

# Design and Analysis of Nonlinear Sampled-Data Control Systems

Dina Shona Laila

*A thesis submitted in total fulfillment of the requirements  
of the degree of Doctor of Philosophy*

April 2003

Department of Electrical and Electronic Engineering  
The University of Melbourne

Produced on acid-free paper

*"The Most Gracious (Allah)! He has taught (you mankind) the Qur'an (by His Mercy).  
He created man. He taught him eloquent speech.  
The sun and the moon run on their fixed courses (exactly) calculated with measured  
out stages for each (for reckoning).  
And the herbs (or stars) and the trees both prostrate themselves (to Allah).  
And the heaven; He has raised it high, and He has set up the Balance.  
In order that you may not transgress (due) balance.  
And observe the weight with equity and do not make the balance deficient."  
(Holy Qur'an, Ar-Rahman (The Most Gracious):1-9)*

# Abstract

Nonlinear sampled-data control systems represent an important system configuration that often arises in engineering practice. This thesis contributes a number of novel analysis and design tools for a quite general class of nonlinear sampled-data systems.

The overall work in this thesis is divided into three parts. In Part I, a general and unified framework for discrete-time controller design using emulation for nonlinear sampled-data systems with disturbances is presented. A preservation of dissipativity properties under sampling and controller emulation is studied. It is shown that if a (dynamic) continuous-time controller, which is designed so that the continuous-time closed-loop system satisfies a certain dissipation inequality, is appropriately discretized and implemented digitally, then the discrete-time model of the closed-loop sampled-data system satisfies a similar dissipation inequality in a semiglobal practical sense.

Part II focuses on the direct discrete-time stabilization of sampled-data systems based on approximate discrete-time models of the plant. A robust stability of nonlinear systems with inputs, which is known as input-to-state stability (ISS), is investigated and a framework for direct discrete-time controller design is proposed. An important ingredient in our approach is finding an appropriate family of Lyapunov functions for testing ISS properties of the family of discrete-time approximate models. We propose two methods for partially constructing a Lyapunov function for quite a general class of nonlinear systems. The first uses techniques of changing supply rates for input-output to state stable (IOSS) discrete-time nonlinear systems, and the second is derived from the Lyapunov based small-gain theorem for ISS discrete-time nonlinear systems.

A case study is presented in Part III, to illustrate an engineering application of results from Part I and Part II. Input to state stabilization problem of a two-link manipulator is considered. Two controllers are designed using the proposed framework. It is demonstrated that the controller designed using direct discrete-time design may outperform the controller designed using emulation. Simulation studies are supporting the theoretical results and this strongly motivates further research in this area.

# Declaration

This is to certify that

- the thesis comprises only my original work towards the PhD,
- due acknowledgement has been made in the text to all other material used,
- the thesis is less than 100,000 words in length, exclusive of tables, bibliographies and appendices.

Dina Shona Laila

Department of Electrical and Electronic Engineering,

University of Melbourne, Australia

April 2003

# Statement of Originality

The contents of this thesis are the results of the original research unless otherwise stated and have not been submitted for a higher degree at any other university or institution. The results presented in this thesis have been obtained during the author's candidature under the supervision of Assoc. Prof. Dragan Nešić. Some results have been obtained in collaboration with Prof. Andrew R. Teel. However, the majority of the work has been done by the author herself.

The material presented in this thesis covers all results from the following journal papers:

1. D. S. Laila, D. Nešić and A. R. Teel, "Open and closed loop dissipation inequalities under sampling and controller emulation", *European Journal of Control*, vol. 8, no. 2, pp. 109-125, 2002.
2. D. Nešić and D. S. Laila, "A note on input-to-state stabilization for nonlinear sampled-data systems", *IEEE Trans. on Automatic Control*, vol. 47, no. 7, pp. 1153-1158, 2002.
3. D. S. Laila and D. Nešić, "Changing supply rates for input-output to state stable discrete-time nonlinear systems with applications", *Automatica*, vol. 39, pp. 821-835, 2003.
4. D. S. Laila and D. Nešić, "Discrete-time Lyapunov based small-gain theorem for parameterized interconnected ISS systems", *IEEE Trans. on Automatic Control*, vol. 48, pp. 1783-1788, 2003.

Also a number of conference papers follows from the results presented in this thesis, some of which overlap with the journal papers.

1. D. Nešić, D. S. Laila and A. R. Teel, "On preservation of dissipation inequalities under sampling", *Proc. 39th IEEE Conference on Decision and Control*, Sydney, Australia, pp. 2472-2477, 2000.

2. D. S. Laila and D. Nešić, "A note on preservation of dissipation inequalities under sampling: the dynamic feedback case", *Proc. American Control Conference*, Arlington, Virginia, pp. 2822-2827, 2001.
3. D. Nešić and D. S. Laila, "Input-to-state stabilization for nonlinear sample-data systems via approximate discrete-time plant models", *Proc. 40th IEEE Conference on Decision and Control*, Orlando, Florida, pp. 887-892, 2001.
4. D. S. Laila and D. Nešić, "Changing supply rates for input-output to state stable discrete-time systems", *Proc. IFAC 15th World Congress*, Barcelona, Spain, 2002.
5. D. S. Laila and D. Nešić, "Lyapunov based small-gain theorem for parameterized discrete-time interconnected ISS systems", *Proc. 41th IEEE Conference on Decision and Control*, Nevada, pp. 2292-2297, 2002.

# Acknowledgements

All the praises and gratefulness due to Allah Subhanahu wa Ta'ala and peace be upon the beloved final Prophet Muhammad Salallahu 'alaihi wa salam. I thank Allah SWT for His guidance and mercy that are always given to me along my life. Only because of His help and blessing that have given me strength and taught me patience, I can finally complete this thesis.

I would like to thank the people who have given their time and support, who have made it possible for me to work on and finish this thesis. On the top of the list of people whose help was indispensable is my supervisor Assoc. Prof. Dragan Nešić. His strong encouragement, support and ideals together with patience and understanding in guiding my effort were essential from the beginning to the final stage of my study in pursuing my PhD. Dragan's high standards and rich ideas about new research have made my experience working with him so meaningful. It has been extremely wonderful and an enjoyable time to work with him. His advice and expertise have helped me learn so much to further develop my understanding on how to do good research in control engineering.

I would like to thank Prof. Andrew R. Teel for having an opportunity to collaborate with him in research. I thank my progress committee members, Prof. Iven Mareels and Dr. Erik Weyer, for all their valuable feedback on my performance in research and study. I am in debt to Prof. Jerry Koliha from the Department of Mathematics and Statistics, who has always welcomed me to attend his classes, that really assisted me in obtaining the necessary mathematical background I needed to conduct my research.

I extend my deepest thanks to my parents and brothers in Indonesia, who always sincerely pray for my success in life and study, for their never ending love and care, without which I may have stopped in the middle of the way.

I also thank all staff of the EEE Department, especially for Debbie, Marie and Nina, all fellow students and my friends, for their support, help and friendship, which have made my study life so colourful.

Finally, I thank AusAID for providing full financial support during my candidature.

# Notation

$\mathbb{R}, \mathbb{N}, \mathbb{Z}$	The sets of real, natural and integer numbers
$\mathbb{R}^n$	The sets of all n-tuples (vectors) of real numbers
$\exists$	Existential quantifier
$\forall$	Universal quantifier
$a \in A$	$a$ is an element of set $A$
$A \subset B$	$A$ is a subset of $B$
$\mathcal{L}_p$	A function space
$\ell_p$	A sequence space
$\dot{x}$	The first derivative of $x$ w.r.t. time $t$
$x^{(n)}$	The $n$ -th derivative of $x$ w.r.t. time $t$
$A^T$	Transpose of matrix or vector $A$
$A^{-1}$	Inverse of matrix $A$
$\text{Id}$	Identity operation, $\text{Id}(a) = a$
$a \circ b$	Composition operation of $a$ and $b$
$a \cdot b$	Multiplication operation of $a$ and $b$
$a \equiv b$	$a$ is equivalent to $b$
$a \approx b$	$a$ approximate $b$



# Abbreviations

A/D	Analog to digital
D/A	Digital to analog
DISS	Derivative input-to-state stability
GAS	Global asymptotic stability
IOSS	Input-output to state stability
ISS	Input-to-state stability
MIMO	Multi Input Multi Output
PID	Proporsional Integral and Derivative
qISS	quasi Input-to-state stability
DOA	Domain of attraction
SPA	Semiglobally practically asymptotically
SP-AS	Semiglobal practical asymptotic stability
SP-ISS	Semiglobal practical input-to-state stability
SP-DISS	Semiglobal practical derivative input-to-state stability
SP-IOSS	Semiglobal practical input-output to state stability
UBT	Uniformly bounded over time sampling $T$
ULISS	Uniformly locally input-to-state stability

# Contents

<b>Abstract</b>	<b>i</b>
<b>Declaration</b>	<b>ii</b>
<b>Statement of Originality</b>	<b>iii</b>
<b>Acknowledgements</b>	<b>v</b>
<b>Notation</b>	<b>vi</b>
<b>Abbreviations</b>	<b>vii</b>
<b>1 Introduction</b>	<b>1</b>
1.1 Nonlinear sampled-data control systems . . . . .	1
1.2 Objectives and motivation . . . . .	5
1.2.1 Research objectives . . . . .	5
1.2.2 Research motivation . . . . .	6
1.3 From analog to digital . . . . .	10
1.3.1 Emulation design . . . . .	12
1.3.2 Direct discrete-time design . . . . .	17
1.3.3 Sampled-data design . . . . .	22
1.4 Overview of the thesis and contributions . . . . .	23
<b>2 Preliminaries</b>	<b>35</b>
2.1 Notation, definitions and fundamental tools . . . . .	35
2.2 Input-to-state stability of nonlinear systems . . . . .	37

2.2.1	Continuous-time input-to-state stability . . . . .	39
2.2.2	Discrete-time input-to-state stability for nonparameterized sys- tems . . . . .	41
2.2.3	Discrete-time input-to-state stability for parameterized systems	42
2.3	Numerical approximation and discretization . . . . .	46
2.3.1	System with piecewise constant inputs . . . . .	46
2.3.2	System with Lebesgue measurable inputs . . . . .	47
<b>I</b>	<b>Design and Analysis using Emulation</b>	<b>51</b>
<b>3</b>	<b>Nonlinear Controller Emulation</b>	<b>53</b>
3.1	Introduction . . . . .	53
3.2	Preliminaries . . . . .	55
3.3	Main results . . . . .	64
3.3.1	Static state feedback results . . . . .	68
3.3.2	Open-loop configuration results . . . . .	69
3.4	Proofs of main results . . . . .	71
3.5	Applications . . . . .	80
3.5.1	Input-to-state stability . . . . .	80
3.5.2	Passivity . . . . .	84
3.6	Conclusion . . . . .	85
<b>4</b>	<b>Illustration and Motivation Example</b>	<b>87</b>
4.1	Introduction . . . . .	87
4.2	Preliminaries . . . . .	89
4.2.1	Continuous-time backstepping design . . . . .	90
4.2.2	Discrete-time backstepping design . . . . .	91
4.3	Jet engine system modeling . . . . .	93
4.4	Jet engine stall and surge control design . . . . .	95
4.4.1	Continuous-time backstepping controller design . . . . .	95
4.4.2	Direct discrete-time backstepping controller design . . . . .	99

4.5	Comparison of the emulation and the Euler based controllers . . . . .	103
4.6	Conclusion . . . . .	104
<b>II Design and Analysis via Approximate Discrete-Time Models</b>		<b>109</b>
<b>5</b>	<b>A Framework for Input-to-State Stabilization</b>	<b>111</b>
5.1	Introduction . . . . .	111
5.2	Motivating examples . . . . .	112
5.2.1	When things go wrong . . . . .	113
5.2.2	When things go right . . . . .	117
5.3	Preliminaries . . . . .	119
5.4	Main results . . . . .	125
5.5	Example . . . . .	131
5.6	Conclusion . . . . .	135
<b>6</b>	<b>Changing Supply Rates for Input-Output to State Stable Systems</b>	<b>137</b>
6.1	Introduction . . . . .	137
6.2	Preliminaries . . . . .	140
6.3	Main results . . . . .	142
6.4	Applications . . . . .	149
6.4.1	A LaSalle criterion for SP-ISS . . . . .	149
6.4.2	SP-ISS of time-varying cascade-connected systems . . . . .	151
6.4.3	SP-ISS via positive semidefinite Lyapunov functions . . . . .	153
6.4.4	Observer-based input-to-state stabilization . . . . .	155
6.5	Proofs of main results . . . . .	158
6.6	Conclusion . . . . .	166
<b>7</b>	<b>Lyapunov Based Small-Gain Theorem for Input-to-State Stability</b>	<b>167</b>
7.1	Introduction . . . . .	167
7.2	Preliminaries . . . . .	169
7.3	Main result . . . . .	172
7.4	Proof of the main result . . . . .	174

7.5	Counter example . . . . .	182
7.6	Conclusion . . . . .	183
<b>III Case Study</b>		<b>185</b>
<b>8</b>	<b>Stabilization Problem for A Two-Link Manipulator</b>	<b>187</b>
8.1	Introduction . . . . .	187
8.2	Preliminaries . . . . .	189
8.3	Manipulator system modeling . . . . .	191
8.4	Sampled-data controller design . . . . .	193
8.4.1	PD plus nonlinear terms controller design . . . . .	194
8.4.2	Lyapunov direct digital redesign . . . . .	195
8.5	Simulation results . . . . .	199
8.6	Conclusion . . . . .	203
<b>9</b>	<b>Conclusions and Further Research</b>	<b>205</b>
9.1	Conclusions . . . . .	206
9.2	Further research . . . . .	209

# List of Figures

1.1	General sampled-data control system configuration. . . . .	2
1.2	Autonomous vehicle control. . . . .	4
1.3	Controller emulation design. . . . .	12
1.4	Sampled-data control system configuration . . . . .	13
1.5	Direct discrete-time design. . . . .	20
1.6	Logical sequence of the chapters. . . . .	33
2.1	Class- $\mathcal{K}$ function. . . . .	36
2.2	Class- $\mathcal{K}_\infty$ function. . . . .	36
2.3	DOA for semiglobal practical stability property . . . . .	44
4.1	Jet engine compression system . . . . .	93
4.2	Lyapunov surface for $V(R, \Phi, \Psi)$ , with $R=1$ . . . . .	97
4.3	Derivative Lyapunov surface for $\dot{V}(R, \Phi, \Psi)$ , with $R=1$ . . . . .	97
4.4	Simulation 4.1a, response of the system with small sampling period $T = 0.01$ sec (— emulation, $\cdots$ continuous). . . . .	100
4.5	Simulation 4.2a, response of the system with large sampling period $T =$ $0.5$ sec (— emulation, $\cdots$ continuous). . . . .	100
4.6	Simulation 4.1b, response of the system with small sampling period $T = 0.01$ sec (___ Euler, — emulation, $\cdots$ continuous). . . . .	105
4.7	Simulation 4.2b, response of the system with large sampling period $T = 0.5$ sec (___ Euler, — emulation, $\cdots$ continuous). . . . .	105

---

4.8	Simulation 4.3, response of the system with nonzero initial states $x_o =$ $(1\ 2\ 7)^T$ (--- Euler, -.- emulation, ... continuous). . . . .	106
4.9	Simulation 4.4, response of the system with large initial states $x_o =$ $(5\ 5\ 10)^T$ (--- Euler, -.- emulation, ... continuous). . . . .	106
5.1	Estimates of DOA with $T=0.5$ sec. . . . .	118
5.2	Simulation 5.1, response of the system using controllers $u_T^1$ , $u_T^2$ and $u_T^3$ . . . . .	134
8.1	A two-link manipulator . . . . .	192
8.2	Simulation 8.1, responses with Euler-based controller (a,b) and emula- tion controller (c,d,e,f) when applying a square wave input. . . . .	200
8.3	Simulation 8.2, responses with Euler-based controller (a,b) and emula- tion controller (c,d), when applying a constant input. . . . .	202

# List of Tables

4.1	Jet engine plant's and controllers' parameters . . . . .	98
5.1	DOAs in disturbance free case. . . . .	132
5.2	Performance with a disturbance . . . . .	132
5.3	Parameters for Simulation 5.1 . . . . .	133
8.1	Manipulator's and controller's parameters . . . . .	199
8.2	Parameters for Simulation 8.1 . . . . .	199
8.3	Parameters for Simulation 8.2 . . . . .	201



# Chapter 1

## Introduction

This thesis explores important issues in design and analysis of nonlinear sampled-data control systems. While nonlinear sampled-data control systems are commonly found in real engineering applications, the analysis and design tools for such systems are still limited. The research covered in this thesis contributes theoretical results for the design and analysis of nonlinear sampled-data control systems, and provides a range of tools for controller design for such systems. The thesis also presents various applications of the results obtained.

In this chapter, the background, objectives, motivations and a short overview of contributions of this thesis are presented.

### 1.1 Nonlinear sampled-data control systems

Technology advances in digital electronics have led to a rapid development in computer technology. Currently, digital computers can be found in most equipment in a range of different applications. Control engineering is one of many areas where digital computer technology has made a great impact. Before the 1950s most control systems were analog, whereas today, most of the systems include a digital computer as their crucial part. Indeed, computer-controlled systems are now a prevalent configuration used in practice.

A sampled-data system involves both continuous-time and discrete-time signals in

its operation. In fact, most plants and processes found in engineering practice are continuous-time. The altitude and speed of an aircraft, fluid level in a vessel, temperature and pressure in a distillation column and voltage between two nodes in an electronic circuit are a few examples of continuous-time signals measured from various continuous-time plants. By controlling a continuous-time plant using a digital controller that operates in a discrete-time environment, we form a sampled-data system. A sampled-data control system therefore consists of a continuous-time plant/process controlled by a digital controller, either as a digital computer or as a simple microchip providing the control action. Consequently, a sampled-data control system is often referred to as computer-controlled system. A general configuration of a sampled-data control system is given schematically in Figure 1.1[8].

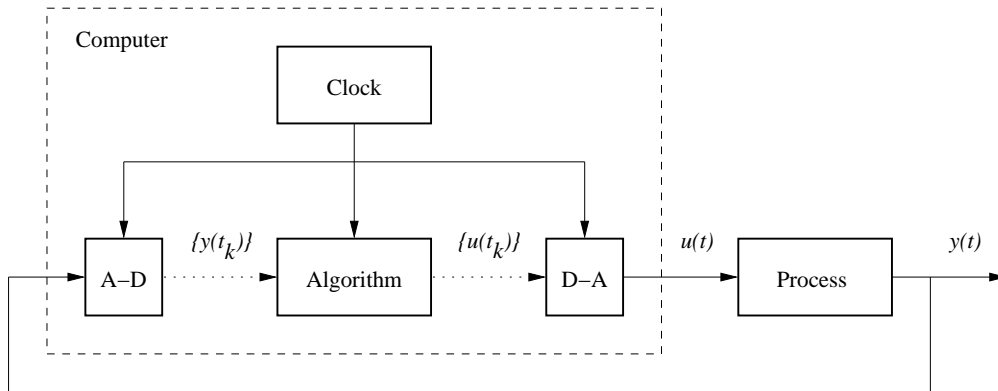


Figure 1.1: General sampled-data control system configuration.

In Figure 1.1, the output from the process  $y(t)$  is a continuous-time signal. The output is converted into digital form by the analog-to-digital (A/D) converter. The conversion is done at the sampling times,  $t_k$ . The control algorithm interprets the converted signal,  $\{y(t_k)\}$ , to calculate the required control sequence,  $\{u(t_k)\}$ . This sequence is converted to an analog signal by a digital-to-analog (D/A) converter. All these events are synchronized by a real-time clock in the computer. The digital computer operates sequentially in time, and each operation takes some time. The D/A converter must, however, produce a continuous-time signal. The simplest way to do this is to keep the control signal constant between each conversion time. In this case

the system runs open-loop in the time interval between the sampling instants because the control signal is constant, irrespective of the value of the plan output  $y(t)$ . In practice, the A/D and D/A converters can be parts of the computer or may be built as separate units.

Most plants and processes are nonlinear in nature. While it is possible to use a linear approximation around a prescribed operating point for analysis and controller design, there are many situations when nonlinearities cannot be neglected. Phenomena such as saturation, hysteresis, deadzone and dry friction are a few examples of common nonlinearities that often arise in practice (see [132] for details). In these cases, a nonlinear model is needed to obtain a more accurate representation of the dynamics of the system. A sampled-data control system which includes a nonlinear plant, controlled by either a linear or a nonlinear controller, is classified as a nonlinear sampled-data control system.

There is a wide area of applications for sampled-data control systems, where nonlinear phenomena cannot be avoided. These applications range from the manoeuvre control of an aircraft, such as a VTOL aircraft systems, ship or submarine vehicle control, position control for robotic systems in a precision manufacturing process, autonomous vehicle systems, biochemical reactors, power plants and many others. Therefore, control of nonlinear sampled-data systems is an important area of control engineering with a range of potential applications.

In many cases, it is possible to use either analog or digital controllers. However, there are certain situations where the complexity and required flexibility of a system can only be achieved using digital technology. An example adopted from [134], that illustrates this situation is presented next.

Figure 1.2 shows a block diagram of an autonomous vehicle system. We identify the elements and operation of this autonomous vehicle and compare it with the general configuration of the sampled-data control system in Figure 1.1. The autonomous vehicle system is dedicated to navigation tasks. The navigation can take place along a coverage pattern or to certain designated points. The process in this case is the vehicle itself, and the control objective is to drive the vehicle to the desired position.

The controller consists of a digital computer, with a software module handling the navigation planning. The navigation planner is assigned a path to track, and then computes the required steering direction based on the vehicle's current position. The steering direction is then sent to the navigation system, which also takes input from an obstacle avoidance module using a navigation sensor. A Global Positioning System (GPS) is one example of commonly used navigation sensors. If the steering direction is not vetoed by the obstacle avoidance module, then the command is sent to the real time system and the actuator to move the vehicle. The output of the system is the actual position of the vehicle, which is a continuous-time variable. The navigation sensor GPS takes a discrete-time measurement of the current position of the vehicle and important information from the environment. Hence, the A/D conversion is done in the GPS, which is a discrete-time sensor. The information from the GPS is then fed to the navigation planner. On the other hand, the steering direction, which is the control signal for the vehicle, is converted to an analog signal to drive the vehicle to the desired position.

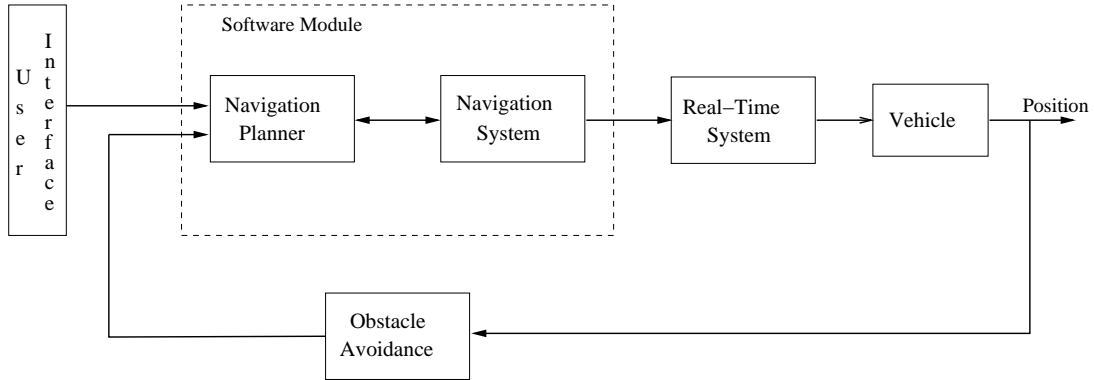


Figure 1.2: Autonomous vehicle control.

Autonomous vehicles are potentially useful for various applications, ranging from mobile robots for hazardous environments such as nuclear power plants and waste management, planetary vehicles, multiple mobile robots controlled by one human operator and submersible for offshore installation maintenance or mine sweeping. They can also provide additional safety for power wheelchair operated by severely disabled people. More tasks than path tracking can be assigned to the vehicle depending on

the application. If tele-operation is required, then communication modules become necessary and the control algorithm then becomes too complicated for analog implementation. Indeed, computer control becomes an absolute necessity to handle this application. This example further motivates investigation in the area of nonlinear sampled-data control systems which is the focus of this thesis.

## 1.2 Objectives and motivation

Although computer-controlled systems have been used widely for various applications, the theory of nonlinear sampled-data systems has not been very well established. Control theory of nonlinear sampled-data systems is considerably lagging behind the corresponding theory for nonlinear continuous-time systems. In this section, the objectives and main motivations of research done in this thesis are presented.

### 1.2.1 Research objectives

So far, there does not exist a comprehensive set of tools for controller design of nonlinear sampled-data systems. Analysis and design tools for continuous-time nonlinear control systems are more developed than their sampled-data counterparts. Yet, sampled-data systems are more prevalent in practice than continuous-time systems. This thesis aims at contributing in filling up this gap by providing novel frameworks and tools for the design and analysis of nonlinear sampled-data control systems. In particular, this thesis aims to:

- provide practical control engineers with a more complete toolbox for controller design of nonlinear sampled-data systems.
- provide mathematically rigorous results that apply under reasonable assumptions to large classes of nonlinear sampled-data systems.
- generalize some available results and to contribute quite a general new design framework for a large class of nonlinear sampled-data systems.
- address controller design problem for some examples arising from engineering practice using tools developed in this thesis.

### 1.2.2 Research motivation

As discussed earlier, this research was triggered by the rapid growth of digital technology, particularly the advances of digital computer technology, that had contributed significantly to the development of control engineering and its applications. Computer-controlled configuration has now become prevalent as a substitute for analog control. Besides this general motivation, each part of this thesis supported by its particular motivation, as presented in the following. Since the thesis focuses on nonlinear systems with inputs, the motivation for considering this class of systems is presented first in this subsection.

#### Nonlinear sampled-data systems with inputs

Nonlinear model of a system generally gives better representation of the dynamics of the system than a linear model. Using a nonlinear model we can gain greater accuracy in modeling. However, to precisely construct a nonlinear model of a system is not easy, since it is almost impossible to include all variables and parameters affecting the dynamic behavior of the system. Therefore, it becomes impossible to represent the exact dynamics of the system. With respect to this problem, often a physical system is modeled as a system with perturbation. The perturbation term included in the model could result from modeling error, aging, uncertainties and disturbances which may exist in any realistic problems.

Analysis and design of systems with disturbances are in general different from systems without disturbances. Since disturbances very often appear in practice, there is a strong motivation to undertake particular research in this direction. In most parts of this thesis, robust stabilization of nonlinear systems, which is stabilization of systems in the presence of disturbances, is considered. The stability property of systems with disturbances that we concentrate on in this thesis is known as the input-to-state stability (ISS) of the system. ISS is introduced for the first time by Sontag in [139]. This property has proven to be natural and very useful in a range of control problems (see for example [31, 70, 74] and references therein). ISS is a particular kind of bounded input bounded state ( $\mathcal{L}_\infty$ ) stability that is fully compatible with Lyapunov techniques.

## Part 1: Design and Analysis using Emulation

Most plants and processes in engineering practice are continuous. Moreover, the early control systems were mainly analog, and this has led to the development of continuous-time control theory. The analog or continuous-time control has also been widely used in applications, and there is a rich variety of tools available for continuous-time control design.

The simplest way to obtain a digital controller is simply by writing the continuous-time control law as a differential equation and approximating the derivatives by differences. For this reason, it makes sense to use continuous-time design tools to design a continuous-time controller, then discretize it using fast sampling to obtain a discrete-time controller. This method for digital controller design is often referred to as emulation. This method is rather popular since it can still achieve a very good performance of the sampled-data systems, although the design is carried out in continuous-time, completely ignoring sampling.

Indeed, in emulation, there is no need to consider sampling when designing the controller. Given a continuous-time model of the plant, we can use any continuous-time design tools to design a controller for the plant and then discretize the controller. There is only concern about the sampling at the implementation stage. The benefit of this is that if the continuous-time plant model possesses a special structure, which can simplify the design, the advantages of the structure can still be used prior to the discretization step in emulation design.

*Sampled-data control systems are now prevalence.*

*While there are a large variety of tools for continuous-time design,*  
*tools for discrete-time controller design are scarce.*

*The simplest way to design a discrete-time controller is using **emulation**.*

The factors described in earlier paragraphs have motivated the use of emulation design, specifically when sample and zero order hold devices are used. This technique is the simplest approach in discrete-time control design that is commonly used in practice, especially when dealing with relatively slow processes or when controlling

plants that do not require rich control signals.

In addition, more advanced approaches for controller discretization have been developed, resulting in more satisfactory designs. New advances in controller discretization techniques have made it possible to preserve certain performance of systems under emulation design. For instance, in [2, 15, 16] optimization based controller discretization techniques have been presented for linear sampled-data systems.

## Part 2: Design and Analysis via Approximate Discrete-Time Models

The main obstacle to emulation design is that in practice, the required sampling period may be too small and hence it is not achievable by using the available hardware. Therefore, the method cannot be used in many situations. A set of comparative examples in [64] illustrate this point. The examples present comparison of various methods of discretizing a continuous-time controller with the variation of sampling frequency. It is shown that stabilization is in general achieved only with very high sampling frequency, in other words, when very fast sampling is used. Moreover, the system may lose the stability with the decrease of the sampling frequency. In such cases, a better alternative is a direct discrete-time design which is based on the discrete-time model of the plant.

Another issue that motivates the use of a direct discrete-time design is the measurement and sensing problem. Sensors that provide a continuous time measurement are usually very expensive, or sometimes there are only digital sensors available to measure certain physical values. For example, a GPS mentioned in the illustration in Section 1.1 is a discrete-time sensor. In addition, some plants, for example a radar, are more naturally represented using a discrete-time model, and hence the information from this kind of plant can only be measured in discrete-time. When measurement is undertaken only in discrete-time, then direct discrete-time design appears to be the only technique to use.

*Emulation design is sometimes not sufficient.*

*An alternative to do a better design is the **direct discrete-time design**.*

*Nevertheless obtaining exact discrete-time model for nonlinear plants is almost impossible, and hence approximate model is used instead.*



Since discretization is in fact an approximation, a discrete-time controller obtained by emulation is an approximation of a continuous-time controller. Therefore, degradation in performance of a system typically occurs when it is controlled by an emulation controller. On the other hand, a better performance may be achieved using a direct discrete-time design. Moreover, arbitrarily larger sampling periods may be applied to the controller obtained using direct discrete-time design. In addition, redesigning an emulation controller using direct discrete-time design to obtain some improvements is possible. For example, it is shown in Chapter 8 that a discrete-time controller that takes form

$$u_T^{dt} = u^{ct} + Tu^r ,$$

where  $u^{ct}$  is the emulated controller and  $u^r$  is an additional term obtained from redesigning  $u^{ct}$ , may outperform the emulated controller.

A stumbling block in a direct discrete-time design for nonlinear sampled-data control systems is the absence of a good model for the design. While for linear systems we can in principle compute the exact discrete-time model of the plant, this is not the case for nonlinear systems. Computing the exact model involves solving an initial value problem. In the case of nonlinear systems, this involves solving analytically a nonlinear differential equation over one sampling interval, which is impossible in general. Instead, various numerical algorithms are used to approximate the solutions. As a result, an approximate discrete-time plant model is used to replace the exact discrete-time model for the design.

*Approximation certainly involves inaccuracy.*

*It could happen that a controller designed using approximation stabilizes the approximate model, but destabilizes the exact model for all sampling periods.*

*Hence, there is a strong concern to build a framework to avoid this situation to occur in practice.*

The issue in direct discrete-time design does not stop as we use an approximate model to design a controller that will be implemented to control the original continuous-time plant. Similar to what happens with discretization of the controller in emulation design, numerical approximation we use to obtain the approximate model

of a plant will certainly cause discrepancy between the exact and the approximate models of the plant . This inaccuracy of modeling can lead to the design failure. It can happen that the controller obtained using approximate based design stabilizes the approximate model of the closed-loop system, but destabilizes the exact model of the system for all sampling periods. Therefore, there is a strong concern to build a framework of direct discrete-time design that provide conditions, which guarantee that the final design objectives are still achieved even though approximate model is used for the design. The problem of finding good approximate models and the guaranteeing conditions have strongly motivated research in direct discrete-time design based on approximate models of a plant.

### 1.3 From analog to digital

Classical control theory was mainly analog and it makes sense that most literature in basic control theory typically provide discussions only about continuous-time systems. Although continuous-time or analog controllers can be used in a range of situations where digital controller can also be applied, technological and functional demands to modern control systems that result in the increase of complexity of the systems, have made digital control be more promising to implement. Several major advantages of using digital controller compared to analog control are listed below [8].

- Cheap: cost to build an analog control configuration increases linearly with the number of control loops while a digital control configuration is cheaper for large installation.
- User friendly: operator interface in analog control consists of a large operator board, while operator communication panel for digital control can only be a simple key pad.
- Flexible: digital control offers more flexibility since it is based on programming instead of wiring as in analog control.
- Expandable: digital control provides easy interaction among control loops, that makes it easier to expand.

- Simple to program: digital control builder can be simplified using a specific language to make it easy for the operator to program .

To state briefly, an analog controller is usually rigid, space consuming, disintegrated and high cost, and the increase of complexity of systems typically makes it inefficient. Therefore, digital controller appears to be superior to analog control.

The digital computer era commenced during the early 1950s. In the beginning, digital computers were very slow, expensive and unreliable. However, rapid development in digital technology has resulted in an impressive improvement in digital computer technology. Currently, the fastest microprocessor can run at 2-GHz, implemented as a microchip, the size of which is no more than 25 cm<sup>2</sup> [118]. Implementing a digital computer to replace an analog controller provides a proper solution to problems caused by the increase in complexity of control systems. Some remarkable results on early computer-controlled system theory are covered in detail in [8].

In the previous section it was stated that a computer-controlled system is a sampled-data control system. Henceforth, the two terms will be used interchangeably throughout the thesis. The process of transforming signals from analog to digital and vice versa becomes crucial when working with this class of systems. Sampling is an essential operation involved in the transformation from analog to digital, and signal reconstruction is important for the converse. It is clear that some information carried by a signal is lost due to sampling, however good the reconstruction process is. Moreover, certain useful properties of continuous-time control systems, such as controllability and observability may be destroyed under sampling. Therefore, developing special tools to carry out analysis and design for nonlinear sampled-data systems is an important research area with a number of open problems.

In general, design of sampled-data control system can be done using three different techniques. The first technique is the so called **emulation design**, where the controller design is done in the continuous-time domain followed by controller discretization to produce a discrete-time controller for digital implementation. The second technique, known as a **direct discrete-time design**, is where a discrete-time controller is designed in discrete-time domain directly, using a discrete-time model of

the plant. With this technique, the inter-sample behavior is ignored during the design process. The third approach, the **sampled-data design**, is done taking into account the inter-sample behavior in the design. Each of these techniques is discussed in more detail in the next subsections.

### 1.3.1 Emulation design

An important controller design method for nonlinear sampled-data control system is the emulation design technique. The use of the emulation technique was initiated by treating discrete-time systems in a continuous-time framework. The technique makes use of continuous-time control design tools that are available in literature. The emulation design procedure follows the steps shown in Figure 1.3. It consists of a three-step design procedure; continuous-time design, controller discretization, and digital implementation. The detailed procedure is described below.

In the first step, a continuous-time controller design is carried out. The obtained continuous-time controller achieves a set of performance and robustness criteria for the closed-loop continuous-time system. At this stage, the sampling is completely ignored.

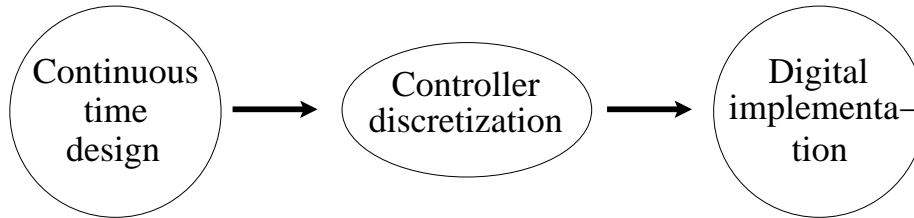


Figure 1.3: Controller emulation design.

The second step is to discretize the continuous-time controller. Discretization is in fact a type of approximation. In general, a sufficiently fast sampling is required, since the discrete-time model is a good approximation of the continuous-time model typically only for small sampling periods. Discrete-time representation of the controller is usually obtained using some numerical integration methods or for linear system, using pole-zero matching from continuous-time to discrete-time domain. For numerical integration, methods such as Euler, Tustin and other Runge-Kutta methods are commonly used, while in the matched pole-zero method the extrapolation from the

relationship between the  $s$ - and the  $z$ - planes is used.

The final step is to implement the controller digitally. This step assumes that a sampler and hold device is placed between the plant and the controller [18]. The configuration of a sampled-data control system, which is obtained from an emulation design is given in Figure 1.4. In this configuration, the discrete-time controller  $K_T$  is a discretization of the originally designed continuous-time controller  $K_C$ , while  $S$  and  $H$  represent respectively sampling and hold devices.

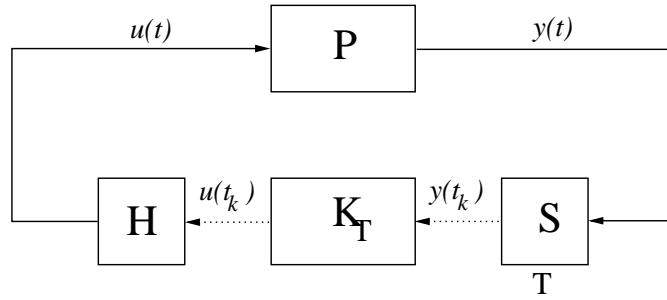


Figure 1.4: Sampled-data control system configuration

The most important issue in the implementation step of an emulation design is the selection of the sampling period  $T$ . It is necessary to determine the sampling period at this stage, so that the property satisfied by the continuous-time system can be preserved in a certain sense, in a sampled-data environment, when an analog controller is replaced by its digital realization. Various techniques of emulation design are available for example in [8, 30, 119]. The following trivial example shows how emulation is applied to design a discrete-time controller for a continuous-time plant, when backstepping is used as the design tool.

**Example 1.3.1** [70] *Consider the plant*

$$\begin{aligned}\dot{x}_1 &= x_1^2 - x_1^3 + x_2 \\ \dot{x}_2 &= u .\end{aligned}\tag{1.1}$$

*Using backstepping [70, 74], we design a continuous-time controller for the plant, and we obtain the control law*

$$u = -(2x_1 + 1)(x_1^2 - x_1^3 + x_2) - x_1 - (x_2 + x_1^2 + x_1) .\tag{1.2}$$

where the following Lyapunov function

$$V_a(x) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + x_1^2 + x_1)^2$$

follows from the design, and with the controller (1.2) we achieve global asymptotic stability for the origin  $x = 0$  of the closed-loop system (1.1), (1.2). Discretizing the control  $u$  using a sample and zero order hold, we have

$$u(k) = -(2x_1(k) + 1)(x_1(k)^2 - x_1(k)^3 + x_2(k)) - x_1(k) - (x_2(k) + x_1(k)^2 + x_1(k)) . \quad (1.3)$$

The discretization is done by holding the control value constant between every sampling period. In this case, the control signal  $u(t) = u(k)$ ,  $t \in [kT, (k+1)T)$ . Choosing a sufficiently small sampling period  $T$ , the implementation of discrete-time controller (1.3) results in asymptotic stability for the closed-loop system (1.1), (1.3), with ultimate boundedness in the states, which is known as the semiglobal practical property (an explanation about semiglobal practical notion in quite a general context is presented later in **Chapter 2**). ■

Emulation controllers usually work well under fast sampling. Reducing the sampling period  $T$  can improve the performance of systems controlled using controller obtained via emulation. Besides, improvement can also be achieved by employing a better discretization technique for the controller. Motivated by for instance [2, 68], a new technique for discretizing a linear time invariant controller was proposed in [15, 16], where a pointwise gap distance was used as one of the design tools.

Design in a frequency domain based on linear models is another approach which is quite commonly found in practice. A technique using describing function is studied in [22]. This technique uses Nyquist criterion to obtain the boundary of instability, so as to further determine the acceptable sampling period for the system. Another approach is given in [126], where  $s$  to  $z$  domain transformation is applied using a combination of modified Tustin, zero-order-hold equivalent and several methods using triangular hold. The delay caused by the zero-order-hold is taken into account in this method.

The  $\mathcal{L}_p$ -stability of linear sampled-data systems is considered in [18, 29]. It is demonstrated that the exponential stability of a sampled-data feedback system is, under a certain nonpathological sampling condition, implied by the closed-loop stability of the corresponding discrete-time system. This also implies the  $\mathcal{L}_p$ -stability of the sampled-data system. To guarantee that the nonpathological sampling condition is satisfied, a low-pass prefilter is introduced into the system. The result is directed to linear time invariant systems. A similar result for the feedback configuration of a nonlinear plant with a static as well as a nonlinear dynamic controller is presented in [77, 82, 107].

In nonlinear context, numerous studies on the behavior of nonlinear systems under emulation design have been conducted. Many of these have important results. The stability of the systems is the property that takes the greatest concern. The effect of emulation on the stability was studied for instance in [18, 19, 124, 128, 154, 163]. These studies concluded that fast sampling is required when applying the emulation design. In [83], the stability property of a nonlinear sampled-data system with dynamic output feedback is analyzed. The system was recast as a sampled-data system with jumps. The jumps represent the dynamics of the system at the sampling instants. The result considered the exponential stability of a class of nonlinear sampled-data system. Quite a similar result for a system with slowly varying exogenous inputs are presented in [84]. In [9], the effect of sampling on the convergence of closed-loop sampled-data systems containing parameter uncertainties was studied. Both results on global and local convergences were presented.

Additional results on stability are presented in [50, 51], where a nonlinear continuous-time plant is controlled by a linear digital controller which is designed using an associated linear model of the real nonlinear plant to be controlled. In this case, the stability analysis of the closed-loop system is derived from the stability condition of the associated linear system. The stability of the original nonlinear sampled-data system achieved from this analysis is in general weaker than the stability condition of the linear system. In these papers, effects of quantization are qualitatively studied. The relationship between the stability of the linearization model of the system around the

equilibrium point and the original system is used. Moreover, the stability property of a sampled-data system without and with a quantizer are compared. The study considered sampled-data systems with single-rate and multi-rate sampling.

The preservation of input-to-state stability under emulation was proved in [154], using a Razumikhin type theorem for ISS of functional differential equation. It is shown that the ISS property of a nonlinear system is preserved under fast sampling, in a semiglobal practical sense. Moreover, if the system is globally exponentially stable and globally Lipschitz, global uniform ISS can be achieved. This result is also valid when an approximate realization of the discrete-time controller, which is obtained using Euler approximation or some higher order approximation, is used and also when a higher-order-hold function is applied instead of the zero-order-hold.

It has been discussed in the previous section that a good performance can be achieved under fast sampling (see [124, 163]). A slow sampling counterpart of this general statement was studied in [164]. That is for a class of systems which has a stable equilibrium point and in which the initial condition at the beginning of the sampling interval lies inside the domain of attraction of the equilibrium point, one may expect that sufficiently slow sampling would still result in a stable closed-loop system. Further studies about emulation design considered the regulation problem [17].

There are situations where a system is stabilizable, but there does not exist a continuous feedback that stabilizes the system. A result of [20] states that every asymptotically controllable system can be stabilized by means of some (discontinuous) feedback law. The result implicitly suggests that a discontinuous-time controller may be a direct result of a continuous-time design. It also suggests that a particular type of sampled control laws can be used in general to stabilize a stabilizable nonlinear systems for which there exists a control Lyapunov function.

In most cases, the properties for nonlinear systems are preserved in a semiglobal practical sense under sampling and controller emulation. The relationship between the sampling rate and the estimate domain of attraction of a class of sampled-data system was studied in [163]. In this paper, the estimate domain of attraction for the



continuous-time system is first determined. It is claimed that if the control is sampled faster than a certain sampling period  $T$ , the stability of the system on the same domain of attraction is preserved. The paper, which is the second part of [124], provides a method for computing a sampling period  $T$ , that guarantees the preservation of the stability condition.

There are many other studies and results on the design and analysis of sampled-data systems under emulation. The 1992 Bode Prize Lecture [2], provides important information on emulation design, especially for linear systems.

In Chapter 3 of this thesis, quite general results on emulation for nonlinear systems are presented. A preservation of dissipativity properties under controller emulation is studied, and various applications to more special cases are presented. In this research, a formula for computing the bound of permissible sampling periods is provided.

There are a number of advantages of the emulation design technique. Tools for controller design in continuous-time domain are numerous and well established. The emulation design separates the controller design problem from the issue of sampling period selection, so that there is no need to specify the time sampling, until the controller is implemented to the system. Since most plants are continuous, this design method becomes a natural and favored method, and is used quite extensively in engineering practice, even when good direct digital controller design methods are available. However, there are some disadvantages that limit the use of this technique. The analog performance can only be recovered using very fast sampling and because of hardware limitations on achievable sampling periods, the required sampling period may not be implementable in practice.

### 1.3.2 Direct discrete-time design

From the previous subsection, we have known that a system controlled using an emulation controller always suffers the degradation of performance compared with its continuous-time counterpart. To reduce the degree of degradation, very fast sampling is needed, which may not be feasible because of hardware limitations. In this case, direct discrete-time design offers an alternative solution, since in this design sampling

is considered from the beginning of the design process.

Parallel with the development of emulation design theory, since the 1950s, studies on the direct discrete-time design have begun to contribute results to control theory. This technique appears as a complement for emulation design. In many cases, this type of design shows potential to obtain controllers that can improve performance of the closed-loop sampled-data systems. Direct discrete-time design is done directly in discrete-time domain, based on the discrete-time model of the plant. There are two approaches taken in literature, regarding the plant model; the first, is when assuming that the exact discrete-time model of the plants are known, while the second assumes that the exact models are unknown.

