Master's thesis

NTNU
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Department of Mathematical Sciences

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A classifying space for principal G-bundles with connection, in the category of simplicial presheaves

Master's thesis in Mathematical Sciences (Topology), MSMNFMA
Supervisor: Gereon Quick
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Abstract

In this master's thesis, we study principal G-bundles on smooth manifolds with connection, and how to make universal objects to classify them, similar to Grassmannians for vector bundles. The constructions take us into the category of simplicial presheaves on smooth manifolds, and result in a theorem that states that the only natural differential forms associated to the connection on principal G-bundles are the ones constructed in the Chern-Weil homomorphism. This was first obtained in this way by Freed and Hopkins in [7]. In the last chapter, we take a short look at what would happen if the manifolds considered were complex instead of smooth.

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1 Introduction

The point of this master's thesis is to study principal G-bundles, connections, and some invariant theory associated to them. When I started out, I had some knowledge of Lie groups and fibre bundles, but not about their combination in principal G-bundles, and the only connection I had encountered was the Levi-Civita-connection, in a class about Riemannian geometry. The important invariant concept of characteristic classes was also something I knew mostly by name. It therefore made sense to start at the definitions, and go through some basic results and constructions in this field, for which I mainly utilized the very helpful lecture notes by Johan Dupont that are cited as [5] and [6]. The main motivation for going into this theory, however, was to be able to understand the paper [7] by Freed and Hopkins, where a categorical approach is taken to construct an interesting classifying space for principal G-bundles with connections. This classifying space is also used to prove what could be considered the main result in [7] as well as in this thesis, which is that all natural differential forms attached to connections on principal G-bundles come from certain invariant symmetric polynomials, and can be obtained for a given principal G-bundle using a map called the Chern-Weil homomorphism. That this map produces natural differential forms – even characteristic classes – is old news, and not covered very thoroughly in this thesis, but that they are the only ones is a more interesting result, discovered by Freed and Hopkins in [7].

By a classifying space for principal G-bundles with connection we mean some bundle $E_\nabla G \to B_\nabla G$ with a universal connection Θ^{univ} that has the following universal property: Any other principal G-bundle with connection Θ can be embedded into the universal one in a way that makes the pullback of the universal connection equal to Θ. It is well known that Grassmannians can be used to make universal bundles that classify vector bundles in this way, but to be able to take the connection into account, we need to move away from the category of smooth manifolds, and into the one containing simplicial presheaves on manifolds. In the same way that the embeddings of manifolds into the Grassmannian bundle are unique only up to homotopy, we will also need to make use of weak equivalences, to arrive at uniqueness in the correct homotopy category.

Since connections are 1-forms on smooth manifolds, it could make sense to look for some universal space of 1-forms when we are already searching for Θ^{univ}, and maybe even a universal de Rham complex. The justify the description "universal" for a space of 1-forms, one would have to identify a neat universal property for it that in a way included all possible 1-forms, and a good candidate is a space $\Omega^\bullet$ for which the maps from a smooth manifold M and into it were in one-to-one correspondence with the 1-forms on M itself. In other words, we would seek the isomorphism

$$\mathrm{Hom}(M, \Omega^\bullet) = \Omega^\bullet(M) \tag{1}$$

for any smooth manifold M. Note that Ω is used for the desired universal object on the left side, and to denote differential forms on M (the de Rham complex) on the right. This is actually also something that is achieved in [7], and which we will take a look at.

To find the universal classifying space $E_\nabla G \to B_\nabla G$, Freed and Hopkins first move away from the category of smooth manifolds – which is where we traditionally construct principal G-bundles – and into the realm of presheaves on manifolds (another term that was rather unknown to me, but will be defined in time). Through an embedding of the former category into the latter, we move the entire classification problem to one of presheaves, but in a way that remembers the smooth structure. The condition in (1) will actually help us on the way to understand why the category of presheaves on manifolds is a good choice, and once there, we also manage to develop $E_\nabla G$, $B_\nabla G$ and Θ^{univ}. To find the right homotopy category, so that the maps into $E_\nabla G \to B_\nabla G$ become unique, we chose to follow the road set out in [7], and go via both groupoids and simplicial sets. Luckily, most of the structures we meet are quite intuitive if viewed in the right way, and result in a very practical situation, where the universal objects can be defined almost directly to have the properties we require of them.

After we have constructed our universal objects quite thoroughly, we move on to two theorems about the de Rham-spaces of $E_\nabla G$ and $B_\nabla G$, and it is here that we get the result about natural differential forms on principal G-bundles associated to a connection. Last of all, we take an outlook at classifying spaces and connections in the setting of holomorphic bundles, which are defined over complex manifolds instead of smooth ones.

The general outline of this thesis is as follows: The second (next) chapter is where we begin with our most basic definitions, and learn more about principal G-bundles in the setting of smooth manifolds. Most is built on results from [5], and we only prove some of the propositions to get a feeling of how one calculates with Lie groups and connections. The only really interesting result here is the Chern-Weil homomorphism, which would require many pages to prove properly, so this is refered to [5].

In the third chapter, we motivate the transition from smooth manifolds to presheaves following [7], and define everything that we need in this new category. The space of universal 1-forms appears of itself by definition, but other neat properties of presheaves on manifolds that result from an easy application of the Yoneda lemma help ensure that we are onto something also when it comes to finding $E_\nabla G$ and $B_\nabla G$. From here and up until the last chapter, we follow the paper [7] closely, as they also build up their theory from the same starting point. There are, however, some parts of the explanations in [7] that were either non-existent or not entirely clear to me, so I have tried to extend these as much as necessary.

In chapter four, we delve deeper into category theory, and try to justify why groupoids and simplicial sets are the appropriate structures to use for our purposes. Some examples are included, but nothing revolutionary happens. We also introduce weak equivalences in these categories, which will later be used to obtain a homotopy category, or localization.

Chapter five is the one where we combine the ideas from the former two chapters into the structure that we are actually going to use, namely simplicial presheaves on manifolds. Some explanations are given that hopefully elaborate on the choices made in [7], and we extend the definition of weak equivalences to simplicial presheaves.

Finally, in chapter six, we construct the simplicial presheaves $E_\nabla G$ and $B_\nabla G$, show how they can be seen as a kind of simplicial G-bundle, and end with the classifying theorem. On the way towards this, we need to work with some weak equivalences that are not covered in great detail in [7], and in fact, we end up completing the proof ideas that are used in that paper to justify the parallel statements of our Theorem 6.3, Proposition 6.6, and Theorem 6.7. The statement in Proposition 6.5 is used entirely without proof in [7], but we mend that as well. This is the chapter where most of the (sometimes quite nasty) computations are done, and where we to the largest degree work independently to complete the ideas found in [7].

The penultimate chapter, seven, begins with a short explanation of what goes on in the homotopy theory induced by our weak equivalences, but this would require a lot of separate theory to cover rigorously. We then define the de Rham complex in the simplicial setting, and state its value for the simplicial presheaves $E_\nabla G$ and $B_\nabla G$. No main theorems are proven in this part, but we go through a property of the newly defined Koszul complex, and calculate the de Rham complex of the simplicial sheaf $\Omega^1(-)$ – something our simplicial setting allows for, even if it seems kind of ridiculous from the viewpoint of ordinary, smooth manifolds. The de Rham complex of $B_\nabla G$ is then investigated, and produces the result that all natural differential forms on principal G-bundles attached to the connection must come from the Chern-Weil homomorphism.

In chapter eight, we take a step back, and consider what would happen if instead of principal G-bundles with connection over smooth manifolds, we were working over complex manifolds with holomorphic connection. This chapter is more speculative, and does not end with any special result, but we introduce the Atiyah classs, which is a sheaf cohomology class associated to holomorphic principal G-bundles that determines whether or not it is possible to endow them with holomorphic connections at all. The fact that not all bundles even have holomorphic connections certainly changes the picture, and we make some remarks on what a holomorphic version of $E_\nabla G$ and $B_\nabla G$ would look like.

Finally, I want to thank my supervisor Gereon Quick for his excellent assistance both in picking out a theme for my thesis, helping me understand the material I was examining, and making it into my own article. I thank also my family and friends for their continued support, especially my fellow students of maths (and physics) at NTNU. My student organization, Delta, also deserves a mention, for having helped me survive the last year by being a great community, where it was possible to take a break when needed the most. A special shout-out to Eiolf Kaspersen, who proof-read almost the entire thesis for me.

2 Principal G-bundles and connections

Even though the main results in this thesis will be obtained in a category of generalized manifolds, which we will define in the following section, it is useful to begin with the classical definitions of principal G-bundles and their connections encountered in the category of smooth manifolds. We will also go through some basic properties, the curvature form, and the classical Chern-Weil homomorphism. We start off with the principal G-bundle, which is a special fibre bundle where each fibre looks like the Lie group G.

Definition 2.1. *If we are given a Lie group G, a principal G-bundle is a triple (E, M, π) where E and M are smooth manifolds and π is a differentiable mapping between them. The group G acts differantiably from the right on E such that the action on each fibre $E_b = \pi^{-1}(b)$ is free and transitive, and we also have local trivializations of E:*

For every point m in M there is a neighbourhood U around it and a diffeomorphism $\phi : \pi^{-1}(U) \to U \times G$, such that

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & U \times G \arrow[dl, "\mathrm{proj}_1"] \\ & U & \end{tikzcd}$$

is commutative, and ϕ is equivariant, which means that

$$\phi(xg) = \phi(x)g \quad \forall x \in \pi^{-1}(U), g \in G.$$

The action of G on $U \times G$ is given by $(x, g)h = (x, gh)$ for $x \in U$, $g, h \in G$.

The space M is called the base space, while E is the total space and is sometimes used instead of (E, M, π) to denote the whole bundle (we also sometimes use $E \to M$, to specify the base space). We note that M is the orbit space of E under the G-action, and that each fibre E_b looks like G even though they are not themselves immediately endowed with any group structure. Trivial G-bundles can always be constructed on any manifold M, simply as $(M \times G, M, \mathrm{proj}_1)$, with the G-action $(p, g) \cdot h = (p, gh)$ for $p \in M$ and $g, h \in G$. A more interesting, motivating example for this construction is the frame bundle, which given any n-dimensional vector bundle $V \to M$ on a smooth manifold, is the fibre bundle that to a point $p \in M$ associates the fibre $\mathrm{Hom}(\mathbb{R}^n, V_p)$, where V_p is the fiber of V at p. This can easily be seen to be a principal $\mathrm{GL}_n(\mathbb{R})$-bundle, and is typically constructed over the tangent bundle $TM \to M$.

Next, we define a bundle map between the principal G-bundles (E, M, π) and (F, N, π') to be a pair $(\bar{f}, f)$ of differentiable maps such that the diagram

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "\pi"'] & F \arrow[d, "\pi'"] \\ M \arrow[r, "\bar{f}"] & N \end{tikzcd}$$

commutes, and $\bar{f}$ is equivariant with respect to the G-action.

An isomorphism of two principal G-bundles (E, M, π) and (F, M, π') over the same base space is obtained when f is the identity on M:

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & F \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

That every fibre in F is hit uniquely by its corresponding fibre in E follows from the commutativity of the diagram, and because ϕ respects the free and transitive G-action, we also get bijectivity on every fibre for free.

It follows from this definition that any automorphism of the trivial bundle $M \times G$ is of the form

$$\phi(p, v) = (p, g(p) \cdot v), p \in M, v \in G, \tag{2}$$

where $g : M \to G$ is a differentiable mapping. If we take any principal G-bundle (E, M, π) and find a cover $\{U_i\}_{i \in I}$ of M corresponding to trivializations

$$\phi_i : \pi^{-1}(U_i) \to U_i \times G, \tag{3}$$

we get a family of isomorphisms

$$\phi_i \circ (\phi_j)^{-1} : (U_i \cap U_j) \times G \to (U_i \cap U_j) \times G \tag{4}$$

that must have the form $\phi_i \circ (\phi_j)^{-1}(p, v) = (p, g_{ij}(p) \cdot v)$, where each $g_{ij} : U_i \cap U_j \to G$ is a differentiable map that we call a transition function for E with respect to the given trivializations $\{\phi_i\}_{i \in I}$. They respect the following so-called co-cycle conditions:

$$\begin{aligned} g_{ij}(p) \cdot g_{jk}(p) = g_{ik}(p) \quad & \forall i, j, k \in I, p \in U_i \cap U_j \cap U_k \\ g_{ii}(p) = 1 \quad & \forall i \in I, p \in U_i, \end{aligned} \tag{5}$$

and conversely, given a system of differentiable maps $f_i : U_i \to G$ that satisfy these, we can construct a principal G-bundle that has the f_i's as transition functions.

2.1 Connections in principal G-bundles

The geometric construction that we call a connection needs some motivation before we define it. Recall first that there to any Lie group G is associated a Lie algebra $\mathfrak{g}$, which as a vector space simply is defined as the tangent space T_eG at the identity element $e \in G$, with a Lie bracket induced canonically from the group structure of G.

Now, if (E, M, π) is any principal G-bundle, we can fix any $x \in E$ and obtain a mapping $f_x : G \to E$, sending $g \mapsto xg$. Since G acts freely this mapping is injective, so the induced tangent map at the identity e is also an injection $v_x = f_{x*} : \mathfrak{g} \to T_xE$. We can also consider the map $\pi_* : T_xE \to T_{\pi(x)}M$, which is surjective as it is induced by the surjective projection, and note that

$$\pi_*(v_x(g)) = (\pi \circ f_x)_*(g) = 0 \quad \forall w \in \mathfrak{g}. \tag{6}$$

The last equation follows because $\pi \circ f_x = \pi(x)$ is a constant function, since the G-action f_x preserves the fibres, and thus the tangent map is trivial. Finally, we can use the fact that E has local trivializations diffeomorphic to $U \times G$, where U is a neighborhood in M, to conclude that

$$\dim(T_xE) = \dim(T_{\pi(x)}M) + \dim(\mathfrak{g}), \tag{7}$$

which shows that the following is a short exact sequence of vector spaces:

$$0 \longrightarrow \mathfrak{g} \xrightarrow{v_x} T_xE \xrightarrow{\pi_*} T_{\pi(x)}M \longrightarrow 0$$

We call the image $v_x(\mathfrak{g}) \subseteq T_xE$ the subspace of vertical tangent vectors of E, while its complimentary subspace is denoted by H_x and contains the horizontal tangent vectors. Note that $H_x \cong T_{\pi(x)}M$ through π_*. These two subspaces change with respect to each other as x moves along E, the vertical one pointing along fibres E_x, and the horizontal one showing the direction of movement that corresponds to actual movement in the base space, and it therefore makes sense to track this change when one wants to understand how E "twists". A connection will be a splitting of the above exact sequence, equivalent to a linear map $\omega_x : T_xE \to \mathfrak{g}$ with

$$\omega_x \circ v_x = \mathrm{id}_{\mathfrak{g}} \quad \text{and} \quad \mathrm{Ker}(\omega_x) = H_x, \tag{8}$$

for every point $x \in E$. Since we also want this map to move differentially along E, it makes sense to choose an element $\omega \in \Omega^1(E; \mathfrak{g})$ with $\omega(x) = \omega_x$. This, however, is not enough for a definition, as it turns out that we further can demand a certain nice behaviour of our connection with respect to the G-action. The following example illustrates what we mean by that in the case of trivializable G-bundles.

Example 2.2. *If we consider the principal G-bundle $E = M \times G$ over M, where M is any manifold, we can define a special 1-form $\omega_{MC} \in \Omega^1(E; \mathfrak{g})$, called the Maurer-Cartan form, using the left multiplication $L_g : G \to G$ as follows:*

$$(\omega_{MC})_{(p,g)} = (L_{g^{-1}} \circ \mathrm{proj}_2)_*, \quad (p,g) \in (M \times G) \tag{9}$$

This form can be shown to be differentiable, so we only need to prove that (8) *holds for it to be a candidate for a connection on* E*. In our trivial case,* $f_{(p,g)}(h) = (p, gh)$*, where* $(p,g) \in E$ *and* $h \in G$*, so*

$$\begin{aligned}(\omega_{MC})_{(p,g)} \circ v_{(p,g)} &= (L_{g^{-1}} \circ \mathrm{proj}_2)_* \circ f_{(p,g)*} \\ &= (L_{g^{-1}} \circ \mathrm{proj}_2 \circ f_{(p,g)})_* \\ &= (\mathrm{id}_{T_eG})_* \\ &= \mathrm{id}_{\mathfrak{g}}\end{aligned}$$

where the second last line can be seen to be equal to the identity on G *by direct evaluation. Now that we have a 1-form splitting our exact sequence, we just need to introduce two important maps to write down the afore-mentioned additional property of* ω_{MC}*.*

Let $R_g : E \to E$ *denote the right action of a fixed* $g \in G$*, in the trivial case* $(p,h) \mapsto (p, hg)$*, and* $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ *be the adjoint map at* g*, induced as the differential at the identity element* $e \in G$ *of the conjugation map* $\mathrm{conj}_g : G \to G$*, where* $x \mapsto gxg^{-1}$ *(*Ad *can be seen as a canonical map from* G *to* $GL(\mathfrak{g})$*, and is called the adjoint representation of* G*). Now we get:*

Lemma 2.3. *The Maurer-Cartan form defined above satisfies the following:*

$$R_g^* \omega_{MC} = \mathrm{Ad}(g^{-1}) \circ \omega_{MC} \quad \forall g \in G \tag{10}$$

Proof. We have that $(\omega_{MC})_{(p,g)} = \mathrm{proj}_2^*(L_{g^{-1}})_*$, and because R_g only acts on the second argument of $(p,g) \in M \times G$ and thus commutes with projection onto G, we get

$$R_h^*(\omega_{MC})_{p,g} = (R_h^*)(\mathrm{proj}_2^*)(L_{g^{-1}})_* = (\mathrm{proj}_2^*)(R_h^*)(L_{g^{-1}})_* \tag{11}$$

Note that here, we use R_h to denote the right action on $M \times G$, but also standard right group multiplication if the domain is G. This works because the bundle we work with is trivial, and therefore the action is exactly right multiplication, only ignoring the first coordinate.

Further, we can also calculate

$$\begin{aligned}R_h^*(L_{g^{-1}})_* &= (L_{(gh)^{-1}})_* \circ (R_h)_* \\ &= (L_{h^{-1}})_* \circ (L_{g^{-1}})_* \circ (R_h)_* \\ &= (L_{h^{-1}})_* \circ (R_h)_* \circ (L_{g^{-1}})_* \\ &= \mathrm{Ad}(h^{-1}) \circ (L_{g^{-1}})_*,\end{aligned}$$

Where the third line follows as right multiplication commutes with left multiplication. Applying proj_2^* to both sides yields the desired result. □

Inspired by this, we finally arrive at the definition of a connection.

