Application of optimal estimator for polynominal systems

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December 18, 2008

Abstract

Based on the interessting work done by Michael Basin et. al. we apply the optimal polynominal filter to a drilling system. Derivations and simulations are included. Simulations show interesting results

1 Optimal observer

1.1 Plant

The plant

$$\begin{aligned} \frac{V_d}{\beta_d}\dot{p}_p &= q_p - q_b, & (1)\\ \frac{V_a}{\beta_a}\dot{p}_c &= q_b - q_o + \dot{V}_a, & (2)\\ M\dot{q}_b &= p_p - p_c - F\,|q_b|\,q_b + (\rho_d - \rho_a)\,gh, & (3) \end{aligned}$$

where q_o is total flow out including choke flow and back pressure pump flow. Assuming

- $p_p, p_c, q_p, q_o, \dot{V}_a, V_a, h$ measured.

1.2 Observer derivations

Define the state vector

$$z = \begin{bmatrix} p_p \\ p_c \\ q_b \\ \frac{1}{M} \\ \frac{F}{M} \\ \frac{(\rho_d - \rho_a)g}{M} \end{bmatrix} \quad (4)$$

and the constants $\tau_d = \frac{V_d}{\beta_d}, \tau_a = \frac{V_a}{\beta_a}$. Assume that the system is affected by noise and that $z_3 = q_b \geq 0$, we then have

$$\begin{aligned} \dot{z} &= \begin{bmatrix} \frac{1}{\tau_d}\left(q_p(t) - z_3\right) \\ \frac{1}{\tau_a}\left(z_3 - q_o(t) + \dot{V}_a(t)\right) \\ z_4 z_1 - z_4 z_2 - z_5 z_3^2 + z_6 h(t) \\ 0 \\ 0 \\ 0 \end{bmatrix} + b_z(t) W_1(t) && (5) \\ &\equiv f_z(z,t) + b(t) W_1(t) && (6) \end{aligned}$$

where $W_1(t) \in \mathbb{R}^6$ is a wiener process (i.e. gaussian white noise) and $b_z(t) \in \mathbb{R}^{6x6}$. Defining the output with measurement noise

$$y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} z \equiv A_z z + B(t) W_2(t), \tag{7}$$

where $W_2(t) \in \mathbb{R}^2$ is a wiener process. and $B(t) \in \mathbb{R}^{2x2}$.$W_1$ and W_2 are assumed independent and to have identity covariance matrices. The scaling of the covariance is done using b_z and B_z. The filter design found in [1, 2] minimizes the Euclidian 2-norm

$$J = E\left[\left(z(t) - \hat{z}(t)\right)^T \left(z(t) - \hat{z}(t)\right) | F_t^Y \right], \tag{8}$$

at each time t. For tuning/scaling purposes it is desireable to minimize the following norm instead

$$J = E\left[\left(z(t) - \hat{z}(t)\right)^T K \left(z(t) - \hat{z}(t)\right) | F_t^Y \right], \tag{9}$$

where $K = H^{-T} H^{-1}$ is a square diagonal positive definite matrix. This can be achieved by considering the change of coordinates

$$z = Hx. \tag{10}$$

Equation (9) can then be written as

$$\begin{aligned} J &= E\left[\left(Hx(t) - H\hat{x}(t)\right)^T H^{-T} H^{-1} \left(Hx(t) - H\hat{x}(t)\right) | F_t^Y \right], && (11) \\ &= E\left[\left(x(t) - \hat{x}(t)\right)^T \left(x(t) - \hat{x}(t)\right) | F_t^Y \right]. && (12) \end{aligned}$$

Now a filter can be derived for the state z instead, giving the ability to weight different state estimates. We will only consider the case where H consists of elements $h_1, h_2 ..., h_n$ along the diagonal. The system (6) and (7)in x coordinates is

$$\begin{aligned} \dot{x} &= H^{-1} f_z(Hx, t) + H^{-1} b_z(t) W_1(t), && (13) \\ &\equiv f(x,t) + b(t) W_1(t), && (14) \\ y &= A_z H x + B(t) W_2(t), && (15) \\ &\equiv Ax + B(t) W_2(t), && (16) \end{aligned}$$

where

$$
\begin{aligned}
f(x,t) &= \begin{bmatrix} \frac{1}{h_1\tau_d}\left(q_p(t) - h_3x_3\right) \\ \frac{1}{h_2\tau_a}\left(h_3x_3 - q_o(t) + \dot{V}_a(t)\right) \\ \frac{1}{h_3}\left(h_4h_1x_4x_1 - h_4h_2x_4x_2 - h_5h_3^2x_5x_3^2 + h_6x_6h(t)\right) \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad (17) \\
b(t) &= H^{-1}b_z, \quad (18) \\
A &= \begin{bmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \end{bmatrix} \quad (19)
\end{aligned}
$$

