New Results on Trajectory Planning for Underactuated Mechanical Systems with Singularities in Dynamics of a Motion Generator*

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***Abstract*— The new approach to the problem of motion planning for underactuated mechanical systems is proposed. The novelty comes from new opportunities to handle singularities of the dynamics of the motion generator provided that the motion is rewritten using the nested representation and kinematic servo-connection between generalized coordinates of the system. The contribution is illustrated by the example of planning oscillations of the Furuta pendulum around the horizontal.**

I. Introduction

Searching for feasible behaviors of controlled mechanical systems with one or several passive degrees of freedom is challenging. Indeed, the dynamics of such variables constitute a continuum set of equality constraints that any motion planning method should comply with. In most of examples such dynamic constraints are non-integrable and cannot be ignored or discretized. Besides various direct shooting methods, an implicit method based on a nested representation of a movement of a mechanical system with two and more degrees of freedom provides the only analytic approach for solving the task[1]. The approach assumes a choice of a scalar coordinate s of an n-DOF mechanical system suitable for representing a given movement

$$q(t) = [q_1(t); \ldots; q_n(t)], \quad t \in [0, T]$$

as a sequence of postures parameterized by that variable

$$q(t) = Q\left(s(t)\right), \quad t \in [0, T]. \qquad (1)$$

Loosely speaking, the coordinate $s(\cdot)$ serves as a motion generator substituting the constant and one-directional flow of time. If the vector function $Q(\cdot)$ is twice differentiable, then Eqn. (1) allows reconstructing velocities and accelerations of the mechanical system along the given motion as well provided that $s(\cdot) \in C^2([0, T])$. The kinematic relation (1) is commonly referred to as *servo-constraint* or *servo-connection* if it is arranged for a given motion by a feed-forward control action.

*The work of S.V. Gusev is supported by the RFBR, the grant No. 17-08-00715. The work of M.O. Surov and A.S. Shiriaev is partly supported by the Norwegian Research Council, the grant No. 262363.

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[1]It is often loosely mentioned as a virtual holonomic constraint (VHC) approach. Such name is partly misleading and partly incorrect taking into account the vocabulary of analytic mechanics (see [1]). Furthermore, the basics of the method are known for a century elaborated in parallel with well accepted terminology, which we plan to adhere and follow.

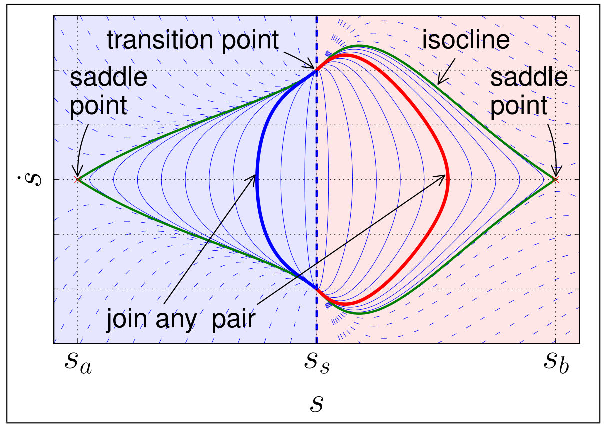

Fig. 1. Phase space of a singular differential equation

For completeness of a motion representation, the kinematic relation (1) should be complemented by a rule that determines a time behavior of the coordinate $s(\cdot)$. For most of underactuated mechanical systems the time profile of $s(\cdot)$ cannot be assigned arbitrary: it should be a solution of a second order differential equation of the form

$$\alpha\left(s\right)\ddot{s} + \beta\left(s\right)\dot{s}^2 + \gamma\left(s\right) = 0 \qquad (2)$$

with the coefficients $\alpha(\cdot)$, $\beta(\cdot)$ and $\gamma(\cdot)$ defined by the system dynamics and by the function $Q(\cdot)$ used in the nested representation (1) of the motion[2]. The reader can think of the system (2) as a counterpart for the time dynamics $\ddot{t} = 0$ in searching an alternative representation of the given motion.

Since the servo-connection of the form (1) is introduced as an attribute of an individual behavior of an underactuated mechanical system, then the equation (2) is also used and necessary for a compact representation of only one of its solutions. This point is typically overlooked in literature and quite a number of publications are focused on exploring the phase portrait of the system (2) as a whole bringing even further assumptions to simplify the analysis. For instance, following the generic view, it is commonly assumed that the variable $s(\cdot)$ is chosen as one of original generalized degrees of freedom of the system. Meanwhile, a possibility to reserve one of the system's excessive coordinates for the motion representation is ignored. Another part of commonly accepted settings for motion planning postulates that for a given motion $q(\cdot)$ and for smooth function $Q(\cdot)$ in (1) the scalar second order system (2) can be resolved with respect to

[2]formulae for these coefficients will be given in section II.

higher derivative becoming an ordinary differential equation for the predefined range of the variable s.