Definition 2.4. *A connection on a principal G-bundle (E, M, π) is a 1-form $\Theta \in \Omega^1(E; \mathfrak{g})$ with the following two properties:*

(1) $\Theta_x \circ v_x = \mathrm{id}_{\mathfrak{g}}$, *where $v_x : \mathfrak{g} \to T_x E$ is the differential at the identity of the map $G \to E$ sending $g \mapsto xg$.*

(2) $R_g^* \Theta = \mathrm{Ad}(g^{-1}) \circ \Theta$ *for all $g \in G$, where R_g and $\mathrm{Ad}(g)$ are the canonical maps defined above.*

It is by definition clear that the Maurer-Cartan form ω_{MC} is a connection on any trivial bundle.

As pointed out in [5], there is a more intuitive version of the second part of this definition, which I wish to include here. It uses the afore-mentioned subspaces of horizontal vectors, $H_x \subseteq T_x E$, which were defined to be equal to $\mathrm{Ker}(\Theta_x)$ for any connection on E.

Proposition 2.5. *For a 1-form $\Theta \in \Omega^1(E; \mathfrak{g})$ that satisfies the first requirement in Definition 2.4, the second one is equivalent to*

(2') $(R_g)_* H_x = H_{xg} \quad \forall x \in E, g \in G$

Proof. The proof is quite direct and can be found in [5]. □

In other words, the horizontal tangent vectors are moved into other horizontal vectors by the tangent map induced by the right G-action.

A nice result about connections is that we can obtain them for any principal G-bundle over a paracompact base space, using a couple of easy results that I will not prove here. First of all, it can be shown that the pullback of a connection Θ on E along a bundle map $f : E' \to E$ gives a connection denoted as $f^*(\Theta)$ on E'. We also have that a sum

$$\Theta' = \sum_{i \in I} \lambda_i \Theta_i \tag{12}$$

of connections $\{\Theta\}_{i \in I}$ on E, where the $\{\lambda_i\}_{i \in I}$ form a partition of unity of the base space M (i.e. each $\lambda_i \in C^\infty(M)$, they are locally finite and have the sum 1 everywhere) is a connection. Since we can always find a covering $\{U_i\}_{i \in I}$ of our paracompact M by trivializing neighbourhoods that admit a partition of unity, we can then construct the Maurer-Cartan form on every $U_i \times G$, pull them all back to local connections on E, and finally tie them together to a global connection using the partition of unity.

Having the connection, we can also define the important curvature form:

Definition 2.6. *Given a principal G-bundle E with connection Θ, the curvature form $F_\Theta \in \Omega^2(E; \mathfrak{g})$ is defined as*

$$F_\Theta = d\Theta + \frac{1}{2}[\Theta, \Theta], \tag{13}$$

where d is the differential, and $[\Theta, \Theta]$ is the 2-form obtained by applying the Lie bracket on $\mathfrak{g}$ to the image of $\Theta \wedge \Theta \in \Omega^2(E; \mathfrak{g} \otimes \mathfrak{g})$.

The name of this special 2-form suggests that it measures some kind of curvature, and indeed, we have the following proposition from [5] that we do not prove, but only include to give the nomenclature a tiny bit of justification:

Proposition 2.7. *Given a principal G-bundle E, the following statements are equivalent*

1. *E can be given a connection θ that produces a trivial curvature form $F_\theta = 0$.*

2. *There exists an open covering $\{U_i\}_i \in I$ of M with trivializations, that produces only constant transition functions $g_{ij} : U_i \cap U_j \to G$.*

The second part of this equivalence certainly seems to suggest that E does not twist too much, if imagined as a bunch of sets set-isomorphic to G, that each sits above a point in the base space M. Of course, constant transitions does not necessarily mean total flatness, which could maybe be defined by demanding transition functions that are all constant and equal to the identity on G. With a flat connection, there could still be some turning going on, but then it would have to be "locally constant". In a similar way, Gaussian curvature on manifolds is defined in such a way that the cylinder has none of it, even if it intuitively seems to turn when embedded in $\mathbb{R}^3$. The 2-sphere, on the other hand, which changes the direction of its turning as one moves along it, has a constant, non-zero Gaussian curvature, because this property is comparable to a sort of derivative of the turning.

This last paragraph is just intuitive speculation, and we do not go deeper into the theory of the curvature form, which would be interesting in itself. It is, however, needed in the next subsection, and was therefore necessary to define. There, we will also need the following two properties:

Definition 2.8. *Given a principal G-bundle E, a vector space V, a k-form $\omega \in \Omega^k(E; V)$, and a group homomorphism $\rho : G \to \mathrm{GL}(V)$ (also called a representation), we say that*

1. *ω is horizontal if at any point $x \in E$, $\omega(v_1, ..., v_k) = 0$ unless all vectors $v_1, ..., v_k$ lie in the horizontal vector space $H_x \subset T_x E$.*

2. *ω is ρ-equivariant if $R_g^* \omega = \rho(g^{-1}) \circ \omega \quad \forall g \in G$.*

A form satisfying both (1) and (2), with ρ being the trivial map $g \mapsto \mathrm{id}_V$, is called basic.

Again refering proofs to [5], we simply state that any curvature form F_Θ is both horizontal and Ad-equivariant, where the representation Ad was defined earlier.

2.2 The Chern-Weil homomorphism

As the final part of our introduction to principal G-bundles with connection, we take a look at the important Chern-Weil homomorphism, because the theory we develop later will result in something that is related to it. It produces a family of so-called characteristic classes, which are important invariants for G-bundles and therefore interesting to study. First, we define what they are.

Definition 2.9. *A characteristic class c for a principal G-bundle is a map that to every principal G-bundle (E, M, π) associates a cohomology class $c(E) \in H^*_{dR}(M)$ (the de Rham cohomology of M) in such a way that any bundle map $(\bar{f}, f) : (E', M', \pi') \to (E, M, \pi)$ from another bundle (E', M', π') satisfies*

$$c(E') = \bar{f}^*(c(E)). \tag{14}$$

Note that the cohomology class $c(E)$ is associated to the bundle E, but sits in the base space.

When we now develop the Chern-Weil homomorphism, we do not dwell on the proofs, but they can all be found in [5] if nothing else is specified. First, we need some definitions.

Given a finite-dimensional, real vector space V, and some $k \in \mathbb{N}$, we call P a symmetric, k-linear function if it is a morphism

$$P : V^k \longrightarrow \mathbb{R} \tag{15}$$

that is linear in all k variables, and satisfies

$$P(v_1, ...v_k) = P(v_{\sigma(1)}, ..., v_{\sigma(k)}) \tag{16}$$

for all $v_i \in V$ and permutations σ on k variables. These functions form a vector space which we call $\operatorname{Sym}^k V$. Setting $\operatorname{Sym}^0 V = \mathbb{R}$, and defining a product $\operatorname{Sym}^k V \times \operatorname{Sym}^l V \to \operatorname{Sym}^{k+l} V$ like this

$$P \odot Q(v_1, ..., v_{k+l}) = \frac{1}{(k+l)!} \sum_{\sigma} P(v_{\sigma(1)}, ..., v_{\sigma(k)}) Q(v_{\sigma(k+1)}, ..., v_{\sigma(k+l)}), \tag{17}$$

where σ runs through all permutations of $(k+l)$ elements, we get a commutative graded algebra

$$\operatorname{Sym}^\bullet V = \bigoplus_{k=0}^{\infty} \operatorname{Sym}^k V. \tag{18}$$

It could be good for intuition to note that each $\operatorname{Sym}^k V$ is isomorphic to the vector space of real, homogeneous polynomials of degree k over the variables $\{x_1, ..., x_n\}$, where n is the dimension of V.

If we set $V = \mathfrak{g}$ for some Lie algebra $\mathfrak{g}$, we can define a right action of the group G on $\operatorname{Sym}^k \mathfrak{g}$ by using the adjoint representation that we defined a while back. We set

$$(P \cdot g)(v_1, ..., v_k) = P(\mathrm{Ad}(g^{-1})(v_1), ..., \mathrm{Ad}(g^{-1})(v_k)) \tag{19}$$

for $P \in \mathrm{Sym}^k \mathfrak{g}$, $g \in G$ and $v_1, ..., v_k \in \mathfrak{g}$. That this is an action is immediate from the fact that $\mathrm{Ad}(g^{-1}) \circ \mathrm{Ad}(h^{-1}) = \mathrm{Ad}((hg)^{-1})$, which follows from the definition of Ad and the chain rule.

Definition 2.10. *An element $P \in \mathrm{Sym}^k \mathfrak{g}$ is called invariant if $P \cdot g = P$ for all $g \in G$. The set of all invariant elements in $\mathrm{Sym}^\bullet \mathfrak{g}$ is denoted by $I^\bullet(G) = \bigoplus_{k=0}^{\infty} I^k(G)$, and is a subalgebra of it.*

We can now combine this new algebra with the curvature forms we defined for principal G-bundles with connection. Given a bundle $E \to M$ with connection Θ, this form F_Θ is an element of $\Omega^2(E; \mathfrak{g})$, so we can define

$$F_\Theta^k = F_\Theta \wedge ... \wedge F_\Theta \in \Omega^{2k}(E, \mathfrak{g} \otimes ... \otimes \mathfrak{g}). \tag{20}$$

Because any $P \in I^k(G)$ defines a k-linear and symmetric map $\mathfrak{g}^k \to \mathbb{R}$, this actually is a map $\mathfrak{g} \otimes ... \otimes \mathfrak{g} \to \mathbb{R}$, and can be composed with the $2k$-form F_Θ^k to obtain $P(F_\Theta^k) \in \Omega^{2k}(E)$. Now, we know that any curvature form is horizontal and Ad-equivariant, and combining this with the fact that P is invariant, one can get that $P(F_\Theta^k)$ is basic, as defined in Definition 2.8. From this, it follows from Corollary 6.13 in [5] that there exists a unique $2k$-form on the base space M that is pulled back to $P(F_\Theta^k)$ via the projection. We usually denote this form by $P(F_\Theta^k)$ as well, and it is called the characteristic form corresponding to P.

Now it can further be shown that $P(F_\Theta^k)$ is a closed form, which places it in $H^{2k}_{\mathrm{dR}}(M)$. Its cohomology class, which we denote by $w(P; E)$, is also independent of the connection chosen on E, and only depends on the isomorphism class of E. Together with the fact that $P(F_\Theta^k)$ is preserved by pullbacks along G-bundle maps $(\bar{f}, f) : (E', M', \pi') \to (E, M, \pi)$ that preserve the connections, we get the following

Proposition 2.11. *Given a Lie group G and some $P \in I^k(G)$, the map $w(-; P)$ that to a principal G-bundle (E, M, π) associates*

$$w(E; P) = [P(F_\Theta^k)] \in H^{2k}_{\mathrm{dR}}(M) \tag{21}$$

is a characteristic class.

The Chern-Weil homomorphism is obtained by fixing the bundle instead of P, namely is the map

$$\begin{aligned} w(E; -) : I^k(G) &\longrightarrow H^{2k}_{\mathrm{dR}}(M) \\ P &\longmapsto w(E; P) \end{aligned}$$

It is a ring homomorphism if we use the product in $I^\bullet(G)$ inherited from $\mathrm{Sym}^\bullet \mathfrak{g}$, and the wedge product in $H^*_{\mathrm{dR}}(M)$.

A main result in Chapter 7 will be that the characteristic classes $w(-; P)$ are the only ones that can be found on principal G-bundles with connection, and actually the only natural differential forms on principal G-bundles associated to connections.

3 Generalized manifolds: Presheaves

From here on the discussion will be significantly more categorical in nature, and we will use **Man** to denote the category of finite-dimensional, smooth manifolds with smooth maps as morphisms. As mentioned, however, we need to move to a slightly different category to construct our classifying space. One of the elegant properties of this category mentioned in the introduction, is that we will be able to find a universal space of differential forms $\Omega^\bullet$ there, which not only can be used to pull back all differential forms on a manifold M from, but does so by a unique map. This can be formulated as

$$\mathrm{Hom}(M, \Omega^\bullet) = \Omega^\bullet(M), \tag{22}$$

where the right hand side denotes the de Rham complex associated to any smooth manifold in **Man**. To begin with, we notice that taking the de Rham complex of a manifold can be thought of as a contravariant functor, because smooth maps $f \in \mathbf{Man}(M, N)$ induce mappings between the associated de Rham complexes,

$$f^* : \Omega^\bullet(N) \longrightarrow \Omega^\bullet(M). \tag{23}$$

This map is constructed simply by taking pullbacks of differential forms, thus the notation f^*, but the details are of little interest here. Since the de Rham complex is a set, we can write

$$\begin{aligned} \Omega^\bullet : \mathbf{Man}^{\mathrm{op}} &\longrightarrow \mathbf{Set} \\ M &\longmapsto \Omega^\bullet(M) \end{aligned}$$

where we have chosen to think of $\Omega^\bullet$ as a covariant functor from $\mathbf{Man}^{\mathrm{op}}$ (which is the standard opposite category where all arrows are reversed) instead of a contravariant one from **Man**.

This means that we could achieve what we want if we manage to construct a category where such functors appear as the objects, and we start by giving them a name.

Definition 3.1. *A covariant functor from* $\boldsymbol{Man}^{\mathrm{op}}$ *to* ***Set*** *is called a presheaf on manifolds.*

Note that this is short for *presheaf of sets on manifolds*, where the fact that the functor goes to **Set** is always implied if nothing else is specified.

This candidate for objects in our category, however, needs a great deal of justification before we can be satisfied with it. First of all, our category is supposed to generalize manifolds, and therefore they have to embed naturally into it. But there is actually a presheaf of manifolds associated to each $M \in \mathbf{Man}$, namely the Hom-functor, which we here denote by $\mathscr{F}_M = \mathrm{Hom}(-, M)$.

$$\begin{aligned} \mathscr{F}_M : \mathbf{Man}^{\mathrm{op}} &\longrightarrow \mathbf{Set} \\ X &\longmapsto \mathrm{Hom}(X, M) \end{aligned}$$

Next, we need to decide what the sets of morphisms will be, and the obvious choice when having functors as objects is to take the natural transformations. In other words, a morphism ϕ between two presheaves $\mathscr{F}$ and $\mathscr{G}$ associates to each $M \in \mathbf{Man}$ a map $\phi(M) : \mathscr{F}(M) \to \mathscr{G}(M)$ such that the following diagram commutes for all choices of $M, N \in \mathbf{Man}$ and $f \in \mathrm{Hom}(M, N)$

$$\begin{tikzcd} \mathscr{F}(N) \arrow[r, "\mathscr{F}(f)"] \arrow[d, "\phi(N)"'] & \mathscr{F}(M) \arrow[d, "\phi(M)"] \\ \mathscr{G}(N) \arrow[r, "\mathscr{G}(f)"] & \mathscr{G}(M) \end{tikzcd}$$

We denote the category of presheaves with natural transformations as $\mathbf{Pre}$. To finally show why it makes sense to move from $\mathbf{Man}$ to this category, we need the contravariant version of the Yoneda lemma.

Lemma 3.2. *If F is a contravariant functor from a category $\mathscr{C}$ to **Set**, and* $\mathrm{Hom}_{\mathscr{C}}(-, C)$ *is the contravariant* Hom*-functor induced by an object $C \in \mathscr{C}$, then the natural transformations $\phi : \mathrm{Hom}_{\mathscr{C}}(-, C) \to F$ are in one-to-one correspondence with the set $F(C)$.*

Proof. The proof is standard, and follows quite easily from associating to any natural transformation ϕ the element $(\phi(C))(\mathrm{id}_C) \in F(C)$, and then reversing the process. $\square$

The important consequence of this lemma here is that we for any $\mathscr{F} \in \mathbf{Pre}$ get a bijection

$$\mathbf{Pre}(\mathscr{F}_X, \mathscr{F}) \cong \mathscr{F}(X), \tag{24}$$

where we remember $\mathscr{F}_X$ to be the functor $\mathrm{Hom}(-, X)$ for some $X \in \mathbf{Man}$. If now both our presheaves are induced by smooth manifolds, X and Y, we obtain

$$\mathbf{Pre}(\mathscr{F}_X, \mathscr{F}_Y) \cong \mathscr{F}_Y(X) = \mathbf{Man}(X, Y), \tag{25}$$

showing that we get exactly the same maps between our presheaves as we had in the category $\mathbf{Man}$. The smooth structure can therefore be said to be preserved in morphisms, even though the codomain of all the functors in $\mathbf{Pre}$ is the almost structure-free category of $\mathbf{Set}$. Another way of phrasing it is that $\mathbf{Man}$ embeds fully faithfully into $\mathbf{Pre}$.

The Yoneda lemma also grants, of course, the consequence we had in mind from the beginning of this section, namely that

$$\mathbf{Pre}(\mathscr{F}_M, \Omega^\bullet) \cong \Omega^\bullet(M) \quad \forall M \in \mathbf{Man} \tag{26}$$

Of course, we can also fix a dimension n and consider the presheaf Ω^n that takes any manifold M to $\Omega^n(M)$. Doing this for all $n \in \mathbb{N}_0$, and using the mappings of presheaves that for any test manifold M become the differential maps $d_n : \Omega^n(M) \to \Omega^{n+1}(M)$, we can also write out the universal de Rham complex as

$$\Omega^0 \xrightarrow{d_0} \Omega^1 \xrightarrow{d_1} \Omega^2 \xrightarrow{d_2} \Omega^3 \xrightarrow{d_3} \cdots$$

namely a chain complex of elements in **Pre**. Inspired by (26), we can actually extend our definition to get de Rham complexes for all elements $\mathscr{F}$ in **Pre**, by setting

$$\Omega^n(\mathscr{F}) = \mathbf{Pre}(\mathscr{F}, \Omega^n). \tag{27}$$

We can thus for example consider q-forms on Ω^n, taking $\Omega^q(\Omega^n)$ for $q \in \mathbb{N}_0$. This is mainly interesting if we actually know about natural transformations $\phi : \Omega^n \to \Omega^q$, but at least when $n = q$, we get the canonical q-form

$$\boldsymbol{\omega}^q = \mathrm{id}_{\Omega^q} : \Omega^q \to \Omega^q. \tag{28}$$

How we extend these sets $\Omega^q(\Omega^n)$ to a chain complex is completely natural, but it might be useful to give a concrete example when working on such a level of abstraction. If we for example want to construct the de Rham complex $\Omega^\bullet(\Omega^1)$ of the presheaf Ω^1, we want something on the form

$$\begin{tikzcd}
\Omega^0(\Omega^1) \arrow[r, "e_0"] \arrow[d, equal] & \Omega^1(\Omega^1) \arrow[r, "e_1"] \arrow[d, equal] & \Omega^2(\Omega^1) \arrow[r, "e_2"] \arrow[d, equal] & \Omega^3(\Omega^1) \arrow[r, "e_3"] \arrow[d, equal] & \cdots \\
\mathbf{Pre}(\Omega^1, \Omega^0) \arrow[r, "e_0"] & \mathbf{Pre}(\Omega^1, \Omega^1) \arrow[r, "e_1"] & \mathbf{Pre}(\Omega^1, \Omega^2) \arrow[r, "e_2"] & \mathbf{Pre}(\Omega^1, \Omega^3) \arrow[r, "e_3"] & \cdots
\end{tikzcd}$$

To see how the differentials e_i work here, we apply all these presheaves to some test manifold M, meaning that any natural transformation $\phi \in \mathbf{Pre}(\Omega^1, \Omega^i)$ is taken to a map $\phi(M) \in \mathbf{Set}(\Omega^1(M), \Omega^i(M))$, for $i \in \mathbb{N}_0$. The differential maps e_i then work on these natural transformations by post-composition of the differentials d_i from the ordinary de Rham complex $\Omega^\bullet(M)$, as shown here in the case $\phi \in \mathbf{Pre}(\Omega^1, \Omega^0)$:

$$\begin{tikzcd}
\Omega^1(M) \arrow[r, "\mathrm{id}"] \arrow[d, "\phi(M)"'] & \Omega^1(M) \arrow[r, "\mathrm{id}"] \arrow[d, "d_0 \circ \phi(M)"'] & \Omega^1(M) \arrow[r, "\mathrm{id}"] \arrow[d, "d_1 \circ d_0 \circ \phi(M)"'] & \Omega^1(M) \arrow[r, "\mathrm{id}"] & \cdots \\
\Omega^0(M) \arrow[r, "d_0"] & \Omega^1(M) \arrow[r, "d_1"] & \Omega^2(M) \arrow[r, "d_2"] & \Omega^3(M) \arrow[r, "d_3"] & \cdots
\end{tikzcd}$$

If we as is commonly done use e to refer to all the maps e_i, the domains being specified when necessary, we see that $e \circ e = 0$ for the presheaf maps follows immediately from the same fact for d in $\Omega^\bullet(M)$, so we indeed have a chain complex. When the presheaf we consider is some $\mathscr{F}_X$, induced by a manifold

X, the construction of this generalized de Rham complex leads to the ordinary, "smooth" one.