The assumption of the availability of the exact discrete-time model is commonly found in literature. This assumption is always used for linear systems, however, it almost never holds for nonlinear systems. Several studies applying this assumption are for example [1, 28, 69, 99, 103, 135, 155]. However, this assumption is practically unrealistic. Assuming that exact models are unknown is in fact a more realistic assumption, especially for nonlinear systems, for which the exact discrete-time model is generally unavailable. Indeed, as mentioned earlier, even if the continuous-time plant model is known, in general the exact discrete-time model of the plant cannot be computed, since it needs an explicit analytic solution of a nonlinear differential equation, which is impossible to obtain in general. In this situation, an approximate model is used and this is the most common approach used in practice.

Numerical methods are used in obtaining an approximate discrete-time model of a continuous-time plant. In fact, even for linear systems, approximate models are always used when numerical methods are applied. A simple Euler and other more general Runge-Kutta methods are examples of approximation techniques that are often used. Hence, design is carried out based on the approximate plant model and stability is checked for the exact model. A problem arises from the use of an approximate model is that there is no guarantee for the stability of the exact model [114, 116]. Therefore, one needs to carefully look at several conditions to check the validity of the design.

This has motivated research on controller design via approximate discrete-time

models for sampled-data nonlinear systems. In the early literature, the focus was on particular problems, where specific plant models, approximation discrete-time models and controllers techniques were considered (see [23, 33, 45, 92]). Early results using this approach can be found in [23, 33], where digital controllers were designed respectively for waste water treatment and pH neutralization processes and for a biochemical reactor based on the Euler approximate model of the plants. An adaptive control scheme based on the Euler approximate model for a larger class of systems was presented in [92].

A drawback of these early results was their limited applicability: the studies investigated a particular class of plant models, a particular approximate discrete-time plant model (usually Euler) and a particular controller. A more general framework for the stabilization of *disturbance-free* sampled-data nonlinear systems via their approximate discrete-time models that is applicable to general plant models, controllers and approximate discrete-time models was first presented in [109, 116]. The first unifying result where a generic discrete-time model is used in the design is presented in [116]. In this paper, a parameterized family of discrete-time approximate plant models is used to carry out the design. The approach taken in this thesis is motivated mostly by this result.

A simple diagram of approximate based direct discrete-time design procedure is given in Figure 1.5. The procedure for approximate based direct discrete-time design consists of three steps, beginning with discretization of the continuous-time plant model, in order to obtain an approximate discrete-time model of the plant. The discrete-time model of a plant is usually parameterized by sampling period  $T$ . Hence, it forms a  $T$  parameterized family of discrete-time models. If the sampling period is fixed, the parameter  $T$  disappears from the discrete-time model. However,  $T$  may be left as a parameter to be determined later. The second step is to design a family of discrete-time controllers based on the family of discrete-time models of the plant. The objective is to construct a discrete-time controller for the discrete-time model, so that the closed-loop discrete-time model satisfies certain stability and robustness criteria that have been set for the design. At this stage,  $T$  is determined so that a satisfactory

performance for the system is achieved.

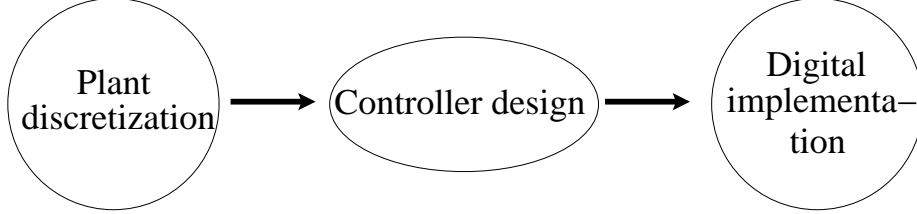


Figure 1.5: Direct discrete-time design.

It is important to note that when using direct discrete-time design, the inter-sample behavior of the system is ignored. As a result, the stability of the approximate discrete-time model does not automatically imply stability of the original sampled-data system. Certain conditions on the discrete-time model, the controller and the property achieved for the discrete-time model determine the stability property of the closed-loop sampled-data system. Therefore, as the final step, design verification needs to be done before implementing the controller to the original continuous-time plant. In Chapter 5 of this thesis, a framework for direct discrete-time design based on approximate models is provided. Together with results of [117], the framework can be used to guarantee that stabilization of the approximate models implies stability for the sampled-data systems. We present in the following, a simple example that shows how approximate based discrete-time design is applied to design a discrete-time controller for a continuous-time plant, when an Euler based discrete-time backstepping is used as the design tool.

**Example 1.3.2** [70, 110] *Consider the plant*

$$\begin{aligned}\dot{x}_1 &= x_1^2 - x_1^3 + x_2 \\ \dot{x}_2 &= u .\end{aligned}\tag{1.4}$$

*We will use a discrete-time backstepping to design a discrete-time controller for the plant, based on the Euler approximate model of the plant:*

$$\begin{aligned}x_1(k+1) &= x_1(k) + T(x_1(k)^2 - x_1(k)^3 + x_2(k)) \\ x_2(k+1) &= x_2(k) + Tu(k) .\end{aligned}\tag{1.5}$$

Applying discrete-time backstepping design proposed in [110], we obtain the control law

$$u(k) = -c(x_2(k) + \phi_T(x_1(k))) - \frac{\widetilde{\Delta W}_T}{T} + \frac{\Delta \phi_T}{T}, \quad (1.6)$$

where

$$\begin{aligned} \phi_T(x_1) &= -x_1^2 - x_1 \\ \widetilde{\Delta W}_T &= Tx_1(x_1^2 + x_1 + x_2) + T^2[(x_1^2 - x_1^3 + x_2)^2 + (x_1^3 + x_1)^2] \\ \Delta \phi_T &= -2Tx_1(x_1^2 - x_1^3 + x_2) - (x_1^2 - x_1^3 + x_2) + T^2(x_1^2 - x_1^3 + x_2)^2, \end{aligned} \quad (1.7)$$

and we have used a composite Lyapunov function

$$V_{aT}(x) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + x_1^2 + x_1)^2$$

for the design. The controller achieves semiglobal practical asymptotic stability for the closed-loop approximate discrete-time model (1.5), (1.6), (1.7), and can be shown further that it also asymptotically stabilizes the closed-loop sampled-data system (1.4), (1.6), (1.7) in a semiglobal practical sense. ■

Most results in direct discrete-time design are initiated by results for the continuous-time domain. Indeed, direct discrete-time design is a natural framework to study digital control. However, it cannot be denied that analysis and design using this method suffer from the fact that discretization of the plant model may destroy some important properties of the continuous-time model. The loss of feedback linearizability [6] and minimum phase properties [97] are some examples of the drawbacks of the discretization effect. Moreover, in [19], it can be shown that feedback linearizability and minimum phase structure are lost generically even with fast sampling. Consequently, analysis and design using direct discrete-time method are usually harder. Part two of this thesis, together with [104, 114, 115, 116], contain a framework for a direct discrete-time design for various classes of nonlinear systems.

Although it is rather ambitious to assume the availability of a discrete-time model of a plant (either exact or approximate) directly without describing how the model is obtained, this assumption is quite commonly used in the research on discrete-time control systems. In fact, the procedure of obtaining a good model for a design purpose

is crucial in practice, since the choice of the discrete-time model used will determine the performance of the sampled-data system. Therefore, a good model cannot be chosen randomly. Tools and techniques that can be utilized to construct approximate discrete-time models of a nonlinear plant are very important for real applications [42, 43]. Related to the issue of constructing approximate models, a research area that supports the development of direct discrete-time design using approximation is the numerical analysis. Ideas from [52] where numerical methods are considered as dynamical systems have significantly contributed to the development of the theory of sampled-data control. Convergence properties of dynamical systems under discretization, which is also important for sampled-data control design, is reviewed in [149]. Intensive research has been done in this area and they inspired the fundamental results in [109, 116], which are the starting points of Part II of this thesis.

A number of applications of numerical algorithms in the design of control systems have been explored. In [42] a systematic method to derive higher order schemes for affinely controlled nonlinear systems was developed. Another application was presented in [73], where a recursive discrete-time controller was designed using spatial discretization of the plant model. Dynamic programming is also a common tool used to solve optimal control problems. In [122], a real time optimization technique for nonlinear receding horizon control is presented. This result modified the earlier result of [121]. Other results on the nonlinear receding horizon control are presented in [67, 93, 161] and references therein. Optimal control using dynamic programming is one method used for direct discrete-time controller design.

### 1.3.3 Sampled-data design

It has been mentioned earlier that direct discrete-time design has potential to improve the design compared to emulation design. Nevertheless, intersample behavior is not taken into account in direct discrete-time design. This limitation becomes a drawback of this technique, in which ripple may occur in the response of the systems. Indeed, careful design and the choice of sampling period have to be done to guarantee good performance of the controlled systems.

Another way to improve the design performance is using the third approach for the controller design of sampled-data systems, where we use the sampled-data model of the plant that takes into account intersample behavior. This approach is known as sampled-data design, and has been developed since 1990's for linear system. However, since the underlying model is time varying, this technique becomes quite complicated and there has not been a similar development for nonlinear systems. This thesis does not pursue this direction, and hence, details on this technique are not provided in this thesis. More details for linear sampled-data controller design can be found in [19].

Besides the issue of computing a proper sampling period and validity of approximation which we focus in this thesis, there are many more issues involving in the design and analysis of sampled-data nonlinear control systems that do not belong to the coverage of this thesis. The issues are such as the effects of quantization, finite word-length implementation of filter coefficient, and the effects of overflow. These issues are usually not taken into account in the controller design step, but they are very important at the implementation stage. Several studies, especially those which concentrate on the implementation of a digital controller, have thoroughly investigated these issues (see [8, 47]).

## 1.4 Overview of the thesis and contributions

A summary of results and contributions of this thesis is presented next. The thesis focuses only on two design approaches for sampled-data systems, as described in the earlier sections. The first part of the thesis concentrates on the emulation results and the second part is dedicated to the direct discrete-time design based on approximate models; finally, a case study is presented. A general class of nonlinear systems with inputs is considered, and various problems in design and analysis for this class of systems are explored. We refer to Chapter 2 for notation and terminology that are not defined in this section.

## Chapter 2: Preliminaries

In this chapter, technical preliminaries are presented. Common notation and def-

initions, which will be used throughout the thesis are presented. Some important results from literature which are needed to prove results in this thesis are cited in this chapter, and a summary of a fundamental stability theory for nonlinear systems with inputs and numerical methods for discretization are presented. Preliminary results cited from literature are included in this chapter, to support the main results presented in this thesis.

### Chapter 3: Nonlinear Controller Emulation

This chapter is dedicated to the emulation design technique. A problem on the preservation of dissipation inequalities under sampling and controller emulation is considered. A general and unified framework for designing nonlinear digital controllers using the emulation method for nonlinear systems with disturbances is presented. It is shown that if a (dynamic) continuous-time controller, which is designed so that the continuous-time closed-loop system satisfies a certain dissipation inequality, is appropriately discretized and implemented using sample and zero-order-hold, then the discrete-time model of the closed-loop sampled-data system satisfies a similar dissipation inequality in a semiglobal practical sense.

Suppose we are given a general nonlinear continuous-time plant with inputs:

$$\begin{aligned}\dot{x} &= f(x, u, d_c, d_s) \\ y &= h(x, u, d_c, d_s) ,\end{aligned}\tag{1.8}$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ ,  $y \in \mathbb{R}^l$ ,  $d_c \in \mathbb{R}^{d_c}$  and  $d_s \in \mathbb{R}^{d_s}$  are respectively the state, input, output and disturbances; and a continuous-time dynamic controller

$$\begin{aligned}\dot{z} &= g(x, z, d_c, d_s) \\ u &= u(x, z, d_c, d_s) ,\end{aligned}\tag{1.9}$$

with state  $z \in \mathbb{R}^z$ . The problem addressed in this chapter can be formulated as follows:

**Problem 1.4.1** *Suppose that for the plant (1.8), we have designed a continuous-time (dynamic) controller (1.9) so that the continuous-time closed-loop system satisfies a certain dissipation inequality. The controller is then appropriately discretized with a sampling period  $T$ , to obtain a discrete-time emulation controller for implementation*



*using a sample and zero-order-hold device to control the plant (1.8). The effect of sampling on the dissipativity property of the discrete-time model of the closed-loop sampled-data system is studied, and two questions are investigated. The first is whether the dissipativity property is preserved under sampling and controller emulation or not. If the answer for the first question is yes, then the second question is under what conditions and in what sense the property is preserved.* ■

The main results are presented as two main theorems, exploring two types of dissipation inequalities, the weak dissipation inequality and the strong dissipation inequality, for the discrete-time model of the closed-loop sampled data system. An approximate model of the discrete-time controller is used for implementation. The approximate model needs to satisfy a certain consistency condition, to be used for proving the results. Consistency is a notion adopted from numerical analysis literature. It measures the closeness between the exact discrete-time model and the approximate discrete-time model of the controller. Two consistency properties are introduced in relation with the two dissipation inequalities to hold. For the weak dissipation result to hold, the discretized controller needs to satisfy the one step weak consistency condition and the disturbances need to be Lipschitz. On the other hand, the strong dissipation inequality result holds if the discretized controller satisfies the one step strong consistency condition and the disturbances are allowed to be only measurable in Lebesgue sense.

The dissipation inequality that we introduce in this chapter is rather general and its special cases are for example the Lyapunov characterization of asymptotic stability, ISS and passivity. Consequently, we are able to state a range of corollaries on preservation of stability, ISS, passivity and other properties under sampling and controller emulation. In this chapter, several numerical examples are also provided to illustrate the results.

## Chapter 4: Illustration and Motivation Example

It is the purpose of this chapter to present an example to apply results of Chapter 3, and to motivate research in the rest of this thesis. It is demonstrated that emulation design is applicable to solve the stabilization problem proposed in the example.

Moreover, the example also shows that design can be improved using direct discrete-time design. It is shown that controller obtained using direct discrete-time design may outperform controller obtained using emulation.

More particularly, in this chapter, a jet engine system rotating stall and surge control problem is considered. A backstepping design technique is used as the design tool to solve this problem. We first design a continuous-time controller for the continuous-time plant, and then emulate the controller to obtain a discrete-time emulation controller for implementation. The continuous-time controller is first designed to globally asymptotically stabilize the continuous-time plant. Simulation results show that with the emulation controller, the asymptotic stability property is preserved with a degree of performance degradation to the closed-loop system. A Lyapunov theory based analysis shows that the preservation is achieved in a semiglobal practical sense.

With the emulation controller, the performance of the continuous-time system cannot be perfectly recovered. Therefore, a different way of design is carried out to improve the achievable performance of the sampled-data system. Another discrete-time controller is designed using approximate-based direct discrete-time design technique, where an Euler based discrete-time backstepping is applied as the design tool. Intensive simulations are carried out, and observations are done to study the performance of the closed-loop system when discrete-time controller obtained from direct discrete-time design is implemented, and then comparing the results with the performance when emulation controller is implemented.

Based on the simulation results, it has been shown that the system controlled using the Euler based direct discrete-time controller outperforms the one controlled by the emulation controller. We avoid presenting detailed and exhaustive theory based analysis or proofs in this chapter. Indeed, the simulation results have shown that direct discrete-time design works better than emulation design. This fact has motivated strongly the research on the direction of direct discrete-time design, which is the main focus of the rest of this thesis.

## Chapter 5: A Framework for Input-to-State Stabilization

This chapter is dedicated to the results on direct discrete-time design based on

approximate plant models. In particular, a general class of nonlinear sampled-data control systems is studied, when a discrete-time controller for a continuous-time plant is designed based on a set of parameterized approximate discrete-time models of the plant.

A framework for the design of  $\mathcal{L}_\infty$  stabilizing controllers via approximate discrete-time models for sampled-data nonlinear systems with disturbances is provided. The property we consider is input-to-state stability (ISS) introduced by Sontag in [139]. In particular, sufficient conditions, under which a discrete-time controller that input-to-state stabilizes an approximate discrete-time model of a nonlinear plant with disturbances, would also input-to-state stabilize (in an appropriate sense) the exact discrete-time plant model, are presented. The results generalize the recent framework for stabilization of sampled-data nonlinear systems without disturbances via their approximate discrete-time models presented in [116]. Conditions for the approximate model of the plant used for designing the controller, the class of controllers, and conditions of the closed-loop discrete-time approximate model of the system that are proposed form a framework for ISS controller design via approximate discrete-time models. The results are established using the ISS Lyapunov functions.

More precisely, consider a continuous-time nonlinear plant in the following form:

$$\dot{x}(t) = f(x(t), u(t), w(t)) , \quad (1.10)$$

where  $w \in \mathbb{R}^d$  is the exogenous disturbance. A crucial issue in direct discrete-time design is to obtain a good discrete-time model of the plant, to be used when carrying out the design. Suppose the exact discrete-time model is

$$x(k+1) = F_T^e(x(k), u(k), w[k]) . \quad (1.11)$$

Note that we use the new notation  $w[k]$  for the disturbance that is related to a certain class of sampled disturbances, which will be explained in more detail in Chapter 5. Since finding an exact model requires solving a nonlinear differential equation, it is generally impossible to obtain the exact discrete-time model for the nonlinear plant explicitly. Hence, an approximate model is used instead. An approximate model  $F_T^a$  of the plant can be obtained using numerical integration methods. The approximate

models need to satisfy a certain consistency property with respect to the exact models, however it does not require an explicit expression of the exact model for checking this property .

We consider two consistency properties, weak consistency and strong consistency, and in parallel with the two consistency properties used, two forms of the family of approximate discrete-time models of the plant are constructed. Related to the weak consistency property, a family of approximate discrete-time models is constructed as follows:

$$x(k+1) = F_T^a(x(k), u(k), w(k)) , \quad (1.12)$$

which is an ordinary difference equation. Moreover, related to the strong consistency, this takes the form

$$x(k+1) = F_T^a(x(k), u(k), w[k]) , \quad (1.13)$$

which is a functional difference equation.

Using an approximate plant model, it is assumed that we have obtained from a direct discrete-time design, a family of discrete-time dynamic controllers

$$\begin{aligned} z(k+1) &= G_T(z(k), x(k)) \\ u(k) &= u_T(z(k), x(k)) \end{aligned} \quad (1.14)$$

that input-to-state stabilizes the closed-loop approximate model of the system. It makes sense to expect some discrepancies between the approximate model and the exact model as a result of approximation, and between the exact model and the sampled-data system itself as a result of discretization. Consequently, the ISS property obtained for the closed-loop approximate model does not guarantee that the same property for the exact model. The problem addressed in this chapter can then be formulated as follows.

**Problem 1.4.2** *Suppose the discrete-time controller (1.14) is obtained from a direct discrete-time design using the approximate model  $F_T^a$  of the plant (1.10). Suppose that the controller will be implemented to control the original continuous-time plant. We are then concerned with the following:*

- *We study the conditions for the continuous-time plant model, the controller and the approximate discrete-time plant model, under which a discrete-time controller that input-to-state stabilizes an approximate discrete-time model of a nonlinear plant with disturbances, would also input-to-state stabilize (in an appropriate sense) the exact discrete-time plant model for a sufficiently small  $T$ .*
- *The condition of the input-to-state stability achieved for the exact discrete-time model, with respect to the input-to-state stability of the approximate model.* ■

The main results are stated in two main theorems. In the first result, systems with Lipschitz disturbances are considered. The result shows that under a set of certain conditions on the approximate model, the controller and the closed-loop approximate model, input-to-state stability of the closed-loop approximate model implies input-to-state stability with derivative for the closed-loop exact model, for the set of disturbances  $w, \dot{w} \in \mathcal{L}_\infty$ . The second result is concerned with measurable disturbance  $w \in \mathcal{L}_\infty$ , and the result obtained stated that input-to-state stability of the closed-loop approximate model implies input-to-state stability for the closed-loop exact model. The results are obtained in a semiglobal practical sense.

In proving these results, the Lyapunov method is used as the main tool. A problem that arises when implementing the framework to a design problem is that it requires knowledge of an explicit formulation of a family of Lyapunov functions for the family of systems concerned, which is in general a complicated task. There is no general procedure for constructing families of Lyapunov functions that are needed within our framework. Therefore, methods of constructing families of Lyapunov functions satisfying conditions of our main theorems is an important problem. We address this problem in the next two chapters of this thesis, where various partial constructions of families of Lyapunov functions are proposed.

## Chapter 6: Changing Supply Rates for Input-Output to State Stable Systems

In this chapter, partial constructions of families of Lyapunov functions are proposed. Results on changing supply rates for families of input-output to state stable

(IOSS) discrete-time nonlinear systems are presented as the tool for the family of Lyapunov functions construction. IOSS with measuring function is quite a general property of a nonlinear system, which covers various specific properties of nonlinear systems such as input-to-state stability (ISS), quasi input-to-state stability (qISS) and detectability. The main result of this chapter is a lemma on changing supply rates for IOSS systems. This result is a discrete-time version and generalization of earlier results presented in [3, 142]. The lemma is basically a result on scaling a family of IOSS Lyapunov functions with a scaling function  $\rho \in K_\infty$ . Using this lemma, two main theorems are derived. These results provide methods of combining two families of Lyapunov functions, none of which can be used to verify that the system has a certain property, into a new composite families of Lyapunov functions, from which the property of interest can be concluded.

Consider a parameterized family of discrete-time nonlinear systems of the following form:

$$\begin{aligned} x(k+1) &= F_T(x(k), u(k)) \\ y(k) &= h(x(k)) . \end{aligned} \tag{1.15}$$

Let  $V_{1T}$  and  $V_{2T}$  be respectively families of “Lyapunov like” functions of (1.15), each characterizing a certain property of the system (1.15). Under certain conditions, we can construct a new family of Lyapunov function using which we can verify a new property of the system that cannot be verified using either  $V_{1T}$  or  $V_{2T}$  only. The family of Lyapunov functions constructed using the proposed methods follows the following formulae. The first construction is as follows:

$$V_T = V_{1T} + \rho(V_{2T}) , \quad \rho \in \mathcal{K}_\infty ,$$

and the second construction is the following:

$$V_T = \rho_1(V_{1T}) + \rho_2(V_{2T}) , \quad \rho_1, \rho_2 \in \mathcal{K}_\infty .$$

Note that the first construction is simpler but it will be presented under different (often stronger) conditions than the second construction.

The results are stated for a parameterized family of discrete-time systems that naturally arises from results in Chapter 5, when an approximate discrete-time model

is used to design a controller for a sampled-data system. They generalize several existing results in a unified framework and also apply to various new applications. Several applications of our results are (i) a LaSalle criterion for input-to-state stability (ISS) of discrete-time systems; (ii) constructing families of ISS Lyapunov functions for time-varying discrete-time cascaded systems; (iii) testing the ISS of discrete-time systems using positive semidefinite Lyapunov functions; (iv) observer-based input-to-state stabilization of discrete-time systems. Later in Chapter 8, we will apply results from this chapter for a case study.

## Chapter 7: Lyapunov Based Small-Gain Theorem for Input-to-State Stability

The input-to-state stability (ISS) of a feedback interconnection of two discrete-time ISS systems satisfying an appropriate small gain condition is investigated via the Lyapunov method. This result, which is a discrete-time version of [60], is the first constructive result in the Lyapunov based small gain theorem for discrete-time systems. In this chapter we also propose a method of constructing a Lyapunov function for feedback connected nonlinear discrete-time systems satisfying a certain small gain condition. In particular, a family of ISS Lyapunov functions for the overall interconnected system is constructed from the families of ISS Lyapunov functions of the two subsystems.

The chapter focuses on a family of parameterized discrete-time interconnected systems in the following form:

$$\begin{aligned}\Sigma_1 : x_1(k+1) &= F_{1T}(x_1(k), x_2(k), u(k)) \\ \Sigma_2 : x_2(k+1) &= F_{2T}(x_1(k), x_2(k), u(k)) .\end{aligned}\tag{1.16}$$

Suppose that each subsystem is ISS with respect to its corresponding inputs, and  $V_{1T}$  and  $V_{2T}$  are respectively the family of ISS Lyapunov functions for each subsystem. The ISS Lyapunov function construction for the interconnected system (1.16) follows the following construction

$$V_T = \max\{V_{1T}, \rho(V_{2T})\} , \quad \rho \in \mathcal{K}_\infty ,\tag{1.17}$$

which can be regarded as a generalization of construction presented in Chapter 5.

## Chapter 8: Stabilization Problem for A Two-Link Manipulator

In this chapter, the theoretical results presented in the earlier chapters are applied to various engineering applications. A case study is conducted, considering a two-link manipulator system. Various controllers are designed for the manipulator, using the two discrete-time design techniques. The results on emulation design from Chapter 3 and the direct discrete-time design of Chapter 5 are applied to the closed-loop system. The advantages and disadvantages of the two design methods when dealing with this particular system are demonstrated.

In order to construct the Lyapunov function used in the design and to establish conclusions of the property of the manipulator system, a result of Chapter 6 on changing supply rates is used. The usefulness of the result is shown on this case study.

Systematic simulations are carried out using the Matlab Simulink, to observe the behaviors of the system under various settings of simulation parameters. The simulation results support our theoretical results obtained in the earlier chapters of this thesis.

## Chapter 9: Conclusions and Further Research

In this chapter, the summary of the obtained results and the conclusions are presented. We propose several directions in which results of this thesis can be further extended and improved.

The logical sequence of the chapters is shown in Figure 1.6. However, each chapter is mostly self contained, to facilitate reading and easy understanding of the results.



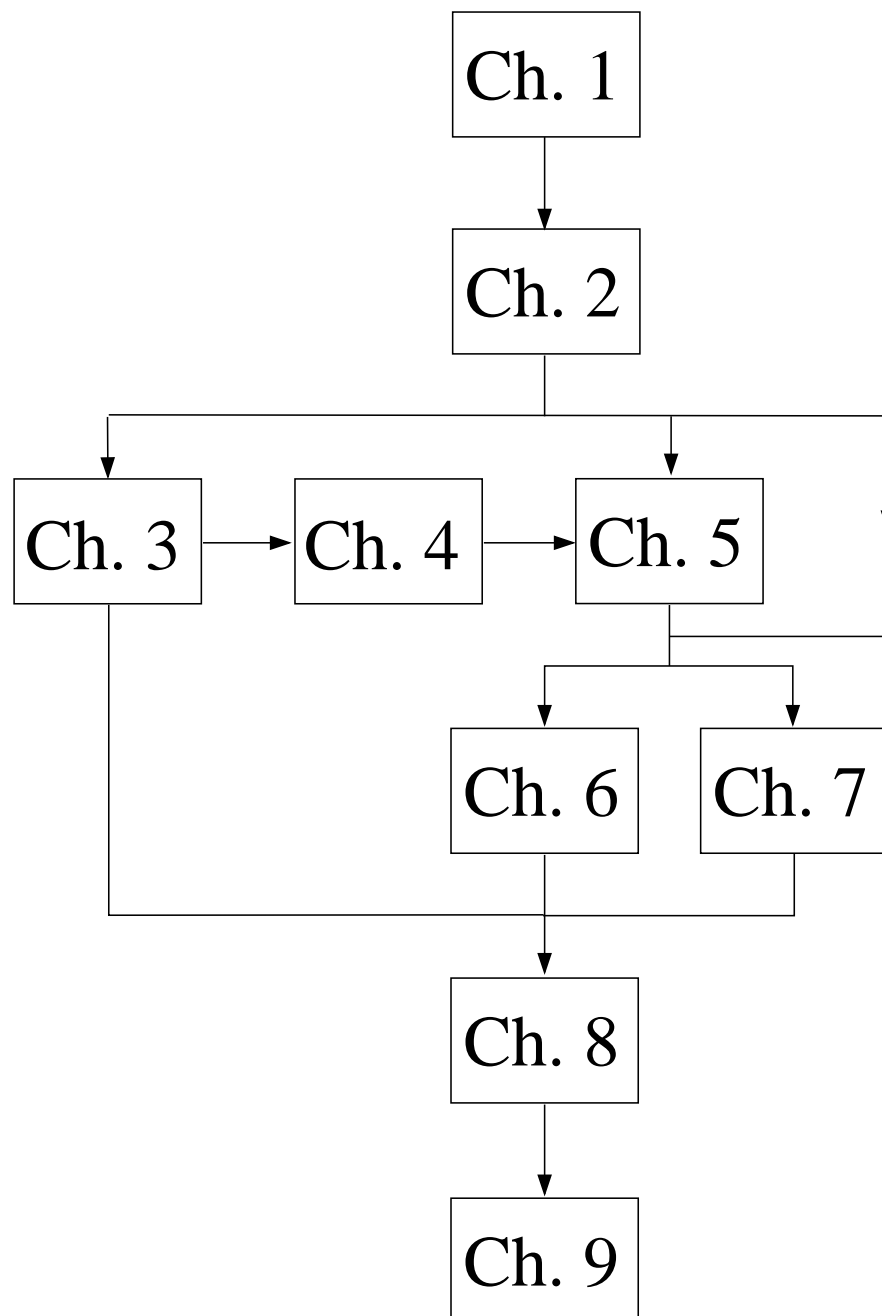


Figure 1.6: Logical sequence of the chapters.



## Chapter 2

# Preliminaries

This chapter provides technical preliminaries that help explain the main results and support the presentation of this thesis. Various important results from the literature that are required to prove results in this thesis are also cited. First, common notation, definitions and fundamental tools that will be used throughout the thesis are presented in Section 2.1. In Section 2.2 important aspects of Lyapunov characterization for input-to-state stability of nonlinear systems are discussed. Unless otherwise stated, the content of these sections follows closely the material presented in [70]. Numerical methods for discretization that are presented in Section 2.3 close the chapter.

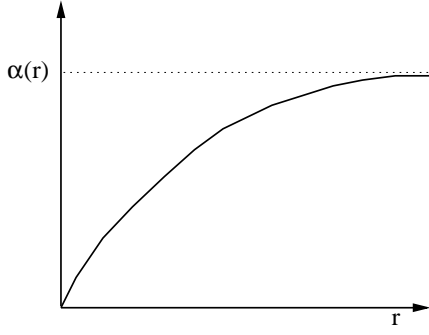
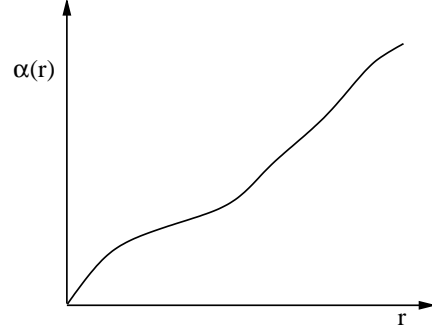
### 2.1 Notation, definitions and fundamental tools

The sets of real and natural numbers (including 0) are denoted respectively by  $\mathbb{R}$  and  $\mathbb{N}$ .  $\mathcal{SN}$  denotes the class of all smooth nondecreasing functions  $q : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ , which satisfy  $q(t) > 0$  for all  $t > 0$ . A function  $\gamma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is of class  $\mathcal{G}$  if it is continuous, nondecreasing and zero at zero. Given two functions  $\alpha(\cdot)$  and  $\gamma(\cdot)$ , we denote their composition and multiplication respectively as  $\alpha \circ \gamma(\cdot)$  and  $\alpha(\cdot) \cdot \gamma(\cdot)$ . Identity function is denoted by  $\text{Id}$ , that is  $\text{Id}(s) := s$ . We define the  $\mathcal{L}_\infty$  norm as  $\|x\|_\infty := \text{ess sup}_{t \geq 0} \|x(t)\|$ , and we say that  $x \in \mathcal{L}_\infty$  when  $\|x\|_\infty < \infty$ .

The classes of functions defined below are very important in characterizing stability properties of nonlinear systems.

**Definition 2.1.1** [70] A continuous function  $\alpha : [0, a) \rightarrow [0, \infty)$  is said to belong to class  $\mathcal{K}$  if it is strictly increasing and  $\alpha(0) = 0$ . It is said to belong to class  $\mathcal{K}_\infty$  if  $a = \infty$  and  $\alpha(r) \rightarrow \infty$  as  $r \rightarrow \infty$ . Functions of class  $\mathcal{K}_\infty$  are invertible. ■

**Definition 2.1.2** [70] A continuous function  $\beta : [0, a) \times [0, \infty) \rightarrow [0, \infty)$  is said to belong to class  $\mathcal{KL}$  if, for each fixed  $s$ , the mapping  $\beta(r, s)$  belongs to class  $\mathcal{K}$  with respect to  $r$  and, for each fixed  $r$ , the mapping  $\beta(r, s)$  is decreasing with respect to  $s$  and  $\beta(r, s) \rightarrow 0$  as  $s \rightarrow \infty$ . ■

Figure 2.1: Class- $\mathcal{K}$  function.Figure 2.2: Class- $\mathcal{K}_\infty$  function.

The following lemma states some important properties of class  $\mathcal{K}$  and  $\mathcal{KL}$  functions.

**Lemma 2.1.1** [70] Let  $\alpha_1(\cdot)$  and  $\alpha_2(\cdot)$  be class  $\mathcal{K}$  functions on  $[0, a)$ ,  $\alpha_3(\cdot)$  and  $\alpha_4(\cdot)$  be class  $\mathcal{K}_\infty$  functions, and  $\beta(\cdot, \cdot)$  be a class  $\mathcal{KL}$  function. Denote the inverse of  $\alpha_i(\cdot)$  by  $\alpha_i^{-1}(\cdot)$ . Then,

- $\alpha_1^{-1}(\cdot)$  is defined on  $[0, \alpha_1(a))$  and belongs to class  $\mathcal{K}$ .
- $\alpha_3^{-1}(\cdot)$  is defined on  $[0, \infty)$  and belongs to class  $\mathcal{K}_\infty$ .
- $\alpha_1 \circ \alpha_2$  belongs to class  $\mathcal{K}$ .
- $\alpha_3 \circ \alpha_4$  belongs to class  $\mathcal{K}_\infty$ .
- $\sigma(r, s) = \alpha_1(\beta(\alpha_2(r), s))$  belongs to class  $\mathcal{KL}$ . ■

Below, a review of some fundamental elements of mathematical analysis that are useful for this thesis is provided.

**Theorem 2.1.1 (Mean Value Theorem)** *Assume that  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuously differentiable at each point  $x$  of an open set  $S \subset \mathbb{R}^n$ . Let  $x$  and  $y$  be two points of  $S$  such that the line segment  $L(x, y) \subset S$ . Then there exists a point  $z$  of  $L(x, y)$  such that*

$$f(y) - f(x) = \left. \frac{\partial f}{\partial x} \right|_{x=z} (y - x) .$$

*The line segment  $L(x, y)$  joining two distinct points  $x$  and  $y$  in  $\mathbb{R}^n$  is*

$$L(x, y) = \{z | z = \theta x + (1 - \theta)y, 0 < \theta < 1\} .$$

■

**Lemma 2.1.2 (Gronwall-Bellman Inequality)** *Let  $\lambda : [a, b] \rightarrow \mathbb{R}$  be continuous and  $\mu : [a, b] \rightarrow \mathbb{R}$  be continuous and nonnegative. If a continuous function  $y : [a, b] \rightarrow \mathbb{R}$  satisfies*

$$y(t) \leq \lambda(t) + \int_a^t \mu(s)y(s)ds$$

*for all  $a \leq t \leq b$ , then on the same interval*

$$y(t) \leq \lambda(t) + \int_a^t \lambda(s)\mu(s) \exp \left[ \int_a^s \mu(\tau)d\tau \right] ds .$$

*In particular, if  $\lambda(t) \equiv \lambda$  is a constant, then*

$$y(t) \leq \lambda \exp \left[ \int_a^t \mu(\tau)d\tau \right] .$$

*If, in addition,  $\mu(t) \equiv \mu \geq 0$  is a constant, then*

$$y(t) \leq \lambda \exp[\mu(t - a)] .$$

■

## 2.2 Input-to-state stability of nonlinear systems

Input-to-state stability is a very important property of nonlinear systems with inputs. This property is one of the central notions studied in this thesis. In this section, a brief review of the input-to-state stability (ISS) property for nonlinear systems is presented. Before starting the description of the input-to-state stability, we first look

at the asymptotic stability notion for autonomous systems. Consider an autonomous system having the form

$$\dot{x} = f(x) , \quad (2.1)$$

where the state  $x \in \mathbb{R}^n$ ,  $f$  is locally Lipschitz and it satisfies  $f(0) = 0$ . The system (2.1) is asymptotically stable, if the following definition hold.

**Definition 2.2.1 (Asymptotic stability)** *System (2.1) is said to be asymptotically stable if there exist a class  $\mathcal{KL}$  function  $\beta(\cdot, \cdot)$  such that for any  $x_\circ \in \mathbb{R}^n$ , the solution  $x(t)$  of (2.1) with initial condition  $x(0) = x_\circ$  satisfies an estimate of the form*

$$|x(t)| \leq \beta(|x_\circ|, t) \quad (2.2)$$

for all  $t \geq 0$ . ■

Definition 2.2.1 suggests that for any initial condition  $x(0)$  the trajectory of the solution  $x(t)$  of (2.1) is asymptotically decayed to 0 as  $t \rightarrow \infty$ . Extending the analysis to the case when the system (2.1) is driven by an input, which is simply a bounded function of time, we no longer can expect that the state  $x(t)$  decays to zero as  $t \rightarrow \infty$ . Rather, we are interested in the case in which  $x(t)$  remains bounded, and the bound on the state can be expressed as a function of the bound on the input. In the special case in which the input tends to zero, we still expect that  $x(t)$  converges to zero as  $t \rightarrow \infty$ . These requirements all together lead to the notion of input-to-state stability.

The ISS property was first introduced by Sontag in his paper [139], as a natural generalization of asymptotic stability. ISS is an important property for systems with inputs, from which we can extract information of robustness of the systems. A system is input-to-state stable if it is globally asymptotically stable in the absence of (exogenous) inputs, and the effect of the inputs on this property is dependent to their magnitude. That is, small disturbances will have small effects on the inherent stability of an ISS system and vice versa.

In the following subsections, description of, respectively, the continuous-time and the discrete-time input-to-state stability are presented. More detail on the input-to-state stability property is presented in the textbooks [31, 57, 70, 74, 133] and research articles [60, 72, 139, 141, 144].

### 2.2.1 Continuous-time input-to-state stability

Consider a nonlinear system with input

$$\dot{x} = f(x, u) \quad (2.3)$$

with state  $x \in \mathbb{R}^n$ , input  $u \in \mathbb{R}^m$ ,  $f(0, 0) = 0$  and  $f(x, u)$  is locally Lipschitz.

The input vector  $u \in \mathbb{R}^m$  of the system may have two different interpretations; first, it can be an internal or external perturbation of the model and second, it can be a control, which is free to be designed, to satisfy specific performance criteria of the system. It can be any Lebesgue integrable function. The set of all such functions endowed with the supremum norm

$$\|u\|_\infty = \operatorname{ess\,sup}_{t \geq 0} \|u(t)\| \quad (2.4)$$

is denoted by  $\mathcal{L}_\infty^m$  or simply  $\mathcal{L}_\infty$ , if the dimension  $m$  is clear from the context. input-to-state stability for the system (2.3) is defined as follows.

**Definition 2.2.2 (Input-to-state stability)** *System (2.3) is said to be input-to-state stable if there exist a class  $\mathcal{KL}$  function  $\beta(\cdot, \cdot)$  and a class  $\mathcal{K}$  function  $\gamma(\cdot)$ , called a gain function, such that for any input  $u \in \mathcal{L}_\infty$  and any  $x_0 \in \mathbb{R}^n$ , the response  $x(t)$  of (2.3) in the initial state  $x(0) = x_0$  satisfies*

$$|x(t)| \leq \beta(|x_0|, t) + \gamma(\|u\|_\infty) \quad (2.5)$$

for all  $t \geq 0$ . ■

**Remark 2.2.1** *Since for any pair  $\beta > 0, \gamma > 0$ ,  $\max\{\beta, \gamma\} \leq \beta + \gamma \leq \max\{2\beta, 2\gamma\}$ , it is seen that an alternative way to say that a system is input-to-state stable is that there exists a class  $\mathcal{KL}$  function  $\beta(\cdot, \cdot)$  and a class  $\mathcal{K}$  function  $\gamma(\cdot)$  such that, for any input  $u(\cdot) \in \mathcal{L}_\infty$  and any  $x_0 \in \mathbb{R}^n$ , the response  $x(t)$  of (2.3) in the initial state  $x(0) = x_0$  satisfies*

$$|x(t)| \leq \max\{\beta(|x_0|, t), \gamma(\|u\|_\infty)\} \quad (2.6)$$

for all  $t \geq 0$ . ■

The input-to-state stability property for a given system can also be characterized using the extension of the well known criterion of Lyapunov for asymptotic stability. The Lyapunov like theorem that gives a sufficient condition for input-to-state stability is stated in the following.

**Theorem 2.2.1** *A continuous and differentiable function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is called an ISS Lyapunov function for system (2.3) if there exist class  $\mathcal{K}_\infty$  functions  $\underline{\alpha}(\cdot)$ ,  $\bar{\alpha}(\cdot)$ ,  $\alpha(\cdot)$ , and a class  $\mathcal{K}$  function  $\chi(\cdot)$  such that*

$$\underline{\alpha}(|x|) \leq V(x) \leq \bar{\alpha}(|x|) , \quad (2.7)$$

$$|x| \geq \chi(|u|) \Rightarrow \frac{\partial V}{\partial x} f(x, u) \leq -\alpha(|x|) , \quad (2.8)$$

for all  $x \in \mathbb{R}^n$  and all  $u \in \mathbb{R}^m$  ■

Similar to the converse Lyapunov theorem for stability, necessary condition is given by the converse Lyapunov theorem for ISS.

**Theorem 2.2.2** *System (2.3) is input-to-state stable if and only if there exists an ISS Lyapunov function for the system.* ■

**Remark 2.2.2** *In Theorem 2.2.1, instead of using (2.8), there is an alternative to check whether or not a function  $V(x)$  is an ISS Lyapunov function. We can replace the inequality (2.8) with the following. There exist functions  $\alpha \in \mathcal{K}_\infty$  and  $\sigma \in \mathcal{K}$  such that*

$$\frac{\partial V}{\partial x} f(x, u) \leq -\alpha(|x|) + \sigma(|u|) \quad (2.9)$$

for all  $x \in \mathbb{R}^n$  and all  $u \in \mathbb{R}^m$ . ■

When dealing with a sampled-data system, it is common to study properties of the system via the properties of its discrete-time model. Properties of the sampled-data system can then be deduced from knowledge about the properties of its discrete-time model, plus other extra conditions. Since this thesis is devoted to research on sampled-data nonlinear systems, it is proper to also discuss the discrete-time version of the input-to-state stability theory.



In the next two subsections, we will present the discrete-time dual of the ISS Lyapunov theory for discrete-time nonlinear systems, respectively for nonparameterized and parameterized systems. Important results regarding the relationship between stability properties of discrete-time systems and sampled-data systems are presented in [117].

### 2.2.2 Discrete-time input-to-state stability for nonparameterized systems

In this subsection, we present a discrete-time version of the ISS Lyapunov characterization for nonparameterized nonlinear system. This kind of representation is used when assuming the existence of the exact discrete-time model of a continuous-time system, or if the sampling periods is assumed constant. Some important results on discrete-time input-to-state stability are cited from [61].

Consider the following discrete-time nonlinear system:

$$x(k+1) = F(x(k), u(k)) \quad (2.10)$$

with state  $x \in \mathbb{R}^n$ , input  $u \in \mathbb{R}^m$ ,  $F(0, 0) = 0$  and  $F(x, u)$  is locally Lipschitz, which can be regarded as a discrete-time counterpart of the system (2.3). The input  $u$  belongs to a sequence space  $\ell_\infty$ , which norm is defined as

$$\|u\|_\infty = \sup_{k \in \mathbb{N}} \|u(k)\| \quad (2.11)$$

**Definition 2.2.3** *The system (2.10) is input-to-state stable if there exist functions  $\beta \in \mathcal{KL}$  and  $\gamma \in \mathcal{K}$  such that for each input  $u \in \ell_\infty$  and each  $x_o \in \mathbb{R}^n$ , it holds that*

$$\|x(k, x_o, u)\| \leq \beta(\|x_o\|, k) + \gamma(\|u\|_\infty) \quad (2.12)$$

for each  $k \in \mathbb{N}$ . ■

The Lyapunov characterization of discrete-time input-to-state stability property for nonparameterized discrete-time systems, which is stated next, is very similar to the characterization of its continuous-time counterpart.

**Theorem 2.2.3** *A continuous and differentiable function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is called an ISS Lyapunov function for system (2.10) if there exist class  $\mathcal{K}_\infty$  functions  $\underline{\alpha}(\cdot)$ ,  $\overline{\alpha}(\cdot)$ ,  $\alpha(\cdot)$ , and a class  $\mathcal{K}$  function  $\chi(\cdot)$  such that*

$$\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|) , \quad (2.13)$$

$$|x| \geq \chi(|u|) \Rightarrow V(F(x, u)) - V(x) \leq -\alpha(|x|) , \quad (2.14)$$

for all  $x \in \mathbb{R}^n$  and all  $u \in \mathbb{R}^m$  ■

The alternative expression for the second inequality of the ISS Lyapunov characterization is also valid for discrete-time systems.

**Remark 2.2.3** *As in the case of continuous-time, the second property in the definition of discrete-time ISS is equivalent to the following property. There exist some  $\alpha \in \mathcal{K}_\infty$  and some  $\sigma \in \mathcal{K}$  such that*

$$V(F(x, u)) - V(x) \leq -\alpha(|x|) + \sigma(|u|) \quad (2.15)$$

for all  $x \in \mathbb{R}^n$  and all  $u \in \mathbb{R}^m$ . ■

### 2.2.3 Discrete-time input-to-state stability for parameterized systems

When discretizing a continuous-time model of a system to obtain its discrete-time model, we will need to specify the sampling period  $T$ . Generically, the sampling period can be chosen later, as a parameter of the discrete-time model. Therefore, it is natural to write a discrete-time model of a system as a  $T$  parameterized discrete-time model. In this subsection, we extend the ISS theory for nonlinear discrete-time systems presented in the previous subsection, by considering a family of  $T$  parameterized discrete-time systems.

