}}) &= L \operatorname{add}(\mathcal{M}_{\mathcal{X}} \cup \{N_A \mid A \in \mathcal{M}\}) \\ &= \operatorname{add}(L(\mathcal{M}_{\mathcal{X}}) \cup \{LN_A \mid A \in \mathcal{X}\}) \\ &= \operatorname{add}(\{LY\} \cup \{LN_A \mid A \in \mathcal{X}\}), \end{aligned}$$

and

$$\begin{aligned} \mu^+(LM; (LM)_{L\mathcal{X}}) &= \operatorname{add}((LM)_{L\mathcal{X}} \cup \{N_U \mid U \in LM\}) \\ &= \operatorname{add}(\{LY\} \cup \{N_U \mid U \in L\mathcal{X}\}) \end{aligned}$$

To prove the two are the same, we show the equality of  $\{LN_A \mid A \in \mathcal{X}\}$  and  $\{N_U \mid U \in L\mathcal{X}\}$ . As  $\mathcal{M}_{\mathcal{X}} = \operatorname{add}\{D \oplus Y\}$ , it is functorially finite in  $\mathcal{T}$  by Proposition 3.4. Then especially, for any  $A \in \mathcal{X}$ , there is a left  $\mathcal{M}_{\mathcal{X}}$ -approximation  $A \xrightarrow{f} B$  and a triangle

$$A \xrightarrow{f} B \longrightarrow N_A \longrightarrow A[1]$$

in  $\mathcal{T}$ . By applying  $L$ , we obtain a triangle

$$LA \xrightarrow{Lf} LB \longrightarrow LN_A \longrightarrow LA[1]$$

in  $\mathcal{U}$ . As an attempt to avoid confusion, the reference to  $L$  has been kept in to remind the reader that these are objects in  $\mathcal{U}$ . Now  $LA \in L\mathcal{X}$  and  $LB \in L(\mathcal{M}_{\mathcal{X}}) = (LM)_{L\mathcal{X}}$ . Thus what remains to prove  $LN_A \in \{N_U \mid U \in L\mathcal{X}\}$  is to see that  $Lf$  is a left  $L(\mathcal{M}_{\mathcal{X}})$ -approximation. To this end, let  $C \in L(\mathcal{M}_{\mathcal{X}}) = \operatorname{add}\{LY\}$  and apply  $\operatorname{Hom}_{\mathcal{T}}(-, C)$  to the previous triangle to get the long exact sequence

$$\cdots \longrightarrow \operatorname{Hom}_{\mathcal{U}}(LB, C) \xrightarrow{- \circ Lf} \operatorname{Hom}_{\mathcal{U}}(LA, C) \longrightarrow \operatorname{Hom}_{\mathcal{U}}(LN_A[-1], C) \longrightarrow \cdots$$

We have that  $N_A \in \mu^+(\mathcal{M}; \mathcal{M}_{\mathcal{X}})$  and  $Y \in \mathcal{M}_{\mathcal{X}} \subseteq \mu^+(\mathcal{M}; \mathcal{M}_{\mathcal{X}})$ , so

$$\operatorname{Hom}_{\mathcal{U}}(LN_A[-1], LY) \cong \operatorname{Hom}_{\mathcal{T}}(N_A, Y[1]) \subseteq \operatorname{Hom}_{\mathcal{T}}(\mu^+(\mathcal{M}; \mathcal{M}_{\mathcal{X}}), \mu^+(\mathcal{M}; \mathcal{M}_{\mathcal{X}})[> 0]) = 0.$$

Then, as  $C \in \operatorname{add}\{LY\}$ , we have  $\operatorname{Hom}_{\mathcal{T}}(LN_A[-1], C) = 0$  as well by  $\operatorname{Hom}_{\mathcal{T}}(LN_A[-1], -)$  being additive. Thus the right term of the exact sequence vanishes, and  $- \circ Lf$  is surjective, so  $Lf$  is a left  $\mathcal{M}_{\mathcal{X}}$ -approximation of  $LB$ .

This shows  $LN_A = N_{LA}$ , so that  $\{LN_A \mid A \in \mathcal{X}\} \subseteq \{N_U \mid U \in L\mathcal{X}\}$ .

The other direction follows directly from this: Let  $U \in L\mathcal{X}$ . Then  $U = LA$  for some  $A \in \mathcal{X}$ , and as shown in the previous case, there is a left  $(LM)_{L\mathcal{X}}$ -approximation  $LA \xrightarrow{Lf} LB$ . Also  $N_{LA} = LN_A$ , so for any  $U \in L\mathcal{X}$ ,  $N_U \in \{LN_A \mid A \in \mathcal{X}\}$ . This proves part (i). Part (ii) is dual.  $\square$

## 5 Transitivity of Iterated Irreducible Silting Mutation

By Theorem 4.3 we saw that a mutation of a silting subcategory is itself silting. A natural question to ask then is if the silting subcategories of a given category  $\mathcal{T}$  can all be mutated into each other. We study this property from Aihara and Iyama's viewpoint of the Krull-Schmidt triangulated categories, and the distinction from the derived categories of hereditary path algebras will be reduced further, as we often will assume the existence of silting objects in  $\mathcal{T}$ .

The transitivity of the silting mutation is visualized through the connectedness of the *silting quiver* of  $\mathcal{T}$ . For the general study, we employ the theory of exceptional objects and -sequences, and use it to show transitivity for the algebras whose derived categories are triangle equivalent to those of hereditary path algebras.

### 5.1 Silting Quivers

Given silting subcategories  $\mathcal{M}$  and  $\mathcal{N}$  of  $\mathcal{T}$ , we wish to see when one is a mutation of the other. More generally, we wish to know if it is possible to perform a sequence of irreducible left- and/or right mutations, where we start with  $\mathcal{M}$  and end with  $\mathcal{N}$ . The concept to be studied is the following:

**Definition 5.1.** Let  $\mathcal{T}$  be a triangulated category, and let  $\mathcal{M}, \mathcal{N} \in \text{silt } \mathcal{T}$ . We say that  $\mathcal{M}$  and  $\mathcal{N}$  are *transitive under iterated irreducible silting mutation* if there is a sequence of left- and/or right irreducible silting mutations which brings us from one to the other. Analogously, we say that silting objects  $M$  and  $N$  are transitive under iterated irreducible silting mutation if  $\text{add } M$  and  $\text{add } N$  are transitive under iterated irreducible silting mutation.

A collection of silting subcategories is said to be transitive under iterated irreducible silting mutation if all pairs of silting subcategories are so.

The moniker *transitive* is due to this relation being, well, transitive. Clearly, a sequence of irreducible mutations taking  $\mathcal{M}$  to  $\mathcal{N}$ , followed by a sequence of irreducible mutations taking  $\mathcal{N}$  to  $\mathcal{L}$ , means there is now a sequence of irreducible mutations taking  $\mathcal{M}$  to  $\mathcal{L}$ . The relation is similarly both symmetrical and reflexive, and so we may ask how many equivalence classes of silting subcategories there are in  $\text{silt } \mathcal{T}$ , and under which conditions there is exactly one.

For a more hands-on approach, we introduce the *silting quiver*:

**Definition 5.2.** For a triangulated category  $\mathcal{T}$ . The *silting quiver* of  $\mathcal{T}$  has as its *vertices* the set of silting subcategories of  $\mathcal{T}$ . It has an arrow from vertex  $\mathcal{M}$  to vertex  $\mathcal{N}$  if  $\mathcal{N}$  is an irreducible *left* mutation of  $\mathcal{M}$ .

Definition 5.1 can then be restated as:

**Definition 5.3.** Let  $\mathcal{T}$  be a triangulated category. The silting subcategories  $\mathcal{M}$  and  $\mathcal{N}$  of  $\mathcal{T}$  are said to be *transitive under iterate irreducible silting mutation* if, in the underlying graph of the silting quiver of  $\mathcal{T}$ ,  $\mathcal{M}$  and  $\mathcal{N}$  belong to the same connected component.

The questions regarding transitivity of silting subcategories of  $\mathcal{T}$  are then questions concerning the connected components of the underlying graph of the silting quiver: How many components are there, and under which conditions is this graph fully connected?

In Section 3 we considered Krull-Schmidt triangulated categories  $\mathcal{T}$  where for any indecomposable object  $X$  of any silting subcategory  $\mathcal{M}$ , the category  $\mathcal{M}_X = \text{add}(\text{ind } \mathcal{M} \setminus \{X\})$  was functorially

finite in  $\mathcal{M}$ . In these cases the silting quiver will coincide with the *Hasse diagram* of the partial ordering  $\geq$  on  $\text{silt } \mathcal{T}$ . Indeed, there is an arrow from  $\mathcal{M}$  to  $\mathcal{N}$  if  $\mathcal{N}$  is an irreducible left mutation of  $\mathcal{M}$ , which by Theorem 4.12 is exactly when  $\mathcal{M} > \mathcal{N}$  and there is no  $\mathcal{L} \in \text{silt } \mathcal{T}$  in between them.

Let  $\mathcal{T}$  be  $K^b(\mathcal{P}(\text{mod } \Lambda))$  for some finite dimensional  $k$ -algebra  $\Lambda$ .  $\mathcal{T}$  is then a Krull-Schmidt triangulated category and has the silting object given by the stalk complex of  $\Lambda$ . Then by Proposition 3.4,  $\mathcal{T}$  is exactly of this form.

**Example 5.4.** Having introduced the silting quiver, it is only fitting that we also include an example of such a quiver. The standard example of  $D^b(\text{mod } kA_3)$  is too big to handle in this situation. We choose therefore to rather consider the simpler algebra  $\Lambda := kA_2$ , where  $A_2$  is the quiver

$$1 \xrightarrow{\alpha} 2.$$

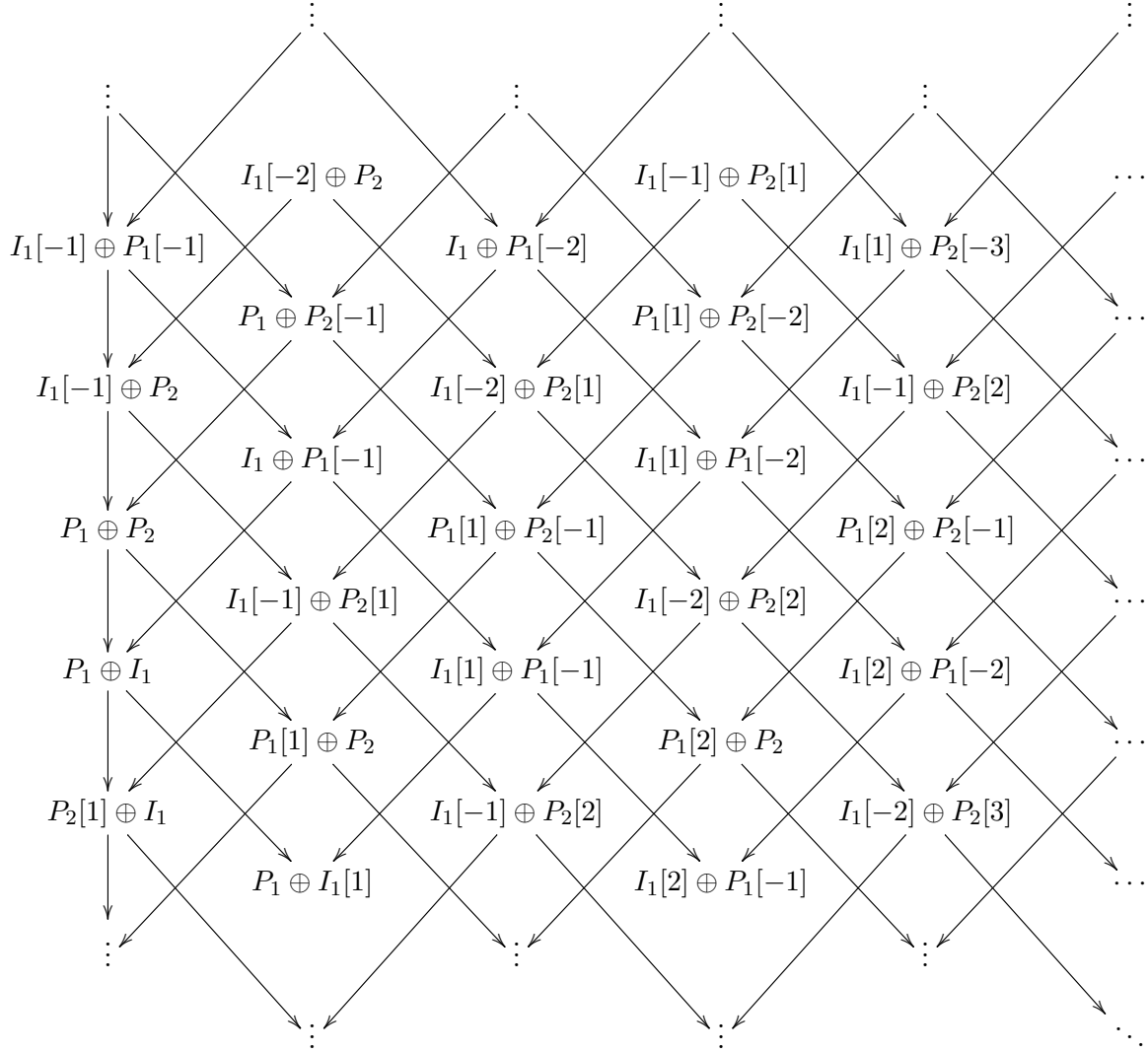
The indecomposable left  $\Lambda$ -modules are then