However, the ability to resolve Eqn. (2) with respect to $\ddot{s}$ is an assumption, and for nested representations of some of behaviors it might not be hold. This point has been recently illustrated by the case study aimed at planning periodic motions of a cart-pendulum system when its passive pendulum was forced to oscillate around the horizontal. Even though such behavior is unrealistic for implementing experiments on most of the laboratory set-ups due to high demands on amplitude of a feed-forward control force and its derivative for creating the motion, planning such motion helps to show severe limitations of the settings admitted as standard in the topic: In particular, the arguments of [2] for the case study have shown the constructive procedure for deriving a smooth periodic solution of the system (2), which do not encircle any of its equilibrium. Detecting a presence of such periodic solutions would contradict to one of assertions of the classical Poincare-Bendixson statement unless one observes that for such successfully found periodic solution of the system (2) the coefficient $\alpha(\cdot)$ becomes zero for some $s = s_s$ within the range the cycle.

This unexpected conclusion drawn for the case study is rather convincing for reconsidering the settings and methods commonly accepted for motion planning under dynamic constraint due to underactuation. However, the example elaborated and explored in [2] was based on structural properties of the cart-pendulum dynamics and cannot be readily generalized to other underactuated systems. The paper provides new generic procedure and its justification aimed at searching feasible motions of under-actuated mechanical systems for the cases of the nested representation of the motion when the assumption of the non-singularity of the system (2) is dropped. In particular, we have investigated the situation where the servo-connection (1) used in the representation leads to the dynamics of the motion generator (2) with the following properties: the singular point s_s with $\alpha(s_s) = 0$ is in between two saddle equilibrium points s_a and s_b, $s_a < s_s < s_b$, see figure 1.

Due to the singularity the classical conditions of existence and uniqueness of solutions of the equation (2) might not anymore be valid. The contribution of the paper shows that solutions exist and intersect the vertical line $s = s_s$ only in two points, which, for convenience, are referred to as "transition points." Furthermore, the rigorous analysis allows establishing the extraordinary property of the singular dynamical system (2): any solution originated on left-hand side of this vertical line can be prolonged and has infinitely many extensions on the right-hand side. By choosing these extensions, one can obtain a rich family of feasible solution, which includes both periodic trajectories encircling the point $(s, \dot{s}) = (s_s, 0)$ on the phase plane and non-periodic behaviors as well.

The paper is organized as follows. Section II contains derivation of dynamics of the motion generator, and gives motivation to study of motion generator with singularities. A geometric meaning of motion generator with singularities is given in section III. Theorem about existence of smooth periodic solutions is given in section IV. As an example, theorem is applied to find horizontal oscillations of Furuta pendulum in section V.

II. Derivation of motion generator equation

For completeness let us shortly present the derivation of dynamics of motion generator for an underactuated Euler-Lagrange system with one passive degree of freedom

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \begin{bmatrix} u \\ 0 \end{bmatrix}, \tag{3}$$

where $q \in \mathcal{M}$ are generalized coordinates, $\dot{q} = \frac{dq}{dt}$ are corresponding generalized velocities, $\mathcal{M}$ is a configuration space of dimension n, $M(q)$ is the inertia tensor, $C(q, \dot{q})$ is Coriolis and centrifugal forces matrix, $G(q)$ is a vector of gravity, u are the control inputs and $\dim u = n - 1$.

Let $q(t)$ be a solution of system (3) in response of the input $u(t)$ both defined for $t \in I \subset \mathbb{R}$. Suppose the solution $q(t)$ can be written in a parametric form (1), where the map $Q : \mathbb{R} \to \mathcal{M}$ and the function $s : \mathbb{R} \to \mathbb{R}$ are twice differentiable, then the function $s(t)$ is one of solutions of the equation

$$\begin{aligned} &M(Q)\left(\frac{d^2Q}{ds^2}\dot{s}^2 + \frac{dQ}{ds}\ddot{s}\right) + \\ &C\left(Q, \frac{dQ}{ds}\right)\frac{dQ}{ds}\dot{s}^2 + G(Q) = \begin{bmatrix} u \\ 0 \end{bmatrix}. \end{aligned} \tag{4}$$

Premultiplying this equation by $B = [0, \ldots, 1] \in \mathbb{R}^{1\times n}$ we get the dynamics of the motion generator

$$\alpha(s)\ddot{s} + \beta(s)\dot{s}^2 + \gamma(s) = 0 \tag{5}$$

with

$$\begin{aligned} \alpha(s) &= B \cdot M(Q(s))\frac{dQ(s)}{ds}, \\ \beta(s) &= B \cdot M(Q(s))\frac{d^2Q(s)}{ds^2} + \\ &\quad B \cdot C\left(Q(s), \frac{dQ(s)}{ds}\right)\frac{dQ(s)}{ds}, \\ \gamma(s) &= B \cdot G(Q(s)) \end{aligned}$$

(see [4] for details).