We are now ready to apply the results found in [1, 2]. The optimal estimate, $\hat{x}$, of x is governed by

$$
\begin{aligned}
\dot{\hat{x}} &= E\left(f(x,t)|F_t^Y\right) + P(t)A^T\left(BB^T\right)^{-1}(y - A\hat{x}), \quad (20) \\
\dot{P} &= E\left([x-\hat{x}]\, f^T(x,t)|F_t^Y\right) + E\left(f(x,t)\,[x-\hat{x}]^T\,|F_t^Y\right) + bb^T - PA^T\left(BB^T\right)^{-1}AP, \quad (21)
\end{aligned}
$$

with initial conditions $\hat{x}(0) = E(x(0)|F_0^Y), P(0) = E\left[\left(x(0)-\hat{x}(0)\right)\left(x(0)-\hat{x}(0)\right)^T|F_0^Y\right]$. P is the error variance matrix. Equations (20) and (21) is not i a closed form due to the dependence on x in several functions. However since $f(x,t)$ is polynomic in x closed form relations can be obtained. We need to find expressions for $E\left(f(x,t)|F_t^Y\right)$ and $E\left([x-\hat{x}]\, f^T(x,t)|F_t^Y\right) + E\left(f(x,t)\,[x-\hat{x}]^T\,|F_t^Y\right)$. First we have the following facts

$$
\begin{aligned}
\hat{x}(t) &= E(x(t)|F_t^Y), \quad (22) \\
P(t) &= E\left[\left(x(t)-\hat{x}(t)\right)\left(x(t)-\hat{x}(t)\right)^T|F_t^Y\right]. \quad (23)
\end{aligned}
$$

For $E\left(f(x,t)|F_t^Y\right)$ we get

$$
\begin{aligned}
E\left(f(x,t)|F_t^Y\right)_1 &= \frac{1}{h_1\tau_d}\left(q_p(t) - h_3E(x_3|F_t^Y)\right), \quad (24) \\
&= \frac{1}{h_1\tau_d}\left(q_p(t) - h_3\hat{x}_3\right). \quad (25) \\
E\left(f(x,t)|F_t^Y\right)_2 &= \frac{1}{h_2\tau_a}\left(h_3\hat{x}_3 - q_o(t) + \dot{V}_a(t)\right), \quad (26) \\
E\left(f(x,t)|F_t^Y\right)_{4-6} &= 0. \quad (27)
\end{aligned}
$$

We will treat $E\left(f(x,t)|F_t^Y\right)_3$ as a special case since it is nonlinear.

$$
\begin{aligned}
E\left(f(x,t)|F_t^Y\right)_3 &= \frac{1}{h_3}E\left(\left[h_4h_1x_4x_1 - h_4h_2x_4x_2 - h_5h_3^2x_5x_3^2 + h_6x_6h(t)\right]|F_t^Y\right), \quad (28) \\
&= \frac{h_4h_1}{h_3}\hat{x}_4\hat{x}_1 + \frac{h_4h_1}{h_3}P_{41} - \frac{h_4h_2}{h_3}\hat{x}_4\hat{x}_2 - \frac{h_4h_2}{h_3}P_{42} + h_6\hat{x}_6h(t) - h_5h_3E(x_5x_3^2|F_t^Y) \quad (29)
\end{aligned}
$$

since $E(x_4 x_1|F_t^Y) = \hat{x}_4\hat{x}_1 + P_{41}$ where P_{41} is the first element of the fourth row of P. For notational simplicity we will use $E(z(t)|F_t^Y) = E(z)$ in the following. The last term is

