The Phase Diagram of the Quark-Meson Model

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Abstract

In this thesis we study the chiral phase transition of two-flavor QCD, employing the quark-meson (QM) effective model. The first part of the text covers some of the fundamentals of thermal field theory. We then introduce the QM model and derive an effective potential for the order parameter of the chiral transition. This is used to map out a phase diagram in the plane of temperature and baryon chemical potential in the chiral limit, using the large-N_c-approximation at one-loop. We find that the results are, qualitatively, in line with the current wisdom on the subject.

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Contents

Notation and Conventions

- The Minkowski metric is $\eta^{\mu\nu} = \text{diag}(1,-1,-1,-1)$.
- The Einstein summation convention is used, where one upper and one lower Greek index imply summation with the Minkowski metric, e.g. $\partial_\mu A^\mu = \eta^{\mu\nu}\partial_\mu A_\nu$.
- Repeated lower Greek indices imply summation with the Euclidean metric $\delta_{\mu\nu}$.
- Repeated Latin indices imply summation of spatial components, e.g. $k_i x_i = \mathbf{k}\cdot\mathbf{x}$.
- Natural units are used, i.e. $k_B = c = \hbar = 1$ where k_B, c and $\hbar$ are the Boltzmann constant, speed of light and reduced Planck constant, respectively.

1. Introduction

The theory of quantum chromodynamics (QCD) turned out to be one of the great successes of 20th century particle physics. For one thing, the theory beautifully makes order out of the seemingly messy multitude of hadrons which have been discovered through the years, classifying them according to their quark content. The theory has been verified time and again by experiment, and holds a firm place in the Standard Model of particle physics [1]. However, while the predictions of the theory seem to hold up, it can be very difficult to extract predictions from it in the first place. One area of research which is currently being explored is the phase structure of QCD [2]. In most of the Universe we see around us, quarks appear only in bound states - in particular in the form of neutrons and protons. However, if matter is heated up to very high temperatures, or compressed to high densities, entirely different phases of matter may result. In the very early universe, it is believed that a phase of matter known as quark-gluon plasma permeated space. It is possible that exotic phases of quark matter still exist, possibly inside the extremely dense cores of neutron stars [3]. Experimentally, the most direct way of studying quark matter is by the use of high-energy particle colliders. For example, at the Relativistic Heavy-Ion Collider at Brookhaven National Laboratory, quark-gluon plasma has been created through the collision of gold ions [4].

Standard perturbation theory, while very useful when working with, e.g., the electroweak interaction, has limited applicability in QCD. Many important advances in the understanding of quark matter have been made through one of two approaches: one of them is lattice QCD, which is a brute-force numerical method operating with discretized spacetime. This method is limited by the computing power which is required, and fails at finite baryon chemical potential due to the so-called sign problem [5]. Another approach is the construction of effective models. In this thesis, we will examine one of the simpler models, the quark-meson (QM) model, and map out its phase diagram in the plane of temperature and baryon chemical potential. But first, we will establish some of the basics of thermal field theory. The derivations given in the coming sections (3-9) are mostly based on refs. [6] and [7]

2. Symmetries in Field Theories

The subject of symmetry is an important one in physics in general, and in field theories in particular. Often, much information about a physical system can be extracted from its symmetry properties alone. This is in large part thanks to a theorem proved by Emmy Noether in 1915, which relates continuous symmetries to conservation laws.

2.1. Noether's Theorem

We will outline a proof of the theorem in classical field theory. For a generalization to quantum field theory, see e.g. ref. [6]. Assume a set of fields $\phi_a(x^\mu)$, $a = 1,2..n$ is described by the Lagrangian density $\mathscr{L}(\phi_a, \partial_\mu \phi_a)$. According to classical field theory, the field configuration which is realized is one which extremizes the action

$$S = \int \mathrm{d}^4 x \mathscr{L}. \tag{1}$$

Using variational calculus, one can derive the Euler-Lagrange equations,

$$\partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_a)} - \frac{\partial \mathscr{L}}{\partial \phi_a} = 0. \tag{2}$$

Assume now that there exists a continuous set of transformations of the fields, labeled by some parameter α, which leaves the action invariant. For an infinitesimal α, we can write such a transformation as

$$\phi_a \to \phi_a + \alpha \delta \phi_a, \tag{3}$$

where $\delta\phi_a$ generally depends on the spatial coordinates x^μ as well as the fields ϕ_a. The corresponding change in the Lagrangian is written

$$\mathscr{L} \to \mathscr{L} + \alpha \delta \mathscr{L}. \tag{4}$$

If $\delta\mathscr{L} = 0$, then clearly S is invariant. A more general condition on $\delta\mathscr{L}$ is that it is a four-divergence, i.e. it can be written as

$$\delta \mathscr{L} = \partial_\mu M^\mu \tag{5}$$

Then , using the divergence theorem, the change in the action is a boundary term,

$$\delta S = \int \mathrm{d}^4 x \partial_\mu M^\mu = \int_{\text{boundary}} M^\mu \mathrm{d}S_\mu, \tag{6}$$

where $\mathrm{d}S_\mu$ is the surface element. When appropriate boundary conditions are imposed (usually the boundary is taken to infinity and the fields are assumed to fall off sufficiently fast so that boundary terms vanish), this term is zero. We have

$$\delta \mathscr{L} = \frac{\partial \mathscr{L}}{\partial \phi_a} \delta \phi_a + \frac{\partial \mathscr{L}}{\partial (\partial_\mu \phi_a)} \delta(\partial_\mu \phi_a). \tag{7}$$

Using partial integration, the last term can be written

$$\partial_\mu \left[\frac{\partial \mathscr{L}}{\partial (\partial_\mu \phi_a)} \delta \phi_a \right] - \partial_\mu \left[\frac{\partial \mathscr{L}}{\partial (\partial_\mu \phi_a)} \right] \delta \phi_a. \tag{8}$$

We then have

$$0 = \delta \mathscr{L} - \partial_\mu M^\mu = \partial_\mu \left[\frac{\partial \mathscr{L}}{\partial (\partial_\mu \phi_a)} \delta \phi_a - M^\mu \right] + \left[\frac{\partial \mathscr{L}}{\partial \phi_a} - \partial_\mu \frac{\partial \mathscr{L}}{\partial (\partial_\mu \phi_a)} \right] \delta \phi_a. \tag{9}$$

Since the last bracket vanishes by the Euler-Lagrange equation, the current

$$J^\mu = \frac{\partial \mathscr{L}}{\partial (\partial_\mu \phi_a)} \delta \phi_a - M^\mu \tag{10}$$

is conserved. The theorem can be generalized to quantum field theory, see for example ref. [6].

2.2. Spontaneous symmetry breaking and Goldstone's theorem

If a symmetry of the action is not respected by the ground state of the system, that symmetry is said to be spontaneously broken. Goldstone's theorem states that in the case of a continuous broken symmetry, massless bosons (called Goldstone bosons) will appear. The number of such bosons is equal to the number of broken generators of the symmetry. We will see an example of this in the section on the quark-meson model. For a detailed discussion and proof of the theorem, see ref. [6].

3. The path integral formulation of the partition function

The canonical partition function of a one-particle system at inverse temperature β is given by

$$Z(\beta) = \mathrm{Tr}\left(\mathrm{e}^{-\beta H}\right) = \int \mathrm{d}q \, \langle q|\mathrm{e}^{-\beta H}|q\rangle \tag{11}$$

where $|q\rangle$ is some orthonormal basis. Here, H is the Hamiltonian of the system, which is assumed to be of the form

$$H = \frac{p^2}{2m} + V(q), \tag{12}$$

where p is the momentum operator conjugate to q. We can write

$$Z(\beta) = \int \mathrm{d}q \, \langle q|\left(\mathrm{e}^{-H\Delta\tau}\right)^N|q\rangle \,, \tag{13}$$

where $\Delta\tau = \beta/N$. Inserting the completeness relation $1 = \int dq_j \, |q_j\rangle \langle q_j|$ $N-1$ times, we obtain

$$Z(\beta) = \prod_{j=1}^{N} \int dq_j \, \langle q_1|\mathrm{e}^{-H\Delta\tau}|q_{N-1}\rangle \, \langle q_{N-1}|\mathrm{e}^{-H\Delta\tau}|q_{N-2}\rangle \ldots \langle q_2|\mathrm{e}^{-H\Delta\tau}|q_1\rangle \,, \tag{14}$$

where $q_1 = q$. We will take the limit $\Delta\tau \to 0$ eventually. Expanding to first order in $\Delta\tau$, we can write

$$\langle q_{j+1}|\mathrm{e}^{-H\Delta\tau}|q_j\rangle = \langle q_{j+1}|1 - H\Delta\tau|q_j\rangle = \langle q_{j+1}|q_j\rangle - \Delta\tau \, \langle q_{j+1}|\frac{p^2}{2m} + V(q_j)|q_j\rangle \,, \tag{15}$$

where we used $V(q)\,|q_j\rangle = V(q_j)\,|q_j\rangle$. We then insert a complete set of momentum states, $1 = \int \frac{dp_j}{2\pi} \, |p_j\rangle \langle p_j|$.

$$\begin{aligned}
\langle q_{j+1}|\mathrm{e}^{-H\Delta\tau}|q_j\rangle &= \int \frac{dp_j}{2\pi} \left[\langle q_{j+1}|p_j\rangle \langle p_j|q_j\rangle - \Delta\tau \, \langle q_{j+1}|\frac{p^2}{2m} + V(q_j)|p_j\rangle \langle p_j|q_j\rangle\right] && (16)\\
&= \int \frac{dp_j}{2\pi} \left[1 - \Delta\tau\left(\frac{p_j^2}{2m} + V(q_j)\right)\right] \langle q_{j+1}|p_j\rangle \langle p_j|q_j\rangle && (17)\\
&= \int \frac{dp_j}{2\pi} \left[1 - \Delta\tau\left(\frac{p_j^2}{2m} + V(q_j)\right)\right] \mathrm{e}^{ip_j(q_{j+1}-q_j)}, && (18)
\end{aligned}$$

using $\langle q|p\rangle = \mathrm{e}^{ipq}$. Using $1 - H\Delta\tau \approx \mathrm{e}^{-H\Delta\tau}$ to first order in $\Delta\tau$, the above expression becomes

$$\langle q_{j+1}|\mathrm{e}^{-H\Delta\tau}|q_j\rangle = \int \frac{dp_j}{2\pi} \exp\left[\Delta\tau\left(ip_j\dot{q}_j - \frac{p_j^2}{2m} - V(q_j)\right)\right], \tag{19}$$

where $\dot{q}_j = (q_{j+1} - q_j)/\Delta\tau$. We now have

$$\begin{aligned}
Z(\beta) = &\prod_{n=1}^{N}\left(\int dq_n\right)\prod_{n=1}^{N}\left(\int \frac{dp_n}{2\pi}\right)\\
&\times \exp\Delta\tau\left(ip_{N-1}\dot{q}_{N-1} - \frac{p_{N-1}^2}{2m} - V(q_{N-1})\right)\ldots\exp\Delta\tau\left(ip_0\dot{q}_0 - \frac{p_0^2}{2m} - V(q_0)\right).
\end{aligned} \tag{20}$$

Putting the exponentials together and renaming $q_j \to q(\tau_j)$ with $\tau_j = (j-1)\Delta\tau$:

$$Z(\beta) = \prod_{n=1}^{N}\left(\int dq(\tau_n)\right)\prod_{n=1}^{N}\left(\int \frac{dp(\tau_n)}{2\pi}\right)\exp\Delta\tau\sum_{n=1}^{N}\left(ip(\tau_n)\dot{q}(\tau_n) - \frac{p(\tau_n)^2}{2m} - V(q(\tau_n))\right) \quad (21)$$

We can now take the limit $\Delta\tau \to 0$, and the argument of the exponential becomes a Riemann integral, with $\dot{q} = \mathrm{d}q/\mathrm{d}\tau$. The integrals over p_n and q_n become a path integral in phase space,

$$\prod_{n=1}^{N}\int dq_n \prod_{m=1}^{N}\int \frac{dp_m}{2\pi} \to \int \mathscr{D}q \int \mathscr{D}p, \quad (22)$$

giving

$$Z(\beta) = \int \mathscr{D}q \int \mathscr{D}p \exp\left\{\int_0^\beta \mathrm{d}\tau\,[ip\dot{q} - H(p,q)]\right\} \quad (23)$$

where the path integral runs over all paths satisfying the boundary condition $q(\beta) = q(0)$. This is the Hamiltonian, or phase space, version of the path integral. The path integral can also be expressed in configuration space, by carrying out the momentum integrals in (20), which are simply Gaussian integrals:

$$\int \frac{dp_n}{2\pi}\exp\Delta\tau\left(ip_n\dot{q}_n - \frac{p_n^2}{2m}\right) = \sqrt{\frac{m}{2\pi\Delta\tau}}\mathrm{e}^{-\Delta\tau\frac{1}{2}m\dot{q}^2}. \quad (24)$$

Inserting (24) into (20) and defining

$$\int \mathscr{D}q = \lim_{\Delta\tau\to 0}\left[\sqrt{\frac{m}{2\pi\Delta\tau}}\prod_{n=1}^{N}\left(\int\sqrt{\frac{m}{2\pi\Delta\tau}}dq_n\right)\right], \quad (25)$$

we obtain

$$Z(\beta) = \int \mathscr{D}q \exp\left\{-\int_0^\beta d\tau\left[\frac{1}{2}m\dot{q}(t)^2 + V(q(t))\right]\right\} \quad (26)$$

We recognise the Euclidean action from classical mechanics,

$$S_E = \int \mathrm{d}t L_E(t), \quad (27)$$

with t the time and

$$L_E = \frac{1}{2}m\dot{q}^2 + V(q) \quad (28)$$

the Euclidean Lagrangian. Thus we can write

$$Z(\beta) = \int \mathscr{D}q \mathrm{e}^{-S_E[q]} \quad (29)$$

The result can be easily generalised to a system of N particles, with

$$H = \sum_{n=1}^{N}\frac{p_n^2}{2m} + U(q_1,\ldots,q_N). \quad (30)$$

The derivation proceeds in the same way, using instead $1 = \prod_{n=1}^{N} (\int dq_n) \, |q1, \ldots, q_N\rangle \langle q_1, ..., q_N|$ etc. The partition function becomes

$$Z(\beta) = \int \prod_{n=1}^{N} \mathscr{D}q_n \exp\left[-\int_0^\beta \mathrm{d}\tau \sum_{j=1}^{N} \frac{1}{2} m\dot{q}_j^2 + U(q, \ldots, q_N) \right] \tag{31}$$

From here we can associate a point in space with each particle, $\hat{q}_n \to \phi(\mathbf{x_n})$, and then take the continuum limit so that every point in space is associated with a field operator $\phi(\mathbf{x})$. The details can be found in standard texts on quantum field theory, such as [8]. The Euclidean Lagrangian of a real scalar field is

$$L_E = \int \mathrm{d}^3\mathbf{x} \mathscr{L}_E = \int \mathrm{d}^3\mathbf{x} \left[\frac{1}{2} \partial_\mu \phi \partial_\mu \phi + \frac{1}{2} m^2 \phi^2 + V(\phi) \right], \tag{32}$$

and the path integral takes the form

$$Z(\beta, j) = \int \mathscr{D}\phi \exp\left\{ -\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3\mathbf{x} [\mathscr{L}_E + J\phi] \right\}. \tag{33}$$

where a source term $J(\tau, \mathbf{x})\phi(\tau, \mathbf{x})$ was added. The path integral runs over all field configurations satisfying the boundary conditions, the first of which is $\phi(\beta, \mathbf{x}) = \phi(0, \mathbf{x})$. We also require that for $|\mathbf{x}| \to \infty$, $\phi(\tau, \mathbf{x})$ vanishes sufficiently fast so that boundary terms vanish when integrating the action by parts. The path integral of Minkowskian quantum field theory can be constructed in the same way from the transition amplitude

$$F(q_f, t_f; q_i, t_i) = \langle q_f | \mathrm{e}^{-iH(t_F - t_i)} | q_i \rangle = \langle q_f, t_f | q_i, t_i \rangle \tag{34}$$

which gives the probability amplitude for a particle in state q_i at time t_i to be found in the state q_f at time t_f. Here, $|q_i\rangle$ is an eigenstate of the operator q in the Schrödinger picture, whereas $|q_i, t_i\rangle$ is an eigenstate of the operator $q(t_i)$ in the Heisenberg picture. The path integral becomes

$$F[J] = \langle 0, \infty | 0, -\infty \rangle_J = \int \mathscr{D}\phi \ \mathrm{e}^{iS + \int \mathrm{d}^4x J\phi}, \tag{35}$$

where S is the Minkowskian action

$$S = \int \mathrm{d}^4x \mathscr{L}(x) = \int \mathrm{d}^4x \left[\frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 - V(\phi) \right] \tag{36}$$

and $\mathscr{L}$ is the Lagrangian density of the field, related to the Euclidean Lagrangian through $\mathscr{L}_E(\tau) = -\mathscr{L}(t \to -i\tau)$. The path integral runs over all field configurations that vanish sufficiently fast at infinity. Comparing (11) and (34), we have

$$Z(\beta) = \int \mathrm{d}q F(q, -i\beta; q, 0). \tag{37}$$

The partition function can be obtained from F by letting $t \to -i\tau$ and imposing periodic boundary conditions on the field. This amounts to replacing the Minkowskian action S with the Euclidean action S_E. For a Lorentz-invariant action the substitution $t \to -i\tau$ will only affect objects like $\partial^\mu \phi \partial_\mu \phi$, so that this change corresponds to replacing the Minkowski metric with the Euclidean one: $\eta_{\mu\nu} \to -\delta_{\mu\nu}$.