We shall actually be able to calculate the de Rham complex of Ω^1 later, as a rather indirect consequence of one of our main theorems, but in general it might be difficult to understand these complexes when our presheaves are not induced by smooth manifolds.

3.1 Sheaves and stalks

Proceeding with the definitions, we now want to decide what we should require of our presheaves to upgrade them to sheaves. The terminology might give it away, since presheaves generally encode local data, and become sheaves if they can be tied together globally. This is translated to our case in the following way.

Definition 3.3. *Any presheaf $\mathscr{F} \in$ **Pre** is a sheaf if, given any manifold $M \in$ **Man** and any open cover $\{U_i\}_{i\in I}$ of M, the sequence*

$$\mathscr{F}(M) \longrightarrow \prod_{i\in I} \mathscr{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathscr{F}(U_i \cap U_j)$$

is an equalizer diagram (i.e. $\mathscr{F}(M)$ is the equalizer of the right part).

Here the morphisms in the diagram are induced by the inclusion morphisms of the open sets, and clearly any intersection $U_i \cap U_j$ has two natural choices, one going into U_i and the other into U_j. This corresponds very well with the normal intuition for a sheaf, since it means that a family of elements in $\{\mathscr{F}(U_i)\}_{i\in I}$ corresponds to exactly one element coming from $\mathscr{F}(M)$ if they agree on all intersecting sets. As this is the case both for functions between manifolds and differential forms, we have that any $\mathscr{F}_X$ is a sheaf, and the same is true for $\Omega^\bullet$.

Notice that the sheaves make up a full subcategory of **Pre**, so whenever we speak of a sheaf map, we just mean a morphism in this category between two objects that happen to be sheaves.

Another important construction when working with presheaves is the stalk, which we also can imitate in our category.

Definition 3.4. *Let $\mathscr{F} \in$ **Pre** be a presheaf. For any natural number $m \in \mathbb{N}_0$ we define the m-dimensional stalk of $\mathscr{F}$ to be the colimit*

$$\operatorname*{colim}_{r\to 0} \mathscr{F}(B^m(r)), \tag{29}$$

where $B^m(r) \subseteq \mathbb{R}^m$ is the open ball around the origin with radius r.

Here we consider all balls $B^m(r)$ as manifolds, so $\mathscr{F}$ sends them to sets. With fixed m, we look at their disjoint union $\amalg_{r>0}\mathscr{F}(B^m(r))$, and taking the colimit means imposing the following equivalence relation on them: $x \in \mathscr{F}(B^m(r))$ is identified with $x' \in \mathscr{F}(B^m(r'))$ where $r \leq r'$ if and only if x' is sent to x by the morphism $\mathscr{F}(B^m(r')) \to \mathscr{F}(B^m(r))$ induced by the inclusion $\iota : B^m(r) \to B^m(r')$.

Remark 3.5. *This definition of stalks differs slightly from the one that is used in other branches of mathematics. Normally, one defines the stalk at a point, and looks at the colimit over all open sets that include this point. When working with manifolds, however, we always have local trivializations, so all open sets around a point can be restricted to a ball of the appropriate dimension. Because of how colimits are defined, restrictions to smaller sets preserve the structure from bigger sets, that themselves might not be homeomorphic to balls, so it is actually enough to consider these n-balls to get all information about what happens at a point. In that sense, all points have the same neighbourhoods if we just get close enough to them, so there is no reason for picking out a particular point for the stalk either.*

It is also worth mentioning about presheaves that we always have a sheaf associated to it by a universal property, namely the sheafification. Given any presheaf $\mathscr{F}$, there exists a sheaf $\boldsymbol{a}\mathscr{F}$ and a so-called universal map $\mathscr{F} \to \boldsymbol{a}\mathscr{F}$, such that any other map from $\mathscr{F}$ to any sheaf $\mathscr{F}'$ lifts uniquely through it. As most universal properties, it is in many ways easiest described with a diagram, in our case the following:

$$\begin{tikzcd} \mathscr{F} \arrow[r] \arrow[dr] & \boldsymbol{a}\mathscr{F} \arrow[d, dashed, "\exists!"] \\ & \mathscr{F}' \end{tikzcd}$$

As a matter of fact, $\boldsymbol{a}$ is a functor, and left adjoint to the forgetful functor that embeds sheaves in **Pre**, but we will not prove why $\boldsymbol{a}\mathscr{F}$ always exists here.

It might be time to take a step back and see how we can apply sheaves to our main problem, namely finding some object from which all principal G-bundles with connection can be pulled back. Sadly, even after all the work so far, we have not yet arrived at a structure that fits it well enough. Following [7], we see what happens if we try to make a classifying space $\mathscr{F}^G$ in the category of presheaves in the following way:

$$\mathscr{F}^G : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Set}, \quad \mathscr{F}^G(M) = \{\text{iso-classes of G-connections on M}\} \tag{30}$$

Here G is a fixed Lie group, with Lie algebra $\mathfrak{g}$, and the elements of the set $\mathscr{F}^G(M)$ are equivalence classes of G-bundles $P \to M$ with connection $\Theta \in \Omega^1(P;\mathfrak{g})$. Two such bundles are considered equivalent if there is a bundle isomorphism covering the base space M, that pulls back the connection in the codomain to the one in the domain. That $\mathscr{F}^G$ is indeed a presheaf is not very difficult to prove, but sadly it is not a sheaf in general, as a simple example shows: Take $M = S^1$, and cover it canonically by two contractible open sets U_1 and U_2 with $U_1 \cap U_2$ diffeomorphic to two open intervals. On a contractible space U, we see almost immediately that any two G-bundles with connections are isomorphic, as we can use exactly the element in $\mathbb{C}^\infty(U, G)$ that tweaks the one connection into the other as an isomorphism. The same holds for the disconnected space $U_1 \cap U_2$, so the sheaf-defining equalizer diagram from Definition 3.3 (which in our case of only two open sets turns into a pullback diagram) should look like this

$$\begin{tikzcd} \mathscr{F}^G(S^1) \arrow[r] \arrow[d] & \mathscr{F}^G(U_1) \arrow[d] \\ \mathscr{F}^G(U_2) \arrow[r] & \mathscr{F}^G(U_1 \cap U_2) \end{tikzcd} \qquad \begin{tikzcd} \mathscr{F}^G(S^1) \arrow[r] \arrow[d] & \{0\} \arrow[d] \\ \{0\} \arrow[r] & \{0\} \end{tikzcd}$$

where $\{0\}$ is the single isomorphism class in the cases where we have only one. This is not a pullback diagram if $\mathscr{F}^G(S^1)$ is non-trivial. If we now consider $G = \mathbb{Z}_2$, we know that the fibres of any $\mathbb{Z}_2$-bundle $P \to S^1$ will consist of two distinct points, and we can prove that all such bundles are double covers. But S^1 can be covered either by two copies of itself, or by a single copy winding around twice as fast. From the first section, we know that both these $\mathbb{Z}_2$-bundles can be endowed with a connection, so they appear as elements in $\mathscr{F}^{\mathbb{Z}_2}(S^1)$, but they are clearly not isomorphic, as they have a different number of connected components. Thus we have found an example where $\mathscr{F}^G(S^1)$ consists of more than one element, and consequently, $\mathscr{F}^G$ is not be a sheaf in general.

The problem we encounter here, which is descriptive of the situation, is that several non-isomorphic bundles with connections can be glued together from the locally trivial ones if the manifold we are working on has interesting topology. We approach this problem in the following section.

4 Groupoids and simplicial sets

The example from the previous section shows that sheaves alone are not enough to tackle our problem, at least not if used as naïvely as first proposed. What we want to do, is to track isomorphism classes without removing all memory of them entirely, and the solution that is arrived upon in [7] is to build groupoids from our presheaves, and then introduce a notion of equivalence that lets us replace the ugliest ones with easier versions. It will soon be clear from an example exactly what we mean by this, but first we need to define these terms properly.

Definition 4.1. *A groupoid is a small category where all morphisms have inverses, i.e. for all objects A and B, every element in $\mathrm{Hom}(A, B)$ has a corresponding element in $\mathrm{Hom}(B, A)$ with which it can be composed (in either order) to form an identity morphism. We can make groupoids into a category of its own, denoted by **Grpd**, by defining $\mathrm{Hom}_{\boldsymbol{Grpd}}(\mathscr{G}, \mathscr{G}')$ to be the set of functors from a groupoid $\mathscr{G}$ to another $\mathscr{G}'$.*

We normally denote a groupoid by $\{\mathscr{G}_0, \mathscr{G}_1\}$, where $\mathscr{G}_0$ is the collection of objects, and $\mathscr{G}_1$ the collection of morphisms, or "arrows", as they are usually called. As is usual when working with categories, we define equivalence slightly laxer than actual isomorphism.

Definition 4.2. *Two groupoids $\mathscr{G}$ and $\mathscr{G}'$ are said to be equivalent if they are equivalent as categories, which means that there exist functors $F : \mathscr{G} \to \mathscr{G}'$ and $G : \mathscr{G}' \to \mathscr{G}$ such that the compositions $G \circ F$ and $F \circ G$ are naturally isomorphic to the identity functors on $\mathscr{G}$ and $\mathscr{G}'$, respectively.*

We recall that this definition is equivalent to having one functor say $F : \mathscr{G} \to \mathscr{G}'$, that is fully faithful (bijective on all Hom-sets) and dense (all objects in $\mathscr{G}'$ are isomorphic to an object hit by F). Notice that in a groupoid, two objects are isomorphic if and only if there is some arrow between them, as all maps are invertible. This simplifies the density criterion.

Just to have a couple of examples in mind, we notice that any set S can be seen as a (boring) groupoid with only identity arrows, i.e. $\mathscr{G}_0 = S$, $\mathscr{G}_1 = \{\mathrm{id}_s\}s \in S$. More interestingly, if we have a group G acting on S from the right, we can take $\mathscr{G}_1$ to be the set $S \times G$ by using the group actions as arrows in the following way: The arrow $(s, g) \in S \times G$ goes from s to $s \cdot g$, and the composition of arrows is defined by the group action. Because of the identity element and inverses in G, we get both identity maps for all $s \in S$ and inverses for all non-identity arrows, so this really does define a groupoid.

A special subcategory of **Grpd** that will be of importance here, is the collection of discrete groupoids.

Definition 4.3. *A groupoid $\mathscr{G}$ is called discrete if all sets of morphisms **Grpd**(A, B) between objects A and B are either empty or contain a single element.*

We observe immediately that two discrete groupoids are equivalent if there is a functor between them that is surjective on objects. This is because the unique morphism between two objects (if they have one) must be taken to some morphism between their target objects, but since this is the only one there, full faithfulness is already guaranteed.

As promised, the advantages of using groupoids will be shown in an example. We now return to the case studied toward the end of last chapter, where tried to classify principal $\mathbb{Z}_2$-bundles with connection. Since the connections have values in the one-element Lie algebra that corresponds to $\mathbb{Z}_2$, there is only one possible connection for any $\mathbb{Z}_2$-bundle, so we only have to find all of these $\mathbb{Z}_2$-bundles. As mentioned, they are the same as double covers of S^1. We can do this classification by defining a groupoid in the following way:

$$\begin{aligned}
\mathscr{G}_0 &= \{P \to S^1 | P \text{ double cover}\} \\
\mathscr{G}_1 &= \{\phi : P \to P' | \phi \text{ isomorphism over } S^1\}
\end{aligned}$$

Notice that this indeed is a groupoid, as all arrows are isomorphisms, and by definition reversible.

Now this approach will give a very big groupoid, as S^1 for example can be covered by any two versions of itself immersed in $\mathbb{R}^2$. Up to isomorphism, on the other hand, there are only two double covers: The trivial and disconnected $S^1 \times \mathbb{Z}_2$, and the cover by a single circle that wraps around the base version of S^1 at twice the speed. If we now define a groupoid $\mathscr{G}'$ based on these isomorphism classes, $\mathscr{G}_0 = \{S^1 \times \mathbb{Z}_2, S^1\}$, which is considerably easier to imagine. All the arrows $\mathscr{G}_1$ come from automorphisms (since no isomorphism can be made from a connected to a disconnected space), and if they are to respect the projection on S^1, we only get a single non-trivial morphism for each double cover, namely the map that switches the elements of every fibre. Notice that they are their own inverses. If we call them f_1 and f_2, we have

$$\mathscr{G}_1 = \{\mathrm{id}_{S^1 \times \mathbb{Z}_2}, \mathrm{id}_{S^1}, f_1, f_2\}. \tag{31}$$

This groupoid $\mathscr{G}'$ of isomorphism classes is so easy that we can draw it, renaming the elements of $\mathscr{G}_0$ as g_1 and g_2:

$$\begin{tikzcd}
g_1 \arrow[loop left, "\mathrm{id}_{g_1}"] \arrow[loop right, "f_1"] & & g_2 \arrow[loop left, "f_2"] \arrow[loop right, "\mathrm{id}_{g_2}"]
\end{tikzcd}$$

We can now define a functor from $\mathscr{G}$ to $\mathscr{G}'$ that sends any double cover to the one representing its isomorphism class (i.e. number of connected components). It does not take much work to convince oneself that this can be done in a way that works well with morphisms, and the resulting functor is dense, full and faithful. We have thus arrived upon a very simple groupoid to classify principal $\mathbb{Z}_2$-bundles over S^1, that is equivalent to the ugly one we started with, but notice that it remembers the non-trivial automorphisms through its arrows f_1 and f_2. This is the main difference from working with simple sets of isomorphism classes, and combined with our notion of equivalence, it is the structure that we will later use for our classification of principal G-bundles with connection.

The groupoid presented in the above paragraphs is actually already very similar to the one we will be working with for the main classification, but before we get to that, we want to learn more about this new category **Grpd**.

4.1 Simplicial sets

To be able to think about our groupoids in a more topological sense, we will now see how we can associate topological spaces to them, by going through the category of simplicial sets. As mentioned in [7], this is not strictly necessary to obtain our theorems, but gives everything a nicer touch for those of us who prefer to work with more geometrical spaces. Equivalent groupoids will translate into weakly homotopy equivalent spaces, so also this notion will become more familiar. All this requires a new series of definitions, which now follows.

Definition 4.4. *The category Δ is the one where the objects are finite, non-empty and totally ordered sets, and the morphisms are order-preserving maps between them. It is equivalent to a category where all objects are of the form $[\boldsymbol{n}] = \{0, 1, ..., n-1\}$, and we will usually just think of these when considering Δ.*

Note that the morphisms need not be strictly order-preserving. In this category, there are essentially two types of maps that need to be understood, because all other maps can be formed by composing these essential ones. The first are the degeneracy maps s_i, that go from an object with $n \geq 2$ elements to the one with one less by halting for one step at the i-th place:

$$\begin{aligned} s_i : & [\boldsymbol{n+1}] \longrightarrow [\boldsymbol{n}] \\ & j \longmapsto \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{cases} \end{aligned}$$

From $[\boldsymbol{n}]$, there are exactly n degeneracy maps, one for each $i \in \{0, ..., n-1\}$. The face maps d_i got the other way, from an object with $n \geq 1$ element(s) to the one above, skipping the i-th element like this:

$$\begin{aligned} d_i : & [\boldsymbol{n}] \longrightarrow [\boldsymbol{n+1}] \\ & j \longmapsto \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases} \end{aligned}$$

From $[\boldsymbol{n}]$, there are $n+1$ such face maps. We do not prove that they can be used to form all morphisms in Δ, which is not very difficult to see, but proceed by defining simplicial sets. A thorough introduction to the the category Δ and the way it is used to develop simplicial sets, can be found in [8].

Definition 4.5. *A simplicial set is a covariant functor $F : \Delta^{\text{op}} \longrightarrow$ **Set**. They form a category denoted by $\boldsymbol{Set}_{\Delta}$ when we use natural transformations as morphisms.*

Since a simplicial set essentially is a sequence of sets, one for each $[\boldsymbol{n}]$ where $n \geq 0$, we often denote them by $F_\bullet$, and set $F_n = F_\bullet([\boldsymbol{n}])$. The set F_n is then called the set of n-simplices, and we get maps between these sets from the face and degeneracy maps in Δ. Often, we illustrate $F_\bullet$ like this

$$F_0 \leftleftarrows\!\!\dashrightarrow F_1 \; \substack{\longleftarrow \\ \dashrightarrow \\ \longleftarrow \\ \dashrightarrow \\ \longleftarrow} \; F_2 \cdots$$

where certain relations hold for the composition of these arrows, induced from Δ. The intuition behind the names "face" and "degeneracy" becomes clear here, because the face maps can be seen as a way of picking out one of the $(n+1)$ $(n-1)$-simplices that can be thought of as the faces of an n-simplex, while the degeneracy maps promote lower simplices to bigger ones by counting one of their nodes twice. To further strengthen this very geometric way of considering simplicial sets, we now state how they all correspond to topological spaces in a very natural way.

First, we need the standard-simplices from Euclidean space, that serve as models for any n-simplex. We define

$$\Delta^n = \{(x_0, x_1, ..., x_n) \in \mathbb{R}^{n+1} | x_i \geq 0, x_0 + x_1 + ... + x_n = 1\} \tag{32}$$

Now as mentioned earlier, any set $I \in \Delta$ has an isomorphism to a unique set $\boldsymbol{n} = \{0, 1, ..., n\}$ for some $n \geq 0$, and we define $\Sigma(I) = \Delta^n$ for the corresponding n. The map Σ can easily be extended to a functor going from Δ to the category of topological spaces, and using this, we can define the geometric realization of simplicial sets as follows.