$$
\begin{aligned}
E(x_5 x_3^2|F_t^Y) &= E(x_5 x_3^2), && (30)\\
&= E((x_5-\hat{x}_5)x_3^2) + E(\hat{x}_5 x_3^2), && (31)\\
&= E((x_5-\hat{x}_5)\left[(x_3-\hat{x}_3)^2 + 2x_3\hat{x}_3 - \hat{x}_3^2\right]) + E(\hat{x}_5 x_3^2), && (32)\\
&= E\left((x_5-\hat{x}_5)(x_3-\hat{x}_3)^2\right) + 2E\left((x_5-\hat{x}_5)x_3\hat{x}_3\right) - E\left((x_5-\hat{x}_5)\hat{x}_3^2\right) + E(\hat{x}_5 x_3^2), && (33)
\end{aligned}
$$

since $E\left((x_5-\hat{x}_5)(x_3-\hat{x}_3)^2\right)$ is an odd moment (3rd) it is zero, we thus have

$$
\begin{aligned}
E(x_5 x_3^2) &= 2E\left((x_5-\hat{x}_5)x_3\hat{x}_3\right) - E\left((x_5-\hat{x}_5)\hat{x}_3^2\right) + E(\hat{x}_5 x_3^2), && (34)\\
&= 2E\left((x_5-\hat{x}_5)(x_3-\hat{x}_3)\hat{x}_3\right) - 2E\left((x_5-\hat{x}_5)\hat{x}_3^2\right) - E\left((x_5-\hat{x}_5)\hat{x}_3^2\right) + E(\hat{x}_5 x_3^2), && (35)\\
&= 2\hat{x}_3 P_{53} - 0 - 0 + \hat{x}_5 E(x_3^2), && (36)\\
&= 2\hat{x}_3 P_{53} + \hat{x}_5 P_{33} + \hat{x}_5\hat{x}_3^2 && (37)
\end{aligned}
$$

where P_{53} is the element in row 5 column 3 of P. And $E(x_3^2) = E\left((x_3-\hat{x}_3)^2 + 2x_3\hat{x}_3 - \hat{x}_3^2\right) = P_{33} + 2E(x_3)\hat{x}_3 - \hat{x}_3^2 = P_{33} + \hat{x}_3^2$. Hence we have

$$
E\left(f(x,t)|F_t^Y\right)_3 = \frac{h_4 h_1}{h_3}\hat{x}_4\hat{x}_1 + \frac{h_4 h_1}{h_3}P_{41} - \frac{h_4 h_2}{h_3}\hat{x}_4\hat{x}_2 - \frac{h_4 h_2}{h_3}P_{42} + \frac{h_6}{h_3}\hat{x}_6 h(t) - h_5 h_3\left(2\hat{x}_3 P_{53} + \hat{x}_5 P_{33} + \hat{x}_5\hat{x}_3^2\right) \tag{38}
$$

Thus we have $E(f(x,t)|F_t^Y)$ in (20) given by (24)–(27) and (38). This can be written compactly as

$$
E(f(x,t)|F_t^Y) = F(\hat{x}, P, t) \tag{39}
$$

where

$$
F(\hat{x}, P, t) = \begin{bmatrix} \frac{1}{h_1\tau_d}\left(q_p(t) - h_3\hat{x}_3\right) \\ \frac{1}{h_2\tau_a}\left(h_3\hat{x}_3 - q_o(t) + \dot{V}_a(t)\right) \\ \frac{h_4 h_1}{h_3}\hat{x}_4\hat{x}_1 + \frac{h_4 h_1}{h_3}P_{41} - \frac{h_4 h_2}{h_3}\hat{x}_4\hat{x}_2 - \frac{h_4 h_2}{h_3}P_{42} + \frac{h_6}{h_3}\hat{x}_6 h(t) - h_5 h_3\left(2\hat{x}_3 P_{53} + \hat{x}_5 P_{33} + \hat{x}_5\hat{x}_3^2\right) \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{40}
$$

We proceed with $E\left([x-\hat{x}]\, f^T(x,t)|F_t^Y\right) + E\left(f(x,t)\,[x-\hat{x}]^T\,|F_t^Y\right)$ in (21).