4. The free scalar field at finite temperature

A free real scalar field is described by the partition function

$$Z(\beta,J)=\int\mathscr{D}\phi\exp-\int_0^\beta\mathrm{d}\tau\int\mathrm{d}^3\mathbf{x}\left[\frac{1}{2}(\partial_\tau\phi)^2+\frac{1}{2}(\nabla\phi)^2+\frac{1}{2}m^2\phi^2+J\phi\right]. \tag{38}$$

After integration by parts, where the boundary terms vanish due to the boundary conditions on ϕ, this becomes

$$Z(\beta,J)=\int\mathscr{D}\phi\exp\left[-\int_0^\beta\mathrm{d}\tau\int\mathrm{d}^3\mathbf{x}\phi\left(-\frac{1}{2}\partial_\tau^2-\frac{1}{2}\nabla^2+\frac{1}{2}m^2\right)\phi+J\phi\right]. \tag{39}$$

Going back to discrete space, viewing ϕ and J as column vectors and $(-\partial_\tau^2-\nabla^2+m^2)$ as a symmetric matrix A, this integral is of the form

$$\int\mathrm{d}^N\phi\mathrm{e}^{-\frac{1}{2}\phi^TA\phi+J^T\phi}. \tag{40}$$

After the substitution $\phi=X-A^{-1}J$, this reads

$$\int\mathrm{d}^NX\mathrm{e}^{-\frac{1}{2}X^TAX+J^TA^{-1}J} \tag{41}$$

which can be solved by a variable transformation $X=OY$ where O is some orthogonal matrix diagonalising A.

$$\int\mathrm{d}^NY\mathrm{e}^{-\frac{1}{2}Y^TDY}=\prod_{j=1}^N\sqrt{\frac{2\pi}{D_{jj}}}=\sqrt{\frac{(2\pi)^N}{\det(A)}}=(2\pi)^{\frac{N}{2}}\mathrm{e}^{-\mathrm{Tr}\ln(A)}, \tag{42}$$

with $D=O^TAO$. The operator in (39) can be diagonalised by an expansion in Fourier modes:

$$\phi(\mathbf{x},\tau)=\sqrt{\frac{V}{\beta}}\sum_{n=-\infty}^{\infty}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\phi(\omega_n,\mathbf{k})\mathrm{e}^{i(\omega_n\tau+\mathbf{kx})}, \tag{43}$$

using box normalization with V the volume. Here the boundary conditions imply $\omega_n=2n\pi/\beta$, known as the bosonic Matsubara frequencies. This is not an orthogonal transformation, but a unitary one. Since $\phi(\mathbf{x},\tau)$ is real, the Fourier coefficients satisfy

$$\phi(-\omega_n,-\mathbf{k})=\phi^*(\omega_n,\mathbf{k}), \tag{44}$$

The action becomes

$$S_E=\frac{V}{2\beta}\int_0^\beta\int\mathrm{d}^3\mathbf{x}\sum_{n,m=-\infty}^{\infty}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\phi(\omega_m,\mathbf{q})\mathrm{e}^{i(\omega_m\tau+\mathbf{qx})}\left[-(i\omega_n)^2-(i\mathbf{k})^2+m^2\right]\phi(\omega_n,\mathbf{k})\mathrm{e}^{i(\omega_n\tau+\mathbf{kx})}. \tag{45}$$

Using (44) and

$$\int_0^\beta\mathrm{e}^{i(\omega_n-\omega_m)\tau}=\beta\delta_{mn},\qquad\frac{1}{(2\pi)^3}\int\mathrm{d}^3\mathbf{x}\mathrm{e}^{i(\mathbf{k}-\mathbf{q})\mathbf{x})}=\delta^{(3)}(\mathbf{k}-\mathbf{q}), \tag{46}$$

gives

$$S_E=\frac{1}{2}V\sum_{n=\infty}^{\infty}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\phi^*(\omega_n,\mathbf{k})\left(\omega_n^2+\omega_{\mathbf{k}}^2\right)\phi(\omega_n,\mathbf{k}). \tag{47}$$

where $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$. Note that the action is independent of the phase of $\phi(\omega_n, \mathbf{k})$. When making the variable transformation (43) in (39), we can essentially use this to integrate out the phase, giving a constant prefactor in the partition function. For a more rigorous treatment, see ref. [7]. The result is

$$Z(\beta,0) \propto \int \mathscr{D}|\phi|(\omega_n,\mathbf{k}) \exp\left[-V \sum_{n=\infty}^{\infty} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \frac{1}{2}(\omega_n,\mathbf{k}) \left(\omega_n^2 + \omega_{\mathbf{k}}^2\right) |\phi|^2(\omega_n,\mathbf{k})\right], \tag{48}$$

We can now use (42), and obtain

$$-\ln Z(\beta) = \frac{1}{2} V \sum_{n=-\infty}^{\infty} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \ln\left(\omega_n^2 + \omega_{\mathbf{k}}^2\right). \tag{49}$$

We will carry out the sum over ω_n first. Clearly, this series is divergent, since the terms grow without bound. However, it can be split into a finite part dependent on β and $\omega_{\mathbf{k}}$, and an infinite constant term which is not physically relevant. To extract the relevant part, we differentiate with respect to $\omega_{\mathbf{k}}$:

$$\frac{\partial}{\partial \omega_{\mathbf{k}}} \sum_{n=-\infty}^{\infty} \ln\left(\omega_n^2 + \omega_{\mathbf{k}}^2\right) = \sum_{n=-\infty}^{\infty} \frac{2\omega_{\mathbf{k}}}{\omega_n^2 + \omega_{\mathbf{k}}^2} \tag{50}$$

In evaluating sums over the bosonic Matsubara frequencies, one can often use

$$\coth(z) = \sum_{n=-\infty}^{\infty} \frac{z}{(\pi n)^2 + z^2} \tag{51}$$

in some way. In this case we get the result directly,

$$\sum_{n=-\infty}^{\infty} \frac{2\omega_{\mathbf{k}}}{(2\pi n/\beta)^2 + \omega_{\mathbf{k}}^2} = \beta \coth\left(\tfrac{1}{2}\beta\omega_{\mathbf{k}}\right). \tag{52}$$

Integrating this with respect to $\omega_{\mathbf{k}}$ gives

$$\frac{1}{2}\beta\omega_{\mathbf{k}} + \ln\left(1 - \mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right) + \text{constant} \tag{53}$$

The constant is dimensionless and independent of $\omega_{\mathbf{k}}$. Since the only parameter it could depend on is β, which is dimensionful, it is also independent on β, and we simply discard it.

$$\mathscr{F} \equiv -\frac{1}{\beta V} \ln Z(\beta) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \left[\frac{\omega_{\mathbf{k}}}{2} + \frac{1}{\beta} \ln\left(1 - \mathrm{e}^{-\beta\omega_{\mathbf{k}}}\right)\right]. \tag{54}$$

The first term, which is divergent, is the zero-point energy density $\mathscr{E}$, i.e. the sum of the ground state energies of all excitation modes. The remaining term determines the thermodynamic properties of the system. In the ultrarelativistic limit $m = 0$, this can be evaluated analytically:

$$\mathscr{F} = \mathscr{E} + \frac{1}{2\pi^2\beta} \int_0^{\infty} k^2 \mathrm{d}k \ln\left(1 - \mathrm{e}^{-\beta k}\right) = \mathscr{E} - \frac{\pi^2}{90\beta^4}. \tag{55}$$

To include the source term, we need the continuous space version of the factor $e^{J^T A^{-1} J}$ in (41). We will only need the special case $\beta \to \infty$. The matrix product then takes the form

$$J^T A^{-1} J \to \int \mathrm{d}^4x \int \mathrm{d}^4x' J(x) D(x,x') J(x'), \tag{56}$$

with $D(x,x')$ a Green function of the operator $-\partial_\tau^2 - \nabla^2 + m^2 = -\partial_\mu\partial_\mu + m^2$. Due to translation invariance, we can write $D(x,x') = D(x-x',0) \equiv D(x-x')$. $D(x)$ satisfies

$$(-\partial_\mu\partial_\mu + m^2)D(x) = \delta^{(4)}(x). \tag{57}$$

Taking the Fourier transform of this equation, we get

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4}(-\partial_\mu\partial_\mu + m^2)\mathrm{e}^{ik_\mu x_\mu}D(k) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{ik_\mu x_\mu} \tag{58}$$

$$\implies (k^2+m^2)D(k) = 1 \tag{59}$$

$$D(k) = \frac{1}{k^2+m^2} \tag{60}$$

Thus,

$$Z(\beta,J) = Z(\beta,0)\exp\left[-\int \mathrm{d}^4x\mathrm{d}^4x' J(x)D(x-x')J(x')\right], \tag{61}$$

or, alternatively,

$$Z(\beta,J) = Z(\beta,0)\exp\left[-\int \frac{\mathrm{d}^4k}{(2\pi)^4}J^*(k)D(k)J(k)\right]. \tag{62}$$

5. The partition function for fermions

To describe particles that obey Fermi statistics, we start again from a simple system. As is well known, the simple harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2, \tag{63}$$

which can be thought of as the 0 + 1 dimensional version of a free scalar field, with $\phi(t,\mathbf{x}) \to \phi(t,\mathbf{0}) = q(t)$, can be described in terms of the bosonic creation and annihilation operators

$$a = \sqrt{\frac{m\omega}{2}}\left(q + \frac{i}{m\omega}p\right) \tag{64}$$

$$a^\dagger = \sqrt{\frac{m\omega}{2}}\left(q - \frac{i}{m\omega}p\right) \tag{65}$$

with the commutation relations

$$[a,a^\dagger] = 1, [a,a] = [a^\dagger,a^\dagger] = 0. \tag{66}$$

The Hamiltonian is

$$H = \frac{1}{2}\omega(a^\dagger a + aa^\dagger) = \omega\left(\frac{1}{2} + a^\dagger a\right), \tag{67}$$

furnishing the orthonormal basis $|n\rangle$ with

$$a|n\rangle = \sqrt{n}\,|n-1\rangle, \tag{68}$$

$$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle, \tag{69}$$

$$a^\dagger a|n\rangle = n|n\rangle. \tag{70}$$

To get the fermionic version of this, we replace the bosonic creation and annilhilation operators with fermionic ones, satisfying instead the anticommutation relations

$$\{c, c^\dagger\} = 1, \{c, c\} = \{c^\dagger, c^\dagger\} = 0. \tag{71}$$

We define the ground state through

$$c\,|0\rangle = 0, \tag{72}$$

and a one-particle state

$$|1\rangle = c^\dagger\,|0\rangle\,. \tag{73}$$

Here we have already exhausted the Hilbert space, since

$$c^\dagger\,|1\rangle = (c^\dagger)^2\,|0\rangle = 0, \tag{74}$$

$$c\,|1\rangle = cc^\dagger\,|0\rangle = (1 - c^\dagger c)\,|0\rangle = |0\rangle\,. \tag{75}$$

To derive a path-integral representation of the partition function of this system, we need completeness relations similar to

$$1 = \int \mathrm{d}q\,|q\rangle\,\langle q|\,. \tag{76}$$

For this we will need to introduce Grassmann variables.

5.1. Grassmann variables

A complex (as opposed to real) Grassmann algebra $\mathscr{A}$ is generated by n Grassmann variables $\eta_1, \ldots, \eta_n$ with the properties:

$$\{\eta_i, \eta_j\} = 0, \tag{77}$$

in particular, $\eta_i^2 = 0$.

$$\alpha\eta_1 + \beta\eta_2 \in \mathscr{A}, \tag{78}$$

where α, β are arbitrary complex numbers.

$$\eta_i\eta_j \in \mathscr{A}. \tag{79}$$

Note that a product of an even number of generators does not in general anticommute with other variables, for example $\{\eta_i\eta_j, \eta_k\} = 2\eta_i\eta_j\eta_k \neq 0$ if i, j and k are distinct.