Definition 4.6. *Take any simplicial set $F : \Delta^{\text{op}} \to \boldsymbol{Set}$. The geometric realization $|F|$ is the topological space obtained as the quotient space of*

$$\coprod_{I \in \Delta} \Sigma(I) \times F(I) \tag{33}$$

when we identify $(\theta_* t, x) \sim (t, \theta^* x)$ for all maps θ in our category Δ.

Intuitively, what the geometric realization does is to make the sets of n-simplices into n-dimensional open disks, and then glue them together along the borders as specified by the face and degeneracy maps. We do not dwell on a longer explanation, as this merely is an intuitive aid, and not a construction we will be using later.

Now that we have seen how simplicial sets can be made into spaces, we go back to groupoids and turn them into simplicial sets. We will then obtain a functor

$$\mathbf{Grpd} \longrightarrow \mathbf{Set}_\Delta \longrightarrow \mathbf{Top} \tag{34}$$

that lets us think geometrically about our groupoids.

If we are given a groupoid $\mathscr{G} = \{\mathscr{G}_0, \mathscr{G}_1\}$, we construct an associated simplicial set $F(\mathscr{G})_\bullet$ as follows: Let $F(\mathscr{G})_0 = \mathscr{G}_0$ be the 0-simplices, and $F(\mathscr{G})_1 = \mathscr{G}_1$ the 1-simplices. The degeneracy map $F(\mathscr{G}_0) \to F(\mathscr{G}_1)$ takes an element in G_0 to its identity map, and the two face maps $F(\mathscr{G}_0) \leftarrow F(\mathscr{G}_1)$ assign either the source or the target to any map in G_1. For $n > 1$, we take $F(\mathscr{G})_n$ to be the compositions of n arrows, and make the face and degeneracy maps in the same way as above. Degeneracy then always corresponds to adding an identity map to some series of compositions, making the series one arrow longer, while face maps skip one of the objects either by considering two arrows f and g in $\mathscr{G}_1$ as the single arrow $g \circ f$, or by dropping the first or last arrow in the composition. To see that this indeed results in a simplicial set is not hard.

The two examples we considered for groupoids give rise to simplicial sets using the above construction. Both are quite important, so we take a closer look at them

Example 4.7. *The groupoids S with only identity arrows, which practically speaking are just sets S_0 of nodes, result in what we call dicrete simplicial sets. We usually write $S = S_\bullet = F(S)_\bullet$ for them, since they consist of sets S_n which all are isomorphic to S itself, the only n-simplices being n copies of the identity map on some element in S. As we see, all simplices except for the ones in degree* 0 *are degenerate.*

Example 4.8. *Our next example was the groupoid created from a set S having a (right) group action from some group G. This time we get something more interesting, with* 1*-simplices coming from the group actions and higher simplices from their compositions. Since each arrow going away from an element $s \in S$ can be represented by a unique element $g \in G$, we get the simplicial set*

$$\begin{tikzcd} S \arrow[r, dashed] & S \times G \arrow[l, shift left=1.5] \arrow[l, shift right=1.5] \arrow[r, dashed, shift left=1] \arrow[r, dashed, shift right=1] & S \times G \times G \cdots \arrow[l, shift right=2.5] \arrow[l] \arrow[l, shift left=2.5] \end{tikzcd}$$

where, as an illustrative example, the element $(s, g) \in S \times G$ can be upgraded by one of the two dashed arrows to either (s, e, g) or (s, g, e). Here $e \in G$ is the unit element and consequently corresponds to all identity arrows. Going the other way, we can send $(s, g, h) \in S \times G \times G$ to either (s, g), (s, gh) or $(s \times g, h)$.

Last of all, we need a notion of equivalence in our new categories, that corresponds to the one we have for groupoids (remember that two groupoids were considered equivalent if they were equivalent as categories). As promised, this will involve good old homotopy theory in the case of the geometric realizations.

Definition 4.9. *A map $f : X \to Y$ between topological spaces is called a weak homotopy equivalence if all the following induced maps on homotopy groups are isomorphisms:*

$$\begin{aligned} f_* :&\pi_0(X) \longrightarrow \pi_0(Y) \\ f_* :&\pi_n(X, x) \longrightarrow \pi_n(Y, f(x)) \quad \forall n > 0, x \in X \end{aligned}$$

If such an f exists, X and Y are said to be weakly homotopy equivalent.

A map (natural transformation) of simplicial sets $F_\bullet \to F'_\bullet$ is called a weak equivalence if the map induced on the geometric realizations, $|F_\bullet| \to |F'_\bullet|$ is a weak homotopy equivalence. If such a map exists, $F_\bullet$ and $F'_\bullet$ are said to be weakly equivalent.

When talking about groupoids, we only referred to equivalences, not weak ones, but we now state formally that all these notions of (weak) (homotopy) equivalence are connected:

Proposition 4.10. *The functors discussed in this chapter and illustrated in* (34)*, going from* ***Grpd*** *via* $\boldsymbol{Set}_\triangle$ *to* ***Top****, send equivalences via weak equivalences to weak homotopy equivalences.*

The proof is omitted as it is not a main focus of this article, but in [12] it is shown that equivalent groupoids map to homotopy equivalent spaces, which is even stronger than the weak homotopy equivalence we demanded. The simplicial sets in the middle are then weakly equivalent by the above definition.

5 Simplicial presheaves and sheaves

Now that we have found a category that seems more suited for dealing with isomorphism classes of principal G-bundles with connection than ordinary sets, namely groupoids, and seen that they can be realized nicely as simplicial sets, we tackle the problem of associating a simplicial set to any given manifold. For, if we cannot embed **Man** in $\mathbf{Set}_\Delta$, there really is no point in trying to find a classifying space there. In the spirit of Section 3, we want to use something similar to presheaves, only using simplicial sets rather than ordinary ones.

Definition 5.1. *A simplicial presheaf on manifolds, often just called a simplicial presheaf, is a covariant functor*

$$\mathscr{F}_\bullet : \boldsymbol{Man}^{\mathrm{op}} \longrightarrow \boldsymbol{Set}_\Delta, \tag{35}$$

and a morphism between two of them is a natural transformation.

Simplicial presehaves on manifolds make up a category, which we denote by **sPre**.

Like we considered simplicial sets as an ordered sequence of sets, we can also consider a simplicial presheaf $\mathscr{F}_\bullet$ as a sequence of presheaves $\mathscr{F}_n$ where $n \in \mathbb{N}_0$, simply by applying $\mathscr{F}_\bullet$ to the elements $\boldsymbol{n} \in \Delta$ before applying it to any manifold. There will be presheaf maps between them, associated to the face and degeneracy maps.

Simple examples of simplicial presheaves are induced from all ordinary presheaves of sets, since sets can be seen as discrete simplicial sets: If we have a presheaf $\mathscr{F}$, we make it into a simplicial presheaf that sends any manifold M to the constant simplicial set $\mathscr{F}(M)$, like the one encountered in Example 4.7. Using the description in the above paragraph, we get a sequence with one copy of the presheaf $\mathscr{F}$ for each $n \in \mathbb{N}_0$, and all face and degeneracy maps equal to the identity.

We also want to be able to speak of simplicial sheaves and stalks, essentially moving our old definitions over from the realm of presheaves:

Definition 5.2. *A simplicial presheaf $\mathscr{F}_\bullet$ on manifolds becomes a simplicial sheaf on manifolds if every presheaf on manifolds $\mathscr{F}_n$ is a sheaf.*

Whether $\mathscr{F}_\bullet$ is a sheaf or not, we can define the m-dimensional stalk of $\mathscr{F}_\bullet$ for any natural number $m \in \mathbb{N}_0$ to be the colimit

$$\operatorname*{colim}_{r\to 0} \mathscr{F}_\bullet(B^m(r)), \tag{36}$$

where $B^m(r) \subseteq \mathbb{R}^m$ is the open ball around the origin with radius r.

Notice that the m-dimensional stalk of $\mathscr{F}_\bullet$ is a simplicial set here, not just a set. Its set of n-simplices is the m-dimensional stalk of $\mathscr{F}_n$.

When we want to translate our weak equivalences from last chapter into the world of simplicial presheaves, a first thought might be to require that a natural transformation between simplicial presheaves has to induce weak equivalences on all simplicial sets, regardless of the manifold input. In other words, the natural transformation $\phi : \mathscr{F}_\bullet \to \mathscr{F}'_\bullet$ should induce weak equivalences of simplicial sets $\phi(M) : \mathscr{F}_\bullet(M) \to \mathscr{F}'_\bullet(M)$ for all $M \in \mathbf{Man}$. This, however, is actually too much, because one of the advantages of working with presheaves is that all information in a way is gathered in the stalks, and as discussed earlier, these can be defined for manifolds using only open balls. When some presheaf sends a manifold to some simplicial set, this has to commute nicely with restricting to any chart domain in the manifold, which can be chosen to be homeomorphic to a ball in $\mathbb{R}^n$ for the appropriate dimension n. This is then picked up by and stored in the n-dimensional stalk, collapsed together in an equivalence class with the information from all other manifolds that restrict in the same way. Since any point in any manifold lies in such a chart domain, the stalks actually capture the entire picture, and it is enough to require that a weak equivalence is induced on them. We therefore make the following definition:

Definition 5.3. *A morphism (natural transformation) of simplicial presheaves is called a weak equivalence if the induced map on all stalks are weak equivalences of simplicial sets. If such a morphism exists between two simplicial presheaves, they are said to be weakly equivalent.*

In the same way that we could embed manifolds in the cateogory of presheaves by using their Hom-functors, we want to construct simplicial presheaves from them. First, we introduce the so-called simplicial manifolds, which are simplicial sets in which every set is a manifold, and each morphism is a smooth map from $\mathbf{Man}$. Equivalently, it is a functor from Δ^{op} to $\mathbf{Man}$. If we call it $X_\bullet$, it can be visualized like this:

$$X_0 \;\substack{\longleftarrow \\ \dashrightarrow \\ \longleftarrow}\; X_1 \;\substack{\longleftarrow \\ \dashrightarrow \\ \longleftarrow \\ \dashrightarrow \\ \longleftarrow}\; X_2 \cdots$$

Now we get a simplicial presheaf $\mathscr{F}_{X_\bullet}$ associated to $X_\bullet$ by assigning to any $M \in \mathbf{Man}$ the simplicial set

$$\mathbf{Man}(M, X_0) \;\substack{\longleftarrow \\ \dashrightarrow \\ \longleftarrow}\; \mathbf{Man}(M, X_1) \;\substack{\longleftarrow \\ \dashrightarrow \\ \longleftarrow \\ \dashrightarrow \\ \longleftarrow}\; \mathbf{Man}(M, X_2) \cdots$$

Here, the morphisms have also been transported by the covariant functor $\mathbf{Man}(M, -)$, so the diagram makes sense. The functoriality makes sure that all relations between the morphisms are preserved, so we do indeed end up with a simplicial set. Now, if we are given any ordinary manifold $X \in \mathbf{Man}$, we can construct a discrete simplicial manifold $X = X_\bullet$ like this

$$X \;\substack{\longleftarrow \\ \dashrightarrow \\ \longleftarrow}\; X \;\substack{\longleftarrow \\ \dashrightarrow \\ \longleftarrow \\ \dashrightarrow \\ \longleftarrow}\; X \cdots$$

where as usual when using the word discrete, all sets of simplices are X, and all morphisms are the identity. Then $\mathscr{F}_X = \mathscr{F}_{X_\bullet}$ is our induced simplicial presheaf, constructed from X's simplicial manifold. Notice that it will be a sequence of copies of the ordinary presheaf $\mathbf{Man}(-, X)$ that we worked with in Chapter 3, with all natural transformations being the identity.

If we take two smooth manifolds X and Y, we notice that all natural transformations ϕ going between the discrete simplicial sheaves $\mathscr{F}_{Y_\bullet}$ and $\mathscr{F}_{X_\bullet}$ have to have a natural transformation of the ordinary presheaves $\mathscr{F}_Y$ and $\mathscr{F}_X$ as the map going between their 0-simplices (since these are exactly $\mathscr{F}_Y$ and $\mathscr{F}_X$). Since all maps within the simplicial presheaves $\mathscr{F}_{Y_\bullet}$ and $\mathscr{F}_{X_\bullet}$ are the identity, ϕ is also completely determined by what happens down at the 0th level, so we actually get a one-to-one correspondence between transformations $\mathscr{F}_Y \to \mathscr{F}_X$ and possible ϕ. In other words

$$\mathbf{sPre}(\mathscr{F}_{Y_\bullet}, \mathscr{F}_{X_\bullet}) \cong \mathbf{Pre}(\mathscr{F}_Y, \mathscr{F}_X) \cong \mathbf{Man}(Y, X), \tag{37}$$

where the right equation is (25).

We note the fact that if we are given a functor from $\mathbf{Man}^{\mathrm{op}}$ to $\mathbf{Grpd}$, we can compose it with our functor taking groupoids to simplicial sets from Chapter 4, and thus end up with a simplicial presheaf. This will be part of our next construction, which finally is the base space $B_\nabla G$ of our classifying space (it will be a simplicial presheaf, of course, not an actual topological space).

6 $E_\nabla G$, $B_\nabla G$ and the classification theorem

Given any Lie group G and smooth manifold M, we can define a groupoid $B_\nabla G'(M)$ similarly to how we did it in the quite trivial case where G was equal to $\mathbb{Z}_2$. The objects $B_\nabla G'(M)_0$ will be the collection of all principal G-bundles P over M with connection Θ, usually written as triples (P, π, Θ), where $\pi : P \to M$ is the projection. The arrows $B_\nabla G'(M)_1$ going from some (P, π, Θ) to (P', π', Θ') will be the collection of isomorphisms $\phi : P \to P'$ that preserve the connection, i.e. $\phi^*(\Theta') = \Theta$.

$$\begin{tikzcd} P \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & P' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

As all such isomorphisms are invertible, this really does define a groupoid, and it tracks automorphisms nicely, as discussed in the previous chapters.

Definition 6.1. *Given any Lie group G, we define the simplicial presheaf $B_\nabla G$ by setting $B_\nabla G(M)$ equal to the simplicial set associated to the groupoid $B_\nabla G'(M)$ defined above.*

Since $B_\nabla G$ is supposed to be a (contravariant) functor, we also need maps $f : M \to N$ in **Man** to be transferred to maps $B_\nabla G'(f) : B_\nabla G'(N) \to B_\nabla G'(M)$ in **Grpd**, which can then be turned into maps of the simplicial sets with our functor **Grpd** $\to$ **Set**$_\Delta$. Given any triple $(P, \pi, \Theta) \in B_\nabla G'(N)_0$, we can send it to the pullback $(f^*(P), f^*(\pi), f^*(\Theta)) \in B_\nabla G'(M)_0$, and this preserves identity maps by the properties of pullbacks. Also our arrows, the isomorphism diagrams $\phi : P \to P'$ over N, can be pulled back by f to isomorphism diagrams $f^*(P) \to f^*(P')$ over M. Because ϕ preserves the connection between P and P', the connections induced on $f^*(P)$ and $f^*(P')$ will also be preserved, so we end up with an arrow in $B_\nabla G'(M)_1$.

There is, unfortunately, a formal problem with this functor when it comes to associativity. Given maps $f : M \to N$ and $g : N \to L$, and a principal G-bundle $P \to L$, we can define both $(g \circ f)^*(P)$ and $f^*(g^*(P))$, but according to [7], they will only be canonically isomorphic, not equal. However, still according to [7], this can be fixed using either Grothendieck's theory of fibred categories or higher categories, so we do not dwell on it in this thesis.

Proposition 6.2. *The simplicial presheaf $B_\nabla G : \boldsymbol{Man}^{\mathrm{op}} \to \boldsymbol{Set}_\Delta$ is a simplicial sheaf.*

Proof. Fix a smooth manifold M. If we are given two open subsets U_1 and U_2 of M, with a principal G-bundle and a connection on both, both the bundles and the connections could be glued uniquely together on $U_1 \cup U_2$ as long as they agreed on the intersection $U_1 \cap U_2$. This follows because both the bundle-structure and the properties of a connection are local. Similarly, if there were given connection-preserving isomorphisms between two bundles on U_1 and U_2, respectively, they could also be glued if they agreed on $U_1 \cap U_2$, for the same reason. Furthermore, this argument could easily be extended to arbitrary families of sets. This shows that the 0- and 1-simplices in $B_\nabla G(M)$ are sheaves, and by definition, $B_\nabla G$ is built entirely from these. $\square$

Now, the way to classify our G-bundles will actually be to make a pair of simplicial sheaves that looks like a bundle itself, and $B_\nabla G$ will have the role as base space. We now go on to define $E_\nabla G$, which can be seen as the total space in this setting.

As with $B_\nabla G$, we want to define $E_\nabla G$ by first making its corresponding groupoid, $E_\nabla G'$. It is very similar to $B_\nabla G'$, but instead of having triples as nodes, $E_\nabla G'(M)_0$ consists of quadruples (P, π, Θ, s), where $\pi : P \to M$ is a G-bundle with connection Θ and a global section $s : M \to P$. Again, the arrows are isomorphisms over M, that this time preserve the section as well as the connection. The simplicial presheaf $E_\nabla G$ then takes manifolds M to the simplicial set built from the above groupoid, and as with $B_\nabla G$ the smooth maps of manifolds induce pullback-maps (the sections work nicely with them, so all of the above arguments still apply). Since sections also are local by nature and can be glued along open sets, $E_\nabla G$ becomes a simplicial sheaf.

The arrows in $E_\nabla G'$ are worth taking a closer look at. If we have one in $E_\nabla G'(M)_1$, for some $M \in \mathbf{Man}$, it will be an isomorphism $\phi : P \to P'$ over M, which by commutativity has to preserve the fibres. Because ϕ also preserves the section, in other words $\phi \circ s = s'$, this determines where one element per fibre is sent, namely $s(m)$ for $m \in M$. But last of all, ϕ has to respect the G-action, which is free and transitive on every fibre, and this takes care of where the rest of the elements in P are being sent. In other words, there can be maximally one choice of isomorphism between two such G-bundles, and thus no more than one arrow between each object in $E_\nabla G'(M)_0$. In particular, there are no non-trivial automorphisms. But this is what defines a discrete groupoid, which means that we should be able to find a groupoid with dramatically fewer objects that is equivalent to $E_\nabla G'(M)$, and from it get a discrete simplicial sheaf weakly equivalent to $E_\nabla G$. The right choice is simply handed to us in [7] as the discrete simplicial sheaf $\Omega^1 \otimes \mathfrak{g}$, but we have to prove the important weak equivalence that we claim:

Theorem 6.3. *The simplicial sheaf $E_\nabla G$ is weakly equivalent to $\Omega^1 \otimes \mathfrak{g}$, the discrete simplicial sheaf which sends a smooth manifold M to $\Omega^1(M; \mathfrak{g})$.*

As discussed previously, taking 1-forms is a sheaf on manifolds, and taking their values in $\mathfrak{g}$ does not change that. We then upgrade $\Omega^1(-; \mathfrak{g})$ to a discrete simplicial sheaf as shown before, by using that one sheaf for all $i \geq 0$ and the identity for all face and degeneracy maps. Now for the proof:

Proof. We start by finding inverse weak equivalences

$$E_\nabla G(M) \underset{\phi}{\overset{\psi}{\rightleftarrows}} (\Omega^1 \otimes \mathfrak{g})(M) \ = \Omega^1(M; \mathfrak{g})$$

for any $M \in \mathbf{Man}$ (it would be easier to use just one of them to prove this theorem, but we will later need to go back and forth between these simplicial sheaves, so both ϕ and ψ are useful). Again, we prefer to work on the level of groupoids, since both of these simplicial sets come from those ($\Omega^1 \otimes \mathfrak{g}$ is originally a sheaf upgraded to a discrete simplicial sheaf, but can also be seen as the simplicial sheaf that at M is generated from the discrete groupoid $\mathscr{G}$ that has $\mathscr{G}_0 = \Omega(M; \mathfrak{g})$, and only identity arrows).