Suppose we discretize the continuous-time model (2.3), and we obtain the following family of parameterized discrete-time models:

$$x(k+1) = F_T(x(k), u(k)) . \quad (2.16)$$

The discrete-time model  $F_T$  is called an **exact discrete-time model** if it is obtained as the exact solution of initial value problem of the continuous-time model over sampling interval and it is called an **approximate discrete-time model** if it is obtained via numerical approximation. In Section 2.3, we will discuss that exact discrete-time model is not available for nonlinear systems and we use numerical algorithm to obtain the approximation of the exact model. Therefore, in general discretization is in fact an approximation, which will obviously involves inaccuracy and this will lead to discrepancies between the exact model and the approximate model. Because of that, sampled-data systems will not be able to attain identical properties as what their continuous-time counterparts have. Indeed, parameterized discrete-time systems (6.1) commonly arise when an approximate discrete-time model is used for designing a digital controller for a nonlinear sampled-data system. Nonparameterized discrete-time systems are a special case of (6.1) when  $T$  is constant (for instance  $T = 1$ ).

If, for continuous-time systems, the properties such as asymptotic stability, input-to-state stability and dissipativity can be achieved for the whole state space and the whole input space (in a global sense), it is in general not the case for sampled-data systems. Sampling might destroy global properties of the systems, so that the properties hold in a weaker (semiglobal practical) sense. Indeed, semiglobal practical property is common in sampled-data systems. For simple illustration, we consider a parameterized family of discrete-time nonlinear autonomous systems

$$x(k+1) = F_T(x(k)) \quad (2.17)$$

as a discrete-time model of the system (2.1). Semiglobal practical asymptotic stability for the system (2.17) is defined as follow.

**Definition 2.2.4** *The family of systems (2.17) is semiglobally practically asymptotically stable if there exists  $\beta \in \mathcal{KL}$  such that for any strictly positive real numbers  $(\Delta, \delta)$  there exists  $T^* > 0$  such that for all  $T \in (0, T^*)$ , all initial states  $x(0) = x_\circ$  with  $|x_\circ| \leq \Delta$ , the solutions of the system satisfy*

$$|x(k)| \leq \beta(|x_\circ|, kT) + \delta, \quad \forall k \in \mathbb{N}. \quad (2.18)$$

■

Indeed, sampling periods  $T$  that guarantee the semiglobal practical asymptotic stability for a family of parameterized systems is dependent on the large outer ball of the initial states with radius  $\Delta$  and the small inner ball with radius  $\delta$ , to which the trajectory of the solutions converge.

The domain of attraction (DOA) for systems with semiglobal practical asymptotic stability property is shown in Figure 2.3. To enlarge the radius of the outer ball and to shrink the radius of the inner ball for obtaining a larger DOA will lead to smaller  $T$  or the requirement for faster sampling. In other words, setting a larger attractivity boundary is possible by using a smaller time sampling. More detailed description of the semiglobal practical stability notion is presented in [20], and more about stability and attractivity of nonlinear systems is presented in [46].

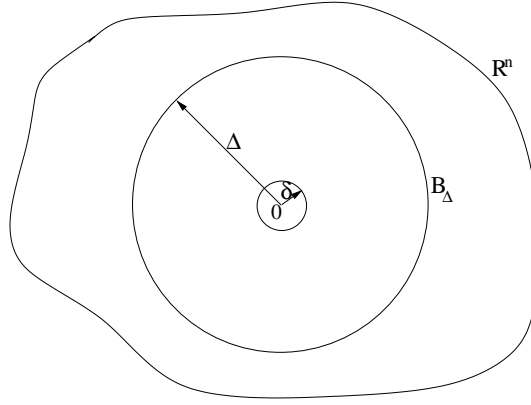


Figure 2.3: DOA for semiglobal practical stability property

Semiglobal practical asymptotic stability property is extended to semiglobal practical input-to-state stability for nonlinear systems with inputs. A Lyapunov characterization of semiglobal practical input-to-state stability (SP-ISS) for the family of parameterized discrete-time systems with inputs (2.16) is defined based on the sets of initial states and inputs as follows.

**Definition 2.2.5** *A continuously differentiable function  $V_T : \mathbb{R}^n \rightarrow \mathbb{R}$  is called a semiglobal practical ISS (SP-ISS) Lyapunov function for the system (2.16) if there exist class  $\mathcal{K}_\infty$  functions  $\underline{\alpha}(\cdot)$ ,  $\overline{\alpha}(\cdot)$ ,  $\alpha(\cdot)$ , and some  $\sigma \in \mathcal{K}$  such that and for any quadruple of strictly positive numbers  $(\Delta_x, \Delta_u, \nu, \mu)$ , there exists  $T^* > 0$  such that for*

all  $T \in (0, T^*)$  and for all  $x \leq \Delta_x$ ,  $u \leq \Delta_u$  and  $T \in (0, T^*)$  the following holds:

$$\underline{\alpha}(|x|) \leq V_T(x) \leq \overline{\alpha}(|x|) , \quad (2.19)$$

$$V_T(F_T(x, u)) - V_T(x) \leq -T\alpha(|x|) + T\sigma(|u|) + T\nu . \quad (2.20)$$

Moreover, the system (2.16) is a semiglobally practically ISS system. ■

**Remark 2.2.4** Definition 2.2.5 of semiglobal practical input-to-state stability property for discrete-time parameterized system presented in this subsection is not standard in literature. Indeed, it is one of the new contributions of this thesis. A trajectory-based definition of SP-ISS can be stated similarly to Definition 2.2.4 where the inputs are affecting the trajectory of solutions. We can also state a Lyapunov characterization of SP-ISS for a family of parameterized systems in a similar way as in Definition 2.2.3. However, it require imposing an extra condition to (2.20) to obtain an equivalent property. We present more detail about this property and its definitions in the preliminary section of the chapters in Part II of this thesis, as we will need to use more particular definition and characterization of SP-ISS in those chapters to prove the results. ■

**Remark 2.2.5** The input-to-state stability theory presented in this subsection can also be presented for a more general class of parameterized systems

$$x(k+1) = F_{T,h}(x(k), u(k)) , \quad (2.21)$$

which naturally arise when a family of approximate discrete-time models of the continuous-time plant is generated by integrating continuous-time plant dynamics over one sampling interval of length  $T > 0$  using a numerical integration scheme with integration period  $h > 0$ . In particular, the results that we stated can be regarded as a special case of this more general situation when  $T = h$  (see [114] for more details). ■

More variants of the Lyapunov's characterizations for various stability notions for nonlinear systems are also defined as some extensions of the Lyapunov stability theorem (see for instance [4, 5, 41, 61]). Some of the notions used in the main results of this thesis will be defined and explained in more detail in the preliminary section of each corresponding chapter.

### 2.3 Numerical approximation and discretization

It has been mentioned in Chapter 1 that numerical methods play an important role in applying the approximate model based direct discrete-time design. Since the focus of Part II of this thesis is on the approximate model based direct discrete-time design, it is appropriate to provide an overview of numerical methods as a preliminary. In this section, several numerical approximation techniques that can be used to obtain a discrete-time representation for continuous-time plants or controllers are presented. There are two approaches to perform discretization with respect to the discretizing space: time discretization and state discretization. In this section, we consider only the time discretization, which is the approach that will also be used in this thesis. Techniques available for this approach are divided into one step and multi step approximations, while we limit our description only to the techniques belonging to the former. Hence, we consider techniques that belong to time, one step discretization with a single step size  $T$ , which henceforth will be referred to simply as a discretization.

#### 2.3.1 System with piecewise constant inputs

Consider affinely controlled nonlinear system

$$\dot{x} = f(x) + g(x)u, \quad (2.22)$$

where  $x \in \mathbb{R}^n$  and  $u \in \mathbb{R}^m$  are respectively the state and the input of the system, and  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $g : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$  are differentiable sufficiently many times. Suppose that the control  $u(t)$  is constant during sampling intervals  $u(t) = u(kT)$ ,  $\forall t \in [kT, (k+1)T)$ . Consider the initial value problem

$$\dot{x} = f(x(k)) + g(x(k))u(k), \quad x_0 = x(k), \quad (2.23)$$

where  $x(k)$  and  $u(k)$  are given. The solution of the initial value problem (2.23) (if it exists) is  $x(t)$ , and the exact discrete-time model of (2.22) is

$$x(k+1) = F_T^e(x(k), u(k)). \quad (2.24)$$

Based on Taylor series expansion, we can write

$$F_T^e = x(k) + x^{(1)}(k)T + x^{(2)}(k)\frac{T^2}{2} + x^{(3)}(k)\frac{T^3}{3!} + \cdots + x^{(n)}(k)\frac{T^n}{n!} + \cdots \quad (2.25)$$

which is an infinite series of  $T$ , where  $x^{(n)}(k)$  is the  $n$ -th derivative of  $x(t)$ , evaluated at time  $kT$ :

$$\begin{aligned} x^{(1)}(k) &:= \dot{x}(t)|_{t=kT} = f(x(k)) + g(x(k))u(k) \\ x^{(2)}(k) &:= \ddot{x}(t)|_{t=kT} = \left[ \frac{\partial f}{\partial x}(x(k)) + \frac{\partial g}{\partial x}(x(k))u(k) \right] [f(x(k)) + g(x(k))u(k)] \\ &\vdots \end{aligned}$$

A finite truncation up to the  $n$ -th order term of the series leads to the approximate model

$$F_T^a = x(k) + x^{(1)}(k)T + x^{(2)}(k)\frac{T^2}{2} + x^{(3)}(k)\frac{T^3}{3!} + \cdots + x^{(n)}(k)\frac{T^n}{n!}, \quad (2.26)$$

which is written as

$$x(k+1) = F_T^a(x(k), u(k)). \quad (2.27)$$

The simplest example for the approximate model is the Euler approximate model, which is obtained by considering only up to the first order term of the series. For the system (2.23), the Euler model follows the form:

$$F_T^a(x(k), u(k)) = x(k) + T(f(x(k)) + g(x(k))u(k)). \quad (2.28)$$

Several numerical expansion presented in for example [95, 151] propose formulae of the exact discrete-time model  $F_T^e$ , and provide methods for generating approximate models  $F_T^a$ . A crucial property that has to hold for the approximate model to be a good approximation of the exact model is *consistency*. The property is measured by the discrepancy between the exact model and the approximate model. One step consistency property is when the exact and the approximate satisfies

$$|F_T^e(x(k), u(k)) - F_T^a(x(k), u(k))| \leq T\rho(T), \quad \rho \in \mathcal{K}. \quad (2.29)$$

on compact sets and for small sampling period  $T$ .

### 2.3.2 System with Lebesgue measurable inputs

When dealing with systems with measurable disturbances, the schemes we have presented in the previous subsection usually fail to maintain their asserted order [42, 43].

In this subsection, we extend our scope to system with measurable disturbance. We consider affinely controlled nonlinear systems

$$\dot{x} = f(x) + g(x)u + g_1(x)w , \quad (2.30)$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$  and  $w \in \mathbb{R}^d$  are respectively the state and the piecewise constant input, and measurable input of the system, and  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $g : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$  and  $g_1 : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}$  are differentiable sufficiently many times.

Suppose that we discretize the system (2.30) by letting the control  $u(t)$  constant during sampling intervals  $u(t) = u(kT)$ ,  $\forall t \in [kT, (k+1)T)$ , and  $w(t)$  be a piece of function  $w_T[k]$  in the  $k$ -th sampling interval  $[kT, (k+1)T)$  ( $w_T[k] := \{w(t) : t \in [kT, (k+1)T)\}$  where  $k \in \mathbb{N}$  and  $T > 0$ ). Consider the initial value problem

$$\dot{x} = f(x(k)) + g(x(k))u(k) + g_1(x(k))w(t) , \quad x_o = x(k) , \quad (2.31)$$

where  $x(k)$ ,  $u(k)$  and  $w_T[k]$  are given. The solution of the initial value problem (2.31) (if it exists) is  $x(t)$ , and the exact discrete-time model of (2.30) is

$$x(k+1) = F_T^e(x(k), u(k), w_T[k]) . \quad (2.32)$$

Several numerical expansion schemes based on Fliess expansion, Volterra series [56], Ferretti expansion [27], and some stochastic numerical expansions [40, 42, 43, 71] propose formulae of the exact discrete-time model  $F_T^e$ , and provide methods for generating approximate models of the form:

$$x(k+1) = F_T^a(x(k), u(k), w_T[k]) \quad (2.33)$$

can be found in the cited references. For example, the following Euler discrete-time approximate model is obtained using [40, 42, 43]:

$$x(k+1) = x(k) + T(f(x(k)) + g(x(k))u(k)) + \int_0^T g_1(x(k))w(\tau)d\tau . \quad (2.34)$$

One step consistency between the exact and the approximate model is then following the form

$$|F_T^e(x(k), u(k), w_T[k]) - F_T^a(x(k), u(k), w_T[k])| \leq T\rho(T), \quad \rho \in \mathcal{K} . \quad (2.35)$$



Using numerical schemes proposed in [27, 40, 42, 43, 71], we are not only able to solve approximation problem for system with measurable disturbances, but also to solve more general problems in higher order stochastic differential systems [42]. These schemes are useful for applying our results presented in Chapter 5, where we use the family of approximate plant models for controller design.



## Part I

# Design and Analysis using Emulation



## Chapter 3

# Nonlinear Controller Emulation

### 3.1 Introduction

This chapter presents a generalization and unification of the known results on emulation design available in the literature, by considering preservation of general dissipation inequalities under sampling in the context of emulation design of dynamic state feedback controllers. The material written in this chapter is based on the results which have appeared in [82].

Emulation is a well-established method to design digital controllers for continuous-time plants (see, for instance [8, 19, 30]). As has been explained in Chapter 1, the first step in the emulation method is to design a continuous-time controller for a continuous-time plant using a certain known continuous-time design method; sampling is completely ignored at this stage. Then, in the second step, the continuous-time controller is discretized and implemented using sample and hold devices. Digital controllers designed using emulation have been proved to perform well for a number of control problems under sufficiently fast sampling. The following problems have been addressed in the literature: stability for linear [18] and nonlinear [9, 98, 124, 128, 163] plants,  $\mathcal{L}_p$  stability of linear systems [18], input-to-state stability (ISS) of nonlinear systems [139, 154] and adaptive stabilization of nonlinear systems [45]. Also, ideas similar to emulation were exploited in [125], where the dissipativity property of continuous-time nonlinear systems is investigated using discrete observation of its storage function. For

more details on dissipation inequalities see [49, 87, 99, 112, 116, 139, 154, 160] and references therein.

The nonlinear plants and dynamic state feedback controllers that we consider to obtain our results only need to satisfy a local Lipschitz condition. Static state feedback and open-loop results follow as corollaries from the dynamic state feedback case. Moreover, the dissipation inequality we consider is rather general and its special cases are dissipation inequalities used to investigate stability,  $L_p$  stability, passivity, input-to-state stability, integral input-to-state stability, forward completeness, detectability, etc. (see for instance [49, 141, 160]). Applications of our results to investigation of input-to-state stability and passivity properties are presented in this chapter to illustrate the generality of our approach.

Since, in general, the discretization of a nonlinear dynamic controller can not be computed exactly, we use an approximate discrete-time model of the controller. In order to obtain a valid approximate model, the discretization of the dynamic controller should be carried out carefully. We introduce properties that the discretized controller should satisfy in order to preserve the dissipation inequality under sampling. These properties, which are called one-step strong and weak consistencies, are specified in Definitions 3.2.2 and 3.2.3 and sufficient conditions for these properties to hold are given in Lemmas 3.2.1 and 3.2.2 respectively.

In our main results we explore two types of dissipation inequalities for the discrete-time model of the closed-loop sampled-data system: the weak and strong form. In Definition 3.2.4, 3.2.5 and 3.2.6, we introduce properties associated with the weak and strong dissipation inequalities. A relationship among the properties is given in Theorem 3.2.1. For the weak dissipation result to hold, the discretized controller needs to satisfy the one-step weak consistency condition (Definition 3.2.2) and the disturbances need to be uniformly Lipschitz (Theorem 3.3.1). It is shown in Proposition 3.3.2 that uniformly Lipschitz disturbances can be obtained by filtering bounded measurable disturbances through a strictly proper input-to-state stable (ISS) filter. The strong dissipation inequality holds if the discretized controller satisfies the one-step strong consistency condition (Definition 3.2.3) and in this case disturbances are allowed to

be only measurable (see Theorem 3.3.3). In general, strong and weak dissipation inequalities do not imply each other and this is illustrated by Example 3.3.1. Similar results then follow for the static feedback and the open loop cases. The generality of our approach is illustrated by two applications of our results to investigation of input-to-state stability of sampled-data systems with emulated controllers and results on preservation of passivity property under sampling. A special case of the input-to-state stability results is a result on preservation of stability under sampling, which is proved for a much general situation than any of the results in the literature that we are aware of (see [9, 18, 124, 163]).

Our main results are semiglobal and practical in nature and their important feature is that the required sampling period can be computed using our method, although it may be conservative (smaller than necessary) which is a consequence of the conservative Lipschitz bounds that are used in the proofs. This is a common problem in numerical analysis literature [151] and emulation design for sampled-data systems [45, 163].

The organization of this chapter is as follows. In Section 3.2 we present preliminaries. Main results are stated and discussed in Section 3.3, while proofs of the main results and technical lemmas are presented in Section 3.4. The application of the main results are presented in Section 3.5, and finally, the conclusion is given in the last section.

## 3.2 Preliminaries

For a given function  $d(\cdot)$ , we use the following notation  $d[t_1, t_2] := \{d(t) : t \in [t_1, t_2]\}$ . If  $t_1 = kT, t_2 = (k+1)T$ , we use the shorter notation  $d[k]$ , and take the norm of  $d[k]$  to be the supremum of  $d(\cdot)$  over  $[kT, (k+1)T]$ , that is  $\|d[k]\|_\infty = \text{ess sup}_{\tau \in [kT, (k+1)T]} |d(\tau)|$ .

Consider the continuous-time nonlinear plant model:

$$\dot{x} = f(x, u, d_c, d_s) \tag{3.1}$$

$$y = h(x, u, d_c, d_s) , \tag{3.2}$$

with the dynamic state feedback controller:

$$\begin{aligned}\dot{z} &= g(x, z, d_c, d_s) \\ u &= u(x, z, d_c, d_s),\end{aligned}\tag{3.3}$$

where  $x \in \mathbb{R}^{n_x}$ ,  $z \in \mathbb{R}^{n_z}$ ,  $u \in \mathbb{R}^m$  and  $y \in \mathbb{R}^p$  are respectively the state of the plant, state of the controller, control input and output of the plant.  $d_c \in \mathbb{R}^{n_c}$  and  $d_s \in \mathbb{R}^{n_s}$  are respectively “continuous” and “sampled” disturbance inputs to the system. The reason for distinguishing between  $d_c$  and  $d_s$  is because the role of each is different in obtaining the discretization of the controller.  $d_c$  is assumed to be a Lebesgue measurable function, while  $d_s$  is assumed to be constant during sampling intervals, when computing the discrete-time model of the controller. For instance,  $d_c$  can be a measurement noise modeled as a Lebesgue measurable function, while  $d_s$  may model the computation errors due to finite word length effects in the digital controller. Moreover, separate investigation of  $d_c$  and  $d_s$  yields different conditions, which explain when it is justified to *assume* (when discretizing the controller) that all disturbances are constant during sampling intervals.

It is assumed that  $f$ ,  $g$ ,  $h$  and  $u$  are locally Lipschitz. We also assume that  $f(0, 0, 0, 0) = 0$ ,  $g(0, 0, 0, 0) = 0$ ,  $h(0, 0, 0, 0) = 0$  and  $u(0, 0, 0, 0) = 0$ . The controller (3.3) covers the case of dynamic output feedback:

$$\begin{aligned}\dot{z} &= \tilde{g}(y, z, d_c, d_s) =: g(x, z, d_c, d_s) \\ u &= \tilde{u}(y, z, d_c, d_s) =: u(x, z, d_c, d_s),\end{aligned}\tag{3.4}$$

where we assume that the feedback system (3.1), (3.2), (3.3) is Lipschitz well posed, that is the equations:

$$\begin{aligned}y &= h(x, u(y, z, d_c, d_s), d_c, d_s) \\ u &= \tilde{u}(h(x, u, d_c, d_s), z, d_c, d_s)\end{aligned}$$

have unique solutions  $y \in \mathbb{R}^p$ ,  $u \in \mathbb{R}^m$  so that (3.1), (3.2) and (3.4) can be written in the form  $\dot{\eta} = \mathcal{F}(\eta, d_c, d_s)$ ,  $\psi = \mathcal{H}(\eta, d_c, d_s)$  where  $\eta := (x^T \ z^T)^T$ ,  $\psi := (y^T \ u^T)^T$  and  $\mathcal{F}$  and  $\mathcal{H}$  are locally Lipschitz.

The following definitions are used in the sequel.



**Definition 3.2.1** *The system (3.1), (3.2), (3.3) is said to be  $(V, w)$ -dissipative if there exist a continuously differentiable function  $V : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \rightarrow \mathbb{R}$ , called the storage function, and a continuous function  $w : \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_s} \rightarrow \mathbb{R}$ , called the dissipation rate, such that for all  $x \in \mathbb{R}^{n_x}, z \in \mathbb{R}^{n_z}, d_c \in \mathbb{R}^{n_c}, d_s \in \mathbb{R}^{n_s}$  the following holds:*

$$\frac{\partial V}{\partial x} f(x, u(x, z, d_c, d_s), d_c, d_s) + \frac{\partial V}{\partial z} g(x, z, d_c, d_s) \leq w(x, z, d_c, d_s) . \quad (3.5)$$

■

**Remark 3.2.1** *Dissipation inequality is sometimes expressed in terms of an integral, the result of integrating (3.5) along the solutions (see, for instance [160]), which takes the following form:*

$$V(x(t), z(t)) - V(x(t_o), z(t_o)) \leq \int_{t_o}^t w(x(\tau), z(\tau), d_c(\tau), d_s(\tau)) d\tau . \quad (3.6)$$

*In this form, no differentiability assumptions are imposed on  $V$  (see, for instance, [160]). We will concentrate mainly on the differential form of dissipation inequalities in this work, but the same proof technique can be used to prove our main results using the integral form (3.6). We also note that it is usually assumed in the literature that  $V$  is positive semidefinite or positive definite. We do not use these conditions on  $V$  in Definition 3.2.1 since they are not needed for the proofs.*

■

**Emulation procedure:** Suppose that, as a first step in the emulation design, we designed a controller (3.3) for the plant (3.1), (3.2) in the continuous-time domain, so that the closed-loop continuous-time system is  $(V, w)$ -dissipative.

As a second step, we discretize the controller and implement it using sample and zero-order-hold devices. The discretization of the controller is carried out as follows. First, we consider an auxiliary system where the state measurements are assumed to be constant during sampling intervals  $x(t) = x(kT) =: x(k)$  and  $d_s(t) = d_s(kT) =: d_s(k)$  for all  $t \in [kT, (k+1)T)$  in the differential equation (3.3), where  $T > 0$  is the sampling period. Consider the following initial value problem:

$$\dot{z}(t) = g(x(k), z(t), d_c(t), d_s(k)) , \quad z_o = z(k) \quad (3.7)$$

where  $x(k), z(k), d_c[k], d_s(k)$  are given. Denote the solution of the initial value problem (3.7) as  $z(t)$ , and then we obtain the exact discretization of the controller (3.3) (see also [18]):

$$\begin{aligned} z(k+1) &= z(k) + \int_{kT}^{(k+1)T} g(x(k), z(\tau), d_c(\tau), d_s(k)) d\tau \\ &=: G_T^e(x(k), z(k), d_c[k], d_s(k)) \\ u(k) &= u(x(k), z(k), d_c(k), d_s(k)) . \end{aligned} \quad (3.8)$$

Note that in general the discretization (3.8) can not be implemented directly since  $G_T^e$  in (3.8) is usually impossible to compute exactly (since we need to solve the nonlinear initial value problem (3.7) explicitly over one sampling interval), so we need to use instead an approximate discrete-time model of the controller:

$$\begin{aligned} z(k+1) &= G_T^a(x(k), z(k), d_c(k), d_s(k)) \\ u(k) &= u(x(k), z(k), d_c(k), d_s(k)) , \end{aligned} \quad (3.9)$$

which is obtained from (3.7) using one of the numerical integration methods (e.g. Runge-Kutta). For instance, if we use the forward Euler method, then we obtain  $G_T^a(x, z, d_c, d_s) := x + Tg(x, z, d_c, d_s)$ . It is obvious that in general we will have to use a sufficiently small sampling period  $T$ , since the approximate discrete-time model (3.9) is usually a good approximation of the exact discrete-time model (3.8) typically only for small  $T$ .

The sampled-data closed-loop system consists of the continuous-time plant (3.1), (3.2) and the controller (3.9), which is between a sample and zero-order-hold device. In the sequel, we use the discrete-time model of this sampled-data system, which consists of (3.9) and the exact discrete-time model of the plant, which is obtained as follows. We assume that  $u(t) = u(kT) =: u(k)$ ,  $d_s(t) = d_s(kT) =: d_s(k)$  for all  $t \in [kT, (k+1)T]$  and consider the initial value problem

$$\dot{x}(t) = f(x(t), u(k), d_c(t), d_s(k)) , \quad x_o = x(k) \quad (3.10)$$

where  $x(k), u(k), d_c[k]$  and  $d_s(k)$  are given. The output  $y$  is measured only at sampling instants  $kT$ ,  $k \geq 0$ . Denote the solution of the initial value problem (3.10) as  $x(t)$ .

Then the exact discrete-time model of the plant can be written as:

$$\begin{aligned}
 x(k+1) &= x(k) + \int_{kT}^{(k+1)T} f(x(\tau), u(k), d_c(\tau), d_s(k)) d\tau \\
 &=: F_T(x(k), u(k), d_c[k], d_s(k)) \\
 y(k) &= h(x(k), u(k), d_c(k), d_s(k)) .
 \end{aligned} \tag{3.11}$$

The discrete-time model of the sampled-data closed-loop system consists of (3.9) and (3.11).

The sampling period  $T$  is assumed to be a design parameter which can be arbitrarily assigned. In practice, the sampling period  $T$  is fixed and our results could be used to determine if it is suitably small. We emphasize that  $F_T$  in (3.11) is not known in most cases, and  $G_T^e$  in (3.8) can not be computed exactly, so we need to use  $G_T^a$  in (3.9) instead. Similarly to [116] we will think of  $F_T$ ,  $G_T^e$  and  $G_T^a$  as being defined globally for all small  $T$ , even though the initial value problem (3.10) and (3.7) may exhibit finite escape times. We do this by defining  $F_T$  and  $G_T^e$  arbitrarily for  $(x(k), z(k), d_c[k], d_s(k))$  corresponding to the finite escapes and noting that such points correspond only to states and inputs of arbitrarily large norm as  $T \rightarrow 0$ , since  $f$  and  $g$  are assumed locally Lipschitz (and hence locally bounded). So, the behavior of  $F_T$  and  $G_T^e$  will reflect the behavior of (3.10) and (3.7) respectively, as long as  $(x(k), z(k), d_c[k], d_s(k))$  remain bounded with a bound that is allowed to grow as  $T \rightarrow 0$ . This is consistent with our main results that guarantee semiglobal dissipativity properties in the sampling period, that is as  $T \rightarrow 0$  the set of states and inputs for which a dissipation inequality for the discrete-time model (3.9), (3.11) holds is guaranteed to contain an arbitrary large neighborhood of the origin.

In order to prove our main results, we need to guarantee that the mismatch between the exact discrete-time model of the controller (3.8) and its approximation (3.9) is small in some sense. We define two consistency properties that are used to limit the mismatch. Different forms of the consistency property are used in numerical analysis literature (see Definition 2 [109], Definition 1 [116] and Definition 3.4.2 [151]).

**Definition 3.2.2 (One-step weak consistency)** *The family  $G_T^a$  is said to be one-step weakly consistent with  $G_T^e$  if given any quintuple of strictly positive real numbers*

$(\Delta_x, \Delta_z, \Delta_{d_c}, \Delta_{\dot{d}_c}, \Delta_{d_s})$ , there exist a function  $\rho \in \mathcal{K}_\infty$  and  $T^* > 0$  such that, for all  $T \in (0, T^*)$ ,  $|x| \leq \Delta_x$ ,  $|z| \leq \Delta_z$ ,  $|d_s| \leq \Delta_{d_s}$  and functions  $d_c(\cdot)$  that are uniformly Lipschitz and satisfy  $\|d_c[0]\|_\infty \leq \Delta_{d_c}$  and  $\|\dot{d}_c[0]\|_\infty \leq \Delta_{\dot{d}_c}$ , we have

$$|G_T^e - G_T^a| \leq T\rho(T) . \quad (3.12)$$

■

A sufficient condition for one-step weak consistency is the following (the proof is given in Section 3.4):

**Lemma 3.2.1** *Consider  $G_T^e$  and  $G_T^a$  of the controller (3.3). If  $G_T^a$  is one-step weakly consistent with  $G_T^{Euler}$ , where  $G_T^{Euler} := z + Tg(x, z, d_c, d_s)$ , then  $G_T^a$  is one-step weakly consistent with  $G_T^e$ .* ■

In the following, we consider a more specific class of controllers that have the following form:

$$\begin{aligned} \dot{z} &= g(x, z, d_s) \\ u &= u(x, z, d_s) . \end{aligned} \quad (3.13)$$

We assume that  $g$  and  $u$  are locally Lipschitz,  $g(0, 0, 0) = 0$  and  $u(0, 0, 0) = 0$ . In a similar manner as for controller (3.3), we define the exact discrete-time model of the controller (3.13) as:

$$\begin{aligned} z(k+1) &= z(k) + \int_{kT}^{(k+1)T} g(x(k), z(\tau), d_s(k)) d\tau =: G_T^e(x(k), z(k), d_s(k)) \\ u(k) &= u(x(k), z(k), d_s(k)) , \end{aligned} \quad (3.14)$$

and its approximate discrete-time model:

$$\begin{aligned} z(k+1) &= G_T^a(x(k), z(k), d_s(k)) \\ u(k) &= u(x(k), z(k), d_s(k)) . \end{aligned} \quad (3.15)$$

**Definition 3.2.3 (One-step strong consistency)** *The family  $G_T^a$  is said to be one-step strongly consistent with  $G_T^e$  if given any quadruple of strictly positive real numbers  $(\Delta_x, \Delta_z, \Delta_{d_c}, \Delta_{d_s})$ , there exists a function  $\rho \in \mathcal{K}_\infty$  and  $T^* > 0$  such that, for all  $T \in (0, T^*)$ ,  $|x| \leq \Delta_x$ ,  $|z| \leq \Delta_z$ ,  $\|d_c[0]\|_\infty \leq \Delta_{d_c}$ ,  $|d_s| \leq \Delta_{d_s}$ , we have*

$$|G_T^e - G_T^a| \leq T\rho(T) . \quad (3.16)$$

■

A sufficient condition for one-step strong consistency is the following (the proof is given in Section 3.4):

**Lemma 3.2.2** *Consider  $G_T^e$  and  $G_T^a$  of the controller (3.13). If  $G_T^a$  is one-step strongly consistent with  $G_T^{Euler}$ , where  $G_T^{Euler} := z + Tg(x, z, d_s)$ , then  $G_T^a$  is one-step strongly consistent with  $G_T^e$ .* ■

**Remark 3.2.2** *Consistency properties specify how the controller should be discretized for the emulation procedure to yield desired results. Lemmas 3.2.1 and 3.2.2 present general checkable conditions under which one-step weak and strong consistency properties hold. It is important to emphasize that if the exact discrete-time model of the controller can be obtained, then we do not have to use an approximate discrete-time model of the controller and consistency definitions become superfluous, i.e., they hold automatically. Two important such cases were considered in the literature: emulation for linear systems was considered in [18] and emulation for static state feedback controllers was considered in [107]. However in linear system case, although the exact discrete-time model is computable, one often implements its approximation. Finally, note that the weak and strong consistency definitions become equivalent when  $G_T^e$  and  $G_T^a$  are independent of  $d_c$ .* ■

**Remark 3.2.3** *Note that the Euler approximation is one-step (weakly or strongly) consistent whenever the second condition in Lemma 3.2.1 or 3.2.2 is satisfied, since the first condition automatically holds. Also, if we want to implement the Euler approximate model of the controller, that is  $G_T^a = z + Tg(x, z, d_s)$ , then we can regard the closed-loop system (3.1), (3.2) and (3.3) as an augmented plant of the form*

$$\begin{aligned}\dot{x} &= f(x, u, d_c, d_s) \\ \dot{z} &= v\end{aligned}$$

*controlled by the static state feedback controller of the form:*

$$\begin{aligned}u &= u(x, z, d_s) \\ v &= g(x, z, d_s)\end{aligned}$$

which is implemented between the sample and zero-order-hold device(s). Note that, this form is valid only when  $g$  is independent of  $d_c$ . In this case, one can use results in [107] on emulation for static state feedback controllers. However, if we want to use an approximate discretization  $G_T^a$  other than Euler, this method is not applicable and we need to use results proved in this chapter that use the notion of consistency for general discretizations. ■

**Remark 3.2.4** *There is a strong motivation to consider controller discretizations other than Euler, although even the simple Euler discretization may sometimes yield satisfactory performance (see for instance [26, 116]). Indeed, a number of studies have shown that the Euler approximation of the controller dynamics is not always appropriate to use. For instance, the Euler approximation is, in general, not recommended to use for singularly perturbed systems that exhibit two-time scale behavior (see [10] and [103]). Using a comparative study in [21], the authors showed that the Tustin (bilinear) approximation is superior to Euler for the particular application. Moreover, even for linear systems, some examples in [2, 64] indicate that if the sampling period is given and fixed, then most of the classical discretization methods (such as Euler) might fail to yield acceptable performance or even stability. For linear systems, this has led to more advanced techniques for controller discretization that obtain the approximate model as a solution of an optimization problem (see [2] for more details). Similar results for nonlinear systems are yet to be proved.*

The consistency properties that we use provide a general and unified framework for investigation of a range of different controller discretizations. Moreover, they generalize in a natural way the consistency definitions commonly found in the numerical analysis literature that apply to ordinary differential equations without inputs (see for instance Definition 3.4.2 in [151]). A range of different consistent discretization can be defined using the results in [95]. Indeed, if the controller dynamics do not depend on  $d_c$  then the results in [95] can be used to write the solution of the initial value problem (3.7) as a series expansion in the sampling period  $T$ . Finite truncations of these expansions give a range of approximate discretization of the controller that are one step consistent. Moreover, classical Runge-Kutta integration schemes can also be

used to obtain one step consistent approximations (see for instance [151]). ■

We also introduce the following properties (Properties P1, P2 and P3), in order to precisely state the main results.

**Definition 3.2.4** Let  $V$  be continuously differentiable and  $w$  be continuous. The system (3.9), (3.11) is said to have Property P1, if given any 6-tuple of strictly positive real numbers  $(\Delta_x, \Delta_z, \Delta_{d_c}, \Delta_{\dot{d}_c}, \Delta_{d_s}, \nu)$ , there exists  $T^* > 0$  such that for all  $T \in (0, T^*)$  and all  $|x| \leq \Delta_x$ ,  $|z| \leq \Delta_z$ ,  $|d_s| \leq \Delta_{d_s}$  and for all disturbances  $d_c(\cdot)$  that satisfy  $\|d_c[0]\|_\infty \leq \Delta_{d_c}$ ,  $\|\dot{d}_c[0]\|_\infty \leq \Delta_{\dot{d}_c}$  the following holds:

$$\begin{aligned} & \frac{V(F_T(x, u(x, z, d_c, d_s), d_c[0], d_s), G_T^a(x, z, d_c, d_s)) - V(x, z)}{T} \\ & \leq \frac{1}{T} \int_0^T w(x, z, d_c(\tau), d_s) d\tau + \nu . \end{aligned} \quad (3.17)$$

■

**Definition 3.2.5** Let  $V$  be continuously differentiable and  $w$  be continuous. The system (3.9), (3.11) is said to have Property P2, if given any 6-tuple of strictly positive real numbers  $(\Delta_x, \Delta_z, \Delta_{d_c}, \Delta_{\dot{d}_c}, \Delta_{d_s}, \nu)$ , there exists  $T^* > 0$  such that for all  $T \in (0, T^*)$  and all  $|x| \leq \Delta_x$ ,  $|z| \leq \Delta_z$ ,  $|d_s| \leq \Delta_{d_s}$  and for all disturbances  $d_c(\cdot)$  that satisfy  $\|d_c[0]\|_\infty \leq \Delta_{d_c}$ ,  $\|\dot{d}_c[0]\|_\infty \leq \Delta_{\dot{d}_c}$  the following holds for the system (3.9), (3.11):

$$\begin{aligned} & \frac{V(F_T(x, u(x, z, d_c, d_s), d_c[0], d_s), G_T^a(x, z, d_c, d_s)) - V(x, z)}{T} \\ & \leq w(x, z, d_c, d_s) + \nu . \end{aligned} \quad (3.18)$$

■

**Definition 3.2.6** Let  $V$  be continuously differentiable and  $w$  be continuous. The system (3.9), (3.11) is said to have Property P3 if given any quintuple of strictly positive real numbers  $(\Delta_x, \Delta_z, \Delta_{d_c}, \Delta_{d_s}, \nu)$ , there exists  $T^* > 0$  such that for all  $T \in (0, T^*)$  and all  $|x| \leq \Delta_x$ ,  $|z| \leq \Delta_z$ ,  $\|d_c[0]\|_\infty \leq \Delta_{d_c}$ ,  $|d_s| \leq \Delta_{d_s}$  the inequality (3.17) holds. ■

**Remark 3.2.5** We defined several different properties (Properties P1, P2 and P3) since each of them may be useful in a particular situation. For instance, Properties P1

or P2 are useful when the input  $d_c$  is filtered through an input-to-state stable filter (see Proposition 3.3.2) or when all inputs are constant during the sampling intervals (see application of our results to preservation of passivity under sampling in Section 3.5). On the other hand, Property P3 is useful when the disturbance  $d_c$  is only assumed to be a measurable function of time, which is important, for instance, in investigation of input-to-state stability (see Section 3.5). ■

The following preliminary result that is proved in Section 3.4 shows that Properties P1 and P2 respectively in Definitions 3.2.4 and 3.2.5 are equivalent.

**Theorem 3.2.1** *The system (3.9), (3.11) has Property P1 if and only if it has Property P2.* ■

The main difference between the Properties P1 and P3 (or P2 and P3, since Properties P1 and P2 are equivalent) is that Property P1 requires the disturbances  $d_c$  to be Lipschitz, uniformly in  $T$ , for the inequality (3.17) to hold, whereas the inequality (3.17) in Property P3 must hold for non-uniformly Lipschitz disturbances as well. The dissipation inequalities in Properties P1 and P2 (since they are equivalent) are said to have the “weak” form (since they hold for a smaller class of disturbances) and the dissipation inequality in Property P3 is said to have the “strong” form (since it holds for a larger class of disturbances).

### 3.3 Main results

In this section we state the main results (Theorem 3.3.1 and 3.3.3) which assume that the continuous-time system is  $(V, w)$ -dissipative. Theorem 3.3.1 states that if one-step weak consistency holds and disturbances  $d_c(\cdot)$  are uniformly Lipschitz, then the (equivalent) Properties P1 and P2 hold for discrete-time model of the sampled-data system. Since in most cases we do not know whether the disturbances are uniformly Lipschitz or not, in Proposition 3.3.2 we prove that if we filter a bounded measurable signal using a strictly proper input-to-state stable filter, we obtain a filtered signal which is bounded and uniformly Lipschitz. If disturbances are only measurable (but



not uniformly Lipschitz) then the inequality (3.18) may not hold in a semiglobal practical sense while the inequality (3.17) still holds (see Example 3.3.1). In Theorem 3.3.3 we show that for a smaller class of controllers, if  $d_c(\cdot)$  are measurable (but not uniformly Lipschitz) and one-step strong consistency holds then the discrete-time model has Property P3.

**Theorem 3.3.1** (*Weak form of dissipativity*) Let  $G_T^a$  (3.9) be any approximate discrete-time model of the controller (3.3), which is one-step weakly consistent with the exact discrete-time model of the controller  $G_T^e$  (3.8). If the system (3.1), (3.2), (3.3) is  $(V, w)$ -dissipative, then the system (3.9), (3.11) has Property P1 (equivalently, Property P2). ■

Note that Properties P1 and P2 require  $d_c(\cdot)$  to be uniformly Lipschitz. The following example shows that indeed the uniformly Lipschitz condition on  $d_c(\cdot)$  is necessary, since the inequality (3.18) may not hold if  $d_c(\cdot)$  is not uniformly Lipschitz.

**Example 3.3.1** [107] Consider the continuous time system:

$$\dot{x} = u(x) + d_c = -x + d_c, \quad (3.19)$$

where  $x, d_c \in \mathbb{R}$ . Using the storage function  $V = \frac{1}{2}x^2$ , the derivative of  $V$  is  $\dot{V} = -x^2 + xd_c \leq -\frac{1}{2}x^2 + \frac{1}{2}d_c^2$ , and (3.19) is ISS. We will show that if  $d_c(t) = \cos\left(\frac{t+2T}{T}\right)$  the claim of Theorem 3.3.1 does not hold since

$$\|d_c\|_\infty = \left\| -\frac{1}{T} \sin\left(\frac{t+2T}{T}\right) \right\|_\infty = \frac{1}{T}, \quad (3.20)$$

which goes to infinity as  $T \rightarrow 0$ . Assume that  $u(x)$  in (3.19) is piecewise constant for the sampled-data system. So, the discrete-time model of the sampled-data system is obtained by integrating  $\dot{x}(t) = -x + d_c(t)$ , with the given  $d_c(t)$  and  $x_0 = x(k)$  over  $[kT, (k+1)T]$  which results in

$$x(k+1) = (1-T)x(k) + \int_{kT}^{(k+1)T} \cos\left(\frac{\tau+2T}{T}\right) d\tau, \quad (3.21)$$

and hence the exact discrete-time model is given by:

$$x(k+1) = (1-T)x(k) + T [\sin(k+3) - \sin(k+2)], \quad \forall k \geq 0. \quad (3.22)$$

Suppose that for any given  $\Delta_x, \Delta_{d_c}$  and  $\nu$ , there exists  $T^* > 0$  such that for all  $T \in (0, T^*)$  and  $k \geq 0$  with  $|x| \leq \Delta_x$  and  $\|d_c[0]\|_\infty \leq \Delta_{d_c}$  we have

$$\frac{\Delta V}{T} \leq -\frac{1}{2}x^2 + \frac{1}{2}d_c^2 + \nu. \quad (3.23)$$

We show by contradiction that the claim is not true for our case. Direct computations yield:

$$\frac{\Delta V}{T} = \frac{1}{2T} \left\{ ((1-T)x + T[\sin(3) - \sin(2)])^2 - x^2 \right\} = -x^2 + x[\sin(3) - \sin(2)] + O(T). \quad (3.24)$$

Let  $\tilde{x} = -0.5$ , (and hence  $\Delta_x = 0.5$ ,  $\Delta_{d_c} = 1$ ). By combining (3.23) and (3.24) we conclude that there should exist  $T^* > 0$  such that  $\forall T \in (0, T^*)$  we obtain:

$$-\frac{1}{2}\tilde{x}^2 + \tilde{x}[\sin(3) - \sin(2)] - \frac{1}{2}\cos(2)^2 - \nu + O(T) \leq 0, \quad (3.25)$$

and since there exists  $\nu^* > 0$  such that  $-\frac{1}{2}\tilde{x}^2 + \tilde{x}[\sin(3) - \sin(2)] - \frac{1}{2}\cos(2)^2 = \nu^*$  we can rewrite (3.25) as  $\nu^* - \nu + O(T) \leq 0$ , which is a contradiction (it does not hold for  $\nu \in (0, \nu^*)$ ).

Therefore, for  $\Delta_x = 0.5$ ,  $\Delta_{d_c} = 1$ ,  $\nu < \nu^*$ , there exists no  $T^* > 0$ , such that  $\forall T \in (0, T^*)$  the condition (3.23) holds. Note that the chosen  $d_c(t)$  does not satisfy the condition  $\|\dot{d}_c\|_\infty \leq \Delta_{\dot{d}_c}$  for some fixed  $\Delta_{\dot{d}_c} > 0$ , which is evident from (3.20). Hence, in this case we can not apply Theorem 3.3.1.  $\blacksquare$

The following result shows that if we can filter any bounded measurable disturbances using a strictly proper input-to-state stable filter, then the filtered disturbances are bounded and uniformly Lipschitz. This further motivates Theorems 3.2.1 and 3.3.1 that require disturbances to be uniformly Lipschitz.

**Proposition 3.3.2** *Consider any nonlinear filter:*

$$\dot{\xi} = f(\xi, d_c) \quad (3.26)$$

$$v = h(\xi), \quad (3.27)$$

which is input-to-state stable with respect to input  $d_c$  and where  $f$  and  $h$  are locally Lipschitz. Then, given any  $d_c(\cdot) \in \mathcal{L}_\infty$  and any  $\xi_0 \in \mathbb{R}^{n_\xi}$  we have that the output  $v(\cdot)$  is bounded, that is  $v(\cdot) \in \mathcal{L}_\infty$ . Moreover,  $\dot{v}(\cdot) \in \mathcal{L}_\infty$ , which implies that there exists  $L > 0$  such that  $|v(t_1) - v(t_2)| \leq L|t_1 - t_2|$ ,  $\forall t_1, t_2$ .  $\blacksquare$

The use of filters in sampled-data systems is standard (see for instance [18]). In particular, filters that are strictly proper, stable, linear and time invariant:

$$\dot{\xi} = A\xi + Bd_c \quad (3.28)$$

$$v = C\xi, \quad (3.29)$$

were considered in [18] in the context of  $\mathcal{L}_p$  stability of linear sampled-data systems. In this case, we have that the filter satisfies all conditions of Proposition 3.3.2 and consequently for any  $\xi_0$  and  $d_c \in \mathcal{L}_\infty$  we have that  $v, \dot{v} \in \mathcal{L}_\infty$ .