Functions of Grassmann variables are given in terms of power series, such as

$$f(\eta_1) = f_0 + f_1\eta_1, \tag{80}$$

$$g(\eta_1, \eta_2) = g_0 + g_1\eta_1 + g_2\eta_2 + g_{a12}\eta_1\eta_2, \tag{81}$$

where all higher order terms vanish. Differentiation can be defined as acting 'from the left', with

$$\frac{\partial g}{\partial \eta_1} = g_1 + g_{12}\eta_2, \tag{82}$$

$$\frac{\partial g}{\partial \eta_2} = g_2 - g_{12}\eta_1. \tag{83}$$

Integration can be defined as equivalent to differentiation, with

$$\int \mathrm{d}\eta \, [af(\eta) + bg(\eta)] = a \int \mathrm{d}\eta f(\eta) + b \int \mathrm{d}\eta g(\eta), \tag{84}$$

$$\int \mathrm{d}\eta = 0, \tag{85}$$

$$\int \mathrm{d}\eta \, \eta = 1, \tag{86}$$

with multiple integrals evaluated as follows:

$$\int \mathrm{d}\eta_2 \mathrm{d}\eta_1 g(\eta_1, \eta_2) = \int \mathrm{d}\eta_2 \left(\int \mathrm{d}\eta_1 g(\eta_1, \eta_2) \right) = g_{12} = \frac{\partial}{\partial \eta_2} \frac{\partial}{\partial \eta_1} g. \tag{87}$$

Thus the infinitesimals $\mathrm{d}\eta_i$ also behave as Grassmann variables, with e.g.

$$\int \mathrm{d}\eta_1 \mathrm{d}\eta_2 = -\int \mathrm{d}\eta_2 \mathrm{d}\eta_1. \tag{88}$$

A useful integral is

$$\int \mathrm{d}\eta^* \mathrm{d}\eta \mathrm{e}^{-a\eta^*\eta} = \int \mathrm{d}\eta^* \mathrm{d}\eta (1 - a\eta^*\eta) = a, \tag{89}$$

which generalises to

$$\int \mathrm{d}^n\eta^* \mathrm{d}^n\eta \mathrm{e}^{-\eta^{*T} A \eta} = \det(A). \tag{90}$$

Returning to the fermionic oscillator, we now introduce a Grassmann algebra generated by the two elements η and η^*. We define these variables to also anticommute with c and $c^\dagger$. We then define

$$|\eta\rangle = \mathrm{e}^{-\eta c^\dagger} |0\rangle = (1 - \eta c^\dagger) |0\rangle, \tag{91}$$

$$\langle\eta| = \langle 0| \mathrm{e}^{-c\eta^*} = \langle 0| (1 - c\eta^*), \tag{92}$$

which are eigenstates of c and $c^\dagger$, respectively:

$$c |\eta\rangle = \eta |0\rangle = \eta |\eta\rangle, \tag{93}$$

$$\langle\eta| c^\dagger = \langle 0| \eta^* = \langle\eta| \eta^*. \tag{94}$$

We then have

$$\begin{aligned} \langle\eta|0\rangle &= \langle 0|\eta\rangle = 1, \\ \langle 1|\eta\rangle &= -\eta, \langle\eta|1\rangle = -\eta^*, \\ \langle\eta_1|\eta_2\rangle &= (\langle 0| - \langle 1| \eta_1^*)(|0\rangle - \eta_2 |1\rangle) = 1 + \eta_1^* \eta_2 = \mathrm{e}^{\eta_1^* \eta_2}. \end{aligned} \tag{95}$$

The identity operator can be written

$$\begin{aligned} \int \mathrm{d}\eta^* \mathrm{d}\eta \mathrm{e}^{-\eta^*\eta} |\eta\rangle \langle\eta| &= \int \mathrm{d}\eta^* \mathrm{d}\eta (1 - \eta^*\eta)(1 - \eta c^\dagger) |0\rangle \langle 0| (1 - c\eta^*) \\ &= |0\rangle \langle 0| \int \mathrm{d}\eta^* \mathrm{d}\eta \, \eta\eta^* + \int \mathrm{d}\eta^* \mathrm{d}\eta \, \eta c^\dagger |0\rangle \langle 0| c\eta^* \\ &= |0\rangle \langle 0| + |1\rangle \langle 1|, \end{aligned} \tag{96}$$

Similarly, the trace of the operator A can be written as

$$\begin{aligned}&\int \mathrm{d}\eta^*\mathrm{d}\eta \mathrm{e}^{-\eta^*\eta}\langle -\eta|A|\eta\rangle \\ =&\int \mathrm{d}\eta^*\mathrm{d}\eta \mathrm{e}^{-\eta^*\eta}\langle -\eta|A|\eta\rangle \langle -\eta|\Big(|0\rangle\langle 0|A_{00}+|0\rangle\langle 1|A_{01}+|1\rangle\langle 0|A_{10}+|1\rangle\langle 1|A_{11}\Big)|\eta\rangle \\ &=A_{00}+A_{11}=\mathrm{Tr}(A),\end{aligned} \tag{97}$$

where the last line is checked by inserting (95). The partition function can now be formulated in the same way as before,

$$Z=\mathrm{Tr}\left(\mathrm{e}^{-\beta H}\right)=\int \mathrm{d}\eta^*\mathrm{d}\eta \mathrm{e}^{-\eta^*\eta}\langle -\eta|\mathrm{e}^{-\beta H}|\eta\rangle=\int \mathrm{d}\eta^*\mathrm{d}\eta \mathrm{e}^{-\eta^*\eta}\langle -\eta|\mathrm{e}^{-H\Delta\tau}\ldots\mathrm{e}^{-H\Delta\tau}|\eta\rangle, \tag{98}$$

with $\Delta\tau=\beta/N$. We then insert $1=\int \mathrm{d}\eta_i^*\mathrm{d}\eta_i \mathrm{e}^{-\eta_i^*\eta_i}|\eta_i\rangle\langle\eta_i|$ between the exponentials:

$$Z=\prod_{i=1}^{N}\int \mathrm{d}\eta_i^*\mathrm{d}\eta_i \langle\eta_{i+1}|\mathrm{e}^{-H\Delta\tau}|\eta_i\rangle \tag{99}$$

where $\eta_1\equiv\eta$, $\eta_{N+1}\equiv-\eta$. Working to first order in $\Delta\tau$,

$$\langle\eta_i|\mathrm{e}^{-H\Delta\tau|\eta_{i-1}}|\eta_{i-1}\rangle=\langle\eta_i|1-\Delta\tau H(c^\dagger,c)|\eta_{i-1}\rangle=\mathrm{e}^{\eta_i^*\eta_{i-1}}(1-\Delta\tau H(\eta_i^*,\eta_{i-1})=\mathrm{e}^{\eta_i^*\eta_{i-1}-\Delta\tau H(\eta_i^*,\eta_{i-1})}, \tag{100}$$

using (95). The exponent here only contains even powers of anticommuting numbers (in general the Hamiltonian cannot contain odd powers of fermionic operators since it must be Hermitian), so that $\mathrm{e}^a\mathrm{e}^b=\mathrm{e}^{a+b}$ holds. Thus,

$$Z=\prod_{i=1}^{N}\int \mathrm{d}\eta_i^*\mathrm{d}\eta_i \exp\left\{\left[-\sum_{j=1}^{N}\left(\eta_{j+1}^*(\eta_{j+1}-\eta_j)+\Delta\tau H(\eta_{j+1}^*\eta_j)\right)\right]\right\}. \tag{101}$$

After taking the limit $\Delta\tau\to 0$ with $\eta_j\to\eta(\tau_j)$, $\tau_j=j\Delta\tau$ etc, we get

$$Z=\int \mathscr{D}\eta^*\mathscr{D}\eta \mathrm{e}^{-S_E} \tag{102}$$

with

$$S_E=\int_0^\beta \mathrm{d}\tau\Big\{\eta^*(\tau)\partial_\tau\eta(\tau)+H\left[\eta^*(\tau),\eta(\tau)\right]\Big\}, \tag{103}$$

where the boundary conditions on the path integral are now

$$\eta(\beta)=-\eta(0) \tag{104}$$

$$\eta^*(\beta)=-\eta^*(0). \tag{105}$$

For a system of several particles the anticommutation relations are

$$\{c_i,c_j\}=\{c_i^\dagger,c_j^\dagger\}=0,\{c_i,c_j^\dagger\}=\delta_{ij}. \tag{106}$$

The generalisation of (102) to several particles and from there to a quantum field is completely analogous to the bosonic case.

6. The free Dirac field at finite temperature

A free fermion field is described by the Lagrangian density

$$\mathscr{L} = \bar{\psi}\left(i\gamma^{\nu}\partial_{\nu} - m\right)\psi, \tag{107}$$

where ψ is a Dirac spinor field, γ^{μ} are gamma matrices, satisfying the anticommutation relations $\{\gamma^{\mu},\gamma^{\nu}\} = 2\eta^{\mu\nu}$, and $\bar{\psi} = \psi^{\dagger}\gamma^{0}$. This Lagrangian density has a continuous symmetry, as it is invariant under the global phase transformation

$$\psi \to \mathrm{e}^{-i\alpha}\psi = \psi - i\alpha\psi + \mathscr{O}(\alpha^2) = \psi + \alpha\delta\psi + \mathscr{O}(\alpha^2), \tag{108}$$

$$\psi^{\dagger} \to \mathrm{e}^{i\alpha}\psi^{\dagger} = \psi^{\dagger} + i\alpha\psi^{\dagger} + \mathscr{O}(\alpha^2) = \psi^{\dagger} + \alpha\delta\psi^{\dagger} + \mathscr{O}(\alpha^2) \tag{109}$$

where α is any real number. Noether's theorem gives the conserved current

$$j^{\nu} = \delta\psi^{\dagger}\frac{\partial\mathscr{L}}{\partial(\partial_{\nu}\psi^{\dagger})} + \frac{\partial\mathscr{L}}{\partial(\partial_{\nu}\psi)}\delta\psi = \bar{\psi}\gamma^{\nu}\psi. \tag{110}$$

The corresponding conserved charge is

$$Q \equiv \int \mathrm{d}^3\mathbf{x}\, j^0 = \int \mathrm{d}^3\mathbf{x}\bar{\psi}\gamma^0\psi, \tag{111}$$

which corresponds to the number of particles minus the number of antiparticles. To study systems with an abundance of particles we must replace the canonical ensemble by the grand canonical one, letting

$$H \to H - \mu Q \tag{112}$$

in (11), where μ is the chemical potential conjugate to Q. The Hamiltonian density is

$$\mathscr{H} = \partial_0(\bar{\psi})\bar{\pi} + \pi\partial_0\psi - \mathscr{L} - \mu j^0, \tag{113}$$

where

$$\pi = \frac{\partial\mathscr{L}}{\partial\left(\partial_0\psi\right)} = i\bar{\psi}\gamma^0 = i\psi^{\dagger} \tag{114}$$

$$\bar{\pi} = \frac{\partial\mathscr{L}}{\partial\left(\partial_0\bar{\psi}\right)} = 0, \tag{115}$$

giving

$$\mathscr{H} = -i\pi\gamma^0\left(i\gamma^j\partial_i + m\right)\psi + i\pi\psi = \bar{\psi}\left(i\gamma^j\partial_i + m - \mu\gamma^0\right)\psi. \tag{116}$$

The Euclidean Lagrangian density corresponding to this Hamiltonian is

$$\mathscr{L}_E = \bar{\psi}\left(\gamma^0\partial_{\tau} + i\gamma^j\partial_i + m - \mu\gamma^0\right)\psi. \tag{117}$$

Defining the Euclidean gamma matrices as

$$\tilde{\gamma}_0 = \gamma^0, \tilde{\gamma}_j = -i\gamma^j, \tag{118}$$

with

$$\{\tilde{\gamma}_{\mu},\tilde{\gamma}_{\nu}\} = 2\delta_{\mu\nu}, \tilde{\gamma}_{\mu}^{\dagger} = \tilde{\gamma}_{\mu}, \tag{119}$$

this can be written

$$\mathscr{L}_E = \bar{\psi}\left(\tilde{\gamma}_\nu \partial_\nu + m - \mu\tilde{\gamma}_0\right)\psi = \bar{\psi}\left(\not{\partial} + m - \mu\tilde{\gamma}_0\right)\psi, \tag{120}$$

where we have introduced the Feynman slash, $\not{a} = \gamma_\mu a_\mu$.

The partition function for the free fermion field is

$$Z(\beta,\mu) = \int \mathscr{D}\psi^\dagger \mathscr{D}\psi \mathrm{e}^{-S_E}, \tag{121}$$

where $\psi, \psi^\dagger$ are now Grassmann valued and antiperiodic in time,

$$\psi(\beta) = -\psi(0), \tag{122}$$

$$\psi^\dagger(\beta) = -\psi^\dagger(0). \tag{123}$$

Taking space to be a cube of volume $V = L^3$ and imposing periodic boundary conditions, $\psi(0,y,z) = \psi(L,y,z)$ etc, we can expand the fields in Fourier modes,

$$\psi(\tau,\mathbf{x}) = \sqrt{\frac{1}{\beta V}} \sum_{n=-\infty}^{\infty} \sum_{\mathbf{k}} \psi(\omega_n,\mathbf{k}) \mathrm{e}^{i(\omega_n\tau + \mathbf{kx})}, \tag{124}$$

$$\psi^\dagger(\tau,\mathbf{x}) = \sqrt{\frac{1}{\beta V}} \sum_{n=-\infty}^{\infty} \sum_{\mathbf{k}} \psi^\dagger(\omega_n,\mathbf{k}) \mathrm{e}^{-i(\omega_n\tau + \mathbf{kx})}. \tag{125}$$

Here ω_n are the fermionic Matsubara frequencies, which are restricted to the form $\omega_n = (2n+1)\pi/\beta$ by (122) and (123). The wavevectors take all values of the form $\mathbf{k} = \frac{2\pi}{L}(n_1,n_2,n_3)$, with $n_i \in \mathbb{Z}$. The Euclidean action can now be written

$$S_E = \sum_n \sum_{\mathbf{k}} \bar{\psi}(\omega_n,\mathbf{k})(i\tilde{\gamma}_0\omega_n + i\tilde{\boldsymbol{\gamma}}\mathbf{k} + m - \mu\tilde{\gamma}_0)\psi(\omega_n,\mathbf{k}) \tag{126}$$

$$= \sum_p \bar{\psi}(p)(i\tilde{\gamma}_\nu p_\nu + m)\psi(p) \equiv \sum_p \bar{\psi}(p) D_p \psi(p), \tag{127}$$

where $p = (\omega_n + i\mu, \mathbf{k})$. Viewing ψ, $\psi^\dagger$ as row and column vectors respectively, the action takes the form

$$S_E = \begin{pmatrix} \bar{\psi}(p_1) & \bar{\psi}(p_2) & \bar{\psi}(p_3) & \dots \end{pmatrix} \begin{pmatrix} D_{p_1} & 0 & 0 & \dots \\ 0 & D_{p_2} & 0 & \dots \\ 0 & 0 & D_{p_3} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \psi(p_1) \\ \psi(p_2) \\ \psi(p_3) \\ \vdots \end{pmatrix} \equiv \bar{\psi} D \psi, \tag{128}$$

in shorthand notation. The determinant of the matrix D is

$$\det(D) = \prod_p \det(D_p). \tag{129}$$

In the Dirac representation, the gamma matrices are given by

$$\tilde{\gamma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \tilde{\gamma}_j = \begin{pmatrix} 0 & -i\sigma^j \\ i\sigma^j & 0 \end{pmatrix}, \tag{130}$$

where σ^j are the Pauli matrices. In this representation, the matrix D_p becomes

$$D_p = \begin{pmatrix} i\omega_n - \mu + m & \mathbf{k}\cdot\boldsymbol{\sigma} \\ -\mathbf{k}\cdot\boldsymbol{\sigma} & -i\omega_n + \mu + m \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

All the 2×2 submatrices in this matrix commute with each other, and it can be shown that the determinant is simply $\det D_p = \det(AD - BC)$. Using $\{\sigma^i, \sigma^j\} = 2\delta_{ij}$,

$$-BC = k_i k_j \sigma^i \sigma^j = \frac{1}{2} k_i k_j (\sigma^i \sigma^j + \sigma^j \sigma^i) = \mathbf{k}^2. \tag{131}$$

Thus,

$$\det(D_p) = \left[(i\omega_n - \mu + m)(-i\omega_n + \mu + m) + \mathbf{k}^2\right]^2 \tag{132}$$

$$= \left[(\omega_n + i\mu)^2 + \omega_{\mathbf{k}}^2\right]^2, \tag{133}$$

where the overall square comes from $\det(a\cdot I_n) = a^n$, where I_n is the $n\times n$ identity matrix. Using (90), the grand potential can now be expressed as

$$-\beta\Omega(\beta,\mu) = \ln(Z) = \ln\Big[\det(D)\Big] = \sum_p \ln\Big[\det(D_p)\Big] \tag{134}$$

$$\to 2V\sum_n \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \ln\left[(\omega_n + i\mu)^2 + \omega_{\mathbf{k}}^2\right], \tag{135}$$

after taking the limit of infinite volume. Comparing with (49) there is an overall minus sign, and a factor of 4 corresponding to the four degrees of freedom of the Dirac field (two particle and antiparticle spin DOFs). The sum now runs over the fermionic Matsubara frequencies, which pick up an imaginary part from the chemical potential. To sum the series we use the same approach as before, using in this case

$$\tanh\left(\frac{z}{2}\right) = \sum_{n=-\infty}^{\infty} \frac{2z}{z^2 + (2n+1)^2\pi^2}. \tag{136}$$