Furthermore, the equation (5) can be a source of new feasible trajectories for underactuated systems. In particular, if a solution of the equation (5) is well-defined for some time interval, then substituting this solution into the servo-connection (1) will define the new feasible trajectory. Such an argument with explicit assumption that the functions $\alpha(s), \beta(s), \gamma(s)$ are continuous and $\alpha(s) \neq 0$ has been used in most of the publications on motion planning for underactuated mechanical systems [10], [11], [12].

However, the settings become limited for recovering some of feasible behaviors. Indeed, the example of [2] has shown that the motion planning based on the servo-connection leading to smooth $\alpha(s), \beta(s), \gamma(s)$ with $\alpha(s) \neq 0$ cannot reconstruct periodic behavior of the cart-pendulum system when the pendulum oscillates around the horizontal. The presence of such oscillations will automatically imply the presence of an equilibrium for non-singular dynamical system (5), which by necessity should be a solution of the equation $\gamma(s_e) = 0$. Meanwhile, as it can be seen from (5), the function γ vanishes only when $B \cdot G(Q(s_e)) = 0$. It is impossible in a vicinity of the horizontal position of the pendulum for the inverted pendulum on a cart, since the original system doesn't have any equilibria in such vicinity for any position of the cart. Therefore, new attempts and considerations of servo-connection leading to singularity of (5) are well-motivated, and in general can lead to description of new feasible behaviors of underactuated mechanical systems.

III. Singular differential equation of 2nd order

Consider the equation (5) with smooth coefficients, if function α is separated from zero, then the right hand side of the equation

$$\ddot{s} = \frac{-\beta(s)\dot{s}^2 - \gamma(s)}{\alpha(s)} \tag{6}$$

is (locally) Lipschitz for any initial condition. In this case any periodic trajectory must encircle at least one equilibrium point s_e given by the equation $\gamma(s_e) = 0$ (see theorem 33 in [5] for details).

On the other hand, if at some point s_s α becomes zero (i.e. $\alpha(s_s) = 0$), then the statement cannot be used, even though the functions α, β, γ in a neighborhood of s_s may satisfy some additional requirements, so the equation have smooth solutions.

For this case one can notice that in the left half-plane $\left\{(s, \dot{s}) \in \mathbb{R}^2 | s < s_s\right\}$ of phase space of the equation (6) the Picard–Lindelöf theorem can be applied, and solutions exist at least on finite time intervals. Consider solution $s(t)$ of (5) defined on $t \in I = (0, \tau) \subset \mathbb{R}$ approaching the critical point s_s from the left: $\lim_{t\to\tau} s(t) = s_s$. If $s(t)$ is a twice differentiable on I, then its time derivative must satisfy: $\lim_{t\to\tau}\left(\beta(s)\dot{s}^2 + \gamma(s)\right) = 0$, which can be rewritten as

$$\lim_{t\to\tau} \dot{s}^2(t) = -\frac{\gamma(s_s)}{\beta(s_s)}.$$

Any of such trajectories therefore will converge from the left to one of the transition points: $(s_s, \pm\dot{s}_s)$, where $\dot{s}_s = \sqrt{-\frac{\gamma(s_s)}{\beta(s_s)}}$ (see figure 1). In these points uniqueness of solutions is then violated. The similar arguments can be applied for trajectories in the right half-plane of the phase plane.

Suppose the point s_s is between two equilibriums s_a and s_b of "saddle" type. Then the isoclines (green curves on figure 1) connect the equilibriums with transition points. Since the uniqueness of solutions of (6) is violated only at the line $s = s_s$, then one can conclude that any trajectory originated from $s \in (s_a, s_s) \cup (s_s, s_b)$ will converge to the points $(s_s, \pm\dot{s}_s)$. This results in two sets of solutions crossing these points from left to the right and from right to the left respectively.

Provided that these solutions crossing the transition points have smooth second derivative $\ddot{s}$ at transition points, one can reconstruct the smooth trajectory well-defined for both right and left hand side of phase space. Such a "concatenated trajectory" will be obviously a solution of (5).

