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String bordism and chromatic characteristics

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Dedicated to Paul Goerss on the occasion of his 60th birthday

Abstract. We introduce characteristics into chromatic homotopy theory. This parallels the prime characteristics in number theory as well as in our earlier work on structured ring spectra and unoriented bordism theory. Here, the $\mathrm{K}(n)$–local Hopkins–Miller classes ζ_n take the place of the prime numbers. Examples from topological and algebraic K-theory, topological modular forms, and higher bordism spectra motivate and illustrate this concept.

Introduction

The classification of manifolds is intimately tied to the homotopy theory of Thom spaces and spectra. If MO denotes the Thom spectrum for the family of orthogonal groups, then its homotopy groups $\pi_d\,\mathrm{MO}$ are given by the groups of bordism classes of d–dimensional closed manifolds. Variants of this correspondence apply to manifolds with extra structure, such as orientations and Spin structures, for instance. Arguably the most relevant of these variants for geometry are ordered into a hierarchy given by the higher connective covers $\mathrm{BO}\langle k\rangle \to \mathrm{BO}$ of BO, and their Thom spectra $\mathrm{MO}\langle k\rangle$. For small values of k, these describe the unoriented ($\mathrm{MO}\langle 1\rangle = \mathrm{MO}$), oriented ($\mathrm{MO}\langle 2\rangle = \mathrm{MSO}$), Spin ($\mathrm{MO}\langle 4\rangle = \mathrm{MSpin}$), and String bordism groups of manifolds ($\mathrm{MO}\langle 8\rangle = \mathrm{MString}$). The name 'string' in this context appears to be due to Miller (see [**34**]). The spectra $\mathrm{MO}\langle k\rangle$ are also interesting as approximations to the sphere spectrum $\mathbb{S}$ itself, in a sense that can be made precise [**26**, Proposition 2.1.1]: There is an equivalence $\mathbb{S} \simeq \lim_k \mathrm{MO}\langle k\rangle$. The geometric relevance of the sphere spectrum stems, of course, from the fact that it is the Thom spectrum for stably framed manifolds.

All the bordism spectra that were just mentioned are canonically commutative ring spectra in the most desirable way, namely E_∞ ring spectra [**39**]. In fact, this concept was more or less invented in order to deal

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with the very examples of Thom spectra [**37**]. The multiplicative structure allows us to study them through their *genera*: multiplicative maps *out of* commutative bordism ring spectra into spectra which are easier to understand. This has been rather successful for small values of k, and the following diagram indicates the situation.

$$\begin{array}{ccc} \vdots & & \\ \downarrow & & \\ \mathrm{MString} & \longrightarrow & \mathrm{tmf} \\ \downarrow & & \\ \mathrm{MSpin} & \longrightarrow & \mathrm{ko} \\ \downarrow & & \\ \mathrm{MSO} & \longrightarrow & \mathrm{H}\mathbb{Z} \\ \downarrow & & \\ \mathrm{MO} & \longrightarrow & \mathrm{H}\mathbb{F}_2 \end{array}$$

Here, the spectra $\mathrm{H}\mathbb{F}_2$ and $\mathrm{H}\mathbb{Z}$ are the Eilenberg–Mac Lane spectra of the indicated rings, and the genera count the number of points mod 2 and with signs, respectively. In the row above, the spectrum ko is the connective real K-theory spectrum that receives the topological $\widehat{A}$–genus (or Atiyah–Bott–Shapiro orientation) (compare [**31**] and [**32**]). Finally, the spectrum tmf is the spectrum of topological modular forms that was constructed in order to refine the Witten genus (or σ–orientation) (see [**20**], [**21**], [**2**], [**3**], and [**1**]).

Characteristics in the sense of the title appear in the approach that is dual to the idea underlying genera. Namely, there are interesting ring spectra that come with maps *into* these bordism spectra. For instance, since the unoriented bordism ring $\pi_*\,\mathrm{MO}$ has characteristic 2, there is a unique (up to homotopy) map $\mathbb{S}/\!\!/2 \to \mathrm{MO}$ of E_∞ ring spectra from the versal E_∞ ring spectrum $\mathbb{S}/\!\!/2$ of characteristic 2. See [**48**], where E_∞ ring spectra of prime characteristics, and their versal examples $\mathbb{S}/\!\!/p$, have been studied from this point of view. However, the fact that $\pi_0\,\mathrm{MO}\langle k\rangle = \mathbb{Z}$ as soon as $k \geqslant 2$ makes it evident that ordinary prime characteristics have nothing to say about higher bordism theories. This is where the present writing sets in. See also [**7**] for a different generalization.

In order to gain a better understanding of higher bordism theories, we propose in this paper to replace the ordinary primes $p \in \mathbb{Z} = \pi_0\mathbb{S}$ by something more elaborate, namely by some classes that only appear after passing to the (Bousfield [**14**]) localization $\widehat{\mathbb{S}}$ of the sphere spectrum $\mathbb{S}$ with respect to any given Morava K-theory $\mathrm{K}(n)$: the classes ζ_n in $\pi_{-1}\widehat{\mathbb{S}}$ which were first defined by Hopkins and Miller. See [**25**], [**16**], and our exposition in Section 1. Just as $\mathbb{S}/\!\!/2$ has been used in [**48**] to study the unoriented bordism spectrum MO, one aim of the present writing is to show that it is the corresponding versal examples $\widehat{\mathbb{S}}/\!\!/\zeta_n$ which are likewise relevant to the study of the chromatic localizations of higher bordism spectra.

Whenever A is any $\mathrm{K}(n)$–local E_∞ ring spectrum with unit $u_A\colon \widehat{\mathbb{S}} \to A$, there is a naturally associated class

$$\zeta_n(A)\colon \mathrm{S}^{-1} \xrightarrow{\zeta_n} \widehat{\mathbb{S}} \xrightarrow{u_A} A$$

in $\pi_{-1}A$. Continuing to use the terminology as in [**48**], we will say that a $\mathrm{K}(n)$–local E_∞ ring spectrum A has *(chromatic) characteristic* ζ_n if there exists a homotopy $\zeta_n(A) \simeq 0$ (compare Definition 1.12 below). We note that this concept only involves the existence of a homotopy, whereas for structural purposes one will want to work with actual choices of homotopies, i.e. with commutative $\widehat{\mathbb{S}}/\!\!/\zeta_n$–algebras. See Section 2.1, and [**48**] again.

There are families of examples of characteristic ζ_n spectra for arbitrary n: the Lubin–Tate spectra E_n (Example 2.2), the Iwasawa extensions B_n of the $\mathrm{K}(n)$–local sphere (Example 2.3), and the versal examples $\widehat{\mathbb{S}}/\!\!/\zeta_n$ that map to all of these (see Proposition 2.5).