For $\mu = 0$ this gives directly

$$\frac{\partial}{\partial\omega_{\mathbf{k}}} \sum_n \ln(\omega_n^2 + \omega_{\mathbf{k}}^2) = \sum_n \frac{2\omega_{\mathbf{k}}}{\omega_n^2 + \omega_{\mathbf{k}}^2} = \beta\tanh\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right). \tag{137}$$

For $\mu \neq 0$ the sum can be evaluated using the following method. We define

$$f(z) = \frac{2\omega_{\mathbf{k}}}{\omega_{\mathbf{k}}^2 - (z-\mu)^2}, \tag{138}$$

$$g(z) = \frac{\beta}{2}\tanh\left(\frac{\beta z}{2}\right) = \sum_{n=\infty}^{\infty} \frac{z}{z^2 + \omega_n^2} \tag{139}$$

As is clear from (139), $g(z)$ has simple poles with residue 1 at $z = i\omega_n$ for all $n \in \mathbb{Z}$, and is analytic everywhere else. Using the residue theorem,

$$S \equiv \sum_{n=\infty}^{\infty} \frac{2\omega_{\mathbf{k}}}{\omega_{\mathbf{k}}^2 + \omega_n^2} = \sum_{n=\infty}^{\infty} f(i\omega_n) = \sum_{z=i\omega_n} Res[f(z)g(z)] = \frac{1}{2\pi i}\oint_C f(z)g(z), \tag{140}$$

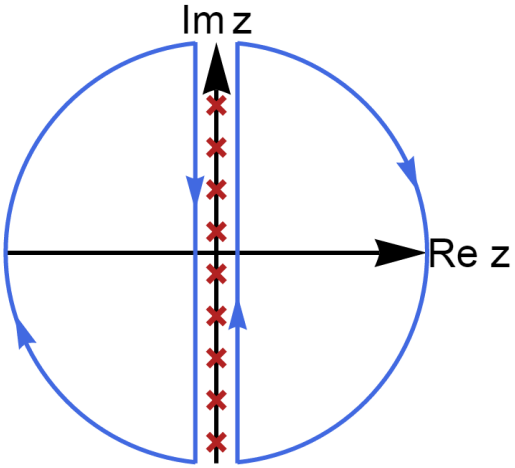

Figure 1: This integration contour, with the radius of the semicircles taken to infinity, can replace a counterclockwise contour "enclosing" the imaginary axis, if the integrand falls off more rapidly than $\frac{1}{|z|}$.

where C is a contour enclosing the imaginary axis. Since $f(z)g(z)$ falls off faster than $1/z$, this contour can be replaced by two clockwise contours as shown in figure 1, where the semicircular parts give vanishing contributions to the integrals when the radius is sent to infinity. $f(z)$ has two poles in the area enclosed by these contours: at $z=\mu\pm\omega_{\mathbf{k}}$, with residues -1 and 1.

$$S=-\sum_{z=\mu\pm\omega_{\mathbf{k}}} Res[fg(z)] = g(\mu+\omega_{\mathbf{k}})-g(\mu-\omega_{\mathbf{k}}) = \frac{\beta}{2}\left[\tanh\frac{\beta(\omega_{\mathbf{k}}+\mu)}{2}+\tanh\frac{\beta(\omega_{\mathbf{k}}-\mu)}{2}\right] \tag{141}$$

This can be integrated:

$$\begin{aligned}\int \mathrm{d}\omega_{\mathbf{k}} S &= \ln\left\{\cosh\left[\frac{\beta}{2}(\omega_{\mathbf{k}}-\mu)\right]\right\}+\ln\left\{\cosh\left[\frac{\beta}{2}(\omega_{\mathbf{k}}+\mu)\right]\right\}+\text{constant}\\ &= \beta\omega_{\mathbf{k}}+\ln\left[1+\mathrm{e}^{-\beta(\omega_{\mathbf{k}}-\mu)}\right]+\ln\left[1+\mathrm{e}^{-\beta(\omega_{\mathbf{k}}+\mu)}\right]+\text{constant},\end{aligned} \tag{142}$$

where we again discard the integration constant. Thus the grand potential per unit of volume is

$$\frac{\Omega}{V}=-\frac{1}{\beta V}\ln Z=-2\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\left\{\omega_{\mathbf{k}}+\frac{1}{\beta}\ln\left[1+\mathrm{e}^{-\beta(\omega_{\mathbf{k}}-\mu)}\right]+\frac{1}{\beta}\ln\left[1+\mathrm{e}^{-\beta(\omega_{\mathbf{k}}+\mu)}\right]\right\} \tag{143}$$

where the first term is again the zero-point energy.

6.1. Source terms

In the same way as in the scalar field case, we can incorporate source terms in the Lagrangian. In this case, we need two independent spinor field sources η and $\bar{\eta}$:

$$\mathscr{L}_E \to \mathscr{L}_E+\bar{\eta}\psi+\bar{\psi}\eta. \tag{144}$$

Analogously to (61) and (62), this leads to

$$Z(\beta,\mu,\eta,\bar{\eta})=Z(\beta,\mu,0,0)\exp\left[\int \mathrm{d}^4x\mathrm{d}^4x'\bar{\eta}(x)S(x-x')\eta(x')\right], \tag{145}$$

if we take the temperature and chemical potential to be zero. The propagator $S(x-x')$ is then a Green function of the operator $\partial\!\!\!/ + m$, whose Fourier transform satisfies

$$(i\not{p}+m)S(p) = 1 \tag{146}$$

$$\implies (-i\not{p}+m)(i\not{p}+m)S(p) = (-i\not{p}+m) \tag{147}$$

$$S(p) = \frac{-i\not{p}+m}{p^2+m^2} \equiv \frac{1}{i\not{p}+m}, \tag{148}$$

where the last expression is a convenient shorthand.

7. Spontaneous symmetry breaking

If a Lagrangian obeys a symmetry that is not respected in the ground state of the system, that symmetry is said to be spontaneously broken. A simple example in field theory is found in the following Lagrangian density:

$$\mathscr{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \equiv \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi) \tag{149}$$

where ϕ is a real scalar field and $m^2 > 0$. We note that the Lagrangian density is invariant under the transformation $\phi \to -\phi$. If $m^2 > 0$, the last term can be treated as a perturbation (for small λ), with the zeroth order expansion in λ corresponding to a free field.For m^2 negative, however, removing the last term in (149) leads to a Hamiltonian density which is not bounded from below, allowing arbitrarily negative energies. Thus, another approach is necessary. $V(\phi)$ has a local maximum at $\phi = 0$, and global minima at $\phi = \pm\sqrt{-6m^2/\lambda} \equiv \pm\phi_0$. Classically, a field with this Lagrangian density has two ground states, $\phi(\mathbf{x}) = \pm\phi_0$. It makes sense to quantise the field about a classical ground state, setting

$$\phi = \phi_0 + \tilde{\phi}. \tag{150}$$

Written in terms of $\tilde{\phi}$, the Lagrangian density becomes

$$\mathscr{L} = -\frac{3}{2}\frac{m^4}{\lambda} + \frac{1}{2}\partial^\mu\tilde{\phi}\partial_\mu\tilde{\phi} + 2m^2\tilde{\phi}^2 - \sqrt{\frac{-\lambda m^2}{6}}\tilde{\phi}^3 - \frac{1}{4!}\lambda\tilde{\phi}^4 \tag{151}$$

$$= -\frac{3}{2}\frac{m^4}{\lambda} + \mathscr{L}_0(\tilde{\phi}) + \mathscr{O}(\sqrt{\lambda}) \tag{152}$$

where

$$\mathscr{L}_0 = \frac{1}{2}\partial^\mu\tilde{\phi}\partial_\mu\tilde{\phi} + 2m^2\tilde{\phi}^2 \tag{153}$$

is the Lagrangian density of a free field with squared mass $-4m^2$. This can be treated perturbatively. In the ground state we have the expectation value $\langle\tilde{\phi}\rangle = 0$ (in general the expectation value of a quantum field obeys the classical field equations (see [6], p. 197-198)). The ground state is then not invariant under the transformation $\phi \to -\phi$.

At finite temperature, one can approximate the partition function by performing the path integral analogue of Laplace's method on Z. If ϕ_0 is a field that extremizes the action (i.e. a classical field), then

the action can be written

$$S_E[\phi] = S_E[\phi_0 + \tilde{\phi}] = \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3\mathbf{x} \left[\frac{1}{2}\partial_\mu \tilde{\phi}\partial_\mu \tilde{\phi} + V(\phi_0) + \frac{1}{2}V''(\phi_0)\tilde{\phi}^2 + \frac{1}{6}V'''(\phi_0)\tilde{\phi}^3 \ldots \right] \tag{154}$$

$$\equiv \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3\mathbf{x} V_{\mathrm{eff}}(\phi_0) = \beta V V_{\mathrm{eff}}(\phi_0) \tag{155}$$

where $V_{eff(\phi_0)}$ is known as the effective potential. In the next-to-lowest order approximation, terms of order $\tilde{\phi}^3$ are neglected, giving

$$Z \approx \int \mathscr{D}\tilde{\phi} \exp\left\{ -\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3\mathbf{x} \left[\partial_\mu \tilde{\phi}\partial_\mu \tilde{\phi} + V(\phi_0) + \frac{1}{2}V''(\phi_0)\tilde{\phi}^2 \right] \right\} \tag{156}$$

In the language of Feynman diagrams, this is the one-loop order of a loop expansion. For details, an introductory textbook on Quantum Field Theory, such as ref. [8], can be consulted. For the potential in (149) we have

$$V''(\phi_0) = m^2 + \frac{1}{2}\lambda\phi_0^2. \tag{157}$$

The effective potential in this approximation can be computed using the result (54). However the divergent term must be dealt with, which we will now proceed to do.

8. The zero-point energy of the free scalar field

We now return to the zero-point energy

$$\begin{aligned} \mathscr{E} &= \frac{1}{2}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\sqrt{m^2+\mathbf{k}^2} = \frac{1}{2}\int \frac{\mathrm{d}\Omega}{(2\pi)^3}\int dk k^2\sqrt{m^2+k^2} \\ &= \frac{4\pi}{2(2\pi)^3}\int \mathrm{d}k k^2\sqrt{m^2+k^2} \end{aligned} \tag{158}$$

of equation (54). This integral is UV divergent. However, there are ways, called regularization schemes, of making sense of such expressions. The idea is to introduce a parameter, called a regulator, so that the divergent integral is a special case of an otherwise finite expression. For example, if we imagine that our theory is valid only up to some finite momentum scale Λ, we could try making the replacement

$$\begin{aligned} \mathscr{E} &= \frac{1}{(2\pi)^2}\int_0^\infty \mathrm{d}k k^2\sqrt{m^2+k^2} \to \frac{1}{(2\pi)^2}\int_0^\Lambda \mathrm{d}k k^2\sqrt{m^2+k^2} \\ &= \frac{\Lambda^4}{(2\pi)^2} f(m/\Lambda), \end{aligned} \tag{159}$$

where

$$f(x) = \int_0^1 \mathrm{d}y y^2\sqrt{y^2+x^2} = \frac{1}{8}\left\{ (2+x^2)\sqrt{1+x^2} - x^4\left[\ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2}+1} \right) \right] \right\}. \tag{160}$$

Another regularization scheme, known as dimensional regularization, is less intuitive but ultimately more useful. The idea, first introduced into quantum field theory by 't Hooft and Veltmann in 1973 [9], is to, by

analytic continuation, promote the dimension of the integral to a continuous complex variable $d = 3-2\varepsilon$. The original integral is viewed as the $\varepsilon \to 0$ limit of this expression.

$$\begin{aligned}\mathscr{E} \to \frac{1}{2}\Lambda^{2\varepsilon}\int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}\sqrt{m^2+\mathbf{k}^2} &= \frac{S_{d-1}}{(2\pi)^d}\frac{1}{2}\Lambda^{2\varepsilon}\int \mathrm{d}k k^{d-1}\sqrt{m^2+k^2} \\ &= \frac{S_{2-2\varepsilon}}{(2\pi)^{3-2\varepsilon}}\frac{1}{2}m^4\left(\frac{\Lambda}{m}\right)^{2\varepsilon}\int \mathrm{d}x x^{2-2\varepsilon}\sqrt{x^2+1} \\ &= \frac{S_{2-2\varepsilon}}{(2\pi)^{3-2\varepsilon}}\frac{1}{4}m^4\left(\frac{\Lambda}{m}\right)^{2\varepsilon}\int \mathrm{d}t\, t^{\frac{1}{2}-\varepsilon}\sqrt{t+1}, \end{aligned} \tag{161}$$

The renormalization scale Λ has dimension mass and ensures that $[\mathscr{E}] = [m^4]$. S_{d-1} is the surface area of a unit sphere in d- dimensional space, which is given by

$$S_{d-1} = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}. \tag{162}$$

The integral in (161) can be computed using an integral representation of Euler's beta function,

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^\infty \mathrm{d}t \frac{t^{x-1}}{(1+t)^{x+y}}, \tag{163}$$

where Γ is the gamma function. We will need the following results:

$$\Gamma(-1/2) = -2\sqrt{\pi}, \tag{164}$$

$$\Gamma(-2+\varepsilon) = \frac{1}{2\varepsilon}+\frac{3}{4}-\frac{1}{2}\gamma_E+\mathscr{O}(\varepsilon), \tag{165}$$

where $\gamma_E \approx 0.578$ is the Euler-Mascheroni constant. This yields,

$$\begin{aligned}\mathscr{E} &= \frac{1}{4}m^4\left(\frac{\Lambda}{m}\right)^{2\varepsilon}\frac{S_{2-2\varepsilon}}{(2\pi)^{3-2\varepsilon}}\int \mathrm{d}t \frac{t^{\frac{1}{2}-\varepsilon}}{(1+t)^{-\frac{1}{2}}} \\ &= \frac{1}{4}m^4\left(\frac{\Lambda}{m}\right)^{2\varepsilon}\frac{2\pi^{\frac{3}{2}-\varepsilon}}{(2\pi)^{3-2\varepsilon}\Gamma\left(\frac{3}{2}-\varepsilon\right)}\frac{\Gamma\left(\frac{3}{2}-\varepsilon\right)\Gamma(-2+\varepsilon)}{\Gamma\left(-\frac{1}{2}\right)} \\ &= -\frac{1}{32\pi^2}m^4\left(\frac{4\pi\Lambda^2}{m^2}\right)^{\varepsilon}\Gamma(-2+\varepsilon). \end{aligned} \tag{166}$$

Expanding in ε, we obtain:

$$\begin{aligned}\mathscr{E} &= -\frac{1}{32\pi^2}m^4\left[1+\varepsilon\ln\left(\frac{4\pi\Lambda^2}{m^2}\right)+\mathscr{O}(\varepsilon^2)\right]\left[\frac{1}{2\varepsilon}+\frac{3}{4}-\frac{1}{2}\gamma_E+\mathscr{O}(\varepsilon)\right] \\ &= -\frac{1}{64\pi^2}m^4\left[\frac{1}{\varepsilon}+\ln\left(\frac{4\pi\Lambda^2}{m^2}\right)-\gamma_E+\frac{3}{2}+\mathscr{O}(\varepsilon)\right]. \end{aligned} \tag{167}$$