IV. Periodic motion planning

The next theorem provides sufficient conditions of existence of smooth solutions of singular equation of type (2).

Theorem 1. *Let there exist 3 values $s_a, s_b, s_s \in \mathbb{R}$, $s_a < s_s < s_b$ such that the functions α, β, γ of equation (5) satisfy the conditions:*

1) $\alpha \in C^3(s_a, s_b)$, *and* $\beta, \gamma \in C^2(s_a, s_b)$,
2) $\alpha(s_s) = 0$, $\alpha'(s_s) > 0$,
3) $\alpha(s) \neq 0\ \forall s \in [s_a, s_b] \setminus \{s_s\}$,
4) $\gamma(s_a) = \gamma(s_b) = 0$, $\gamma'(s_a) > 0$, $\gamma'(s_b) < 0$,
5) $\gamma(s) > 0\ \forall s \in (s_a, s_b)$,
6) $\frac{\beta(s_s)}{\alpha'(s_s)} < -\frac{1}{2}$.

Then the following statements hold:

1) *for any solution $s_l(t)$ of initial value problem (5) with $s_l(0) \in (s_a, s_s)$, $\dot{s}_l(0) = 0$ there is a time moment $\tau_l \in (0, \infty)$ such that*
$$\lim_{t\to\tau_l - 0} s_l(t) = s_s,$$
$$\lim_{t\to\tau_l - 0} \dot{s}_l(t) = \sqrt{-\frac{\gamma(s_s)}{\beta(s_s)}};$$
2) *for any solution $s_r(t)$ with $s_r(0) \in (s_s, s_b)$, $\dot{s}_r(0) = 0$, there is a time moment $\tau_r \in (0, \infty)$ such that*
$$\lim_{t\to -\tau_r + 0} s_r(t) = s_s,$$
$$\lim_{t\to -\tau_r + 0} \dot{s}_r(t) = \sqrt{-\frac{\gamma(s_s)}{\beta(s_s)}}; \tag{7}$$
3) *the twice differentiable function $s(t)$ defined as*
$$s(t) := \begin{cases} s_l(t) & t \in [0, \tau_l) \\ s_s & t = \tau_l \\ s_r(t - T) & t \in (\tau_l, T], \end{cases}$$
$$s(T + t) = s(T - t) \quad \forall t \in \mathbb{R},$$
where $T = \tau_l + \tau_r$, is a periodic solution of the equation (5) with period $2T$;
4) *the trajectory $q(t) = Q(s(t))$ is a periodic solution of (3) defined in response with control input given by (4).*

Remark 2. Since the solution $s(t)$ is twice differentiable, then the corresponding control input $u_\star(t)$ is a continuous function. Furthermore, according to (4) the nominal control input $u_\star$ can be expressed as a function of s.

Remark 3. The statement can be applied to solve the problem of trajectory planning for point-to-point movement. Indeed, for this case one needs to find a trajectory which begins at point s_1 and ends at s_2. One way to solve the problem is to find such a map Q that is defined on $s_1 < s_s < s_2$ and corresponding conditions of theorem 1 hold. Then, as stated, the pair of solutions $s_l(t)$, $s_r(t)$ of initial value problem with $s_l(0) = s_1$, $\dot{s}_l = 0$ and $s_r(0) = s_2$, $\dot{s}_r(0) = 0$ can be concatenated into the function $s(t)$, which solves the original problem.

V. Example: Furuta Pendulum Oscillations Around a Horizontal

As shown in [2], the cart-pendulum system possesses a periodic solution $q_\star(t)$ in response to $u_\star(t)$, for which the pendulum oscillates in a small neighborhood of the horizontal. The proof of existence of such a solution was based on the fact that the right-hand side of dynamic equations does not depend on cart displacement and its velocity. Unfortunately, such a solution cannot be readily generalized for other systems, like Furuta pendulum [6] or Pendubot [7]. Nevertheless, theorem 1 gives alternative criterion and can be applied for shaping new class of oscillations of such underactuated systems.