We will use the notation $E\left([x-\hat{x}]\, f^T(x,t)\right) = E\left([x-\hat{x}]\, f^T(x,t)|F_t^Y\right)$.

$$\begin{aligned}
E\left([x-\hat{x}]\, f_1(x,t)\right) &= E\left([x-\hat{x}]\left[\frac{1}{h_1\tau_d}\left(q_p(t)-h_3x_3\right)\right]\right), && (41)\\
&= E\left([x-\hat{x}]\,\frac{1}{h_1\tau_d}q_p(t)\right) - \frac{h_3}{h_1\tau_d}E([x-\hat{x}]\,x_3), && (42)\\
&= 0 - \frac{h_3}{h_1\tau_d}E([x-\hat{x}]\,[x_3-\hat{x}_3] + \hat{x}_3\,[x-\hat{x}]) && (43)\\
&= -\frac{h_3}{h_1\tau_d}E([x-\hat{x}]\,[x_3-\hat{x}_3]), && (44)\\
&= -\frac{h_3}{h_1\tau_d}P_{*3}, && (45)
\end{aligned}$$

where P_{*3} is the third column of the variance matrix P. Similarly for

$$\begin{aligned}
E\left([x-\hat{x}]\, f_2(x,t)\right) &= E\left([x-\hat{x}]\left[\frac{1}{h_2\tau_a}\left(h_3x_3 - q_o(t) + \dot{V}_a(t)\right)\right]\right), && (46)\\
&= \frac{h_3}{h_2\tau_a}P_{*3}, && (47)\\
E\left([x-\hat{x}]\, f_{4-6}(x,t)\right) &= 0. && (48)
\end{aligned}$$

The special nonlinear case of $E\left([x-\hat{x}]\, f_3(x,t)\right)$ will be delt with now

$$\begin{aligned}
E\left([x-\hat{x}]\, f_3(x,t)\right) &= E\left([x-\hat{x}]\left[\frac{1}{h_3}\left(h_4h_1x_4x_1 - h_4h_2x_4x_2 - h_5h_3^2x_5x_3^2 + h_6x_6h(t)\right)\right]\right), && (49)\\
&= \frac{h_4h_1}{h_3}E\left([x-\hat{x}]\,[x_4x_1]\right) - \frac{h_4h_2}{h_3}E\left([x-\hat{x}]\,[x_4x_2]\right)... && (50)\\
&\quad -\frac{h_5h_3^2}{h_3}E\left([x-\hat{x}]\,\left[x_5x_3^2\right]\right) + \frac{h_6}{h_3}h(t)E\left([x-\hat{x}]\,[x_6]\right). && (51)
\end{aligned}$$

Dealing with each term separately

$$\begin{aligned}
E\left([x-\hat{x}]\,[x_4x_1]\right) &= E\left([x-\hat{x}]\,[(x_4-\hat{x}_4)\,x_1 + \hat{x}_4x_1]\right), && (52)\\
&= E\left([x-\hat{x}]\,[(x_4-\hat{x}_4)\,x_1]\right) + E\left([x-\hat{x}]\,[\hat{x}_4\,(x_1-\hat{x}_1) + \hat{x}_4\hat{x}_1]\right), && (53)\\
&= E\left([x-\hat{x}]\,[(x_4-\hat{x}_4)\,(x_1-\hat{x}_1) + \hat{x}_1\,(x_4-\hat{x}_4)]\right) + E\left([x-\hat{x}]\,[\hat{x}_4\,(x_1-\hat{x}_1) + \hat{x}_4\hat{x}_1]\right),\\
&= E\left([x-\hat{x}]\,(x_4-\hat{x}_4)\right) + \hat{x}_1E\left([x-\hat{x}]\,(x_4-\hat{x}_4)\right) + ... && (54)\\
&\quad ... + \hat{x}_4E\left([x-\hat{x}]\,(x_1-\hat{x}_1)\right) + \hat{x}_4\hat{x}_1E\left([x-\hat{x}]\right), && (55)\\
&= P_{*4} + \hat{x}_1P_{*4} + \hat{x}_4P_{*1}. && (56)
\end{aligned}$$

Similarly

$$\begin{aligned}
E\left([x-\hat{x}]\,[x_4x_2]\right) &= P_{*4} + \hat{x}_2P_{*4} + \hat{x}_4P_{*2}, && (57)\\
E\left([x-\hat{x}]\,[x_6]\right) &= E\left([x-\hat{x}]\,[x_6-\hat{x}_6] - [x-\hat{x}]\,[\hat{x}_6]\right), && (58)\\
&= P_{*6}. && (59)
\end{aligned}$$