$$\begin{aligned} S_2 = P_2 &: 0 \longrightarrow k \\ P_1 = I_2 &: k \xrightarrow{1} k \\ S_1 = I_1 &: k \longrightarrow 0, \end{aligned}$$

and the AR-quiver

$$\begin{array}{cccccccc} \cdots & & I_1[-1] & & P_1 & & P_2[1] & & I_1[1] & & \cdots \\ & \nearrow & & \searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow & \searrow & \\ \cdots & & & & P_2 & & I_1 & & P_1[1] & & \cdots \end{array}$$

is found using the construction from Happel [7]. The mutation here is even simpler than in the previous examples. The object  $P_1 \oplus P_2$  is a tilting object by Proposition 2.2, and the triangles in  $D^b(\text{mod } \Lambda)$  are given by any sequence of successive objects and morphisms in the AR-quiver. The silting quiver is too large to include in its entirety. We instead do as in Aihara and Iyama [1] and present a part of a component of the quiver and provide an explanation of its general structure. As there is a bijection between the silting subcategories and -objects of  $D^b(\text{mod } kA_2)$ , we use the silting objects as vertices.



The objects to the left in this silting quiver are the silting objects where the indecomposable direct summands are *next to each other* in the AR-quiver, and the vertical arrows then represent the right mutations done by using the approximations from the AR-triangles. We observe then that *upwards* movement in the silting quiver amounts to passing to silting objects further to the *left* in the AR-quiver, and vice versa for *downwards* movement - as expected by the definition of left- and right silting mutation. Also, moving to the *right* in the silting quiver amounts to passing to silting objects which has summands further away from each other, while moving to the *left* passes to objects whose summands are closer.

As we will see in Theorem 5.26, this category is transitive under iterated irreducible silting mutation, and this component describes the entire silting quiver of  $D^b(\text{mod } \Lambda)$

The first result concerning transitivity is the following corollary to Lemma 3.19.

**Corollary 5.5.** *Let  $\mathcal{T}$  be a Krull-Schmidt triangulated category with an indecomposable silting object, then iterated irreducible silting mutation on  $\text{silt } \mathcal{T}$  is transitive.*



*Proof.* Assume  $M \in \mathcal{T}$  is an indecomposable silting object in  $\mathcal{T}$ . By Lemma 3.19, the silting subcategories of  $\mathcal{T}$  are then all of the form  $\text{add}\{M[i]\}$  for  $i \in \mathbb{Z}$ .

As  $M$  is indecomposable, there is only one choice of indecomposable object in  $\mathcal{M}$ , and thus only one irreducible left mutation of  $\mathcal{M}$ , namely

$$\mu^+(\mathcal{M}; \mathcal{M}_M) = \mu^+(\mathcal{M}; 0) = \text{add}\{N_M\}.$$

Since  $\mathcal{M}_M = 0$ , a left  $\mathcal{M}_M$ -approximation of  $M$  is the 0-map. I.e. we have the triangle

$$M \longrightarrow 0 \longrightarrow N_M \longrightarrow M[1],$$

which means  $N_M \cong M[1]$  and  $\mu^+(\mathcal{M}; \mathcal{M}_M) = \text{add}\{M[1]\}$ . Similarly,  $\mu^-(\mathcal{M}; \mathcal{M}_M) = \text{add}\{M[-1]\}$ . As  $M[n]$  is indecomposable for any  $n \in \mathbb{Z}$ , we can do this over and over to irreducibly mutate from any silting subcategory of  $\mathcal{T}$  to any other – as asserted.  $\square$

**Example 5.6.** Let  $A$  be a finite dimensional local  $k$ -algebra and  $\mathcal{T} = \text{K}^b(\mathcal{P}(\text{mod } A))$ . The endomorphism ring of the stalk complex of  $A$  is isomorphic to the endomorphism ring of  $A$ , as the only null-homotopic map is 0. By the equivalence

$$\text{mod } A \xrightarrow{\text{Hom}_{\text{mod } A}(A, -)} \mathcal{P}(\text{mod } \Gamma)$$

from Proposition 3.2 we get that  $\text{End}_{\text{mod } A}(A)$  is local, and as we saw in the beginning of Section 3, this means  $A$  is indecomposable (in this case both in  $\text{mod } A$  and  $\mathcal{T}$ ). Thus  $\mathcal{T}$  has the indecomposable silting object  $A$ , and iterated irreducible silting mutation in  $\mathcal{T}$  is transitive.

Consider again the situation where for any  $\mathcal{M} \in \text{silt } \mathcal{T}$  and any indecomposable  $M \in \mathcal{M}$ , the category  $\mathcal{M}_M$  is functorially finite in  $\mathcal{M}$ . Such an  $\mathcal{M}_M$  is not itself a silting subcategory of  $\mathcal{T}$ , but is contained in one, and they are exactly one indecomposable object away from each other. This property identifies an interesting class of subcategories of  $\mathcal{T}$ , defined below.

**Definition 5.7.** Let  $\mathcal{T}$  be a Krull-Schmidt triangulated category, and  $\mathcal{D}$  a subcategory of  $\mathcal{T}$ .  $\mathcal{D}$  is said to be *almost complete silting* if it is contained in some silting subcategory  $\mathcal{M}$  of  $\mathcal{T}$ , and  $|\text{ind } \mathcal{M} \setminus \text{ind } \mathcal{D}| = 1$ .

**Example 5.8.** Let  $\mathcal{T}$  be a Krull-Schmidt triangulated category with silting objects, and let  $\mathcal{M} \in \text{silt } \mathcal{T}$ . Then, by Proposition 2.28, there is a basic object  $M$  such that  $\mathcal{M} = \text{add}\{M\}$ . If  $M$  has the decomposition  $\bigoplus_{i=1}^n M_i$  with  $M_i$  indecomposable, we get that for  $1 \leq j \leq n$ , the category

$$\mathcal{M}_{M_j} = \text{add} \left\{ \bigoplus_{\substack{i=1 \\ i \neq j}}^n M_i \right\}$$

is almost complete silting.

Note that if  $\mathcal{T}$  has an indecomposable silting object  $M$ , then the subcategory  $\mathcal{D}$ , only containing the zero object, is almost complete silting. Corollary 5.5 then asserts that the silting subcategories of  $\mathcal{T}$  which contain  $\mathcal{D}$  are transitive under iterated irreducible silting mutation. Theorem 5.9 presents conditions under which this holds in general.

**Theorem 5.9.** *Let  $\mathcal{T}$  be a Krull-Schmidt triangulated category with silting objects, and  $\mathcal{D}$  an almost complete silting subcategory of  $\mathcal{T}$ . If  $\mathcal{D}$  is functorially finite in  $\mathcal{T}$ , then the set  $\{\mathcal{M} \in \text{silt } \mathcal{T} \mid \mathcal{D} \subseteq \mathcal{M}\}$  is transitive under iterated silting mutation.*

*Proof.*  $\mathcal{T}$  has silting objects, so for any silting subcategory  $\mathcal{M}$  of  $\mathcal{T}$ , there is by Proposition 2.28 a (basic)  $M \in \mathcal{T}$  such that  $\mathcal{M} = \text{add}\{M\}$ . As  $\mathcal{D}$  is almost complete silting, there is an  $\mathcal{N} \in \text{silt } \mathcal{T}$  such that  $\mathcal{D} \subseteq \mathcal{N}$  and  $|\text{ind } \mathcal{N} \setminus \text{ind } \mathcal{D}| = 1$ . This means that there are indecomposable and pairwise non-isomorphic objects  $X_1, \dots, X_n \in \mathcal{T}$ , where

$$\mathcal{N} = \text{add}\{X_1 \oplus \dots \oplus X_n\}$$

and

$$\mathcal{D} = \text{add}\{X_1 \oplus \dots \oplus X_{n-1}\}.$$

Let  $X := X_n$  be the unique object in  $\text{ind } \mathcal{N} \setminus \text{ind } \mathcal{D}$ . Set  $\mathcal{U} := \mathcal{T}/\text{thick } \mathcal{D}$ . By Proposition 4.21, the canonical functor  $\mathcal{T} \xrightarrow{F} \mathcal{U}$  induces an equivalence  $\mathcal{N}/[\mathcal{D}] \simeq F\mathcal{N}$ , and a bijection between  $\text{ind } \mathcal{N} \setminus \text{ind } \mathcal{D}$  and  $\text{ind } F\mathcal{N}$ . This means  $FX$  is the unique (up to isomorphism) indecomposable object in  $F\mathcal{N}$ , and so  $F\mathcal{N} = \text{add}(FX)$ . By Theorem 4.19,  $F\mathcal{N} \in \text{silt } \mathcal{U}$ , so  $\mathcal{U}$  has the indecomposable silting object  $FX$ , and  $\text{silt } \mathcal{U}$  is transitive under iterated irreducible silting mutation by Corollary 5.5.

As  $\mathcal{S}$  is functorially finite in  $\mathcal{T}$ , apply Theorem 4.19 (ii) to get that  $F$  induces a bijection between the silting subcategories of  $\mathcal{T}$  containing  $\mathcal{D}$  and the silting subcategories of  $\mathcal{U}$ .