So to start with ψ, it takes an element $(P, \pi, \Theta, s) \in E_\nabla G'(M)_0$ to the pulled-back 1-form $s^*(\Theta)$. Remember that connections are 1-forms on the total space with values in $\mathfrak{g}$, so this becomes a similar form living on M, the domain of s, and hence an element in $\Omega^1(M; \mathfrak{g})$. We wait before we define what ψ does to the arrows in $E_\nabla G'(M)_1$. The map ϕ takes a form $\alpha \in \Omega^1(M; \mathfrak{g})$ to the trivial bundle $\mathrm{proj}_1 : M \times G \to M$, with the identity section

$$s : M \longrightarrow M \times G \quad m \longmapsto (m, e) \tag{38}$$

where e is the identity element in G. We also need a connection, which we set to be $\Theta = \mathrm{Ad}(\bullet^{-1}) \circ \alpha + \omega_{MC}$, where ω_{MC} is the Maurer-Cartan form on $M \times G$, defined for all trivial bundles in Section 2.1.[1] We use the notation $\mathrm{Ad}(\bullet^{-1})$ to mean the map $\mathrm{Ad}(g^{-1})$ at a point (g, m), and as earlier, it is the differential of the $G \to G$-mapping $x \to gxg^{-1}$ at e. Note that we here consider the first summand, $\mathrm{Ad}(\bullet^{-1}) \circ \alpha$, as a 1-form on all of $TM \times TG$ even though it only depends on input from TM. This quadruple $(M \times G, \mathrm{proj}_1, \Theta, s)$ is certainly in $E_\nabla G'(M)_0$ if we can prove that Θ is a connection. Only identity arrows exist in the groupoid $\Omega^1(M; \mathfrak{g})$, and these are obviously sent to identities in $E_\nabla G'(M)$.

Now to prove that Θ is a connection. First of all, it takes any point $(m, g) \in M \times G$ to the function

$$\Theta_{(m,g)} = \alpha_m + \omega_{MC(m,g)} \in \mathrm{Hom}(T_m M \times T_g G, \mathfrak{g}), \tag{39}$$

and this depends smoothly on the base point, so Θ is indeed an element in $\Omega^1(M \times G, \mathfrak{g})$. For the first part of Definition 2.4, we remember that the G-action is closed on any fibre, which means that the differential function $v_{(m,g)} : \mathfrak{g} \to T_m M \times T_g G$ (of the map $h \to (m, gh)$, taken at the identity e) must map to 0 in the first coordinate (since the function that it is the derivative of never moves along M, only G). This means that

$$\begin{aligned}
\Theta_{(m,g)} \circ v_{(m,g)} &= (\alpha_m + \omega_{MC(m,g)}) \circ v_{(m,g)} \\
&= \alpha_m \circ \circ v_{(m,g)} + \omega_{MC(m,g)} \circ v_{(m,g)} \\
&= 0 + \mathrm{id}_{\mathfrak{g}} \\
&= \mathrm{id}_{\mathfrak{g}},
\end{aligned}$$

[1]The connection used in [7] is actually just $\Theta = \alpha + \omega_{MC}$, but this did not seem to fit the rest of their discussion. Inspired by Proposition 6.8 in [5], I tried composing with $\mathrm{Ad}(\bullet^{-1})$, and this yielded the desired weak equivalence, and also seems to fit better with the later discussion about how the group $\mathbf{Man}(M, G)$ acts on $\Omega^1 \otimes \mathfrak{g}$. It is of course possible that everything is computed slightly differently in [7], including using a less intuitive group action, but I could not see it.

as required.

For the second part of our definition, we use the equivalent statement $(2')$, found in Proposition 2.5 (notice that we can use it, because we already have proven the first part of Definition 2.4). It requires that the differential $(R_g)_*$ of the right action preserves the kernel of Θ, so we first start with any tangent vector $[v_1, v_2] \in T_mM \times T_gG$ that satisfies

$$\Theta_{(m,g)}([v_1, v_2]) = \mathrm{Ad}(g^{-1}) \circ \alpha(v_1) + \omega_{MC(m,g)}([v_1, v_2]) = 0, \tag{40}$$

or, equivalently

$$\mathrm{Ad}(h^{-1}g^{-1}) \circ \alpha(v_1) = -\,\mathrm{Ad}(h^{-1}) \circ \omega_{MC(m,g)}([v_1, v_2]), \tag{41}$$

where we have composed on both sides with $\mathrm{Ad}(h^{-1})$ for some $h \in G$, which is always a diffeomorphism with inverse $\mathrm{Ad}(h)$. Both this last fact and the composition rules for the adjoint are immediate from its definition.

We now want to see what Θ does to $(R_h)_*([v_1, v_2])$, for any $h \in G$. The map R_h is as always the right action with the element h fixed, going from $M \times G$ to $M \times G$, but since the action only affects the second coordinate in trivial bundles, the differential $(R_h)_*$ must be the identity on TM. In other words, $(R_h)_*([v_1, v_2]) = ([v_1, (R_h)_*(v_2)])$, and we compute:

$$\begin{aligned}
&\Theta_{(m,gh)}((R_h)_*([v_1, v_2])) \\
&\quad = \mathrm{Ad}(h^{-1}g^{-1}) \circ \alpha(v_1) + \omega_{MC(m,gh)}([v_1, (R_h)_*(v_2)]) \\
&\quad = -\,\mathrm{Ad}(h^{-1}) \circ \omega_{MC(m,g)}([v_1, v_2]) + \omega_{MC(m,gh)}([v_1, (R_h)_*(v_2)]) \\
&\quad = -(L_{h^{-1}} \circ R_h)_* \circ \mathrm{proj}_2^*((L_{g^{-1}})_*)([v_1, v_2]) + \mathrm{proj}_2^*((L_{h^{-1}g^{-1}})_*) \circ (R_h)_*([v_1, v_2]) \\
&\quad = -(L_{h^{-1}} \circ R_h \circ L_{g^{-1}})_*(v_2) + (L_{h^{-1}} \circ L_{g^{-1}} \circ R_h)_*(v_2) \\
&\quad = 0
\end{aligned}$$

Here, the map $L_g : G \to G$ is the left multiplication, but is defined inside of G, whereas R_g denotes the right group action. In our trivial case, however, R_g has the exact same effect if viewed as the right multiplication within G, and is therefore considered that way in the penultimate line above. Because multiplying from the right commutes with multiplying from the left, the differential maps also commute, so we really get 0 in the last line. To get from the second to the third line, (41) was used. Everything here is reversible, so we end up with

$$(R_h)_*(\mathrm{Ker}(\Theta)) = \mathrm{Ker}(\Theta) \quad \forall h \in G, \tag{42}$$

as desired. Consequently, Θ is a connection on $M \times G$.

Now, we need to show that ϕ and ψ are inverse weak equivalences. We start with $\psi \circ \phi$, which is relatively easy: Our definitions give us that

$$\psi \circ \phi(\alpha) = s^*(\mathrm{Ad}(\bullet^{-1}) \circ \alpha + \omega_{MC}), \tag{43}$$

where $s : M \to M \times G$ is the identity section on the trivial bundle, sending $m \longmapsto (m, e)$. The differential becomes $s_* : v \longmapsto (v, 0)$, so this is a 1-form which at an element $m \in M$ returns the map

$$\begin{aligned}(\psi \circ \phi(\alpha))_m : T_m M &\longrightarrow \mathfrak{g} \\ v &\longmapsto \mathrm{Ad}(e) \circ \alpha_m(v) + \omega_{MC(m,e)}(v,0) = \alpha_m(v),\end{aligned}$$

i.e. $\psi \circ \phi(\alpha)$ is equal to α. Again we use that ω_{MC} only depends on the second coordinate, and basic properties of the adjoint mapping.

When we turn to $\phi \circ \psi$, we will not be able to obtain the identity functor, but since a weak equivalence of groupoids is the same as an equivalence of categories, an equivalence is all we need. Starting with a quadruple $(P, \pi, \Theta, s) \in E_\nabla G(M)_0$, $\phi \circ \psi$ takes it to $(M \times G, \mathrm{proj}_1, \mathrm{Ad}(\bullet^{-1}) \circ s^*(\Theta) + \omega_{MC}, i)$, where $i : M \to M \times G$ is the identity section. We now show that this new G-bundle is isomorphic to the one we started with, by an isomorphism that preserves both the section and the connection. It is the following:

$$\begin{tikzcd} M \times G \arrow[rr, "f"] \arrow[dr, shift right, "\mathrm{proj}_1"'] & & P \arrow[dl, shift left, "\pi"] \\ & M \arrow[ul, shift right, "i"'] \arrow[ur, shift left, "s"] & \end{tikzcd} \qquad f(m,g) = s(m) \cdot g$$

We see immediately that f is smooth, as a composition of the smooth section and the action map, and equivariant with respect to the G-action. As mentioned in the first chapter, this is enough to get an isomorphism, since equivariance guarantees bijectivity on the fibres. We also get the section preserved, since

$$f \circ i(m) = f(m,e) = s(m) \quad \forall m \in M. \tag{44}$$

The connection, however, is not as easy. To see what the pullback of Θ is, we first need to break f up in a slightly strange way that will help us find its differential map:

$$M \times G \xrightarrow{\begin{bmatrix} s & 0 \\ 0 & \mathrm{id} \end{bmatrix}} P \times G \xrightarrow{\begin{bmatrix} \mathrm{id} & 0 \\ 0 & L_{h^{-1}} \end{bmatrix}} P \times G \xrightarrow{\begin{bmatrix} \mathrm{id} & 0 \\ 0 & L_h \end{bmatrix}} P \times G \xrightarrow{a} P$$

where $a : P \times G \to P$ is the G-action on P, and h is any element in G. If we now fix a point (m,g) in $M \times G$, $f^*(\Theta)$ will yield the function

$$\begin{aligned}(f^*\Theta)_{(m,g)} : T_m M \times T_g G &\longrightarrow \mathfrak{g} \\ [v_1, v_2] &\longmapsto \Theta_{s(m)g}(f_*(v_1) + f_*(v_2)),\end{aligned}$$

so we need to get a nice expression for f_*. The first three parts of the composition we made out of f above are easy to differentiate, and if we fix g, we see that a becomes $a(-,g) : P \to P$, which is simply R_g from before. The first summand is therefore

$$\begin{aligned}\Theta_{s(m)g}(f^*(v_1)) &= \Theta_{s(m)g}((R_g \circ s)_*(v_1)) \\ &= (R_g^*\Theta)_{s(m)}(s_*(v_1)) \\ &= \mathrm{Ad}(g^{-1}) \circ \Theta_{s(m)}(s_*(v_1)) \\ &= \mathrm{Ad}(g^{-1}) \circ (s^*\Theta)_m(v_1)\end{aligned}$$

where the third line follows from the second by definition since Θ is a connection. We also use the chain rule a lot without pointing it out.

For the second summand, we treat the last two maps in our chain as a single one, and if we set $h = g$ and fix some $s(m) \in P$, this composition $a(s(m), -) \circ L_g$ will be a map $G \to P$, sending $k \mapsto s(m)gk$, where $k \in G$. Starting f at our base point (m, g), we notice that it changes to $(s(m), e)$ after the two first maps, so when we differentiate, $(a(s(m), -) \cdot L_g)_*$ will be going from $\mathfrak{g}$ to $T_{s(m)g}P$, and be exactly the map $v_{s(m)g}$, which shows up in the definition of a connection. This results in

$$\begin{aligned}\Theta_{s(m)g}(f^*(v_2)) &= \Theta_{s(m)g}(v_{s(m)g} \circ (L_{g^{-1}})_*(v_2)) \\ &= \Theta_{s(m)g} \circ v_{s(m)g} \circ (L_{g^{-1}})_*(v_2) \\ &= \mathrm{id}_{\mathfrak{g}} \circ (L_{g^{-1}})_*(v_2) \\ &= \omega_{MC(m,g)}(v_2)\end{aligned}$$

where Θ is the left inverse of $v_{s(m)g}$ because of the defining properties of connections, and the last part merely is the definition of the Maurer-Cartan form on any trivial bundle (technically, the projection should be in there, but we have already simplified by only inputting the part of the tangent vector that comes from TG, so this works out nicely).

Summing up, we see that

$$(f^*\Theta)_{(m,g)} = \mathrm{Ad}(g^{-1}) \circ (s^*(\Theta))_m(v_1) + \omega_{MC(m,g)}(v_2) \tag{45}$$

holds for every choice of (m, g), but this is exactly the connection we have on our G-bundle $M \times G$. Thus f really is an isomorphism of G-bundles that preserves both section and connection.

With this done, we can finally make ψ into a functor. Any arrow in $E_\nabla G'(M)_1$ is an isomorphism, say between the bundles P and P' (ignoring for now the rest of the quadruples), and since $\phi \circ \psi$ sends them both to trivial bundles $(M \times G) \cong P$ and $(M \times G)' \cong P'$, these trivial bundles also have to be isomorphic. But they both have identity sections, so this determines that their isomorphism must be the identity, again forcing them to have the same connection (since their isomorphism preserves it). Therefore, they come from the same element $\alpha \in \Omega^1(M; \mathfrak{g})$ (the earlier fact that $\psi \circ \phi = \mathrm{id}$, shows that ϕ is injective), so P and P' are sent to the same element by ψ. Then their isomorphism can be sent to the identity on α.

That means that we finally have two functors, whose one composition results in the identity functor, while the other sends all objects to isomorphic ones. It is easy to show that this is naturally isomorphic to the identity functor on $E_\nabla G'(M)$, so we get what we claimed. The weakly equivalent groupoids $E_\nabla G'(M)$ and $\Omega^1(M;\mathfrak{g})$ induce weakly equivalent simplicial sets $E_\nabla G(M)$ and $\Omega^1(M;\mathfrak{g})$ for any $M \in \mathbf{Man}$, and in particular when we choose M to be an n-dimensional ball $B^n(r) \in \mathbb{R}^n$ of radius $r > 0$ around the origin. To get weakly equivalent simplicial sheaves, we need a weak equivalence on the colimits of these balls, namely the stalks, but here we simply reference a result saying that the colimit of weak equivalences always is a weak equivalence for filtered colimits, which is what we have here (this is not part of what I have worked with in this thesis, but the result can be found in [4]).

□

Now that we have both a base space $B_\nabla G$ and total space $E_\nabla G$, we want to make them into something similar to a principal G-bundle, but in the category of simplicial sheaves. To do this, we first define what is meant by a group action in this setting, and then find a base space weakly equivalent to $B_\nabla G$ that is easier to work with.

Definition 6.4. *If $\mathscr{F}$ is a sheaf and G a Lie group, a G-action on $\mathscr{F}$ is a sheaf map (natural transformation)*

$$a : \mathscr{F}_G \times \mathscr{F} \longrightarrow \mathscr{F}, \tag{46}$$

where $\mathscr{F}_G$ as usual is the sheaf $\boldsymbol{Man}(-,G)$, where G for the moment is considered simply a manifold. When inputting a test manifold M, we require the resulting map

$$a(M) : \boldsymbol{Man}(M,G) \times \mathscr{F}(M) \longrightarrow \mathscr{F}(M) \tag{47}$$

to be an action of the group $\boldsymbol{Man}(M,G)$ on the set $\mathscr{F}(M)$ (the set $\boldsymbol{Man}(M,G)$ can easily be seen to be a group where the operation is multiplication of the functions, which is easy to define as G already is a group).

Just like we constructed a groupoid from an ordinary group action on a set, we can make one here whenever we input some manifold M. As before, it will have the set $\mathscr{F}(M)$ as objects, and $\mathbf{Man}(M,G)$ as arrows. From this groupoid, we know how to obtain a simplicial set by considering compositions of maps as higher simplices, and if all this is done without first fixing any M, we obtain a simplicial sheaf that can be expressed in the following way:

$$\begin{tikzcd} \mathscr{F} \arrow[r, dashed] & \mathscr{F}_G \times \mathscr{F} \arrow[l, shift right=2, "p_1"'] \arrow[l, shift left=2, "p_2"] \arrow[r, dashed, shift left=1] \arrow[r, dashed, shift right=1] & \mathscr{F}_G \times \mathscr{F}_G \times \mathscr{F} \cdots \arrow[l, shift right=3] \arrow[l] \arrow[l, shift left=3] \end{tikzcd}$$

As before, the two leftmost left-going maps are the projection and the group action. We will not show explicitly that this construction really results in a simplicial sheaf, but since $\mathscr{F}$ is a sheaf and the G-action was defined naturally, it is not difficult to see that maps of manifolds will result in maps between the lower simplices. Since the whole simplicial set is built from the 0-simplices and the arrows, this suffices.

Now, to get back to the problem of considering $E_\nabla G$ as a total space over $B_\nabla G$, we remember that the only thing separating these two simplicial sheaves is the choice of a section in $E_\nabla G$, which is equivalent to a trivialization of the bundle by basic G-bundle theory. It would therefore maybe make more sense to work with a base space $B_\nabla^{triv} G$ of trivializable bundles, but without any specific trivialization being chosen. Then, for any smooth manifold M, $B_\nabla^{triv} G(M)$ would be a triple like the ones found in $B_\nabla G(M)$, but with the added condition that the bundles could all be trivialized. This, however, does not result in a simplicial sheaf, only a simplicial presheaf, since trivializability is a very local property (in fact, local instead of global trivialization is more or less the whole point of working with manifolds). There is, however, another alternative, as presented in [7]: If we take a principal G-bundle $P \to M$ with a given trivialization $s : M \to P$, any other trivialization $s' : M \to P$ must be given by $s \cdot g = s'$, where $g : M \to G$ is a smooth map and the multiplication is the right group action of G on P. This g is possible to find since the G-action is transitive on all fibres, and unique because it is free. But the set of such maps g is exactly $\mathscr{F}_G(M) = \mathbf{Man}(M, G)$, so we can think of $\mathscr{F}_G(M)$ as acting on the 0-simplices of $E_\nabla G(M)$. The orbits of this action will represent trivializable G-bundles with connection over M, but with the specific trivialization (given by the section) "moded" out. As it turns out, it is easier to work with our weakly equivalent simplicial sheaf $\Omega^1 \otimes \mathfrak{g}$, which we can consider as an ordinary sheaf since it is discrete. For those, we have already defined group actions, in Definition 6.4. We next show that we really get a $\mathscr{F}_G$-action on $\Omega^1 \otimes \mathfrak{g}$, using our weak equivalences from Theorem 6.3.