The rescaling

$$\Lambda^2 \to \frac{\mathrm{e}^{\gamma_E}}{4\pi}\Lambda^2 \tag{168}$$

is convenient, and known as the modified minimal subtraction (or $\overline{\text{MS}}$) scheme. With this, we get the more compact expression

$$\mathscr{E} = -\frac{1}{64\pi^2} m^4 \left[\frac{1}{\varepsilon} + \ln\left(\frac{\Lambda^2}{m^2}\right) + \frac{3}{2} \right], \tag{169}$$

where we have discarded terms of order ε. At this point we are ready to compute the effective potential for the field from the previous section, by making the substitution $m^2 \to V''(\phi_0)$ in (54). The full expression is

$$\begin{aligned} V_{\text{eff}}(\phi_0, \Lambda) = {} & \frac{1}{2} m^2 \phi_0^2 + \frac{1}{4!} \lambda \phi_0^4 - \frac{1}{64\pi^2} \left[m^4 + \lambda m^2 \phi_0^2 + \frac{1}{4} \lambda^2 \phi_0^4 \right] \left[\frac{1}{\varepsilon} + \ln\left(\frac{\Lambda^2}{m^2 + \frac{1}{2}\lambda\phi_0^2} \right) + \frac{3}{2} \right] \\ & + \frac{1}{\beta} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \ln\left(1 - \mathrm{e}^{-\beta\tilde{\omega}_{\mathbf{k}}}\right), \end{aligned} \tag{170}$$

where $\tilde{\omega}_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2 + \frac{1}{2}\lambda\phi_0^2}$. It can be shown [6] that the approximation made in (156) amounts to making a series expansion to first order in the Planck constant. If the Planck constant was included explicitly in (156), the quantum corrections to the classical potential, $V_{\text{eff}}(\phi_0) - V(\phi_0)$, would carry a factor of $\hbar$. To eliminate the divergent terms in (170), the mass and coupling constant are renormalised as follows:

$$m^2 \to m^2 + \delta m^2 = m^2 + \frac{\lambda m^2 \hbar}{32\pi^2 \varepsilon} + \mathscr{O}(\hbar^2), \tag{171}$$

$$\lambda \to \lambda + \delta\lambda = \lambda + \frac{3\lambda^2 \hbar}{32\pi^2 \varepsilon} + \mathscr{O}(\hbar^2). \tag{172}$$

We did not previously include a constant term in the Lagrangian, but the last divergent term is removed by adding a constant vacuum energy term $\mathscr{E}_0$ to the Lagrangian density and renormalizing this as

$$\mathscr{E}_0 \to \mathscr{E}_0 + \delta\mathscr{E}_0 = \mathscr{E}_+ \frac{m^4 \hbar}{64\pi^2 \varepsilon} + \mathscr{O}(\hbar^2) \tag{173}$$

When the renormalized parameters are inserted in the Lagrangian, the divergent terms in (170) disappear, whereas the remainder of the expression remains unchanged to first order in $\hbar$.

9. Quantum Chromodynamics

The theory known as Quantum Chromodynamics is regarded as a fundamental theory of the strong interaction. In this section we will give a minimal summary of some of its properties; in particular the symmetries of the theory, which provide the justification for the use of the model which will be studied in the next section.

The building blocks of the theory are quarks, which are fermions, and gluons, which are gauge bosons. The gauge group of QCD is $SU(3)$ (or more generally, $SU(N_c)$ if one allows for an arbitrary number of colors). That is to say, a quark field has three (N_c) components,

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}. \tag{174}$$

The Lagrangian is invariant under global $SU(3)$ transformations on these components, and by introducing a gauge field, the gluon field, it is made to be invariant under local transformations as well. The gluon field has eight components, one for each of the generators of $SU(3)$. There are six different flavors (discovered so far) of quarks, divided into three generations: the up and down quarks are relatively light (their masses are approximately 2 and 5 MeV, respectively), and are the constituents of protons and neutrons. Progressively heavier are the strange, charm, bottom and top quarks, with masses ranging from ~ 95 MeV to ~ 170 GeV [10]. Since the mimimum energy required to create a particle is proportional to its mass, it is possible to neglect the heavier quarks in a low-energy approximation. The (Minkowskian) Lagrangian density of the theory can be written as [11]

$$\mathscr{L}_{\mathrm{QCD}} = \bar{\psi}_i \left[i(\gamma^\mu D_\mu)_{ij} - m\delta_{ij} \right] \psi_j - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a. \tag{175}$$

Here, the indices i, j correspond to the three components of the quark field. The index a runs from 1 to 8, corresponding to the eight components of the gluon field. D_μ is the covariant derivative

$$(D_\mu)_{ij} = \partial_\mu \delta_{ij} - ig(T_a)_{ij} A^a, \tag{176}$$

with A^a the gluon field, T_a the eight generators of $SU(3)$, and g a dimensionless coupling constant. $G^a_{\mu\nu}$ is the gluon field strength tensor, defined as

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu, \tag{177}$$

with f^{abc} the structure constants of $SU(3)$, satisfying

$$[T^a, T^b] = f^{abc} T^c. \tag{178}$$

The Lagrangian is invariant under simultaneous transformations

$$\begin{aligned} \psi &\to \mathrm{e}^{i\alpha_a(x)T^a} \psi \\ \bar{\psi} &\to \bar{\psi} \mathrm{e}^{-i\alpha_a(x)T^a} \\ A^a_\mu &\to A^a_\mu - \frac{1}{g} \partial_\mu \alpha^a(x), \end{aligned} \tag{179}$$

where $\alpha(x)$ is an arbitrary phase.

9.1. Color charge, confinement and asymptotic freedom

We can associate a conserved charge with the strong interaction, known as color charge. The three components of the quark field each correspond to one of the colors red, green and blue. Quarks carry one unit of color, whereas antiquarks carry one unit of "anticolor" ("antired", etc.). Gluons carry one unit of color and one unit of anticolor. To get a color-neutral composite, we can either combine one unit of each color ("red + green + blue = white"), or a unit of one color with a unit of the corresponding anticolor. It turns out that QCD has a property known as confinement, which means that quarks and gluons never appear as isolated particles, and are only found in color-neutral bound states. Three quarks make a baryon, such as the proton and neutron, whereas a quark-antiquark pair constitute a meson. This is related to the property known as asymptotic freedom: The coupling between quarks and gluons increases at low energies (i.e. when the center-of-mass energy of two interacting particles is low), and approaches zero as the energy approaches infinity. This means that the attractive force between two quarks in a hadron becomes increasingly strong when the distance between them increases (large distance corresponds to low energy), and an infinite amount of energy would be required to remove a single quark completely from a hadron, or completely separate the quark-antiquark pair in a pion.

9.2. Flavor symmetry

If we have several quark flavors, the quark field ψ becomes a multiplet,

$$\psi = \begin{pmatrix} u \\ d \\ \vdots \end{pmatrix}, \tag{180}$$

and the mass term in (175) becomes a diagonal matrix of quark masses, $m = \text{diag}(m_u, m_d, \dots,)$. We will be concerned with the low-energy approximation, including only the up and down quarks. In this case, the Lagrangian has an approximate flavor symmetry: if the masses of the up and down quarks were precisely equal, $U(2)$ transformations of the flavor doublet would leave the Lagrangian invariant. There is also an approximate chiral symmetry: We can decompose a fermion field into left-handed and right-handed components, $\psi = \psi_R + \psi_L$, where

$$\begin{aligned} \psi_R &= \frac{1}{2}(1+\gamma_5)\psi, \\ \psi_L &= \frac{1}{2}(1-\gamma_5)\psi. \end{aligned} \tag{181}$$

where $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_0$. For massless fermions, these components correspond to particles with positive and negative helicity, respectively [6]. Writing (175) in terms of these components, we have

$$\mathscr{L}_{\text{QCD}} = \bar{\psi}_{Ri} i(\gamma^\mu D_\mu)_{ij}\psi_{Rj} + \bar{\psi}_{Li} i(\gamma^\mu D_\mu)_{ij}\psi_{Lj} - m(\bar{\psi}_{Ri}\psi_{Li} + \bar{\psi}_{Li}\psi_{Ri}) - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a. \tag{182}$$

If not for the mass term, the chiral components would decouple in the Lagrangian, and we could make $U(2)$ transformations on the chiral components independently. The symmetry group $U(2)_L \times U(2)_R$ can be decomposed into $SU(2)_L \times SU(2)_R \times U(1)_V \times U(1)_A$ (V for vector and A for axial), when the chiral components transform independently under $SU(2)$. Under $U(1)_V$, the fields transform as follows:

$$\psi_L \to e^{i\alpha}\psi_L, \qquad \psi_R \to e^{i\alpha}\psi_R. \tag{183}$$

And under $U(1)_A$,

$$\psi_L \to e^{i\beta}\psi_L, \qquad \psi_R \to e^{-i\beta}\psi_R. \tag{184}$$

The $U(1)_A$ symmetry, while an exact symmetry at the classical level (in the case of vanishing quark masses), turns out not to be an exact symmetry of the quantum field theory due to a so-called anomaly [12]. Therefore, the flavor symmetry group of this model in the limit of vanishing quark masses is $SU(2)_L \times SU(2)_R \times U(1)_V$.

While the masses of the up and down quarks are in reality finite and different, they are small enough to consider chiral symmetry an approximate symmetry of the Lagrangian. As it turns out, this symmetry is spontaneously broken down to $SU(2)_V \times U(1)_V$ in the vacuum by the formation of a quark condensate, a nonzero expectation value $\langle\bar{\psi}\psi\rangle$. According to Goldstone's theorem, there should appear massless bosons in the energy spectrum. They are known as pions. However, since the symmetry which is being broken is not an exact one, the pions are not in fact massless; they do, however have relatively small masses, and are classified as pseudo-Goldstone bosons.

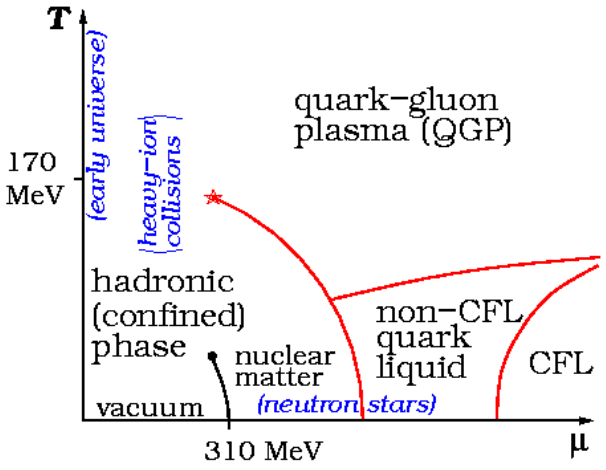

Figure 2: This diagram shows what the qualitative form of the QCD phase diagram, in the plane of temperature and baryon chemical potential, is thought to be, as pieced together by different means of research.

9.3. The phases of QCD

The current understanding of the QCD phase diagram is summarized in figure 2 [13]. At low temperature and quark chemical potential, quarks and gluons are confined in bound states (hadrons), and chiral symmetry is spontaneously broken. When the temperature increases, the coupling between the particles decreases, and it is expected that a phase transition will occur at some point, leading to a phase known as quark-gluon plasma, where the quarks and gluons are deconfined and asymptotically free, and approximate chiral symmetry is restored. At low densities (corresponding to low baryon chemical potential), this transition is a crossover, whereas at a sufficiently high density one expects to find a critical point, beyond which the phase transition is discontinuous (first-order). At low temperature and high density, the hadrons become increasingly closely packed together. At sufficiently high densities, the hadrons are expected to merge with a each other, leading to a phase of deconfined quarks and gluons. It is expected that a phase of matter may result in which the quarks form Cooper-pairs, leading to a color-superconducting phase of matter known as the color-flavor-locked (CFL) phase [14]. In the area between the hadronic and CFL phase, several distinct phases of QCD have been hypothesized. This is currently a blooming area of research [2].

It is generally very difficult to treat QCD analytically. Therefore, many of the results on its phase structure are obtained in other ways. An important method is known as lattice QCD, which is a numerical method based on the discretization of spacetime. One of its main limitations, known as the sign problem, makes this method unfeasible at large baryon chemical potential [5]. Another approach is the use of effective models. In the next section we turn to such a model.

10. Quark-Meson Model

In this section we will use the main results we have obtained so far to study the so-called quark-meson (QM) model (also known as the linear sigma model coupled to quarks, or $\mathrm{LSM_q}$). It was originally proposed by Yukawa [15] as a model of the interactions between nucleons through exchange of pions. In later years, this model has also been used as an effective model of QCD at low energies. By construction, it has an $SU(2)\times SU(2)\times U(1)$ symmetry which is broken down to $SU(2)\times U(1)$in the vacuum, as is the

case in two-flavor QCD. Therefore, it can be used to model the chiral phase transition of QCD, i.e. the transition between phases of broken and unbroken chiral symmetry. The model includes the quarks as massless fermions coupled to a four-component scalar field. In this model, the quarks are not confined, but there exists an extension of the model, the Polyakov-quark-meson model, which incorporates quark confinement [16].

The Euclidean Lagrangian density is given by

$$\mathscr{L}_E = \mathscr{L}_{\text{meson}} + \mathscr{L}_{\text{quark}} + \mathscr{L}_{\text{Yukawa}}, \tag{185}$$

where

$$\mathscr{L}_{\text{meson}} = \frac{1}{2}\partial_\mu \phi_i \partial_\mu \phi_i + \frac{1}{2}m^2 \phi_i \phi_i + \frac{1}{4!}\lambda(\phi_i \phi_i)^2 - h\sigma, \tag{186}$$

$$\mathscr{L}_{\text{quark}} = \bar{\psi}(\partial\!\!\!/ - \mu\gamma_0)\psi, \tag{187}$$

$$\mathscr{L}_{\text{Yukawa}} = g\bar{\psi}(\sigma + i\gamma_5 \vec{\tau}\cdot\vec{\pi})\psi, \tag{188}$$

where $i = 1,2,3,4$, and

$$(\phi_1, \phi_2, \phi_3, \phi_4) = (\sigma, \pi_0, \pi_1, \pi_2) = (\sigma, \vec{\pi}). \tag{189}$$

$$\psi = \begin{pmatrix} u \\ d \end{pmatrix} \tag{190}$$

is an isospin doublet, with u and d the up and down quarks (in Yukawa's model, the proton and neutron), with no bare mass. $\mu = \text{diag}(\mu_u, \mu_d)$ is the quark chemical potential. $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ are the Pauli matrices, acting on the isospin indices of ψ.

The quark sector of the Lagrangian is invariant under $U(2)$ transformations, which decompose into $SU(2) \times U(1)$, in isospin space. Without a quark mass term, the left- and right-chiral components decouple, and $\mathscr{L}_{\text{quark}}$ becomes invariant under $U(2)_L \times U(2)_R \sim SU(2)_L \times SU(2)_R \times U(1)_V \times U(1)_A$. In addition, the quark fields belong to the fundamental representation of the group $SU(N_c)$, where N_c is the number of colors. while in QCD this symmetry is a local one, we leave it as a global symmetry in this model.