Hopkins [**22**] and Laures [**36**] have given useful descriptions of the $\mathrm{K}(1)$–local E_∞ ring spectra $\mathrm{KO}_{\mathrm{K}(1)}$ and $\mathrm{tmf}_{\mathrm{K}(1)}$ at the prime $p = 2$. The first step in these cases is to kill the class ζ_1 in $\widehat{\mathbb{S}}$ in an E_∞ manner so as to obtain the versal example $\widehat{\mathbb{S}}/\!\!/\zeta_1$ above. The second (and already last) step in either case is to kill another class in the latter. This underlines the importance of an understanding of the versal examples $\widehat{\mathbb{S}}/\!\!/\zeta_n$, and since $\mathrm{B}_1 = \mathrm{KO}_{\mathrm{K}(1)}$, it naturally leads one to ask for a similar description of the higher Iwasawa extensions B_n. I hope this will be pursued elsewhere (see Remark 2.7).

The $\mathrm{K}(1)$–localizations of many algebraic K-theory spectra are not of characteristic ζ_1, and the behavior of multiplication with ζ_1 on the homotopy groups is connected to open number theoretic conjectures (see Remark 3.3). In contrast to that, the work of Laures [**35**] and his student Reeker [**44**] shows that the $\mathrm{K}(1)$–localizations of MSpin, and MSU all have characteristic ζ_1.

In some genuinely new examples dealt with here, we take the natural next step: The $\mathrm{K}(2)$–localizations of the topological modular forms spectrum tmf, the String bordism spectrum MString, and $\mathrm{MU}\langle 6\rangle$ have characteristic ζ_2 almost everywhere (see Propositions 4.1 and 5.4).

The paper is organized into five sections. In Section 1, we briefly review the basic context for chromatic homotopy theory, establish the notation that we are going to use here, and define chromatic characteristics. Section 2 introduces the versal examples and presents the higher Iwasawa extensions. Section 3 contains our discussion of topological and algebraic K-theory spectra. In Section 4, we show how to deal with spectra related to topological modular forms, and bordism spectra are examined in the final Section 5.

1. Characteristics in chromatic homotopy theory

In this section we will review some chromatic homotopy theory as far as it is needed for our purposes, and introduce the basic concept of chromatic characteristics (see Section 1.7). The case $n = 1$ will be mentioned as an accompanying example throughout, but we emphasize that this case is always somewhat atypical, and the general case is the one we are interested in. Also, in the spirit of [**30**], we have chosen notation that avoids having to say anything special when $p = 2$. Nevertheless, we do so, if it seems appropriate for the examples at hand, in particular in Section 5 when it comes to bordism theories.

We will use the following conventions: All spectra are implicitly $\mathrm{K}(n)$–localized. In particular, the notation $X \wedge Y$ will refer to the $\mathrm{K}(n)$–localization of the usual smash product, and the homology $X_0 Y$ is defined as π_0 of that. As an exception to these rules, we will write $\widehat{\mathbb{S}}$ for the $\mathrm{K}(n)$–local sphere to emphasize the idea that it is a completed form of the sphere spectrum $\mathbb{S}$, and $\mathrm{S}^n = \Sigma^n \widehat{\mathbb{S}}$ denotes its (de)suspensions.

1.1. The Lubin–Tate spectra. Let p be a prime number, and n a positive integer. We will denote by E_n the corresponding Lubin–Tate spectrum. The coefficient ring is isomorphic to

$$\pi_* \mathrm{E}_n \cong \mathrm{W}(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]][u^{\pm 1}],$$

where W is the Witt vector functor from commutative rings to commutative rings, and the generators sit in degrees $|u_j| = 0$ and $|u| = -2$. This coefficient ring (or rather its formal spectrum) is a base for the universal formal deformation of the Honda formal group of height n.

EXAMPLE 1.1. If $n = 1$, then the Lubin–Tate spectrum E_1 is the p–adic completion KU_p of the complex topological K-theory spectrum KU.

1.2. The Morava groups. The n–th Morava stabilizer group S_n and the Galois group of $\mathbb{F}_{p^n}$ over $\mathbb{F}_p$ both act on E_n such that their semi-direct product G_n, the *extended* Morava stabilizer group, also acts on E_n.

EXAMPLE 1.2. If $n = 1$, then the Morava stabilizer group $\mathrm{G}_1 = \mathrm{S}_1$ is the group $\mathbb{Z}_p^\times$ of p–adic units which acts on $\mathrm{E}_1 = \mathrm{KU}_p$ via Adams operations.

1.3. Devinatz–Hopkins fixed point spectra. If $K \leqslant \mathrm{G}_n$ is a closed subgroup of the extended Morava stabilizer group G_n, then $\mathrm{E}_n^{\mathrm{h}K}$ will denote the corresponding Devinatz–Hopkins fixed point spectrum [**16**]. For instance, in the maximal case $K = \mathrm{G}_n$, we have $\mathrm{E}_n^{\mathrm{h}G_n} \simeq \widehat{\mathbb{S}}$ (see Thm. 1(iii) of *loc. cit.*), as a reflection of Morava's change-of-rings theorem. See also [**11**] for a different approach.

The Devinatz–Hopkins fixed point spectra are well under control in the optic of their Morava modules: There are isomorphisms

$$(\mathrm{E}_n)_*(\mathrm{E}_n^{\mathrm{h}K}) = \pi_*(\mathrm{E}_n \wedge \mathrm{E}_n^{\mathrm{h}K}) \cong \mathcal{C}(\mathrm{G}/K, \pi_* \mathrm{E}_n),$$

where $\mathcal{C}(\mathrm{G}/K, \pi_* \mathrm{E}_n)$ is the ring of continuous functions from the coset space G/K to $\pi_* \mathrm{E}_n$ with its $(p, u_1, ..., u_{n-1})$–adic topology. For the trivial group $K = e$ this has been known to Morava (and certainly others) for a long time. See [**28**] for the history and a careful exposition.

1.4. Some subgroups of the Morava stabilizer group. The Morava stabilizer group acts on the Dieudonné module of the Honda formal group of height n, which is free of rank n over $\mathrm{W}(\mathbb{F}_{p^n})$. The determinant gives a homomorphism $\mathrm{S}_n \to \mathrm{W}(\mathbb{F}_{p^n})^\times$. This extends over G_n and factors through $\mathbb{Z}_p^\times$. The subgroup SG_n is defined as the kernel of the (surjective) determinant, so that we have an extension

$$1 \longrightarrow \mathrm{SG}_n \longrightarrow \mathrm{G}_n \longrightarrow \mathbb{Z}_p^\times \longrightarrow 1$$

of groups. Let $\Delta \leqslant \mathbb{Z}_p^\times$ denote the torsion subgroup. If $p = 2$, then this subgroup is cyclic of order 2, and if $p \neq 2$, then it is cyclic of order $p - 1$. The pre-image of Δ under the determinant is customarily denoted by G_n^1. In other words, there is an extension

$$1 \longrightarrow \mathrm{G}_n^1 \longrightarrow \mathrm{G}_n \longrightarrow \mathbb{Z}_p^\times/\Delta \longrightarrow 1, \tag{1.1}$$

and the groups SG_n and G_n^1 are then also related by a short exact sequence.

$$1 \longrightarrow \mathrm{SG}_n \longrightarrow \mathrm{G}_n^1 \longrightarrow \Delta \longrightarrow 1$$

We remark that there are (abstract) isomorphisms $\mathbb{Z}_p^\times/\Delta \cong \mathbb{Z}_p$ of groups, but no canonical choice seems to be available.