Example 3.3.1 showed that if disturbances  $d_c(\cdot)$  are not uniformly Lipschitz, then Properties P2 may not hold. It is of interest to investigate conditions, under which Property P3 still holds, for the case when  $d_c(\cdot)$  are not uniformly Lipschitz. To prove a general result for this case it is necessary to restrict our attention to the controllers of the form (3.13) (see Example 3.3.2 below). Note that the controller (3.13) does not have  $d_c(\cdot)$  as its input and the following example shows that this is necessary in general if we want to prove that the discrete-time model of the sampled-data system has Property P3.

**Example 3.3.2** [107] *Consider the system  $\dot{x} = u$ , where  $u = -d_c$ , where  $d_c(0) = 0$  and  $d_c(t) = 1$ ,  $\forall t > 0$ . The storage function that we consider is  $V(x) = x$ , so that the derivative:  $\frac{\partial V}{\partial x}(-d_c) = -d_c$ , and hence the dissipation rate is  $w(x, d_c, d_s) = -d_c$ . Since  $u$  is sampled and  $d_c(0) = 0$ , we have that  $x(t) = 0, \forall t \in [0, T]$  and so  $\Delta V/T = 0$ . On the other hand  $\int_0^T w(d_c(\tau))d\tau = -T$ . Hence, if Property P3 held, then we would obtain  $0 \leq -1 + \nu$ , which is not true for small  $\nu$ . ■*

Compared to Theorem 3.3.1, the following result on strong form of dissipativity considers a larger class of measurable disturbances  $d_c$ .

**Theorem 3.3.3** (Strong form of dissipativity) *Let  $G_T^a$  (3.15) be any approximate discrete-time model of the controller (3.13), which is one-step strongly consistent with the exact discrete-time model of the controller  $G_T^e$  (3.14). If the system (3.1), (3.2), (3.13) is  $(V, w)$ -dissipative, then the system (3.11), (3.15) has Property P3. ■*

Two important special cases of our main results are the static state feedback and open-loop system. All of the results given below follow directly from the more general case of dynamic state feedback and we describe below the connections.

### 3.3.1 Static state feedback results

The static state feedback:

$$u = u(x, d_c, d_s) \quad (3.30)$$

is a special case of (3.3), where  $n_z = 0$ . Similarly, the controller:

$$u = u(x, d_s) \quad (3.31)$$

is a special case of the controller (3.13). Obvious changes are introduced in definitions of Properties P1, P2 and P3 to cover the static state feedback case and we list them below for ease of reference. The inequality (3.5) in the  $(V, w)$ -dissipativity property is replaced by

$$\frac{\partial V}{\partial x} f(x, u(x, d_c, d_s), d_c, d_s) \leq w(x, d_c, d_s) . \quad (3.32)$$

The discretized controllers of (3.30) and (3.31) take respectively the following forms:

$$u(k) = u(x(k), d_c(k), d_s(k)), \quad k \geq 0 , \quad (3.33)$$

$$u(k) = u(x(k), d_s(k)), \quad k \geq 0 , \quad (3.34)$$

and they are implemented using a sample and zero-order-hold. As already indicated in Remark 3.2.2, the consistency properties are always satisfied since the controller has no dynamics. Since  $n_z = 0$ , we omit all conditions on  $z$  variable in Properties P1, P2 and P3. Consequently, the inequalities (3.17) and (3.18) are respectively replaced by the following inequalities:

$$\frac{V(F_T(x, u(x, d_c, d_s), d_c[0], d_s)) - V(x)}{T} \leq \frac{1}{T} \int_0^T w(x, d_c(\tau), d_s) d\tau + \nu , \quad (3.35)$$

and

$$\frac{V(F_T(x, u(x, d_c, d_s), d_c[0], d_s)) - V(x)}{T} \leq w(x, d_c, d_s) + \nu . \quad (3.36)$$

Direct consequences of Theorems 3.3.1 and 3.3.3 are the following corollaries.

**Corollary 3.3.1** *If the system (3.1), (3.2), (3.30) is  $(V, w)$ -dissipative, then the exact discrete-time model (3.11), (3.33) of the system has Property P1 (equivalently, Property P2).* ■

**Corollary 3.3.2** *If the system (3.1), (3.2), (3.31) is  $(V, w)$ -dissipative, then the exact discrete-time model (3.11), (3.34) of the system has Property P3.* ■

**Example 3.3.1 (cont'd)** *Note that since the state feedback of the system in Example 3.3.1 is static and it does not depend on  $d_c$ , all conditions of Corollary 3.3.2 are satisfied and the exact discrete-time model has Property P3.* ■

### 3.3.2 Open-loop configuration results

Besides the static feedback results, the results on preservation of dissipation inequalities under sampling for open-loop systems are also a direct consequence of our main results on dynamics state feedback controllers. Indeed, the open-loop systems can be viewed as a special case of “closed-loop” systems, with  $m = 0$  and  $n_z = 0$ . The continuous-time system (3.1), (3.2) can be rewritten as

$$\dot{x} = \tilde{f}(x, d_c, \tilde{d}_s) := f(x, u, d_c, d_s) \quad (3.37)$$

$$y = \tilde{h}(x, d_c, \tilde{d}_s) := h(x, u, d_c, d_s) , \quad (3.38)$$

where  $\tilde{d}_s := (u^T \quad d_s^T)^T$  and the control  $u$  can be treated in the same way as the disturbance  $d_s$ .

For ease of reference we list the changes needed in Properties P1, P2 and P3 to cover the open-loop case. We replace (3.5) of the  $(V, w)$ -dissipativity property with

$$\frac{\partial V}{\partial x} f(x, u, d_c, d_s) \leq w(x, u, d_c, d_s) . \quad (3.39)$$

Since there is no controller in this case, the consistency properties are superfluous. The exact discrete-time model of the open-loop system is given by (3.11). The statements of Properties P1, P2 and P3 are changed in the following way: “... given any quintuple of strictly positive numbers  $(\Delta_x, \Delta_u, \Delta_{d_c}, \Delta_{d_s}, \nu)$  there exists  $T^* > 0$  such that ...”. The inequalities (3.17) and (3.18) are respectively replaced by the following inequalities:

$$\frac{V(F_T(x, u, d_c[0], d_s)) - V(x)}{T} \leq \frac{1}{T} \int_0^T w(x, u, d_c(\tau), d_s) d\tau + \nu , \quad (3.40)$$

and

$$\frac{V(F_T(x, u, d_c[0], d_s)) - V(x)}{T} \leq w(x, u, d_c, d_s) + \nu. \quad (3.41)$$

The following results are direct consequences of our main results.

**Corollary 3.3.3** *If the system (3.37), (3.38) is  $(V, w)$ -dissipative, then the exact discrete-time model (3.11) of the system has Property P1 (equivalently, Property P2).* ■

Under slightly stronger conditions we can prove a stronger result that is useful in some situations:

**Proposition 3.3.4** *If the system (3.37), (3.38) is  $(V, w)$ -dissipative, with  $\frac{\partial V}{\partial x}$  being locally Lipschitz and  $\frac{\partial V}{\partial x}(0) = 0$ , then given any quintuple of strictly positive real numbers  $(\Delta_x, \Delta_u, \Delta_{d_c}, \Delta_{\dot{d}_c}, \Delta_{d_s})$ , there exist  $T^* > 0$  and positive constants  $K_1, K_2, K_3, K_4, K_5$  such that for all  $T \in (0, T^*)$  and all  $|x| \leq \Delta_x, |u| \leq \Delta_u, |d_s| \leq \Delta_{d_s}$  and functions  $d_c(\cdot)$  that are uniformly Lipschitz and satisfy  $\|d_c[0]\|_\infty \leq \Delta_{d_c}, \|\dot{d}_c[0]\|_\infty \leq \Delta_{\dot{d}_c}$ , we have for the exact discrete-time model (3.11) of the system:*

$$\begin{aligned} \frac{V(F_T(x, u, d_c[0], d_s)) - V(x)}{T} &\leq w(x, u, d_c, d_s) \\ &+ T \left( K_1|x|^2 + K_2|u|^2 + K_3|d_s|^2 + K_4\|d_c[0]\|_\infty^2 + K_5\|\dot{d}_c[0]\|_\infty^2 \right). \end{aligned}$$

■

Analogous to Theorem 3.3.1, we need the uniformly Lipschitz condition on  $d_c(\cdot)$  for Corollary 3.3.3 and Proposition 3.3.4 to hold. For the case when  $d_c(\cdot)$  is not uniformly Lipschitz, results similar to Theorem 3.3.3 are stated in the following. Note that in this open-loop case, for either the weak or strong dissipativity result, there is no dependency of control on  $d_c$ , since the control is an external input.

**Corollary 3.3.4** *If the system (3.37), (3.38) is  $(V, w)$ -dissipative, and  $d_c(\cdot)$  is measurable but not necessarily uniformly Lipschitz, then the exact discrete-time model (3.11) of the system has Property P3.* ■

**Proposition 3.3.5** *If the system (3.37), (3.38) is  $(V, w)$ -dissipative, with  $\frac{\partial V}{\partial x}$  being locally Lipschitz and  $\frac{\partial V}{\partial x}(0) = 0$ , then given any quadruple of strictly positive real numbers  $(\Delta_x, \Delta_u, \Delta_{d_c}, \Delta_{d_s})$  there exist  $T^* > 0$  and positive constants  $K_1, K_2, K_3, K_4$  such that for all  $T \in (0, T^*)$  and all  $|x| \leq \Delta_x$ ,  $|u| \leq \Delta_u$ ,  $\|d_c[0]\|_\infty \leq \Delta_{d_c}$ , and  $|d_s| \leq \Delta_{d_s}$  we have for the exact discrete-time model (3.11) of the system:*

$$\begin{aligned} & \frac{V(F_T(x, u, d_c[0], d_s)) - V(x)}{T} \\ & \leq \frac{1}{T} \int_0^T w(x, u, d_c(\tau), d_s) d\tau + T \left( K_1 |x|^2 + K_2 |u|^2 + K_3 \|d_c[0]\|_\infty^2 + K_4 |d_s|^2 \right). \end{aligned}$$

■

### 3.4 Proofs of main results

**Proof of Lemma 3.2.1:** Let strictly positive real numbers  $(\Delta_x, \Delta_z, \Delta_{d_c}, \Delta_{\dot{d}_c}, \Delta_{d_s})$  be given. Let  $R_z = \Delta_z + 1$ , and let  $(\Delta_x, R_z, \Delta_{d_c}, \Delta_{\dot{d}_c}, \Delta_{d_s})$  generate  $T^* > 0$  from the weak consistency of  $G_T^a$  and  $G_T^{Euler}$ . Let  $L > 0$  be the Lipschitz constant of  $g$  on the set where  $|x| \leq \Delta_x$ ,  $|z| \leq R_z$ ,  $|d_c| \leq \Delta_{d_c}$ ,  $|d_s| \leq \Delta_{d_s}$ . Since  $g$  is locally Lipschitz and  $g(0, 0, 0, 0) = 0$ , there exists  $M > 0$ , such that for all  $|x| \leq \Delta_x$ ,  $|z| \leq R_z$ ,  $|d_c| \leq \Delta_{d_c}$ ,  $|d_s| \leq \Delta_{d_s}$ , the following holds:

$$|g(x, z, d_c, d_s)| \leq M. \quad (3.42)$$

Let  $T_1^* := \min\{T^*, 1/M\}$ . It follows from (3.42) that, for each  $|x| \leq \Delta_x$ ,  $|z| \leq \Delta_z$ ,  $\|d_c[0]\|_\infty \leq \Delta_{d_c}$ ,  $|d_s| \leq \Delta_{d_s}$  and all  $t \in [0, T]$ , where  $T \in (0, T_1^*)$ , the solution  $z(t)$  of

$$\dot{z}(t) = g(x, z(t), d_c(t), d_s), \quad z(0) = z \quad (3.43)$$

satisfies  $|z(t)| \leq R_z$  and  $|z(t) - z| \leq Mt$ . It also follows from the Lipschitz property of  $g$  that for all  $|z| \leq R_z$ ,  $|x| \leq \Delta_x$ ,  $\|d_c[0]\|_\infty \leq \Delta_{d_c}$ ,  $\|\dot{d}_c[0]\|_\infty \leq \Delta_{\dot{d}_c}$ ,  $|d_s| \leq \Delta_{d_s}$  and all  $T \in (0, T_1^*)$ , we have

$$\begin{aligned} & \left| \int_0^T [g(x, z(\tau), d_c(\tau), d_s) - g(x, z, d_c, d_s)] d\tau \right| \\ & \leq \int_0^T L(|z(\tau) - z| + |d_c(\tau) - d_c|) d\tau \leq \frac{1}{2} T^2 L(M + \Delta_{\dot{d}_c}) = T^2 \tilde{L}, \quad (3.44) \end{aligned}$$

where  $\tilde{L} := \frac{1}{2}L(M + \Delta_{d_c})$ . Since

$$\begin{aligned} G_T^e(x, z, d_c[0], d_s) \\ = z + Tg(x, z, d_c, d_s) + \int_0^T [g(x, z(\tau), d_c(\tau), d_s) - g(x, z, d_c, d_s)]d\tau, \end{aligned} \quad (3.45)$$

the result follows from (3.44) and the fact that  $G_T^a$  is one step weakly consistent with  $G_T^{Euler}$ , which implies the existence of  $\tilde{\rho}_1 \in \mathcal{K}_\infty$ , such that

$$\left| G_T^a - G_T^{Euler} \right| \leq T\tilde{\rho}_1(T).$$

Finally, by letting  $\rho(s) = \tilde{L}s + \tilde{\rho}_1(s)$  we prove that  $G_T^a$  is one-step weakly consistent with  $G_T^e$ . ■

**Proof of Lemma 3.2.2:** Let strictly positive real numbers  $(\Delta_x, \Delta_z, \Delta_{d_s})$  be given. Let  $R_z = \Delta_z + 1$ , and let  $(\Delta_x, R_z, \Delta_{d_s})$  generate  $T^* > 0$  from the strong consistency of  $G_T^a$  and  $G_T^{Euler}$ . Let  $L > 0$  be the Lipschitz constant of  $g$  on the set where  $|x| \leq \Delta_x$ ,  $|z| \leq R_z$ ,  $|d_s| \leq \Delta_{d_s}$ . Since  $g$  is locally Lipschitz and  $g(0, 0, 0) = 0$ , there exists  $M > 0$ , such that for all  $|x| \leq \Delta_x$ ,  $|z| \leq R_z$ ,  $|d_s| \leq \Delta_{d_s}$ , the following holds:

$$|g(x, z, d_s)| \leq M. \quad (3.46)$$

Let  $T_1^* := \min\{T^*, 1/M\}$ . It follows from (3.46) that, for each  $|x| \leq \Delta_x$ ,  $|z| \leq \Delta_z$ ,  $|d_s| \leq \Delta_{d_s}$  and all  $t \in [0, T]$ , where  $T \in (0, T_1^*)$ , the solution  $z(t)$  of

$$\dot{z}(t) = g(x, z(t), d_s), \quad z(0) = z \quad (3.47)$$

satisfies  $|z(t)| \leq R_z$  and  $|z(t) - z| \leq Mt$ . It also follows from the Lipschitz property of  $g$  that for all  $|z| \leq R_z$ ,  $|x| \leq \Delta_x$ ,  $|d_s| \leq \Delta_{d_s}$  and all  $T \in (0, T_1^*)$ , we have

$$\left| \int_0^T [g(x, z(\tau), d_s) - g(x, z, d_s)]d\tau \right| \leq \int_0^T L(|z(\tau) - z|)d\tau \leq \frac{1}{2}T^2LM = T^2\tilde{L}, \quad (3.48)$$

where  $\tilde{L} := \frac{1}{2}LM$ . Since

$$G_T^e(x, z, d_s) = z + Tg(x, z, d_s) + \int_0^T [g(x, z(\tau), d_s) - g(x, z, d_s)]d\tau, \quad (3.49)$$

the result follows from (3.48) and the fact that  $G_T^a$  is one step strongly consistent with  $G_T^{Euler}$ , which implies the existence of  $\tilde{\rho}_1 \in \mathcal{K}_\infty$ , such that

$$\left| G_T^a - G_T^{Euler} \right| \leq T\tilde{\rho}_1(T).$$



Finally, by letting  $\rho(s) = \tilde{L}s + \tilde{\rho}_1(s)$  we prove that  $G_T^a$  is one-step strongly consistent with  $G_T^e$ . ■

**Proof of Theorem 3.2.1:**

(P1)  $\implies$  (P2) Suppose that Property P1 holds. Let  $(\Delta_x, \Delta_z, \Delta_{d_c}, \Delta_{\dot{d}_c}, \Delta_{d_s}, \nu_w)$  be given and let  $T_s^* > 0$  (from Property P1) be such that for all  $|x| \leq \Delta_x$ ,  $|z| \leq \Delta_z$ ,  $\|d_c[0]\|_\infty \leq \Delta_{d_c}$ ,  $\|\dot{d}_c[0]\|_\infty \leq \Delta_{\dot{d}_c}$ ,  $|d_s| \leq \Delta_{d_s}$  and all  $T \in (0, T_s^*)$  the following holds:

$$\begin{aligned} \frac{\Delta V}{T} &\leq \frac{1}{T} \int_0^T w(x, z, d_c(\tau), d_s) d\tau + \frac{\nu_w}{2} \\ &\leq w(x, z, d_c, d_s) + \frac{\nu_w}{2} + \frac{1}{T} \int_0^T |w(x, z, d_c(\tau), d_s) - w(x, z, d_c, d_s)| d\tau, \end{aligned} \quad (3.50)$$

where the second inequality was obtained by adding and subtracting  $w(x, z, d_c, d_s)$ . Since  $d_c(\cdot)$  is uniformly Lipschitz with Lipschitz constant  $\Delta_{\dot{d}_c}$ , we can write  $|d_c(\tau) - d_c| \leq \Delta_{\dot{d}_c} \tau$ . Moreover, since  $w$  is continuous, it is uniformly continuous on compact sets, and given any  $\varepsilon > 0$  there exists  $T_s > 0$  such that for any  $\tau \in [0, T_s]$ ,  $|x| \leq \Delta_x$ ,  $|z| \leq \Delta_z$ ,  $\|d_c[0]\|_\infty \leq \Delta_{d_c}$ ,  $\|\dot{d}_c[0]\|_\infty \leq \Delta_{\dot{d}_c}$ , and  $|d_s| \leq \Delta_{d_s}$ , we have that  $|w(x, z, d_c(\tau), d_s) - w(x, z, d_c, d_s)| \leq \varepsilon$ . Let  $\varepsilon = \frac{\nu_w}{2}$  and let this fix  $T_s$ . Let  $T_w^* = \min\{T_s, T_s^*\}$ . Then using (3.50) we have that for all  $T \in (0, T_w^*)$ ,  $|x| \leq \Delta_x$ ,  $|z| \leq \Delta_z$ ,  $\|d_c[0]\|_\infty \leq \Delta_{d_c}$ ,  $\|\dot{d}_c[0]\|_\infty \leq \Delta_{\dot{d}_c}$ ,  $|d_s| \leq \Delta_{d_s}$ :

$$\frac{\Delta V}{T} \leq w(x, z, d_c, d_s) + \frac{\nu_w}{2} + \frac{1}{T} \int_0^T \frac{\nu_w}{2} d\tau = w(x, z, d_c, d_s) + \frac{\nu_w}{2} + \frac{\nu_w}{2}, \quad (3.51)$$

which shows that Property P2 holds.

(P2)  $\implies$  (P1) follows a similar way as the proof for (P1)  $\implies$  (P2), to show that if Property P2 holds, then Property P1 holds. ■

**Proof of Theorem 3.3.1:** To shorten the notation we define  $u := u(x, z, d_c, d_s)$ ,  $f := f(x, u, d_c, d_s)$ ,  $g := g(x, z, d_c, d_s)$ ,  $F_T := F_T(x, u, d_c[0], d_s)$ ,  $G_T^e := G_T^e(x, z, d_c[0], d_s)$  and  $G_T^a := G_T^a(x, z, d_c, d_s)$ .

**Definition of  $T^*$ :** Suppose that the continuous-time system (3.1), (3.2), (3.3) is  $(V, w)$ -dissipative, that is for all  $x \in \mathbb{R}^{n_x}$ ,  $z \in \mathbb{R}^{n_z}$ ,  $d_c \in \mathbb{R}^{n_c}$ ,  $d_s \in \mathbb{R}^{n_s}$ , the inequality (3.5) holds. Let  $G_T^a$  be one-step weakly consistent with  $G_T^e$ , and let a 6-tuple of strictly positive real numbers  $(\Delta_x, \Delta_z, \Delta_{d_c}, \Delta_{\dot{d}_c}, \Delta_{d_s}, \nu)$  be given. Let these data generate

$\rho \in \mathcal{K}_\infty$  from the definition of one-step weak consistency. Define  $R_x := \Delta_x + 1$  and  $R_z := \Delta_z + 1$ . Let  $L > 0$  be the Lipschitz constant of  $f$  and  $g$  on the sets where  $|x| \leq R_x$ ,  $|z| \leq R_z$ ,  $|d_c| \leq \Delta_{d_c}$ ,  $|d_s| \leq \Delta_{d_s}$ , and let  $b > 0$  be a number that satisfies  $\max \left\{ \left| \frac{\partial V}{\partial x} \right|, \left| \frac{\partial V}{\partial z} \right|, |f|, |g| \right\} \leq b$  for all  $|x| \leq R_x$ ,  $|z| \leq R_z$ ,  $|d_c| \leq \Delta_{d_c}$ ,  $|d_s| \leq \Delta_{d_s}$ . Define  $\Delta := \Delta_x + \Delta_z + \Delta_{d_c} + \Delta_{d_s}$ .

We assume without loss of generality that  $\nu \leq 1$  and  $b \geq 1$  and define

$$T_1^* := \min \left\{ \frac{1}{2b}, \rho^{-1} \left( \frac{\nu}{2b} \right) \right\}. \quad (3.52)$$

Note that  $T_1^* \leq \frac{1}{2b} \leq \frac{1}{2} < 1$ . Let  $T_2^* > 0$  be such that the following holds:

$$bL \left[ (\Delta + 1) \frac{\exp(LT) - 1 - LT}{LT} + \frac{1}{2} \Delta_{d_c} T \right] \leq \frac{\nu}{8}, \quad \forall T \in (0, T_2^*). \quad (3.53)$$

It is easy to see that such a  $T_2^*$  always exists. Let  $x_1 := x + \theta_1 T f$  and  $z_1 := z + \theta_2 T g$  where  $\theta_1, \theta_2 \in (0, 1)$ . Let  $T_3^* > 0$  be such that:

$$b \left| \frac{\partial V}{\partial x} \Big|_{(x_1, z+Tg)} - \frac{\partial V}{\partial x} \Big|_{(x, z)} \right| \leq \frac{\nu}{8}, \quad (3.54)$$

for all  $T \in (0, T_3^*)$ ,  $|x| \leq R_x$ ,  $|z| \leq R_z$ ,  $|d_s| \leq \Delta_{d_s}$ , and  $d_c(\cdot)$  such that  $\|d_c[0]\|_\infty \leq \Delta_{d_c}$ , and  $\|\dot{d}_c[0]\|_\infty \leq \Delta_{d_c}$ . The required  $T_3^*$  always exists, which can be proved as follows. From the continuity of  $\frac{\partial V}{\partial x}$ , which implies that  $\frac{\partial V}{\partial x}$  is uniformly continuous on the compact sets, and since  $|x_1 - x| \leq T|f| \leq Tb$  and  $|(z + Tg) - z| = T|g| \leq Tb$ , it follows that given any  $\epsilon > 0$  there exists  $T_\epsilon > 0$  such that  $\left| \frac{\partial V}{\partial x} \Big|_{(x_1, z+Tg)} - \frac{\partial V}{\partial x} \Big|_{(x, z)} \right| \leq \epsilon$ ,  $\forall T \in (0, T_\epsilon)$ ,  $|x| \leq R_x$ ,  $|z| \leq R_z$ ,  $|d_c| \leq \Delta_{d_c}$  and  $|d_s| \leq \Delta_{d_s}$ . Hence, we can choose  $\epsilon^* := \nu/(8b)$  and let this fix the desired  $T_3^* := T_{\epsilon^*}$  for which (3.54) holds.

In exactly the same way we choose  $T_4^* > 0$  such that

$$b \left| \frac{\partial V}{\partial z} \Big|_{(x, z_1)} - \frac{\partial V}{\partial z} \Big|_{(x, z)} \right| \leq \frac{\nu}{8}, \quad (3.55)$$

for all  $T \in (0, T_4^*)$ ,  $|x| \leq R_x$ ,  $|z| \leq R_z$ ,  $|d_s| \leq \Delta_{d_s}$ , and  $d_c(\cdot)$  such that  $\|d_c[0]\|_\infty \leq \Delta_{d_c}$ , and  $\|\dot{d}_c[0]\|_\infty \leq \Delta_{d_c}$ . Finally, we define

$$T^* := \min\{T_1^*, T_2^*, T_3^*, T_4^*\}. \quad (3.56)$$

**Proof that Property P1 (P2) holds:** We will show first, that Property P2 holds.

Consider arbitrary  $T \in (0, T^*)$ ,  $|x| \leq \Delta_x$ ,  $|z| \leq \Delta_z$ ,  $|d_s| \leq \Delta_{d_s}$ , and  $d_c(\cdot)$  such that  $\|d_c[0]\|_\infty \leq \Delta_{d_c}$ , and  $\|\dot{d}_c[0]\|_\infty \leq \Delta_{d_c}$ .

Since  $T < T^* \leq \frac{1}{2b}$ , the solutions  $x(t)$  and  $z(t)$  of the initial value problems (3.10) and (3.7) exist and  $|x(t)| \leq \Delta_x + \frac{1}{2}$ ,  $|z(t)| \leq \Delta_z + \frac{1}{2}$ ,  $\forall t \in [0, T]$ , which implies

$$\begin{aligned} |F_T| &\leq \Delta_x + \frac{1}{2} < R_x, \\ |G_T^e| &\leq \Delta_z + \frac{1}{2} < R_z. \end{aligned} \quad (3.57)$$

From the second inequality in (3.57), one-step weak consistency and the choice of  $T_1^*$  we have:

$$\begin{aligned} |G_T^a| &\leq |G_T^e| + |G_T^a - G_T^e| \\ &< \Delta_z + \frac{1}{2} + \rho(T_1^*) \\ &\leq \Delta_z + \frac{1}{2} + \frac{1}{2} \\ &= R_z. \end{aligned} \quad (3.58)$$

From the local Lipschitz properties of  $f$  and  $g$  and the fact that they are zero at zero, we can write

$$|x(\tau) - x| \leq (\Delta + 1)[\exp(L\tau) - 1], \quad \forall \tau \in [0, T] \quad (3.59)$$

$$|z(\tau) - z| \leq (\Delta + 1)[\exp(L\tau) - 1], \quad \forall \tau \in [0, T] \quad (3.60)$$

and since  $d_c(\cdot)$  is uniformly Lipschitz, with Lipschitz constant  $\Delta_{d_c}$ , we can write that for all  $\tau$

$$|d_c(\tau) - d_c| = |d_c(\tau) - d_c(0)| \leq \Delta_{d_c} \tau. \quad (3.61)$$

We consider

$$\begin{aligned} \frac{\Delta V}{T} &= \frac{V(F_T, G_T^a) - V(x, z)}{T} \\ &= \underbrace{\frac{\partial V}{\partial x} \Big|_{(x,z)} f + \frac{\partial V}{\partial z} \Big|_{(x,z)} g}_{\mathbf{1}} + \underbrace{\frac{1}{T} \left\{ V(F_T, G_T^a) - V(x + Tf, z + Tg) \right\}}_{\mathbf{2}} \\ &\quad + \underbrace{\frac{1}{T} \left\{ V(x + Tf, z + Tg) - V(x, z) - \frac{\partial V}{\partial x} \Big|_{(x,z)} Tf - \frac{\partial V}{\partial z} \Big|_{(x,z)} Tg \right\}}_{\mathbf{3}}, \end{aligned} \quad (3.62)$$

where the second equality holds since we just added and subtracted  $\frac{1}{T}V(x+Tf, z+Tg)$ ,  $\frac{\partial V}{\partial x} \Big|_{(x,z)} f$  and  $\frac{\partial V}{\partial z} \Big|_{(x,z)} g$ . Now we bound each term in (3.62).

**Term 1:** It follows from  $(V, w)$ -dissipativity of the continuous-time system (3.1), (3.2), (3.3) that:

$$\frac{\partial V}{\partial x} \Big|_{(x,z)} f + \frac{\partial V}{\partial z} \Big|_{(x,z)} g \leq w(x, z, d_c, d_s) . \quad (3.63)$$

**Term 2:** Applying the Mean Value Theorem to the Term 2, we have by adding and subtracting  $\frac{1}{T}V(x + Tf, G_T^a)$ :

$$\begin{aligned} & \frac{1}{T} \left\{ V(F_T, G_T^a) - V(x + Tf, z + Tg) \right\} \\ & \leq \underbrace{\frac{1}{T} \left| \frac{\partial V}{\partial x} \Big|_{(x_2, G_T^a)} \right| |F_T - (x + Tf)|}_{\mathbf{2a}} + \underbrace{\frac{1}{T} \left| \frac{\partial V}{\partial z} \Big|_{(x+Tf, z_2)} \right| |G_T^a - (z + Tg)|}_{\mathbf{2b}} , \end{aligned} \quad (3.64)$$

where  $x_2 = \theta_3 F_T + (1 - \theta_3)(x + Tf)$  and  $z_2 = \theta_4 G_T^a + (1 - \theta_4)(z + Tg)$  and  $\theta_3, \theta_4 \in (0, 1)$ .

Since  $\max\{|F_T|, |x + Tf|\} \leq R_x$  (see (3.57)), then  $|x_2| \leq R_x$ . Moreover, since  $\max\{|G_T^a|, |z + Tg|\} \leq R_z$  (see (3.57) and (3.58)), this implies  $|z_2| \leq R_z$ . Hence, we have that  $\left| \frac{\partial V}{\partial x} \Big|_{(x_2, G_T^a)} \right| \leq b$  and  $\left| \frac{\partial V}{\partial z} \Big|_{(x+Tf, z_2)} \right| \leq b$ .

**Term 2a:** Since  $\left| \frac{\partial V}{\partial x} \Big|_{(x_2, G_T^a)} \right| \leq b$  and  $f$  is locally Lipschitz, we can write

$$\begin{aligned} & \frac{1}{T} \left| \frac{\partial V}{\partial x} \Big|_{(x_2, G_T^a)} \right| |F_T - (x + Tf)| \\ & \leq \frac{b}{T} |F_T - (x + Tf)| \\ & = \frac{b}{T} \left| \int_0^T f(x(\tau), u, d_c(\tau), d_s) d\tau - \int_0^T f(x, u, d_c, d_s) d\tau \right| \\ & \leq \frac{b}{T} \left\{ L \int_0^T |x(\tau) - x| d\tau + L \int_0^T |d_c(\tau) - d_c| d\tau \right\} \\ & \leq \frac{bL}{T} \left\{ (\Delta + 1) \int_0^T [\exp(L\tau) - 1] d\tau + \Delta_{\dot{d}_c} \int_0^T \tau d\tau \right\} \\ & = bL \left\{ (\Delta + 1) \frac{\exp(LT) - 1 - LT}{LT} + \frac{1}{2} \Delta_{\dot{d}_c} T \right\} \\ & \leq \frac{\nu}{8} , \end{aligned} \quad (3.65)$$

where we first added and subtracted  $\frac{b}{T} \int_0^T f(x, u, d_c(\tau), d_s) d\tau$ , then used the local Lipschitz property of  $f$ , then used bounds (3.59) and (3.61) and finally exploited the definition of  $T_2^*$ .

**Term 2b:** We use the fact that  $\left| \frac{\partial V}{\partial z} \Big|_{(x+Tf, z_2)} \right| \leq b$ , then add and subtract  $G_T^e$  to the last factor of Term 2b to obtain:

$$\begin{aligned}
& \frac{1}{T} \left| \frac{\partial V}{\partial z} \right|_{(x+Tf, z_2)} \left| G_T^a - (z + Tg) \right| \\
& \leq \frac{b}{T} |G_T^a - z - Tg| \\
& \leq \frac{b}{T} |G_T^a - G_T^e| + \frac{b}{T} |G_T^e - z - Tg| \\
& \leq b\rho(T) + \frac{b}{T} \left| \int_0^T g(x, z(\tau), d_c(\tau), d_s) d\tau - Tg(x, z, d_c, d_s) \right| \quad (3.66) \\
& \leq b\rho(T) + \frac{b}{T} \int_0^T L |z(\tau) - z| d\tau + \frac{b}{T} \int_0^T L |d_c(\tau) - d_c| d\tau \\
& \leq b\rho(T) + bL \left[ (\Delta + 1) \frac{\exp(LT) - 1 - LT}{LT} + \frac{1}{2} \Delta_{d_c} T \right] \\
& \leq \frac{\nu}{2} + \frac{\nu}{8},
\end{aligned}$$

where we first used one-step weak consistency and definition of  $T_1^*$ , then the local Lipschitz property of  $g$ , then inequalities (3.60) and (3.61) and finally the definition of  $T_2^*$ .

**Term 3:** From the differentiability of  $V$ , we apply the Mean Value Theorem to Term 3 (where  $x_1$  and  $z_1$  are defined just before (3.54)) to obtain:

$$\begin{aligned}
& \frac{1}{T} \left\{ V(x + Tf, z + Tg) - V(x, z) - \frac{\partial V}{\partial x} \Big|_{(x, z)} Tf - \frac{\partial V}{\partial z} \Big|_{(x, z)} Tg \right\} \\
& \leq \frac{\partial V}{\partial x} \Big|_{(x_1, z+Tg)} f + \frac{\partial V}{\partial z} \Big|_{(x, z_1)} g - \frac{\partial V}{\partial x} \Big|_{(x, z)} f - \frac{\partial V}{\partial z} \Big|_{(x, z)} g \\
& \leq |f| \cdot \left| \frac{\partial V}{\partial x} \Big|_{(x_1, z+Tg)} - \frac{\partial V}{\partial x} \Big|_{(x, z)} \right| + |g| \cdot \left| \frac{\partial V}{\partial z} \Big|_{(x, z_1)} - \frac{\partial V}{\partial z} \Big|_{(x, z)} \right| \quad (3.67) \\
& \leq b \left| \frac{\partial V}{\partial x} \Big|_{(x_1, z+Tg)} - \frac{\partial V}{\partial x} \Big|_{(x, z)} \right| + b \left| \frac{\partial V}{\partial z} \Big|_{(x, z_1)} - \frac{\partial V}{\partial z} \Big|_{(x, z)} \right| \\
& \leq \frac{\nu}{8} + \frac{\nu}{8}.
\end{aligned}$$

In deriving (3.67) we first used the definition of  $b$  and then definitions of  $T_3^*$  and  $T_4^*$ . Combining (3.62), (3.63), (3.65), (3.66) and (3.67) complete the proof that Property P2 holds. The proof for Property P1 to hold follows directly from Theorem 3.2.1. ■

**Proof of Proposition 3.3.2:** It is trivial; since  $d_c \in \mathcal{L}_\infty$  and (3.26) is ISS, then  $\xi \in \mathcal{L}_\infty$ . Since  $f$  and  $h$  are continuous, then  $\dot{\xi} \in \mathcal{L}_\infty$  and  $v \in \mathcal{L}_\infty$ . Finally, since  $h$  is

locally Lipschitz, then

$$\begin{aligned}
 |\dot{v}| &= \left| \lim_{\delta \rightarrow 0} \frac{h(\xi(t+\delta)) - h(\xi(t))}{\delta} \right| \\
 &\leq L \lim_{\delta \rightarrow 0} \left| \frac{\xi(t+\delta) - \xi(t)}{\delta} \right| \\
 &\leq L \left| \dot{\xi} \right|,
 \end{aligned}$$

which implies  $\dot{v} \in \mathcal{L}_\infty$ . ■

#### Proof of Proposition 3.3.4:

The proof of the proposition follows the same steps as the proof of Theorem 3.3.1. Using the idea from the theorem, we first take any number  $\nu > 0$ , and do the computation of  $T^*$  in the same way as we have done in the proof of Theorem 3.3.1. Then, we show how we can further reduce  $T^*$  to obtain  $K_1, K_2, K_3, K_4, K_5$  so that the desired bound holds.

We arrive at the following, which comes from (3.62) after some changes to match the open-loop case:

$$\begin{aligned}
 \frac{\Delta V}{T} &= \frac{V(F_T) - V(x)}{T} \\
 &= \underbrace{\frac{\partial V}{\partial x} \Big|_x f}_1 + \underbrace{\frac{1}{T} \{V(F_T) - V(x + Tf)\}}_2 \\
 &\quad + \underbrace{\frac{1}{T} \left\{ V(x + Tf) - V(x) - T \frac{\partial V}{\partial x} \Big|_x f \right\}}_3,
 \end{aligned} \tag{3.68}$$

where the second equality holds since we just added and subtracted  $V(x + Tf)/T$  and  $\frac{\partial V}{\partial x} \Big|_x f$  to  $\Delta V/T$ . The following changes are then used in the proof. Since  $\frac{\partial V}{\partial x}$  is locally Lipschitz and  $\frac{\partial V}{\partial x}(0) = 0$ , we can write for all  $|x| \leq \Delta_x + 1$ ,  $|u| \leq \Delta_u$ ,  $|d_c| \leq \Delta_{d_c}$ ,  $|d_s| \leq \Delta_{d_s}$  that  $\left| \frac{\partial V}{\partial x} \right| \leq L|x|$ . Also, since  $f$  is locally Lipschitz and  $f(0, 0, 0, 0) = 0$ , we can write for all  $|x| \leq \Delta_x + 1$ ,  $|u| \leq \Delta_u$ ,  $|d_c| \leq \Delta_{d_c}$ ,  $|d_s| \leq \Delta_{d_s}$ :

$$|f(x, u, d_c, d_s)| \leq L(|x| + |u| + |d_c| + |d_s|). \tag{3.69}$$

Since  $\left| \frac{\partial V}{\partial x} \Big|_{x_2} \right| \leq L|x_2|$ , where  $x_2 = \theta_3 F_T + (1 - \theta_3)(x + Tf)$ ,  $\theta_3 \in (0, 1)$ , then we have

that Term **2** in (3.68) can be bounded as:

$$\begin{aligned}
& \frac{1}{T} \{V(F_T) - V(x + Tf)\} \\
& \leq \frac{1}{T} \left| \frac{\partial V}{\partial x} \right|_{x_2} |F_T - (x + Tf)| \\
& = \frac{1}{T} L |x_2| \left| \int_0^T f(x(\tau), u, d_c(\tau), d_s) d\tau - \int_0^T f(x, u, d_c, d_s) d\tau \right| \\
& \leq \frac{1}{T} L |x_2| \left\{ L \int_0^T |x(\tau) - x| d\tau + L \int_0^T |d_c(\tau) - d_c| d\tau \right\} \quad (3.70) \\
& \leq \frac{1}{T} L^2 |x_2| \left\{ D_o \int_0^T [\exp(L\tau) - 1] d\tau + \|\dot{d}_c[0]\|_\infty \int_0^T \tau d\tau \right\} \\
& = L^2 |x_2| \left\{ D_o \frac{\exp(LT) - 1 - LT}{LT} + \frac{1}{2} \|\dot{d}_c[0]\|_\infty T \right\} \\
& \leq TL^2 |x_2| \left[ D_o K + \frac{1}{2} \|\dot{d}_c[0]\|_\infty \right],
\end{aligned}$$

for some  $K \geq \frac{\exp(LT) - 1 - LT}{LT^2}$ ,  $\forall T \in (0, T^*)$ , where  $D_o := |x| + |u| + \|d_c[0]\|_\infty + |d_s|$ . We can write

$$|x_2| \leq |x| + L \left( \int_0^T |x(\tau) - x| d\tau + \int_0^T |d_c(\tau) - d_c| d\tau \right) + T |f(x, u, d_c, d_s)|. \quad (3.71)$$

Using calculations similar to (3.59) and (3.61), we obtain:

$$\begin{aligned}
& \int_0^T |x(\tau) - x| d\tau + \int_0^T |d_c(\tau) - d_c| d\tau \\
& \leq \int_0^T \left( D_o (\exp(L\tau) - 1) + \|\dot{d}_c[0]\|_\infty \tau \right) d\tau \quad (3.72) \\
& \leq D_o \frac{\exp(LT) - 1 - LT}{L} + \frac{T^2}{2} \|\dot{d}_c[0]\|_\infty \\
& \leq T^2 \left[ D_o K + \frac{1}{2} \|\dot{d}_c[0]\|_\infty \right],
\end{aligned}$$

and substitute (3.69) and (3.72) into (3.71) to obtain

$$\begin{aligned}
|x_2| & \leq |x| + LT^2 \left[ D_o K + \frac{1}{2} \|\dot{d}_c[0]\|_\infty \right] \\
& \leq |x| + LTD_o(TK + 1) + \frac{1}{2} LT^2 \|\dot{d}_c[0]\|_\infty. \quad (3.73)
\end{aligned}$$

Hence, there exists  $\bar{K} > 0$  such that for all sufficiently small  $T$  we can write:

$$|x_2| \leq (1 + \bar{K}) |x| + \bar{K} \left( |u| + \|d_c[0]\|_\infty + \|\dot{d}_c[0]\|_\infty + |d_s| \right).$$

Since  $x_1 = x + \theta_1 T f$ , where  $\theta_1 \in (0, 1)$ , then  $|x_1 - x| \leq T |f(x, u, d_c, d_s)|$ . By referring to (3.67), Term **3** in (3.68) can be bounded by:

$$L |x_1 - x| |f(x, u, d_c, d_s)| \leq T L^3 (|x| + |u| + \|d_c[0]\|_\infty + |d_s|)^2 .$$

Direct but lengthy calculations show the existence of  $K_1, K_2, K_3, K_4, K_5$ . ■

The proof of Theorem 3.3.3 is omitted, since it follows the same steps as that of Theorem 3.3.1. The only difference is that instead of using one-step weak consistency, we use one-step strong consistency. Corollaries 3.3.1 and 3.3.3 follow directly from Theorem 3.3.1 and Remark 3.2.2. The proofs for Corollaries 3.3.2 and 3.3.4 and Proposition 3.3.5 are carried out similarly as the proofs of Corollaries 3.3.1 and 3.3.3 and Proposition 3.3.4 respectively, by using Theorem 3.3.3.

## 3.5 Applications

We present now two applications of our results. First, we consider ISS with respect to non-sampled inputs. It is interesting to see that we have to use strong dissipation inequalities in this case, since the use of weak dissipation inequalities would yield a weaker conclusion. Second, we consider preservation of passivity under sampling where the inputs are assumed to be controls that are constant during the sampling intervals. In the first and second applications we apply our results on, respectively, the dynamic feedback case and open-loop case. An asymptotic stability result is stated as a special case of the ISS result (see [107]). Further applications of our results to  $L_p$  stability, integral ISS, etc. are possible and are left for later exposition.