$\mathscr{L}_{\text{meson}}$ alone is known as the $O(4)$ linear sigma model. Note that when $h = 0$ this Lagrangian is invariant under $O(4)$ transformations. When m^2 is negative, this symmetry is broken in the vacuum, which only respects $O(3)$ symmetry. When $h \neq 0$ the symmetry is explicitly broken in the Lagrangian. $O(4)$ is locally isomorphic to $SU(2) \times SU(2)$. For $U, V \in SU(2)$ the transformation

$$\sigma + i\vec{\pi}\cdot\vec{\tau} \to V(\sigma + i\vec{\pi}\cdot\vec{\tau})U^{-1} = \sigma' + i\vec{\pi}'\cdot\vec{\tau} \tag{191}$$

leaves $\mathscr{L}_{\text{meson}}$ invariant, since

$$\sigma'^2 + \vec{\pi}'^2 \propto \det(\sigma' + i\vec{\pi}'\cdot\vec{\tau}) = \det(V)\det(\sigma + i\vec{\pi}\cdot\vec{\tau})\det(U^{-1}) = \det(\sigma + i\vec{\pi}\cdot\vec{\tau}) \tag{192}$$

with $\sigma', \vec{\pi}'$ real given real $\sigma, \vec{\pi}$ [1]. In terms of the chiral fields, the interaction term is

$$\mathscr{L}_{\text{Yukawa}} = g\bar{\psi}_L(\sigma - i\gamma_5\vec{\tau}\cdot\vec{\pi})\psi_R + g\bar{\psi}_R(\sigma + i\gamma_5\vec{\tau}\cdot\vec{\pi})\psi_L \tag{193}$$

[1] This follows from the fact that any 2×2 unitary matrix can be written $U = aI + \vec{b}\cdot\vec{\tau}$ with $a, \vec{b}$ real, and the fact that the product of two unitary matrices is unitary.

Clearly, the transformation

$$\begin{aligned} \psi_R &\to U\psi_R, \\ \psi_L &\to V\psi_L, \\ \sigma + i\vec{\pi}\cdot\vec{\tau} &\to V(\sigma + i\vec{\pi}\cdot\vec{\tau})U^{-1} \end{aligned} \tag{194}$$

leaves the full Lagrangian invariant when $h = 0$. We can also make $U(1)_V$ transformations, however the $U(1)_A$ symmetry is not respected by the interaction term. Thus the symmetry group of the full Lagrangian with $h = 0$ is $SU(2)_L \times SU(2)_R \times U(1)_V$.

We will treat this model in the same way as the single scalar field. In general, the expectation value of the four-component scalar field can point in any direction in ϕ-space. However, by making the appropriate transformation (191), we can always ensure that it points in the σ direction. Thus, without loss of generality, we can set

$$\sigma = \phi_0 + \tilde{\sigma}, \tag{195}$$

such that

$$\langle\tilde{\sigma}\rangle = \langle\vec{\pi}\rangle = 0. \tag{196}$$

The expectation value of the σ field in this model corresponds to the chiral condensate $\langle\bar{\psi}\psi\rangle$ of QCD [17]. Making the same approximation as earlier, we keep only the terms quadratic in the quantum fields $\tilde{\sigma}$ and $\vec{\pi}$. This leaves us with

$$\begin{aligned} \mathscr{L}_{\text{meson}} &\approx \frac{1}{2}m^2\phi_0^2 + \frac{1}{4!}\lambda\phi_0^4 - h\phi_0 + \frac{1}{2}\partial_\mu\tilde{\sigma}\partial_\mu\tilde{\sigma} + \frac{1}{2}\left(m^2 + \frac{1}{2}\lambda\phi_0^2\right)\tilde{\sigma}^2 + \frac{1}{2}\partial_\mu\vec{\pi}\partial_\mu\vec{\pi} + \frac{1}{2}\left(m^2 + \frac{1}{6}\lambda\phi_0^2\right)\vec{\pi}^2 \\ &\equiv V(\phi_0) + \frac{1}{2}\partial_\mu\tilde{\sigma}\partial_\mu\tilde{\sigma} + \frac{1}{2}m_\sigma^2\tilde{\sigma}^2 + \frac{1}{2}\partial_\mu\vec{\pi}\partial_\mu\vec{\pi} + \frac{1}{2}m_\pi^2\vec{\pi}^2, \end{aligned} \tag{197}$$

and

$$\mathscr{L}_{\text{Yukawa}} \approx g\bar{\psi}\phi_0\psi, \tag{198}$$

so that

$$\begin{aligned} \mathscr{L}_{\text{quark}} + \mathscr{L}_{\text{Yukawa}} &\approx \bar{\psi}(\gamma_\mu\partial_\mu - \mu\gamma_0 + g\phi_0)\psi \\ &= \bar{\psi}(\gamma_\mu\partial_\mu - \mu\gamma_0 + m_q)\psi. \end{aligned} \tag{199}$$

When $\phi_0 \neq 0$, the quarks acquire an effective mass $m_q = g\phi_0$, which mixes the left and right chiral components and leads to the breaking of chiral symmetry. That is, $SU(2)_L \times SU(2)_R \times U(1)_V$ is broken down to $SU(2) \times U(1)_V$. Note that if $h = 0$, then in the vacuum, we have $\langle\sigma\rangle = \sqrt{-6m^2}$, which is the minimum of the tree-level potential. Thus the pion mass $m_\pi^2 = m^2 + \frac{1}{6}\phi_0^2$ is zero. Since $SU(2)$ has three generators, and there are three pions, this is in accordance with Goldstone's theorem.

Using equations (**??**) and (170), the effective potential for ϕ_0 can now be found. The contribution from the sigma field is

$$V_\sigma(\phi_0, \Lambda) = -\frac{1}{64\pi^2}m_\sigma^4\left[\frac{1}{\varepsilon} + \ln\left(\frac{\Lambda^2}{m_\sigma^2}\right) + \frac{3}{2}\right] + \frac{1}{\beta}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\ln\Big\{1 - \exp\left[-\beta\omega_\sigma(\mathbf{k})\right]\Big\}, \tag{200}$$

where the dispersion relation is

$$\omega_\sigma(\mathbf{k}) = \sqrt{m_\sigma^2 + \mathbf{k}^2}. \tag{201}$$

The contribution from each pion field is identical, and found by making the substitution $m_\sigma \to m_\pi$ in (200). Their sum is

$$V_\pi(\phi_0,\Lambda) = -\frac{3}{64\pi^2}m_\pi^4\left[\frac{1}{\varepsilon}+\ln\left(\frac{\Lambda^2}{m_\pi^2}\right)+\frac{3}{2}\right] + \frac{3}{\beta}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\ln\left\{1-\exp\left[-\beta\omega_\pi(\mathbf{k})\right]\right\}. \tag{202}$$

From (**??**) and (169) we get the fermion contribution. The factor of N_c comes from the contribution of each color component.

$$V_q(\phi_0,\Lambda) = \frac{N_c}{8\pi^2}m_q^4\left[\frac{1}{\varepsilon}+\ln\left(\frac{\Lambda^2}{m_q^2}\right)+\frac{3}{2}\right] - \frac{2N_c}{\beta}\sum_{f=u,d}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\left(\ln\left\{1+\exp\left[-\beta(\omega_\psi(\mathbf{k})-\mu_f)\right]\right\}+\ln\left\{1+\exp\left[-\beta(\omega_\psi(\mathbf{k})+\mu_f)\right]\right\}\right). \tag{203}$$

Collecting the divergent terms, we get

$$V_{\mathrm{eff}}(\phi_0,\Lambda) = -\frac{1}{64\pi^2\varepsilon}m_\sigma^4 - \frac{3}{64\pi^2\varepsilon}m_\pi^4 + \frac{N_c}{8\pi^2\varepsilon}m_q^4 + \mathscr{O}(\varepsilon^0) \tag{204}$$

$$= -\frac{m^4}{16\pi^2\varepsilon} - \frac{\lambda m^2}{32\pi^2\varepsilon}\phi_0^2 + \left[-\frac{\lambda^2}{192\pi^2\varepsilon}+\frac{N_c g^4}{8\pi^2\varepsilon}\right]\phi_0^4 + \mathscr{O}(\varepsilon^0) \tag{205}$$

These are, as before, eliminated by renormalizing the vacuum energy, meson masses and the coupling constant, with the counterterms

$$\delta\mathscr{E} = \frac{m^4}{16\pi^2\varepsilon}, \tag{206}$$

$$\delta m^2 = \frac{\lambda m^2}{16\pi^2\varepsilon}, \tag{207}$$

$$\delta\lambda = \frac{\lambda^2}{8\pi^2\varepsilon} - \frac{3N_c g^4}{2\pi^2\varepsilon}. \tag{208}$$

After renormalization, the one-loop effective potential of the model is

$$V_{\mathrm{eff}}(\phi_0,\Lambda) = \frac{1}{2}m^2\phi_0^2 + \frac{\lambda}{4!}\phi_0^4 - h\phi_0 - \frac{1}{64\pi^2}m_\sigma^4\left[\ln\left(\frac{\Lambda^2}{m_\sigma^2}\right)+\frac{3}{2}\right] - \frac{3}{64\pi^2}m_\pi^4\left[\ln\left(\frac{\Lambda^2}{m_\pi^2}\right)+\frac{3}{2}\right] + \frac{N_c}{8\pi^2}m_q^4\left[\ln\left(\frac{\Lambda^2}{m_q^2}\right)+\frac{3}{2}\right] + \frac{1}{\beta}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\left(\ln\left\{1-\exp\left[-\beta\omega_\sigma(\mathbf{k})\right]\right\}3\ln\left\{1-\exp\left[-\beta\omega_\pi(\mathbf{k})\right]\right\}\right) - \frac{2N_c}{\beta}\sum_{f=u,d}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\left(\ln\left\{1+\exp\left[-\beta(\omega_\psi(\mathbf{k})-\mu_f)\right]\right\}+\ln\left\{1+\exp\left[-\beta(\omega_\psi(\mathbf{k})+\mu_f)\right]\right\}\right). \tag{209}$$

10.1. Large-N_c-approximation

One problem we now face is that the meson masses

$$m_\sigma^2 = m^2 + \frac{1}{2}\lambda\phi_0^2,$$
$$m_\pi^2 = m^2 + \frac{1}{6}\lambda\phi_0^2,$$

are negative for values of ϕ_0 below the classical minimum of the potential, saddling the potential in (209) with an imaginary part. The interpretation of this is related to the instability of certain states, as is discussed in ref. [18]. The simplest way to handle this problem is to neglect the imaginary terms altogether. One approximation one could make in order to make (209) less unwieldy is called the large-N_c-approximation. In the limit of large N_c, the terms in (209) coming from the quantum and thermal fluctuations of the meson fields should become negligible, and we are left with

$$\begin{aligned} V_{\text{eff}}(\phi_0,\Lambda) = {} & \frac{1}{2}m^2\phi_0^2 + \frac{\lambda}{4!}\phi_0^4 - h\phi_0 + \frac{N_c}{8\pi^2}m_q^4\left[\ln\left(\frac{\Lambda^2}{m_q^2}\right) + \frac{3}{2}\right] \\ & - \frac{2N_c}{\beta}\sum_{f=u,d}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\ln\left\{1+\exp\left[-\beta(\omega_\psi(\mathbf{k})-\mu_f)\right]\right\} \\ & - \frac{2N_c}{\beta}\sum_{f=u,d}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\ln\left\{1+\exp\left[-\beta(\omega_\psi(\mathbf{k})+\mu_f)\right]\right\}. \end{aligned} \tag{210}$$

It should be noted that when the meson fluctuations are neglected, the potential is unbounded from below due to the term

$$\frac{N_c}{8\pi^2}m_q^4\ln\left(\frac{\Lambda^2}{m_q^2}\right) \sim -\phi^4\ln\phi_0^2, \tag{211}$$

which dominates for large ϕ_0 and becomes arbitrarily large and negative. When the meson fluctuations are included there are terms of the same form of the opposite sign, and depending on the parameters used, these terms may stabilize the potential. In any case, if the potential is calculated to all orders, it must be bounded from below for the vacuum to be stable. In general, the quantum corrections become large for increasing ϕ_0, and we expect that higher order corrections are needed at higher ϕ_0. We will assume that our approximation is valid for low ϕ_0, say $0 < \phi_0 < \phi_m$. For our purposes, the quantity of interest is the value of ϕ_0 which minimizes the effective potential. If this value is less than ϕ_m, the approximation can be used.

In the vacuum, $T = \mu_u = \mu_d = 0$, we have $f_\pi = \phi_0$, where $f_\pi = 93$ MeV is the pion decay constant [19]. From the tree level potential, the four independent parameters of the model, m^2, λ, g and h can be related to physical quantities as follows:

$$\begin{aligned} m^2 &= \frac{1}{2}\left(3m_\pi^2 - m_\sigma^2\right), \\ \lambda &= \frac{3(m_\sigma^2 - m_\pi^2)}{f_\pi^2}, \\ g &= \frac{m_q}{f_\pi}, \\ h &= f_\pi m_\pi^2. \end{aligned} \tag{212}$$

Matching parameters at tree level when working at one-loop order in the effective potential, as in eqs. (212) is commonly done in the literature. However, this is not quite correct, since the expressions for the sigma and pion masses, in tems of the parameters of the model, are modified at one-loop order. For a discussion of this, see ref. [20]. With this approximation, the minimum of the effective potential in the vacuum is dependent on the renormalization scale Λ. Since we require that the minimum remains at $\phi_0 = f_\pi$, this fixes Λ at a particular value:

$$\left.\frac{\mathrm{d}V_{\mathrm{eff}}(\phi_0)}{\mathrm{d}\phi_0}\right|_{\phi_0=f_\pi} = \left.\frac{\mathrm{d}V_0(\phi_0)}{\mathrm{d}\phi_0}\right|_{\phi_0=f_\pi} + \frac{gN_c}{2\pi^2}m_q^3\left[\ln\frac{\Lambda^2}{m_q^2}+1\right] = 0. \tag{213}$$

Since the tree-level potential already has its minimum at $\phi_0 = f_\pi$, the first term vanishes, and the condition on Λ is

$$\ln\frac{\Lambda^2}{g^2 f_\pi^2} + 1 = 0, \tag{214}$$

or $\Lambda = \mathrm{e}^{-\frac{1}{2}} m_q$.

Determining the values of the input parameters m_q and m_σ is not trivial. m_σ is, in principle, a measurable physical quantity. However, several candidate scalar particles have been observed experimentally, and it is not clear which one should be identified with the σ particle [21]. In some sense, therefore, the sigma mass can be regarded as a free parameter of the model. We will take as a starting point the value $m_\sigma = 800$ MeV, which is often used in this model, and look at how varying this value affects the results.

m_q, the effective quark mass, does not correspond directly to a measurable physical quantity. It can be related to the value of the quark condensate in the vacuum [22], which, although not measurable, can, in principle, be computed in lattice simulations. A reasonable estimate is 300 MeV [23], about one third of the mass of a nucleon.

Since up and down quarks are not quite massless, pions have finite but relatively small masses of approximately 140 MeV, earning them the name pseudo-Goldstone bosons. Had the up and down quarks been massless, the pions would have been proper massless Goldstone bosons. Rather than using the physical pion masses, we will work in the *chiral limit*, $h = 0$, which means the pions are massless in the vacuum. From this point, we will set $N_c = 3$, as is the case in QCD. We will work with a finite quark chemical potential $\mu = \frac{1}{2}(\mu_d + \mu_u) = \frac{1}{3}\mu_B$, where μ_B is the baryon chemical potential, and set the isospin chemical potential $\mu_I = \frac{1}{2}(\mu_d - \mu_u)$ to zero.