1.5. The Iwasawa extensions of the local spheres. An Iwasawa extension is a (pro-)Galois extension (for instance of number fields) with Galois group isomorphic to the additive group $\mathbb{Z}_p$ of the p–adic integers for some prime number p. The canonical Iwasawa extension of the $\mathrm{K}(n)$–local sphere is the Devinatz–Hopkins fixed point spectrum

$$\mathrm{B}_n = \mathrm{E}_n^{\mathrm{hG}_n^1}$$

with respect to the closed subgroup G_n^1. This spectrum is sometimes referred to as *Mahowald's half-sphere*, in particular in the case $n = 2$.

EXAMPLE 1.3. If $n = 1$, then the spectrum B_1 is either the 2–completion KO_2 of the real topological K-theory spectrum KO (when $p = 2$) or the Adams summand L_p of the p–completion of the complex topological K-theory spectrum KU (when $p \neq 2$).

The spectra B_n are well under control in the optic of their Morava modules: There are isomorphisms

$$(\mathrm{E}_n)_*(\mathrm{B}_n) = \pi_*(\mathrm{E}_n \wedge \mathrm{B}_n) \cong \mathcal{C}(\mathbb{Z}_p^\times/\Delta, \pi_*\mathrm{E}_n),$$

and the right hand side can be identified (non-canonically) with the ring of continuous functions on the p–adic affine line $\mathbb{Z}_p$.

From (1.1) we infer that the spectrum B_n carries a residual action of the group $\mathbb{Z}_p^\times/\Delta \cong \mathbb{Z}_p$, and this makes $\widehat{\mathbb{S}} \to \mathrm{B}_n$ into an Iwasawa extension of the $\mathrm{K}(n)$–local sphere (see [**16**, bottom of p. 5]). Whenever we choose a topological generator of this group, this yields an automorphism $g\colon \mathrm{B}_n \to \mathrm{B}_n$.

PROPOSITION 1.4. ([**16**, Proposition 8.1]) *There is a homotopy fibration sequence*

$$\mathrm{S}^0 \longrightarrow \mathrm{B}_n \xrightarrow{g-\mathrm{id}} \mathrm{B}_n \xrightarrow{\delta} \mathrm{S}^1 \tag{1.2}$$

of $\mathrm{K}(n)$*–local spectra.*

For each p and n, we fix one such fibration sequence once and for all.

EXAMPLE 1.5. If $n = 1$, then the fibration sequence

$$\mathrm{S}^0 \longrightarrow \mathrm{B}_1 \xrightarrow{g-\mathrm{id}} \mathrm{B}_1 \tag{1.3}$$

has been known for a long time. It can be extracted from [**14**], which in turn relies on work of Mahowald ($p = 2$) and Miller ($p \neq 2$).

1.6. The Hopkins–Miller classes. We are now ready to introduce the Hopkins–Miller classes ζ_n that are to play the role of the integral primes p in the chromatic context.

DEFINITION 1.6. The homotopy class $\zeta_n \in \pi_{-1}\widehat{\mathbb{S}}$ is defined as the (de-suspension of) the composition

$$\mathrm{S}^0 \longrightarrow \mathrm{B}_n \xrightarrow{\delta} \mathrm{S}^1$$

of the outer maps in the homotopy fibration sequence (1.2).

On the face of it, this definition seems to depend upon the choice of a topological generator g of the group $\mathbb{Z}_p^\times/\Delta$. But every other generator h has the form $h = g^\alpha$ for a p–adic unit α.

LEMMA 1.7. *For any p–adic unit $\alpha \in \mathbb{Z}_p^\times$, we can write*

$$(\mathrm{T}+1)^\alpha - 1 = \epsilon(\mathrm{T}) \cdot \mathrm{T}$$

for some unit $\epsilon(\mathrm{T})$ in the Iwasawa ring $\mathbb{Z}_p[[\mathrm{T}]]$.

PROOF. Consider the function $f(\mathrm{T}) = (\mathrm{T}+1)^\alpha - 1$ and observe that we have $f(0) = 0$, so that f is divisible by T. And $f'(0) = \alpha$ is a unit in the coefficient ring $\mathbb{Z}_p$ by assumption. □

If $\mathrm{T} = g - 1$, then $h - 1 = g^\alpha - 1 = (\mathrm{T}+1)^\alpha - 1$, and the lemma implies that two choices g, h of generators of $\mathbb{Z}_p^\times/\Delta$ yield self-maps of B_n that only differ by an equivalence.

Similarly, it also makes no essential difference whether we have $g - \mathrm{id}$ or its negative $\mathrm{id} - g$ in (1.2): it changes δ (and therefore ζ_n) by at most a sign. The convention in [**16**, §8] is different from ours.

REMARK 1.8. On a more conceptual level, one might be tempted to describe ζ_n using the canonical map $\mathrm{S}^0 = (\mathrm{B}_n)^{\mathrm{h}\mathbb{Z}_p} \to \mathrm{B}_n \to (\mathrm{B}_n)_{\mathrm{h}\mathbb{Z}_p}$ from the homotopy fixed points to the homotopy orbits, and duality. Since this point of view has, so far, not led to computational advances, we refrain from doing so.

If we map S^0 into the fibration sequence (1.2), then we obtain a long exact sequence

$$[\mathrm{S}^0,\mathrm{B}_n] \xrightarrow{g_*-\mathrm{id}} [\mathrm{S}^0,\mathrm{B}_n] \xrightarrow{\delta_*} [\mathrm{S}^0,\mathrm{S}^1] \longrightarrow [\mathrm{S}^{-1},\mathrm{B}_n], \tag{1.4}$$

and ζ_n is, by definition, the image of the unit u under the map δ_*.

Using the defining homomorphism (1.1) of G_n^1, we obtain a homomorphism

$$c_n\colon \mathrm{G}_n \longrightarrow \mathbb{Z}_p^\times/\Delta \cong \mathbb{Z}_p \subseteq \pi_0\mathrm{E}_n$$

that we can think of as a twisted homomorphism, or 1-cocycle, and as such it defines a class in the first continuous cohomology $\mathrm{H}^1(\mathrm{G}_n;\pi_0\mathrm{E}_n)$.

Proposition 1.9. ([**16**, Proposition 8.2]) *The Hopkins–Miller class ζ_n is detected by $\pm c_n$ in the $\mathrm{K}(n)$–local E_n–based Adams–Novikov spectral sequence.*

The class c_n is non-zero and ζ_n is non-zero in $\pi_{-1}\mathrm{S}^0$. In fact, it generates a subgroup isomorphic to $\mathbb{Z}_p$ in $\pi_{-1}\mathrm{S}^0$. However, the class ζ_n becomes zero in B_n:

Proposition 1.10. *The composition*

$$\mathrm{S}^{-1} \xrightarrow{\zeta_n} \mathrm{S}^0 \longrightarrow \mathrm{B}_n$$

of ζ_n with the unit u of B_n is zero.