Proposition 6.5. *If G is any Lie group, the natural transformation*

$$T : \mathscr{F}_G \times (\Omega^1 \otimes \mathfrak{g}) \longrightarrow \Omega^1 \otimes \mathfrak{g}, \tag{48}$$

which for any smooth manifold M first sends the element $\alpha \in \Omega^1(M; \mathfrak{g})$ to $E_\nabla G_0$ with ϕ, then multiplies $g \in \mathscr{F}_G(M) = \boldsymbol{Man}(M, G)$ with the section, and finally moves back to $\Omega^1(M; \mathfrak{g})$ with ψ, is a G-action of sheaves.

Proof. First, we find a more explicit expression for what T does to an element $(g, \alpha) \in \mathbf{Man}(M, G) \times \Omega^1(M; \mathfrak{g})$, for some $M \in \mathbf{Man}$ (we also denote $T(g, \alpha)$ as $\alpha \cdot g$, of course, since it is an action). Following our three steps, we get:

$$\begin{aligned}
\alpha &\mapsto (M \times G, \mathrm{proj}_1, \mathrm{Ad}(\bullet^{-1}) \circ \alpha + \omega_{MC}, i) \\
&\mapsto (M \times G, \mathrm{proj}_1, \mathrm{Ad}(\bullet^{-1}) \circ \alpha + \omega_{MC}, i \cdot g) \\
&\mapsto (i \cdot g)^*(\mathrm{Ad}(\bullet^{-1}) \circ \alpha + \omega_{MC}) \\
&= \mathrm{Ad}(g^{-1}) \circ \alpha + g^* \omega_{MC}
\end{aligned}$$

Again, i is the identity section, and notice that Ad now depends on the function g, not a single group element. We use ω_{MC} for the Maurer-Cartan form on

$M \times G$, but it is in the last line regarded as the Maurer-Cartan form restricted to G. Being slightly sloppy with this canonical form's domain should not lead to any problems, as it is determined solely by the G-coordinate, and also sends vectors from TM to 0. It is preserved nicely along inclusions and projections when working with trivial bundles. The last equality is easily obtainable by noticing that $i \cdot g$ sends m to $(m, g(m))$, which gives the differential $(i \cdot s)_* = (\mathrm{id}_T\, M, g_*)$, and remembering that $\alpha_{(m,h)}$ only depends on input from $T_m M$.

To show that T is a natural transformation, we take another smooth manifold N, and a smooth map $f : N \to M$. Remembering that $\Omega^1 \times \mathfrak{g}$ takes maps to pullbacks, and $\mathscr{F}_G$ uses pre-compositions, we get the commutative diagram

$$\begin{tikzcd}
\mathbf{Man}(M, \mathfrak{g}) \times \Omega^1(M; \mathfrak{g}) \arrow[r, "T(M)"] \arrow[d, "- \circ f \times f^*(-)"'] & \Omega^1(M; \mathfrak{g}) \arrow[d, "f^*(-)"] \\
\mathbf{Man}(N, \mathfrak{g}) \times \Omega^1(N; \mathfrak{g}) \arrow[r, "T(N)"] & \Omega^1(N; \mathfrak{g})
\end{tikzcd}
\qquad
\begin{tikzcd}
(g, \alpha) \arrow[r, mapsto] \arrow[d, mapsto] & (\mathrm{Ad}(g^{-1}) \circ \alpha + g^*\omega_{MC}) \arrow[d, mapsto] \\
(g \circ f, f^*(\alpha)) \arrow[r, mapsto] & f^*(\mathrm{Ad}(g^{-1}) \circ \alpha + g^*\omega_{MC})
\end{tikzcd}$$

The only transition that is not immediate from the definition is the bottom one, but

$$\begin{aligned}
[T(N)](g \circ f, f^*(\alpha)) &= \mathrm{Ad}(g \circ f) \circ f^*(\alpha) + (g \circ f)^*(\omega_{MC}) \\
&= f^*(\mathrm{Ad}(g) \circ \alpha) + f^*(g^*(\omega_{MC}))
\end{aligned}$$

using basic properties of pullbacks. We therefore have a natural transformation, but still need to see that the action of $\mathbf{Man}(M, G)$ on $\Omega^1(M; \mathfrak{g})$ is a group action for any $M \in \mathbf{Man}$. First of all, the unit function $e : M \to G$, which maps all m to the unit e in G, has differential 0 like any constant function. We therefore get

$$\alpha \cdot e = \mathrm{Ad}(e^{-1}) \circ \alpha + e^*\omega_{MC} = \mathrm{id}_{\mathfrak{g}} \circ \alpha + 0 = \alpha, \tag{49}$$

as desired. Compatibility is a tad nastier, but we compute

$$\begin{aligned}
\alpha \cdot (gh) &= \mathrm{Ad}((gh)^{-1}) \circ \alpha + (gh)^*(\omega_{MC}) \\
(\alpha \cdot g) \cdot h &= \mathrm{Ad}(h^{-1}) \circ \mathrm{Ad}(g^{-1}) \circ \alpha + \mathrm{Ad}(h^{-1}) \circ g^*(\omega_{MC}) + h^*(\omega_{MC}),
\end{aligned}$$

where the first summand in both expressions are equal by properties of the adjoint. Cancelling those, we take a closer look at $(gh)^*(\omega_{MC})$, starting by splitting $gh : M \to G$ into two parts:

$$M \xrightarrow{\begin{bmatrix} g \\ h \end{bmatrix}} G \times G \xrightarrow{- \cdot -} G$$

Here, the last part is group multiplication. The differential of this multiplication is that of right multiplication for our left element, and vice versa. We therefore obtain

$$(gh)_* = (R_h)_* \circ g_* + (L_g)_* \circ h_* \tag{50}$$

If we now remember the definition of the Maurer-Cartan form, we see that

$$\begin{aligned}(gh)^*\omega_{MC} &= (L_{(gh)^{-1}})_*((R_h)_* \circ g_* + (L_g)_* \circ h_*) \\ &= (L_{h^{-1}g^{-1}})_*((R_h)_* \circ g_* + (L_g)_* \circ h_*) \\ &= (L_{h^{-1}} \circ R_h \circ L_{g^{-1}} \circ g)_* + (L_{h^{-1}} \circ \mathrm{id}_{TG} \circ h)_* \\ &= \mathrm{Ad}(h^{-1}) \circ g^*(\omega_{MC}) + h^*(\omega_{MC}),\end{aligned}$$

which is exactly what we needed. The base points in the above calculations would be horrible to juggle, but luckily, the global formalism suffices, and we can ignore them.

□

With this in place, we can define $B_\triangledown^{triv}G$ as the simplicial sheaf induced by this group action, namely

$$\Omega^1 \otimes \mathfrak{g} \underset{p_2}{\overset{p_1}{\leftleftarrows}} \mathscr{F}_G \times \Omega^1 \otimes \mathfrak{g} \leftleftarrows \mathscr{F}_G \times \mathscr{F}_G \times \Omega^1 \otimes \mathfrak{g} \cdots$$

It is a sheaf because both $\Omega^1 \otimes \mathfrak{g}$ and $\mathbf{Man}(-, G)$ are, just like we wanted. The whole idea was, however, to find a more understandable simplicial sheaf to replace, $B_\triangledown G$, so we really need a weak equivalence for our new construction to have any value. Again we do this slightly differently from [7], because it seems to work better with the calculations. We define the map

$$\zeta : B_\triangledown^{triv}G' \longrightarrow B_\triangledown G', \tag{51}$$

again using groupoids since both these simplicial sheaves are built from them (we use $B_\triangledown^{triv}G'$ to denote the groupoid behind $B_\triangledown^{triv}G$). Given a fixed smooth manifold M, ζ_M maps an element $\alpha \in \Omega^1(M; \mathfrak{g})$ to the triple $(M \times G, \mathrm{proj}_1, \mathrm{Ad}(\bullet^{-1})) \circ \alpha + \omega_{MC})$. An arrow in $B_\triangledown^{triv}G'$ is a group action by some element $g \in \mathbf{Man}(M, G)$, and this is sent to the isomorphism $\zeta_M(g) : M \times G \to M \times G$ that sends $(m, h) \mapsto (m, g(m)^{-1}h)$, which is an arrow in $B_\triangledown G'(M)$ if it preserves the connection. We prove that everything works out nicely.

Proposition 6.6. *The map $\zeta_M : B_\triangledown^{triv}G'(M) \longrightarrow B_\triangledown G'(M)$ of groupoids described above is a weak equivalence for every $M \in$ **Man**, and therefore induces a weak equivalence of the simplicial sheaves $B_\triangledown^{triv}G$ and $B_\triangledown G$.*

Proof. We first show that ζ_M actually maps arrows g from $\mathbf{Man}(M, G)$ to the right kind of isomorphisms (that they are isomorphisms is evident by the now well-known argument of the free and transitive group action). Starting with α, the action from g sends it to $\mathrm{Ad}(g^{-1}) \circ \alpha + g^*\omega_{MC}$, and we want that the isomorphism $\zeta_M(g)$ should go between the two copies of the trivial bundle in the way that works well with pullbacks of the two connections. As it turns out, it is easier to show this using the inverse function $\zeta_M^{-1}(g) : (m, h) \mapsto (m, g(m)h)$, which means that we want to prove

$$(\zeta_M^{-1}(g))(\mathrm{Ad}(\bullet^{-1}) \circ \alpha + \omega_{MC}) = \mathrm{Ad}(\bullet^{-1}) \circ (\mathrm{Ad}(g^{-1}) \circ \alpha + g^*\omega_{MC}) + \omega_{MC} \tag{52}$$

Fixing a point $(m, h) \in M \times G$, we get the differential

$$(\zeta_M^{-1}(g)_*)_{(m,h)} = \begin{bmatrix} \mathrm{id} & 0 \\ (R_h \circ g)_* & (L_{g(m)})_* \end{bmatrix} \tag{53}$$

Taking $[v_1, v_2] \in T_m M \times T_g G$ and remembering that (m, h) lands in $(m, g(m)h)$, we compute the pullback:

$$\begin{aligned}
&(\zeta_M^{-1}(g))^*(\mathrm{Ad}(\bullet^{-1}) \circ \alpha + \omega_{MC})([v_1, v_2]) \\
&\qquad = \mathrm{Ad}((gh)^{-1})\alpha(v_1) + (L_{(gh)^{-1}})_* \circ (R_h \circ g)_*(v_1) + (L_{(gh)^{-1}})_* \circ (L_{g(m)})_*(v_2) \\
&\qquad = \mathrm{Ad}(h^{-1}) \circ \mathrm{Ad}(g^{-1}) \circ \alpha(v_1) + \mathrm{Ad}(h^{-1}) \circ (L_{g^{-1}})_* \circ g_*(v_1) + (L_{h^{-1}})_*(v_2) \\
&\qquad = \mathrm{Ad}(h^{-1}) \circ \mathrm{Ad}(g^{-1}) \circ \alpha(v_1) + \mathrm{Ad}(h^{-1}) \circ (g^*\omega_{MC})(v_1) + \omega_{MC}(\alpha)
\end{aligned}$$

We see that this agrees with Equation (52). Note that g in this calculation is a function dependant on M, even if we do not write $g(m)$ everywhere. The base points have also been ignored for the sake of simplicity, but everything should be completely unambiguous, and works out nicely.

Now that we know that ζ_M is a functor for all $M \in \mathbf{Man}$, we take a look at what happens when the manifold input is an n-dimensional ball $B = B^n(r)$ of radius $r > 0$ in $\mathbb{R}^n$. Because B is contractible, any principal G-bundle over it is trivializable, so any object in $B_\nabla G'(B)$ is isomorphic to a trivial one, $(B \times G, \mathrm{proj}_1, \Theta)$. But in the proof of Theorem 6.3, we saw that trivial principal G-bundles with connection were preserved by the map $\phi \circ \psi$, which means that ϕ is surjective on them. But ζ is just ϕ without construction of the identity section, so that means that any $(B \times G, \mathrm{proj}_2, \Theta)$ can be hit with ζ_B. In other words, ζ_B can hit something isomorphic to anything in $B_\nabla G'(B)$, and is therefore dense. Now for the arrow sets, we know from the first chapter that any automorphism of a trivial principal G-bundle $B \times G$ is given by a unique $g \in \mathbf{Man}(M, G)$ as $(m, h) \mapsto (m, g(m)h)$, which means that ζ_B is full and faithful as well. In conclusion, ζ_B becomes an equivalence of the groupoids $B_\nabla^{triv} G'(B)$ and $B_\nabla G'(B)$, and thus induces a weak equivalence between the simplicial sets $B_\nabla^{triv} G(B)$ and $B_\nabla G(B)$. As mentioned towards the end of the proof of Theorem 6.3, this is enough to get a weak equivalence on the stalks, so $B_\nabla^{triv} G$ is weakly equivalent to $B_\nabla G$ as simplicial sheaves.

□

6.1 The classification theorem

All this means that we, up to our weak equivalences, can consider $E_\nabla G$ as some sort of principal G-bundle of simplicial sheaves over $B_\nabla G$, where the projection map just takes a quadruple to the triple where the section has been left out (remember how $B_\nabla G$ and $E_\nabla G$ are defined). To mirror manifolds even closer, we even construct a connection, which should of course be a $\mathfrak{g}$-valued 1-form on $E_\nabla G$, or in symbols, a map

$$\Theta^{\mathrm{univ}} : E_\nabla G \to \Omega^1 \otimes \mathfrak{g}. \tag{54}$$

Fortunately, we already have a map like this, namely the ψ from the proof of Theorem 6.3, and simply set $\Theta^{\text{univ}} = \psi$. We will call it the universal connection because of the following theorem, which shows that the simplicial sheaves we have constructed in this chapter really make up a classifying space.

Theorem 6.7. *Given a Lie group G and a principal G-bundle $\pi : P \to X$ with connection $\Theta \in \Omega^1(P;\mathfrak{g})$, there is a unique (up to homotopy) classifying map*

$$\begin{tikzcd} \mathscr{F}_P \arrow[r, "f"] \arrow[d, "\pi"'] & E_\nabla G \arrow[d] \\ \mathscr{F}_X \arrow[r, "\bar{f}"] & B_\nabla G \end{tikzcd}$$

such that $f^*(\Theta^{univ}) = \Theta$.

Here, $\mathscr{F}_P$ and $\mathscr{F}_X$ of course denote the discrete simplicial sheaves constructed from the sheaves $\mathbf{Man}(-,P)$ and $\mathbf{Man}(-,X)$. The expression $f^*(\Theta^{\text{univ}})$ means the composition of f and Θ^{univ}, which becomes a natural transformation

$$f^*(\Theta^{\text{univ}}) : \mathscr{F}_P \longrightarrow \Omega^1 \otimes \mathfrak{g}. \tag{55}$$

Using (24), or essentially the Yoneda lemma, we know that these natural transformations correspond bijectively to the set $(\Omega^1 \otimes \mathfrak{g})(P) = \Omega^1(P;\mathfrak{g})$, which is where we find Θ. It is under this bijection that we claim $f^*(\Theta^{\text{univ}}) = \Theta$.[2]

Proof. To define f, we begin by taking the pullback of our bundle $\pi : P \to X$ along itself, resulting in a principal G-bundle $\pi' : P' \to P$ with connection $\Theta' = \pi^*\Theta$. Using the universal property of pullback diagrams in a clever way, we obtain a canonical section $s' : P \to P'$, as illustrated in the diagram below.

$$\begin{tikzcd} P \arrow[dr, dashed, "s'"] \arrow[drr, bend left, "\text{id}"] \arrow[ddr, bend right, "\text{id}"'] & & \\ & P' \arrow[r, "\pi'"] \arrow[d, "\pi'"'] & P \arrow[d, "\pi"] \\ & P \arrow[r, "\pi"'] & X \end{tikzcd}$$

Note that the outer diagram easily commutes, so there has to be a unique morphism s' to complete it. Commutativity of the diagram makes $\pi' \circ s' = \text{id}_P$, so s' indeed is a section of the bundle $P' \to P$. This means that (P', π', Θ', s') is a 0-simplex in $E_\nabla G(P)$, so for any test manifold M, we can define a map

$$\begin{aligned} f(M) : \mathbf{Man}(M,P) &\longrightarrow E_\nabla G(M) \\ (\phi : M \to P) &\longmapsto \phi^*(P', \pi', \Theta', s') \end{aligned}$$

[2]This is actually not entirely clear from [7], but the only way I could make the claim made in the corresponding Proposition 5.26 work.