By Corollary 3.25, all basic silting objects in  $\mathcal{T}$  have the same number of indecomposable summands. It follows that if  $\mathcal{M} \in \text{silt } \mathcal{T}$  with  $\mathcal{D} \subseteq \mathcal{M}$ , then for some indecomposable  $Y_{\mathcal{M}}$ ,

$$\mathcal{M} = \text{add}\{X_1 \oplus \dots \oplus X_{n-1} \oplus Y_{\mathcal{M}}\},$$

and that mutation on  $\{\mathcal{M} \in \text{silt } \mathcal{T} \mid \mathcal{D} \subseteq \mathcal{M}\}$  amounts to mutating with respect to  $\mathcal{D}$ , i.e. interchanging the singular indecomposable direct summand  $Y_{\mathcal{M}}$ . This means mutation on  $\{\mathcal{M} \in \text{silt } \mathcal{T} \mid \mathcal{D} \subseteq \mathcal{M}\}$  is mutation  $\mu^{\pm}(\mathcal{M}; \mathcal{D}) = \mu^{\pm}(\mathcal{M}; \mathcal{M}_{Y_{\mathcal{M}}})$ , and from Lemma 4.22, such mutation commutes with  $F$ , and  $\{\mathcal{M} \in \text{silt } \mathcal{T} \mid \mathcal{D} \subseteq \mathcal{M}\}$  is transitive under iterated irreducible silting mutation.  $\square$

## 5.2 Transitivity for Derived Categories of Finite Dimensional Piecewise Hereditary Algebras

In this final section, we show that if  $A$  is a finite dimensional *piecewise hereditary algebra*, then  $D^b(\text{mod } A)$  is transitive under iterated irreducible silting mutation. This is done by following a similar strategy to that of Aihara and Iyama, in that we employ the theory of *exceptional sequences* and *braid group actions* on these to prove the results.

Recall that an abelian category  $\mathcal{H}$  is a *hereditary category* if

$$\text{Ext}_{\mathcal{H}}^i(X, Y) = \text{Hom}_{D(\mathcal{H})}(X, Y[i]) = 0$$

for all  $X, Y \in \mathcal{H}$  and  $i \geq 2$ . In particular, if  $\mathcal{H}$  has enough projectives, the length of any projective resolution is at most 1. Having this, we say that an algebra  $\Lambda$  is a *hereditary algebra* if  $\text{mod } \Lambda$  is a hereditary abelian category.

In [19] is defined the *canonical algebras* which are *one-point extensions* of certain path algebras. These algebras are not necessarily hereditary, but have global dimension  $\leq 2$ . Sticking to the notation from [19], we have the following definition.

**Definition 5.10.** Let  $\mathcal{H}$  be a hereditary abelian  $k$ -finite category. A finitely generated  $k$ -algebra  $A$  is said to be *piecewise hereditary of type  $\mathcal{H}$*  if there is a triangle equivalence

$$D^b(\text{mod } A) \xrightarrow{\Delta} D^b(\mathcal{H}).$$

We assume  $k$  is algebraically closed.

In addition, the following is true due to Happel and Reiten [8, 9].

**Theorem 5.11.** *Let  $A$  be a finite dimensional piecewise hereditary  $k$ -algebra. Then there is a finite dimensional hereditary  $k$ -algebra or a finite dimensional canonical  $k$ -algebra  $\Lambda$  and a triangle equivalence*

$$D^b(\text{mod } A) \xrightarrow{\Delta} D^b(\text{mod } \Lambda).$$

Thus clearly, both the finite dimensional hereditary  $k$ -algebras and the finite dimensional canonical  $k$ -algebras are piecewise hereditary, and up derived equivalence, these are the only ones. The study of the piecewise hereditary algebras thus encompass the 'nice' hereditary path algebras we usually want to consider. Also included among these is now the algebra  $kA_3/\langle\beta\alpha\rangle$ , as its bounded derived category is seen to be triangle equivalent to that of the hereditary  $kA_3$ .

**Remark 5.12.** *We note the following facts about piecewise hereditary algebras.*

- (i) *Both the hereditary and canonical algebras are of finite global dimension. Thus we get that if  $\Lambda$  is a finite dimensional hereditary or -canonical  $k$ -algebra over some field  $k$*

$$D^b(\text{mod } \Lambda) = K^b(\mathcal{P}(\text{mod } \Lambda)).$$

*by Theorem A.1.*

- (ii) *Let  $A$  be a finite dimensional piecewise hereditary  $k$ -algebra and  $\mathcal{T} = D^b(\text{mod } A)$ . Since  $\mathcal{T} \simeq K^b(\mathcal{P}(\text{mod } \Lambda))$  for a finite-dimensional  $k$ -algebra  $\Lambda$ , it follows from Propositions 2.2 and 3.3 that  $\mathcal{T}$  is a Krull-Schmidt triangulated category with silting objects.*

To prove transitivity of iterated irreducible silting mutation, we will, as Aihara and Iyama, make use of the theory of exceptional sequences.

**Definition 5.13.** Let  $\mathcal{T}$  be any Krull-Schmidt triangulated category. An object  $T \in \mathcal{T}$  is called an *exceptional object* if

- (i)  $\text{End}_{\mathcal{T}}(T)$  is a division algebra over  $k$ , and
- (ii)  $\text{Hom}_{\mathcal{T}}(T, T[\neq 0]) = 0$ .

A sequence  $(X_1, \dots, X_n)$  in  $\mathcal{T}$  is called an *exceptional sequence* if

- (iii)  $X_i$  is a nonzero exceptional object in  $\mathcal{T}$  for all  $i$ , and
- (iv) For all  $j < i$  and any  $n \in \mathbb{Z}$ , we have  $\text{Hom}_{\mathcal{T}}(X_i, X_j[n]) = 0$ .

Furthermore, an exceptional sequence  $(X_1, \dots, X_n)$  is said to be *full* if  $\text{thick}\{X_1 \oplus \dots \oplus X_n\} = \mathcal{T}$ . The set of isomorphism classes of full exceptional sequences in  $\mathcal{T}$  is denoted by  $\text{exp } \mathcal{T}$ .

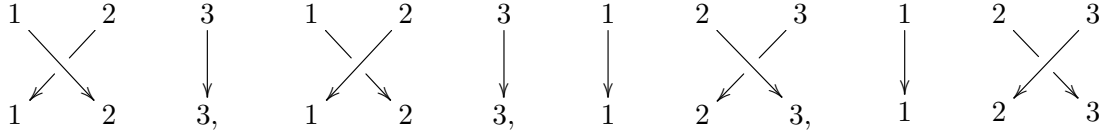
Since an exceptional object  $X \in \mathcal{T}$  has  $\text{End}_{\mathcal{T}}(X)$  a division algebra, it has a *local endomorphism ring*. We saw in Section 3 that this amounts to  $X$  being indecomposable.

If we let  $\mathbf{X} = (X_1, \dots, X_n)$  be a full exceptional sequence in  $\mathcal{T}$  and let  $\mathbf{l} = (l_1, \dots, l_n) \in \mathbb{Z}^n$ , we define

$$\mathbf{lX} := (X_1[l_1], \dots, X_n[l_n]).$$

It is straight forward to check the axioms and see that this produces a full exceptional sequence in  $\mathcal{T}$  again. We also see that this defines an action of  $\mathbb{Z}^n$  on the set of full exceptional sequences in  $\mathcal{T}$  of length  $n$ .

Furthermore, we let  $B_n$  be the *braid group* on  $n$  indices. That is,  $B_n$  is generated by the *elementary braids*  $\sigma_1, \dots, \sigma_{n-1}$ . For clarification, in  $B_3$ , the elements  $\sigma_1, \sigma_1^{-1}, \sigma_2$  and  $\sigma_2^{-1}$  would be



respectively.

In  $B_n$ , we have the relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i$$

for all  $1 \leq i, j < n$  with  $|i - j| > 1$ , and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

for all  $1 \leq i < n$ .

Similarly to how we let  $\mathbb{Z}^n$  act on the full exceptional sequences, we define an action of  $B_n$  on the full exceptional sequences on  $\mathcal{T}$ : For a full exceptional sequence  $\mathbf{X} = (X_1, \dots, X_n)$ , we define, for  $1 \leq i < n$  the object  $L_{X_{i+1}} X_i$  by the triangle

$$X_i \longrightarrow \coprod_{l \in \mathbb{Z}} (\text{D Hom}_{\mathcal{T}}(X_i, X_{i+1}[l]) \otimes_k X_{i+1}[l]) \longrightarrow L_{X_{i+1}} X_i[1] \longrightarrow X_i[1].$$

Similarly, we define the object  $R_{X_i} X_{i+1}$  by the triangle

$$R_{X_i} X_{i+1} \longrightarrow \coprod_{l \in \mathbb{Z}} (\text{Hom}_{\mathcal{T}}(X_i[l], X_{i+1}) \otimes_k X_i[l]) \longrightarrow X_{i+1} \longrightarrow R_{X_i} X_{i+1}[1].$$

We then define the action of the braids  $\sigma_i, \sigma_i^{-1} \in B_n$  on  $\mathbf{X} = (X_1, \dots, X_n)$  by

$$\sigma_i \mathbf{X} = (X_1, \dots, X_{i-1}, R_{X_i} X_{i+1}, X_i, X_{i+2}, \dots, X_n),$$

and

$$\sigma_i^{-1} \mathbf{X} = (X_1, \dots, X_{i-1}, X_{i+1}, L_{X_{i+1}} X_i, X_{i+2}, \dots, X_n)$$

respectively.

This action will not be studied in detail here. Instead we will refer to the following theorem by [6, 18, 24].

**Theorem 5.14.** *Let  $A$  be a finite dimensional piecewise hereditary  $k$ -algebra and  $\mathcal{T} = \mathrm{D}^b(\mathrm{mod} A)$ . Then  $B_n \times \mathbb{Z}^n$  acts transitively on the set of full exceptional sequences in  $\mathcal{T}$  of length  $n$ . We denote the action of  $(\sigma, \mathbf{v}) \in B_n \times \mathbb{Z}^n$  on  $\mathbf{X} \in \exp \mathcal{T}$  by  $(\sigma, \mathbf{v})\mathbf{X}$ .*

This theorem will be applied to prove the main result of this section, which asserts the transitivity of iterated irreducible silting mutation within these kinds of triangulated categories. To get there, we prove some intermediate results. Note that the property  $\mathrm{Hom}_{\mathcal{T}}(X, Y[>> 0]) = 0$  for all  $X, Y \in \mathcal{T}$  holds whenever  $\mathcal{T}$  is the derived category of an abelian category of finite global dimension.

**Proposition 5.15.** *Let  $\mathcal{T}$  be a Krull-Schmidt triangulated category such that for any  $X, Y \in \mathcal{T}$  we have  $\mathrm{Hom}_{\mathcal{T}}(X, Y[>> 0]) = 0$ . For any full exceptional sequence  $\mathbf{X} = (X_1, \dots, X_n)$  in  $\mathcal{T}$ , there exists an integer  $a \geq 0$  such that for any sequence  $(l_1, \dots, l_n) \in \mathbb{Z}^n$  satisfying  $l_i + a \leq l_{i+1}$  for all  $i < n$ , we have that  $X_1[l_1] \oplus \dots \oplus X_n[l_n]$  is a silting object in  $\mathcal{T}$ .*

*Proof.* By assumption we have that for any  $X_i, X_j$  in  $\mathbf{X}$ , there exists some integer  $n_{X_i X_j}$  such that  $\mathrm{Hom}_{\mathcal{T}}(X_i, X_j[n]) = 0$  for all  $n \geq n_{X_i X_j}$ . We set  $a := \max\{n_{X_i X_j} \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq n\}$ , and show that this  $a$  satisfies the proposition.

Let  $l_i, \dots, l_n$  satisfy  $l_i + a \leq l_{i+1}$ , and set

$$S := X_1[l_1] \oplus \dots \oplus X_n[l_n].$$

As  $\mathcal{T}$  is a Krull-Schmidt triangulated category, it follows from Remark 3.5 that we only need to check  $\mathrm{Hom}_{\mathcal{T}}(S, S[> 0]) = 0$  and  $\mathrm{thick}\{S\} = \mathcal{T}$ .