Proof. We have $u\zeta_n = u\delta u$, and there is already a homotopy $u\delta \simeq 0$ as part of the fibration sequence (1.2). □

Remark 1.11. The reader might wonder if perhaps $\pi_{-1}\mathrm{B}_n = 0$, which is a stronger statement than the one in Proposition 1.10. But this is not always true. In fact, it is false for $n = 2$ and $p = 2$ by recent work of Beaudry, Goerss, and Henn (see [**8**, Cor. 8.1.6], which gives $\pi_{-1}\mathrm{B}_2 = \mathbb{Z}/2$ at $p = 2$). However, even in that case it is still true that $g_* = \mathrm{id}$ on $\pi_{-1}\mathrm{B}_n$, and this allows us to prove the surjectivity of $\delta_*\colon \pi_0\mathrm{B}_n \to \pi_{-1}\widehat{\mathbb{S}}$ in (1.4) for $n \leqslant 2$ at all primes. Injectivity follows from $g_* = \mathrm{id}$ on $\pi_0\mathrm{B}_n$, which also holds for $n \leqslant 2$ at all primes. See [**18**] and [**19**] for the case $n = 2$ and $p = 3$. It follows that δ_* is an isomorphism for $n \leqslant 2$ at all primes. It appears to be open if these groups are isomorphic for $n \geqslant 3$.

1.7. Chromatic characteristics. In the predecessor [**48**] of this paper, we have defined the notion of an E_∞ ring spectrum A of prime characteristic. If p is the prime number in question, then this means that there is a null-homotopy $p \simeq 0$ in A. We will now work $\mathrm{K}(n)$–locally and replace the prime numbers p by the Hopkins–Miller classes ζ_n.

Definition 1.12. If A is a $\mathrm{K}(n)$–local E_∞ ring spectrum, and if we let $u_A\colon \widehat{\mathbb{S}} \to A$ denote its unit, then

$$\zeta_n(A)\colon \mathrm{S}^{-1} \xrightarrow{\zeta_n} \widehat{\mathbb{S}} \xrightarrow{u_A} A$$

is the associated class in $\pi_{-1}A$. If A is a $\mathrm{K}(n)$–local E_∞ ring spectrum such that there exists a null-homotopy $\zeta_n(A) \simeq 0$, then we will say that A has *characteristic* ζ_n. If we write $\mathrm{Char}(\zeta_n)$ for the class of all $\mathrm{K}(n)$–local E_∞ ring spectra that have characteristic ζ_n, then we may also write

$$A \in \mathrm{Char}(\zeta_n)$$

in that case.

Remark 1.13. By definition, being of characteristic ζ_n is a property of $\mathrm{K}(n)$–local E_∞ ring spectra. Definition 1.12 applies more generally to $\mathrm{K}(n)$–local ring spectra up to homotopy, but the examples of interest to us always come with an E_∞ structure.

Proposition 1.14. *If A is a $\mathrm{K}(n)$–local E_∞ ring spectrum of characteristic ζ_n, then so is every $\mathrm{K}(n)$–local commutative A–algebra B.*

Proof. The unit of any A–algebra B factors through the unit of A. □

2. Chromatic and versal examples

First of all, here is an example which shows that not all $\mathrm{K}(n)$–local E_∞ ring spectra have characteristic ζ_n.

Example 2.1. In the initial example $A = \widehat{\mathbb{S}}$ of the $\mathrm{K}(n)$–local sphere, the unit is the identity, so that we have $\zeta_n(\widehat{\mathbb{S}}) = \zeta_n$, and this is non-zero as a consequence of Proposition 1.9. Therefore,

$$\widehat{\mathbb{S}} \notin \mathrm{Char}(\zeta_n).$$

This result is analogous to the fact that $\mathbb{S} \notin \mathrm{Char}(p)$ for the (un-localized) ring of spheres.

Clearly, if A is a $\mathrm{K}(n)$–local E_∞ ring spectrum such that $\pi_{-1}A$ vanishes, then the element $\zeta_n(A) \in \pi_{-1}A = 0$ is automatically null-homotopic. Let us mention a couple of interesting examples of this type.

Example 2.2. Because the Lubin–Tate spectra E_n are even spectra, we have $\pi_{-1}\mathrm{E}_n = 0$, so that $\zeta_n(\mathrm{E}_n) \simeq 0$, and this implies

$$\mathrm{E}_n \in \mathrm{Char}(\zeta_n).$$

In other words, the Lubin–Tate spectra E_n all have characteristic ζ_n.

Even if we have $\pi_{-1}A \neq 0$, or if we are perhaps in a situation when we do not know yet whether this or $\pi_{-1}A = 0$ holds, we might still be able to decide if $\zeta_n(A)$ is null-homotopic. This is the case in the following examples.

Example 2.3. We have

$$\mathrm{B}_n \in \mathrm{Char}(\zeta_n)$$

for all heights n by Proposition 1.10.

2.1. The versal examples. An important theoretical role in the theory of $\mathrm{K}(n)$–local E_∞ ring spectra of characteristic ζ_n is played by the versal examples. These will be introduced now.

Let $\mathbb{P}X$ denote the free $\mathrm{K}(n)$–local E_∞ ring spectrum on a $\mathrm{K}(n)$–local spectrum X. There is an adjunction

$$\mathcal{E}_\infty^{\mathrm{K}(n)}(\mathbb{P}X, A) \cong \mathcal{S}^{\mathrm{K}(n)}(X, A)$$

between the space of $\mathrm{K}(n)$–local E_∞ ring maps and the space of maps of $\mathrm{K}(n)$–local spectra. In one direction, the bijection sends an E_∞ map $\mathbb{P}X \to A$ to its restriction along the unit $X \to \mathbb{P}X$ of the adjunction. (The unit of the E_∞ ring spectrum $\mathbb{P}X$ is a map $\widehat{\mathbb{S}} \to \mathbb{P}X$, of course.) The inverse is denoted by $x \mapsto \mathrm{ev}(x)$ for any given class $x\colon X \to A$.

Definition 2.4. The $\mathrm{K}(n)$–local E_∞ ring spectrum $\widehat{\mathbb{S}}/\!\!/\zeta_n$ is defined as a homotopy pushout

$$\begin{array}{ccc} \mathbb{P}\mathrm{S}^{-1} & \xrightarrow{\mathrm{ev}(0)} & \widehat{\mathbb{S}} \\ {\scriptstyle \mathrm{ev}(\zeta_n)}\downarrow & & \downarrow \\ \widehat{\mathbb{S}} & \longrightarrow & \widehat{\mathbb{S}}/\!\!/\zeta_n \end{array}$$

in the category of $\mathrm{K}(n)$–local E_∞ ring spectra.

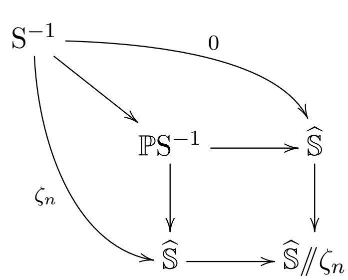
There are various ways of producing such a homotopy pushout diagram. The easiest one might be to start with a cofibrant model of $\widehat{\mathbb{S}}$, replacing the morphism $\mathrm{ev}(0) = \mathbb{P}(\mathrm{S}^{-1} \to \mathrm{D}^0)$ by $\mathbb{P}$ of a cofibration $\mathrm{S}^{-1} \to K$ for some contractible K, for instance the cone on S^{-1}, and then taking the actual pushout. See [**38**] for suitable notions of cofibrancy in the relevant model categories.