This is extendable to a map between the whole simplicial sets because $\mathbf{Man}(M, P)$ is discrete, so all its upward-going identity maps can be sent to the degeneracy maps that add more and more identity automorphism diagrams to the quadruples in $E_\nabla G(M)$, while the downward going identity maps go to their left inverses, that delete the last identity automorphism diagram. That f is a map of simplicial sheaves (i.e. a natural transformation) follows because pullbacks behave well with respect to compositions, so given a map $\xi \in \mathbf{Man}(N, M)$, we get the following commutative diagram

$$\begin{tikzcd} \mathbf{Man}(M,P) \arrow[r, "f(M)"] \arrow[d, "-\circ\xi"'] & E_\nabla G(M) \arrow[d, "\xi^*(-)"] \\ \mathbf{Man}(N,P) \arrow[r, "f(N)"] & E_\nabla G(N) \end{tikzcd} \qquad \begin{tikzcd} (\phi : M \to P) \arrow[r, mapsto] \arrow[d, mapsto] & \phi^*(P', \pi', \Theta', s') \arrow[d, mapsto] \\ (\phi \circ \xi : N \to P) \arrow[r, mapsto] & \xi^*(\phi^*(P', \pi', \Theta', s')) \end{tikzcd}$$

Now that we know that f is a functor, we check that $f^*(\Theta^{\text{univ}}) = \Theta$. First of all, using the Yoneda lemma, we know that $\Theta \in \Omega^1(P; \mathfrak{g})$ corresponds to a natural transformation $\zeta^\Theta : \mathscr{F}_P \to \Omega^1 \otimes \mathfrak{g}$, which evaluated at a smooth manifold M gives

$$\begin{aligned} \zeta^\Theta_M : \mathbf{Man}(M, P) &\longrightarrow \Omega^1(M; \mathfrak{g}) \\ \phi &\longmapsto \Omega^1 \otimes \mathfrak{g}(\phi)(\Theta) = \phi^*\Theta \end{aligned}$$

since $\Omega^1 \otimes \mathfrak{g}$ takes smooth maps to pullbacks. Now we just need to see that the composition of f and Θ^{univ} is the same map at every M. Remembering how Θ^{univ} works, we get

$$\begin{aligned} f^*(\Theta^{\text{univ}})_M : \mathbf{Man}(M, P) &\longrightarrow E_\nabla G(M) \longrightarrow \Omega^1(M; \mathfrak{g}) \\ \phi &\longmapsto \phi^*(P', \pi', \Theta', s') \longmapsto (\phi^* s')^*(\phi^*\Theta') \end{aligned}$$

Now, we need to carefully unravel what this means. First of all, $\Theta' = \pi^*\Theta$, but pulling back a connection along a map of the base spaces actually corresponds to pulling it back along the induced map from the pullback-bundle to the original total space (the domains would not make sense if we tried to do a more direct pullback along π). This means that $\Theta' = \pi'^*\Theta$, since π' is what we called this induced map in our above pullback diagram. The same goes for $\phi^*\Theta'$, which becomes $\phi'^*\Theta'$, where ϕ' is the map of total spaces, as illustrated below.

$$\begin{tikzcd} \phi^* P' \arrow[r, "\phi'"] \arrow[d] & P' \arrow[d, "\pi'"] \\ M \arrow[r, "\phi"] & P \end{tikzcd}$$

The pullback-section $\phi^* s'$ is defined as $(\phi')^{-1} \circ s' \circ \phi$, which works because ϕ' bijectively hits the fibres represented by basepoints hit by ϕ. All in all, this yields

$$
\begin{aligned}
(\phi^* s')^*(\phi^*\Theta') &= ((\phi')^{-1} \circ s' \circ \phi)^* \phi'^*(\pi'^*\Theta) \\
&= \phi^* \circ s'^* \circ ((\phi')^{-1})^* \circ \phi'^* \circ \pi'^*\Theta \\
&= \phi^* \circ s'^* \circ \pi'^*\Theta \\
&= \phi^* \circ \mathrm{id}_P^* \Theta \\
&= \phi^*\Theta
\end{aligned}
$$

which is exacty what we wanted. Note that we use $\pi' \circ s' = \mathrm{id}_P$ from above. All pullbacks in the above calculation are ordinary ones, of 1-forms, so we can use the composition rules.

Now, only uniqueness remains. Let us assume that there exist two maps $f_1, f_2 : \mathscr{F}_P \to E_\nabla G$ which also yield Θ when composed with Θ^{univ}. On some $M \in \mathbf{Man}$, we would then get maps that sent some $\phi \in \mathbf{Man}(M, P)$ to quadruples in $E_\nabla G(M)$, say $(M \times G, \mathrm{proj}_1, \Theta_1, s_1)$ and $(M \times G, \mathrm{proj}_1, \Theta_2, s_s)$ (remember that all principal G-bundles in $E_\nabla G(M)$ are trivializable, because of their global section). Composing with Θ^{univ} produces

$$
s_1^*(\Theta_1) = \Theta^{\mathrm{univ}}(f_1(\phi)) = (f_1^*\Theta^{\mathrm{univ}})(\phi) = \phi^*\Theta = (f_2^*\Theta^{\mathrm{univ}})(\phi) = \Theta^{\mathrm{univ}}(f_2(\phi)) = s_2^*(\Theta_2)
$$

by assumption. We now show that the two quadruples are actually isomorphic, by constructing a map ξ:

$$
\begin{tikzcd}
M \times G \arrow[rr, "\xi"] \arrow[dr, shift left, "\mathrm{proj}_1"'] & & M \times G \arrow[dl, shift left, "\mathrm{proj}_1"] \\
& M \arrow[ul, shift left, "s_1"] \arrow[ur, shift left, "s_2"'] &
\end{tikzcd}
\qquad \xi(m, g) = (m, s_2(m) \cdot s_1(m)^{-1} \cdot g)
$$

We see immediately that ξ is smooth, equivariant and bijective, and commutes with both the projections and the sections. We can construct the pullback connection $\xi^*\Theta_2 = \Theta_1$ on the left version of $M \times G$ to make ξ into a section- and connection-preserving isomorphism, i.e. an arrow in $E_\nabla G$. Since $\xi \circ s_1 = s_2$, we have

$$
s_1^*(\xi^*\Theta_2) = s_2^*\Theta_2 = s_1^*(\Theta_1), \tag{56}
$$

where the rightmost equation was found above. This means that the trivial bundle with section s_1 and connection $\xi^*\Theta_2$ is taken to the same element in $\Omega^1(M; \mathfrak{g})$ as $(M \times G, \mathrm{proj}_1, \Theta_1, s_1)$ by Θ^{univ}, but we saw in the proof of Theorem 6.3 that this makes them isomorphic by a section- and connection-preserving isomorphism, or another arrow in $E_\nabla G$. Then we can compose it with ξ to get an arrow in $E_\nabla G$ between $(M \times G, \mathrm{proj}_1, \Theta_1, s_1)$ and $(M \times G, \mathrm{proj}_1, \Theta_2, s_2)$, so they are isomorphic. In other words, any other choice of $f' : \mathscr{F}_P \to E_\nabla G$ that fulfils the hypothesis of the theorem must map to quadruples isomorphic to the ones we hit with f, but then f and f' would be naturally isomorphic as functors. The simplicial sheaf $\Omega^1 \otimes \mathfrak{g}$ is weakly equivalent to $E_\nabla G$ and would be hit in the same place by the images of f and f', so the map f is unique up to homotopy, as claimed.

To finish up, we define $\bar{f}$ as well. To make the diagram shown in the statement of the theorem commute, we set

$$\begin{aligned} \bar{f}(M) : \mathbf{Man}(M, X) &\longrightarrow B_{\nabla}G(M) \\ \phi &\mapsto \phi^*(P, \pi, \Theta), \end{aligned}$$

which looks a lot like f, but this time just taking a pullback-bundle instead of first constructing a special bundle with section over P. Notice that, if we send an element $\phi \in \mathbf{Man}(M, P)$ down to $\mathbf{Man}(M, X)$, this results in $\pi \circ \phi$, and

$$(\pi \circ \phi)^*(P, \pi, \Theta) = \phi^*(\pi^*(P, \pi, \Theta)) = \phi^*(P', \pi', \Theta) \tag{57}$$

which is exactly taking the image of ϕ by f, and then dropping the section. Therefore, this choice for $\bar{f}$ makes for a commutative diagram. We do not bother to show that $\bar{f}$ is a map of simplicial sheaves, as it mirrors the proof for f itself.

$\square$

With this proof, we have completed our search for a classifying space for principal G-bundles with connection, and seen that our notion of weak equivalence is the correct one to get unique classifying maps. Most constructions have been quite tautological, so using these categories to generalize smooth manifolds seems to be a very good way to obtain this result. Proceeding, we want to study the simplicial sheaves $E_{\nabla}G$ and $B_{\nabla}G$ a little closer, now that we know that they are useful. The approach will be to try to find what corresponds to de Rham complexes in the simplicial case, and computing them for $E_{\nabla}G$ and $B_{\nabla}G$.

7 De Rham complexes of $E_\nabla G$ and $B_\nabla G$

To define de Rham complexes in our setting, which is not only the category **sPre**, but also takes weak equivalences into account, we actually need to go to a homotopy category, which is a category where inverses have been added to all weak equivalences, making them isomorphisms. We do not dwell on the details here, as the prerequisites for understanding this properly are multiple, but it is discussed in Chapter 6 of [7]. What is arrived upon, is a category ho **sPre** and functor $L : \mathbf{sPre} \to \mathrm{ho}\,\mathbf{sPre}$ which sends a weak equivalence $X \to Y$ to an isomorphism $LX \to LY$, and has the universal property that any other "homotopy" category $\mathscr{C}$ with functor $K : \mathbf{sPre} \to \mathscr{C}$ that does the same, can be lifted uniquely through it:

$$\begin{tikzcd} \mathbf{sPre} \arrow[r, "L"] \arrow[dr, "K"'] & \mathrm{ho}\,\mathbf{sPre} \arrow[d, dashed, "!\exists"] \\ & \mathscr{C} \end{tikzcd}$$

The category ho **sPre** is also called the localization of **sPre**. Since the point of this new category is to keep the same objects as in **sPre**, only with many more morphisms between them than before, we will refer to the image of a simplicial presheaf $\mathscr{F}_\bullet$ by L simply as $\mathscr{F}_\bullet$. This is just like how one often refers to equivalence classes by an example element from the relevant class. A problem that arises when we want to work in ho **sPre** is to understand the new Hom-sets that arise between non-isomorphic objects in there, but we get the following result from [7]:

Proposition 7.1. *If $\mathscr{F}_\bullet$ is any simplicial presheaf, and $\mathscr{F}'$ a sheaf regarded as a constant simplicial presheaf, the map*

$$\boldsymbol{sPre}(\mathscr{F}_\bullet, \mathscr{F}') \longrightarrow ho\,\boldsymbol{sPre}(\mathscr{F}_\bullet, \mathscr{F}') \tag{58}$$

is an isomorphism.

Furthermore, with the same condition on $\mathscr{F}'$, we actually get that

$$\mathbf{sPre}(\mathscr{F}_\bullet, \mathscr{F}') \cong \ker\{\ \mathbf{Pre}(\mathscr{F}_0, \mathscr{F}') \rightrightarrows \mathbf{Pre}(\mathscr{F}_1, \mathscr{F}')\ \}, \tag{59}$$

where the ker-part is standard notation for the equalizer of the diagram within. The two maps in this diagram are the ones induced from the two face maps going from $\mathscr{F}_1$ to $\mathscr{F}_0$. We do not prove this in detail either, but remember that a map between these simplicial presheaves is a set of presheaf-maps $\mathscr{F}_i \to \mathscr{F}'_i$ indexed by $i \in \mathbb{N}_0$, which commute with the face and degeneracy maps. Because $\mathscr{F}'$ is constant, $\mathscr{F}'_i$ is equal to $\mathscr{F}'$ for all $i \in \mathbb{N}_0$, and all its internal maps are the identity. Thus, the bottom map $\mathscr{F}_0 \to \mathscr{F}'$ has to be one that, post-composed to either of the two above face maps, results in the same map $\mathscr{F}_1 \to \mathscr{F}'$. Going in the other direction, an entire map of simplicial presheaves can be built uniquely from a map $\mathscr{F}_0 \to \mathscr{F}'$, if it belongs to the above equalizer.

We can now define de Rham complexes for simplicial presheaves, and compute them for $E_\nabla G$ and $B_\nabla G$. We here use Ω^n to denote the discrete simplicial presheaf built from the sheaf Ω^n, for $n \in \mathbb{N}_0$.

Definition 7.2. *Given a simplicial presheaf $\mathscr{F}_\bullet$, we define its de Rham complex to be*

$$ho\, \boldsymbol{sPre}(\mathscr{F}_\bullet, \Omega^0 \to \Omega^1 \to ...) \cong ho\, \boldsymbol{sPre}(\mathscr{F}_\bullet, \Omega^0) \to ho\, \boldsymbol{sPre}(\mathscr{F}_\bullet, \Omega^1) \to ... \tag{60}$$

Since all Ω^n fulfil the requirements in Proposition 7.1, and using (59), these terms can be computed as the equalizers

$$\mathrm{ho}\, \mathbf{sPre}(\mathscr{F}_\bullet, \Omega^n) = \ker\{\ \mathbf{Pre}(\mathscr{F}_0, \Omega^n) \overset{\rho_0}{\underset{\rho_1}{\rightrightarrows}} \mathbf{Pre}(\mathscr{F}_1, \Omega^n)\ \}, \tag{61}$$

where the maps ρ_0 and ρ_1 are induced from the face maps $\mathscr{F}_0 \overset{p_1}{\underset{p_0}{\leftleftarrows}} \mathscr{F}_1$. If $\mathscr{F} = \mathscr{F}_X = \mathrm{Hom}(-, X)$ for some $X \in \mathbf{Man}$, we remember that $\mathbf{Pre}(\mathscr{F}_X, \Omega^n) = \Omega^n(X)$, which produces an even nicer result.

Now, we also need a certain differential graded algebra, the Koszul complex, which is designed using the canonical graded algebras $\bigwedge^\bullet V$ and $\mathrm{Sym}^\bullet V$, with V a real vector space. We do not repeat the whole constructions, but the first of these consists of all alternating k-linear functions from V^k to $\mathbb{R}$, while the second collects all of the symmetric k-linear functions. These functions are graded by the V-dimension of their domains, and equipped with a standard multiplication that preserves alternation and symmetry, respectively.

Definition 7.3. *Given a real vector space V, the Koszul complex* $\mathrm{Kos}^\bullet V$ *is the differential graded algebra*

$$\mathrm{Kos}^\bullet V = \bigwedge\nolimits^\bullet V \otimes \mathrm{Sym}^\bullet V, \tag{62}$$

where for now $\bigwedge^\bullet V$ is graded normally, while $\mathrm{Sym}^n V$ *has degree $2n$ for all $n \geq 0$ (add zeros for odd degrees). The differential d_K of this graded algebra is defined by the relations*

$$d_K(v) = v', \quad d_K(v') = 0, \quad v \in V = \bigwedge\nolimits^1 V, \quad v' \in V = \mathrm{Sym}^1 V \tag{63}$$

The only difference between v and v' in the definition is that they are considered elements in different versions of the same space, but d_K does not really change v. We do not bother to show that the tensor product results in a new graded algebra, but check that our differential behaves as it should. Note that this definition works with the degree of differentials being 1, because we have graded $\mathrm{Sym}^\bullet V$ with the even numbers. It is also enough to define d_K on these elements as $\mathrm{Kos}^\bullet V$ is generated by $V = \bigwedge^1 V$: the wedge part directly from it, and the symmetric part by $\mathrm{Sym}^1 V = V = \mathrm{Im}(d_K)$. Because all differentials on graded algebras must satisfy the graded Leibniz rule, we then get

$$d_K(a \cdot b) = d_K(a) \cdot b + (-1)^{\deg(a)} a \cdot d_K(b), \tag{64}$$

which determines d_K on all elements, and keeps the degree as 1. The last property to check is that $d_K \circ d_K = 0$, but this is obvious on both $\bigwedge^1 V$ and $\mathrm{Sym}^1 V$ from the definition, and if we assume that it holds for all elements of degree less than that of an element $a \cdot b$, where both a and b are of positive degree, we can compute

$$\begin{aligned} d_K \circ d_K(a \cdot b) &= d_K(d_K(a) \cdot b + (-1)^{\deg(a)} a \cdot d_K(b)) \\ &= d_K \circ d_K(a) \cdot b + (-1)^{\deg(d_K(a))} d_K(a) \cdot d_K(b) \\ &\quad + (-1)^{\deg(a)}(d_K(a) \cdot d_K(b) + (-1)^{\deg(a)} a \cdot d_K \circ d_K(b)) \\ &= 0 \cdot b + (1-1) d_K(a) \cdot d_K(b) + a \cdot 0 \\ &= 0 \end{aligned}$$

since $\deg(d_K(a)) = \deg(a) + 1$. Because all elements in $\mathrm{Kos}^\bullet V$ are generated from below, this is enough. We now know that this is a differential graded algebra, and it looks like this:

$$\mathrm{Kos}^\bullet V = \mathbb{R} \to \bigwedge^1 V \to \bigwedge^2 V \oplus \mathrm{Sym}^1 V \to \bigwedge^3 V \oplus \big(\bigwedge^1 V \otimes \mathrm{Sym}^1 V\big) \to \ldots \tag{65}$$

The cohomology of Koszul complexes will later be of interest, and it is actually quite easy to compute. We start by taking a look at the simplest case, $\mathrm{Kos}^\bullet \mathbb{R}$.

Example 7.4. *To get to* $\mathrm{Kos}^\bullet \mathbb{R}$*, we need the two graded algebras that it consists of. Any k-linear function* $f : \mathbb{R}^k \to \mathbb{R}$ *is obviously determined by its value at the point* $(1, \ldots, 1) \in \mathbb{R}$*, and is symmetric because*

$$f(x_1, \ldots, x_i, \ldots, x_j, \ldots x_k) = x_i x_j f(x_1, \ldots, 1, \ldots, 1, \ldots x_k) = f(x_1, \ldots, x_j, \ldots, x_i, \ldots x_k) \tag{66}$$

Therefore, $\mathrm{Sym}^k \mathbb{R} = \mathbb{R}$ *for all* $k \geq 0$*, while* $\bigwedge^k \mathbb{R} = \mathbb{R}$ *only for* $k \in \{0, 1\}$*, and is equal to* 0 *everywhere else. The non-trivial part of* $\bigwedge^\bullet \mathbb{R}$ *of course comes from the fact that alternation not really is a thing in the two lowest dimensions. It is now obvious that* $\mathrm{Kos}^k \mathbb{R} = \mathbb{R}$ *in all dimensions, because we get one copy from* $\mathrm{Sym}^k \mathbb{R}$ *in all even dimensions, and one from* $(\mathrm{Sym}^{\frac{k-1}{2}} \mathbb{R}) \otimes (\bigwedge^1 \mathbb{R})$ *in the odd ones. From the definition of* d_K*, this results in*

$$\mathrm{Kos}^\bullet \mathbb{R} = \mathbb{R} \xrightarrow{0} \mathbb{R} \xrightarrow{1} \mathbb{R} \xrightarrow{0} \mathbb{R} \xrightarrow{1} \mathbb{R} \xrightarrow{0} \mathbb{R} \xrightarrow{1} \cdots$$

which gives the cohomology directly as that of a contractible space:

$$H^k(\mathrm{Kos}^\bullet \mathbb{R}) = \begin{cases} \mathbb{R}, & k = 0 \\ 0 & k > 0 \end{cases} \tag{67}$$

Having this result, we state without going through the bothersome and irrelevant proof that $\mathrm{Kos}^\bullet(V_1 \oplus V_2) \cong \mathrm{Kos}^\bullet V_1 \otimes \mathrm{Kos}^\bullet V_2$ for any finite-dimensional, real vector spaces V_1 and V_2. Using the Kunneth formula, we then get inductively that

$$H^k(\mathrm{Kos}^\bullet V) = \begin{cases} \mathbb{R}, & k = 0 \\ 0 & k > 0 \end{cases} \tag{68}$$

for finite-dimensional vector spaces V.

We can now state two of our main results:

Theorem 7.5. *Given a Lie group G with Lie algebra $\mathfrak{g}$, the de Rham complex of $E_\nabla G$ is* $(\mathrm{Kos}^\bullet \mathfrak{g}^*, d_K)$

Proof. The proof of this is quite long and requires a bunch of new theory, so we do not cover it here. It can be found in Chapter 8 of [7]. □

This graded differential algebra is also called the Weil algebra. Since we only work with finite-dimensional Lie groups, and therefore Lie algebras, we get from the above discussion that $E_\nabla G$ always has trivial de Rham cohomology. Note also the interesting fact that the Koszul algebra is constructed using only the vector space properties of $\mathfrak{g}$ (or, really, its dual), which means that all Lie groups of the same dimension yield identical de Rham complexes, totally disregarding the Lie brackets involved.