Since  $\mathrm{thick}\{S\}$  is closed under direct summands, shifts and finite coproducts, it contains  $X_1 \oplus \dots \oplus X_n$ , and so it is all of  $\mathcal{T}$  by  $\mathbf{X}$  being full. Furthermore, if  $m > 0$ , we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{T}}(S, S[m]) &\cong \bigoplus_{i,j=0}^n \mathrm{Hom}_{\mathcal{T}}(X_i[l_i], X_j[l_j][m]) \\ &\cong \bigoplus_{i,j=0}^n \mathrm{Hom}_{\mathcal{T}}(X_i, X_j[l_j - l_i + m]). \end{aligned}$$

$j < i$ : Then  $\mathrm{Hom}_{\mathcal{T}}(X_i, X_j[l_j - l_i + m]) = 0$  by  $\mathbf{X}$  being exceptional.

$j = i$ : Then  $\mathrm{Hom}_{\mathcal{T}}(X_i, X_j[l_j - l_i + m]) = \mathrm{Hom}_{\mathcal{T}}(X_i, X_i[m]) = 0$  by  $X_i$  being exceptional.

$j > i$ : Then  $\mathrm{Hom}_{\mathcal{T}}(X_i, X_j[l_j - l_i + m]) = 0$  since  $l_j - l_i + m > a \geq n_{X_i X_j}$ .

Thus all summands are 0, so  $\mathrm{Hom}_{\mathcal{T}}(S, S[> 0]) = 0$ . □

This next lemma generalizes both the base step and the induction step of the proof of Aihara and Iyama's Lemma 5.17.

**Lemma 5.16.** *Let  $\mathcal{T}$  be a Krull-Schmidt triangulated category and  $X_1, \dots, X_n \in \mathcal{T}$  indecomposable objects such that  $\mathcal{M} := \mathrm{add}\{X_1 \oplus \dots \oplus X_n\} \in \mathrm{silt} \mathcal{T}$ . Fix an  $i \in \{1, \dots, n\}$ . If for all  $j \neq i$ , we have  $\mathrm{Hom}_{\mathcal{T}}(X_j, X_i) = 0$ , then*

$$\mu^{\pm}(\mathcal{M}; \mathcal{M}_{X_i}) = \mathrm{add}\{X_1 \oplus \dots \oplus X_i[\pm 1] \oplus \dots \oplus X_n\}.$$

*In other words,  $\mathrm{add}\{X_1 \oplus \dots \oplus X_i[\mp 1] \oplus \dots \oplus X_n\}$  is obtainable by irreducible mutation from  $\mathrm{add}\{X_1 \oplus \dots \oplus X_n\}$ .*

*Proof.* We show that

$$\mu^-(\mathcal{M}; \mathcal{M}_{X_i}) = \text{add}\{X_1 \oplus \cdots \oplus X_i[-1] \oplus \cdots \oplus X_n\},$$

as the other part is dual.

The indecomposable objects in  $\mathcal{M}$  are exactly the  $X_j$  for  $1 \leq j \leq n$ , so  $\mathcal{M}_{X_i}$  is the additive closure of the other  $X_j$ :

$$\mathcal{M}_{X_i} = \text{add}(\text{ind } \mathcal{M} \setminus \{X_i\}) = \text{add}\left\{\bigoplus_{\substack{j=1 \\ j \neq i}}^n X_j\right\}.$$

Let  $M \in \mathcal{M}$  be some object. Then for some  $d_i \geq 0$ , we have  $M = X_1^{d_1} \oplus \cdots \oplus X_n^{d_n}$ . We set

$$D := \bigoplus_{\substack{j=1 \\ j \neq i}}^n X_j^{d_j} \in \mathcal{M}_{X_i}$$

to be the direct summand of  $M$  obtained by removing all copies of  $X_i$ . In addition, we set  $D \xrightarrow{\iota} M$  to be the natural inclusion. Our first goal is to show this is an  $\mathcal{M}_{X_i}$ -approximation of  $M$ . To do this, we consider any morphism  $D' \xrightarrow{g'} M$  with  $D' \in \mathcal{M}_{X_i}$ , and show it factors through  $\iota$ . The object  $D'$  is now of the form

$$D' = \bigoplus_{\substack{j=1 \\ j \neq i}}^n X_j^{d'_j}$$

for some  $d'_j \geq 0$ , and we obtain from this the solid part of the diagram

$$\begin{array}{ccc} D = \bigoplus_{\substack{j=1 \\ j \neq i}}^n X_j^{d_j} & \xrightarrow{\iota} & \bigoplus_{j=1}^n X_j^{d_j} = M \\ \uparrow \text{---} & \nearrow g' & \\ D' = \bigoplus_{\substack{j=1 \\ j \neq i}}^n X_j^{d'_j} & & \end{array}$$

By assumption,  $\text{Hom}_{\mathcal{T}}(X_j, X_i) = 0$  for all  $j \neq i$ . Thus the morphism  $g'$  is one from  $\bigoplus_{j=1, j \neq i}^n X_j^{d'_j}$  to  $\bigoplus_{j=1, j \neq i}^n X_j^{d_j}$ . I.e. it factors through  $\iota$ , and  $\iota$  is a right  $\mathcal{M}_{X_i}$ -approximation of  $M$ .

As the mutation is independent on the choice of approximation, we can find our  $N'_M$  by completing  $\iota$  to a triangle

$$N'_M \longrightarrow D \xrightarrow{\iota} M \longrightarrow N'_M[1].$$

Here we know the right side splits by [11] as the inclusion is a split monomorphism. That is,  $M \cong D \oplus N'_M[1]$ , which by uniqueness of decomposition in  $\mathcal{T}$  and by definition of  $D$  means that  $N'_M[1] \cong X_i^{d_i}$ , and so  $N'_M \cong X_i[-1]^{d_i}$ . It follows that

$$\begin{aligned} \mu^-(\mathcal{M}; \mathcal{M}_{X_i}) &= \text{add}(\mathcal{M}_{X_i} \cup \{N'_M \mid M \in \mathcal{M}\}) \\ &= \text{add}\left(\left\{\bigoplus_{\substack{j=1 \\ j \neq i}}^n X_j\right\} \cup \{X_i[-1]\}\right) \\ &= \text{add}\{X_1 \oplus \cdots \oplus X_i[-1] \oplus \cdots \oplus X_n\} \end{aligned}$$

as claimed.  $\square$

We apply Lemma 5.16 in the proof of Lemma 5.17 (ii).

**Lemma 5.17.** *Let  $\mathcal{T}$  be a Krull-Schmidt triangulated category and  $\mathbf{X} = (X_1, \dots, X_n)$  a full exceptional sequence in  $\mathcal{T}$  such that  $X_1 \oplus \dots \oplus X_n$  is a silting object. For any set of integers  $l_1 \leq \dots \leq l_n$ , we have the following:*

- (i)  $X_1[l_1] \oplus \dots \oplus X_n[l_n]$  is a silting object.
- (ii) If  $l_1 \geq 0$ , then  $X_1[l_1] \oplus \dots \oplus X_n[l_n]$  and  $X_1 \oplus \dots \oplus X_n$  are transitive under iterated irreducible silting mutation.

*Proof.* For ease of notation, set  $S = X_1[l_1] \oplus \dots \oplus X_n[l_n]$ , and  $\mathcal{M} := \text{add}\{S\}$ .

(i) From Remarks 5.12 and 3.5 we only need to check  $\text{silt}\{S\} = \mathcal{T}$  and  $\text{Hom}_{\mathcal{T}}(S, S[> 0]) = 0$ . We already saw that  $\text{thick}\{S\} = \text{thick}\{X_1 \oplus \dots \oplus X_n\} = \mathcal{T}$ . Also, for  $m > 0$

$$\text{Hom}_{\mathcal{T}}(S, S[m]) \cong \bigoplus_{i,j=0}^n \text{Hom}_{\mathcal{T}}(X_i, X_j[l_j - l_i + m]),$$

and the direct summands where  $j < i$  vanish as  $X_i$  comes before  $X_j$  in the exceptional sequence  $\mathbf{X}$ . Those where  $j = i$  vanish as the  $X_i$  are exceptional objects. If  $j > i$ , then  $l_j - l_i \geq 0$ , so  $l_j - l_i + m > 0$ . As  $X_i$  and  $X_j$  are in the silting category  $\text{add}\{X_1 \oplus \dots \oplus X_n\}$ , these summands disappear as well, and  $S$  is a silting object.

(ii) We start this part by performing right mutation on  $S$  with respect to  $X_1[l_1]$ . That is, we find  $\mu^-(\mathcal{M}; \mathcal{M}_{X_1[l_1]})$ .

For any  $j > 1$ , we have that  $\text{Hom}_{\mathcal{T}}(X_j[l_j], X_1[l_1]) = 0$  as  $X_1$  comes before  $X_j$  in  $\mathbf{X}$ . Then we have by Lemma 5.16 that

$$\mu^-(\mathcal{M}; \mathcal{M}_{X_1[l_1]}) = \text{add}\{X_1[l_1 - 1] \oplus \dots \oplus X_n[l_n]\}.$$

Note especially that we can do this step  $l_1 \geq 0$  times to arrive at  $\text{add}\{X_1 \oplus X_2[l_2] \oplus \dots \oplus X_n[l_n]\}$ . Also, if  $l_1 > 0$ , then  $l_2 > 0$  as well.

For the general step, assume there is an  $i \in \{2, \dots, n\}$  such that  $l_{i-1} < l_i$ . We perform left mutation on  $S$  with respect to the direct summand  $X_i[l_i]$ . For  $1 \leq j \leq n$ , we have

$j < i$ : Then  $l_i - l_j > 0$ , and  $\text{Hom}_{\mathcal{T}}(X_j[l_j], X_i[l_i]) \cong \text{Hom}_{\mathcal{T}}(X_j, X_i[l_i - l_j]) = 0$ , as  $X_i$  and  $X_j$  are in the silting category  $\text{add}\{X_1 \oplus \dots \oplus X_n\}$ .

$j > i$ : Then  $\text{Hom}_{\mathcal{T}}(X_j[l_j], X_i[l_i]) = 0$  as  $X_i$  comes before  $X_j$  in  $\mathbf{X}$ .

So  $\text{Hom}_{\mathcal{T}}(X_j[l_j], X_i[l_i]) = 0$  for all  $j \neq i$ , and it follows from part (i) and Lemma 5.16 that

$$\mu^-(\mathcal{M}; \mathcal{M}_{X_i[l_i]}) = \text{add}\{X_1[l_1] \oplus \dots \oplus X_i[l_i - 1] \oplus \dots \oplus X_n[l_n]\}$$

We now combine these two steps. By first performing the base step  $l_1$  times we obtain  $\text{add}\{X_1 \oplus X_2[l_2] \oplus \dots \oplus X_n[l_n]\}$ . Then, we do the general step for  $i = 2$  a total number of  $l_2$  times, then  $l_3$  times for  $i = 3$  etc. It is clear that this sequence of irreducible right silting mutation takes us from  $\text{add}\{S\}$  to  $\text{add}\{X_1 \oplus \dots \oplus X_n\}$ , as asserted.  $\square$

Note that the assumption  $l_1 \geq 0$  is due to the fact that this process only ever decreases the value of  $l_i$ . Before moving on to strictly considering derived categories of piecewise hereditary algebras, Lemma 5.17 is applied to show the following transitivity result:

**Proposition 5.18.** *Let  $\mathcal{T}$  be a Krull-Schmidt triangulated category,  $\mathbf{X} = (X_1, \dots, X_n)$  a full exceptional sequence in  $\mathcal{T}$  and  $l_1, \dots, l_n$  a sequence of integers. If  $X_1 \oplus \dots \oplus X_n$  and  $X_1[l_1] \oplus \dots \oplus X_n[l_n]$  are silting objects, then the two are transitive under iterated irreducible silting mutation.*

*Proof.* Pick  $m_1 \geq \max\{0, -l_1\}$ , and for each  $1 < i \leq n$ , iteratively pick  $m_i \geq \max\{m_{i-1}, m_{i-1} + l_{i-1} - l_i\}$ . Then we have

$$0 \leq m_1 \leq \dots \leq m_n$$

with

$$0 \leq m_1 + l_1 \leq \dots \leq m_n + l_n.$$

By Lemma 5.17,  $X_1 \oplus \dots \oplus X_n$  and  $X_1[m_1 + l_1] \oplus \dots \oplus X_n[m_n + l_n]$  are transitive. Furthermore, since  $(X_1[l_1], \dots, X_n[l_n])$  is a full exceptional sequence in  $\mathcal{T}$  such that  $X_1[l_1] \oplus \dots \oplus X_n[l_n]$  is a silting object, Lemma 5.17 gives us that  $X_1[l_1] \oplus \dots \oplus X_n[l_n]$  and  $X_1[l_1 + m_1] \oplus \dots \oplus X_n[l_n + m_n]$  are transitive as well. By the transitivity of being transitive,  $X_1[l_1] \oplus \dots \oplus X_n[l_n]$  and  $X_1 \oplus \dots \oplus X_n$  are transitive under iterated irreducible silting mutation.  $\square$

The following Lemmas are due to Krause [17] and Happel [7], respectively. They are stated here for reference, but not proved.