In figure 3, the normalized effective potential

$$\tilde{V}_{\mathrm{eff}}(\phi_0) \equiv [V_{\mathrm{eff}}(\phi_0) - V_{\mathrm{eff}}(0)]/f_\pi^4 \tag{215}$$

is plotted for $\mu = T = 0$. The function has a local minimum at $\phi_0 = 93$MeV, followed by an increase, before it eventually turns around. We will focus only on the local minimum, assuming that it corresponds to the global minimum of the true potential when all corrections are included. Whether or not, and under which circumstances, this can be justified is not a trivial question. If we lower the value of m_σ, the resulting effective potential looks discouraging (figure 4). At $m_\sigma = 600$ MeV, the potential just barely admits a local minimum at $\phi_0 = 93$ MeV, and at $m_\sigma = 500$ MeV this minimum has been replaced by a local maximum, so the condition that V_{eff} has a minimum at 93 MeV cannot be enforced. From this it can be concluded that, to the extent that our approximation is useful, its validity depends on the parameters used.

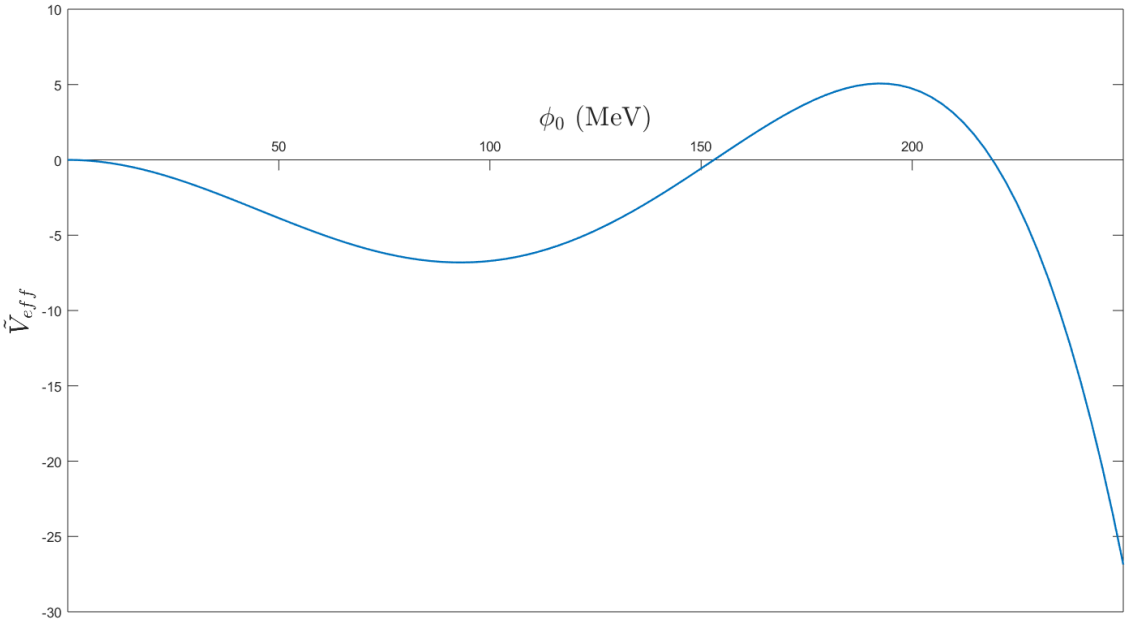

Figure 3: The normalized one-loop effective potential $\tilde{V}_{\text{eff}}(\phi_0)$ at zero temperature.

10.2. The phase diagram

As mentioned, the symmetry group $SU(2)_L \times SU(2)_R \times U(1)_V$ of the model in the chiral limit, i.e. with $h = 0$, is broken down to $SU(2)_V \times U(1)_V$ in the vacuum due to the nonvanishing expectation value of the σ field. If we increase the temperature, the system becomes increasingly disordered and we should expect this expectation value to vanish at some point, restoring the symmetry. Numerically, it was found that the temperature T_c at which this occurs is 168.1 MeV. The phase transition is continuous, as $\langle\sigma\rangle$, given by the minimum of the potential, approaches zero continuously. Figure 5 shows the normalized potential $V'_{\text{eff}}(\phi_0)$ plotted as a function of ϕ_0 for increasing values of T. For $T < T_c$ the potential has a minimum at finite ϕ_0, which moves toward the origin while becoming shallower as T increases. At $T = T_c$ this minimum reaches the origin.

It may be interesting at this point to compare these findings with the behavior of the corresponding classical field, i.e. the effective potential without the quantum corrections. When this term is not included, it is found that a *discontinuous*, or first-order phase transition occurs at $T = 173.6$ MeV. This can be seen in figure 6: At the critical temperature, the potential has two degenerate minima, one at the origin and one at $\phi_0 \approx 50$ MeV. Apparently, quantum fluctuations in this case have the effect of displacing the critical temperature slightly and changing the order of the transition.

Next, we investigate what happens at $T = 0$, $\mu \neq 0$. It turns out that in this case, the phase transition is first-order whether quantum fluctuations are included or not. The behavior of the one-loop potential around the critical chemical potential is shown in figure 7. The critical chemical potential at zero temperature is found to be $\mu_c = 316.7$ MeV. When the vacuum fluctuation term is not included, the critical chemical potential is $\mu_c = 343.4$ MeV. Again, quantum fluctuations cause the system to more easily

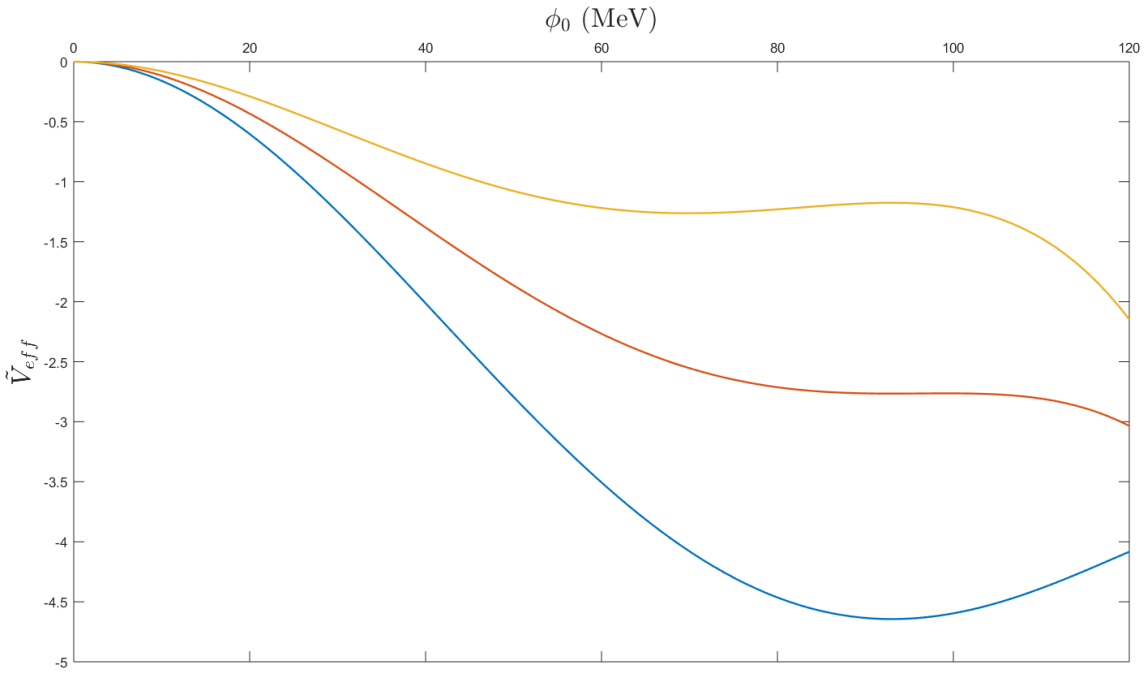

Figure 4: The normalized one-loop effective potential $\tilde{V}_{\text{eff}}(\phi_0)$ at zero temperature, for three different values of m_σ: $m_\sigma = 700$ MeV (blue curve), $m_\sigma = 600$ MeV (red curve), and $m_\sigma = 500$ MeV (yellow curve).

transition into a less ordered phase, as one might expect.

The next step is to map out the phase diagram in the entire $\mu - T$ plane. By numerically computing the critical chemical potential at a set of temperatures, the phase diagram in figure 8 (blue line) was produced. Solid and dashed lines correspond to first- and second-order phase transitions, respectively. The point where the solid and dashed line meet (known as a critical point) is at $(T,\mu) = (69,278)$ MeV. If the critical point is approached along the full line, the value of $\langle\sigma\rangle$ just inside the line decreases continuously from its value at zero temperature, which is about 93 MeV, to zero. The phase diagram which results when vacuum fluctuations are excluded is shown as the red line in fig. 8. Comparing this to the blue line, the main differences are in the order of the phase transition, and the location of the phase line. In the former case, the phase line is located slightly further from the origin throughout the diagram.

Figure 9 shows phase diagrams computed at various values of the sigma mass. Clearly, the value of m_σ strongly influences the shape of the phase line. At sufficiently high m_σ, the phase transition is second-order along the entire phase line.

10.3. Comparison with results from lattice QCD

In order to judge the results we have obtained, the ideal test would be comparison with experiment. However, it is at present not feasible to create quark matter in thermal equilibrium in a laboratory. The next best thing, then, is to turn to lattice QCD. As mentioned, this method is limited by the sign problem, and cannot be used at large baryon chemical potential. However, the chiral transition at $\mu_B = 0$ has been studied extensively. In ref. [24], the chiral transition of two-flavor QCD in the chiral limit (corresponding to a vanishing mass term in eq. (175)) was studied, and the transition was found to be of second order

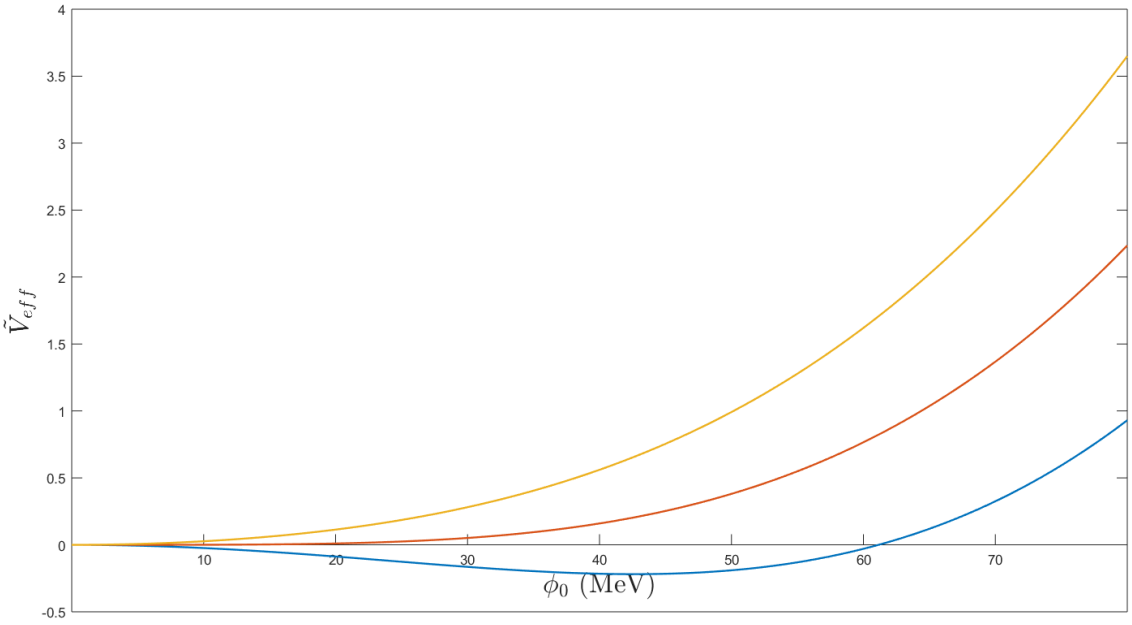

Figure 5: The normalized effective potential $\tilde{V}_{\text{eff}}$ as a function of ϕ_0, at zero chemical potential and three different temperatures: $T = T_c = 168.1$ MeV (red), $T = T_c - 10$ MeV (blue) and $T = T_c + 10$ MeV (yellow). A second-order phase transition occurs at $T = T_c$.

with an estimated transition temperature of 171 MeV. When compared with our results, this suggests a value of m_σ close to 800 MeV; in fact, perfect agreement is found by adjusting m_σ to the value 814 MeV.

When physical quark masses are used, the transition at $\mu_B = 0$ is found to be a crossover. That is to say, the chiral condensate approaches zero as the temperature increases, but never vanishes completely. In this case there is no critical temperature, but a "pseudocritical" temperature is commonly defined as the inflection point of the chiral condensate as a function of temperature. With three quark flavors (up, down and strange) with physical quark masses, the pseudocritical temperature defined this way was computed in refs. [25] and [26], finding the values 157 and 155 MeV, respectively.

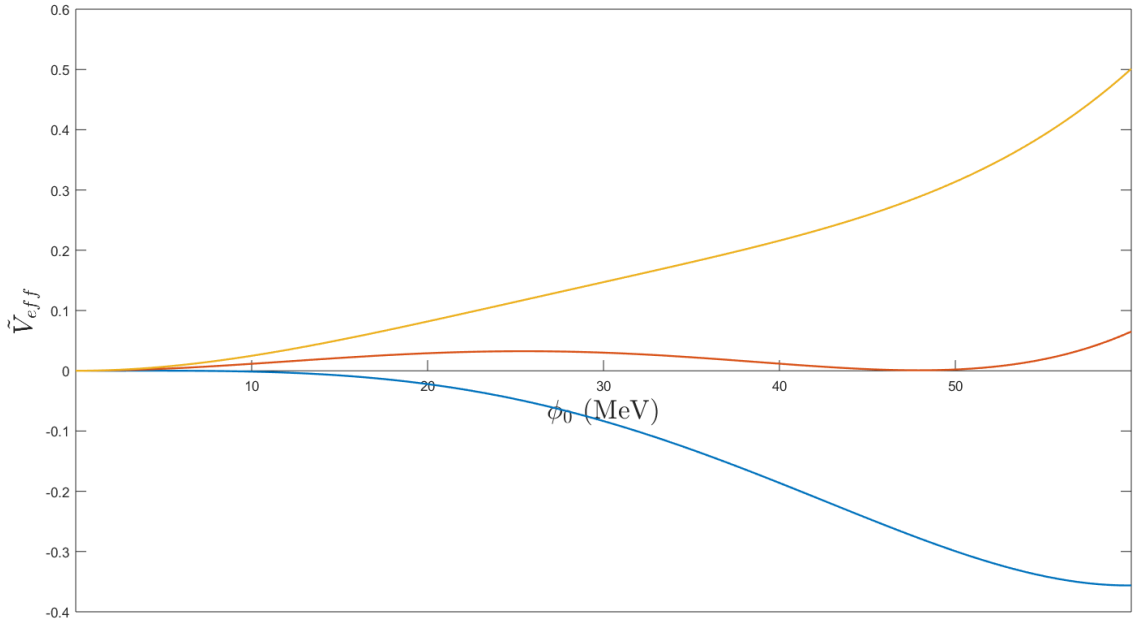

Figure 6: The normalized potential $\tilde{V}_{\text{eff}}$, without the vacuum fluctuation term, as a function of ϕ_0, at zero chemical potential and three different temperatures: $T = T_c = 173.6$ MeV (red), $T = T_c - 5$ MeV (blue), and $T = T_c + 5$ MeV (yellow). A first-order phase transition occurs at $T = T_c$.