The most complex term is

$$
\begin{aligned}
E\left([x-\hat{x}]\left[x_5 x_3^2\right]\right) &= E\left([x-\hat{x}]\left[(x_5-\hat{x}_5)\,x_3^2+\hat{x}_5 x_3^2\right]\right), && (60)\\
&= E\left([x-\hat{x}]\left[(x_5-\hat{x}_5)\,x_3^2\right]\right)+E\left([x-\hat{x}]\left[\hat{x}_5 x_3^2\right]\right), && (61)\\
E\left([x-\hat{x}]\left[(x_5-\hat{x}_5)\,x_3^2\right]\right) &= E\left([x-\hat{x}]\left[(x_5-\hat{x}_5)(x_3-\hat{x}_3)^2+2x_3\hat{x}_3(x_5-\hat{x}_5)-\hat{x}_3^2(x_5-\hat{x}_5)\right]\right), && (62)\\
&= E\Big([x-\hat{x}]\Big[(x_5-\hat{x}_5)(x_3-\hat{x}_3)^2+2(x_3-\hat{x}_3)\,\hat{x}_3(x_5-\hat{x}_5)+...\\
&\quad ...+2\hat{x}_3^2(x_5-\hat{x}_5)-\hat{x}_3^2(x_5-\hat{x}_5)\Big]\Big), && (63)\\
&= E\left([x-\hat{x}]\left[(x_5-\hat{x}_5)(x_3-\hat{x}_3)^2\right]\right)+E\left([x-\hat{x}]\left[2(x_3-\hat{x}_3)\,\hat{x}_3(x_5-\hat{x}_5)\right]\right)...\\
&\quad ...+E\left([x-\hat{x}]\left[2\hat{x}_3^2(x_5-\hat{x}_5)\right]\right)-E\left([x-\hat{x}]\left[\hat{x}_3^2(x_5-\hat{x}_5)\right]\right), && (64)\\
&= 0+0+2\hat{x}_3^2E\left([x-\hat{x}]\left[(x_5-\hat{x}_5)\right]\right)-\hat{x}_3^2E\left([x-\hat{x}]\left[(x_5-\hat{x}_5)\right]\right), && (65)\\
&= 2\hat{x}_3^2P_{*5}-\hat{x}_3^2P_{*5}, && (66)\\
&= \hat{x}_3^2P_{*5}. && (67)
\end{aligned}
$$

Where the fact that moments of odd powers are zero has been used.

$$
\begin{aligned}
E\left([x-\hat{x}]\left[\hat{x}_5 x_3^2\right]\right) &= E\left([x-\hat{x}]\left[(x_3-\hat{x}_3)^2\hat{x}_5+2\hat{x}_3x_3\hat{x}_5-\hat{x}_3^2\hat{x}_5\right]\right) && (68)\\
&= E\left([x-\hat{x}]\left[(x_3-\hat{x}_3)^2\hat{x}_5+2\hat{x}_3(x_3-\hat{x}_3)\,\hat{x}_5+2\hat{x}_3^2\hat{x}_5-\hat{x}_3^2\hat{x}_5\right]\right) && (69)\\
&= 2\hat{x}_3\hat{x}_5E\left([x-\hat{x}]\,(x_3-\hat{x}_3)\right), && (70)\\
&= 2\hat{x}_3\hat{x}_5P_{*3}. && (71)
\end{aligned}
$$

Insterting (56), (57), (59),(61),(67) and (71) into (50) gives

$$
\begin{aligned}
E\left([x-\hat{x}]\,f_3(x,t)\right) &= \frac{h_4h_1}{h_3}\left(P_{*4}+\hat{x}_1P_{*4}+\hat{x}_4P_{*1}\right)-\frac{h_4h_2}{h_3}\left(P_{*4}+\hat{x}_2P_{*4}+\hat{x}_4P_{*2}\right)... && (72)\\
&\quad ...-\frac{h_5h_3^2}{h_3}\left[\hat{x}_3^2P_{*5}.+2\hat{x}_3\hat{x}_5P_{*3}\right]+\frac{h_6}{h_3}h(t)P_{*6} && (73)\\
&= \frac{h_4}{h_3}(h_1-h_2)\,P_{*4}+\frac{h_4}{h_3}(h_1\hat{x}_1-h_2\hat{x}_2)\,P_{*4}+\frac{h_4h_1}{h_3}\hat{x}_4P_{*1}-\frac{h_4h_2}{h_3}\hat{x}_4P_{*2}... && (74)\\
&\quad ...-h_5h_3\left[\hat{x}_3^2P_{*5}+2\hat{x}_3\hat{x}_5P_{*3}\right]+\frac{h_6}{h_3}h(t)P_{*6}. && (75)
\end{aligned}
$$