**Lemma 5.19.** *Let  $\mathcal{H}$  be a hereditary abelian category. Then*

- (i) *The indecomposable objects in  $D(\mathcal{H})$  are stalk complexes of indecomposable objects in  $\mathcal{H}$ .*
- (ii) *Nonzero morphisms  $X \xrightarrow{f} Y$  between indecomposable objects in  $D(\mathcal{H})$  can only exist when they are concentrated in either the same degree or the codomain is concentrated in degree one lower than the domain. I.e. if*

$$\begin{aligned} X &= \dots \longrightarrow 0 \longrightarrow X^n \longrightarrow 0 \longrightarrow \dots && \in \mathcal{H}[n] \\ Y &= \dots \longrightarrow Y^{n-1} \longrightarrow 0 \longrightarrow 0 \longrightarrow \dots && \in \mathcal{H}[n+1]. \end{aligned}$$

**Lemma 5.20.** *Let  $A$  be a finite dimensional piecewise hereditary  $k$ -algebra and  $\mathcal{T} = D^b(\text{mod } A)$ . Let  $\mathcal{H}$  be a hereditary abelian category such that  $T \simeq D^b(\mathcal{H})$ .*

- (i) *Assume  $X, Y \in \mathcal{H}$  are indecomposable objects such that  $\text{Ext}_{\mathcal{H}}^1(Y, X) = 0$ . Then any nonzero morphism  $X \xrightarrow{f} Y$  is either a monomorphism or an epimorphism.*
- (ii) *Assume  $X, Y \in \mathcal{H}$  are such that their stalk complexes are exceptional objects in  $D^b(\mathcal{H})$ . If  $\text{Hom}_{\mathcal{H}}(X, Y) \neq 0$  and  $\text{Ext}_{\mathcal{H}}^1(X, Y) \neq 0$ , then  $\text{Ext}_{\mathcal{H}}^1(Y, X) \neq 0$ .*

In this next link in our chain of lemmas, we show that in this case of piecewise hereditary algebras, there are conditions under which the Hom-Ext-sequences obtained from  $\mathbf{R}\text{Hom}_{\mathcal{H}}(-, \sim)$  [26] are stalk complexes.



**Lemma 5.21.** *Let  $A$  be a finite dimensional piecewise hereditary  $k$ -algebra and  $\mathcal{T} = \mathrm{D}^b(\mathrm{mod} A)$ . Assume  $X, Y \in \mathcal{T}$  are indecomposable objects such that  $\mathrm{Hom}_{\mathcal{T}}(X \oplus Y, (X \oplus Y)[> 0]) = 0$ . Then  $\mathrm{Hom}_{\mathcal{T}}(X, Y[l]) = 0$  for all but possibly one  $l \in \mathbb{Z}$ .*

*Proof.* As  $A$  is piecewise hereditary, we know there is an abelian hereditary category  $\mathcal{H}$  and a triangle equivalence  $\mathcal{T} \xrightarrow{F} \mathrm{D}^b(\mathcal{H})$ .

Since  $F$  is a triangle equivalence, it maps triangles to triangles, so for any  $T \in \mathcal{T}$ , we get the triangle

$$FT \longrightarrow FT \longrightarrow 0 \longrightarrow F(T[1])$$

in  $\mathrm{D}^b(\mathcal{H})$ , which means  $F(T[1]) \cong (FT)[1]$ , so  $F$  commutes with shift.

Since  $X$  is indecomposable, so is  $FX$ , and for any  $n \in \mathbb{Z}$ , we have

$$\mathrm{Hom}_{\mathcal{T}}(X, X[n]) \cong \mathrm{Hom}_{\mathrm{D}^b(\mathcal{H})}(FX, FX[n]).$$

From the assumption on  $X \oplus Y$  we get that  $\mathrm{Hom}_{\mathcal{T}}(X, X[n]) = 0$  for all  $n > 0$ , and by Lemma 5.19 (ii) the same is true for all  $n > 0$ . In addition, it follows from Lemma 5.20 (i) that the endomorphism ring  $\mathrm{End}_{\mathrm{D}^b(\mathcal{H})}(FX)$  is a division algebra, so that  $FX$  is an *exceptional object* in  $\mathrm{D}^b(\mathcal{H})$ . Clearly, this is also true for  $FY$ , and for any shifts of  $FX$  and  $FY$ .

In particular, we have by Lemma 5.19 (i) that both  $FX$  and  $FY$  are stalk complexes of indecomposable objects in  $\mathcal{H}$ . We then let  $a \in \mathbb{Z}$  be such that  $FX$  and  $FY[a]$  are concentrated in the same degree, and let  $X', Y' \in \mathcal{H}$  be such that  $FX$  is a stalk complex of  $X'$  and  $FY[a]$  is a stalk complex of  $Y'$ . By Lemma 5.19 (ii), we then know that

$$\mathrm{Hom}_{\mathcal{T}}(X, Y[a+l]) \cong \mathrm{Hom}_{\mathrm{D}^b(\mathcal{H})}(FX, FY[a+l]) \cong \mathrm{Ext}_{\mathcal{H}}^l(X', Y') = 0$$

for all  $l$  except possibly  $l \in \{0, 1\}$ .

With the goal being to arrive at a contradiction, we assume both  $\mathrm{Hom}_{\mathcal{T}}(X, Y[a])$  and  $\mathrm{Hom}_{\mathcal{T}}(X, Y[a+1]) \cong \mathrm{Ext}_{\mathcal{H}}^1(X', Y') \neq 0$ . Then by Lemma 5.20 (ii), we have  $\mathrm{Ext}_{\mathcal{H}}^1(Y', X') \cong \mathrm{Hom}_{\mathcal{T}}(Y[a], X[1]) \neq 0$ .

Since  $\mathrm{Hom}_{\mathcal{T}}(X, Y[a+1]) \neq 0$ , we have that  $\mathrm{Hom}_{\mathcal{T}}(X \oplus Y, (X \oplus Y)[a+1])$  has a nonzero direct summand, and is nonzero. By assumption of  $X$  and  $Y$ , this means  $1+a \leq 0$  so  $a < 0$ .

On the other hand, since  $\mathrm{Hom}_{\mathcal{T}}(Y[a], X[1]) \neq 0$ , we get that  $\mathrm{Hom}_{\mathcal{T}}(X \oplus Y, (X \oplus Y)[1-a])$  has a nonzero direct summand, and is nonzero. Again by assumption of  $X$  and  $Y$ , this means  $1-a \leq 0$  so  $a > 0$ , and we arrive at our contradiction, completing the proof.  $\square$

Now we will show that basic pre-silting objects in which the summands exist in a cyclic sequence where any one summand has a nonzero morphism to the next, are indecomposable.

**Lemma 5.22.** *Let  $A$  be a finite dimensional piecewise hereditary  $k$ -algebra and  $\mathcal{T} = \mathrm{D}^b(\mathrm{mod} A)$ . Assume  $X_1, \dots, X_n, X_{n+1} = X_1 \in \mathcal{T}$  are indecomposable and  $X_i \not\cong X_j$  for all  $i \neq j$ . If*

(i)  $\mathrm{Hom}_{\mathcal{T}}(\bigoplus_{i=1}^n X_i, \bigoplus_{i=1}^n X_i[> 0]) = 0$ , and

(ii) there are integers  $l_1, \dots, l_n \in \mathbb{Z}$  such that for  $1 \leq i \leq n$ ,  $\mathrm{Hom}_{\mathcal{T}}(X_i, X_{i+1}[l_i]) \neq 0$ ,

then  $n = 1$ .

*Proof.* By (i), any  $l > 0$  implies

$$\mathrm{Hom}_{\mathcal{T}}\left(\bigoplus_{i=1}^n X_i, \bigoplus_{i=1}^n X_i[l]\right) \cong \bigoplus_{i,j=0}^n \mathrm{Hom}_{\mathcal{T}}(X_i, X_j[l]) = 0, \quad (20)$$

and so also  $\mathrm{Hom}_{\mathcal{T}}(X_i, X_j[> 0]) = 0$  for all  $i$  and  $j$ . It then follows from  $\mathrm{Hom}_{\mathcal{T}}(X_i, X_{i+1}[l_i]) \neq 0$  that  $l_i \leq 0$  for all  $i$ .

As  $A$  is piecewise hereditary, we know there is a hereditary abelian category  $\mathcal{H}$  and a triangle equivalence  $\mathcal{T} \xrightarrow{F} \mathrm{D}^b(\mathcal{H})$ . Then, as  $X_i$  is indecomposable,  $FX_i$  is indecomposable in  $\mathrm{D}^b(\mathcal{H})$ . By Lemma 5.19 (i) this means that  $FX_i$  is a stalk complex of some object  $H_i \in \mathcal{H}$ , i.e.  $FX_i \in \mathcal{H}[a_i]$  for some  $a_i \in \mathbb{Z}$ . We have

$$\mathrm{Hom}_{\mathcal{T}}(X_i, X_{i+1}[l_i]) \cong \mathrm{Hom}_{\mathrm{D}^b(\mathcal{H})}(FX_i, FX_{i+1}[l_i]) \cong \mathrm{Ext}_{\mathcal{H}}^{a_{i+1}-a_i+l_i}(H_i, H_{i+1}) \neq 0,$$

and as we know from Lemma 5.19 (ii), this means that  $a_{i+1} - a_i + l_i \in \{0, 1\}$  for all  $i$ . Also,  $\sum_{i=1}^n (a_{i+1} - a_i + l_i) = \sum_{i=1}^n l_i$  since  $a_{n+1} = a_1$ , and since we saw that  $l_i \leq 0$  for all  $i$ , we obtain

$$0 \leq \sum_{i=1}^n l_i = \sum_{i=1}^n (a_{i+1} - a_i + l_i) = \sum_{i=1}^n (0 \text{ or } 1) \geq 0.$$

This means that both  $l_i$  and  $a_{i+1} - a_i + l_i = 0$  for all  $i$ . I.e.  $a_{i+1} - a_i = 0$  for all  $i$ , so  $a_1 = \dots = a_n$ .

From this we have that the  $FX_i$  are all stalk complexes in  $\mathrm{D}^b(\mathcal{H})$  in the same degree, so

$$\mathrm{Hom}_{\mathcal{T}}(X_i, X_j[l]) \cong \mathrm{Hom}_{\mathrm{D}^b(\mathcal{H})}(FX_i, FX_j[l]) \cong \mathrm{Ext}_{\mathcal{H}}^l(H_i, H_j)$$

for all  $i$  and  $j$ , and all  $l \in \mathbb{Z}$ .