### 3.5.1 Input-to-state stability

It was shown in [154] that if an ISS controller is emulated then the ISS property is preserved in a semiglobal practical sense for the sampled-data system. Detailed proofs were given in [154] only for the case when Euler method was used to find the approximate discrete-time model of the controller (see Remark 3.2.3), while the case of higher order approximation was only commented on. Below, the main results of this chapter are applied to provide a sketch of proof for the case of emulation of dynamic



ISS controllers, when any one-step strongly consistent approximation is used. Suppose that the nonlinear plant

$$\dot{x} = f(x, u, d_c) \quad (3.74)$$

can be rendered ISS using the dynamic feedback controller

$$\begin{aligned} \dot{z} &= g(x, z) \\ u &= u(x, z) , \end{aligned} \quad (3.75)$$

where  $f$ ,  $g$ , and  $u$  are locally Lipschitz. Suppose that the dynamic feedback controller is emulated and then implemented digitally using a sample and zero-order-hold, where we use an approximation of the dynamic controller, so that:

$$\begin{aligned} z(k+1) &= G_T^a(x(k), z(k)) \\ u(k) &= u(x(k), z(k)) , \end{aligned} \quad (3.76)$$

Assume that the approximate discrete-time model of the dynamic controller  $G_T^a$  is one-step strongly consistent with the exact discrete-time model  $G_T^e$  (see Definition 3.2.3 and Lemma 3.2.2). Motivated by discussions in [18, 117] we introduce the state of the sampled-data system  $\chi(t) := (x^T(t) \ x^T(k) \ z^T(k))^T$  for  $t \in [kT, (k+1)T)$ . We write  $(x, z)$  to denote the vector  $(x^T \ z^T)^T$ . We also assume that:

**Assumption 3.5.1** *There exists  $\gamma_g \in \mathcal{K}_\infty$  such that given any  $\Delta > 0$  there exists  $T^* > 0$  such that for all  $|(x, z)| \leq \Delta$  and  $T \in (0, T^*)$  we have:*

$$|G_T^a(x, z)| \leq \gamma_g(|(x, z)|) . \quad (3.77)$$

■

**Remark 3.5.1** *Note that since  $f$  and  $g$  are assumed to be locally Lipschitz and zero at zero, if we let  $L > 0$  be the Lipschitz constant on the set  $|(x, z)| \leq 2\Delta$ , then we can write that for all  $|(x, z)| \leq \Delta$  and all  $T \in (0, \frac{\ln(2)}{L})$  that*

$$|G_T^e(x, z)| \leq 2|(x, z)| .$$

*If, in addition, a slightly stronger consistency holds in the following sense: given any  $\Delta > 0$  there exist  $T^* > 0$  and  $\gamma_1 \in \mathcal{K}_\infty$  such that for all  $|(x, z)| \leq \Delta$  and  $T \in (0, T^*)$  we have:*

$$|G_T^e(x, z) - G_T^a(x, z)| \leq \gamma_1(|(x, z)|) ,$$

then Assumption 3.5.1 holds (just apply the triangular inequality). This stronger form of consistency is known to hold for a large class of Runge-Kutta methods (see for instance Theorem 4.6.7 in [151]). ■

**Remark 3.5.2** Since  $f$  and  $u$  are locally Lipschitz and zero at zero, and Assumption 3.5.1 holds, the following is true: there exist  $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$  such that given any strictly positive numbers  $\Delta_1, \Delta_2$ , there exists  $T^* > 0$  such that for all  $T \in (0, T^*)$  and  $t_o \geq 0$  the solutions of the sampled-data system (3.74), (3.76) satisfy:

$$|\chi(t)| \leq \gamma_1(|\chi(t_o)|) + \gamma_2(\|d_c\|_\infty), \quad \forall t \in [t_o, t_o + T],$$

whenever  $|\chi(t_o)| \leq \Delta_1$  and  $\|d_c\|_\infty \leq \Delta_2$ . This conditions is referred to as uniform boundedness over  $T$  (UBT) in [117]. ■

We can state and prove the following result using Theorem 3.3.3:

**Corollary 3.5.1** If the continuous time system (3.74), (3.75) with  $f$ ,  $g$  and  $u$  locally Lipschitz is ISS, then given any approximate discrete-time model  $G_T^a$  of the dynamic controller which satisfies Assumption 3.5.1 and is one-step strongly consistent with the exact discrete-time model of the dynamic controller  $G_T^e$ , there exist  $\beta \in \mathcal{KL}$ ,  $\gamma \in \mathcal{K}$  such that given any triple of strictly positive real numbers  $(\Delta_\chi, \Delta_{d_c}, \nu)$ , there exists  $T^* > 0$  such that  $\forall T \in (0, T^*)$ ,  $|\chi(t_o)| \leq \Delta_\chi$ ,  $\|d_c\|_\infty \leq \Delta_{d_c}$ , the solutions of the sampled-data system (3.74), (3.76) satisfy:

$$|\chi(t)| \leq \beta(|\chi(t_o)|, t - t_o) + \gamma(\|d_c\|_\infty) + \nu, \quad \forall t \geq t_o \geq 0. \quad (3.78)$$

■

**Sketch of proof of Corollary 3.5.1:** Since the continuous time system (3.74), (3.75) is ISS, it implies (see Theorem 1 in [143]) that the system (3.74), (3.75) is  $(V, w)$ -dissipative, where  $V$  is smooth and there exist  $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathcal{K}_\infty$ ,  $\gamma_1 \in \mathcal{K}$  such that

$$\begin{aligned} \alpha_1(|(x, z)|) &\leq V(x, z) \leq \alpha_2(|(x, z)|) \\ w(x, z, d_c) &= -\alpha_3(|(x, z)|) + \gamma_1(|d_c|) \\ \left| \left( \frac{\partial V}{\partial x}, \frac{\partial V}{\partial z} \right) \right| &\leq \alpha_4(|(x, z)|). \end{aligned} \quad (3.79)$$

Then it follows from Theorem 3.3.3, that given any  $G_T^a$  which is one-step strongly consistent with  $G_T^e$ , and given any  $(\Delta_1, \Delta_2, \Delta_3, \nu_1)$  there exists  $T_1^* > 0$  such that for all  $T \in (0, T_1^*)$  and  $|x| \leq \Delta_1, |z| \leq \Delta_2, \|d_c[0]\|_\infty \leq \Delta_3$ , the discrete-time model of (3.74), (3.76) satisfies:

$$\begin{aligned} \frac{\Delta V}{T} &\leq \frac{1}{T} \int_0^T [-\alpha_3(|(x, z)|) + \gamma_1(|d_c(\tau)|)] d\tau + \nu_1 \\ &\leq -\alpha_3(|(x, z)|) + \gamma_1(\|d_c[0]\|_\infty) + \nu_1 . \end{aligned} \quad (3.80)$$

This implies (see Lemma 4 of [112]) that there exists  $\beta_2 \in \mathcal{KL}, \gamma_2 \in \mathcal{K}$  such that if all the assumptions on  $G_T^a$  hold and given any  $(\Delta_4, \Delta_5, \Delta_6, \nu_2)$  there exists  $T_2^* > 0$  such that for all  $T \in (0, T_2^*)$  and  $|x(0)| \leq \Delta_4, |z(0)| \leq \Delta_5, \|d_c\|_\infty \leq \Delta_6$ , the discrete-time model of (3.74), (3.76) satisfies:

$$|(x(k), z(k))| \leq \beta_2(|(x(0), z(0))|, kT) + \gamma_2(\|d_c\|_\infty) + \nu_2, \quad \forall k \geq 0 . \quad (3.81)$$

From Lemma 2 in [117] it follows that there exist  $\beta_3 \in \mathcal{KL}$  and  $\gamma_3 \in \mathcal{K}$  such that given any strictly positive  $(\Delta_7, \Delta_8, \nu_3)$  there exists  $T_3^* > 0$  such that for all  $T \in (0, T_3^*)$  and  $|\chi(0)| \leq \Delta_7, \|d_c\|_\infty \leq \Delta_8$ , the solutions of the sampled-data system satisfy:

$$|\chi(k)| \leq \beta_3(|\chi(0)|, kT) + \gamma_3(\|d_c\|_\infty) + \nu_3, \quad \forall k \geq 0 . \quad (3.82)$$

Finally, from Assumption 3.5.1 it follows that solutions of the sampled-data system are UBT (see Remark 3.5.2 and Definition 2 in [117]) and then using results in Section 3 in [117], there exists  $\beta \in \mathcal{KL}, \gamma \in \mathcal{K}$  such that given any  $G_T^a$  which is one-step strongly consistent with  $G_T^a$  and any  $(\Delta_\chi, \Delta_{d_c}, \nu)$  there exists  $T^* > 0$  such that for all  $T \in (0, T^*)$  and  $|\chi(t_o)| \leq \Delta_\chi, \|d_c\|_\infty \leq \Delta_{d_c}$ , the solutions of (3.74), (3.76) satisfy:

$$|\chi(t)| \leq \beta(|\chi(t_o)|, t - t_o) + \gamma(\|d_c\|_\infty) + \nu, \quad \forall t \geq t_o \geq 0 , \quad (3.83)$$

which completes the proof. ■

It is important to note that we can not use Theorem 3.3.1 instead of Theorem 3.3.3 to prove semiglobal practical ISS of the sampled-data system in Corollary 3.5.1. Indeed, Theorem 3.3.1 requires an additional condition on disturbances to be uniformly Lipschitz and hence the bound (3.83) would hold for a smaller set of disturbances

(bounded and uniformly Lipschitz) than measurable bounded disturbances for which the ISS property is defined.

A direct consequence of the ISS result is a result on semiglobal practical asymptotic stability, which is stated in the following corollary. Note that since we will consider the systems which has no external input or disturbances, by Remark 3.2.2, one step weak and strong consistency are the same.

**Corollary 3.5.2** *If the origin of the continuous time system*

$$\begin{aligned}\dot{x} &= f(x, u(x, z)) \\ \dot{z} &= g(x, z)\end{aligned}\tag{3.84}$$

*is GAS, then given any approximate discrete-time model  $G_T^a$  of the dynamic controller which satisfies Assumption 3.5.1 and is one-step weakly/strongly consistent with the exact discrete-time model of the dynamic controller  $G_T^e$ , there exists  $\beta \in \mathcal{KL}$  such that given any pair of strictly positive numbers  $(\Delta_\chi, \nu)$ , there exists  $T^* > 0$  such that  $\forall T \in (0, T^*)$ ,  $|\chi(t_o)| \leq \Delta_\chi$ , the solutions of the sampled-data system satisfy:*

$$|\chi(t)| \leq \beta(|\chi(t_o)|, t - t_o) + \nu, \quad \forall t \geq t_o \geq 0.\tag{3.85}$$

■

### 3.5.2 Passivity

Consider the continuous time system with outputs

$$\dot{x} = f(x, u), \quad y = h(x, u),\tag{3.86}$$

where  $x \in \mathbb{R}^n, y, u \in \mathbb{R}^m$  and assume that the system is passive, that is  $(V, w)$ -dissipative, where  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  and  $w = y^T u$ . We can apply either results of Theorem 3.3.1 or 3.3.3 since  $u$  is a piecewise constant input, to obtain that the discrete-time model satisfies: for any  $(\Delta_x, \Delta_u, \nu)$  there exists  $T^* > 0$  such that  $\forall T \in (0, T^*)$ ,  $|x| \leq \Delta_x, |u| \leq \Delta_u$  we have:

$$\frac{\Delta V}{T} \leq y^T u + \nu.\tag{3.87}$$

In ISS applications, adding  $\nu$  in the dissipation inequality deteriorated the property, but the deterioration was gradual. However, in (3.87)  $\nu$  acts as an infinite energy storage (constant power source) and hence it contradicts the definition of a passive system as one that can not generate power internally. As a result, conditions which guarantee that  $\nu$  in (3.87) can be set to zero are very important. These conditions are spelled out in the next corollary:

**Corollary 3.5.3** *Suppose that the system (3.86) is strictly input and state passive in the following sense: the dissipation rate can be taken as  $w(x, y, u) = y^T u - \psi_1(x) - \psi_2(u)$ , where  $\psi_1$  and  $\psi_2$  are positive definite functions that are locally quadratic. Then given any pair of strictly positive numbers  $(\Delta_x, \Delta_u)$  there exists  $T^* > 0$  such that for all  $T \in (0, T^*)$ ,  $|x| \leq \Delta_x$ ,  $|u| \leq \Delta_u$  we have:*

$$\frac{\Delta V}{T} \leq y^T u - \frac{1}{2}\psi_1(x) - \frac{1}{2}\psi_2(u) \quad (3.88)$$

■

**Sketch of proof of Corollary 3.5.3:** Using Proposition 3.3.4, we see that given any  $(\Delta_x, \Delta_u)$  there exists  $T_1^* > 0$  such that  $\forall T \in (0, T_1^*)$ ,  $|x| \leq \Delta_x$ ,  $|u| \leq \Delta_u$  we have:

$$\frac{\Delta V}{T} \leq y^T u - \psi_1(x) - \psi_2(u) + TK_1|x|^2 + TK_2|u|^2,$$

and from properties of  $\psi_1$  and  $\psi_2$ , it follows that there exists  $T^* \leq T_1^*$  such that  $\forall T \in (0, T^*)$ ,  $|x| \leq \Delta_x$ ,  $|u| \leq \Delta_u$  we have that (3.88) holds. ■

We emphasize that the above approach can be used for more general properties than passivity to cancel  $\nu$  in the dissipation inequality for the discrete-time system.

### 3.6 Conclusion

We have presented general results on preservation of general dissipation inequalities under sampling in the emulation controller design. We have covered the closed-loop and open-loop cases. These results generalize all available results on emulation design in the nonlinear sampled-data literature that we are aware of (see [18, 107, 124, 128, 139, 154, 163]) and provide a unified framework for digital controller design using the emulation method for general nonlinear systems.

In the next chapter, we will apply our results, in particular Corollary 3.5.1, to solve stabilization problem of an engineering system. The stability property will be the main concern, since it can be considered as a special case of input-to-state stability with zero input. We will consider a rotating stall and surge control for a jet engine system. We study the performance of the system when a controller is designed using the emulation technique, following the framework presented in this chapter. Moreover, ideas on how to improve the design and to handle design constraints are also raised in Chapter 4. These then become strong motivations for the research presented in Part II of this thesis.

## Chapter 4

# Illustration and Motivation Example

### 4.1 Introduction

In Chapter 3 we presented results on emulation design. The main results concerned the preservation of dissipativity properties under controller emulation. The results basically state that dissipativity properties of a continuous-time system are preserved when the continuous-time controller acting on the system is replaced by a discrete-time controller obtained via emulation. It was shown that the results apply to a wide range of more particular problems, such as stability, input-to-state stability and passivity.

In this chapter, an engineering example applying our results on emulation design from Chapter 3 is presented. A jet engine stall and surge control adopted from [74] is presented as the problem under consideration. Asymptotic stabilization problem is the main issue explored in this example, and the backstepping technique (see [74, 110]), which is a systematic design technique for nonlinear control systems, is used as the design tool.

We apply continuous-time backstepping technique [74] to design a discrete-time controller via emulation for the plant, study the preservation of the asymptotic stability property under sampling and observe the behavior of the closed-loop sampled-data system applying the emulation controller. We show that our results on emulation are

applicable to this example. However, as was indicated in Chapter 3, emulation design causes a certain degree of degradation to the system performance compared to its continuous-time counterpart.

In this chapter, we also motivate another way of designing discrete-time controllers that can be used to potentially improve the performance of sampled-data systems. In this case, we use a direct discrete-time design which is an alternative for sampled-data system design besides emulation. Unfortunately, in contrast to emulation design that is based on a large collection of tools from continuous-time control theory, tools for approximate based direct discrete-time design are scarce, and there is no tool available to do a counterpart design we need for the examples in this chapter. The only discrete-time backstepping scheme that is close to our need is the Euler-based discrete-time backstepping technique [110]. However, this tool does not support a certain condition that arises in our examples (we will provide the explanation in more detail later). Therefore, we do an *ad hoc* design for each example, by imitating the Euler-based discrete-time backstepping technique [110] to obtain “Euler-based like” discrete-time controllers.

We emphasize that because there is no theoretical background for the direct discrete-time design we do in this chapter, we do not refer to any particular theory to draw a conclusion of what we achieve from the design. In addition, consistent with the aim of this chapter, which is to motivate research in the rest of this thesis, although it is possible, we avoid being exhausted to prove the theoretical aspects of the design we use. Based only on intuition and simulation results, we show that the Euler-based discrete-time controller can be design to work better than the emulation controller. Nevertheless, the finding of this simulation study is important as a strong motivation to the research presented in Part II of this thesis, where we will develop a range of tools for direct discrete-time controller design based on approximate models.

This chapter is organized as follows. In Section 4.2, we present the preliminaries. In Section 4.3, we derive the model for the jet engine system. In Sections 4.4 two discrete-time controllers are designed respective using the continuous-time and the Euler-based backstepping techniques. Comparison of the closed-loop system performance when



applying each controller is presented in Section 4.5. Finally, the conclusion is presented in Section 4.6.

## 4.2 Preliminaries

In this chapter, the primary tool utilized for designing the controllers is the backstepping technique. Backstepping is a recursive procedure to design a feedback controller based on a Lyapunov function. It breaks down the design problem for the full system into a sequence of design problems for lower order subsystems. Backstepping is a useful technique to solve stabilization, tracking and robust control problems, and it is applicable to a quite large class of systems.

To carry out the emulation design, the continuous-time backstepping procedure introduced in [74] is applied, while for the direct discrete-time design, a similar technique adopted from the discrete-time backstepping technique presented in [110] is used. For that reason, some important results from [74] and [110] that are useful for the design are cited in this section.

Consider a parameterized family of discrete-time nonlinear systems  $x(k+1) = F_T(x(k), u_T(x(k)))$ , we state the following definitions:

**Definition 4.2.1** [110] *The family of controllers  $u_T$  renders semiglobal practical asymptotic (SPA) stability for  $F_T$  if there exists  $\beta \in \mathcal{KL}$  such that for any pair of strictly positive real numbers  $(D, \nu)$  there exists  $T^* > 0$  such that for each  $T \in (0, T^*)$  the solutions of  $x(k+1) = F_T(x(k), u_T(x(k)))$  with  $x(0) = x_o$  satisfy:*

$$|x(k, x_o)| \leq \beta(|x_o|, kT) + \nu, \quad k \geq 0, \quad (4.1)$$

whenever  $|x_o| \leq D$ . ■

**Definition 4.2.2** [110] *Let  $\hat{T} > 0$  be given and for each  $T \in (0, \hat{T})$  let the functions  $V_T : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  and  $u_T : \mathbb{R}^n \rightarrow \mathbb{R}$  be defined. We say that the pair  $(u_T, V_T)$  is a semiglobally practically asymptotically (SPA) stabilizing pair for  $F_T$  if there exist  $\underline{\alpha}$ ,  $\bar{\alpha}$ ,  $\alpha \in \mathcal{K}_\infty$ , such that for any pair of strictly positive real numbers  $(\Delta, \nu)$  there exist a triple of strictly positive real numbers  $(T^*, L, M)$ , with  $T^* \leq \hat{T}$ , such that for all*

$x, z \in \mathbb{R}^n$  with  $\max\{|x|, |z|\} \leq \Delta$ , and  $T \in (0, T^*)$  we have:

$$\underline{\alpha}(|x|) \leq V_T(x) \leq \overline{\alpha}(|x|) \quad (4.2)$$

$$V_T(F_T(x, u)) - V_T(x) \leq -T\alpha(|x|) + T\nu. \quad (4.3)$$

$$|V_T(x) - V_T(z)| \leq L|x - z| \quad (4.4)$$

$$|u_T(x)| \leq M. \quad (4.5)$$

■

#### 4.2.1 Continuous-time backstepping design

Consider a continuous-time plant of the strict feedback form:

$$\begin{aligned} \dot{x} &= f(x) + g(x)\xi \\ \dot{\xi} &= u. \end{aligned} \quad (4.6)$$

An integrator backstepping technique can be used to design a state feedback controller to stabilize the origin of the system. We use the following assumption.

**Assumption 4.2.1** *Consider the system*

$$\dot{x} = f(x) + g(x)u, \quad f(0) = 0, \quad (4.7)$$

where  $x \in \mathbb{R}^n$  is the state and  $u \in \mathbb{R}$  is the control input. There exist a continuously differentiable feedback control law

$$u = \alpha(x), \quad \alpha(0) = 0, \quad (4.8)$$

and a smooth, positive definite, radially unbounded function  $W : \mathbb{R}^n \rightarrow \mathbb{R}$  such that

$$\frac{\partial W}{\partial x}(x)[f(x) + g(x)\alpha(x)] \leq -\Omega(x), \quad \forall x \in \mathbb{R}^n, \quad (4.9)$$

where  $\Omega : \mathbb{R}^n \rightarrow \mathbb{R}$  is positive definite. ■

Assumption 4.2.1 is slightly stronger than Assumption 2.7 of [74], since we require  $\Omega$  to be positive definite. Under this assumption, we state the following lemma on integrator backstepping, cited from [74]. The proof of this lemma is provided in [74].

**Lemma 4.2.1** *Consider the system (4.6), which is an augmentation of (4.7) with an integrator. Suppose that all conditions on Assumption 4.2.1 are satisfied by the upper subsystem of (4.6) with control  $\xi \in \mathbb{R}$ . Then*

$$V(x, \xi) = W(x) + \frac{1}{2}[\xi - \alpha(x)]^2 \quad (4.10)$$

*is a control Lyapunov function (clf) for the full system (4.6). That is, there exists a feedback control  $u = \alpha^a(x, \xi)$  which renders  $x = 0, \xi = 0$  the global asymptotic stability (GAS) equilibrium of (4.6). One such control is*

$$u = -c(\xi - \alpha(x)) + \frac{\partial \alpha}{\partial x}(x)[f(x) + g(x)\xi] - \frac{\partial W}{\partial x}(x)g(x), \quad c > 0. \quad (4.11)$$

■

The continuous-time backstepping controller achieves global asymptotic stabilization for the continuous-time system. The backstepping technique does not apply only for systems in strict feedback form, but also to a more general feedback form or even for a larger class of systems that do not follow any formal feedback forms. The detailed procedure for backstepping design is presented in [74] and various examples of its application can be obtained for instance in [14, 62, 63].

#### 4.2.2 Discrete-time backstepping design

In this subsection, the result from [110] is cited, which gives a discrete-time counterpart for the continuous-time result cited in Subsection 4.2.1. This discrete-time backstepping scheme is based on the Euler approximate model of the continuous-time plant. The Euler model is used, since this approximation preserves the strict feedback structure of the plant that is needed to apply a backstepping technique.

The Euler approximate model of (4.6) has the following form:

$$x(k+1) = x(k) + T(f(x(k)) + g(x(k))\xi(k)) \quad (4.12)$$

$$\xi(k+1) = \xi(k) + Tu(k). \quad (4.13)$$

**Theorem 4.2.1** *Consider the Euler approximate model (4.12), (4.13). Suppose that there exists  $\hat{T} \geq 0$  and a pair  $(\alpha_T, W_T)$  that is defined for all  $T \in (0, \hat{T})$  and that is*

a SPA stabilizing pair for the subsystem (4.12), with  $\xi \in \mathbb{R}$  regarded as its control. Moreover, suppose that the pair  $(\alpha_T, W_T)$  has the following properties:

1.  $\alpha_T$  and  $W_T$  are continuously differentiable for any  $T \in (0, \hat{T})$ ;
2. there exists  $\tilde{\varphi} \in \mathcal{K}_\infty$  such that

$$|\alpha_T(x)| \leq \tilde{\varphi}(|x|) . \quad (4.14)$$

3. for any  $\tilde{\Delta} > 0$  there exist a pair of strictly positive numbers  $(\tilde{T}, \tilde{M})$  such that for all  $T \in (0, \tilde{T})$  and  $|x| \leq \tilde{\Delta}$  we have

$$\max \left\{ \left| \frac{\partial W_T}{\partial x} \right|, \left| \frac{\partial \alpha_T}{\partial x} \right| \right\} \leq \tilde{M} . \quad (4.15)$$

Then there exists a SPA stabilizing pair  $(u_T, V_T)$  for the Euler model (4.12), (4.13). In particular, we can take:

$$u_T = -c(\xi - \alpha_T(x)) - \frac{\widetilde{\Delta W}_T}{T} + \frac{\Delta \alpha_T}{T} \quad (4.16)$$

where  $c > 0$  is arbitrary, and

$$\Delta \alpha_T := \alpha_T(x + T(f + g\xi)) - \alpha_T(x) \quad (4.17)$$

$$\widetilde{\Delta W}_T := \begin{cases} \frac{\overline{\Delta W}_T}{(\xi - \alpha_T(x))}, & \xi \neq \alpha_T(x) \\ T \frac{\partial W_T}{\partial x}(x + T(f + g\xi))g, & \xi = \alpha_T(x) \end{cases} \quad (4.18)$$

$$\overline{\Delta W}_T := W_T(x + T(f + g\xi)) - W_T(x + T(f + g\alpha_T)) \quad (4.19)$$

and the Lyapunov function  $V_T$  is

$$V_T(x, \xi) = W_T(x) + \frac{1}{2}(\xi - \alpha_T(x))^2 . \quad (4.20)$$

■

Complete proof of this theorem is provided in [110]. The Euler-based discrete-time backstepping achieves semiglobal practical asymptotic stabilization for the sampled-data system.

The design procedure for the discrete-time backstepping follows the same steps as the continuous-time one, with an addition that the Euler discrete-time model needs to

be formulated at the beginning of the design. We need to emphasize here that although the two design procedures are the same, the controller obtained following Theorem 4.2.1 is in general different from the controller obtained using Lemma 4.2.1. However it makes sense to compare the discrete-time controller obtained using Theorem 4.2.1 with the one resulted via emulation of the continuous-time controller of Lemma 4.2.1, since the Euler-based controller in general takes form

$$u_T^{Euler}(k) = u^{ct}(k) + Tu^r(k) \quad (4.21)$$

where  $u^{ct}$  is the emulated controller and  $u_r$  groups the extra terms with higher order  $T$ . We will show that this relationship holds for the jet engine control problem studied in this chapter.

### 4.3 Jet engine system modeling

Jet engine compression systems have recently become the subject of intensive control studies. The systems play an important role in turbomachinery, both in heavy industries and in turbojet engines. The example presented in this section was considered in [74]. Figure 4.1 shows a schematic diagram of the jet engine, where the flow enters the compressor from an inlet duct and exits into a plenum volume representing the combustion chamber. It is then exhausted to the atmosphere through a throttle valve modeling the turbine [94].

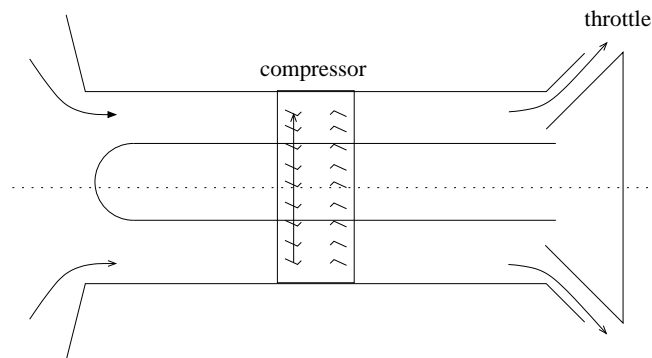


Figure 4.1: Jet engine compression system

Jet engine compression systems are designed to operate in steady axisymmetric

flow and their function is to increase the pressure of the flow. If the mass flow through the compressor is decreased, the pressure rise increases and this improves the power output. Unfortunately, a critical value of mass flow is reached, beyond which the steady design flow is no longer stable. One would like to operate the engine as close as possible to this critical point, but in such cases a small change in the flow may be enough to push the operation point into the unstable region.

Intensive studies of the control of jet engine compression systems have aimed at understanding and preventing two types of instability: rotating stall and surge. Rotating stall manifests itself as a region of severely reduced flow that rotates at a fraction of the rotor speed. Surge is an axisymmetric pumping oscillation which can cause flameout and engine damage.

The simplest model that describes these instabilities is a three-state Galerkin approximation of the nonlinear PDE model by Moore and Greitzer [74, 94]. The model is the following:

$$\dot{\Phi} = -\Psi + \Psi_C(\Phi) - 3\Phi R \quad (4.22)$$

$$\dot{\Psi} = \frac{1}{\beta^2}(\Phi - \Phi_T) \quad (4.23)$$

$$\dot{R} = \sigma R(1 - \Phi^2 - R) \quad (4.24)$$

where  $\Phi$  is the mass flow,  $\Psi$  is the pressure rise and  $R$  is the normalized stall squared amplitude. Both  $\Psi$  and  $R$  can assume only positive values; the former because of physical constraints and the later because it is a squared quantity.  $\Phi_T$  is the mass flow through the throttle, and  $\sigma$  and  $\beta$  are constant positive parameters. The compressor and throttle characteristic, respectively  $\Psi_C(\Phi)$  and  $\Phi_T(\Psi)$  are

$$\Psi_C(\Phi) = \Psi_{C0} + 1 + \frac{3}{2}\Phi - \frac{1}{2}\Phi^3 \quad (4.25)$$

$$\Psi = \frac{1}{\gamma^2}(1 + \Phi_T(\Psi))^2 \quad (4.26)$$

where  $\Psi_{C0}$  is a constant and  $\gamma$  is variable that changes by varying the throttle opening.

Recall the system model (4.22), using the flow through the throttle  $\Phi_T$  as the control input. Our objective is to stabilize the equilibrium point  $R^e = 0$ ,  $\Phi^e = 1$  and  $\Psi^e = \Psi_C(\Phi^e) = \Psi_{C0} + 2$ . Translating the origin to the desired equilibrium point

$\phi = \Phi - 1$ ,  $\psi = \Psi - \Psi_{C0} - 2$ , and letting the control variable be

$$u = \frac{1}{\beta^2}(\phi + 1 - \Phi_T), \quad (4.27)$$

the model (4.22) can now be written as a three state equation:

$$\dot{R} = \sigma R^2 - \sigma R(2\phi + \phi^2) \quad (4.28)$$

$$\dot{\phi} = -\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 - 3R\phi - 3R \quad (4.29)$$

$$\dot{\psi} = -u. \quad (4.30)$$

The equations are reordered to reveal the similarity with the strict feedback form. The only discrepancy is in the first equation, which is not affine in the second variable  $\phi$ .

We now have the jet engine model that is ready to use for a backstepping design. In the next section, we design a controller for the plant using the backstepping technique [74] and then design a discrete-time counterpart of the controller using discrete-time backstepping, which is introduced in [110].

## 4.4 Jet engine stall and surge control design

### 4.4.1 Continuous-time backstepping controller design

The continuous-time backstepping controller design performed in this subsection follows the procedure presented in [74], applying Theorem 4.2.1 cited in the previous section. In standard backstepping design, the number of steps needed for the recursion generally follows the number of subsystems constructing the plant. According to this, there should be three design steps for this jet engine example. However, since  $R \geq 0$  and hence the stabilization of the subsystem (4.28) is obvious by inspection, choosing simply a virtual control  $\phi = \alpha(R) = 0$  yields  $\dot{R} = -\sigma R^2$ , which means that  $R(t) \rightarrow 0$  as  $t \rightarrow \infty$ . Therefore the procedure can be shortened from a three-step design to a two-step design.

We first consider (4.28), (4.29) as the upper subsystem, and (4.30) as the lower subsystem. For  $\psi$  as the virtual control for the upper subsystem, we choose

$$\alpha(\phi, R) = c_1\phi - \frac{3}{2}\phi^2 - 3R. \quad (4.31)$$

This choice has allowed us to avoid cancellation of the useful nonlinearities  $-\frac{1}{2}\phi^3$  and  $-3R\phi$ . Define  $\tilde{\psi} = \psi - \alpha(\phi, R)$ , and substitute it to (4.29). We get

$$\dot{R} = \sigma R^2 - \sigma R(2\phi + \phi^2) \quad (4.32)$$

$$\dot{\phi} = -c_1\phi - \frac{1}{2}\phi^3 - 3R\phi - \tilde{\psi} . \quad (4.33)$$

The Lyapunov function  $W(\phi, R) = \frac{1}{2}\phi^2$  satisfies

$$\dot{W}(\phi, R) = \phi\dot{\phi} = -c_1\phi^2 - \frac{1}{2}\phi^4 - 3R\phi^2 - \tilde{\psi}\phi , \quad (4.34)$$

which is negative definite with respect to  $\phi$ , for  $\psi = \alpha$ . At the second step, we consider

$$\begin{aligned} \dot{R} &= \sigma R^2 - \sigma R(2\phi + \phi^2) \\ \dot{\phi} &= -c_1\phi - \frac{1}{2}\phi^3 - 3R\phi - \tilde{\psi} \\ \dot{\tilde{\psi}} &= -u - \dot{\alpha}(\phi, R) =: \nu . \end{aligned} \quad (4.35)$$

Using  $V = W + \frac{1}{2}\tilde{\psi}^2$  as the Lyapunov function candidate for the total system, we obtain the derivative

$$\begin{aligned} \dot{V} &= \frac{\partial W}{\partial \phi}(-c_1\phi - \frac{1}{2}\phi^3 - 3R\phi - \alpha) + \frac{\partial W}{\partial \phi}(-\tilde{\psi}) + \tilde{\psi}\nu \\ &= -c_1\phi^2 - \frac{1}{2}\phi^4 - 3R\phi^2 - \phi\tilde{\psi} + \tilde{\psi}\nu . \end{aligned} \quad (4.36)$$

We choose  $\nu = \phi - c_2\tilde{\psi}$ , and obtain

$$\begin{aligned} u &= c_2\tilde{\psi} - \phi - \dot{\alpha} \\ &= c_2\tilde{\psi} - \phi - (c_1 - 3\phi) \left( -\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 - 3R\phi - 3R \right) \\ &\quad + 3\sigma R(-2\phi - \phi^2 - R) . \end{aligned} \quad (4.37)$$

With this control, the resulting close-loop feedback system is

$$\begin{aligned} \dot{R} &= \sigma R^2 - \sigma R(2\phi + \phi^2) \\ \dot{\phi} &= -c_1\phi - \frac{1}{2}\phi^3 - 3R\phi - \tilde{\psi} \\ \dot{\tilde{\psi}} &= \phi - c_2\tilde{\psi} \end{aligned} \quad (4.38)$$

and the derivative of the Lyapunov function  $V$  satisfies

$$\dot{V} \leq -c_1\phi^2 - \frac{1}{2}\phi^4 - 3R\phi^2 - c_2\tilde{\psi}^2 , \quad (4.39)$$



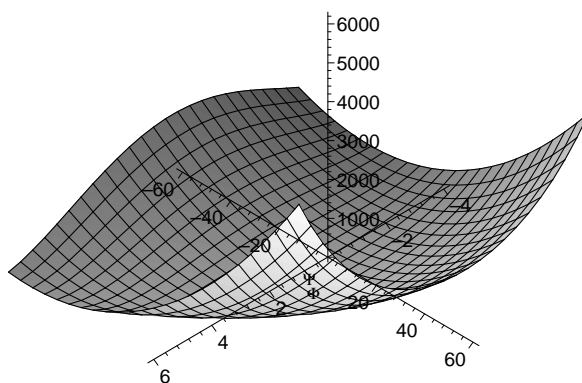


Figure 4.2: Lyapunov surface for  $V(R, \Phi, \Psi)$ , with  $R=1$ .

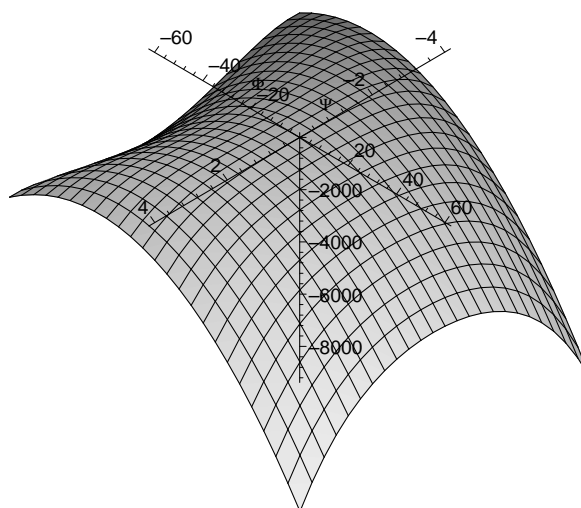


Figure 4.3: Derivative Lyapunov surface for  $\dot{V}(R, \Phi, \Psi)$ , with  $R=1$ .

which is negative definite with respect to  $\phi$  and  $\tilde{\psi}$ .

Therefore, the equilibrium point  $(\phi, \tilde{\psi}) = 0$  of the upper subsystem is GAS for all  $R \geq 0$ . In addition,  $R(t) \rightarrow 0$  because of the ISS property of (4.28). This means that surge and stall are suppressed within the region of validity of this engine model. Following the Lemma 4.2.1, we can conclude that the controller (4.37) renders GAS for the overall closed-loop system. The Lyapunov surface is shown in Figure 4.2 and its derivative is shown in Figure 4.3.

The controller (4.37) is then implemented digitally, assuming that the controller is put between a sampler and zero-order-hold device. With this, the control is constant between every sampling interval. Applying Corollary 3.5.1 of Chapter 3, which is a consequence of Theorem 3.3.3, we can define the inclusion sets of states and inputs, and then compute the upper bound for the sampling period  $T^* > 0$  by using the formula (3.56), which derivation is given in the proof of Theorem 3.3.1. Moreover, we can conclude that for any sampling period  $T \in (0, T^*)$ , the closed-loop sampled-data system is semiglobally practically asymptotically stable.

Table 4.1: Jet engine plant's and controllers' parameters

Parameter	Notation	Value
Physical dimensions and modeling parameter	$\sigma$	7
Number of stages in the compressor	$\Psi_{C0}$	4
Controller gain 1	$c_1$	1
Controller gain 2	$c_2$	1

It remains only to transform back the states into the original states of the jet engine system. This can be done directly by substituting back the appropriate terms to the control law and the dynamic equation of the plant. We then simulate the system when the emulation controller is implemented to control the continuous-time plant. We compare the responses with the case when the original continuous-time controller is used. The parameters of the jet engine plant and the controllers that are used through out all simulations in this chapter follow [94] and are listed in Table 4.1.

For the Simulation 4.1a, we use sampling period  $T = 0.01$  sec and initial condition  $x_o = (0 \ 0 \ 0)^T$ . The results of Simulation 4.1a are shown in Figure 4.4.

The figure shows that under the chosen simulation parameters, in which we choose a zero initial condition and a relatively fast sampling, the emulation controller is able to bring all the states to the equilibrium with good performance, and the time response of each state is very close to the responses of the continuous-time system. This behavior is as expected, since as shown in Figure 4.4(c), with small sampling the discrete-time control signal is very close to the continuous-time control signal.

For the second simulation, the Simulation 4.2a, we increase the time sampling to  $T = 0.5$  sec, while the initial condition is still zero. With this slower sampling, the performance of the system is degraded and the closed-loop system becomes oscillatory. The response of the system is shown in Figure 4.5, where we can show that the controller cannot stabilize the closed-loop system when the sampling is quite large.

The disadvantage of emulation is that sampling is ignored during the design, whereas the change of sampling rate is affecting significantly the control signal. Therefore, the behavior of the closed-loop system may be sensitive to the increase of sampling rate. In the next subsection, we design another controller, where sampling is taken into account, expecting better performance of the closed-loop sampled-data system.

#### 4.4.2 Direct discrete-time backstepping controller design

In the previous subsection, we have designed a discrete-time emulation controller for the jet engine and have observed the closed-loop behavior of the system with the emulation controller. In this subsection, we design another controller, expecting to achieve some improvement to the design. This second controller is designed using the same technique, backstepping. However, this time the design is done in direct discrete-time mode, based on the Euler model of the plant. The design procedure follows similar steps as the continuous-time backstepping. We apply an ad hoc design similar to the result of [110] cited in Theorem 4.2.1 to obtain the control law.

Recall the system (4.28)-(4.30), and we build the Euler discretization of the system. The difference between the technique we use and the Theorem 4.2.1 is the fact that we do not apply the design step to the first subsystem (4.28). We let the second subsystem (4.29) inherit its stability to the first subsystem (4.28) instead, following the same

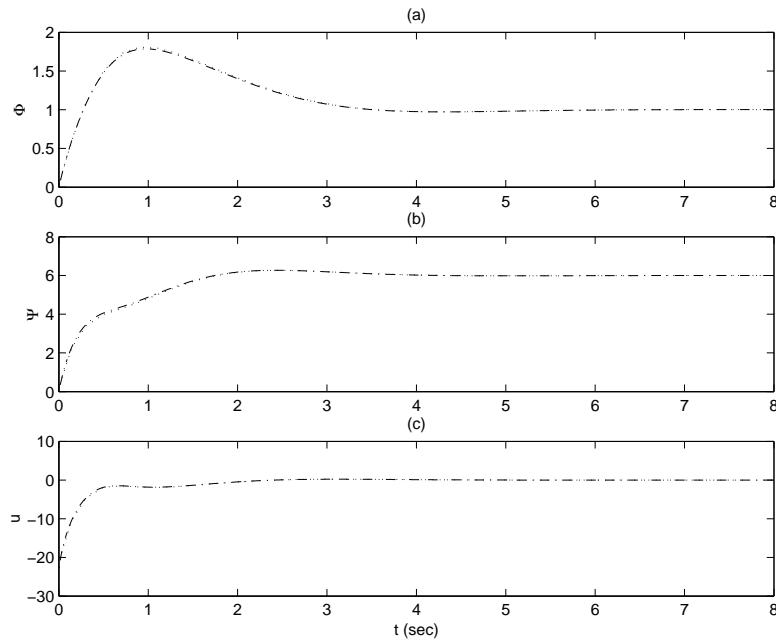


Figure 4.4: Simulation 4.1a, response of the system with small sampling period  $T = 0.01$  sec (— emulation,  $\cdots$  continuous).

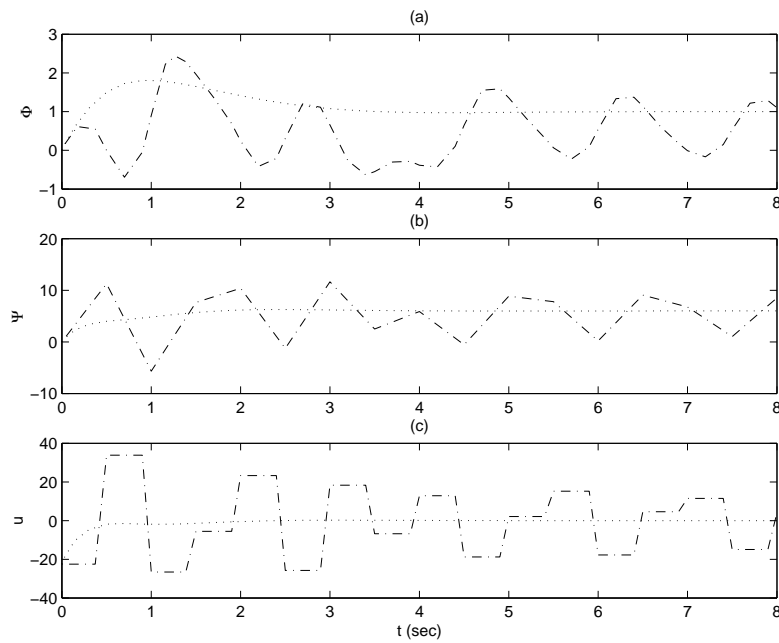


Figure 4.5: Simulation 4.2a, response of the system with large sampling period  $T = 0.5$  sec (— emulation,  $\cdots$  continuous).

argument we used in the continuous-time design. This leads to the use of Lyapunov function that does not depend on the first state variable  $R$ , which causes negative semidefiniteness of the derivative of the Lyapunov function (this condition is normally not supported by Theorem 4.2.1).

Relying on the intuition that the design steps given by [110] would still hold ignoring the above mentioned difference, we continue the discrete-time design to obtain the “Euler-based like” controller. Suppose that we have done the state transformation to the model (4.22), and obtain the strict feedback form for the model. The Euler approximate model of the system is written as follows:

$$\begin{aligned} R(k+1) &= R(k) + T(\sigma R(k)^2 - \sigma R(k)(2\phi(k) + \phi(k)^2)) \\ \phi(k+1) &= \phi(k) + T(-\psi(k) - \frac{3}{2}\phi(k)^2 - \frac{1}{2}\phi(k)^3 - 3R(k)\phi(k) - 3R(k)) \\ \psi(k+1) &= \psi(k) - Tu(k) . \end{aligned} \quad (4.40)$$

The discretization is obtained using sampler and zero-order-hold, where we keep the control constant during each sampling interval. Therefore we have  $u(t) = u(kT) =: u(k)$ ,  $\forall t \in [kT, (k+1)T)$ ,  $k \in \mathbb{N}$ , where  $T > 0$  is the sampling period. The same treatment is also applied to the state measurements. Henceforth, we drop the discrete-time argument  $k$  from the state variables and the input since it is now clear from the context.

We apply the result adopted from [110] to obtain the discrete-time controller. First, we substitute the state equation of the jet engine into (4.12),(4.13), and then apply Theorem 4.2.1 to design the controller. We use formula (4.16) to construct the control law, with

$$\alpha_T(\phi, R) = c_1\phi - \frac{3}{2}\phi^2 - 3R \quad (4.41)$$

$$W_T(\phi, R) = \frac{1}{2}\phi^2 . \quad (4.42)$$

We compute

$$\begin{aligned}
\Delta\alpha_T &= c_1\phi(k+1) - \frac{3}{2}\phi^2(k+1) - 3R(k+1) - (c_1\phi - \frac{3}{2}\phi^2 - 3R) \\
&= c_1T(-\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 - 3R\phi - 3R) - 3T\phi(-\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 \\
&\quad - 3R\phi - 3R) - \frac{3}{2}T^2(-\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 - 3R\phi - 3R)^2 \\
&\quad - 3(\sigma R^2 - \sigma R(2\phi + \phi^2)) ,
\end{aligned} \tag{4.43}$$

and get

$$\begin{aligned}
\frac{\Delta\alpha_T}{T} &= (c_1 - 3\phi) \left( -\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 - 3R\phi - 3R \right) + 3\sigma R(-2\phi - \phi^2 - R) \\
&\quad - \frac{3}{2}T \left( -\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 - 3R\phi - 3R \right)^2 .
\end{aligned} \tag{4.44}$$

Moreover, we compute

$$\begin{aligned}
\overline{\Delta W}_T &= \frac{1}{2}(\phi + T(-\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 - 3R\phi - 3R))^2 \\
&\quad - \frac{1}{2}(\phi + T(-\frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 - 3R\phi - 3R - \alpha))^2 \\
&= \frac{1}{2}(-2T(\psi - \alpha)(-\frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 - 3R\phi - 3R) + T^2(\psi^2 - \alpha^2)) ,
\end{aligned} \tag{4.45}$$

and, using (4.18) we obtain

$$\frac{\widetilde{\Delta W}_T}{T} = -\phi + \frac{1}{2}T(3\phi^2 + \phi^3 + 6R\phi + 6R + \psi + \alpha) . \tag{4.46}$$

Hence, the controller is obtained as the following:

$$\begin{aligned}
-u_T(k) &= -c_2(\psi - \alpha_T) + \phi - \frac{1}{2}T(\frac{3}{2}\phi^2 + \phi^3 + 6R\phi + 3R + \psi + c_1\phi) \\
&\quad + (c_1 - 3\phi)(-\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 - 3R\phi - 3R) \\
&\quad - 3(\sigma R^2 - \sigma R(2\phi + \phi^2)) - \frac{3}{2}T(-\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 - 3R\phi - 3R)^2 ,
\end{aligned} \tag{4.47}$$

that can be written as

$$\begin{aligned}
u_T(k) &= u_c + T(\frac{3}{2}(-\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 - 3R\phi - 3R)^2 + \frac{1}{2}(\frac{3}{2}\phi^2 + \phi^3 \\
&\quad + 6R\phi + 3R + \psi + c_1\phi)) \\
&= u_c + T\frac{1}{2}(3(-\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 - 3R\phi - 3R)^2 + (\frac{3}{2}\phi^2 + \phi^3 \\
&\quad + 6R\phi + 3R + \psi + c_1\phi)) .
\end{aligned} \tag{4.48}$$

With this control, the discrete-time closed-loop system satisfies the following

$$\begin{aligned}
 R(k+1) &= R + T(\sigma R^2 - \sigma R(2\phi + \phi^2)) \\
 \phi(k+1) &= \phi + T(-c_1\phi - \frac{1}{2}\phi^3 - 3R\phi - \tilde{\psi}) \\
 \psi(k+1) &= \psi + T(\phi - c_2\tilde{\psi}) + T^2\frac{1}{2}(3(-\psi - \frac{3}{2}\phi^2 - \frac{1}{2}\phi^3 - 3R\phi - 3R)^2 \\
 &\quad + (\frac{3}{2}\phi^2 + \phi^3 + 6R\phi + 3R + \psi + c_1\phi))
 \end{aligned} \tag{4.49}$$

and the Lyapunov difference

$$\Delta V_T \leq T(-c_1\phi^2 - \frac{1}{2}\phi^4 - 3R\phi^4 - c_2\tilde{\psi}^2) + O(T^2) . \tag{4.50}$$

Since all the states are bounded, then the higher order term  $O(T^2)$  can be made very small for a small sampling period. We can pick a small positive constant  $\nu$  and find a sampling period  $T$  so that the Lyapunov difference satisfies

$$\Delta V_T \leq T(-c_1\phi^2 - \frac{1}{2}\phi^4 - 3R\phi^4 - c_2\tilde{\psi}^2 + \nu) . \tag{4.51}$$

Finally, we apply Theorem 4.2.1 to conclude that the controller renders semiglobal practical asymptotic stability to the closed-loop system for some sampling periods  $T > 0$ .