A. Equations of Motion of Furuta Pendulum

The equations of motion of Furuta pendulum are [8]:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \begin{bmatrix} u \\ 0 \end{bmatrix} \tag{8}$$

with

$$M = \begin{bmatrix} m_1 l_1^2 + m_2\left(l_1^2 + l_2^2 \sin^2\theta\right) & m_2 l_1 l_2 \cos\theta \\ m_2 l_1 l_2 \cos\theta & m_2 l_2^2 \end{bmatrix}$$

$$C = m_2 \begin{bmatrix} l_2^2 \dot{\theta}\cos\theta\sin\theta & \left(l_2^2\dot{\phi}\cos\theta - l_1 l_2\dot{\theta}\right)\sin\theta \\ -l_2^2\dot{\phi}\cos\theta\sin\theta & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} 0 \\ -g m_2 l_2 \sin\theta \end{bmatrix}$$

where $q = (\phi, \theta)$ are generalized coordinates with ϕ being the angle (azimuth) of the first link, θ being the angle between the upward vertical and the second link, m_1, m_2 are masses of the links, l_1, l_2 are lengths of the links, g is the gravitational acceleration, u is the torque applied to the first link.

B. Periodic Trajectories of Furuta Pendulum Around the Horizontal

Here we will explore conditions of theorem 1 to the system (8) and prove existence of a periodic solution when the second link of the Furuta pendulum remains all the time in a neighborhood of the horizontal. The problem can be formulated as follows:

Problem 4. Find a solution $q_\star(t)$ with corresponding control input $u_\star(t)$ of system (8) satisfying the properties

- the trajectory $q_\star(t)$ is twice differentiable;
- the control input $u_\star(t)$ is continuous;
- the functions are periodic with some period T: $\exists T \in (0,\infty)$: $q_\star(t+T) = q_\star(t)$, $u_\star(t+T) = u_\star(t)$ $\forall t \in \mathbb{R}$;
- the function $\theta_\star(t)$ stays within the interval $\left(-\frac{3}{4}\pi, -\frac{1}{4}\pi\right)$

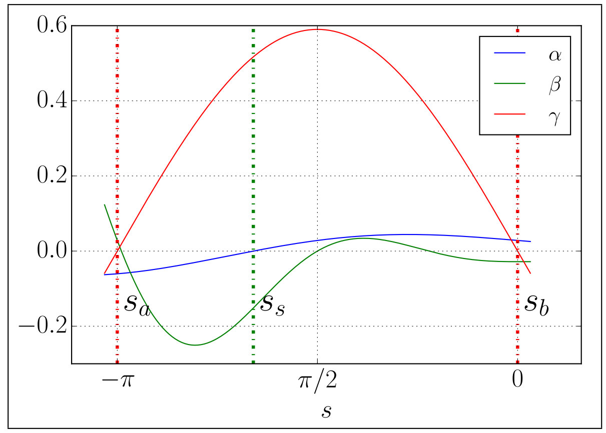

Fig. 2. Functions $\alpha(s)$, $\beta(s)$, $\gamma(s)$

when the parameters are [3]: $l_1 = 0.262$m, $l_2 = 0.470$m, $m_1 = 0.431$kg, $m_2 = 0.128$kg, $g = 9.81$m/s^2.

Let the map Q in (1) be a polynomial

$$Q(s) = \begin{pmatrix} -\frac{l_2}{2l_1}s^2 \\ s \end{pmatrix},$$

then after dividing by $m_2 l_2$ the equation (5) will be

$$\alpha(s)\ddot{s} + \beta(s)\dot{s}^2 + \gamma(s) = 0$$

with

$$\alpha = l_1\cos s\left(-\frac{l_2}{l_1}s\right) + l_2,$$
$$\beta = -l_2\cos s\sin s\left(-\frac{l_2}{l_1}s\right)^2 + l_1\cos s\left(-\frac{l_2}{l_1}\right),$$
$$\gamma = -g\sin s. \tag{9}$$

According to theorem 1, one needs to find two saddle points which for our case can be defined as $s_a = -\pi$, $s_b = 0$. Let s_s be a solution of $s_s \cos s_s = 1$ on $[s_a, s_b]$. It's approximate value is given by $s_s \approx -2.074$. It can be easily checked that the functions (9) meet the conditions 1-5 of theorem. The condition 6 gives the restriction for the system parameters $l_2 > 0.948 \cdot l_1$. It is valid for the system under consideration (actually, this restriction can be eliminated by choosing a different Q). Thus, all the conditions of theorem 1 are valid. The phase portrait of equation (9) is depicted on figure 3. This plot shows trajectories in a neighborhood of singularity where the uniqueness of solutions is violated: all the trajectories are crossing the vertical line $s = s_s$ with the same velocity. While, the phase coordinates stay continuous in the neighborhood of transition points as well as corresponding control input (see red curves on figure 7 and figure 4).