With the intention of arriving at a contradiction, let's assume  $n > 1$ . By assumption (ii), we can pick a nonzero morphism

$$f_i \in \mathrm{Hom}_{\mathcal{H}}(H_i, H_{i+1}) \cong \mathrm{Ext}_{\mathcal{H}}^0(H_i, H_{i+1}) \cong \mathrm{Hom}_{\mathcal{T}}(X_i, X_{i+1})$$

for all  $1 \leq i \leq n$ . Since the  $X_i$  are non-isomorphic,  $f_i$  is not an isomorphism. Furthermore,

$$\mathrm{Hom}_{\mathcal{T}}(X_{i+1}, X_i[1]) \cong \mathrm{Ext}_{\mathcal{H}}^1(H_{i+1}, H_i) = 0$$

by (20), so Lemma 5.20 (i) gives  $f_i$  either a monomorphism or an epimorphism.

Set  $f := f_n \cdots f_1 : H_1 \rightarrow H_1$ . If  $f$  is an isomorphism,  $H_1 \xrightarrow{f_1} H_2$  is a split monomorphism, and as the  $H_i$  are indecomposable, this means that either  $f_1$  is an isomorphism or  $H_1 = 0$ . Neither of these are true, as the latter implies  $X_1 = 0$ , so we conclude that  $f$  is *not* an isomorphism.

Assume then that all the  $f_i$  are monomorphisms, so that  $H_1 \xrightarrow{f} H_1$  is also a monomorphism. The endomorphism ring for  $H_1$  is local since

$$\mathrm{End}_{\mathcal{H}}(H_1) \cong \mathrm{End}_{\mathcal{T}}(X_1),$$

and  $X_1$  is indecomposable in the Krull-Schmidt category  $\mathcal{T}$ , so  $f$  is nilpotent. Then  $f^n = 0$  for some  $n \geq 0$ , and so  $f = 0$  by it being a monomorphism. This again means  $0 = \ker(f) \cong H_1$ , which is not true. Hence we have a contradiction, and not all  $f_i$  are monomorphisms. An analogous line of reasoning shows that not all the  $f_i$  are epimorphisms either.

To summarize; each  $f_i$  is shown to be either a monomorphism or an epimorphism, but not all are of the same type. By the circular nature of our  $X_i$ , this means that there is some  $i \in \{1, \dots, n\}$  where  $f_i$  is an epimorphism and  $f_{i+1}$  is a monomorphism. The composition  $f_{i+1}f_i$  is then nonzero, and by Lemma 5.20 (i) it is itself either a monomorphism or an epimorphism. In the first case,  $f_i$  is then both a monomorphism and an epimorphism – which is impossible. Similarly, the latter case is also impossible as neither  $f_{i+1}$  is both a monomorphism and an epimorphism.

This shows it is impossible for  $n$  to be greater than 1.  $\square$

Proposition 5.15 asserts that within the realm of piecewise hereditary algebras, any full exceptional sequence gives rise to a silting object. The following asserts the converse; that any silting object gives rise to a full exceptional sequence.

**Proposition 5.23.** *Let  $A$  be a finite dimensional piecewise hereditary  $k$ -algebra,  $\mathcal{T} = \mathcal{D}^b(\text{mod } A)$  and let  $X_1, \dots, X_n \in \mathcal{T}$  be nonzero, indecomposable objects such that  $M := X_1 \oplus \dots \oplus X_n$  is a basic silting object in  $\mathcal{T}$ . Then there is a permutation of the indices  $\{X'_1, \dots, X'_n\} = \{X_1, \dots, X_n\}$ , such that  $(X'_1, \dots, X'_n)$  is a full exceptional sequence in  $\mathcal{T}$ .*

*Proof.* We define a partial ordering on the indecomposable summands of  $M$  as follows: Say  $X_i \leq X_j$  if there is a sequence  $X_i = X_{i_1}, \dots, X_{i_m} = X_j$  of indecomposable summands of  $M$  such that

$$\text{Hom}_{\mathcal{T}}(X_{i_a}, X_{i_{a+1}}[\mathbb{Z}]) \neq 0$$

for all  $1 \leq a < m$ .  $X_i \leq X_j$  and  $X_j \leq X_k$  means we have two sequences as above. By connecting them at  $X_j$  we clearly get such a sequence from  $X_i$  to  $X_k$ , showing  $X_i \leq X_k$ . Since  $X_i \neq 0$ , we also have  $X_i \leq X_i$ . Finally, if we have  $X_i$  and  $X_j$  summands of  $M$  with  $X_i \leq X_j$  and  $X_j \leq X_i$ , we get the setup as in Lemma 5.22, telling us that  $X_i = X_j$ . This proves  $\leq$  gives a partial ordering.

The key to continue now is to realize that if  $X_i \not\leq X_j$ , then  $\text{Hom}_{\mathcal{T}}(X_i, X_j[\mathbb{Z}]) = 0$ . This includes both the cases where  $X_j < X_i$  and the cases where the two are not related by  $\leq$  at all. We rename  $\{X_1, \dots, X_n\}$  as  $\{X'_1, \dots, X'_n\}$  in such a way that all the  $X_j$  with  $X_i \leq X_j$  are placed *after*  $X_i$ . This way we guarantee that  $j < i$  gives  $\text{Hom}_{\mathcal{T}}(X'_i, X'_j[\mathbb{Z}]) = 0$ . The properties of the partial ordering allow us to do this in a well-defined manner. Our claim then is that  $\mathbf{X}' := (X'_1, \dots, X'_n)$  is a full exceptional sequence in  $\mathcal{T}$ .

As usual, there is a hereditary abelian category  $\mathcal{H}$  and a triangle equivalence  $\mathcal{T} \xrightarrow{F} \mathcal{D}^b(\mathcal{H})$ . Since  $X_i \in \mathcal{T}$  is indecomposable, there is from Lemma 5.19 (i) an  $n_i \in \mathbb{Z}$  and an  $H_i \in \mathcal{H}$  for each  $i$  such that  $FX_i[n_i]$  is isomorphic to the stalk complex of  $H_i$  concentrated in degree 0. Then

$$\text{Hom}_{\mathcal{T}}(X_i, X_i[l]) \cong \text{Hom}_{\mathcal{D}^b(\mathcal{H})}(FX_i[n_i], FX_i[n_i + l]) \cong \text{Ext}_{\mathcal{H}}^l(H_i, H_i)$$

Since

$$\text{Ext}_{\mathcal{H}}^1(H_i, H_i) \cong \text{Hom}_{\mathcal{T}}(X_i, X_i[1]) = 0,$$

Lemma 5.20 (i), gives us that any nonzero morphism  $f \in \text{End}_{\mathcal{H}}(H_i) \cong \text{End}_{\mathcal{T}}(X_i)$  is either a monomorphism or an epimorphism. As  $X_i$  is an indecomposable in the Krull-Schmidt category  $\mathcal{T}$ , any non-zero  $f$  is then an isomorphism. This means any nonzero  $f$  has a multiplicative inverse - i.e. the endomorphism ring is a division algebra. Furthermore, by Lemma 5.21, we also get  $\text{Hom}_{\mathcal{T}}(X_i, X_i[\neq 0]) = 0$ , and so  $X_i$  is an exceptional object.

By construction,  $\mathbf{X}'$  is then an exceptional sequence, and as  $\text{thick}\{\bigoplus_{i=1}^n X'_i\} = \text{thick}\{M\} = \mathcal{T}$ , it is also shown to be full.  $\square$

In order to make some notation more compact, we introduce a simple naming convention.

**Definition 5.24.** Let  $\mathcal{T}$  be a Krull-Schmidt triangulated category, and  $\mathbf{X} = (X_1, \dots, X_n)$  an exceptional sequence in  $\mathcal{T}$ . We define

$$[[\mathbf{X}]] := X_1 \oplus \cdots \oplus X_n.$$

Using this new notation, an exceptional sequence  $\mathbf{X}$  is full if  $\text{thick}\{[[\mathbf{X}]]\} = \mathcal{T}$ .

**Lemma 5.25.** *Let  $A$  be a finite dimensional piecewise hereditary  $k$ -algebra,  $\mathcal{T} = \text{D}^b(\text{mod } A)$  and  $\mathbf{X} = (X_1, \dots, X_n) \in \text{exp } \mathcal{T}$  be such that  $[[\mathbf{X}]]$  is a silting object. Then, for any  $1 \leq i < n$ , there exist  $\mathbf{l}, \mathbf{m} \in \mathbb{Z}^n$  such that  $[(\sigma_i, \mathbf{l})\mathbf{X}]$  and  $[(\sigma_i^{-1}, \mathbf{m})\mathbf{X}]$  are iterated irreducible silting mutations of  $[[\mathbf{X}]]$ .*

*Proof.* We fix an  $i \in \{1, \dots, n-1\}$  and apply Lemma 5.21 to  $X_i \oplus X_{i+1}$  to we get that  $\text{Hom}_{\mathcal{T}}(X_i, X_{i+1}[a]) \neq 0$  for at most one  $a \in \mathbb{Z}$ . Since  $[[\mathbf{X}]]$  is silting,  $a \leq 0$ . In the case where such an integer does not exist, we set  $a := 0$ .

For any  $j \in \{1, 2, \dots, i-1\}$  and any  $l \in \{a, a+1, \dots, 0\}$ , there is an integer  $b_{jl} \geq 0$  such that  $\text{Hom}_{\mathcal{T}}(X_j, X_{i+1}[l+n]) = 0$  for all  $n \geq b_{jl}$ . By letting  $-b := \max\{b_{jl}\} \geq 0$ , we then have an integer  $b \leq 0$  such that

$$\text{Hom}_{\mathcal{T}}(X_j[b], X_{i+1}[l]) = 0 \tag{21}$$

for all  $1 \leq j < i$  and all  $a \leq l \leq 0$ . Then, we define the full exceptional sequence  $\mathbf{Y}$  as

$$\mathbf{Y} := (X_1[b], \dots, X_{i-1}[b], X_i, \dots, X_n),$$

and note that  $[[\mathbf{Y}]]$  is a silting object by Lemma 5.17 (i). The first goal is to show that  $[[\mathbf{Y}]]$  and  $[[\mathbf{X}]]$  are transitive under iterated irreducible silting mutation.