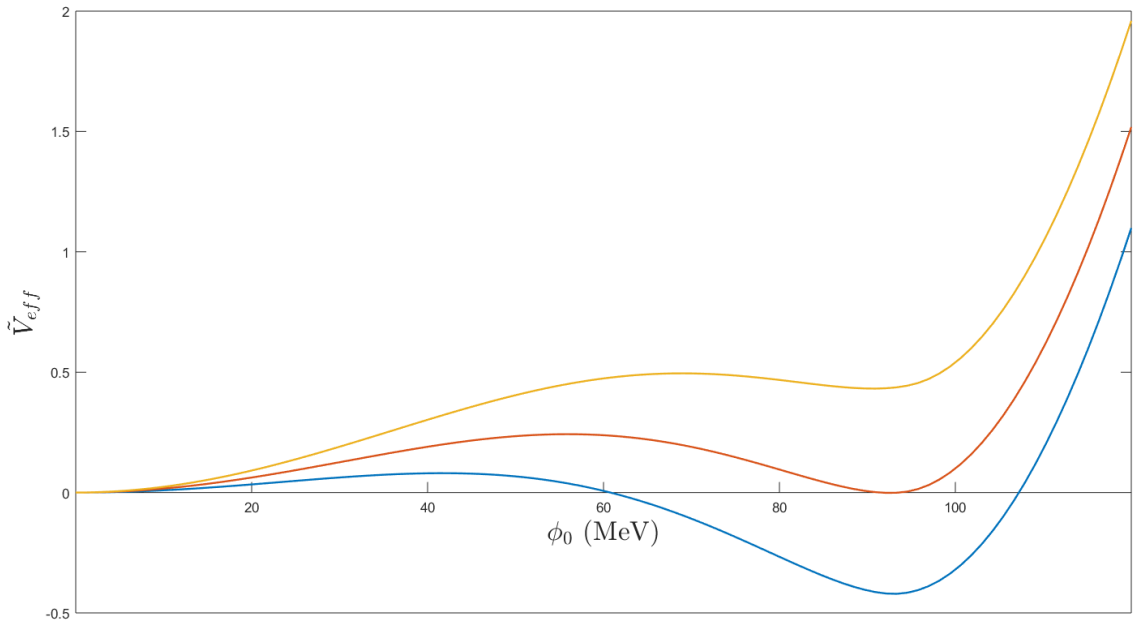

Figure 7: The normalized effective potential $\tilde{V}_{\text{eff}}$ as a function of ϕ_0, at zero temperature and three different chemical potentials: $\mu = \mu_c = 316.7$ MeV (red), $\mu = \mu_c - 5$ MeV (blue) and $\mu = \mu_c + 5$ MeV (yellow).

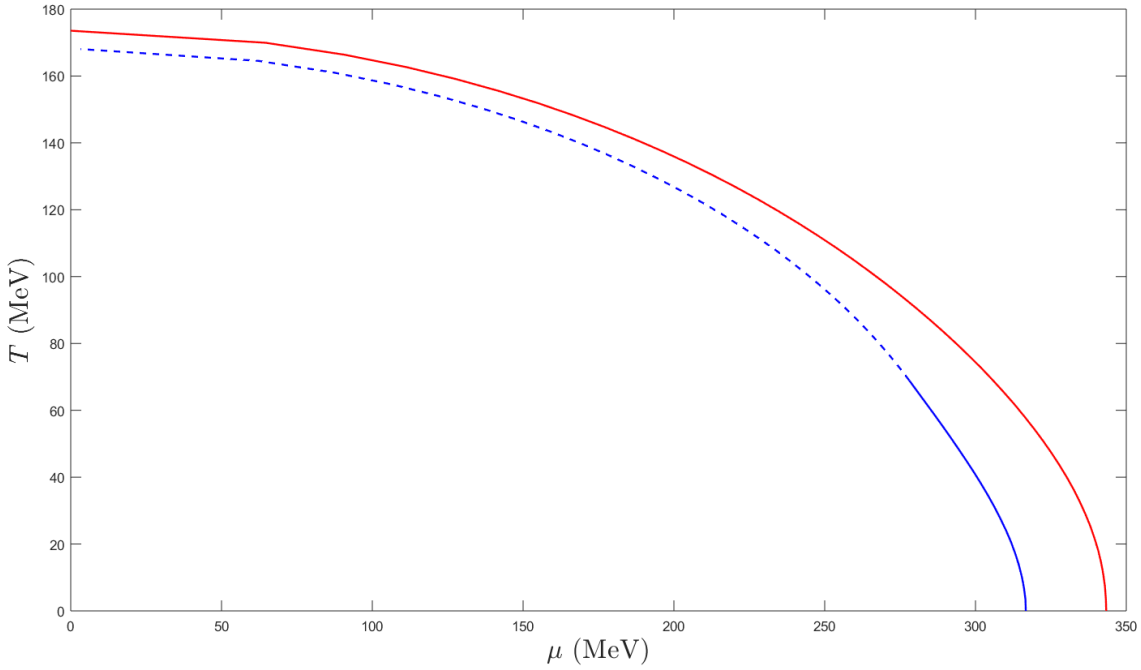

Figure 8: The phase diagram in the μ-T plane in the one-loop (blue) and classical (red) approximation. under the phase line, the expectation value of the σ field is nonzero, and chiral symmetry is broken. A dashed line indicates a continuous phase transition, whereas a full line indicates a first-order transition. The two meet at a critical point at $(T, \mu) = (69, 278)$ MeV

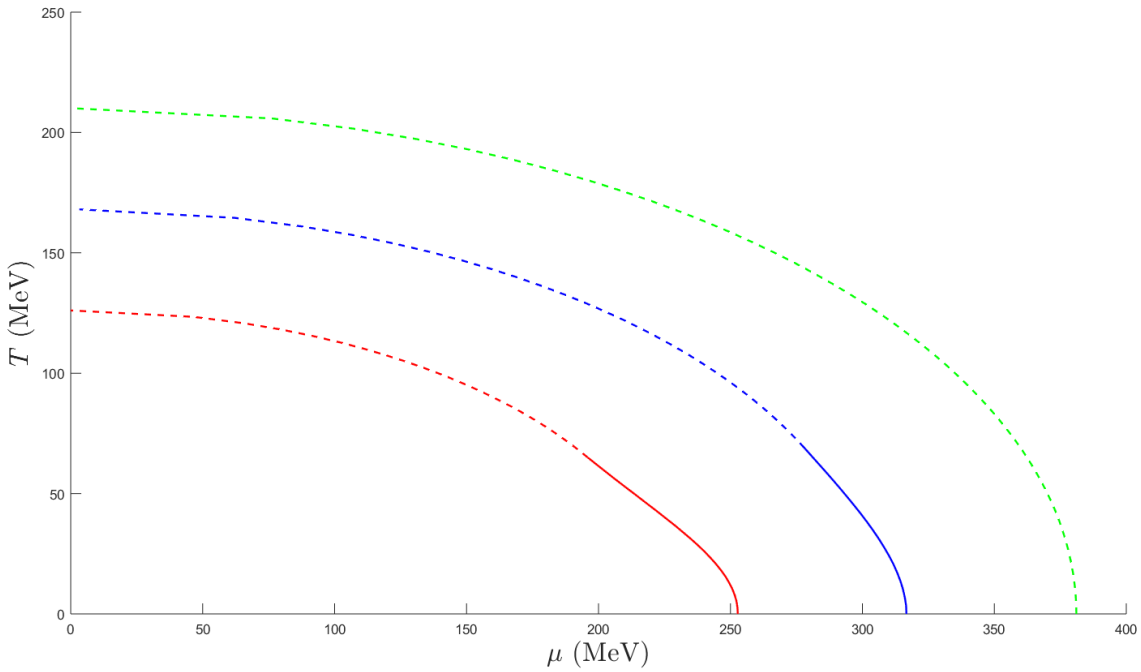

Figure 9: Phase lines computed from the one-loop effective potential with $m_\sigma = 1000$ MeV (green), $m_\sigma = 800$ MeV (blue), and $m_\sigma = 600$ MeV (red).

11. Conclusion and Outlook

In this thesis we have studied the quark-meson model in one of its simplest forms; at one-loop order in the chiral limit and in the large-N_c approximation. The phase diagrams that were produced appear to qualitatively agree with the one shown in figure 2, with a second-order transition in place of the crossover. A natural next step would be to move away from the chiral limit, and use the physical value for the pion mass. We can then expect to find a crossover where there was previously a second-order transition, as can be seen by making a Taylor expansion of the effective potential in the chiral limit:

$$V_{\text{eff}}(\phi_0) = a(T,\mu)\phi_0^2 + b(T,\mu)\phi_0^4 \dots. \tag{216}$$

A second-order phase transition occurs at a point where $a(T,\mu)$ changes sign. In the case of finite pion mass, the potential takes the form

$$V_{\text{eff}}(\phi_0) = -h\phi_0 + a(T,\mu)\phi_0^2 + b(T,\mu)\phi_0^4 \dots, \tag{217}$$

and the minimum of the potential will always be at finite ϕ_0. first-order transitions can still occur in this case, with $\langle\sigma\rangle$ jumping from a large to smaller but finite value. We have seen an example of how quantum fluctuations can influence the properties of the phase diagram, in particular the order of the phase transition. This indicates that - not unexpectedly - a classical treatment of the model is insufficient to obtain a qualitatively accurate representation of the phase diagram. By neglecting the vacuum fluctuations of the mesons, we may have discarded important information, and so incorporating these could change the picture.

The situation regarding the mass of the sigma particle is still somewhat unclear, and this is a question which may be clarified in the coming years. Until then, one could in the context of the QM model regard the sigma mass as a free parameter, which can be adjusted to fit the data. Our results suggest a value around 814 MeV, when compared to relevant results from lattice QCD.

A clear weakness of the QM model is that it does not incorporate quark confinement in any way. An important extension of the QM model is known as the Polyakov-quark-meson (PQM) model. For an introduction, see ref. [16]. In QCD, is possible to construct an operator whose expectation value serves as an order parameter for confinement; i.e. the confined phase is characterized by a finite value of this parameter, whereas it is zero in a deconfined phase. This operator is known as the (color trace of the) Polyakov loop. This can be incorporated into the QM model, and the chiral and deconfinement transitions can be modeled simultaneously. Results indicate that these two transitions roughly coincide in the entire $\mu - T$ plane [16][27].

12. Appendix: MATLAB code

This appendix contains the code which was used for numerical calculations.

The following function computes the normalized effective potential, inputs are the value of ϕ_0, the quark chemical potentials μ_u and μ_d, a boolean value which determines whether the vacuum fluctuation term is included, and the sigma mass.

```
function V = Veff(fi,T,mu1,mu2,vacuum,msigma)
fpi = 93;
mq = 300;
Lambda = mq*exp(1/2);
mpi = 0; % The script was written so as to be able to accommodate a
    finite pion mass, this was not used.
m2 = 1/2*(msigma^2 3*mpi^2);
lambda = 3/fpi^2*(msigma^2 mpi^2);
h = fpi*mpi^2;
g = mq/fpi;
if fi> 0
V = 1/2*m2*fi^2 + lambda/24*fi^4 h*fi + vacuum*3/8/pi^2*g^4*fi^4*(
    log(Lambda^2/(g^2*fi^2))+3/2) + Vtermisk(T,mu1,fi,g) + Vtermisk(T,
    mu2,fi,g);
else
    V = Vtermisk(T,mu1,fi,g) + Vtermisk(T,mu2,fi,g);
end
V = V/fpi^4;
end
```

The following function computes the thermal contribution to the effective potential:

```
function V = Vtermisk(T,mu,fi,g)
argument = @(k) 3*T/pi^2*k.^2.*log(1+exp( (sqrt(k.^2+g^2.*fi^2) mu)/
    T)) 3.*T/pi^2.*k.^2.*log(1+exp( (sqrt(k.^2+g^2.*fi^2)+mu)/T));
argument2 = @(k) 3/pi^2*k.^2.*(sqrt(mu.^2) sqrt(k.^2+g^2*fi^2));
fermivektor = 0;
if mu^2 > g^2*fi^2
fermivektor = sqrt(mu^2 g^2*fi^2);
end
if T > 0
V = quad(argument,0,Inf);
else
    V = quad(argument2,0,fermivektor);
end
end
```

The following function plots $\tilde{V}_{\text{eff}}(\phi_0)$ for given values of μ and T.

```
function Veffplott(mu,T,start,slutt,N,vacuum,msigma) %start, slutt, N
    give the range of fi and the number of points to plot.
fi = linspace(start,slutt,N);
```

```
V = linspace(start,slutt,N);
V0 = Veff(0,T,mu,mu,vacuum,msigma);
for i = 1:N
    V(i) = Veff(fi(i),T,mu,mu,vacuum,msigma);
end
plot(fi,V    V0,'LineWidth',1.4);
ax = gca;
ax.XAxisLocation = 'origin';
ax.YAxisLocation = 'origin';
xlabel('$\phi_0$ (MeV)','interpreter','LaTex','FontSize',20)
ylabel('$\tilde{V}_{eff}$','interpreter','LaTex','FontSize',20)

end
```

This recursive function computes the critical chemical potential for a given value of T:

```
function mu_c = muc(T,vacuum,mi,max,resolution,msigma)
mu_c =  1;
if max    mi < 0.1
    mu_c = (mi+max)/2;
else
fi = linspace(0,93,resolution);
V = fi;
mu = (mi + max)/2;
for i = 1:resolution
    V(i) = Veff(fi(i),T,mu,mu,vacuum,msigma);
end
for j = 2:resolution
    if V(j) < V(1)
        mu_c = muc(T,vacuum,mu,max,resolution,msigma);
    break;
    end
end
if mu_c ==  1
mu_c = muc(T,vacuum,mi,mu,resolution,msigma);
end
end
end
```

This function computes the critical temperature for a given value of μ:

```
function T_c = tc(mu,vacuum,mi,max,resolution,msigma)
T_c =  1;
if max    mi < 0.1
    T_c = (mi+max)/2;
else
fi = linspace(0,93,resolution);
V = fi;
```

```
T = (mi + max)/2;
for i = 1:resolution
    V(i) = Veff(fi(i),T,mu,mu,vacuum,msigma);
end
for j = 2:resolution
    if V(j) < V(1)
        T_c = tc(mu,vacuum,T,max,resolution,msigma);
    break;
    end
end
if T_c == 1
T_c = tc(mu,vacuum,mi,T,resolution,msigma);
end
end
end
```

The following function plots the phase diagram for a given value of m_σ. The temperature at the critical point is given as input, and is found by plotting the potential at various points along the phase line and judging the order of the transition by inspection.

```
function phasediagram(n,T_c,vacuum,msigma,color)
Tmax = tc(0,vacuum,0,1000,50,msigma);
N = ceil(n*T_c/Tmax);
M = n   N;
Ts = linspace(0,T_c,N);
Tf = linspace(T_c,Tmax,M);
mus = zeros(1,N);
muf = zeros(1,M);
res = 50;
for i = 1:N
    mus(i) = muc(Ts(i),vacuum,0,1000,res,msigma);
end
for i = 1:M
    muf(i) = muc(Tf(i),vacuum,0,1000,res,msigma);
end
plot(mus,Ts,color,'LineWidth',1.4)
hold on
plot(muf,Tf,strcat(color,'  '),'LineWidth',1.4)
ax = gca;
ax.XAxisLocation = 'origin';
ax.YAxisLocation = 'origin';
xlabel('$\mu$ (MeV)','interpreter','LaTex','FontSize',20)
ylabel('$T$ (MeV)','interpreter','LaTex','FontSize',20)
```

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