Proposition 2.5. *We have*

$$\widehat{\mathbb{S}}/\!\!/\zeta_n \in \mathrm{Char}(\zeta_n)$$

for all primes p.

Proof. The homotopy commutativity of the enlarged diagram



immediately shows that $\zeta_n(\widehat{\mathbb{S}}/\!\!/\zeta_n)$ is homotopic to zero. □

Remark 2.6. The $\mathrm{K}(n)$–local E_∞ ring spectrum $\widehat{\mathbb{S}}/\!\!/\zeta_n$ has the usual property of any homotopy pushout: a null-homotopy of $\zeta_n(A)$ gives rise to a map $\widehat{\mathbb{S}}/\!\!/\zeta_n \to A$, and conversely. In fact, this allows us to add upon the preceding proposition: Any choice of homotopy pushout $\widehat{\mathbb{S}}/\!\!/\zeta_n$ comes with a *preferred* homotopy $\zeta_n(\widehat{\mathbb{S}}/\!\!/\zeta_n) \simeq 0$ (that corresponds to the identity map). It also implies that there is a map

$$\widehat{\mathbb{S}}/\!\!/\zeta_n \longrightarrow A \tag{2.1}$$

of $\mathrm{K}(n)$–local E_∞ ring spectra if and only if $A \in \mathrm{Char}(\zeta_n)$. There is no reason why a map (2.1), once it exists, should be unique. In fact, there will usually be many such maps, even up to homotopy. This explains our use of Artin's term 'versal' (from [**4**]) rather than 'universal.'

Remark 2.7. As a consequence of the versal property, we have an E_∞ map $\widehat{\mathbb{S}}/\!\!/\zeta_n \to \mathrm{B}_n$, and it is tempting to try to fit it into a pushout square

$$\begin{array}{ccc} \widehat{\mathbb{S}}/\!\!/\zeta_n & \longrightarrow & \mathrm{B}_n \\ \uparrow & & \uparrow \\ X & \longrightarrow & \widehat{\mathbb{S}} \end{array}$$

of $\mathrm{K}(n)$–local E_∞ ring spectra. Two more requirements are on my wish list for that. First, the morphism $X \to \widehat{\mathbb{S}}/\!\!/\zeta_n$ on the left is an Iwasawa extension, just like the one on the right. In particular, the spectrum X can be described as the homotopy fixed points of a Galois action on $\widehat{\mathbb{S}}/\!\!/\zeta_n$ of a group isomorphic to $\mathbb{Z}_p$. Second, the spectrum X is free as an E_∞ ring spectrum ($X \simeq \mathbb{P}Y$ for some small, $\mathrm{K}(n)$–local Y), so that the E_∞ map $X \to \widehat{\mathbb{S}}/\!\!/\zeta_n$ is adjoint to a map $Y \to \widehat{\mathbb{S}}/\!\!/\zeta_n$ of spectra, and hence easier to construct. For $n = 1$ this rediscovers Hopkins' cell decomposition of B_1 from [**22**] (with $Y = \mathrm{S}^0$ the $\mathrm{K}(n)$–local sphere).

In order to demonstrate the relevance of the concept of (chromatic) characteristics outside of chromatic homotopy theory itself, we will, in the rest of this paper, give many examples of naturally occurring $\mathrm{K}(n)$–local E_∞ ring spectra of characteristic ζ_n, in particular for $n = 1$ and $n = 2$.

3. K-theories

Let us start with the topological K-theory spectra. There are equivalences $\mathrm{ko}_{\mathrm{K}(1)} \simeq \mathrm{KO}_{\mathrm{K}(1)}$ and $\mathrm{ku}_{\mathrm{K}(1)} \simeq \mathrm{KU}_{\mathrm{K}(1)}$ so that it is sufficient to state the results for the connective versions. Note that $\mathrm{ko}_{\mathrm{K}(n)}$ and $\mathrm{ku}_{\mathrm{K}(n)}$ are contractible when $n \geqslant 2$, so that $n = 1$ is the canonical height of choice.

Proposition 3.1. *We have*

$$\mathrm{ko}_{\mathrm{K}(1)}, \mathrm{ku}_{\mathrm{K}(1)} \in \mathrm{Char}(\zeta_1)$$

at all primes.

Proof. The K(1)–localizations (at p) agree with the p–completions of the periodic versions, compare [**26**, Lemma 2.3.5]. The complete periodic theories are well known to have vanishing π_{-1}. □

The situation for algebraic K-theory spectra is different: Let $\overline{\mathbb{F}}_q$ be an algebraic closure of a finite field $\mathbb{F}_q$ with q elements. If q is a power of the prime p we are working at, then the algebraic K-theory spectra $\mathrm{K}(\overline{\mathbb{F}}_q)_{\mathrm{K}(1)}$ and $\mathrm{K}(\mathbb{F}_q)_{\mathrm{K}(1)}$ are contractible by Quillen's work [**43**]. (To lift his space level statements to spectra, use the Bousfield–Kuhn functor, or [**39**]; see [**16**].)

We can therefore assume that the characteristic of $\mathbb{F}_q$ is different from p from now on, so that $q \in \mathbb{Z}_p^\times$. Then, again by Quillen, there is an equivalence $\mathrm{K}(\overline{\mathbb{F}}_q)_{\mathrm{K}(1)} \simeq \mathrm{E}_1$, and $\mathrm{K}(\mathbb{F}_q)_{\mathrm{K}(1)}$ can be identified with the homotopy fiber $\mathrm{E}_1^{\mathrm{h}\langle q\rangle}$ of the self-map $q - \mathrm{id}$ on E_1.

Proposition 3.2. *We have*

$$\mathrm{K}(\mathbb{F}_q)_{\mathrm{K}(1)} \notin \mathrm{Char}(\zeta_1)$$

at all primes different from q.

Proof. As we have remarked before, we have $q \in \mathbb{Z}_p^\times$, and this element has infinite order. It generates an infinite closed subgroup $\langle q\rangle$ such that the quotient $\mathbb{Z}_p^\times/\langle q\rangle$ is finite. The long exact sequence induced by (1.3) shows that ζ_1 is zero in $\pi_{-1}\mathrm{E}_1^{\mathrm{h}\langle q\rangle}$ if and only if there is an element f in

$$(\mathrm{B}_1)_0(\mathrm{E}_1^{\mathrm{h}\langle q\rangle}) \cong \mathcal{C}(\mathbb{Z}_p^\times/\langle q\rangle, \mathbb{Z}_p)$$

such that $f(gu) = f(u) + 1$ for all $u \in \mathbb{Z}_p^\times$, where g is as in (1.3). Since g has finite order in the finite group $\mathbb{Z}_p^\times/\langle q\rangle$, but 1 has infinite order in $\mathbb{Z}_p$, such an element f cannot exist. □