By using the sequence of integers  $(-b, \dots, -b)$  of length  $n$ , we get by Lemma 5.17 that  $[[\mathbf{Y}]]$  and

$$X_1 \oplus \cdots \oplus X_{i-1} \oplus X_i[-b] \oplus \cdots \oplus X_n[-b]$$

are transitive under iterated irreducible silting mutation. If it just so happens that  $-b = 0$ , we are done. If, however  $-b > 0$ , we get by Lemma 5.17 that

$$X_1 \oplus \cdots \oplus X_{i-1} \oplus X_i[-b] \oplus \cdots \oplus X_n[-b]$$

and

$$[[\mathbf{X}]] = X_1 \oplus \cdots \oplus X_i \oplus \cdots \oplus X_n$$

are transitive under iterated irreducible silting mutation. Combine these, and we get that  $[[\mathbf{X}]]$  and  $[[\mathbf{Y}]]$  are transitive under iterated irreducible silting mutation. Next, set

$$\mathbf{Z} := (X_1[b], \dots, X_{i-1}[b], X_i, X_{i+1}[a], X_{i+2}, \dots, X_n),$$

so that

$$\sigma_i \mathbf{Z} := (X_1[b], \dots, X_{i-1}[b], R_{X_i}(X_{i+1}[a]), X_i, X_{i+2}, \dots, X_n).$$

The next goal is to show that  $[[\mathbf{Z}]]$  and  $[[\mathbf{Y}]]$  are transitive under iterated irreducible silting mutation. Note that if  $a = 0$ ,  $[[\mathbf{Z}]] = [[\mathbf{Y}]]$ , and we are done. Assume  $a < 0$ , and consider the right mutation  $\mu_{X_{i+1}}^-(\text{add}\{[[\mathbf{Y}]]\})$ . We have that

$$\text{Hom}_{\mathcal{T}}(X_j[b], X_{i+1}) = 0$$

for all  $j < i$ ,

$$\mathrm{Hom}_{\mathcal{T}}(X_i, X_{i+1}) = 0$$

by assumption that  $a < 0$ , and

$$\mathrm{Hom}_{\mathcal{T}}(X_j, X_{i+1}) = 0$$

for  $j > i + 1$  as  $\mathbf{Y}$  is an exceptional sequence. Thus it follows from Lemma 5.16 that

$$\mu_{X_{i+1}}^{\bar{}}(\mathrm{add}\{[[\mathbf{Y}]]\}) = \mathrm{add}\{X_1[b] \oplus \cdots \oplus X_{i-1}[b] \oplus X_i \oplus X_{i+1}[-1] \oplus \cdots \oplus X_n\}$$

If  $a < -1$ , it is possible to repeat the process – mutating the new category with respect to  $X_{i+1}[-1]$ . Let  $\mu_{X_{i+1}}^{\bar{(-n)}}$ ( $\mathrm{add}\{[[\mathbf{Y}]]\}$ ) be the result after  $n$  iterated irreducible silting mutations done with respect to the  $i + 1$ 'st summand. Then it is clear that  $\mu_{X_{i+1}}^{\bar{(-a)}}(\mathrm{add}\{[[\mathbf{Y}]]\}) = \mathrm{add}\{[[\mathbf{Z}]]\}$ , so that  $[[\mathbf{Y}]]$  and  $[[\mathbf{Z}]]$  are transitive under iterated irreducible silting mutation.

We perform one additional mutation, to obtain  $\mu_{X_{i+1}[a]}^{\bar{}}(\mathrm{add}\{[[\mathbf{Z}]]\})$ . Set

$$M := X_1[b]^{t_1} \oplus \cdots \oplus X_{i-1}[b]^{t_{i-1}} \oplus X_i^{t_i} \oplus X_{i+1}[a]^{t_{i+1}} \oplus X_{i+2}^{t_{i+2}} \oplus \cdots \oplus X_n^{t_n}$$

to be a general object in  $\mathrm{add}\{[[\mathbf{Z}]]\}$ , and define  $D$  to be the object

$$D := X_1[b]^{t_1} \oplus \cdots \oplus X_i^{t_i} \oplus \left( \prod_{l \in \mathbb{Z}} (\mathrm{Hom}_{\mathcal{T}}(X_i[l], X_{i+1}[a]) \otimes_k X_i[l]) \right)^{t_{i+1}} \oplus X_{i+2}^{t_{i+2}} \oplus \cdots \oplus X_n^{t_n}$$

obtained from  $M$  by interchanging every copy of  $X_{i+1}[a]$  by  $\prod_{l \in \mathbb{Z}} (\mathrm{Hom}_{\mathcal{T}}(X_i[l], X_{i+1}[a]) \otimes_k X_i[l])$ . Since  $\mathrm{Hom}_{\mathcal{T}}(X_i[l], X_{i+1}[a]) = 0$  for all  $l \neq 0$ , we get that

$$D = X_1[b]^{t_1} \oplus \cdots \oplus X_{i-1}[b]^{t_{i-1}} \oplus X_i^{t_i} \oplus (\mathrm{Hom}_{\mathcal{T}}(X_i, X_{i+1}[a]) \otimes_k X_i)^{t_{i+1}} \oplus X_{i+2}^{t_{i+2}} \oplus \cdots \oplus X_n^{t_n}$$

Note that  $\mathrm{Hom}_{\mathcal{T}}(X_i, X_{i+1}[a])$  is a finite-dimensional  $k$ -vector space, say  $\mathrm{Hom}_{\mathcal{T}}(X_i, X_{i+1}[a]) \cong k^s$ . This means that

$$\mathrm{Hom}_{\mathcal{T}}(X_i, X_{i+1}[a]) \otimes_k X_i \cong k^s \otimes_k X_i \cong X_i^s,$$

and so  $D \in \mathrm{add}\{[[\mathbf{Y}]]\}_{X_{i+1}[a]}$ . Furthermore, let  $D \xrightarrow{f} M$  be  $f = \iota_1 \oplus \cdots \oplus \iota_i \oplus \varphi^{t_{i+1}} \oplus \iota_{i+2} \oplus \cdots \oplus \iota_n$ . I.e. the inclusion into the biproduct for all summands except no.  $i + 1$ , where we use the second morphism in the triangle

$$R_{X_i}(X_{i+1}[a]) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(X_i, X_{i+1}[a]) \otimes_k X_i \xrightarrow{\varphi} X_{i+1}[a] \longrightarrow R_{X_i}(X_{i+1}[a])[1]. \quad (22)$$

from the definition of  $R_{X_i}(X_{i+1}[a])$ . We show this is a right  $\mathrm{add}\{[[\mathbf{Y}]]\}_{X_{i+1}[a]}$ -approximation of  $M$ . Let

$$D' := X_1[b]^{t'_1} \oplus \cdots \oplus X_{i-1}[b]^{t'_{i-1}} \oplus X_i^{t'_i} \oplus X_{i+2}^{t'_{i+2}} \oplus \cdots \oplus X_n^{t'_n}$$

be some object in  $\mathrm{add}\{[[\mathbf{Y}]]\}_{X_{i+1}[a]}$ , and  $D \xrightarrow{f'} M$  any morphism. As before

$$\mathrm{Hom}_{\mathcal{T}}(X_j[b], X_{i+1}[a]) = 0$$

for  $1 \leq j < i$  by (21), and

$$\mathrm{Hom}_{\mathcal{T}}(X_j, X_{i+1}[a]) = 0$$

for  $j > i + 1$  by  $[[\mathbf{X}]]$  silting. If  $a = 0$  and  $\text{Hom}_{\mathcal{T}}(X_i, X_{i+1}[a]) = 0$ , then  $D$  is a summand of  $M$  and  $f'$  clearly factors through  $f$ . Else,  $\text{Hom}_{\mathcal{T}}(X_i, X_{i+1}[a]) \neq 0$ , and we see that

$$\begin{array}{ccc} D & \xrightarrow{f} & M \\ g \uparrow & \nearrow f' & \\ D' & & \end{array}$$

commutes, where  $g = \iota_1 \oplus \cdots \oplus \iota_{i-1} \oplus (f' \otimes_k -) \oplus \iota_{i+2} \oplus \cdots \oplus \iota_n$ . Thus  $f$  is a right  $\text{add}\{[[\mathbf{Y}]]\}_{X_{i+1}[a]}$ -approximation of  $M$ . Then by Lemma 1.6,  $\varphi$  is a right  $\text{add}\{[[\mathbf{Y}]]\}_{X_{i+1}[a]}$ -approximation of  $X_{i+1}[a]$ , and by (22),  $N_M = R_{X_i}(X_{i+1}[a])^{t_{i+1}}$ . This means that

$$\mu_{i+1}^{-(-a+1)}(\text{add}\{[[\mathbf{Y}]]\}) = \text{add}\{[[\sigma_i \mathbf{Z}]]\}$$

so  $[[\mathbf{X}]]$  and  $[[\sigma_i \mathbf{Z}]]$  are transitive under iterated irreducible silting mutation. Let

$$\mathbf{1} := (b, \dots, b, 0, a, 0, \dots, 0).$$

Then

$$\begin{aligned} (\sigma_i, \mathbf{1})\mathbf{X} &= \sigma_i(X_1[b], \dots, X_{i-1}[b], X_i, X_{i+1}[a], X_{i+2}, \dots, X_n) \\ &= \sigma_i \mathbf{Z}, \end{aligned}$$

and the proof is half complete. The other half is dual.  $\square$

We now have the necessary tools to prove the main result:

**Theorem 5.26.** *Let  $A$  be a finite dimensional piecewise hereditary  $k$ -algebra and  $\mathcal{T} := \text{D}^b(\text{mod } A)$ . Then iterated irreducible silting mutation on  $\text{silt } \mathcal{T}$  is transitive.*

*Proof.* We know there is an finite dimensional hereditary or -canonical algebra  $\Lambda$  such that  $\mathcal{T} \simeq \text{K}^b(\mathcal{P}(\text{mod } \Lambda))$ , and  $\Lambda$  is a silting object in  $\text{K}^b(\mathcal{P}(\text{mod } \Lambda))$  by Lemma 2.2. Thus by Proposition 3.6, we only need to consider the basic silting objects in  $\mathcal{T}$ . To that end, let

$$T := X_1 \oplus \cdots \oplus X_n$$

and

$$U := Y_1 \oplus \cdots \oplus Y_n$$

be basic silting objects in  $\mathcal{T}$ , these direct sums being their respective *basic representations*. We permute the indices as per Proposition 5.23 so that  $\mathbf{X} := (X_1, \dots, X_n)$  and  $\mathbf{Y} := (Y_1, \dots, Y_n)$  are full exceptional sequences. By Theorem 5.14  $B_n \times \mathbb{Z}^n$  acts transitively on  $\text{exp } \mathcal{T}$ , so for some  $(\sigma, \mathbf{1}) \in B_n \times \mathbb{Z}^n$ , we have  $(\sigma, \mathbf{1})\mathbf{X} = \mathbf{Y}$ . The braid  $\sigma \in B_n$  is a product of the elementary braids  $\sigma_1^\pm, \dots, \sigma_n^\pm$ , say

$$\sigma = \sigma_{i_m}^\pm \cdots \sigma_{i_1}^\pm.$$

By Lemma 5.25, there is an  $\mathbf{r}_1 \in \mathbb{Z}^n$  such that  $[[\sigma_{i_1}^\pm, \mathbf{r}_1]\mathbf{X}]]$  is an iterated irreducible silting mutation of  $[[\mathbf{X}]]$ . Now  $(\sigma_{i_1}^\pm, \mathbf{r}_1)\mathbf{X}$  is a full exceptional sequence in  $\mathcal{T}$  with  $[[\sigma_{i_1}^\pm, \mathbf{r}_1]\mathbf{X}]]$  a silting object. Thus by Lemma 5.25 again there exists an  $\mathbf{r}_2 \in \mathbb{Z}^n$  such that  $[[\sigma_{i_2}^\pm, \mathbf{r}_2](\sigma_{i_1}^\pm, \mathbf{r}_1)\mathbf{X}]] = [[\sigma_{i_2}^\pm \sigma_{i_1}^\pm, \mathbf{r}_2 + \mathbf{r}_1]\mathbf{X}]]$  is

an iterated irreducible silting mutation of  $[(\sigma_{i_1}^\pm, \mathbf{r}_1)\mathbf{X}]$  – i.e. an iterated irreducible silting mutation of  $[\mathbf{X}]$ . We keep going, and get that there are  $\mathbf{r}_1, \dots, \mathbf{r}_m \in \mathbb{Z}^n$  such that

$$[(\sigma, \mathbf{r})\mathbf{X}] := [(\sigma_{i_m}^\pm \cdots \sigma_{i_1}^\pm, \mathbf{r}_m + \cdots + \mathbf{r}_1)\mathbf{X}]$$

is an iterated irreducible silting mutation of  $[\mathbf{X}]$ . We also have that

$$(\sigma, \mathbf{r})\mathbf{X} = (\sigma\sigma^{-1}\sigma, \mathbf{r} - \mathbf{1} + \mathbf{1})\mathbf{X} = (0, \mathbf{r} - \mathbf{1})\mathbf{Y} = (\mathbf{r} - \mathbf{1})\mathbf{Y},$$

and by Proposition 5.18  $[(\mathbf{r} - \mathbf{1})\mathbf{Y}]$  and  $[\mathbf{Y}]$  are transitive under iterated irreducible silting mutation. This completes the chain, and  $T = [\mathbf{X}]$  and  $U = [\mathbf{Y}]$  are transitive under iterated irreducible silting mutation.  $\square$

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## A Appendix

The following is an important result which lets us work with our derived categories using chain maps in the homotopy category. It is shown here as it will allow us to work with examples using AR-theory.