C. Orbital Stabilization

Here we indicate the method for orbital stabilization of the found trajectory of the system (8). To this end we rewrite the system (3) in affine form with state vector $x = \left(\phi, \theta, \dot{\phi}, \dot{\theta}\right)$:

$$\dot{x} = f(x) + g(x)u \tag{10}$$

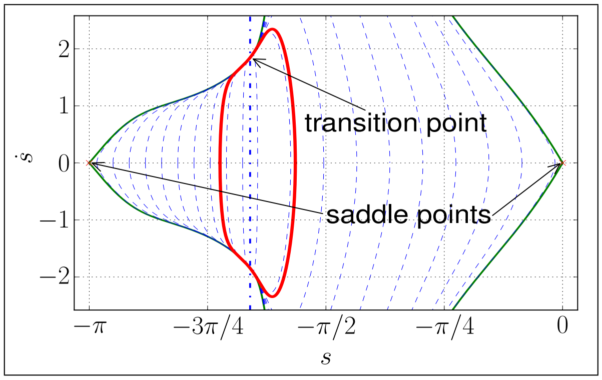

Fig. 3. Phase portrait of equation 9

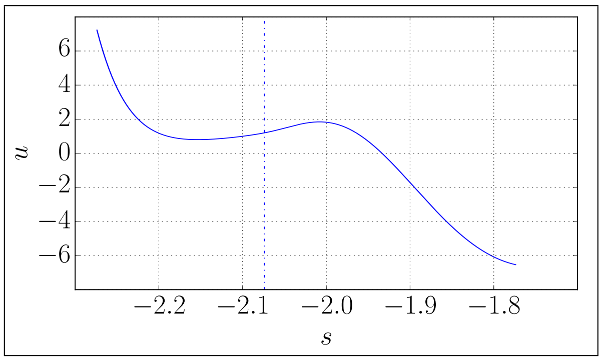

Fig. 4. Control input as a function of s

where

$$f(x) = \begin{pmatrix} \dot{q} \\ -M^{-1}(q)\left(C(q,\dot{q})\,\dot{q} + G(q)\right) \end{pmatrix},$$
$$g(x) = \begin{pmatrix} 0 \\ \mathrm{col}_1 M^{-1}(q) \end{pmatrix}$$

are smooth functions.

Let the desired trajectory $x_\star(t)$ of the system (10) be constructed as described in previous subsection. It means that there exists the input $u_\star(t)$ such that

$$\dot{x}_\star = f(x_\star) + g(x_\star)\, u_\star.$$

The aim is to find such a feedback $u(x)$ that the given trajectory $x_\star(t)$ be orbitally stable for the closed loop system (10). For this purpose let us represent the desired input $u_\star(t)$ as a function of state $u_\star(x)$, such that $u_\star(t) = u_\star(x_\star(t))$. This function can be directly obtained from the equation (4) (see figure 4).

For the synthesis of feedback controller we reuse the procedure presented in [2]. It is based on the transverse linearization approach, where for a given trajectory $x_\star(t)$ of underactuated mechanical system (considered as a curve in the phase space of the system) one reconstruct a family of moving orthogonal hyperplanes Π_t. On each hyperplane we consider a local coordinate frame with basis $P(t) \in \mathbb{R}^{4\times 3}$ originated at the intersection of Π_t and $x_\star(t)$. Here matrix $P(t)$ consists of an orthogonal complement of the unit tangent vector $\frac{\dot{x}_\star(t)}{\|\dot{x}_\star(t)\|}$ (see [2] for more details).

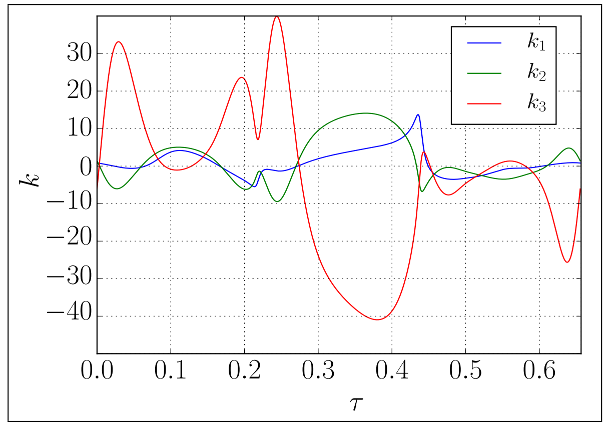

Fig. 5. LQR coefficients

Let us introduce also the new control input v as

$$v = u - u_\star(x),$$

and define $h(x) \equiv f(x) + g(x)\, u_\star(x)$. Then the system (10) can be written as

$$\dot{x} = h(x) + g(x)\, v. \tag{11}$$

In a small tubular neighborhood of curve $x_\star(t)$ we define local coordinates (τ, ξ) by the rule