## 4.5 Comparison of the emulation and the Euler based controllers

Up to this stage, we have designed two backstepping controller for the jet engine system, and have shown that both controllers asymptotically stabilize the jet engine in a semiglobal practical sense. We have also observed the performance of the closed-loop system when implementing the emulation controller.

In this section, we implement the “Euler-based like” controller to control the jet engine, and observe the performance of the closed-loop sampled-data system with this controller, compared with the performance when applying the emulation controller and use the continuous-time system performance as the reference.

It makes sense to compare the controllers (4.37) and (4.48) since the controller (4.48) can in fact be seen as the refined form of (4.37), where the relationship between (4.37) and (4.48) follows the formula (4.21).

Figure 4.6 shows the result of Simulation 4.1b, where we use the same set of parameters as in Simulation 4.1a, which is a quite standard set of parameters, where we set a relatively small initial condition and relatively fast sampling. Comparing the two discrete-time controllers with the continuous-time controller, we can see that the responses of the closed-loop system with the three controllers almost overlap each other.

For Simulation 4.2b, the set of parameters that have been used in Simulation 4.2a are applied. The simulation result is shown in Figure 4.7. With this setting, it is shown that while the emulation controller causes the system to oscillate, the Euler-based controller still performs very well, even when compared to the continuous-time controller.

The next two simulations are carried out when the initial conditions are not zero. In Simulation 4.3 we set  $T = 0.1$  sec and  $x_o = (1 \ 2 \ 7)^T$ , which is quite close to the equilibrium point of the system. The simulation result of Simulation 4.3 is shown in Figure 4.8, where we again see that the Euler-based controller gives a better response.

For the last simulation, the Simulation 4.4, we set  $T = 0.05$  sec with large initial condition  $x_o = (5 \ 5 \ 10)^T$ . The simulation result is shown in Figure 4.9. From this simulation, we observe that the Euler-based controller again outperforms the emulation controller, in terms of the closeness of responses to the continuous-time controller.

From the four sets of simulation presented above, we see that for a very small sampling period and zero or very small initial conditions, all controllers perform quite similarly. For the rest, in general the system with the Euler-based controller can always achieve better performance than with the emulation controller. This has shown us that direct discrete-time design gives a possible way to improve the quality of sampled-data design.

## 4.6 Conclusion

We have applied our emulation design framework to the problem of rotating stall and surge control for a jet engine system. We have shown that emulation design works for this example. However, the main drawback in emulation design, that sampling is



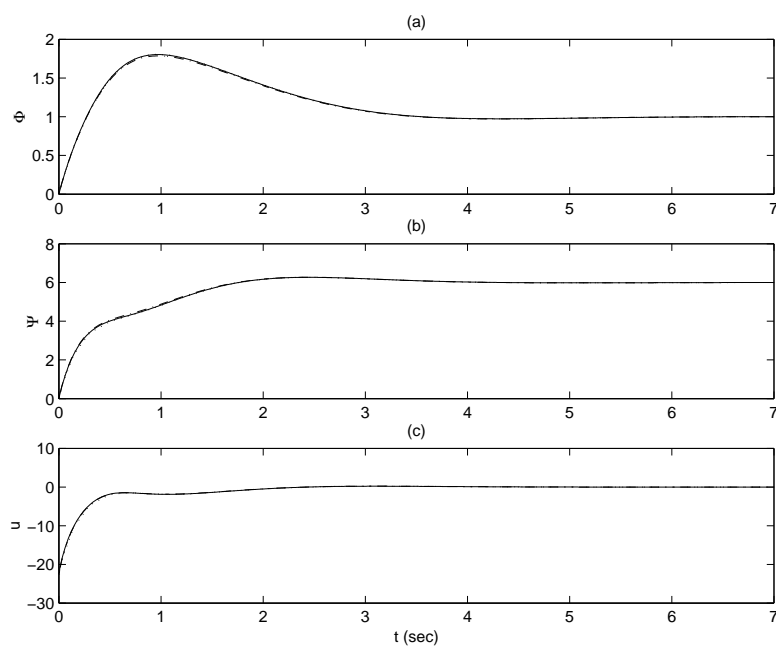


Figure 4.6: Simulation 4.1b, response of the system with small sampling period  $T = 0.01$  sec (--- Euler, -.- emulation,  $\cdots$  continuous).

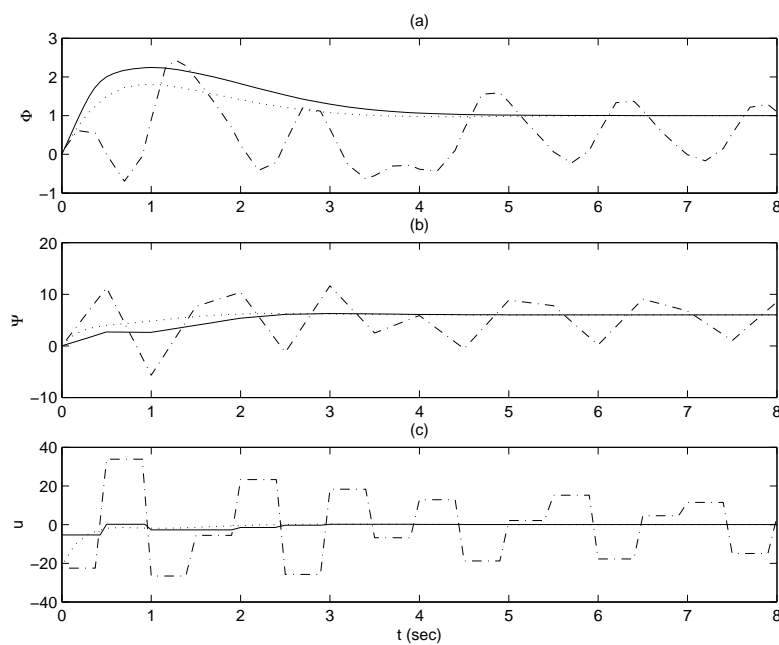


Figure 4.7: Simulation 4.2b, response of the system with large sampling period  $T = 0.5$  sec (--- Euler, -.- emulation,  $\cdots$  continuous).

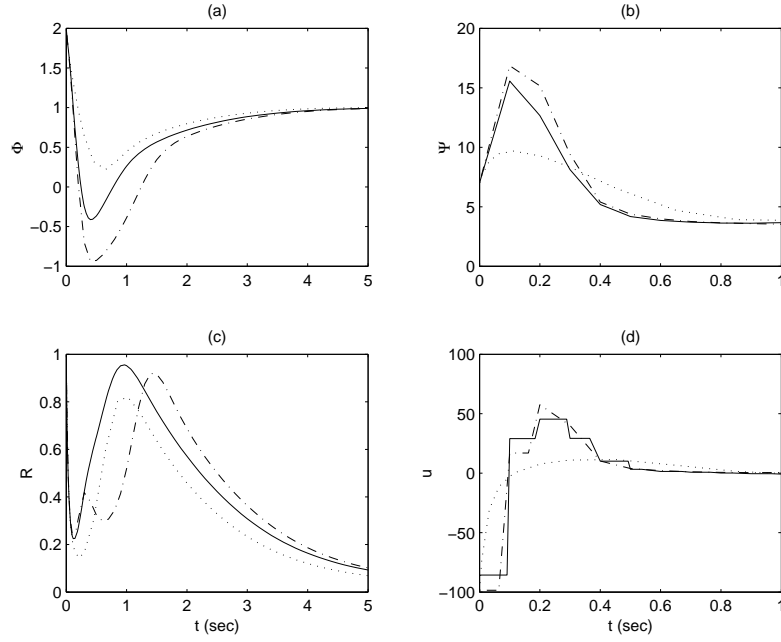


Figure 4.8: Simulation 4.3, response of the system with nonzero initial states  $x_o = (1 \ 2 \ 7)^T$  (--- Euler, -.- emulation,  $\cdots$  continuous).

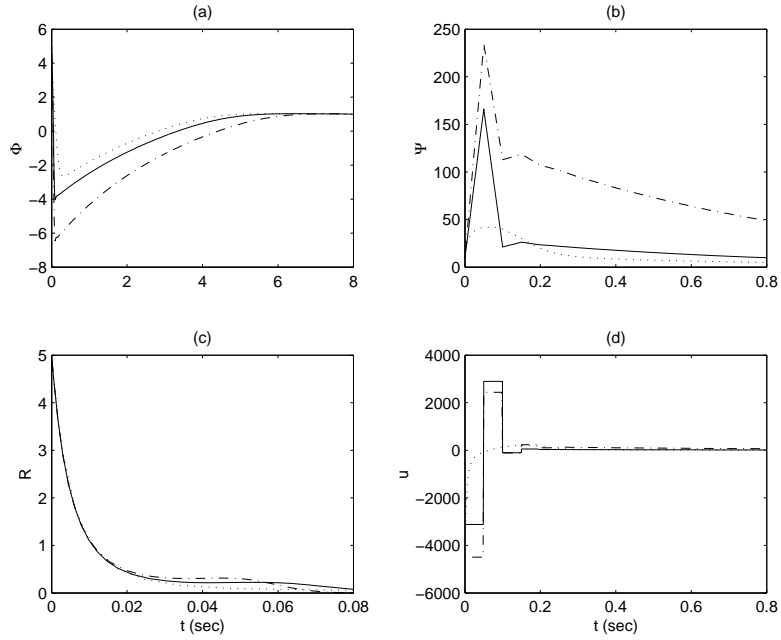


Figure 4.9: Simulation 4.4, response of the system with large initial states  $x_o = (5 \ 5 \ 10)^T$  (--- Euler, -.- emulation,  $\cdots$  continuous).

ignored prior to the implementation stage has caused the closed-loop sampled-data system to suffer some degradation of performance as a result of discretization. Moreover, a very fast sampling is needed when implementing the emulation controller. Although in Chapter 3, proofs of the main results also provide bounds on the sampling rate allowed, these bounds are however very conservative. In reality, much larger sampling periods can be applied to the sampled-data system.

On the other hand, we have shown that discrete-time controller obtained by direct discrete-time design can give some improvement to the closed-loop performance, although the design was done in an *ad hoc* way. The findings from the example in this chapter is not the only cases where direct discrete-time controller outperform the emulation controller. Example presented in [110] has exhibited similar phenomenon, where the direct discrete-time controller consistently shows larger domain of attraction than the emulation controller. Also in [158, 159], a direct discrete-time controller guarantees asymptotic stability of the closed-loop system that is not achievable by the emulation controller for a two link manipulator system with Slotine and Lie controller. Except the example in [110], all design were done in an *ad hoc* approach.

From the jet engine example presented in this chapter, together with two other examples presented in [110, 158, 159], it is reasonable to assume that if sampling is taken into account from the beginning of the design, then better performance can potentially be achieved. This has motivated the research on direct discrete-time design for nonlinear sampled-data system. The rest of this thesis contributes to more procedural way to do a direct discrete-time design. Part II of this thesis is dedicated to the research where methods that use approximate discrete-time model of the plant for controller design are developed, by considering a more general class of nonlinear systems with inputs.



## Part II

# Design and Analysis via Approximate Discrete-Time Models



## Chapter 5

# A Framework for Input-to-State Stabilization

### 5.1 Introduction

In Chapter 3, general results on emulation design for sampled-data systems were presented. An engineering example was explored in Chapter 4, to show the applicability of the results from Chapter 3. Moreover, the example has indicated that design can be improved if sampling is taken into account in the design process. This has motivated the research in this chapter, where we provide a framework of direct discrete-time stabilization for nonlinear sampled-data systems with inputs.

The results presented in this chapter are based on results that have appeared in [105, 106], which focus on the direct discrete-time design technique for sampled-data systems. The design technique is known as another alternative to designing controllers for sampled-data systems.

A stumbling block in direct discrete-time controller design for nonlinear sampled-data control systems is the absence of a good model for the design. Indeed, even if the continuous-time plant model is known, we can not in general compute the exact discrete-time model of the plant since this requires an explicit analytic solution of a nonlinear differential equation. This has motivated research on controller design via approximate discrete-time models for sampled-data nonlinear systems [23, 33, 92].

A drawback of these early results was their limited applicability: they investigate a particular class of plant models, a particular approximate discrete-time plant model (usually Euler) and a particular controller.

A more general framework for stabilization of *disturbance-free* sampled-data nonlinear systems via their approximate discrete-time models that is applicable to general plant models, controllers and approximate discrete-time models was first presented in [109, 116]. Example in Chapter 4, where asymptotic stability becomes the main concern, dealt with this disturbance-free situation.

In this chapter, we generalize results in [116] by: (i) considering sampled-data nonlinear systems *with disturbances*; (ii) providing a framework for the design of input-to-state stabilizing (ISS) controllers based on approximate discrete-time plant models (for more details on ISS see [74, 139, 141, 144]). In particular, we provide sufficient conditions for the continuous-time plant model, the controller and the approximate discrete-time model, which guarantee that if the controller input-to-state stabilizes the approximate discrete-time plant model it would also input-to-state stabilize the exact discrete-time plant model. Our results apply to dynamic controllers and our approach benefits from the results on numerical integration schemes in [151] and [27, 37]. Our interest in input-to-state stabilization is motivated by numerous applications of this robust stability property that have appeared in the literature [57, 74, 139, 141].

The chapter is organized as follows. Section 5.2 provides some motivating examples to start the presentation of this chapter. In Section 5.3 we present preliminaries. The main results are stated and proved in Section 5.4, follows with an example in Section 5.5. Finally, the conclusion is given in the last section.

## 5.2 Motivating examples

In Chapter 4, it has been shown that approximate based discrete-time design has potential to improve the design performance compared with emulation design. However, in that chapter, design was done based on an *ad hoc* approach that does not work in general. In this section, we will show some examples of the cases when controller design based on approximate discrete-time models fails and the cases when the design



succeeds. These examples motivate careful investigation of direct discrete-time design based on approximate discrete-time models.

This section is completely taken from [114]. We cite this part in this chapter since it is fully relevant to motivate stronger the conditions presented in this chapter.

### 5.2.1 When things go wrong

In the following three examples, we design control laws for several continuous-time processes based on an approximate discrete-time model. The approximate models are “consistent” in the sense that they approach the exact discrete-time model in the limit as a modeling parameter tends to zero. Moreover, the control laws, which are also parameterized by the discrete-time modeling parameter, globally exponentially or asymptotically stabilize the origin of the approximate model. Nevertheless, the origin of the closed-loop system with the exact discrete-time model is exponentially unstable, or at least not approximately attractive, no matter how small the modeling parameter is. The problem is that the family of discrete-time closed-loop systems does not have the proper robustness condition to account for the mismatch between the exact and approximate discrete-time plant model. Each example which is discussed has a different indicator of insufficient robustness. The main contribution of results in this chapter will be to show if these indicators are ruled out then robustness to the mismatch between approximate and discrete-time models can be guaranteed.

**Example 5.2.1 (Control with excessive force)** *We consider the sampled-data control of the triple integrator*

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u .\end{aligned}\tag{5.1}$$

*Although the exact discrete-time model of this system can be computed, we base our control algorithm on the family of the Euler approximate discrete-time models in order to illustrate possible pitfalls in control design based on approximate discrete-time*

models. The family of Euler approximate discrete-time models is

$$\begin{aligned} x_1(k+1) &= x_1(k) + Tx_2(k) \\ x_2(k+1) &= x_2(k) + Tx_3(k) \\ x_3(k+1) &= x_3(k) + Tu(k) . \end{aligned} \tag{5.2}$$

A minimum time dead beat controller for the Euler discrete-time model is given by

$$u = \alpha_T(x) = \left( -\frac{x_1}{T^3} - \frac{3x_2}{T^2} - \frac{3x_3}{T} \right) . \tag{5.3}$$

The closed-loop system (5.2), (5.3) has all poles equal to zero for all  $T > 0$  and hence this discrete-time Euler based closed-loop system is asymptotically stable for all  $T > 0$ . On the other hand, the closed-loop system consisting of the exact discrete-time model of the triple integrator and controller (5.3) has a pole at  $\approx -2.644$  for all  $T > 0$ . Hence, the closed-loop sampled-data control system is unstable for all  $T > 0$ . ■

In the Example 5.2.1, the approximate closed-loop system contains two indicators that its robustness may not be sufficient to account for the mismatch between the approximate and exact discrete-time plant models, i.e.:

1. **Nonuniform bound on overshoot.** The solutions of the family of approximate models with the given controller satisfy for all  $T > 0$  an estimate of the following type:

$$|\phi_T(k, x_\circ)| \leq b_T e^{-kT} |x_\circ|, \quad k \in \mathbb{N}$$

and  $b_T \rightarrow \infty$  as  $T \rightarrow 0$ . Hence, the overshoot in the stability estimate for the family of approximate models is not uniformly bounded in  $T$ .

2. **Nonuniform bound on control.** The control is not uniformly bounded on compact sets with respect to the parameter  $T$  and in particular we have for all  $x \neq 0$  that  $|\alpha_T(x)| \rightarrow \infty$  as  $T \rightarrow 0$ .

**Example 5.2.2 (Control with excessive finesse)** Consider the system

$$\dot{x} = x + u . \tag{5.4}$$

Again, the exact discrete-time model of the system can be computed, but we consider a control design based on the “partial Euler” model

$$x(k+1) = (1+T)x(k) + (e^T - 1)u(k) . \quad (5.5)$$

The control

$$u = \alpha_T(x) = -\frac{T(1 + \frac{1}{2}T)x}{e^T - 1} \quad (5.6)$$

stabilizes the family of approximate models (for  $T \in (0, 2)$ ) by placing the pole of the closed-loop at  $1 - \frac{1}{2}T^2$ . On the other hand, the pole of the exact discrete-time closed-loop is located at

$$e^T - T - \frac{1}{2}T^2 > 1, \quad \forall T > 0 .$$

■

In the Example 5.2.2, the approximate closed-loop system contains the following indicator that its robustness may not be sufficient to account for the mismatch between the approximate and the exact discrete-time plant models:

- **Nonuniform attractive rate.** For all  $T > 0$ , the family of approximate discrete-time models satisfies

$$|\phi_T(k, x_o)| \leq b e^{-kT^2} |x_o|, \quad k \in \mathbb{N} ,$$

where  $b > 0$  is independent of  $T$ . Therefore the overshoot is uniformly bounded with  $T$ . However, if we think of  $kT = t$  as “continuous-time”, then as  $T \rightarrow 0$ , the rate of convergence of solutions satisfies that for any  $t > 0$  we have  $e^{-tT} \rightarrow 1$ . In other words, the rate of convergence in continuous-time is not uniform in the parameter  $T$ .

### Example 5.2.3 (Control without a continuous Lyapunov function certificate)

Consider a single integrator

$$\dot{x} = u , \quad (5.7)$$

and a fixed sampling period  $T$ . We build a controller based on the approximate discrete-time model

$$x(k+1) = x(k) + (T + \varepsilon)u(k) , \quad (5.8)$$

where  $\varepsilon > 0$ . As  $\varepsilon \rightarrow 0$  we approach the exact discrete-time model. We choose

$$u = \alpha_\varepsilon(x) = \frac{1}{T + \varepsilon} [x + f(x)] , \quad (5.9)$$

where  $f$  is defined as follows:

$$f(0) = 0, \quad f(x) = \operatorname{sgn}(x)j(x) \quad x \neq 0 \quad (5.10)$$

and  $j(x)$  is the integer  $j$  satisfying  $j < |x| \leq j+1$ . The approximate closed-loop system is

$$x(k+1) = f(x) , \quad (5.11)$$

which origin is locally exponentially and globally asymptotically stable. The exact discrete-time closed-loop system is

$$x(k+1) = f(x) + \frac{\varepsilon}{T + \varepsilon} [x - f(x)] , \quad (5.12)$$

which has the set  $\mathbb{R}_{>j} := \{x \in \mathbb{R} : x > j\}$ , for each positive integer  $j$ , forward invariant for all  $\varepsilon > 0$ . Thus, the origin of the exact discrete-time closed-loop system is not even approximately attractive. The problem is that the asymptotic stability of the origin for (5.11) has no robustness. ■

In the Example 5.2.3, while it would be easy to attribute this to the discontinuous nature of the right-hand side of (5.11), we would like to develop results that permit discontinuous controllers. Indeed, discontinuous controllers are generic in stabilization algorithms based on optimal control formulations (see, for example [20]). The property that gives such algorithms some limited robustness is that they come with a continuous Lyapunov function that certifies asymptotic stability. While discontinuous Lyapunov functions are generic (see, for example [116]), continuous Lyapunov functions are not. It can be shown that the system (5.11) admits no continuous Lyapunov function. We take this as the indicator in the approximate closed-loop system that the robustness (it actually does not have any) may not be sufficient to account for the mismatch between the approximate and the exact discrete-time plant models:

- **No continuous Lyapunov function certificate.** If there existed a continuous Lyapunov function for (5.11), then the origin of (5.12) would be semiglobally

practically asymptotically stable in  $\varepsilon$ . Since this is not the case, there does not exist a continuous Lyapunov function that certifies the asymptotic stability of the origin of (5.11).

Note that loss of robustness of the parameterized discrete-time systems is similar to lack of robustness of time varying systems that lose robustness if they are non-uniformly asymptotically stable [101]. In our work, we will rule out all of the indicators that we have seen in the above examples by assuming that the feedback control is uniformly bounded in the modeling parameter and that there exists a parameterized family of continuous Lyapunov functions that are positive definite, decrescent and decreasing along trajectories with all of there properties uniform, in an appropriate sense, in the modeling parameter(s). The example in the next subsection illustrate what is sufficient.

### 5.2.2 When things go right

In the next example which is taken from [110], we demonstrate how the procedure we have described can out-perform sample and hold implementation of a continuous control algorithm.

**Example 5.2.4** *Consider the continuous-time plant:*

$$\begin{aligned}\dot{\eta} &= \eta^2 + \xi \\ \dot{\xi} &= u .\end{aligned}\tag{5.13}$$

*First, we design a continuous-time backstepping controller based on result in [74]. Note that the first subsystem can be stabilized with the “control”  $\phi(\eta) = -\eta^2 - \eta$  with the Lyapunov function  $W(\eta) = \frac{1}{2}\eta^2$ . Using this information and applying [74, Lemma 2.8 with  $c=1$ ], we obtain:*

$$u^{ct}(\eta, \xi) = -2\eta - \eta^2 - \xi - (2\eta + 1)(\xi + \eta^2) ,\tag{5.14}$$

*which globally asymptotically stabilizes the continuous-time model (5.13).*

*Assume now that the plant (5.13) is between a sampler and a zero-order-hold and*

consider its Euler approximate model:

$$\begin{aligned}\eta(k+1) &= \eta(k) + T(\eta^2(k) + \xi(k)) \\ \xi(k+1) &= \xi(k) + Tu(k) .\end{aligned}\tag{5.15}$$

Again, the control law  $\phi(\eta) = -\eta^2 - \eta$  globally asymptotically stabilizes the  $\eta$ -subsystem of (5.15) with the Lyapunov function  $W(\eta) = \frac{1}{2}\eta^2$ . Using results in [110], we obtain the controller:

$$u_T^{Euler}(\eta, \xi) = u^{ct}(\eta, \xi) - T[0.5\eta^2 + 0.5\xi - 0.5\eta + (\xi + \eta^2)^2] ,\tag{5.16}$$

which semiglobally practically asymptotically stabilizes the Euler model (5.15). This can be proven with the Lyapunov function  $V(\eta, \xi) = \frac{1}{2}\eta^2 + \frac{1}{2}(\xi + \eta + \eta^2)^2$ , which serves as a Lyapunov certificate of stability that is uniform in  $T$  in an appropriate way. ■

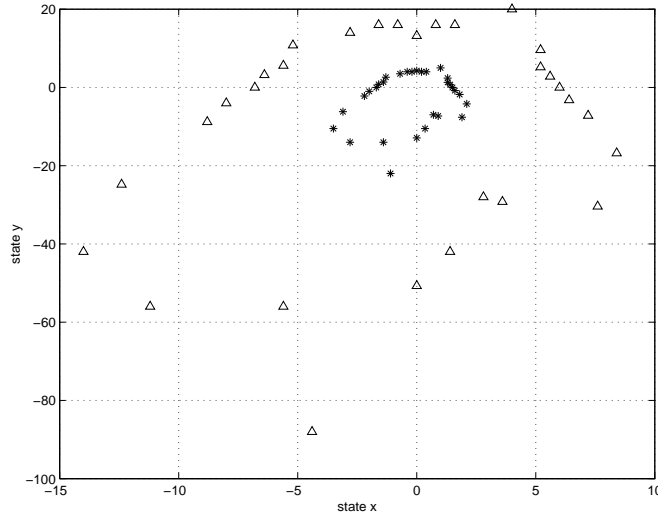


Figure 5.1: Estimates of DOA with  $T=0.5$  sec.

Note that the term  $-T[0.5\eta^2 + 0.5\xi - 0.5\eta + (\xi + \eta^2)^2]$  in (5.16) can be regarded as a modification to the controller (5.14). Moreover, for  $T = 0$  we have that  $u_0^{Euler}(\eta, \xi) = u^{ct}(\eta, \xi)$ . It turns out that  $u_T^{Euler}$  consistently yielded at least 4 times larger domain of attraction than  $u^{ct}$  for all tested sampling periods. Estimates domain of attraction (DOA) with the two controllers for the sampling period  $T = 0.5$  sec were obtained using simulations, which details can be found in [110]. Figure 5.1 indicates the boundary of

the estimate domain of attraction (DOA) for the closed-loop system with respectively  $u^{ct} (*)$  and  $u_T^{Euler} (\triangle)$ . It is important to mention that not all backstepping controllers that are based on Euler model will stabilize the exact model. Indeed, the controller (5.3) in Example 5.2.1 can be obtained using a backstepping procedure similar to the one used in this example.

### 5.3 Preliminaries

In this section we present technical preliminaries that will be used to state and prove our main results. For the sake of clarity and easy reading, we may repeat some notions and definitions that have been introduced in Chapter 3.

For a given function  $w : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ , we use the following notation:  $w_T[k] := \{w(t) : t \in [kT, (k+1)T]\}$  where  $k \in \mathbb{N}$  and  $T > 0$  (in other words  $w_T[k]$  is a piece of function  $w(t)$  in the  $k$ -th sampling interval  $[kT, (k+1)T]$ ); and  $w(k)$  is the value of the function  $w(\cdot)$  at  $t = kT, k \in \mathbb{N}$ . We denote the norms  $\|w_T[k]\|_\infty = \sup_{\tau \in [kT, (k+1)T]} |w(\tau)|$  and  $\|w\|_\infty := \sup_{\tau \geq 0} |w(\tau)|$  and in the case when  $w(\cdot)$  is a measurable function (in the Lebesgue sense) we use the essential supremum in the definitions. If  $\|w\|_\infty < \infty$ , then we write  $w \in \mathcal{L}_\infty$ . Consider a continuous-time nonlinear plant with disturbances:

$$\dot{x}(t) = f(x(t), u(t), w(t)) , \quad (5.17)$$

where  $x \in \mathbb{R}^{n_x}$ ,  $u \in \mathbb{R}^m$  and  $w \in \mathbb{R}^p$  are respectively the state, control input and exogenous disturbance. It is assumed that  $f$  is locally Lipschitz and  $f(0, 0, 0) = 0$ . We will consider two cases:  $w(\cdot)$  are measurable functions (in the Lebesgue sense); and  $w(\cdot)$  are continuously differentiable functions. We will always make precise which case we consider. The control is taken to be a piecewise constant signal  $u(t) = u(kT) =: u(k)$ ,  $\forall t \in [kT, (k+1)T)$ ,  $k \in \mathbb{N}$ , where  $T > 0$  is the sampling period. Also, we assume that some combination (output) or all of the states ( $x(k) := x(kT)$ ) are available at sampling instant  $kT, k \in \mathbb{N}$ . The exact discrete-time model for the plant (5.17), which describes the plant behavior at sampling instants  $kT$ , is obtained by integrating the initial value problem

$$\dot{x}(t) = f(x(t), u(k), w(t)) , \quad (5.18)$$

with given  $w_T[k]$ ,  $u(k)$  and  $x_0 = x(k)$ , over the sampling interval  $[kT, (k+1)T]$ . If we denote by  $x(t)$  the solution of the initial value problem (5.18) at time  $t$  with given  $x_0 = x(k)$ ,  $u(k)$  and  $w_T[k]$ , then the exact discrete-time model of (5.17) can be written as:

$$x(k+1) = x(k) + \int_{kT}^{(k+1)T} f(x(\tau), u(k), w(\tau)) d\tau =: F_T^e(x(k), u(k), w_T[k]) . \quad (5.19)$$

We refer to (5.19) as a *functional difference equation* since it depends on  $w_T[k]$ . We emphasize that  $F_T^e$  is not known in most cases. Indeed, in order to compute  $F_T^e$  we have to solve the initial value problem (5.18) analytically and this is usually impossible since  $f$  in (5.17) is nonlinear. Hence, we will use an approximate discrete-time model of the plant to design a controller.

Different approximate discrete-time models can be obtained using different methods. For example, we may first assume that the disturbances  $w(\cdot)$  are constant during sampling intervals,  $w(t) = w(k)$ ,  $\forall t \in [kT, (k+1)T]$  and then use a classical Runge-Kutta numerical integration scheme (such as Euler) for the initial value problem (5.18). In this case, the approximate discrete-time model can be written as

$$x(k+1) = F_T^a(x(k), u(k), w(k)) . \quad (5.20)$$

We refer to the approximate model (5.20) as an *ordinary difference equation* since  $F_T^a$  does not depend on  $w_T[k]$  but on  $w(k)$ . For instance, the Euler approximate model is  $x(k+1) = x(k) + T f(x(k), u(k), w(k))$ . Recently, numerical integration schemes for systems with measurable disturbances were considered in [27, 37]. Using these numerical integration techniques we can obtain an approximate discrete-time model

$$x(k+1) = F_T^a(x(k), u(k), w_T[k]) , \quad (5.21)$$

which is in general a functional difference equation. For instance, the simplest such approximate discrete-time model, which is analogous to Euler model, has the following form (see [37]):

$$x(k+1) = x(k) + \int_{kT}^{(k+1)T} f(x(k), u(k), w(s)) ds .$$



Since we will consider semiglobal ISS (see Definition 5.3.2), we will think of  $F_T^e$  and  $F_T^a$  as being defined globally for all small  $T$ , even though the initial value problem (5.18) may exhibit finite escape times.

The sampling period  $T$  is assumed to be a design parameter which can be arbitrarily assigned. Since we are dealing with a family of approximate discrete-time models  $F_T^a$ , parameterized by  $T$ , in order to achieve a certain objective we need in general to obtain a family of controllers, parameterized by  $T$ . We consider a family of dynamic feedback controllers

$$\begin{aligned} z(k+1) &= G_T(x(k), z(k)) \\ u(k) &= u_T(x(k), z(k)) , \end{aligned} \quad (5.22)$$

where  $z \in \mathbb{R}^{n_z}$ . To shorten notation, we introduce  $\tilde{x} := (x^T \ z^T)^T$ ,  $\tilde{x} \in \mathbb{R}^{n_{\tilde{x}}}$ , where  $n_{\tilde{x}} := n_x + n_z$  and

$$\mathcal{F}_T^i(\tilde{x}(k), \cdot) := \begin{pmatrix} F_T^i(x(k), u_T(x(k), z(k)), \cdot) \\ G_T(x(k), z(k)) \end{pmatrix} . \quad (5.23)$$

The superscript  $i$  may be either  $e$  or  $a$ , where  $e$  stands for *exact* model,  $a$  for *approximate* model. We omit the superscript if we refer to a general model. The second argument of  $\mathcal{F}_T^i(\tilde{x}, \cdot)$  (third argument of  $F_T^i$ ) is either a vector  $w(k)$  or a piece of function  $w_T[k]$ . Similar to [109], we define the following:

**Definition 5.3.1 (Lyapunov-SP-ISS)** *The family of systems  $\tilde{x}(k+1) = \mathcal{F}_T(\tilde{x}(k), w_T[k])$  is Lyapunov semiglobally practically input-to-state stable (Lyapunov-SP-ISS) if there exist functions  $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$  and  $\tilde{\gamma} \in \mathcal{K}$ , and for any strictly positive real numbers  $(\Delta_1, \Delta_2, \delta_1, \delta_2)$  there exist strictly positive real numbers  $T^*$  and  $L$  such that for all  $T \in (0, T^*)$  there exists a function  $V_T : \mathbb{R}^{n_{\tilde{x}}} \rightarrow \mathbb{R}_{\geq 0}$  such that for all  $\tilde{x} \in \mathbb{R}^{n_{\tilde{x}}}$  with  $|\tilde{x}| \leq \Delta_1$  and all  $w \in \mathcal{L}_\infty$  with  $\|w\|_\infty \leq \Delta_2$  the following holds:*

$$\alpha_1(|\tilde{x}|) \leq V_T(\tilde{x}) \leq \alpha_2(|\tilde{x}|) \quad (5.24)$$

$$\frac{1}{T}[V_T(\mathcal{F}_T(\tilde{x}, w_T)) - V_T(\tilde{x})] \leq -\alpha_3(|\tilde{x}|) + \tilde{\gamma}(\|w_T\|_\infty) + \delta_1 , \quad (5.25)$$

and, moreover, for all  $x_1, x_2, z$  with  $|(x_1^T \ z^T)^T|, |(x_2^T \ z^T)^T| \in [\delta_2, \Delta_1]$  and all  $T \in (0, T^*)$ , we have

$$|V_T(x_1, z) - V_T(x_2, z)| \leq L|x_1 - x_2| . \quad (5.26)$$

The function  $V_T$  is called an ISS-Lyapunov function for the family  $\mathcal{F}_T$ . ■

**Remark 5.3.1** *In the case when the family of parameterized closed-loop discrete-time nonlinear systems is an ordinary difference equation  $\tilde{x}(k+1) = \mathcal{F}_T(\tilde{x}(k), w(k))$ , the condition (5.25) is replaced by: for all  $T \in (0, T^*)$ , all  $\tilde{x} \in \mathbb{R}^{n_{\tilde{x}}}$  with  $|\tilde{x}| \leq \Delta_1$  and all  $w \in \mathbb{R}^p$  with  $|w| \leq \Delta_2$  we have*

$$\frac{1}{T}[V_T(\mathcal{F}_T(\tilde{x}, w)) - V_T(\tilde{x})] \leq -\alpha_3(|\tilde{x}|) + \tilde{\gamma}(|w|) + \delta_1, \quad (5.27)$$

and  $V_T$  is called an ISS-Lyapunov function for the family  $\mathcal{F}_T$ . ■

The following definition is a semiglobal-practical version of the ISS property used in [139, 141] and we use it in the case when we consider measurable disturbances  $w$ .

**Definition 5.3.2 (Semiglobal practical ISS)** *The family of systems  $\tilde{x}(k+1) = \mathcal{F}_T(\tilde{x}(k), w_T[k])$  is semiglobally practically input-to-state stable (SP-ISS) if there exist  $\beta \in \mathcal{KL}$  and  $\gamma \in \mathcal{K}_\infty$  such that for any strictly positive real numbers  $(\Delta_{\tilde{x}}, \Delta_w, \delta)$  there exists  $T^* > 0$  such that for all  $T \in (0, T^*)$ ,  $|\tilde{x}(0)| \leq \Delta_{\tilde{x}}$  and  $w(\cdot)$  with  $\|w\|_\infty \leq \Delta_w$ , the solutions of the system satisfy  $|\tilde{x}(k)| \leq \beta(|\tilde{x}(0)|, kT) + \gamma(\|w\|_\infty) + \delta, \forall k \in \mathbb{N}$ . ■*

The following semiglobal practical “ISS-like property” was used in [112] and we use it when the disturbances are continuously differentiable.

**Definition 5.3.3 (Semiglobal practical derivative ISS)** *The family of systems  $\tilde{x}(k+1) = \mathcal{F}_T(\tilde{x}(k), w_T[k])$  is semiglobally practically derivative input-to-state stable (SP-DISS) if there exist  $\beta \in \mathcal{KL}$  and  $\gamma \in \mathcal{K}_\infty$  such that for any strictly positive real numbers  $(\Delta_{\tilde{x}}, \Delta_w, \Delta_{\dot{w}}, \delta)$  there exists  $T^* > 0$  such that for all  $T \in (0, T^*)$ ,  $|\tilde{x}(0)| \leq \Delta_{\tilde{x}}$  and all continuously differentiable  $w(\cdot)$  such that  $\|w\|_\infty \leq \Delta_w$  and  $\|\dot{w}\|_\infty \leq \Delta_{\dot{w}}$ , the solutions of the family  $\mathcal{F}_T$  satisfy  $|\tilde{x}(k)| \leq \beta(|\tilde{x}(0)|, kT) + \gamma(\|w\|_\infty) + \delta, \forall k \in \mathbb{N}$ . ■*

Note that a similar property to SP-ISS, called input-to-state practical stability (ISpS) was defined in [60, 144] when considering non-parameterized systems.

**Definition 5.3.4 (Uniformly locally bounded)**  *$u_T$  is said to be uniformly locally bounded if for any  $\Delta_{\tilde{x}} > 0$  there exist strictly positive numbers  $T^*$  and  $\Delta_u$  such that for all  $T \in (0, T^*)$  and all  $|\tilde{x}| \leq \Delta_{\tilde{x}}$  we have  $|u_T(\tilde{x})| \leq \Delta_u$ . ■*

In order to prove our main results, we need to guarantee that the mismatch between  $F_T^e$  and  $F_T^a$  is small in some sense. To limit the mismatch, two consistency properties as what have been used in Chapter 3 are utilized. Due to convenient reading, we recall Definitions 3.2.2 and 3.2.3, adapting the notation used in the context of this chapter. In the sequel we use the notation  $x = x(k)$ ,  $u = u(k)$ ,  $w = w(k)$ ,  $w_T = w_T[k]$ .

**Definition 5.3.5 (One-step weak consistency)** *The family  $F_T^a$  is said to be one-step weakly consistent with  $F_T^e$  if given any strictly positive real numbers  $(\Delta_x, \Delta_u, \Delta_w, \Delta_{\dot{w}})$ , there exist a function  $\rho \in \mathcal{K}_\infty$  and  $T^* > 0$  such that, for all  $T \in (0, T^*)$ , all  $x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^m$  with  $|x| \leq \Delta_x, |u| \leq \Delta_u$  and continuously differentiable functions  $w(\cdot)$  that satisfy  $\|w_T\|_\infty \leq \Delta_w$  and  $\|\dot{w}_f\|_\infty \leq \Delta_{\dot{w}}$ , we have  $|F_T^e - F_T^a| \leq T\rho(T)$ . ■*

**Definition 5.3.6 (One-step strong consistency)** *The family  $F_T^a$  is said to be one-step strongly consistent with  $F_T^e$  if given any strictly positive real numbers  $(\Delta_x, \Delta_u, \Delta_w)$ , there exist a function  $\rho \in \mathcal{K}_\infty$  and  $T^* > 0$  such that, for all  $T \in (0, T^*)$ , all  $x \in \mathbb{R}^{n_x}, u \in \mathbb{R}^m, w \in \mathcal{L}_\infty$  with  $|x| \leq \Delta_x, |u| \leq \Delta_u, \|w_T\|_\infty \leq \Delta_w$ , we have  $|F_T^e - F_T^a| \leq T\rho(T)$ . ■*

Sufficient checkable conditions for one-step weak and strong consistency are given next.

**Lemma 5.3.1**  *$F_T^a$  is one-step weakly consistent with  $F_T^e$  if the following conditions hold:*

1.  $F_T^a$  is one-step weakly consistent with

$$F_T^{Euler}(x, u, w) := x + Tf(x, u, w) .$$

2. *Given any strictly positive real numbers  $(\Delta_x, \Delta_u, \Delta_w, \Delta_{\dot{w}})$ , there exist  $\rho_1 \in \mathcal{K}_\infty, \rho_2 \in \mathcal{K}_\infty, T^* > 0$ , such that, for all  $T \in (0, T^*)$ , all  $x_1, x_2 \in \mathbb{R}^{n_x}$  with  $\max\{|x_1|, |x_2|\} \leq \Delta_x$ , all  $u \in \mathbb{R}^m$  with  $|u| \leq \Delta_u$  and all  $w_1, w_2 \in \mathbb{R}^p$  with  $\max\{|w_1|, |w_2|\} \leq \Delta_w$ , the following holds*

$$|f(x_1, u, w_1) - f(x_2, u, w_2)| \leq \rho_1(|x_1 - x_2|) + \rho_2(|w_1 - w_2|) .$$

■

**Lemma 5.3.2**  $F_T^a$  is one-step strongly consistent with  $F_T^e$  if the following conditions hold:

1.  $F_T^a$  is one-step strongly consistent with

$$\tilde{F}_T^{Euler}(x, u, w_T) := x + \int_{kT}^{(k+1)T} f(x, u, w(s)) ds .$$

2. Given any strictly positive real numbers  $(\Delta_x, \Delta_u, \Delta_w)$ , there exist  $\rho_1 \in \mathcal{K}_\infty$ ,  $T^* > 0$ , such that, for all  $T \in (0, T^*)$  and for all  $x_1, x_2 \in \mathbb{R}^{n_x}$  with  $\max\{|x_1|, |x_2|\} \leq \Delta_x$ , all  $u \in \mathbb{R}^m$  with  $|u| \leq \Delta_u$  and all  $w \in \mathbb{R}^p$  with  $|w| \leq \Delta_w$ , the following holds

$$|f(x_1, u, w) - f(x_2, u, w)| \leq \rho_1(|x_1 - x_2|) .$$

■

**Proof of Lemma 5.3.1:** Let strictly positive real numbers  $(\Delta_x, \Delta_u, \Delta_w, \Delta_{\dot{w}})$  be given. Using the numbers  $(R_x, \Delta_u, \Delta_w, \Delta_{\dot{w}})$ , where  $R_x = \Delta_x + 1$ , let the second condition of the lemma generate  $T_1^* > 0$ ,  $\rho_1 \in \mathcal{K}_\infty$  and  $\rho_2 \in \mathcal{K}_\infty$ . Since  $f$  is locally Lipschitz, it is locally bounded and there exists a number  $M > 0$  such that for all  $|x| \leq R_x$ ,  $|u| \leq \Delta_u$ ,  $|w|_\infty \leq \Delta_w$  we have  $|f(x, u, w)| \leq M$ . Let  $T^* := \min\{T_1^*, 1/M\}$ . It follows that, for each  $|x| \leq \Delta_x$ ,  $\|w_T\|_\infty \leq \Delta_w$  and all  $t \in [kT, (k+1)T]$ , where  $T \in (0, T^*)$ , the solution  $x(t)$  of

$$\dot{x}(t) = f(x(t), u, w(t)) , \quad x_0 = x(k) = x \quad (5.28)$$

satisfies  $|x(t)| \leq R_x$  and  $|x(t) - x| \leq M(t - kT) \leq MT$  and since  $w(\cdot)$  is continuously differentiable by definition, we have  $|w(t) - w(k)| \leq \Delta_{\dot{w}}(t - kT) \leq \Delta_{\dot{w}}T$ , for all  $t \in [kT, (k+1)T]$  and  $T \in (0, T^*)$ . It then follows from condition 2 of the lemma that, for all  $|x| \leq \Delta_x$ ,  $|u| \leq \Delta_u$ ,  $\|w_T\|_\infty \leq \Delta_w$ ,  $\|\dot{w}_f[k]\|_\infty \leq \Delta_{\dot{w}}$ , and all  $T \in (0, T^*)$ ,

$$\begin{aligned} & \left| \int_{kT}^{(k+1)T} [f(x(\tau), u, w(\tau)) - f(x, u, w)] d\tau \right| \\ & \leq \int_{kT}^{(k+1)T} \rho_1(|x(\tau) - x|) d\tau + \int_{kT}^{(k+1)T} \rho_2(|w(\tau) - w|) d\tau \\ & \leq T\rho_1(MT) + T\rho_2(\Delta_{\dot{w}}T) \leq T\tilde{\rho}(T) , \end{aligned} \quad (5.29)$$

where  $\tilde{\rho}(s) := \rho_1(Ms) + \rho_2(\Delta_{\dot{w}}s)$  is a  $\mathcal{K}_\infty$  function since  $\rho_1$  and  $\rho_2$  are  $\mathcal{K}_\infty$ . Since

$$F_T^e = \underbrace{x + Tf(x, u, w)}_{F_T^{Euler}} + \int_{kT}^{(k+1)T} [f(x(\tau), u, w(\tau)) - f(x, u, w)] d\tau, \quad (5.30)$$

the result follows from (5.29) and the first condition of the lemma, which implies the existence of  $\tilde{\rho}_1 \in \mathcal{K}_\infty$ , such that  $|F_T^a - F_T^{Euler}| \leq T\tilde{\rho}_1(T)$ . Finally, by letting  $\rho = \tilde{\rho} + \tilde{\rho}_1$  we prove that  $F_T^a$  is one-step weakly consistent with  $F_T^e$ . Proof of Lemma 5.3.2 is omitted since it follows closely the proof of Lemma 5.3.1. ■

## 5.4 Main results

In this section, we state and prove our main results (Theorems 5.4.1 and 5.4.2). The results specify conditions on the approximate model, the controller and the plant, which guarantee that the family of controllers  $(G_T, u_T)$  that input-to-state stabilize  $F_T^a$  would also input-to-state stabilize  $F_T^e$  for sufficiently small  $T$ . We emphasize that our results are given for general approximate discrete-time models  $F_T^a$  (not only for the Euler approximation). We remark that under certain mild conditions on the plant and the controller, our results can be extended to include inter-sample behavior, to conclude SP-ISS results for the closed-loop sampled-data systems (see results in [117]).