As an unexpected treat, we can use this to compute the de Rham complex of the simplicial sheaf Ω^1:

Example 7.6. *Since $\Omega^1 \otimes \mathfrak{g}$ is weakly equivalent to $E_\nabla G$ for any Lie group G, we can pick $G = S^1$, which of course has $\mathbb{R}$ as Lie algebra (since there is only one choice of Lie bracket in the 1-dimensional case, this statement is actually unambiguous, but we do not use the bracket right now anyway). In this case, $E_\nabla S^1$ becomes weakly equivalent to $\Omega^1 \otimes \mathbb{R} = \Omega^1$, so they all have the same de Rham complex. This is equal to* $\mathrm{Kos}^\bullet \mathbb{R}$, *and was computed in the above example together with its cohomology.*

Sadly, this indirect technique is in no obvious way extendable to an algorithm for finding the de Rham complexes of the simplicial sheaves Ω^n for arbitrary $n \in \mathbb{N}_0$.

We also have a result for our simplicial base space, but again refer its proof to [7].

Theorem 7.7. *Given a Lie group G with Lie algebra $\mathfrak{g}$, the de Rham complex of $B_\nabla G$ is $(I_2^\bullet(G), d = 0)$.*

Here, $I_2^\bullet(G)$ denotes the algebra $I^\bullet(G)$ of invariant elements in $\mathrm{Sym}^\bullet \mathfrak{g}^*$, just like we defined it in Definition 2.10, but this time graded by twice the degree. The result is very interesting, for the following reason: If we look at the simplicial version of the Chern-Weil homomorphism, and fix our simplicial sheaf $E_\nabla G$, we get an injective[3] map

$$w(E_\nabla G; -) : I^k(G) \longrightarrow H^{2k}_{\mathrm{dR}}(B_\nabla G) = I_2^{2k}(G) \quad \forall k \in \mathbb{N}. \tag{69}$$

Because this is an injective homomorphism of finite dimensional vector spaces, and Theorem 7.7 states that the codomain equals the domain, it becomes an isomorphism. All the different cohomology classes $w(E_\nabla G; P)$, one for each $P \in I^\bullet(G)$, are the ones associated to $E_\nabla G$ by the characteristic class $w(-; P)$. They can be pulled back via diagrams like the one below:

$$\begin{tikzcd} \mathscr{F}_E \arrow[r, "f"] \arrow[d, "\pi"'] & E_\nabla G \arrow[d] \\ \mathscr{F}_X \arrow[r, "\bar{f}"] & B_\nabla G \end{tikzcd}$$

Here, $\pi : E \to X$ is a principal G-bundle, and when we pull back via $\bar{f}$, we obtain an element in $H^{2k}_{\mathrm{dR}}(\mathscr{F}_X)$, a space which is equal to

$$\begin{aligned} \mathrm{ho}\,\mathbf{sPre}(\mathscr{F}_X, \Omega^{2k}) &= \ker\{\ \mathbf{Pre}(\mathscr{F}_X, \Omega^{2k}) \underset{\mathrm{id}}{\overset{\mathrm{id}}{\rightrightarrows}} \mathbf{Pre}(\mathscr{F}_X, \Omega^{2k})\ \} \\ &= \mathbf{Pre}(\mathscr{F}_X, \Omega^{2k}) \\ &= \Omega^{2k}(X) \end{aligned}$$

These equalities come from the de Rham-definition, the fact that the equalizer in a diagram like the one above always equals the domain, and finally (24). Everything is natural all the way, so the element $\bar{f}^*(w(E_\nabla G; P))$ is equal to $w(E; P)$, which comes from the ordinary Chern-Weil homomorphism on the bundle E (this is how characteristic classes are supposed to work). Now, according to Theorem 6.7, there is exactly one connection-preserving map $(f, \bar{f})$ into $E_\nabla G \to B_\nabla G$ induced from any bundle $E \to X$, so $H^*_{\mathrm{dR}}(B_\nabla G) = I_2^\bullet(G)$ actually sets a cap on the number of characteristic classes attached to connections: If a characteristic class c is defined from a connection, like the Chern-Weil forms, its definition can be extended naturally to $E_\nabla G$ with Θ^{univ}, and thus c's value at E must be equal to $c(E) = \bar{f}^*(c(E_\nabla G))$ for the appropriate map $\bar{f}$. Therefore, there can be no more characteristic classes than there are cohomology classes in $H^\bullet_{\mathrm{dR}}(B_\nabla G)$. But the whole argument actually extends to all natural differential forms on G-bundles attached to connections: Ability to be defined at $E_\nabla G$ and naturality is enough to ensure that the value on $E_\nabla G$ determines

[3] See Remark 7.9

the differential form on any other principal G-bundle with connection, so the de Rham complex of $B_\nabla G$ is a cap also here.

On the other hand, we know from the Chern-Weil theory presented in Chapter 2 that all polynomials $P \in I^\bullet(G)$ give rise to a characteristic class $w(-;P)$, and they are all different classes since (69) is an isomorphism. In conclusion, the characteristic classes constructed as in Proposition 2.11 with the Chern-Weil homomorphism are the only ones that exist for principal G-bundles when attached to a connection, even the only natural differential forms on principal G-bundles attached to connections. As it can be seen as the most important result in this thesis, we state it properly.

Corollary 7.8. *Let G be any Lie group. Then the only natural differential forms on principal G-bundles that are attached to connections, are the characteristic classes constructed from polynomials in $I^\bullet(G)$ through the Chern-Weil homomorphism.*

Remark 7.9. *Actually, the fact that this Chern-Weil homomorphism is an injection is not shown in [7], but refered to [3]. To get this result, one might go back to the definition of curvature forms and the Chern-Weil homomorphism and do everything in a simplicial setting. This would also require defining basic forms for $E_\nabla G$ and showing that they can be pulled back to forms on $B_\nabla G$. The definition of the basic subcomplex is done in Chapter* 7 *of [7] for simplicial sheaves coming from simplicial group actions, which is exactly what $B_\nabla^{triv} G$ is (we defined it by letting $\mathscr{F}_G$ act on $\Omega^1 \otimes \mathfrak{g} = E_\nabla G$). Up to weak equivalence, $B_\nabla^{triv} G$ is exactly $B_\nabla G$, as we showed. From this, one could show that different polynomials in $I^\bullet(G)$ give rise to different, basic cohomology classes in $H^\bullet_{\mathrm{dR}}(E_\nabla G)$ when applied to the curvature form obtained from Θ^{univ}. They would then correspond uniquely to cohomology classes in $H^\bullet_{\mathrm{dR}}(B_\nabla^{triv} G) = H^\bullet_{\mathrm{dR}}(B_\nabla G)$ by Proposition* 7.14 *in [7], and we would have our injection.*

Another way of getting the injectivity, is by showing that any two polynomials P and P' in $I^\bullet(G)$ give rise to different characteristic classes c and c', i.e. $c(E) \neq c'(E)$ for at least some principal G-bundle (E, M, π). Then, if the polynomials would have $w(E_\nabla G; P) = w(E_\nabla G; P')$, they would be pulled back to the same class by the classifying map $(\bar{f}, f) : (E, M) \to (E_\nabla G, B_\nabla G)$, so we would also get $c(E) = \bar{f}^*(w(E_\nabla G; P)) = \bar{f}^*(w(E_\nabla G; P')) = c'(E)$, a contradiction.

Remark 7.10. *In [6], a principal G-bundle $EG \to BG$ is constructed for which there is a one-to-one correspondence between characteristic classes c for principal G-bundles, and its own cohomology classes $c(EG) \in H^*_{\mathrm{dR}}(BG)$ (Theorem* 5.5, *[6]). It is in other words a bundle containing exactly enough structure to capture all possible characteristic classes, and nothing more, just like the simplicial sheaves $E_\nabla G$ and $B_\nabla G$. The advantages of the method presented in this thesis, however, is that we also get $E_\nabla G \to B_\nabla G$ as a classifying space, by Theorem 6.7, from which all other principal G-bundles can be pulled back, even uniquely. We therefore arrive at a construction that seems to capture a very important part of invariant theory for principal G-bundles, even though some of its properties can be found in other places.*

8 The holomorphic case

So far, we have only been studying principal G-bundles on smooth manifolds, but it is worth taking a look at what happens if we try to consider them over complex manifolds instead. Much works the same way when we define the bundles, but we restrict the number of available maps by requiring them to be holomorphic. This added structure actually makes it a lot harder to find connections on the holomorphic principal G-bundles, as we will name them, with some having no possible connections at all, so the notion of a universal classifying object similar to $E_\nabla G \to B_\nabla G$ becomes a little different. We define the most important concepts, but as this chapter is meant mostly as a holomorphic exposition building on what we have seen in the previous ones, we do not prove much. We also assume knowledge about some concepts that have not been discussed earlier, like germs and sheaf cohomology.

Definition 8.1. *Let M be a complex manifold. Given a complex Lie group G, a holomorphic principal G-bundle is an ordinary principal G-bundle (E, M, π) where we also require that E is a complex manifold, and that both π and the right G-action $E \times G \to E$ are holomorphic maps.*

Accordingly, maps between holomorphic principal G-bundles are defined to be maps between ordinary principal G-bundles, that are holomorphic when taking the complex structures of the manifolds into account.

As in the smooth case, we can get canonical principal $\mathrm{GL}_n(\mathbb{C})$-bundles by taking the frame bundle of any holomorphic vector bundle $V \to M$, which means the bundle over M that to each $p \in M$ associates the vector space of linear maps $R^n \to V_m$, where $n = \dim(V)$.

Now, these new holomorphic principal G-bundles are of course special cases of our ordinary principal G-bundles, and can therefore be endowed with connections in the sense of Definition 2.4. Carrying these, they get unique classification maps into the universal object $E_\nabla G \to B_\nabla G$, from where they can pull back their connections. But none of this theory takes the holomorphic structure into account, not even the definition of the connections, and we therefore turn to another version of the connection, which is the one belonging to holomorphic bundles.

When we now want to define holomorphic connections, we take inspiration from [1] and [2] (they both have longer and more rigorous explanations for the results we simply claim in the following paragraphs). Taking a holomorphic principal G-bundle (E, M, π), it is possible to construct two new holomorphic vector bundles over M: The first is the Atiyah bundle, $\mathrm{At}(E)$, which is defined as the quotient TE/G, where the G-action on TE comes by differentiating the action on E. One can see that the natural projection $TE \to E$ is G-equivariant, so we get a well-defined projection for $\mathrm{At}(E)$ as well:

$$\mathrm{At}(E) = TE/G \to E/G = M \tag{70}$$

To make the other bundle, we start with the product $E \times \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. Remembering the adjoint representation, we can define a G-action on this product in the following way:

$$(x, v) \cdot g = (x \cdot g, \mathrm{Ad}(g^{-1})(v)) \in (E \times \mathfrak{g}) \quad \forall (x, v) \in E \times G, g \in G \tag{71}$$

Then it is possible to form the space $(E \times_G \mathfrak{g})$, which is the quotient space under this action. Using the projection $E \to M$, $(E \times_G \mathfrak{g})$ can also be shown to be a holomorphic vector bundle, which we call $L \to M$.

These holomorphic vector bundles over M can, if the maps are chosen correctly, be put together with the canonical tangent bundle $TM \to M$ in the following exact sequence:

$$0 \longrightarrow L \xrightarrow{i} \mathrm{At}(E) \xrightarrow{\rho} TM \longrightarrow 0 \tag{72}$$

Complying with the category in which we work, all arrows in the sequence are maps of holomorphic vector bundles (they are not principal G-bundles, so we do not demand any equivariance), and exactness means that

$$0 \longrightarrow L_m \longrightarrow \mathrm{At}(E)_m \longrightarrow T_m M \longrightarrow 0 \tag{73}$$

is an exact sequence of vector spaces for all elements $m \in M$. It is not hard to see that the fibres L_m are isomorphic to $\mathfrak{g}$, at least as vector spaces, and then the dimension of $\mathrm{At}(E)_m$ must be $\dim(\mathfrak{g}) + \dim(T_m M)$ for the exactness to work (this can also be seen directly). In other words, we have an exact sequence that looks a lot like the one used for defining the connection in Chapter 2, with TM as cokernel and a kernel looking like $\mathfrak{g}$. The difference is that we this time work with bundle maps over a manifold M, while we in the smooth case did not bother to define bundle structures for our vector spaces. Anyway, the definition of a holomorphic connection follows the same pattern:

Definition 8.2. *A holomorphic connection for the holomorphic principal bundle* (E, M, π) *is a splitting of the exact sequence given in* (72).

Such a splitting is of course always possible to obtain when working on a single fibre, like in (73), because vector spaces are torsion-free.

A splitting of the whole exact sequence of vector bundles, on the other hand, is only obtainable in some special cases, which according to [1] are determined by a special sheaf-cohomology class called the Atiyah class. If we let $\tilde{L}$ mean the sheaf of germs of holomorphic sections $M \to L$, and $\tilde{\Omega}^1$ the sheaf of germs of holomorphic differential 1-forms on M, we can obtain a new sheaf, $\mathrm{Hom}(\tilde{L}, \tilde{\Omega}^1)$ on M by a standard construction. The Atiyah class $a(E)$ is then a class living in the first sheaf cohomology $H^1(M, \mathrm{Hom}(\tilde{L}, \tilde{\Omega}^1))$, obtainable from any holomorphic principal G-bundle E through a process detailed in [1]. A main result from the same paper states that the exact sequence (72) splits if and only if $a(E) = 0$.

This condition is actually not that strict; for example, all Stein manifolds have $\mathrm{Hom}(\tilde{L}, \tilde{\Omega}^1) = 0$, and therefore allow holomorphic connections in general. There are, however, holomorphic principal G-bundles that have non-trivial values for $a(E)$, and these can never be endowed with holomorphic connections.

We can now try to repeat the whole process behind $E_\nabla G$ and $B_\nabla G$, but in a holomorphic setting. This means seeking universal objects $E_\nabla^{\mathbb{C}}G$ and $B_\nabla^{\mathbb{C}}G$ that all holomorphic bundles with holomorphic connection can be pulled back from. The Chapters 2-5 of this thesis do not really rely on the fact that we were working in a smooth setting, so all the category theoretical preparations could go as before. If we denote the category of complex manifolds and holomorphic mappings as $\mathbf{Man}_{\mathbb{C}}$, we could also define holomorphic simplicial presheveaves to be functors

$$F : \mathbf{Man}_{\mathbb{C}}^{\mathrm{op}} \longrightarrow \mathbf{Set}_{\Delta}, \tag{74}$$

which would then form a category $\mathbf{sPre}_{\mathbb{C}}$. The category $\mathbf{Man}_{\mathbb{C}}$ would be embedded in $\mathbf{sPre}_{\mathbb{C}}$ by the map taking $X \in \mathbf{Man}_{\mathbb{C}}$ to $\mathbf{Man}_{\mathbb{C}}(-, X)$ (considered as a constant holomorphic simplicial presheaf), again as a consequence of the Yoneda lemma.

We can now try to construct the holomorphic simplicial presehaves $E_\nabla^{\mathbb{C}}G$ and $B_\nabla^{\mathbb{C}}G$ from groupoids where the objects are holomorphic principal G-bundles with holomorphic connection (and a global holomorphic section, in the case of $E_\nabla^{\mathbb{C}}G$), and the arrows isomorphism diagrams that preserve everything. Over the complex manifold M, an arrow would look like this:

$$\begin{tikzcd} P \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & P' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

A problem appears immediately, that we did not have for $E_\nabla G$ and $B_\nabla G$, namely that not all holomorphic bundles over M appear in $B_\nabla^{\mathbb{C}}G(M)$, only the ones that allow a holomorphic connection. If there should exist an M such that none of its bundles had a trivial Atiyah class, that would set $B_\nabla^{\mathbb{C}}G(M) = \emptyset$, and threaten $B_\nabla^{\mathbb{C}}G$'s status as a functor. Luckily for us, all trivial bundles $M \times G$ allow holomorphic connections: The map ρ in the exact sequence (72) would in this trivial case just be the projection from $(TE/G) \times TM$ onto TM, which is holomorphically reversible with the inclusion. So long, the smooth method seems to work for $E_\nabla^{\mathbb{C}}G$ and $B_\nabla^{\mathbb{C}}G$ as well.[4]

To proceed from this, though, we would need some result similar to Theorem 6.3, and that would require a greater understanding of holomorphic connections. If they too can be given as some fundamental holomorphic connection (similar to the Maurer-Cartan form) plus a $\mathfrak{g}$-valued 1-form over M, i.e. be seen as a kind of affine space, that would be good. We could then develop a holomorphic version of $B_\nabla^{triv}G$, since a holomorphic section $s : M \to E$ also would be moved bijectively into another one by multiplication with an element in $\mathbf{Man}_{\mathbb{C}}(M, G)$, and end up with a bundle $E_\nabla^{\mathbb{C}}G \to B_\nabla^{\mathbb{C}}G$.

[4]Since holomorphic connections always exist on trivial bundles, one could ask why it is not possible to glue together local connecitons on trivializing open sets to a global holomorphic connection, like we did for ordinary principal G-bundles. The problem is to construct holomorphic partitions of unity, which is not possible because of the identity theorem on complex manifolds. Without them, the process from the smooth case cannot be repreated.

If we indeed could construct such a classifying space, find a universal holomorphic connection, and embed all holomorphic G-bundles with holomorphic connections into it, it would be a little strange. If we then took a complex manifold M that allowed for some holomorphic principapl G-bundle $P \to M$ with non-trivial Atiyah class, and therefore no holomorphic connection, there would not exist any map from $\mathbf{Man}_{\mathbb{C}}(-, P) \to \mathbf{Man}_{\mathbb{C}}(-, M)$ into $E^{\mathbb{C}}_{\nabla}G \to B^{\mathbb{C}}_{\nabla}G$, as this would endow P with a holomorphic connection. There would, however, be a base map, as one could use the one that appears when the bundle $M \times G \to M$ is embedded (remember that all trivial holomorphic principal G-bundles have holomorphic connections). Thus the only thing missing in the case $P \to M$ would be the mapping of the total spaces, completing the commutative diagram.

To conclude, the possible holomorphic version of Theorem 6.7 could be slightly stranger than the smooth case, but might be achievable with a little more insight into how holomorphic connections work. I have not had time to study them deep enough in this thesis, but the result might not be that far away if one finds a nice holomorphic simplicial sheaf weakly equivalent to $E^{\mathbb{C}}_{\nabla}G$. If so, computing the de Rham cohomology of both $E^{\mathbb{C}}_{\nabla}G$ and $B^{\mathbb{C}}_{\nabla}G$ would be natural further steps, and maybe result in something useful. These computations are already quite challenging for $E_{\nabla}G$ and $B_{\nabla}G$, but as we know, going to the complex world can either greatly simplify or complicate everything. It is certainly worth trying, if one wants to learn more about holomorphic principal G-bundles and their holomorphic connections.

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