It follows immediately that many other algebraic K-theory spectra do not have characteristic ζ_1, for instance $\mathrm{K}(\mathbb{Z})_{\mathrm{K}(1)}$ and $\mathrm{K}(\mathbb{Z}_\ell)_{\mathrm{K}(1)}$ for primes $\ell \neq p$. The same is true for $\mathrm{K}(\mathbb{Z}_p)_{\mathrm{K}(1)}$, but this requires results of Bökstedt–Madsen (for odd primes p) or Rognes (for $p = 2$). In the former case, there is a p–adic splitting

$$\mathrm{K}(\mathbb{Z}_p) \simeq \mathrm{j} \vee \Sigma\mathrm{j} \vee \Sigma\,\mathrm{bu},$$

so that there is K(1)–local splitting

$$\mathrm{K}(\mathbb{Z}_p) \simeq \mathbb{S} \vee \Sigma\mathbb{S} \vee \Sigma\mathrm{E}_1,$$

and $\zeta_1 \neq 0$. In the latter case, the situation is the same up to extensions: We have

$$\Sigma\mathrm{j} \longrightarrow X \longrightarrow \Sigma\,\mathrm{ku}$$

for some spectrum X that is in

$$X \longrightarrow \mathrm{K}(\mathbb{Z}_2) \longrightarrow \mathrm{j}$$

(see [**12**] and [**45**]).

Remark 3.3. We have been concentrating on establishing only the non-triviality of the class ζ_1 for some algebraic K-theory spectra. In fact, Mitchell's work [**41**] explains that several unsolved conjectures in number theory are related to the K(1)–localization of algebraic K-theory spectra, and the behavior of ζ_1 on them. For instance, let F be a number field with ring $\mathcal{O}_F$ of integers. Let ℓ be an odd prime, and assume that $\mathcal{O}_F[1/\ell]$ contains the ℓ–th roots of unity. The $\mathbb{Z}_\ell$–rank of $\pi_0\mathrm{K}(\mathcal{O}_F[1/\ell])_{\mathrm{K}(1)}$ is the number s of primes dividing ℓ in $\mathcal{O}_F$. The image of multiplication with ζ_1,

$$\zeta_1\colon \pi_1\mathrm{K}(\mathcal{O}_F[1/\ell])_{\mathrm{K}(1)} \longrightarrow \pi_0\mathrm{K}(\mathcal{O}_F[1/\ell])_{\mathrm{K}(1)},$$

lies in the Adams filtration 1 subgroup $\mathrm{H}^2_{\text{ét}}(\mathcal{O}_F[1/\ell];\mathbb{Z}_\ell(1))$ of rank $s-1$. It turns out that the image has maximal rank $s-1$ if and only if an algebraic version of Gross' conjecture holds (see [**41**, 3.6.1]).

Remark 3.4. Thanks to Mitchell's earlier work [**40**], we know that the algebraic K-theory spectra $\mathrm{K}(R)_{\mathrm{K}(n)}$ are contractible for all (discrete) rings R and all heights $n \geqslant 2$. This does not hold if we are willing to work with ring *spectra* E instead: We have $\zeta_n \neq 0$ in $\mathrm{K}(\mathbb{S})_{\mathrm{K}(n)}$, because $\mathrm{K}(\mathbb{S})$ is equivalent to Waldhausen's A-theory of a point, and that splits off the sphere spectrum. It might be more interesting to study $\mathrm{K}(E)$ for $E = \mathrm{ko}$ or $E = \mathrm{ku}$ instead of $E = \mathbb{S}$. See the work [**5, 6**] of Ausoni–Rognes.

4. Topological modular forms

In this section, we discuss the spectrum tmf of topological modular forms. See the ICM talks [**20, 21**], [**23, 24**], and the Bourbaki seminar [**17**], for instance.

Proposition 4.1. *We have*

$$\mathrm{tmf}_{\mathrm{K}(2)} \in \mathrm{Char}(\zeta_2)$$

at all primes.

Proof. For $n = 2$, Behrens [**9**, Remark 1.7.3] has given an argument for the identification of the K(2)–localization of the spectrum of topological modular forms with EO_2, the homotopy fixed point spectrum of E_2 with respect to the maximal finite subgroup M of the extended Morava group G_2, that holds for the prime $p = 3$. His argument can be adapted to the case $p = 2$ as well.

Since the maximal finite subgroup M sits inside the subgroup G^1_2, the K(2)–localization of the topological modular forms spectrum is a commutative B_2–algebra. By Proposition 1.14 and Example 2.3, we know that $T \in \mathrm{Char}(\zeta_n)$ for all commutative B_n–algebras T.

The situation at large primes $p \geqslant 5$ is similar, but less well represented in the published literature. The K(2)–localization of tmf is the spectrum of global sections of the derived structure sheaf of the completion of the moduli stack of generalized elliptic curves in characteristic p at the complement of the ordinary locus. (See Behrens' notes [**10**], for instance.) This sheaf can be constructed using the Goerss–Hopkins–Miller theory of Lubin–Tate spectra. The upshot is that the spectra of sections are again given by homotopy fixed points of Lubin–Tate spectra with respect to finite subgroups. These lie in the kernel of any homomorphism to a torsion-free group. (A difference is that this time their orders are co-prime to the characteristic, but this does not play a role here.) In any case, we see that the same argument as for $p = 2$ and $p = 3$ can be applied. □

Remark 4.2. For $n \geqslant 3$, we trivially have $\mathrm{tmf}_{\mathrm{K}(n)} \in \mathrm{Char}(\zeta_n)$ as well, at all primes, because the spectrum tmf is $\mathrm{K}(n)$–acyclic in that case, so that the localization vanishes.

Remark 4.3. The case $n = 1$ is non-trivial and interesting. Hopkins has studied $\mathrm{tmf}_{\mathrm{K}(1)}$ at all primes, constructed a nullhomotopy of ζ_1 on $\mathrm{tmf}_{\mathrm{K}(1)}$ and used it to describe the latter as an E_∞ algebra over the versal example $\widehat{\mathbb{S}}/\!\!/\zeta_1$ with one more cell attached. See [**22**], [**36**], and [**10**].

Remark 4.4. The K(1)–local K3 spectra from [**46, 47**] are even hence obviously of characteristic ζ_1. No presentation as an E_∞ algebra over the versal example $\widehat{\mathbb{S}}/\!\!/\zeta_1$ is known in these cases.

Remark 4.5. The strategy for the proof of Proposition 4.1 can also be pursued to show that the higher real K-theories EO_{p-1} have chromatic characteristics.

5. Bordism theories

Since the sphere spectrum represents framed bordism, it is clear that not all bordism spectra have chromatic characteristics. In this section we discuss the bordism spectra MSpin and MString as well as their complex cousins MSU and $\mathrm{MU}\langle 6\rangle$.

Proposition 5.1. *We have*

$$\mathrm{MSpin}_{\mathrm{K}(n)} \in \mathrm{Char}(\zeta_n)$$

at all primes and all heights $n \geqslant 1$.