**Theorem 5.4.1** *Suppose that:*

1. *The family of approximate discrete-time models  $\mathcal{F}_T^a(\tilde{x}, \cdot)$  is Lyapunov-SP-ISS (where either (5.25) or (5.27) holds);*
2.  *$F_T^a$  is one-step weakly consistent with  $F_T^e$ ;*
3.  *$u_T$  is uniformly locally bounded.*

*Then, the family of exact discrete-time models  $\mathcal{F}_T^e(\tilde{x}, w_T)$  is SP-DISS.* ■

**Theorem 5.4.2** *Suppose that:*

1. *The family of approximate discrete-time models  $\mathcal{F}_T^a(\tilde{x}, w_T)$  is Lyapunov-SP-ISS (where (5.25) holds);*

2.  $F_T^a$  is one-step strongly consistent with  $F_T^e$ ;

3.  $u_T$  is uniformly locally bounded.

Then, the family of exact discrete-time models  $\mathcal{F}_T^e(\tilde{x}, w_T)$  is SP-ISS. ■

The following lemmas are needed to complete proofs of both theorems. We prove only Lemma 5.4.1 for the case of ordinary difference equations (i.e., when (5.27) holds) and then comment on the changes in the proof for the case of functional difference equations (i.e., when (5.25) holds) and the proof of Lemma 5.4.2.

**Lemma 5.4.1** *If all conditions in Theorem 5.4.1 are satisfied, then there exist  $\hat{\gamma} \in \mathcal{K}_\infty$  such that for any strictly positive numbers  $(C_{\tilde{x}}, C_w, C_{\dot{w}}, \nu)$ , there exists  $T^* > 0$  such that for all  $T \in (0, T^*)$ , we have*

$$\left\{ \begin{array}{l} |\tilde{x}| \leq C_{\tilde{x}} , \quad \|w\|_\infty \leq C_w , \quad \|\dot{w}\|_\infty \leq C_{\dot{w}} \\ \max\{V_T(\mathcal{F}_T^e(\tilde{x}, w_T)), V_T(\tilde{x})\} \geq \hat{\gamma}(\|w\|_\infty) + \nu \end{array} \right\} \implies \frac{V_T(\mathcal{F}_T^e(\tilde{x}, w_T)) - V_T(\tilde{x})}{T} \leq -\frac{1}{4}\alpha_3(|\tilde{x}|) . \quad (5.31)$$
■

**Lemma 5.4.2** *If all conditions in Theorem 5.4.2 are satisfied, then there exist  $\hat{\gamma} \in \mathcal{K}_\infty$  such that for any strictly positive numbers  $(C_{\tilde{x}}, C_w, \nu)$ , there exists  $T^* > 0$  such that for all  $T \in (0, T^*)$ , we have*

$$\left\{ \begin{array}{l} |\tilde{x}| \leq C_{\tilde{x}} , \quad \|w\|_\infty \leq C_w \\ \max\{V_T(\mathcal{F}_T^e(\tilde{x}, w_T)), V_T(\tilde{x})\} \geq \hat{\gamma}(\|w\|_\infty) + \nu \end{array} \right\} \implies \frac{V_T(\mathcal{F}_T^e(\tilde{x}, w_T)) - V_T(\tilde{x})}{T} \leq -\frac{1}{4}\alpha_3(|\tilde{x}|) . \quad (5.32)$$
■

**Proof of Lemma 5.4.1:** First, we prove the following fact:

**Fact 1:** Suppose that for any strictly positive numbers  $(\tilde{\Delta}_1, \tilde{\Delta}_2, \tilde{\delta}_1)$  there exists  $T_w^* > 0$  such that for all  $T \in (0, T_w^*)$ ,  $|\tilde{x}| \leq \tilde{\Delta}_1$  and  $|w| \leq \tilde{\Delta}_2$  we have that (5.27) holds. Then, for any strictly positive numbers  $(\Delta_1, \Delta_2, \Delta_3, \delta_1)$  there exists  $T_s^* > 0$  such that for all

$T \in (0, T_s^*)$ ,  $|\tilde{x}| \leq \Delta_1$  and continuously differentiable disturbances with  $\|w\|_\infty \leq \Delta_2$  and  $\|\dot{w}\|_\infty \leq \Delta_3$  we have that

$$\frac{V_T(\mathcal{F}_T(\tilde{x}, w)) - V_T(\tilde{x})}{T} \leq -\alpha_3(|\tilde{x}|) + \tilde{\gamma}(\|w_T\|_\infty) + \delta_1. \quad (5.33)$$

**Proof of Fact 1:** Let the numbers  $(\Delta_1, \Delta_2, \Delta_3, \delta_1)$  be given. Let  $\tilde{\delta}$  be such that  $\sup_{s \in [0, \Delta_2]} |\tilde{\gamma}(s + \tilde{\delta}) - \tilde{\gamma}(s)| \leq \frac{\delta_1}{2}$ . Let  $\tilde{\Delta}_1 := \Delta_1$ ,  $\tilde{\Delta}_2 := \Delta_2$ ,  $\tilde{\delta}_1 := \frac{\delta_1}{2}$  and let the numbers  $(\tilde{\Delta}_1, \tilde{\Delta}_2, \tilde{\delta}_1)$  generate  $T_w^* > 0$  from the condition of Fact 1. Let  $T_s^* := \min\left(T_w^*, \frac{\tilde{\delta}}{\Delta_3}\right)$ . Consider arbitrary  $T \in (0, T_s^*)$ ,  $|\tilde{x}| \leq \Delta_1$  and any continuously differentiable disturbance with  $\|w\|_\infty \leq \Delta_2$  and  $\|\dot{w}\|_\infty \leq \Delta_3$ . From the Mean Value Theorem and our choice of  $T_s^*$  it follows that for all  $t \in [kT, (k+1)T]$ ,  $k \in \mathbb{N}$  we have that  $|w| = |w(k)| \leq |w(t) - w(kT)| + |w(t)| \leq \|\dot{w}_f\|_\infty (t - kT) + \|w_T\|_\infty \leq \Delta_3 T + \|w_T\|_\infty \leq \Delta_3 T_s^* + \|w_T\|_\infty \leq \tilde{\delta} + \|w_T\|_\infty$ . Finally, using our definitions of  $\tilde{\delta}, \tilde{\delta}_1$  we can write:

$$\begin{aligned} \frac{V_T(\mathcal{F}_T(\tilde{x}, w)) - V_T(\tilde{x})}{T} &\leq -\alpha_3(|\tilde{x}|) + \tilde{\gamma}(|w|) + \tilde{\delta}_1 \\ &= -\alpha_3(|\tilde{x}|) + \tilde{\gamma}(\|w_T\|_\infty) + \tilde{\gamma}(|w|) - \tilde{\gamma}(\|w_T\|_\infty) + \frac{\delta_1}{2} \\ &\leq -\alpha_3(|\tilde{x}|) + \tilde{\gamma}(\|w_T\|_\infty) + \tilde{\gamma}(\tilde{\delta} + \|w_T\|_\infty) \\ &\quad - \tilde{\gamma}(\|w_T\|_\infty) + \frac{\delta_1}{2} \\ &\leq -\alpha_3(|\tilde{x}|) + \tilde{\gamma}(\|w_T\|_\infty) + \frac{\delta_1}{2} + \frac{\delta_1}{2}, \end{aligned} \quad (5.34)$$

which completes the proof of the fact. Now we continue the proof of Lemma 5.4.1.

Suppose that all conditions in Theorem 5.4.1 (where (5.27) holds) are satisfied. Using Fact 1 it follows that all conditions in Theorem 5.4.1 (where (5.33) holds) are also satisfied. Let  $\hat{\gamma}(s) := \alpha_2 \circ \alpha_3^{-1}(4\tilde{\gamma}(s))$ . Let arbitrary strictly positive numbers  $(C_{\tilde{x}}, C_w, C_{\dot{w}}, \nu)$  be given. Using  $(C_{\tilde{x}}, C_w, C_{\dot{w}}, \nu)$ , we define:  $\epsilon := \frac{1}{2}\alpha_2^{-1}(\frac{\nu}{2})$ ;  $\delta_1 := \min\left\{\frac{1}{4}\alpha_1\left(\frac{\epsilon}{4}\right), \frac{1}{4}\alpha_3 \circ \alpha_2^{-1}\left(\frac{1}{2}\alpha_1(\epsilon)\right)\right\}$ ;  $\delta_2 := \alpha_2^{-1}\left(\frac{1}{2}\alpha_1(\epsilon)\right)$ ; and  $\Delta := \alpha_1^{-1}(\alpha_2(C_{\tilde{x}}) + \tilde{\gamma}(C_w) + \delta_1) + \epsilon$ . Let the numbers  $(\delta_1, \delta_2, \Delta, \epsilon)$  generate the numbers  $T_1^* > 0$  and  $L > 0$  from condition (i) of Theorem 5.4.1 (where (5.33) holds). Let  $\Delta$  generate  $\Delta_u > 0$  and  $T_2^* > 0$  from condition (iii) of Theorem 5.4.1. Let the quadruple  $(\Delta, \Delta_u, C_w, C_{\dot{w}})$  generate  $T_3^* > 0$  and  $\rho$  from condition (ii) of Theorem 5.4.1. Let strictly positive numbers  $T_4^*, T_5^*, T_6^*, T_7^*$  be such that:  $L\rho(T_4^*) \leq \frac{1}{4}\alpha_3(\delta_2)$ ;  $T_5^*\rho(T_5^*) \leq \epsilon$ ;

$T_6^* \tilde{\gamma}(C_w) \leq \frac{1}{2} \alpha_1 \left( \frac{1}{4} \epsilon \right)$ ; and  $T_7^* \left( \frac{1}{4} \alpha_3(C_{\tilde{x}}) + \tilde{\gamma}(C_w) + \delta_1 + L\rho(T_7^*) \right) \leq \frac{\nu}{2}$ . Finally, we take  $T^* = \min\{T_1^*, T_2^*, T_3^*, T_4^*, T_5^*, T_6^*, T_7^*, 1\}$ .

In the calculations that follow, we consider arbitrary  $T \in (0, T^*)$ ,  $|\tilde{x}| \leq C_{\tilde{x}}$ ,  $\|w\|_\infty \leq C_w$  and  $\|\dot{w}\|_\infty \leq C_{\dot{w}}$ . From (5.24), (5.25) and definition of  $\Delta$  and the fact that  $T^* \leq 1$ , we have that

$$\begin{aligned} |\mathcal{F}_T^a(\tilde{x}, w)| &\leq \alpha_1^{-1}(V_T(\mathcal{F}_T^a(\tilde{x}, w))) \\ &\leq \alpha_1^{-1}(V_T(\tilde{x}) + T\tilde{\gamma}(\|w\|_\infty) + T\delta_1) \\ &\leq \alpha_1^{-1}(\alpha_2(C_{\tilde{x}}) + \tilde{\gamma}(C_w) + \delta_1) \\ &< \Delta . \end{aligned} \tag{5.35}$$

Using the condition (ii) of Theorem 5.4.1, inequality (5.35) and our choice of  $\Delta$  and  $T^*$  (in particular the choice of  $T_5^*$ ), we can write:

$$\begin{aligned} |\mathcal{F}_T^e(\tilde{x}, w_T)| &\leq |\mathcal{F}_T^a(\tilde{x}, w)| + |\mathcal{F}_T^e(\tilde{x}, w_T) - \mathcal{F}_T^a(\tilde{x}, w)| \\ &\leq \alpha_1^{-1}(\alpha_2(C_{\tilde{x}}) + \tilde{\gamma}(C_w) + \delta_1) + T\rho(T) \\ &\leq \alpha_1^{-1}(\alpha_2(C_{\tilde{x}}) + \tilde{\gamma}(C_w) + \delta_1) + \epsilon = \Delta . \end{aligned} \tag{5.36}$$

Suppose that  $V_T(\mathcal{F}_T^e(\tilde{x}, w_T)) \geq \hat{\gamma}(C_w) + \frac{\nu}{2}$ . From (5.24), the definition of  $\epsilon$  and the choice of  $T^*$ , we have

$$|\mathcal{F}_T^e(\tilde{x}, w_T)| \geq \alpha_2^{-1} \left( \frac{\nu}{2} \right) = 2\epsilon > \epsilon , \tag{5.37}$$

and then using the condition (ii) of Theorem 5.4.1 and our choice of  $T_5^*$ , we have

$$\begin{aligned} |\mathcal{F}_T^a(\tilde{x}, w)| &\geq -|\mathcal{F}_T^e(\tilde{x}, w_T) - \mathcal{F}_T^a(\tilde{x}, w)| + |\mathcal{F}_T^e(\tilde{x}, w_T)| \\ &\geq -T\rho(T) + \alpha_2^{-1} \left( \frac{\nu}{2} \right) \\ &\geq -\epsilon + 2\epsilon = \epsilon . \end{aligned} \tag{5.38}$$

From our choice of  $T^* \leq 1$ ,  $T_6^*$ ,  $\delta_1$ , and  $\epsilon$  and using the inequality (5.33) it follows that:

$$\begin{aligned} \frac{1}{2} \alpha_1(\epsilon) &\leq \frac{1}{2} \alpha_1(\epsilon) + \frac{1}{2} \alpha_1(\epsilon) - \frac{1}{4} \alpha_1 \left( \frac{\epsilon}{4} \right) - \frac{1}{4} \alpha_1 \left( \frac{\epsilon}{4} \right) \\ &\leq \alpha_1(\epsilon) - T\tilde{\gamma}(C_w) - T\delta_1 \\ &\leq \alpha_1(|\mathcal{F}_T^a(\tilde{x}, w)|) - T\tilde{\gamma}(C_w) - T\delta_1 \\ &\leq V_T(\mathcal{F}_T^a(\tilde{x}, w)) - T\tilde{\gamma}(\|w\|_\infty) - T\delta_1 \\ &\leq V_T(\tilde{x}) \leq \alpha_2(|\tilde{x}|) , \end{aligned} \tag{5.39}$$



which implies:

$$|\tilde{x}| \geq \alpha_2^{-1} \left( \frac{1}{2} \alpha_1(\epsilon) \right) = \delta_2 . \quad (5.40)$$

Note that  $\delta_2 \leq \epsilon$ . From the conditions (i) and (ii) of Theorem 5.4.1 and from the choice of  $T^*$  (in particular the choice of  $T_4^*$  and  $T_7^*$ ), the choice of  $\delta_1$  and  $\delta_2$  and using (5.35)-(5.40) we deduce that  $V_T(\mathcal{F}_T^e) \geq \hat{\gamma}(C_w) + \frac{\nu}{2}$  implies

$$\begin{aligned} \hat{\gamma}(C_w) + \frac{\nu}{2} &\leq V_T(\mathcal{F}_T^e) - V_T(\tilde{x}) + V_T(\tilde{x}) + V_T(\mathcal{F}_T^a) - V_T(\mathcal{F}_T^a) \\ &\leq V_T(\mathcal{F}_T^a) - V_T(\tilde{x}) + |V_T(\mathcal{F}_T^e) - V_T(\mathcal{F}_T^a)| + V_T(\tilde{x}) \\ &\leq T\tilde{\gamma}(C_w) + T\delta_1 + LT\rho(T) + V_T(\tilde{x}) \\ &\leq \frac{\nu}{2} + V_T(\tilde{x}). \end{aligned} \quad (5.41)$$

Hence, we can conclude that

$$V_T(\mathcal{F}_T^e) \geq \hat{\gamma}(C_w) + \frac{\nu}{2} \implies V_T(\tilde{x}) \geq \hat{\gamma}(C_w). \quad (5.42)$$

Again using the conditions (i) and (ii) of Theorem 5.4.1 and from the choice of  $T^*$  (in particular the choice of  $T_4^*$ ), the choice of  $\delta_1$  and  $\delta_2$  and using (5.35)-(5.42) we can write:

$$\begin{aligned} &V_T(\mathcal{F}_T^e(\tilde{x}, w_T)) - V_T(\tilde{x}) \\ &\leq V_T(\mathcal{F}_T^a(\tilde{x}, w)) - V_T(\tilde{x}) + |V_T(\mathcal{F}_T^e(\tilde{x}, w_T)) - V_T(\mathcal{F}_T^a(\tilde{x}, w))| \\ &\leq -T\alpha_3(|\tilde{x}|) + T\tilde{\gamma}(\|w\|_\infty) + T\delta_1 + LT\rho(T) \\ &\leq -\frac{T}{4}\alpha_3(|\tilde{x}|) - \frac{3T}{4}\alpha_3(|\tilde{x}|) + T\tilde{\gamma}(C_w) + \frac{T}{4}\alpha_3(\delta_2) + \frac{T}{4}\alpha_3(\delta_2) \\ &\leq -\frac{T}{4}\alpha_3(|\tilde{x}|) - \underbrace{\frac{T}{4}\alpha_3 \circ \alpha_2^{-1}(V_T(\tilde{x}))}_{\leq 0} + T\tilde{\gamma}(C_w) - \underbrace{\frac{T}{2}\alpha_3(|\tilde{x}|) + \frac{T}{2}\alpha_3(\delta_2)}_{\leq 0} \\ &\leq -\frac{T}{4}\alpha_3(|\tilde{x}|) . \end{aligned} \quad (5.43)$$

Suppose now that  $V_T(\mathcal{F}_T^e(\tilde{x}, w_T)) \leq \hat{\gamma}(C_w) + \frac{\nu}{2}$  and  $V_T(\tilde{x}) \geq \hat{\gamma}(C_w) + \nu$ . From our choice of  $T^*$  (in particular the choice of  $T_7^*$ ), it follows that:

$$\begin{aligned} V_T(\mathcal{F}_T^e(\tilde{x}, w_T)) - V_T(\tilde{x}) &\leq \underbrace{\hat{\gamma}(C_w) + \frac{\nu}{2} - V_T(\tilde{x})}_{\leq 0} + \frac{\nu}{2} - \frac{\nu}{2} \\ &\leq -\frac{\nu}{2} \leq -\frac{T}{4}\alpha_3(|\tilde{x}|) , \end{aligned} \quad (5.44)$$

which shows that (5.31) is valid, and this completes the proof of Lemma 5.4.1.  $\blacksquare$

The proof of Lemma 5.4.1 for the case of functional difference equations and the proof of Lemma 5.4.3 follow the same steps as above except that we do not need to use Fact 1 since (5.25) holds. Also, in the case of functional difference equations of Lemma 5.4.1 we use one-step weak consistency and in the case of Lemma 5.4.3 we use one-step strong consistency. The next lemma is needed in proofs of Theorems 5.4.1 and 5.4.2 and it was proved as a part of the proof of Theorem 2 in [116].

**Lemma 5.4.3** *Let  $\mathcal{W} \subset \mathcal{L}_\infty$  and let  $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ . Let strictly positive real numbers  $(d, D)$  be such that  $\alpha_1(D) > d$  and let  $T^* > 0$  be such that for any  $T \in (0, T^*)$  there exists a function  $V_T : \mathbb{R}^{n_{\tilde{x}}} \rightarrow \mathbb{R}_{\geq 0}$  such that for all  $T \in (0, T^*)$  and all  $\tilde{x} \in \mathbb{R}^{n_{\tilde{x}}}$  we have  $\alpha_1(|\tilde{x}|) \leq V_T(\tilde{x}) \leq \alpha_2(|\tilde{x}|)$  and, moreover, for all  $\tilde{x} \in \mathbb{R}^{n_{\tilde{x}}}$  with  $\max\{V_T(\mathcal{F}_T(\tilde{x}, w_T)), V_T(\tilde{x})\} \geq d$  and  $|\tilde{x}| \leq D$ , all  $w \in \mathcal{W}$  and all  $T \in (0, T^*)$  the following holds  $\frac{V_T(\mathcal{F}_T(\tilde{x}, w_T)) - V_T(\tilde{x})}{T} \leq -\frac{1}{4}\alpha_3(|\tilde{x}|)$ . Then, there exist a function  $\beta \in \mathcal{KL}$  such that for all  $T \in (0, T^*)$ ,  $|\tilde{x}(0)| \leq \alpha_2^{-1} \circ \alpha_1(D)$  and  $w \in \mathcal{W}$  and all  $k \in \mathbb{N}$  the solutions of the family of discrete-time models  $\tilde{x}(k+1) = \mathcal{F}_T(\tilde{x}(k), w_T[k])$  exist and satisfy  $|\tilde{x}(k)| \leq \beta(|\tilde{x}(0)|, kT) + \alpha_1^{-1}(d)$ .  $\blacksquare$*

**Proof of Theorem 5.4.1:** Let arbitrary strictly positive real numbers  $(\Delta_{\tilde{x}}, \Delta_w, \Delta_{\dot{w}}, \delta)$  be given and let all conditions in Theorem 5.4.1 hold. Let  $\hat{\gamma} \in \mathcal{K}$  come from Lemma 5.4.1. We define  $(C_{\tilde{x}}, C_w, C_{\dot{w}}, \nu)$  as:  $C_w := \Delta_w$ ,  $C_{\dot{w}} := \Delta_{\dot{w}}$ ,  $\nu > 0$  is such that  $\sup_{s \in [0, \Delta_w]} [\alpha_1^{-1}(\hat{\gamma}(s) + \nu) - \alpha_1^{-1} \circ \hat{\gamma}(s)] \leq \delta$ , and the number  $C_{\tilde{x}} := \max\{\alpha_1^{-1}(\hat{\gamma}(\Delta_w) + \nu) + 1, \alpha_1^{-1} \circ \alpha_2(\Delta_{\tilde{x}})\}$ .

Using Lemma 5.4.1, let  $(C_{\tilde{x}}, C_w, C_{\dot{w}}, \nu)$  generate  $T^* > 0$ , such that (5.31) holds. Introduce  $D := C_{\tilde{x}}$  and  $d := \hat{\gamma}(\|w\|_\infty) + \nu$ , and from the choice of  $(C_{\tilde{x}}, C_w, C_{\dot{w}}, \nu)$  we have that  $\alpha_1(D) > d$ . Let  $\mathcal{W}$  be a set of continuously differentiable functions defined as follows  $\mathcal{W} := \{w \in \mathcal{L}_\infty \mid \|w\|_\infty \leq C_w, \|\dot{w}\|_\infty \leq C_{\dot{w}}\}$ . With these definitions of  $(D, d)$  and  $\mathcal{W}$ , together with (5.24), we have that all conditions of Lemma 5.4.3 hold. Hence, we can conclude that for all  $T \in (0, T^*)$ ,  $\tilde{x}(0) \in \mathbb{R}^{n_{\tilde{x}}}$ ,  $|\tilde{x}(0)| \leq \Delta_{\tilde{x}}$  and  $w \in \mathcal{L}_\infty$  with  $\|w\|_\infty \leq \Delta_w$ ,  $\|\dot{w}\|_\infty \leq \Delta_{\dot{w}}$  and all  $k \geq 0$  we have that the solutions of  $\mathcal{F}_T^e(\tilde{x}, w_T)$  exist

and satisfy

$$\begin{aligned}
|\tilde{x}(k)| &\leq \beta(|\tilde{x}(0)|, kT) + \alpha_1^{-1}(d) \\
&\leq \beta(|\tilde{x}(0)|, kT) + \alpha_1^{-1}(\hat{\gamma}(\|w\|_\infty) + \nu) \\
&\leq \beta(|\tilde{x}(0)|, kT) + \alpha_1^{-1} \circ \hat{\gamma}(\|w\|_\infty) + \delta \\
&= \beta(|\tilde{x}(0)|, kT) + \gamma(\|w\|_\infty) + \delta,
\end{aligned} \tag{5.45}$$

where  $\gamma(s) := \alpha_1^{-1} \circ \hat{\gamma}(s)$ . This completes the proof of Theorem 5.4.1. The proof of Theorem 5.4.2 is omitted since it follows closely the proof of Theorem 5.4.1. ■

## 5.5 Example

In this section, we illustrate our results via an example. Consider the scalar continuous-time plant

$$\dot{x}(t) = x^3(t) + u(t) + w(t), \tag{5.46}$$

and its approximate discrete-time model

$$\begin{aligned}
x(k+1) &= x(k) + T(x^3(k) + u(k)) + \int_{kT}^{(k+1)T} w(s)ds \\
&=: F_T^a(x(k), u(k), w_T[k]),
\end{aligned} \tag{5.47}$$

which can be obtained from numerical integration schemes described in [37]. The following three controllers:

$$\begin{aligned}
u_T^1(x) &= -x^3 - x \\
u_T^2(x) &= -x^3 - x - Tx \\
u_T^3(x) &= -\frac{1}{2T} [1 + 2Tx^2 - \sqrt{1 - 4T}] x
\end{aligned} \tag{5.48}$$

can be shown to yield respectively the following three dissipation inequalities with  $V_T(x) = \frac{1}{2}x^2$ :

$$\begin{aligned}
\frac{V_T(F_T^a(x, u_T^1(x), w_T)) - V_T(x)}{T} &\leq -\frac{1}{2}x^2 + \frac{1}{2}\|w_T\|_\infty^2 + T\|w_T\|_\infty^2 + Tx^2 \\
\frac{V_T(F_T^a(x, u_T^2(x), w_T)) - V_T(x)}{T} &\leq -\frac{1}{2}x^2 + \frac{1}{2}\|w_T\|_\infty^2 + (T + \frac{T^2}{2})\|w_T\|_\infty^2 \\
&\quad + (T + \frac{T^2}{2} + \frac{T^3}{2})x^2 \\
\frac{V_T(F_T^a(x, u_T^3(x), w_T)) - V_T(x)}{T} &\leq -\frac{1}{2}x^2 + \frac{1}{2}\|w_T\|_\infty^2 + T\|w_T\|_\infty^2.
\end{aligned} \tag{5.49}$$

From our choice of  $V_T(x)$  and (5.49) it follows that the approximate discrete-time model with any of the controllers (5.48) is Lyapunov SP-ISS. Moreover, since the approximate discrete-time model is the same as  $\tilde{F}_T^{Euler}$  in the first condition of Lemma 5.3.2, it follows that  $F_T^a$  is one-step strongly consistent with  $F_T^e$ . Finally, all of the controllers in (5.48) are locally uniformly bounded (for  $u_T^1$  and  $u_T^2$  this is obvious and for  $u_T^3$  this can be seen by using the Taylor series expansion  $\sqrt{1-4T} = 1-2T+O(T^2)$ ). Therefore, for  $F_T^a$ ,  $V_T(x)$  and any controller in (5.48) we have that all conditions of Theorem 5.4.2 hold. Hence, we can conclude using Theorem 5.4.2 that each of controllers (5.48) semiglobally practically input-to-state stabilizes the exact discrete-time plant model.

Table 5.1: DOAs in disturbance free case.

T[s]	DOA estimate		
	$u_T^1(k)$	$u_T^2(k)$	$u_T^3(k)$
0.25	[-2.99,2.99]	[-2.90,2.90]	[-2.66,2.66]
0.15	[-4.10,4.10]	[-4.04,4.04]	[-4.01,4.01]
0.05	[-7.78,7.78]	[-7.75,7.75]	[-7.75,7.75]
0.001	[-67.81,67.81]	[-67.81,67.81]	[-67.81,67.81]

Table 5.2: Performance with a disturbance

T[s]	$x_o$	Amplitude of disturbance		
		$u_T^1(k)$	$u_T^2(k)$	$u_T^3(k)$
0.25	2.66	2.50	3.04	4.37
0.15	4.01	3.48	3.95	4.20
0.05	7.75	6.84	7.12	7.15
0.001	67.81	63.62	63.70	63.70

We applied the controllers (5.48) via a sampler and zero-order-hold to the continuous-time plant model and compared the performance of the three controllers via simulations in SIMULINK. We used the following parameters in simulations: variable step size; ode-45; relative tolerance  $10^{-3}$ , absolute tolerance  $10^{-6}$ ; max step size *auto*; initial step size *auto*. Note that the controller  $u_T^1(x)$  may be obtained using a continuous-time design (using  $V := V_T$  and obtain  $\dot{V} \leq -\frac{1}{2}x^2 + \frac{1}{2}w^2$  for the continuous-time closed-

loop) and controller discretization. In Table 5.1 we estimated domains of attraction (DOAs) of the closed-loop sampled-data system with controllers (5.48) for different sampling periods. The controller  $u_T^1$  gives the largest DOA for all tested sampling periods. In Table 5.2 we summarize simulations for different sampling periods and fixed initial states with a sinusoidal disturbance of frequency  $1 \frac{rad}{sec}$ .

Table 5.3: Parameters for Simulation 5.1

Parameter	Value
Sampling period (T)	0.15s
Initial state	4.0
Amplitude of disturbance	3.95

The values of amplitude of the sinusoidal disturbance recorded in Table 5.2 are the largest values for which solutions of the sampled-data closed-loop system stay bounded. It is obvious that the controller  $u_T^3$  is the most robust with respect to the test disturbance for all tested sampling periods. Similar observations were obtained for other initial states and disturbances that are not presented in Table 5.2. From Tables 5.1 and 5.2 we see that the performance of all controllers (5.48) becomes very similar for small sampling periods which can be expected since the dissipation inequalities in (5.49) differ only in terms of order  $T$ , which can be made arbitrarily small on compact sets by reducing  $T$ . Difference in performance of controllers (5.48) is more pronounced for larger sampling periods (see Tables 5.1 and 5.2).

A set of simulations are also done using the set of parameters given in Table 5.3. The time responses of the states and the control signals of the closed-loop system with the three controllers are shown in Figure 5.2. Figure 5.2a shows the time responses when implementing the controller  $u_T^1$ , Figure 5.2b with the controller  $u_T^2$ , and Figure 5.2c with the controller  $u_T^3$ . These simulations show that with the given set of parameters, the controller  $u_T^2$  and  $u_T^3$  which are designed following the approximate based design framework presented in this chapter, perform better than the emulation controller  $u_T^1$ . It is observed that while the plant becomes unstable with the emulation controller, the other two controllers can still stabilize the plant.

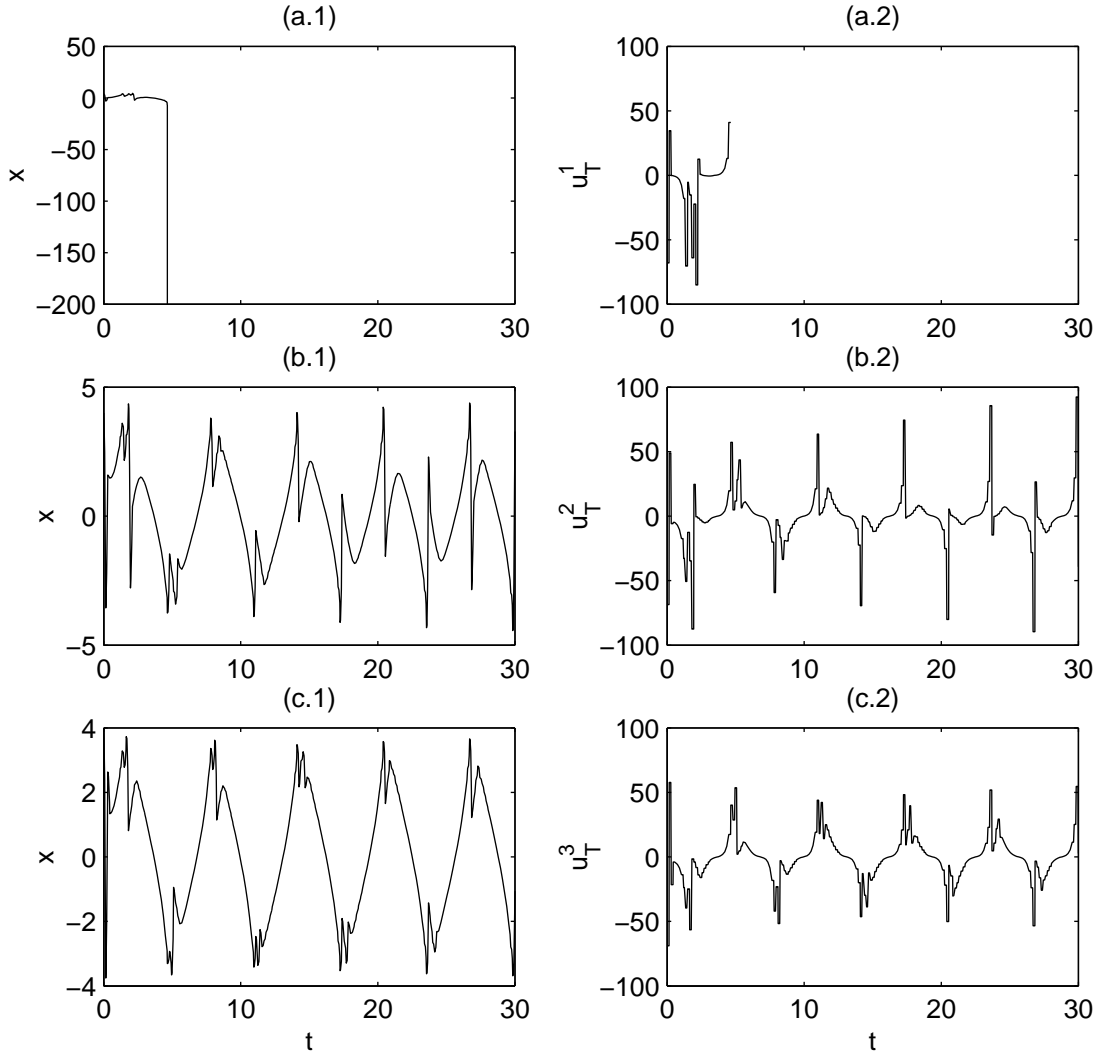


Figure 5.2: Simulation 5.1, response of the system using controllers  $u_T^1$ ,  $u_T^2$  and  $u_T^3$ .

## 5.6 Conclusion

We have presented quite general results that prescribe the conditions for designing nonlinear discrete-time controllers that input-to-state stabilize the exact discrete-time model of closed-loop sampled-data systems. We covered the case of dynamic feedback controllers, where the design is based on an approximate discrete-time model of the plant, with sufficiently fast sampling. The results provide a generalized framework for digital controller design when approximate discrete-time model is used.

Results of this chapter provide a prescriptive framework that can be extended to the development of design tools within the framework. The result provide a framework for input-to-state stabilizing controller design based on approximate discrete-time model of the continuous-time plant, where Lyapunov functions play a central role in applying the framework. On the other hand, there is no general procedure to construct a family of Lyapunov functions for nonlinear systems. In the next two chapters, the Chapter 6 and Chapter 7, we sequentially present a number of techniques for constructing a family of Lyapunov functions for certain classes of parameterized family of discrete-time nonlinear systems. Also, various Lyapunov based tools that facilitate the controller design within the framework presented in this chapter are developed in the next two chapters.





## Chapter 6

# Changing Supply Rates for Input-Output to State Stable Systems

### 6.1 Introduction

The framework for ISS controller design presented in Chapter 5 is prescriptive. Indeed, Theorems 5.4.1 and 5.4.2 specify conditions for the controller, the plant and the closed-loop approximate model, assuming that the controller has already been designed (using arbitrary appropriate design methods which are not specified in the results). One of the conditions to satisfy is a Lyapunov characterization for the closed-loop approximate model, for which an explicit expression of a Lyapunov function is required. Unfortunately, there is no generic method to construct a Lyapunov function for nonlinear systems in general. In this chapter, as well as in the next chapter, development of tools to partially construct Lyapunov functions that facilitate controller design within the framework of Chapter 5 is addressed.

The Lyapunov method is one of the most important and useful methods in stability analysis and design of nonlinear control systems (see for example [70, 129]). Lyapunov functions, which are the main tool in this method, can be used to characterize various stability properties of control systems. Because there is no general systematic way of

finding Lyapunov functions, therefore, developing methods for constructing Lyapunov functions is of utmost importance.

A very useful method for a partial construction of Lyapunov functions was introduced in [142] where it was shown how it is possible to combine two Lyapunov functions, none of which can be used to conclude a property of interest, into a new composite Lyapunov function from which the desired property follows. Results in [142] apply to the analysis of input to state stability (ISS) property of continuous-time cascade-connected systems. In [3] a similar proof technique was used to combine a Lyapunov function whose derivative is negative semidefinite and another Lyapunov function that characterizes a detectability property, which is called input-output to state stability (IOSS) (see [146]), into a new Lyapunov function from which ISS of a continuous-time system follows. A discrete-time counterpart of results in [142] was presented in [111]. These results and proof techniques were used in discrete-time backstepping [110], stability of continuous-time cascades [7, 142], stability of discrete-time cascades [111], continuous-time stabilization of robot manipulators [3] and  $L_p$  stability of time-varying nonlinear sampled-data systems [162]. A related Lyapunov based method for interconnected ISS continuous-time systems satisfying a small-gain condition can be found in [59].

The results presented in this chapter are based on results that have appeared in [78, 80]. The purpose is to present a general and unifying framework for partial constructions of families of Lyapunov functions for families of discrete-time systems parameterized by a positive parameter (sampling period) that were introduced in Chapter 5. We consider semiglobal practical stability properties of these systems that arise naturally when approximate discrete-time models are used to design controllers for sampled-data nonlinear systems. The results of Chapter 5 strongly motivate the material presented in this chapter. More motivations can also be found in [106, 110, 116, 117].

It is the main purpose of the work in this chapter to further contribute to the approach that was pursued in [106, 116]. We emphasize that different from the earlier results where only one Lyapunov function was usually constructed, in this research we

consider families of discrete-time systems parameterized by the sampling period, and we construct families of Lyapunov functions for these families of systems.

Our main technical result is contained in Lemma 6.3.1 where we prove a general result on changing supply rates for a generalized notion of input-output to state stability (IOSS) for discrete-time nonlinear systems. We refer to this property as “IOSS with measuring functions” (see Definition 6.2.1) and show that many important properties considered in the literature are special cases of this general property. The general result of Lemma 6.3.1 is central to this work since it allows us to prove several new results and generalize several existing results in the literature in a unified framework. Lemma 6.3.1 is a discrete-time version (as well as generalization) of the continuous-time result in [3] and a generalization of the discrete-time result in [111].

Using Lemma 6.3.1 we present two partial constructions of a family of Lyapunov functions from two other families of Lyapunov functions in Theorems 6.3.1 and 6.3.2. The construction in Theorem 6.3.1 was used in [3] for continuous-time systems, whereas the construction in Theorem 6.3.2 was used in [142] and [111] for continuous-time and discrete-time systems respectively. However, because of the generality of the “IOSS with measuring functions property” that we use, we obtain more general results by using the same Lyapunov function constructions as in the cited references. While the statements of our main results in discrete-time are very similar to continuous-time results of [3, 142], the proof technique is notably different and it requires a judicious use of the Mean Value Theorem (see the proof of Lemma 6.3.1).

Finally, we apply our results in a unified manner to several problems: (i) a LaSalle criterion for ISS of discrete-time systems (see also [3]); (ii) constructing ISS Lyapunov functions for time-varying discrete-time cascade-connected systems (see also [57, 60, 111, 142]); (iii) testing ISS of discrete-time systems using positive semidefinite Lyapunov functions (see also [13, 35]); (iv) observer-based input-to-state stabilization of discrete-time systems (see also [65, 66]). We emphasize that our results have potential for further important applications. An engineering application of our results is presented in Chapter 8, a case study of a two link manipulator, which illustrates the usefulness of our results via simulations.

The chapter is organized as follows. In Section 6.2 we introduce notation and definitions. Main results are stated in Section 6.3. Four applications of the main results are presented respectively in Section 6.4. The proofs of main results are provided in Section 6.5. Conclusion is presented in Section 6.6.

## 6.2 Preliminaries

We refer to Chapter 2 for general notation definitions.  $\mathcal{SN}$  denotes the class of all smooth nondecreasing functions  $q : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ , which satisfy  $q(t) > 0$  for all  $t > 0$ .  $|x|$  denotes the 1-norm of a vector  $x \in \mathbb{R}^n$ , that is  $|x| := \sum_{i=1}^n |x_i|$ .

We consider a parameterized family of discrete-time nonlinear systems of the following form:

$$\begin{aligned} x(k+1) &= F_T(x(k), u(k)) \\ y(k) &= h(x(k)) \end{aligned} \tag{6.1}$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ ,  $y \in \mathbb{R}^l$  are respectively the state, input and output of the system. It is assumed that  $F_T$  is well defined for all  $x, u$  and sufficiently small  $T$ ,  $F_T(0, 0) = 0$  for all  $T$  for which  $F_T$  is defined,  $h(0) = 0$  and  $F_T$  and  $h$  are continuous.  $T > 0$  is the sampling period, which parameterizes the system and can be arbitrarily assigned. We use the following definition.

**Definition 6.2.1** *The system (6.1) is  $(V_T, \underline{\alpha}, \bar{\alpha}, \alpha, \lambda, \sigma)$ -semiglobally practically input-output to state stable  $((V_T, \underline{\alpha}, \bar{\alpha}, \alpha, \lambda, \sigma)$ -SP-IOSS) with measuring functions, if there exist functions  $\underline{\alpha}, \bar{\alpha}, \alpha \in \mathcal{K}_{\infty}$ , and  $\lambda, \sigma \in \mathcal{G}$ , functions  $w_{\underline{\alpha}} : \mathbb{R}^n \rightarrow \mathbb{R}^{n_{\underline{\alpha}}}$ ,  $w_{\bar{\alpha}} : \mathbb{R}^n \rightarrow \mathbb{R}^{n_{\bar{\alpha}}}$ ,  $w_{\alpha} : \mathbb{R}^n \rightarrow \mathbb{R}^{n_{\alpha}}$ ,  $w_{\lambda} : \mathbb{R}^n \rightarrow \mathbb{R}^{n_{\lambda}}$ ,  $w_{\sigma} : \mathbb{R}^m \rightarrow \mathbb{R}^{n_{\sigma}}$ ,  $w_x : \mathbb{R}^n \rightarrow \mathbb{R}^{n_x}$ ,  $w_u : \mathbb{R}^m \rightarrow \mathbb{R}^{n_u}$ , and for any triple of strictly positive real numbers  $\Delta_x, \Delta_u, \nu$ , there exists  $T^* > 0$  and for all  $T \in (0, T^*)$  there exists a smooth function  $V_T : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  such that for all  $|w_x(x)| \leq \Delta_x$ ,  $|w_u(u)| \leq \Delta_u$  the following holds:*

$$\underline{\alpha}(|w_{\underline{\alpha}}(x)|) \leq V_T(x) \leq \bar{\alpha}(|w_{\bar{\alpha}}(x)|) \tag{6.2}$$

$$V_T(F_T(x, u)) - V_T(x) \leq -T\alpha(|w_{\alpha}(x)|) + T\lambda(|w_{\lambda}(x)|) + T\sigma(|w_{\sigma}(u)|) + T\nu. \tag{6.3}$$

The functions  $w_{\underline{\alpha}}$ ,  $w_{\overline{\alpha}}$ ,  $w_{\alpha}$ ,  $w_{\lambda}$ ,  $w_{\sigma}$ ,  $w_x$  and  $w_u$  are called measuring functions;  $\underline{\alpha}$ ,  $\overline{\alpha}$ ,  $\alpha$ ,  $\lambda$ ,  $\sigma$  are called bounding functions;  $\alpha$ ,  $\lambda$ ,  $\sigma$  are called supply functions; and  $V_T$  is called a SP-IOSS Lyapunov function. If  $T^* > 0$  exists such that (6.2) and (6.3), with  $\nu = 0$ , hold for all  $T \in (0, T^*)$ ,  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ , the property holds globally and the system (6.1) is  $(V_T, \underline{\alpha}, \overline{\alpha}, \alpha, \lambda, \sigma)$ -IOSS with measuring functions. ■

Often, when all functions are clear from the context, we refer to the property defined in Definition 6.2.1 as SP-IOSS (or IOSS if the property holds globally). Moreover, if the system is SP-IOSS (respectively IOSS) with  $\lambda = 0$  then we say that the system is SP-ISS (respectively ISS). SP-IOSS with measuring functions is quite a general notion that covers a range of different properties of nonlinear discrete-time systems, such as stability, detectability, output to state stability, etc. For example, by letting  $\lambda = 0$ ,  $\sigma = 0$  and  $w_{\underline{\alpha}}(x) = w_{\overline{\alpha}}(x) = w_{\alpha}(x) = x$ , we obtain the standard Lyapunov characterization for asymptotic stability of (6.1). By letting  $\lambda = 0$ ,  $w_{\underline{\alpha}}(x) = w_{\overline{\alpha}}(x) = w_{\alpha}(x) = x$ , and  $w_{\sigma}(u) = u$ , we obtain a Lyapunov characterization for (semiglobal practical) ISS. The reason for introducing such a general property in Definition 6.2.1 is that we will apply our results to a range of its different special cases (see Section 4) for particular choices of  $\lambda$ ,  $\sigma$  and the measuring functions. Hence, Definition 6.2.1 is a very compact way of defining various different properties to which our results apply.