$$\tau = \arg\min_t \|x - x_\star(t)\|,$$
$$x = x_\star(\tau) + P(\tau)\,\xi.$$

The variable τ plays the role of the new time for the linearized system. The vector ξ represents deviations of real trajectory from desired one. The dynamics of (11) in coordinates τ, ξ can be written as

$$\dot{\xi} = \frac{dP^T}{d\tau} P \xi \dot{\tau} + P^T\left(h(x) - h(x_\star)\,\dot{\tau} + g(x)\, v\right)$$

and it's linearization is

$$\frac{d\xi}{d\tau} \approx A(\tau)\,\xi + b(\tau)\, v \tag{12}$$

with $A(\tau) = \frac{dP^T}{d\tau} P + P^T \frac{\partial h}{\partial x}\big|_{x_\star} P$, and $b(\tau) = P^T g(x_\star)$.

To stabilize the given system we designed an LQR of the form $v = k(\tau)\,\xi$, where $k \in \mathbb{R}^{1\times 3}$. The numerically computed coefficients $k(\tau)$ are depicted on the figure 5. The behavior of phase coordinates of the Furuta pendulum augmented with corresponding nonlinear controller is depicted on figure 7, the corresponding control input as a function of time is given on figure 6.

VI. Discussions and Future Work

A new general method for constructing the periodic and aperiodic motions for underactuated systems is proposed. The idea of the method consists in a targeted choice of servo-connections, leading to a singular dynamics of motion generator. We have formulated sufficient conditions, under which solutions of singular equation and their derivatives are continuous functions, and the phase trajectories pass through singularity with the same velocity and acceleration. This