Proof. At odd primes, the spectrum MSpin is equivalent to a wedge of even suspensions of BP. (The splitting as a wedge of Brown–Peterson spectra is well-known [**26**]. It can be deduced from one of Steinberger's general splitting results [**15**, Theorem III.4.3]. We can then work rationally in order to see that only even suspensions are necessary. And rationally, both MSpin and BP are even. Compare a similar argument in the proof of Proposition 5.4 below.) Consequently, the $\mathrm{K}(n)$–localizations of MSpin are well understood at odd primes. The $\mathrm{K}(n)$–localization of BP has

$$\pi_* \mathrm{BP}_{\mathrm{K}(n)} = (v_n^{-1}\pi_* \mathrm{BP})_{p,v_1,\dots,v_{n-1}}.$$

This has been explained by Hovey [**25**, Lemma 2.3], for instance. We see that the homotopy groups $\pi_* \mathrm{BP}_{\mathrm{K}(n)}$ are concentrated in even degrees. This clearly implies $\pi_{-1} = 0$ for the $\mathrm{K}(n)$–localizations of MSpin, and *a fortiori* these have characteristic ζ_n.

The even prime $p = 2$ affords some extra arguments. Since the Spin bordism spectrum MSpin, as any Thom spectrum, is connective, we in particular have $\pi_{-1}\,\mathrm{MSpin} = 0$. This does not imply the result for the $\mathrm{K}(1)$–localization, however (think of $\mathbb{S}$). But, the Anderson–Brown–Peterson (ABP) splitting shows that this still holds after $\mathrm{K}(1)$–localization, since the spectrum MSpin splits $\mathrm{K}(1)$–locally at the prime $p = 2$ as a wedge of (unsuspended) localizations of copies of the spectrum ko (compare [**26**, Proposition 2.3.6]).

$$\mathrm{MSpin}_{\mathrm{K}(1)} \simeq \left(\bigvee_j \mathrm{KO}\right)_2$$

Therefore, the result follows from what we have said for the K-theories in Section 3, Proposition 3.1. For heights $n \geqslant 2$, the spectrum MSpin is $\mathrm{K}(n)$–acyclic, again by the ABP splitting. □

Remark 5.2. At odd primes, the spectrum MSU also decomposes into even suspensions of BP. (This time, the splitting is explicitly stated by Steinberger [**15**, Remarks III.4.4], and 'even' follows again by rational considerations.) We can similarly conclude that

$$\mathrm{MSU}_{\mathrm{K}(n)} \in \mathrm{Char}(\zeta_n)$$

at odd primes and all heights $n \geqslant 1$. At the prime $p = 2$, the situation is substantially different, since no simple ABP-type splitting is known. See Pengelley [**42**], who found the BoP summands. According to Reeker's thesis [**44**], we have at least $\mathrm{MSU}_{\mathrm{K}(1)} \in \mathrm{Char}(\zeta_1)$.

The canonical maps $\mathrm{MString} \to \mathrm{MSpin}$ and $\mathrm{MU}\langle 6\rangle \to \mathrm{MSU}$ of bordism spectra are both $\mathrm{K}(1)$–local equivalences [**26**, Prop. 2.3.1]. Therefore, we immediately get:

Corollary 5.3. *We have*

$$\mathrm{MString}_{\mathrm{K}(1)}, \mathrm{MU}\langle 6\rangle_{\mathrm{K}(1)} \in \mathrm{Char}(\zeta_1)$$

at all primes p.

For $n \geqslant 2$ we can offer the following result.

Proposition 5.4. *We have*

$$\mathrm{MString}_{\mathrm{K}(n)}, \mathrm{MU}\langle 6\rangle_{\mathrm{K}(n)} \in \mathrm{Char}(\zeta_n)$$

at all primes $p \geqslant 5$ and all heights $n \geqslant 2$.

Proof. If $p \geqslant 5$, then both MString and $\mathrm{MU}\langle 6\rangle$ split p–locally as wedges of suspensions of BP by [**29**, Corollary 2.2]. See also [**27**].

$$\mathrm{MU}\langle 6\rangle_{(p)} \simeq \bigvee_j \Sigma^{m_j}\,\mathrm{BP}_{(p)}$$

$$\mathrm{MString}_{(p)} \simeq \bigvee_j \Sigma^{n_j}\,\mathrm{BP}_{(p)}$$

We can work rationally in order to obtain information about the suspensions m_j and n_j needed, and that is easy: Since $\pi_*\,\mathrm{MU}\langle 6\rangle_{\mathbb{Q}} \cong \mathbb{Q}[c_2, c_3, \dots]$ with $|c_n| = 2n$, and $\pi_*\,\mathrm{BP}_{\mathbb{Q}} \cong \mathbb{Q}[v_1, v_2, \dots]$ with $|v_n| = 2(p^n - 1)$, we see that the m_j are even. Similarly for the n_j, using $\pi_*\,\mathrm{MString}_{\mathbb{Q}} \cong \mathbb{Q}[p_2, p_3, \dots]$ and $|p_n| = 4n$.

A fortiori, these additive decompositions exist also $\mathrm{K}(n)$–locally at the prime in question. The $\mathrm{K}(n)$–localization of BP has

$$\pi_*\,\mathrm{BP}_{\mathrm{K}(n)} = (v_n^{-1}\pi_*\,\mathrm{BP})_{p,v_1,\dots,v_{n-1}}, \tag{5.1}$$

see [**25**, Lemma 2.3] again. Therefore, the homotopy groups of both of the spectra $\mathrm{MU}\langle 6\rangle_{\mathrm{K}(n)}$ and $\mathrm{MString}_{\mathrm{K}(n)}$ are concentrated in even degrees. We can deduce that both of the groups $\pi_{-1}\,\mathrm{MString}_{\mathrm{K}(n)}$ and $\pi_{-1}\,\mathrm{MU}\langle 6\rangle_{\mathrm{K}(n)}$ vanish for primes $p \geqslant 5$, from which the statement follows. □

Remark 5.5. For the small primes $p = 2$ and $p = 3$ it is still true that we are able to find finite complexes F (depending on p) with cells only in even dimensions such that $\mathrm{MString} \wedge F$ and $\mathrm{MU}\langle 6\rangle \wedge F$ split as wedges of (even) suspensions of BP (see [**29**, Corollary 2.2]). Strictly speaking, this excludes the case MString at the prime $p = 2$. But, since there is a map

$$\mathrm{MU}\langle 6\rangle \longrightarrow \mathrm{MString},$$

Proposition 1.14 guarantees that it would be sufficient to prove that $\mathrm{MU}\langle 6\rangle$ has chromatic characteristics to be able to infer that for the string bordism spectrum as well.

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References

[1] M. Ando, M.J. Hopkins, C. Rezk. Multiplicative orientatons of KO–theory and the spectrum of topological modular forms. Preprint. www.math.uiuc.edu/∼mando/papers/koandtmf.pdf

[2] M. Ando, M.J. Hopkins, N.P. Strickland. Elliptic spectra, the Witten genus and the theorem of the cube. Invent. Math. 146 (2001) 595–687.

[3] M. Ando, M.J. Hopkins, N.P. Strickland. The sigma orientation is an H_∞ map. Amer. J. Math. 126 (2004) 247–334.

[4] M. Artin. Algebraization of formal moduli. I. Global Analysis (Papers in Honor of K. Kodaira) 21–71. Univ. Tokyo Press, Tokyo, 1969.

[5] C. Ausoni. On the algebraic K-theory of the complex K-theory spectrum. Invent. Math. 180 (2010) 611–668.