**Theorem A.1** (Equivalence of  $D^b(\text{mod } \Lambda)$  and  $K^{-,b}(\mathcal{P}(\text{mod } \Lambda))$ ). *Let  $\Lambda$  be a finite dimensional algebra over a field  $k$ . Then*

$$D^b(\text{mod } \Lambda) \simeq K^{-,b}(\mathcal{P}(\text{mod } \Lambda)).$$

As we work directly with complexes, we will use the notation  $X^\cdot$  for objects which are explicitly complexes. This to avoid confusion. Note that in the case where  $\Lambda$  is in addition hereditary,  $K^{-,b}(\mathcal{P}(\text{mod } \Lambda)) = K^b(\mathcal{P}(\text{mod } \Lambda))$ .

*Proof.* To show this we find a functor  $F : K^{-,b}(\mathcal{P}(\text{mod } \Lambda)) \rightarrow D^b(\text{mod } \Lambda)$  which is *full, faithful* and *dense*.

The first step is to show that any object of  $D^b(\text{mod } \Lambda)$  is isomorphic to a complex of projectives, bounded to the right, and bounded in homology in both directions. We show this by a modified induction proof: Let  $A^\cdot \in D^b(\text{mod } \Lambda)$ . We assume  $A^\cdot$  is of the form

$$A^\cdot = \quad \dots \longrightarrow A^{n-2} \xrightarrow{d^{n-2}} A^{n-1} \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} P^{n+1} \xrightarrow{d^{n+1}} P^{n+2} \longrightarrow \dots$$

For  $P^i$  projective. As 0 is projective, this covers both the situation where  $A^n$  is the last nonzero term, and the term where there are more nonzero, but projective, terms to the right of  $A^n$ .

We now produce a quasi-isomorphism from a new complex, replacing two terms, as follows: By taking the projective cover  $P^n \twoheadrightarrow A^n$ , we get the commutative diagram

$$\begin{array}{ccccccccccc} p(A^\cdot) := & \dots & \longrightarrow & A^{n-2} \xrightarrow{\begin{pmatrix} d^{n-2} \\ 0 \end{pmatrix}} & X & \xrightarrow{(0 \ 1)} & P^n & \xrightarrow{d^n \circ \pi^n} & P^{n+1} & \xrightarrow{d^{n+1}} & P^{n+2} & \longrightarrow & \dots \\ f \downarrow & & & \downarrow 1 & \downarrow (1 \ 0) & & \downarrow \pi^n & & \downarrow 1 & & \downarrow 1 & & \\ A^\cdot = & \dots & \longrightarrow & A^{n-2} & \xrightarrow{d^{n-2}} & A^{n-1} & \xrightarrow{d^{n-1}} & A^n & \xrightarrow{d^n} & P^{n+1} & \xrightarrow{d^{n+1}} & P^{n+2} & \longrightarrow & \dots \end{array}$$

Where  $X$  is the pullback

$$X = A^{n-1} \coprod_{A^n} P^n = \{(a, p) \in A^{n-1} \oplus P^n \mid d^{n-1}(a) = \pi^n(p)\}.$$

The map  $\begin{pmatrix} d^{n-2} \\ 0 \end{pmatrix}$  is the unique map such that the left square commutes and the top composes to 0, by the pullback property of  $X$ .

Clearly, for  $i \geq n+2$ , the homologies of these complexes match up, and the identity on these terms induce identity on homology. This is also true for  $i \leq n-3$ . For  $n-2 \leq i \leq n+1$ , we check manually.

$$H^{n-2}(p(A^\cdot)) = \ker(d^{n-2}_0) / \text{im}(d^{n-3}).$$

$$\ker(d^{n-2}_0) = \left\{ a \in A^{n-2} \mid \ker \begin{pmatrix} d^{n-2}(a) \\ 0 \end{pmatrix} = \ker \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} \cong \ker(d^{n-2}).$$

Then  $H^{n-2}(p(A^\cdot)) = \ker(d^{n-2}) / \text{im}(d^{n-3}) = H^{n-2}(A^\cdot)$ , and the induced map is the identity map. That is,  $H^{n-2}(f^\cdot)$  is an isomorphism.

Next we have

$$H^{n-1}(p(A^\cdot)) = \ker\left(\begin{smallmatrix} 0 & 1 \\ & 0 \end{smallmatrix}\right) / \operatorname{im}\left(\begin{smallmatrix} d^{n-2} \\ 0 \end{smallmatrix}\right).$$

We have:

$$\begin{aligned} \ker\left(\begin{smallmatrix} 0 & 1 \\ & 0 \end{smallmatrix}\right) &= \{(a, b) \in X \mid (0, b) = (0, 0)\} \\ &\cong \{a \in A^{n-1} \mid d^{n-1}(a) = \pi^n(0) = 0\} \\ &= \ker(d^{n-1}) \end{aligned}$$

and

$$\begin{aligned} \operatorname{im}\left(\begin{smallmatrix} d^{n-2} \\ 0 \end{smallmatrix}\right) &= \left\{ \begin{pmatrix} d^{n-2}(a) \\ 0 \end{pmatrix} \mid a \in A^{n-2} \right\} \\ &\cong \{d^{n-2}(a) \mid a \in A^{n-2}\} \\ &= \operatorname{im}(d^{n-2}). \end{aligned}$$

Thus

$$H^{n-1}(p(A^\cdot)) = \ker\left(\begin{smallmatrix} 0 & 1 \\ & 0 \end{smallmatrix}\right) / \operatorname{im}\left(\begin{smallmatrix} d^{n-2} \\ 0 \end{smallmatrix}\right) \cong \ker(d^{n-1}) / \operatorname{im}(d^{n-2}) = H^{n-1}(A^\cdot).$$

The induced map

$$\ker\left(\begin{smallmatrix} 0 & 1 \\ & 0 \end{smallmatrix}\right) / \operatorname{im}\left(\begin{smallmatrix} d^{n-2} \\ 0 \end{smallmatrix}\right) \rightarrow \ker(d^{n-1}) / \operatorname{im}(d^{n-2}),$$

given by

$$(a, 0) + \operatorname{im}\left(\begin{smallmatrix} d^{n-2} \\ 0 \end{smallmatrix}\right) \mapsto \pi^n(p) + \operatorname{im}(d^{n-1})$$

is well-defined, injective and surjective, and so  $H^{n-1}(f^\cdot)$  is an isomorphism.

$$H^n(p(A^\cdot)) = \ker(d^n \circ \pi^n) / \operatorname{im}\left(\begin{smallmatrix} 0 & 1 \\ & 0 \end{smallmatrix}\right).$$

The induced map  $H^n(f) : \ker(d^n \circ \pi^n) / \operatorname{im}\left(\begin{smallmatrix} 0 & 1 \\ & 0 \end{smallmatrix}\right) \rightarrow \ker(d^n) / \operatorname{im}(d^{n-1})$  is given by  $p + \operatorname{im}\left(\begin{smallmatrix} 0 & 1 \\ & 0 \end{smallmatrix}\right) \mapsto \pi^n(p) + \operatorname{im}(d^{n-1})$ .

$$\begin{aligned} \pi^n(p) + \operatorname{im}(d^{n-1}) = 0 &\Leftrightarrow \pi^n(p) \in \operatorname{im}(d^{n-1}) \\ &\Leftrightarrow \pi^n(p) = d^{n-1}(a) \text{ for some } a \in A^{n-1} \\ &\Leftrightarrow (a, p) \in X \text{ for some } a \in A^{n-1} \\ &\Leftrightarrow p \in \operatorname{im}\left(\begin{smallmatrix} 0 & 1 \\ & 0 \end{smallmatrix}\right) \\ &\Leftrightarrow p + \operatorname{im}\left(\begin{smallmatrix} 0 & 1 \\ & 0 \end{smallmatrix}\right) = 0, \end{aligned}$$

so its kernel is 0. Also,  $\pi^n$  is surjective, so  $\forall a \in \ker(d^n) \exists p \in P^n$  with  $a = \pi^n(p)$ . Furthermore  $d^n(a) = (d^n \circ \pi^n)(p) = 0$  implies  $p \in \ker(d^n \circ \pi^n)$ .

So for any  $s + \operatorname{im}(d^{n-1}) \in \ker(d^n) / \operatorname{im}(d^{n-1})$  there is a  $p + \operatorname{im}\left(\begin{smallmatrix} 0 & 1 \\ & 0 \end{smallmatrix}\right) \in \ker(d^n \circ \pi^n) / \operatorname{im}\left(\begin{smallmatrix} 0 & 1 \\ & 0 \end{smallmatrix}\right)$  such that the first is mapped to the second by  $H^n(f^\cdot)$ .

I.e  $H^n(f^\cdot)$  is an isomorphism.

$$H^{n+1}(p(A^\cdot)) = \ker(d^{n+1}) / \operatorname{im}(d^n \circ \pi^n).$$

The image is

$$\begin{aligned} \operatorname{im}(d^n \circ \pi^n) &= \{p \in P^{n+1} \mid p = d^n(\pi^n(p')) \text{ for some } p' \in P^n\} \\ &= \{p \in P^{n+1} \mid p = d^n(a) \text{ for some } a \in A^n\} \\ &= \operatorname{im}(d^n). \end{aligned}$$

So  $H^{n+1}(p(A^\cdot)) = \ker(d^{n+1}) / \operatorname{im}(d^n) = H^{n+1}(A^\cdot)$ , and the induced morphism  $H^{n+1}(f^\cdot)$  is the identity - an isomorphism.

Thus we have proved, by brute force, that  $p(A) \xrightarrow{f} A$  is a quasi-isomorphism.

Repeat this process sequentially to replace all terms in  $A$  by projective objects. Note that at the leftmost term of  $A$ , this process becomes the process of taking projective resolution. There is thus no guarantee that the complex of projectives will ever terminate as we move leftwards. The complex will however be bounded in homology by its quasi-isomorphism to  $A$ . In the case where our algebra has finite global dimension, this process will terminate, yielding a bounded complex of projectives quasi-isomorphic to  $A$ .

Now let  $P \in K^{-,b}(\mathcal{P}(\text{mod } \Lambda))$ . Define the functor  $F$  to  $D^-(\text{mod } \Lambda)$  by  $F(P) = P$ . That is, the obvious functor taking a morphism to its corresponding roof.

This induces, by [22] and by our subcategories being full, an isomorphism

$$\text{Hom}_{K^{-,b}(\mathcal{P}(\text{mod } \Lambda))}(P, Q) \rightarrow \text{Hom}_{D^b(\text{mod } \Lambda)}(P, Q).$$

Hence our functor is fully faithful.

$F$  needs not be dense in  $D^-(\text{mod } \Lambda)$ . However, let  $A \in D^b(\text{mod } \Lambda)$ . Then we have  $p(A) \in K^{-,b}(\mathcal{P}(\text{mod } \Lambda))$ , and so  $\forall A \in D^b(\text{mod } \Lambda) \exists p(A) \in K^{-,b}(\mathcal{P}(\text{mod } \Lambda))$  such that  $F(p(A)) = p(A) \cong A$ . We have an equivalence

$$F : K^-(\mathcal{P}(\text{mod } \Lambda)) \rightarrow D^-(\text{mod } \Lambda),$$

such that the restriction of  $F$  to  $K^{-,b}(\mathcal{P}(\text{mod } \Lambda))$  is dense in  $D^b(\text{mod } \Lambda)$ . This proves the statement.  $\square$