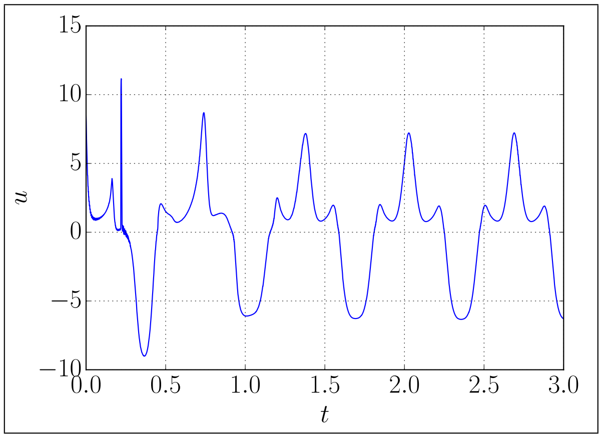

Fig. 6. Orbital stabilization: control input

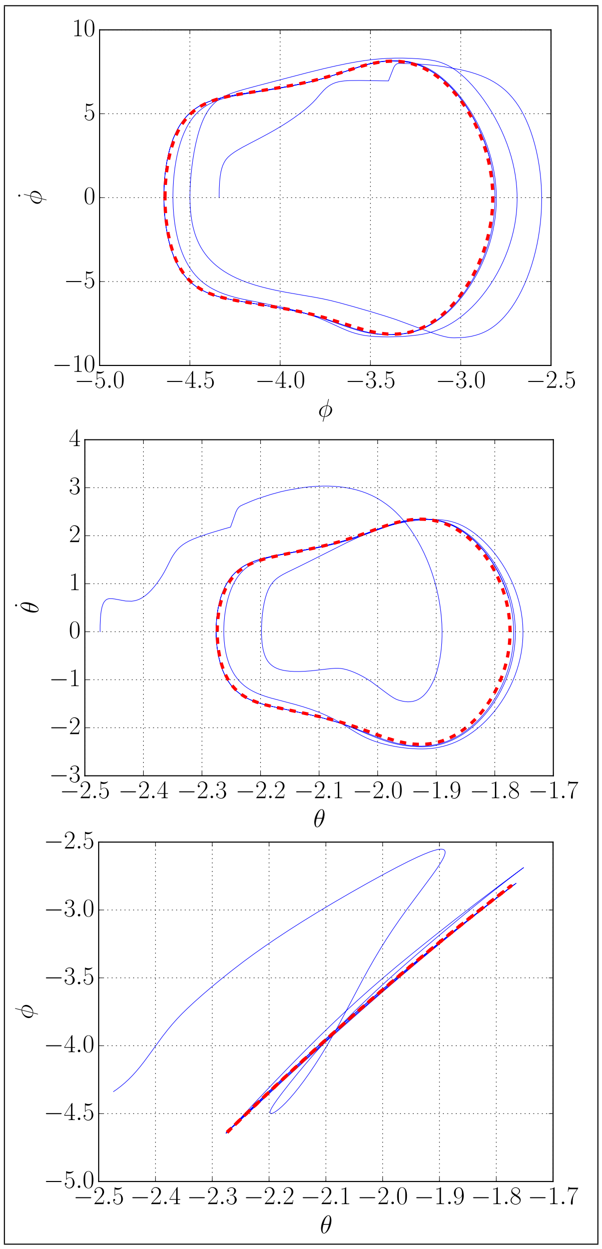

Fig. 7. Orbital stabilization: phase coordinates

allows concatenating any segments of trajectories belonging to opposite sides of the singularity into the solution. Joining the different segments one can obtain periodic trajectories with arbitrarily small amplitude or trajectories passing the given points. The phase portrait of such equation has an extraordinary form such as depicted on figure 1.

The results are illustrated by the example of constructing a periodic behavior of the Furuta pendulum when the pendulum oscillates around the horizontal. This problem is chosen because it cannot be solved by known methods of constructing motions for underactuated systems. Based on the previous results of the authors [2], the constructed motion is stabilized. The designed feedback controller provides orbital exponential stability for the found cycle. The simulation can be fount at `https://youtu.be/pzyIWb7szew`. The results validate the robustness of the constructed nonlinear control system.

It is worth to emphasize that the conditions of theorem 1 describing the smooth prolongation of a solution of the equation as it passes through a singular point are expressed in the form of a system of equations and inequalities imposed on the parameters of mechanical system.

References

[1] Kirgetov V. I. The Motion of Controlled Mechanical Systems with Prescribed Constraints (Servoconstraints), J. Appl. Math. Mech., 21, 433–466, 1967.

[2] Maksim O. Surov, Sergei V. Gusev, Anton S. Shiriaev. Shaping Stable Oscillation of a Pendulum on a Cart around the Horizontal. IFAC-PapersOnLine, Volume 50, Issue 1, July 2017, Pages 7621-7626.

[3] T. Ortega-Montiel,1 R. Villafuerte-Segura,1 C. Vázquez-Aguilera,2 and L. Freidovich: Proportional Retarded Controller to Stabilize Underactuated Systems with Measurement Delays: Furuta Pendulum Case Study. Mathematical Problems in Engineering Volume 2017 (2017), Article ID 2505086, 12 pages.

[4] Anton Shiriaev, John W. Perram, and Carlos Canudas-de-Wit. Constructive Tool for Orbital Stabilization of Underactuated Nonlinear Systems: Virtual Constraints Approach. IEEE Transactions on Automatic Control, 50(8):1164-1176, 2005.

[5] A. A. Andronov, E. A. Leontovich, I. I. Gordon, and A. G. Maier. Qualitative Theory of Second-Order Dynamic Systems. Nauka, Moscow 1966.

[6] K Furuta, M Yamakita, S Kobayashi. Swing-up Control of Inverted Pendulum Using Pseudo-State Feedback. Proceedings of the Institution of Mechanical Engineers, Part I: Journal of Systems and Control Engineering. Volume: 206 issue: 4, page(s): 263-269.

[7] Spong, M. W., & Block, D. J. (1995). The Pendubot: A mechanical system for control research and education. In Proceedings of the 34th conference on decision and control (pp. 555–557), New Orleans, USA.

[8] Anton Shiriaev, L. Paramonov, A. Sørensen, Anders Robertsson. Stabilization of Rotational Modes for the Furuta Pendulum. Nonlinear Control Systems 2001. IFAC Nonlinear Control Systems, St. Petersburg, Russia 2001.

[9] Spong M.W. (1998) Underactuated mechanical systems. In: Siciliano B., Valavanis K.P. (eds) Control Problems in Robotics and Automation. Lecture Notes in Control and Information Sciences, vol 230. pp. 135–150. Springer, Berlin, Heidelberg

[10] Luca Consolini, Manfredi Maggiore. Virtual Holonomic Constraints for Euler-Lagrange Systems. IFAC Proceedings Volumes Volume 43, Issue 14, September 2010, Pages 1193-1198.

[11] Brent Griffin and Jessy Grizzle. \Nonholonomic Virtual Constraints for Dynamic Walking," in the Proceedings of the 54th IEEE Conference of Decision and Control (CDC), Dec. 2015.

[12] Maksim Surov, Anton Shiriaev, Leonid Freidovich, Sergei Gusev, Leonid Paramonov. Case study in non-prehensile manipulation: planning and orbital stabilization of one-directional rollings for the "Butterfly" robot. 2015 IEEE International Conference on Robotics and Automation (ICRA), Seattle, WA, USA.