Stacking the row vectors (45),(47),(74),(48) after each other gives

$$
E\left([x-\hat{x}]\,f^T(x,t)|F_t^Y\right)=PG(\hat{x},t) \qquad (76)
$$

where

$$
G(\hat{x},t)=\begin{bmatrix}
0 & 0 & \frac{h_4h_1}{h_3}\hat{x}_4 & 0 & 0 & 0\\
0 & 0 & -\frac{h_4h_2}{h_3}\hat{x}_4 & 0 & 0 & 0\\
-\frac{h_3}{h_1\tau_d} & \frac{h_3}{h_2\tau_a} & -2h_5h_3\hat{x}_3\hat{x}_5 & 0 & 0 & 0\\
0 & 0 & \frac{h_4}{h_3}(h_1-h_2)+\frac{h_4}{h_3}(h_1\hat{x}_1-h_2\hat{x}_2) & 0 & 0 & 0\\
0 & 0 & -h_5h_3\hat{x}_3^2 & 0 & 0 & 0\\
0 & 0 & \frac{h_6}{h_3}h(t) & 0 & 0 & 0
\end{bmatrix}. \qquad (77)
$$

To summarize, the equations (20) and (21) can be written in the closed form

$$\dot{\hat{x}} = F(\hat{x}, P, t) + PA^T \left(BB^T\right)^{-1} (y - A\hat{x}), \tag{78}$$

$$\dot{P} = PG(\hat{x}, t) + G^T(\hat{x}, t)P^T + bb^T - PA^T \left(BB^T\right)^{-1} AP, \tag{79}$$

where $F(\hat{x}, P, t)$ is defined in (40) and $G(\hat{x}, t)$ is defined in (77). Note the resemblence to a standard Kalman filter, the difference is that $F(\hat{x}, P, t)$ depends on the covariance matrix and that $G(\hat{x}, t)$ contains state estimates that appear nonlinearly.

2 Simulations

Simulated the filter and model. Noise was added to measurements. Results are shown below. A simple PI controller is used to keep choke pressure "constant" during pump variations. Very little tuning of H, $b(t)$ and $B(t)$ was done.

One long simulation to test if parameters converged during PE was also performed.