[6] C. Ausoni, J. Rognes. Algebraic K-theory of topological K-theory. Acta Math. 188 (2002) 1–39.

[7] A. Baker. Characteristics for E_∞ ring spectra. An alpine bouquet of algebraic topology, 1–17. Contemp. Math. 708. Amer. Math. Soc., Providence, RI, 2018.
[8] A. Beaudry, P.G. Goerss, H.-W. Henn. Chromatic splitting for the $K(2)$-local sphere at $p = 2$. Preprint.
arxiv.org/abs/1712.08182
[9] M. Behrens. A modular description of the K(2)–local sphere at the prime 3. Topology 45 (2006) 343–402.
[10] M. Behrens. The construction of tmf. Topological modular forms (Talbot Workshop 2007) 131–188. Mathematical Surveys and Monographs 201. American Mathematical Society, 2014.
[11] M. Behrens, D.G. Davis. The homotopy fixed point spectra of profinite Galois extensions. Trans. Amer. Math. Soc. 362 (2010) 4983–5042.
[12] M. Bökstedt, I. Madsen. Topological cyclic homology of the integers. K-theory (Strasbourg 1992). Astérisque 226 (1994) 57–143.
[13] B.I. Botvinnik. Structure of the ring MSU_*. Math. USSR-Sb. 69 (1991) 581–596.
[14] A.K. Bousfield. The localization of spectra with respect to homology. Topology 18 (1979) 257–281.
[15] R.R. Bruner, J.P. May, J.E. McClure, M. Steinberger. H_∞ ring spectra and their applications. Lecture Notes in Mathematics 1176. Springer-Verlag, Berlin, 1986.
[16] E.S. Devinatz, M.J. Hopkins. Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups. Topology 43 (2004) 1–47.
[17] P.G. Goerss. Topological modular forms [after Hopkins, Miller and Lurie]. Séminaire Bourbaki. Volume 2008/2009. Exposés 997–1011. Astérisque 332 (2010), Exp. No. 1005, viii, 221–255.
[18] P.G. Goerss, H.-W. Henn, M. Mahowald. The rational homotopy of the K(2)–local sphere and the chromatic splitting conjecture for the prime 3 and level 2. Doc. Math. 19 (2014) 1271–1290.
[19] P.G. Goerss, H.-W. Henn, M. Mahowald, C. Rezk. On Hopkins' Picard groups for the prime 3 and chromatic level 2. J. Topol. 8 (2015) 267–294.
[20] M.J. Hopkins. Topological modular forms, the Witten genus, and the theorem of the cube. Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994) 554–565. Birkhäuser, Basel, 1995.
[21] M.J. Hopkins. Algebraic topology and modular forms. Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002) 291–317. Higher Ed. Press, Beijing, 2002.
[22] M.J. Hopkins. K(1)–local E_∞–ring spectra. Topological modular forms (Talbot Workshop 2007) 287–302. Mathematical Surveys and Monographs 201. American Mathematical Society, 2014.
[23] M.J. Hopkins and M. Mahowald. From elliptic curves to homotopy theory. Topological modular forms (Talbot Workshop 2007) 287–302. Mathematical Surveys and Monographs 201. American Mathematical Society, 2014.
[24] M.J. Hopkins and H.R. Miller. Elliptic curves and stable homotopy I. Topological modular forms (Talbot Workshop 2007) 287–302. Mathematical Surveys and Monographs 201. American Mathematical Society, 2014.
[25] M.A. Hovey. Bousfield localization functors and Hopkins' chromatic splitting conjecture. The Čech centennial (Boston, MA, 1993) 225–250. Contemp. Math. 181. Amer. Math. Soc., Providence, AI, 1995.
[26] M.A. Hovey. v_n–elements in ring spectra and applications to bordism theory. Duke Math. J. 88 (1997) 327–356.
[27] M.A. Hovey. The homotopy of MString and $MU\langle 6\rangle$ at large primes. Algebr. Geom. Topol. 8 (2008) 2401–2414.
[28] M.A. Hovey. Operations and co-operations in Morava E-theory. Homology Homotopy Appl. 6 (2004) 201–236.

[29] M.A. Hovey, D.C. Ravenel. The 7-connected cobordism ring at $p = 3$. Trans. Amer. Math. Soc. 347 (1995) 3473–3502.
[30] M.A. Hovey, N.P. Strickland. Morava K-theories and localisation. Mem. Amer. Math. Soc. 139 (1999) 666.
[31] M. Joachim. A symmetric ring spectrum representing KO–theory. Topology 40 (2001) 299–308.
[32] M. Joachim. Higher coherences for equivariant K-theory. Structured ring spectra, 87–114. London Math. Soc. Lecture Note Ser. 315. Cambridge Univ. Press, Cambridge, 2004.
[33] S.O. Kochman. The ring structure of BoP. Algebraic topology (Oaxtepec, 1991), 171–197. Contemp. Math. 146. Amer. Math. Soc., Providence, RI, 1993.
[34] G. Laures. The topological q–expansion principle. Topology 38 (1999) 387–425.
[35] G. Laures. E_∞ splitting of spin bordism. Amer. J. Math. 125 (2003) 977–1027.T
[36] G. Laures. K(1)–local topological modular forms. Invent. Math. 157 (2004) 371–403.
[37] L.G. Lewis, J.P. May, M. Steinberger. Equivariant stable homotopy theory. With contributions by J.E. McClure. Lecture Notes in Mathematics 1213. Springer-Verlag, Berlin, 1986.
[38] M.A. Mandell, J.P. May, S. Schwede, B. Shipley. Model categories of diagram spectra. Proc. London Math. Soc. 82 (2001) 441–512.
[39] J.P. May. E_∞ ring spaces and E_∞ ring spectra. With contributions by Frank Quinn, Nigel Ray, and Jørgen Tornehave. Lecture Notes in Mathematics 577. Springer-Verlag, Berlin-New York, 1977.
[40] S.A. Mitchell. The Morava K-theory of algebraic K-theory spectra. K-Theory 3 (1990) 607–626.
[41] S.A. Mitchell. K(1)–local homotopy theory, Iwasawa theory and algebraic K-theory. Handbook of K-theory. Vol. 1, 2, 955–1010. Springer, Berlin, 2005.
[42] D.J. Pengelley. The homotopy type of MSU. Amer. J. Math. 104 (1982) 1101–1123.
[43] D. Quillen. On the cohomology and K-theory of the general linear groups over a finite field. Ann. of Math. 96 (1972) 552–586.
[44] H. Reeker. On K(1)–local SU–bordism. Dissertation, Bochum, 2009.
[45] J. Rognes. Algebraic K-theory of the two-adic integers. J. Pure Appl. Algebra 134 (1999) 287–326.
[46] M. Szymik. K3 spectra. Bull. Lond. Math. Soc. 42 (2010) 137–148.
[47] M. Szymik. Crystals and derived local moduli for ordinary K3 surfaces. Adv. Math. 228 (2011) 1–21.
[48] M. Szymik. Commutative $\mathbb{S}$–algebras of prime characteristics and applications to unoriented bordism. Algebr. Geom. Topol. 14 (2014) 3717–3743.

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