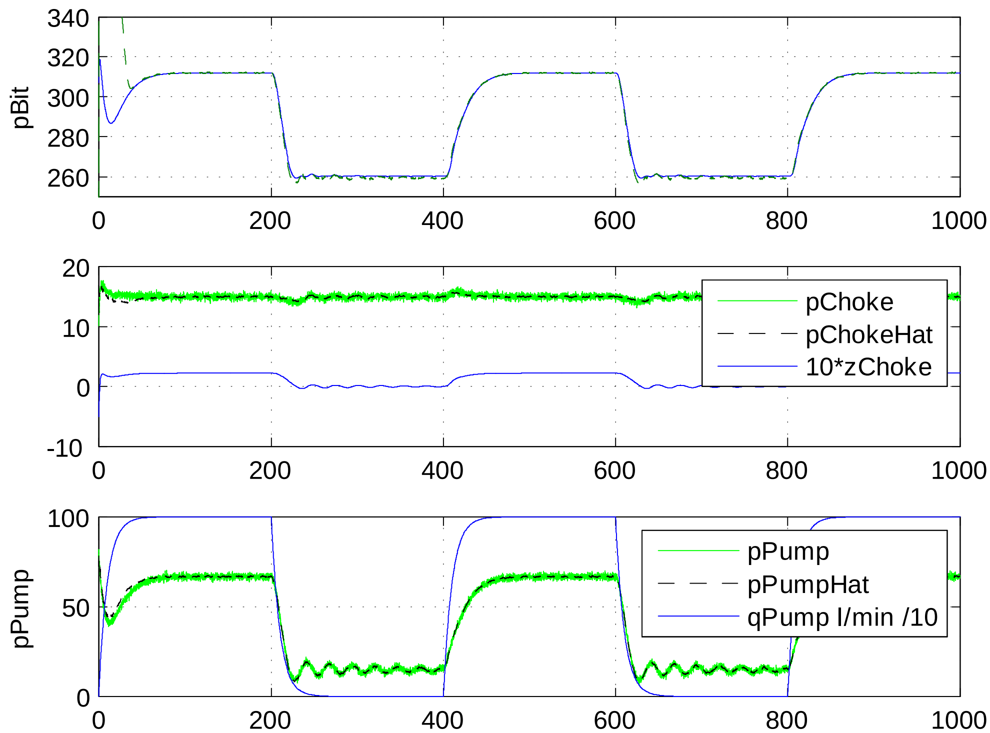

Figure 1: Simulation with varying pump rates. Noise on pressure measurements. No noise on pBit.

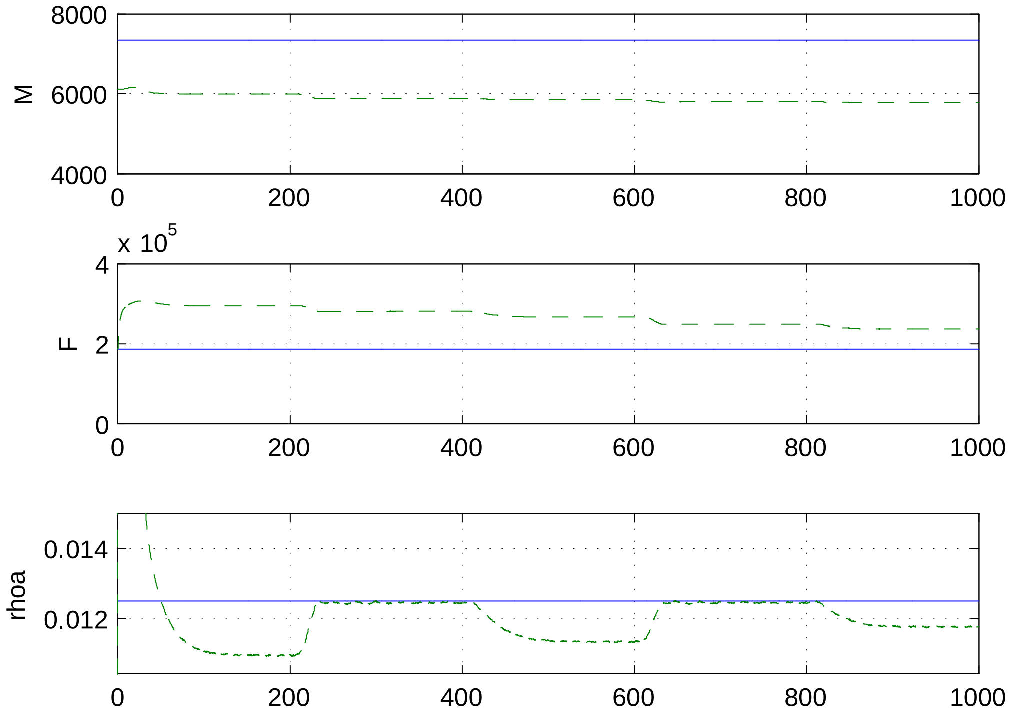

Figure 2: Simulation with varying pump rates. Noise on pressure measurements

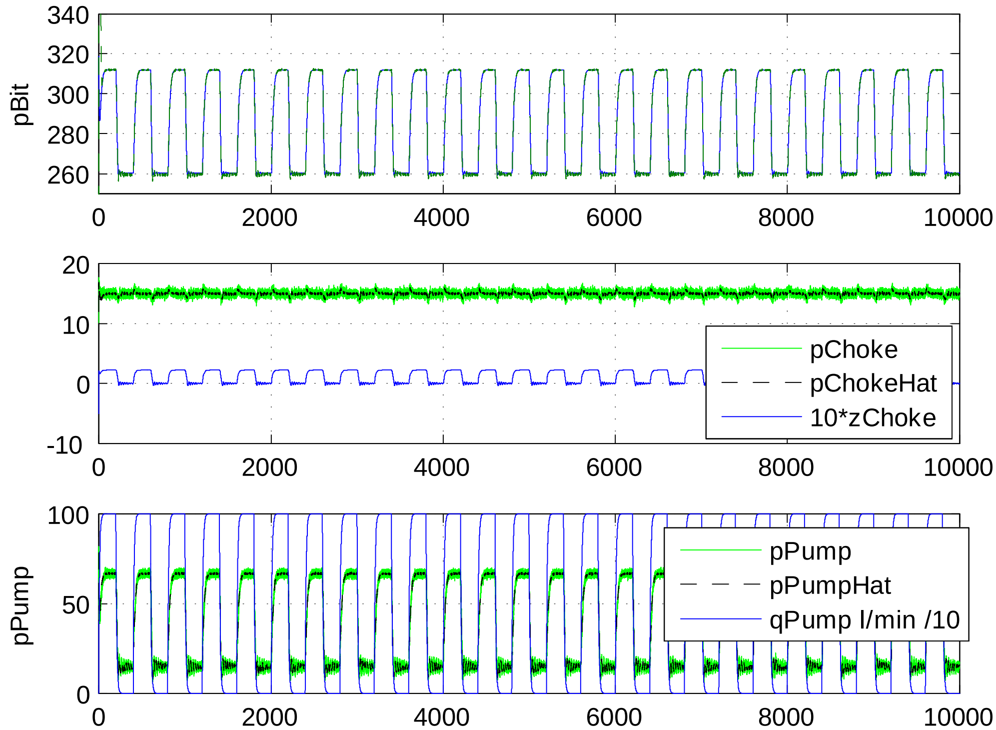

Figure 3: Long PE simulation

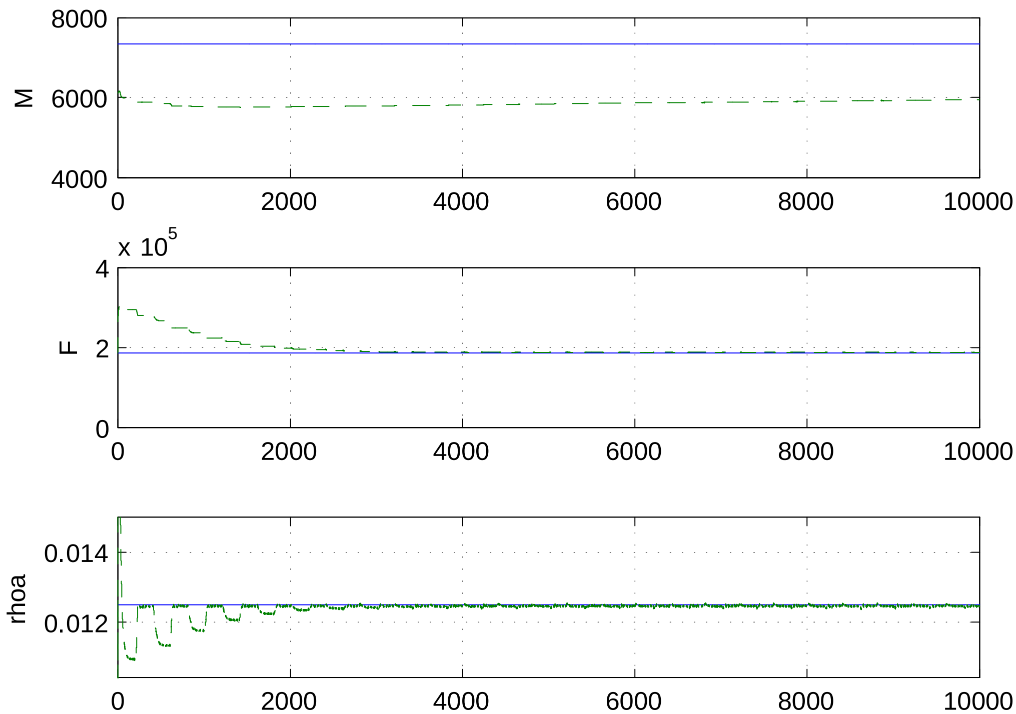

Figure 4: Long PE simulation, dashed lines are estimates. Note that F and rhoa converge to their true values while M needs more time.

3 Conclusion

The results found in [1, 2] seem applicable to the drilling system model. This could be tested further e.g. in a master thesis.

- Filter handles noise
- Optimal w.r.t. given cost function
- Can deal with parameteric uncertainty
- Can deal with time varying deterministic functions.
- Very similar to Kalman filter

  Future work

- Stability and convergence properties
- PE conditions
- Constraint handling

In addition the fact that there exists an optimal filter for the system with noise implies that there should exist an optimal filter for the system without noise.

References

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