The following two lemmas and remark are used in proving our main results (Theorems 6.3.1 and 6.3.2).

**Lemma 6.2.1** [142] *Assume that the functions  $\beta, \beta' \in \mathcal{K}$  are such that  $\beta'(s) = O[\beta(s)]$  as  $s \rightarrow 0^+$ . Then there exists a function  $q \in \mathcal{SN}$  so that  $\beta'(s) \leq q(s)\beta(s)$ ,  $\forall s \geq 0$ . ■*

**Lemma 6.2.2** [142] *Assume that the functions  $\beta, \beta' \in \mathcal{K}$  are such that  $\beta(r) = O[\beta'(r)]$  as  $r \rightarrow +\infty$ . Then there exists a function  $q \in \mathcal{SN}$  so that  $q(r)\beta(r) \leq \beta'(r)$ ,  $\forall r \geq 0$ . ■*

**Remark 6.2.1** *Since for any  $\alpha \in \mathcal{K}$  we have  $\alpha(s_1 + s_2) \leq \alpha(2s_1) + \alpha(2s_2)$  for all  $s_1 \geq 0, s_2 \geq 0$ , then for any  $\alpha_1, \alpha_2 \in \mathcal{K}$ , there exist  $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}$  such that the following*

holds:

$$\underline{\alpha}(s_1 + s_2) \leq \alpha_1(s_1) + \alpha_2(s_2) \leq \overline{\alpha}(s_1 + s_2), \quad \forall s_1 \geq 0, s_2 \geq 0, \quad (6.4)$$

where  $\underline{\alpha}(s) := \min\{\alpha_1(\frac{s}{2}), \alpha_2(\frac{s}{2})\}$  and  $\overline{\alpha}(s) := \max\{2\alpha_1(s), 2\alpha_2(s)\}$ . ■

### 6.3 Main results

In this section, we state the main results of this chapter, which consist of two main theorems (Theorems 6.3.1 and 6.3.2), where we show two partial constructions of a SP-IOSS Lyapunov function from two auxiliary Lyapunov functions. Some corollaries following from our main results are also presented. First, we discuss our approach in more detail.

When using the SP-IOSS property of Definition 6.2.1 to check if a certain property (such as stability, input-to-state stability or some other special case of SP-IOSS property) holds, one usually needs to have that all bounding functions and the corresponding measuring functions satisfy appropriate conditions. For example, if we want to check global asymptotic stability of the origin of the input-free system (6.1) then we need to have:

$$\begin{aligned} \underline{\alpha}(|w_{\underline{\alpha}}(x)|) &\leq V_{1T}(x) \leq \overline{\alpha}(|w_{\overline{\alpha}}(x)|) \\ V_{1T}(F_T(x, 0)) - V_{1T}(x) &\leq -T\alpha(|w_{\alpha}(x)|), \end{aligned} \quad (6.5)$$

for all  $x \in \mathbb{R}^n$  and  $T \in (0, T^*)$ , for some  $T^* > 0$ ,  $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_{\infty}$  and  $\alpha$  is positive definite,  $|w_{\underline{\alpha}}(x)|$  is positive definite and radially unbounded and  $|w_{\alpha}(x)|$  is positive definite.

However, it is often the case that some of the desired conditions are not satisfied by a candidate SP-IOSS Lyapunov function  $V_{1T}$ . For instance, in the above example it may happen that  $|w_{\underline{\alpha}}(x)|$  and/or  $|w_{\alpha}(x)|$  is positive semidefinite. If this happens, then one possibility is trying to prove that the appropriate property holds with weaker conditions. For instance, in the above example when  $|w_{\underline{\alpha}}(x)|$  is positive semidefinite one can use the results of [35] to check stability of the system. If, on the other hand,  $|w_{\alpha}(x)|$  is positive semidefinite one can use the celebrated LaSalle's invariance principle to check stability. Another approach that is taken in this work is to construct a new SP-IOSS Lyapunov function that satisfies all the required conditions. This construction

is carried out by first introducing an auxiliary SP-IOSS Lyapunov function  $V_{2T}$  and then combining the two functions into a new SP-IOSS Lyapunov function  $V_T$ . We note that neither  $V_{1T}$  nor  $V_{2T}$  can be used alone to conclude the desired property, whereas their appropriate combination can.

We present in Theorems 6.3.1 and 6.3.2 two constructions that can be used (under different conditions on bounding and measuring functions) to construct a new SP-IOSS Lyapunov function  $V_T$  from two SP-IOSS Lyapunov functions  $V_{1T}$  and  $V_{2T}$ . In particular, in Theorem 6.3.1 we construct  $V_T$  from  $V_{1T}$  and  $V_{2T}$ , by using a scaling function  $\rho \in \mathcal{K}_\infty$  of a special form, in the following way:

$$V_T = V_{1T} + \rho(V_{2T}) , \quad (6.6)$$

while in Theorem 6.3.2 we use two scaling functions  $\rho_1, \rho_2 \in \mathcal{K}_\infty$  of a special form to construct  $V_T$  as follow:

$$V_T = \rho_1(V_{1T}) + \rho_2(V_{2T}) . \quad (6.7)$$

In Theorem 6.3.1 we use weaker conditions on the measuring functions of  $V_{1T}$  than in Theorem 6.3.2. This leads to a less general construction (6.6) than (6.7). The important point to be made is that the measuring functions for the new function  $V_T$  are different from measuring functions of either  $V_{1T}$  or  $V_{2T}$ . It is this fact that allows us to conclude that the system has a property which was impossible to conclude by using either  $V_{1T}$  or  $V_{2T}$  alone.

We first present Lemma 6.3.1, which is instrumental in proving our main results. The lemma is a discrete-time version, as well as a generalization, of the lemma on changing supply rates for IOSS continuous-time systems in [3]. Lemma 6.3.1 also generalizes the result of [111] on changing supply rates for ISS discrete-time systems. We use the following construction that was introduced in [3, 142]. Given an arbitrary  $q \in \mathcal{SN}$ , we define:

$$\rho(s) := \int_0^s q(\tau) d\tau , \quad (6.8)$$

where it is easy to see that  $\rho \in \mathcal{K}_\infty$  and  $\rho$  is smooth. Suppose that we have a SP-IOSS Lyapunov function  $V_T$  for a system, and then consider a new function  $\rho(V_T)$ . In Lemma 6.3.1, we state conditions under which the new function is also a SP-IOSS Lyapunov function for the system.

**Lemma 6.3.1** *Let the following conditions be satisfied:*

1. *System (6.1) is  $(V_T, \underline{\alpha}, \bar{\alpha}, \alpha, \lambda, \sigma)$ -SP-IOSS with measuring functions  $w_{\underline{\alpha}}$ ,  $w_{\bar{\alpha}}$ ,  $w_{\alpha}$ ,  $w_{\lambda}$ ,  $w_{\sigma}$ ,  $w_x$  and  $w_u$ .*
2. *There exist  $\underline{\kappa}, \bar{\kappa} \in \mathcal{K}_{\infty}$  such that  $\underline{\kappa}(|w_{\alpha}(x)|) \leq |w_{\underline{\alpha}}(x)|$  and  $|w_{\bar{\alpha}}(x)| \leq \bar{\kappa}(|w_{\alpha}(x)|)$ , for all  $x \in \mathbb{R}^n$ .*
3. *For any strictly positive real numbers  $\Delta_x, \Delta_u$  there exist strictly positive real numbers  $M$  and  $T^*$  such that*

$$|w_x(x)| \leq \Delta_x, |w_u(u)| \leq \Delta_u, T \in (0, T^*) \implies \max\{|w_{\bar{\alpha}}(F_T(x, u))|, |w_{\bar{\alpha}}(x)|, |w_{\lambda}(x)|, |w_{\sigma}(u)|\} \leq M. \quad (6.9)$$

*Then for any  $q \in \mathcal{SN}$  and  $\rho \in \mathcal{K}_{\infty}$  defined by (6.8) there exist  $\underline{\alpha}'$ ,  $\bar{\alpha}'$ ,  $\alpha'$ ,  $\lambda'$ ,  $\sigma'$  such that the system (6.1) is  $(\rho(V_T), \underline{\alpha}', \bar{\alpha}', \alpha', \lambda', \sigma')$ -SP-IOSS with the same measuring functions, where  $\underline{\alpha}'(s) = \rho \circ \underline{\alpha}(s)$ ,  $\bar{\alpha}'(s) = \rho \circ \bar{\alpha}(s)$ ,  $\alpha'(s) = \frac{1}{4}q \circ \frac{1}{2}\underline{\alpha} \circ \underline{\kappa}(s) \cdot \alpha(s)$ ,  $\lambda'(s) = 2q \circ \theta_{\lambda}(s) \cdot \lambda(s)$ ,  $\sigma'(s) = 2q \circ \theta_{\sigma}(s) \cdot \sigma(s)$ , and*

$$\theta_{\sigma}(s) := \bar{\alpha} \circ \bar{\kappa} \circ \alpha^{-1} \circ 4\sigma(s) + 2\sigma(s) \quad (6.10)$$

$$\theta_{\lambda}(s) := \bar{\alpha} \circ \bar{\kappa} \circ \alpha^{-1} \circ 4\lambda(s) + 2\lambda(s). \quad (6.11)$$

■

Lemma 6.3.1 provides us with some flexibility when constructing a SP-IOSS Lyapunov function  $V_T$  from two Lyapunov functions using (6.6) and (6.7). We prove the result for semiglobal practical IOSS since this is a property that naturally arises when an approximate discrete-time model is used for controller design of a sampled-data nonlinear systems (see the case study in Chapter 7). Some of the conditions of Lemma 6.3.1 are rather technical but they were considered in order to prove the result in a considerable generality.

**Remark 6.3.1** *It is instructive to discuss the third condition of Lemma 6.3.1 since it appears to be the least intuitive. We consider its two special cases for stability of the origin and stability of arbitrary (not necessarily compact) sets.*



Let us first consider the stability of the origin of the input-free system (6.1). In this case, the conditions (6.5) need to hold and we can assume without loss of generality that  $w_{\underline{\alpha}}(x) = w_{\bar{\alpha}}(x) = w_{\alpha}(x) = w_x(x) = x$ . In this case the third condition of Lemma 6.3.1 holds if  $F_T(x, 0)$  is bounded on compact sets, uniformly in  $T \in (0, T^*)$ . This holds if  $F_T(0, 0) = 0$  for all  $T \in (0, T^*)$  and  $F_T(x, 0)$  is continuous in  $x$ , uniformly in  $T \in (0, T^*)$ . This condition is rather natural to use and it is often assumed in the literature (see for instance [60]).

Suppose now that (6.5) hold with  $w_{\underline{\alpha}}(x) = w_{\bar{\alpha}}(x) = w_{\alpha}(x) = w_x(x) = |x|_{\mathcal{A}}$ , where  $\mathcal{A}$  is a non-empty closed set. In this case, the condition 3 of Lemma 6.3.1 requires that for any  $\Delta_x$  there exists  $M$  and  $T^*$  such that

$$|x|_{\mathcal{A}} \leq \Delta_x, T \in (0, T^*) \implies |F_T(x, 0)|_{\mathcal{A}} \leq M.$$

This condition also appears to be natural and similar conditions have been used in the literature [153]. ■

We can also state a similar result to Lemma 6.3.1, when the IOSS property holds globally, that is when the system (6.1) is  $(V_T, \underline{\alpha}, \bar{\alpha}, \alpha, \lambda, \sigma)$ -IOSS with measuring functions. It is interesting that in this case the third condition of Lemma 6.3.1 is not needed to prove the result.

**Corollary 6.3.1** *Let the following conditions be satisfied:*

1. *System (6.1) is  $(V_T, \underline{\alpha}, \bar{\alpha}, \alpha, \lambda, \sigma)$ -IOSS with measuring functions  $w_{\underline{\alpha}}$ ,  $w_{\bar{\alpha}}$ ,  $w_{\alpha}$ ,  $w_{\lambda}$  and  $w_{\sigma}$ .*
2. *There exist  $\underline{\kappa}, \bar{\kappa} \in \mathcal{K}_{\infty}$  such that  $\underline{\kappa}(|w_{\alpha}(x)|) \leq |w_{\underline{\alpha}}(x)|$  and  $|w_{\bar{\alpha}}(x)| \leq \bar{\kappa}(|w_{\alpha}(x)|)$  for all  $x \in \mathbb{R}^n$ .*

*Then for any  $q \in \mathcal{SN}$  and  $\rho \in \mathcal{K}_{\infty}$  defined by (6.8) there exist  $\underline{\alpha}'$ ,  $\bar{\alpha}'$ ,  $\alpha'$ ,  $\lambda'$ ,  $\sigma'$  such that the system (6.1) is  $(\rho(V_T), \underline{\alpha}', \bar{\alpha}', \alpha', \lambda', \sigma')$ -IOSS with the same measuring functions, where  $\underline{\alpha}'$ ,  $\bar{\alpha}'$ ,  $\alpha'$ ,  $\lambda'$ ,  $\sigma'$  are the same as in Lemma 6.3.1.* ■

We present our main results below. Note that Theorem 6.3.1 is a discrete-time version, as well as generalization, of the continuous-time results in [3], whereas Theorem 6.3.2 has appeared in a simpler form in [111], which is a discrete-time version of

[142], when  $\lambda = 0$ ,  $w_{\underline{\alpha}}(x) = w_{\bar{\alpha}}(x) = w_{\alpha}(x) = x$ ,  $w_{\sigma}(u) = u$  and all properties hold globally.

**Theorem 6.3.1** *Suppose that:*

1. *the system (6.1) is  $(V_{1T}, \underline{\alpha}_1, \bar{\alpha}_1, \alpha_1, \sigma_1)$ -SP-ISS with measuring functions  $w_{\underline{\alpha}_1}$ ,  $w_{\bar{\alpha}_1}$ ,  $w_{\alpha_1}$ ,  $w_{\sigma_1}$ ,  $w_{x_1}$ ,  $w_{u_1}$ ;*
2. *the system (6.1) is  $(V_{2T}, \underline{\alpha}_2, \bar{\alpha}_2, \alpha_2, \lambda_2, \sigma_2)$ -SP-IOSS with measuring functions  $w_{\underline{\alpha}_2}$ ,  $w_{\bar{\alpha}_2}$ ,  $w_{\alpha_2}$ ,  $w_{\lambda_2}$ ,  $w_{\sigma_2}$ ,  $w_{x_2}$ ,  $w_{u_2}$ , and there exist  $\underline{\kappa}_2, \bar{\kappa}_2 \in \mathcal{K}_{\infty}$ , such that the second and third conditions of Lemma 6.3.1 hold;*
3. *there exist  $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_{\infty}$  such that  $|w_{\lambda_2}(x)| \leq \gamma_1(|w_{\alpha_1}(x)|)$ ,  $|w_{x_2}(x)| \leq \gamma_2(|w_{x_1}(x)|)$ ,  $|w_{u_2}(u)| \leq \gamma_3(|w_{u_1}(u)|)$  for all  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ ;*
4.  $\limsup_{s \rightarrow +\infty} \frac{\lambda_2(s)}{\alpha_1(s)} < +\infty$ .

*Then there exists  $\rho \in \mathcal{K}_{\infty}$  such that the system (6.1) is  $(V_T, \underline{\alpha}, \bar{\alpha}, \alpha, \sigma)$ -SP-ISS with new measuring functions  $w_{\underline{\alpha}}$ ,  $w_{\bar{\alpha}}$ ,  $w_{\alpha}$ ,  $w_{\sigma}$ ,  $w_x$ ,  $w_u$  where*

$$V_T = V_{1T} + \rho(V_{2T}) , \quad (6.12)$$

*and the new measuring functions are*

$$\begin{aligned} w_{\underline{\alpha}}(x) &:= |w_{\underline{\alpha}_1}(x)| + |w_{\underline{\alpha}_2}(x)|, & w_{\bar{\alpha}}(x) &:= |w_{\bar{\alpha}_1}(x)| + |w_{\bar{\alpha}_2}(x)|, \\ w_x(x) &:= w_{x_1}(x), & w_{\alpha}(x) &:= |w_{\alpha_2}(x)|, \\ w_{\sigma}(u) &:= |w_{\sigma_1}(u)| + |w_{\sigma_2}(u)|, & w_u(u) &:= w_{u_1}(u). \end{aligned} \quad (6.13)$$

■

**Remark 6.3.2** *We note that in Theorems 6.3.1 and 6.3.2 we concentrate only on verifying conditions similar to (5.24), (5.25). However, we note that if the functions  $V_{1T}$  and  $V_{2T}$  satisfy the local Lipschitz condition (5.26), then the new Lyapunov function constructed using either (6.6) or (6.7) would also satisfy the local Lipschitz condition. Hence, results of Theorems 6.3.1 and 6.3.2 can be used to verify the first condition of Theorem 5.4.1.*

■

In the next result, we consider a stronger condition for the Lyapunov function  $V_{1T}$ , so that we can relax the condition 4 of the Theorem 6.3.1.

**Theorem 6.3.2** *Suppose that:*

1. *the system (6.1) is  $(V_{1T}, \underline{\alpha}_1, \bar{\alpha}_1, \alpha_1, \sigma_1)$ -SP-ISS with measuring functions  $w_{\underline{\alpha}_1}$ ,  $w_{\bar{\alpha}_1}$ ,  $w_{\alpha_1}$ ,  $w_{\sigma_1}$ ,  $w_{x_1}$ ,  $w_{u_1}$  and there exist  $\underline{\kappa}_1, \bar{\kappa}_1 \in \mathcal{K}_\infty$ , such that the second and third conditions of Lemma 6.3.1 hold;*
2. *the system (6.1) is  $(V_{2T}, \underline{\alpha}_2, \bar{\alpha}_2, \alpha_2, \lambda_2, \sigma_2)$ -SP-IOSS with measuring functions  $w_{\underline{\alpha}_2}$ ,  $w_{\bar{\alpha}_2}$ ,  $w_{\alpha_2}$ ,  $w_{\lambda_2}$ ,  $w_{\sigma_2}$ ,  $w_{x_2}$ ,  $w_{u_2}$  and there exist  $\underline{\kappa}_2, \bar{\kappa}_2 \in \mathcal{K}_\infty$ , such that the second and third conditions of Lemma 6.3.1 hold;*
3. *there exist  $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$  such that  $|w_{\lambda_2}(x)| \leq \gamma_1(|w_{\alpha_1}(x)|)$ ,  $|w_{x_2}(x)| \leq \gamma_2(|w_{x_1}(x)|)$ ,  $|w_{u_2}(u)| \leq \gamma_3(|w_{u_1}(u)|)$  for all  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ .*

*Then there exist  $\rho_1, \rho_2 \in \mathcal{K}_\infty$  such that the system (6.1) is  $(V_T, \underline{\alpha}, \bar{\alpha}, \alpha, \sigma)$ -SP-ISS with new measuring functions  $w_{\underline{\alpha}}$ ,  $w_{\bar{\alpha}}$ ,  $w_{\alpha}$ ,  $w_{\sigma}$ ,  $w_x$ ,  $w_u$ , where*

$$V_T = \rho_1(V_{1T}) + \rho_2(V_{2T}) , \quad (6.14)$$

*and the new measuring functions are*

$$\begin{aligned} w_{\underline{\alpha}}(x) &:= |w_{\underline{\alpha}_1}(x)| + |w_{\underline{\alpha}_2}(x)|, & w_{\bar{\alpha}}(x) &:= |w_{\bar{\alpha}_1}(x)| + |w_{\bar{\alpha}_2}(x)|, \\ w_x(x) &:= w_{x_1}(x), & w_{\alpha}(x) &:= |w_{\alpha_1}(x)| + |w_{\alpha_2}(x)|, \\ w_{\sigma}(u) &:= |w_{\sigma_1}(u)| + |w_{\sigma_2}(u)|, & w_u(u) &:= w_{u_1}(u). \end{aligned} \quad (6.15)$$

■

Note that the main difference between Theorems 6.3.1 and 6.3.2 is that in Theorem 6.3.1 we cannot apply Lemma 6.3.1 to the Lyapunov function  $V_{T1}$ , since the second and third conditions of the lemma do not hold. Consequently, we need an extra condition on the bounding functions (condition 4 in Theorem 6.3.1) and we use a less general construction (6.12) than in Theorem 6.3.2 where we use (6.14).

As a consequence of Corollary 6.3.1, we can also state global results of the Theorems 6.3.1 and 6.3.2, if both  $V_{1T}$  and  $V_{2T}$  characterize IOSS properties of the system (6.1) in a global sense. The following corollary is derived from Theorem 6.3.1.

**Corollary 6.3.2** *Suppose that:*

1. the system (6.1) is  $(V_{1T}, \underline{\alpha}_1, \bar{\alpha}_1, \alpha_1, \sigma_1)$ -ISS with measuring functions  $w_{\underline{\alpha}_1}, w_{\bar{\alpha}_1}, w_{\alpha_1}, w_{\sigma_1}, w_{x_1}, w_{u_1}$ ;
2. the system (6.1) is  $(V_{2T}, \underline{\alpha}_2, \bar{\alpha}_2, \alpha_2, \lambda_2, \sigma_2)$ -IOSS with measuring functions  $w_{\underline{\alpha}_2}, w_{\bar{\alpha}_2}, w_{\alpha_2}, w_{\lambda_2}, w_{\sigma_2}, w_{x_2}, w_{u_2}$  and there exist  $\underline{\kappa}_2, \bar{\kappa}_2 \in \mathcal{K}_\infty$ , such that the second condition of Corollary 6.3.1 holds;
3. there exist  $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$  such that  $|w_{\lambda_2}(x)| \leq \gamma_1(|w_{\alpha_1}(x)|), |w_{x_2}(x)| \leq \gamma_2(|w_{x_1}(x)|), |w_{u_2}(u)| \leq \gamma_3(|w_{u_1}(u)|)$  for all  $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ ;
4.  $\limsup_{s \rightarrow +\infty} \frac{\lambda_2(s)}{\alpha_1(s)} < +\infty$ .

Then there exists  $\rho \in \mathcal{K}_\infty$  such that the system (6.1) is  $(V_T, \underline{\alpha}, \bar{\alpha}, \alpha, \sigma)$ -ISS with  $V_T$  is given by (6.12) and new measuring functions  $w_{\underline{\alpha}}, w_{\bar{\alpha}}, w_{\alpha}, w_{\sigma}, w_x, w_u$  are given by (6.13). ■

The next corollary, which is derived from Theorem 6.3.2 considers the same conditions as those of Theorem 6.3.2, when all properties hold globally for the system (6.1).

**Corollary 6.3.3** *Suppose that:*

1. the system (6.1) is  $(V_{1T}, \underline{\alpha}_1, \bar{\alpha}_1, \alpha_1, \sigma_1)$ -ISS with measuring functions  $w_{\underline{\alpha}_1}, w_{\bar{\alpha}_1}, w_{\alpha_1}, w_{\sigma_1}, w_{x_1}, w_{u_1}$ , and there exist  $\underline{\kappa}_1, \bar{\kappa}_1 \in \mathcal{K}_\infty$ , such that the second condition of Corollary 6.3.1 holds;
2. the system (6.1) is  $(V_{2T}, \underline{\alpha}_2, \bar{\alpha}_2, \alpha_2, \lambda_2, \sigma_2)$ -IOSS with measuring functions  $w_{\underline{\alpha}_2}, w_{\bar{\alpha}_2}, w_{\alpha_2}, w_{\lambda_2}, w_{\sigma_2}, w_{x_2}, w_{u_2}$  and there exist  $\underline{\kappa}_2, \bar{\kappa}_2 \in \mathcal{K}_\infty$ , such that the second condition of Corollary 6.3.1 holds;
3. there exist  $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$  such that  $|w_{\lambda_2}(x)| \leq \gamma_1(|w_{\alpha_1}(x)|), |w_{x_2}(x)| \leq \gamma_2(|w_{x_1}(x)|), |w_{u_2}(u)| \leq \gamma_3(|w_{u_1}(u)|)$  for all  $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ .

Then there exist  $\rho_1, \rho_2 \in \mathcal{K}_\infty$  such that the system (6.1) is  $(V_T, \underline{\alpha}, \bar{\alpha}, \alpha, \sigma)$ -ISS with  $V_T$  is given by (6.14) and new measuring functions  $w_{\underline{\alpha}}, w_{\bar{\alpha}}, w_{\alpha}, w_{\sigma}, w_x, w_u$  are given by (6.15). ■

## 6.4 Applications

In this section we show how our results can be specialized to deal with several important situations:

- (i) a LaSalle criterion for SP-ISS of parameterized discrete-time systems;
- (ii) SP-ISS of parameterized time-varying discrete-time cascaded systems;
- (iii) SP-ISS via positive semidefinite Lyapunov functions for parameterized discrete-time systems;
- (iv) observer-based ISS controller design for parameterized discrete-time systems.

We emphasize that our results are quite general and they have potential for other applications. This section also illustrates the generality of Definition 6.2.1, since we show that a range of properties considered in the literature are in fact special cases of the SP-ISS property with measuring functions.

### 6.4.1 A LaSalle criterion for SP-ISS

In this subsection, we present a novel result which is a discrete-time version of the continuous-time result presented in [3]. This result is a direct consequence of Theorem 6.3.1. We use this result for the case study in Chapter 8 to design a digital controller for a two link manipulator via its Euler approximate model.

We recall the quasi input-to-state stability (qISS) property and input output to state stability (IOSS) property from [3], and the condition

$$\limsup_{s \rightarrow +\infty} \frac{\lambda_2(s)}{\alpha_1(s)} < +\infty, \quad (6.16)$$

that has been used in the result of [3]. Using Theorem 6.3.1 we can state a semiglobal practical version of this result for parameterized discrete-time systems (6.1). In particular, we show that semiglobal practical qISS, semiglobal practical IOSS and the condition (6.16) imply semiglobal practical ISS. We use the following assumption:

**Assumption 6.4.1** *For any strictly positive real numbers  $\Delta_x, \Delta_u$  there exist strictly positive real numbers  $M$  and  $T^*$  such that  $|x| \leq \Delta_x, |u| \leq \Delta_u, T \in (0, T^*)$  implies  $|F_T(x, u)| \leq M$ .* ■

We state now a discrete-time version of the result in [3].

**Corollary 6.4.1** *Consider the system (6.1) and suppose that Assumption 6.4.1 holds. Suppose that there exist  $\underline{\alpha}_1, \bar{\alpha}_1, \alpha_1, \underline{\alpha}_2, \bar{\alpha}_2, \alpha_2 \in \mathcal{K}_\infty$ , and  $\sigma_1, \lambda_2, \sigma_2 \in \mathcal{G}$  such that:*

1. *for any triple of strictly positive real numbers  $(\Delta_x, \Delta_u, \nu)$  there exists  $T^* > 0$  and for any  $T \in (0, T^*)$  there exist  $V_{1T} : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  and  $V_{2T} : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  such that for all  $|x| \leq \Delta_x, |u| \leq \Delta_u, T \in (0, T^*)$  we have the following:*

- *SP-qISS:*

$$\begin{aligned} \underline{\alpha}_1(|x|) &\leq V_{1T}(x) \leq \bar{\alpha}_1(|x|) \\ V_{1T}(F_T(x, u)) - V_{1T}(x) &\leq T \left( -\alpha_1(|y|) + \sigma_1(|u|) + \nu \right) \end{aligned} \quad (6.17)$$

- *SP-IOSS:*

$$\begin{aligned} \underline{\alpha}_2(|x|) &\leq V_{2T}(x) \leq \bar{\alpha}_2(|x|) \\ V_{2T}(F_T(x, u)) - V_{2T}(x) &\leq T \left( -\alpha_2(|x|) + \lambda_2(|y|) + \sigma_2(|u|) + \nu \right) \end{aligned} \quad (6.18)$$

2. *the condition (6.16) holds.*

Then, there exist  $\underline{\alpha}, \bar{\alpha}, \alpha \in \mathcal{K}_\infty$  and  $\sigma \in \mathcal{G}$  such that for any triple of strictly positive real numbers  $(\tilde{\Delta}_x, \tilde{\Delta}_u, \tilde{\nu})$  there exists  $\tilde{T} > 0$  and for any  $T \in (0, \tilde{T})$  there exist  $V_T : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  such that for all  $|x| \leq \tilde{\Delta}_x, |u| \leq \tilde{\Delta}_u, T \in (0, \tilde{T})$  we have:

$$SP - ISS \left\{ \begin{array}{l} \underline{\alpha}(|x|) \leq V_T(x) \leq \bar{\alpha}(|x|) \\ V_T(F_T(x, u)) - V_T(x) \leq T \left( -\alpha(|x|) + \sigma(|u|) + \tilde{\nu} \right) \end{array} \right. \quad (6.19)$$

■

**Proof of Corollary 6.4.1:** It can be seen immediately that all conditions of Theorem 6.3.1 hold, by noting that: (i) the system (6.1) is  $(V_{1T}, \underline{\alpha}_1, \bar{\alpha}_1, \alpha_1, \sigma_1)$ -SP-ISS with measuring functions  $w_{\underline{\alpha}_1}(x) = w_{\bar{\alpha}_1}(x) = w_{\alpha_1}(x) = x$ ,  $w_{\sigma_1}(u) = w_{u_1}(u) = u$ ; (ii) the system (6.1) is  $(V_{2T}, \underline{\alpha}_2, \bar{\alpha}_2, \alpha_2, \lambda_2, \sigma_2)$ -SP-IOSS with measuring functions  $w_{\underline{\alpha}_2}(x) = w_{\bar{\alpha}_2}(x) = w_{\alpha_2}(x) = w_{\lambda_2}(x) = x$ ,  $w_{\lambda_2}(x) = h(x) = y$  and  $w_{\sigma_2}(u) = w_{u_2}(u) = u$ ; the second condition of Lemma 6.3.1 holds since  $w_{\underline{\alpha}_2}(x) = w_{\bar{\alpha}_2}(x) = w_{\alpha_2}(x)$ ; from Assumption 6.4.1 and Remark 6.3.1 we have that the third condition

of Lemma 6.3.1 holds; hence, the second condition of Theorem 6.3.1 holds; (iii) the third condition of Theorem 6.3.1 holds since  $w_{\alpha_1}(x) = w_{\lambda_2}(x) = h(x) = y$ ,  $w_{x_1}(x) = w_{x_2}(x) = x$  and  $w_{u_1}(u) = w_{u_2}(u) = u$  for all  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ ; (iv) the fourth condition of Theorem 6.3.1 follows trivially from the second condition of the corollary.

Therefore, applying Theorem 6.3.1 and defining the new SP-ISS Lyapunov function  $V_T$  as in (6.12), we obtain that the system (6.1) is SP-ISS with measuring functions  $w_{\underline{\alpha}}(x) = w_{\overline{\alpha}}(x) = w_{\alpha}(x) = |x|$ ,  $w_x(x) = x$ ,  $w_{\sigma}(u) = |u|$ , and  $w_u(u) = u$ . It is obvious that  $\gamma_2 = \gamma_3 = Id$ . Since  $h$  is continuous and  $h(0) = 0$ , there exists  $\gamma_1 \in \mathcal{K}_{\infty}$  such that  $|y| \leq \gamma_1(|x|)$ , and this completes the proof. ■

#### 6.4.2 SP-ISS of time-varying cascade-connected systems

A novel result on SP-ISS for time-varying discrete-time cascade-connected system is presented in this subsection. This result is a direct consequence of Theorem 6.3.2 and it generalizes the main result of [111] in two directions:

- (i) the result is stated for semiglobal practical ISS (only global stability was considered in [111]);
- (ii) the result is stated for time-varying cascade-connected systems (only time-invariant cascade-connected systems were considered in [111]).

We note that similar non Lyapunov based proof of the same result can be found in [60] for non parameterized discrete-time systems.

Consider the time-varying discrete-time system:

$$\begin{aligned} x(k+1) &= F_T(k, x(k), z(k), u(k)) \\ z(k+1) &= G_T(k, z(k), u(k)) , \end{aligned} \tag{6.20}$$

where  $x \in \mathbb{R}^{n_x}$ ,  $z \in \mathbb{R}^{n_z}$  and  $u \in \mathbb{R}^m$ . The state of the overall system is denoted as  $\tilde{x} := (x^T \ z^T)^T$ ,  $\tilde{x} \in \mathbb{R}^n$ , where  $n := n_x + n_z$ . We will assume the following:

**Assumption 6.4.2** *For any strictly positive real numbers  $\Delta_{\tilde{x}}, \Delta_u$  there exist strictly*

positive real numbers  $M$  and  $T^*$  such that

$$\begin{aligned} |\tilde{x}| \leq \Delta_{\tilde{x}}, |u| \leq \Delta_u, T \in (0, T^*), k \geq 0 \\ \implies \max\{|F_T(k, x, z, u)|, |G_T(k, z, u)|\} \leq M . \end{aligned} \quad (6.21)$$

■

The family of systems (6.20) is not in the form (6.1) which is time invariant. However, we can still apply results of our paper in the following way. We introduce an augmented *time-invariant* system in the following way:

$$\begin{aligned} x(k+1) &= F_T(p(k), x(k), z(k), u(k)) \\ z(k+1) &= G_T(p(k), z(k), u(k)) \\ p(k+1) &= p(k) + 1 , \end{aligned} \quad (6.22)$$

where  $p \in \mathbb{R}$  is a new state variable. Then it is standard to show that semiglobal practical uniform ISS of the time-varying system (6.20) with respect to the origin  $(x, z) = (0, 0)$  can be deduced from semiglobal practical ISS of the time-invariant system (6.22) with respect to a non-compact set  $\mathcal{A} := \{(\tilde{x}, p) : \tilde{x} = 0\}$ . Note also that we can write  $|\tilde{x}| = |(\tilde{x}, p)|_{\mathcal{A}}$ .

In the next result we show that SP-ISS Lyapunov function for the overall system (6.22) can be constructed from Lyapunov functions for individual subsystems in (6.22). In particular, we can state the following:

**Corollary 6.4.2** *Consider the system (6.20) and suppose that Assumption 6.4.2 holds. Suppose that there exist  $\underline{\alpha}_1, \bar{\alpha}_1, \alpha_1, \underline{\alpha}_2, \bar{\alpha}_2, \alpha_2 \in \mathcal{K}_\infty$ , and  $\sigma_1, \lambda_1, \sigma_2 \in \mathcal{G}$  such that for any triple of strictly positive real numbers  $(\Delta_{\tilde{x}}, \Delta_u, \nu)$  there exists  $T^* > 0$  and for any  $T \in (0, T^*)$  there exist  $V_{1T} : \mathbb{R} \times \mathbb{R}^{n_x} \rightarrow \mathbb{R}_{\geq 0}$  and  $V_{2T} : \mathbb{R} \times \mathbb{R}^{n_z} \rightarrow \mathbb{R}_{\geq 0}$  such that for all  $|\tilde{x}| \leq \Delta_{\tilde{x}}, |u| \leq \Delta_u, p \geq 0, T \in (0, T^*)$  we have the following:*

$$\begin{aligned} \underline{\alpha}_1(|x|) &\leq V_{1T}(p, x) \leq \bar{\alpha}_1(|x|) \\ V_{1T}(p+1, F_T(p, x, z, u)) - V_{1T}(p, x) &\leq T \left( -\alpha_1(|x|) + \lambda_1(|z|) + \sigma_1(|u|) + \nu \right) \end{aligned} \quad (6.23)$$

$$\begin{aligned} \underline{\alpha}_2(|z|) &\leq V_{2T}(p, z) \leq \bar{\alpha}_2(|z|) \\ V_{2T}(p+1, G_T(p, z, u)) - V_{2T}(p, z) &\leq T \left( -\alpha_2(|z|) + \sigma_2(|u|) + \nu \right) . \end{aligned} \quad (6.24)$$



Then, there exist  $\underline{\alpha}, \bar{\alpha}, \alpha \in \mathcal{K}_\infty$  and  $\sigma \in \mathcal{G}$  such that for any triple of strictly positive real numbers  $(\tilde{\Delta}_{\tilde{x}}, \tilde{\Delta}_u, \tilde{\nu})$  there exists  $\tilde{T} > 0$  and for any  $T \in (0, \tilde{T})$  there exist  $V_T : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  such that for all  $|\tilde{x}| \leq \tilde{\Delta}_{\tilde{x}}, |u| \leq \tilde{\Delta}_u, p \geq 0, T \in (0, \tilde{T})$  we have:

$$SP - ISS \left\{ \begin{array}{l} \underline{\alpha}(|\tilde{x}|) \leq V_T(p, x, z) \leq \bar{\alpha}(|\tilde{x}|) \\ V_T(p+1, F_T(p, x, z, u), G_T(p, z, u)) - V_T(p, x, z) \\ \leq T \left( -\alpha(|\tilde{x}|) + \sigma(|u|) + \tilde{\nu} \right). \end{array} \right. \quad (6.25)$$

■

**Proof of Corollary 6.4.2:** It follows directly from Assumption 6.4.2 and conditions of the corollary that all conditions of Theorem 6.3.2 hold. Indeed, we have that: (i) the system is  $(V_{1T}, \underline{\alpha}_1, \bar{\alpha}_1, \alpha_1, \sigma_1, \lambda_1)$ -SP-IOSS with measuring functions  $w_{\underline{\alpha}_1}(\tilde{x}, p) = w_{\bar{\alpha}_1}(\tilde{x}, p) = w_{\alpha_1}(\tilde{x}, p) = x$ ,  $w_{\lambda_1}(\tilde{x}, p) = z$ ,  $w_{x_1}(\tilde{x}, p) = \tilde{x}$ ,  $w_{\sigma_1}(u) = w_{u_1}(u) = u$ , so that  $\underline{\kappa}_1, \bar{\kappa}_1$  exist; moreover, from Assumption 6.4.2 we have that the third condition of Lemma 6.3.1 holds; hence, condition 2 of Theorem 6.3.2 holds; (ii) the system is  $(V_{2T}, \underline{\alpha}_2, \bar{\alpha}_2, \alpha_2, \sigma_2)$ -SP-ISS with measuring functions  $w_{\underline{\alpha}_2}(\tilde{x}, p) = w_{\bar{\alpha}_2}(\tilde{x}, p) = w_{\alpha_2}(\tilde{x}, p) = z$ ,  $w_{x_2}(\tilde{x}, p) = \tilde{x}$ ,  $w_{\sigma_2}(u) = w_{u_2}(u) = u$  and  $\lambda_2 = 0$ , so that  $\underline{\kappa}_2, \bar{\kappa}_2$  exist; moreover, from Assumption 6.4.2 we have that the third condition of Lemma 6.3.1 holds; hence, condition 1 of Theorem 6.3.2 holds; (iii)  $|w_{\lambda_1}(\tilde{x}, p)| = |w_{\alpha_2}(\tilde{x}, p)| = |z|$ ,  $|w_{x_1}(\tilde{x}, p)| = |w_{x_2}(\tilde{x}, p)| = |\tilde{x}|$  and  $|w_{u_1}(u)| = |w_{u_2}(u)| = |u|$  for all  $\tilde{x} \in \mathbb{R}^n, u \in \mathbb{R}^m$ ; hence, condition 3 of Theorem 6.3.2 holds.

Therefore, applying Theorem 6.3.2 and defining the new SP-ISS Lyapunov function  $V_T$  as in (6.14), we obtain that the system (6.20) is SP-ISS with measuring functions  $w_{\underline{\alpha}}(x) = w_{\bar{\alpha}}(x) = w_{\alpha}(x) = |x| + |z| = |\tilde{x}|$ ,  $w_x(x) = \tilde{x}$ ,  $w_\sigma(u) = |u|$ , and  $w_u(u) = u$ . ■

### 6.4.3 SP-ISS via positive semidefinite Lyapunov functions

The problem of checking stability using positive semidefinite Lyapunov functions has been considered in [13] for continuous-time systems and in [35] for discrete-time systems. The idea is to use a Lyapunov function  $V(x)$ , which is positive semidefinite, to check stability of a system. An approach taken in [13, 35] was to use a trajectory-based technique to prove stability of the origin of the system. In particular, besides appropriate conditions on the Lyapunov function, it was required in [13, 35] that all

trajectories in the maximal invariant subset of the set  $Z := \{x : V(x) = 0\}$  satisfy the  $\epsilon - \delta$  definition of asymptotic stability (this property was referred to as conditional stability to the set  $Z$ ).

We note that the results on stability of cascade-connected systems in [111, 142] and in the previous subsection can be interpreted as a special case of testing ISS using positive semidefinite Lyapunov functions. However, this approach is different from the one in [13, 35] since an ISS Lyapunov function is constructed explicitly from ISS and IOSS Lyapunov functions of each subsystem. The advantage of the approach of [111, 142] is that it leads to a construction of a Lyapunov function for the overall system, whereas the disadvantage is that it requires usually stronger conditions and it appears to apply only to a special class of cascade-connected systems. However, we show here that the same approach can be used with few modifications to test semiglobal practical ISS of general parameterized discrete-time systems (6.1) that are not in the cascade form. In particular, we can state:

**Corollary 6.4.3** *Consider the family of systems (6.1) and suppose that Assumption 6.4.1 holds. Suppose that there exist  $\underline{\alpha}_1, \bar{\alpha}_1, \alpha_1, \underline{\alpha}_2, \bar{\alpha}_2, \alpha_2 \in \mathcal{K}_\infty$ ,  $\sigma_1, \lambda_1, \sigma_2 \in \mathcal{G}$  and positive semidefinite functions  $W_1 : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  and  $W_2 : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ , with  $W_1(x) + W_2(x)$  is positive definite and radially unbounded, such that for any triple of strictly positive real numbers  $(\Delta_x, \Delta_u, \nu)$  there exists  $T^* > 0$  and for any  $T \in (0, T^*)$  there exist  $V_{1T} : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  and  $V_{2T} : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  such that for all  $|x| \leq \Delta_x$ ,  $|u| \leq \Delta_u$ ,  $T \in (0, T^*)$  we have the following:*

$$\begin{aligned} \underline{\alpha}_1(W_1(x)) &\leq V_{1T}(x) \leq \bar{\alpha}_1(W_1(x)) \\ V_{1T}(F_T(x, u)) - V_{1T}(x) &\leq T \left( -\alpha_1(W_1(x)) + \lambda_1(W_2(x)) + \sigma_1(|u|) + \nu \right) \end{aligned} \quad (6.26)$$

$$\begin{aligned} \underline{\alpha}_2(W_2(x)) &\leq V_{2T}(x) \leq \bar{\alpha}_2(W_2(x)) \\ V_{2T}(F_T(x, u)) - V_{2T}(x) &\leq T \left( -\alpha_2(W_2(x)) + \sigma_2(|u|) + \nu \right). \end{aligned} \quad (6.27)$$

Then, there exist  $\underline{\alpha}, \bar{\alpha}, \alpha \in \mathcal{K}_\infty$  and  $\sigma \in \mathcal{G}$  such that for any triple of strictly positive real numbers  $(\tilde{\Delta}_x, \tilde{\Delta}_u, \tilde{\nu})$  there exists  $\tilde{T} > 0$  and for any  $T \in (0, \tilde{T})$  there exist  $V_T :$

$\mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  such that for all  $|x| \leq \tilde{\Delta}_x$ ,  $|u| \leq \tilde{\Delta}_u$ ,  $T \in (0, \tilde{T})$  we have:

$$SP-ISS \left\{ \begin{array}{l} \underline{\alpha}(|x|) \leq V_T(x) \leq \bar{\alpha}(|x|) \\ V_T(F_T(x, u)) - V_T(x) \leq T \left( -\alpha(|x|) + \sigma(|u|) + \tilde{\nu} \right) \end{array} \right. \quad (6.28)$$

■

**Proof of Corollary 6.4.3:** It can be seen immediately that all conditions of Theorem 6.3.2 hold, by noting that: (i) the system (6.1) is  $(V_{1T}, \underline{\alpha}_1, \bar{\alpha}_1, \alpha_1, \lambda_1, \sigma_1)$ -SP-ISS with measuring functions  $w_{\underline{\alpha}_1}(x) = w_{\bar{\alpha}_1}(x) = w_{\alpha_1}(x) = W_1(x)$ ,  $w_{x_1}(x) = x$ ,  $w_{\lambda_1}(x) = W_2(x)$ ,  $w_{\sigma_1}(u) = w_{u_1}(u) = u$ , so that  $\underline{\kappa}_1$ ,  $\bar{\kappa}_1$  exist; moreover, from Assumption 6.4.1 and Remark 6.3.1 we have that the third condition of Lemma 6.3.1 holds; hence, condition 2 of Theorem 6.3.2 holds; (ii) the system (6.1) is  $(V_{2T}, \underline{\alpha}_2, \bar{\alpha}_2, \alpha_2, \sigma_2)$ -SP-ISS with measuring functions  $w_{\underline{\alpha}_2}(x) = w_{\bar{\alpha}_2}(x) = w_{\alpha_2}(x) = W_2(x)$ ,  $w_{x_2}(x) = x$  and  $w_{\sigma_2}(u) = w_{u_2}(u) = u$ , so that  $\underline{\kappa}_2$ ,  $\bar{\kappa}_2$  exist; moreover, from Assumption 6.4.1 and Remark 6.3.1 we have that the third condition of Lemma 6.3.1 holds; hence, condition 1 of Theorem 6.3.2 holds; (iii) the third condition of Theorem 6.3.2 holds since  $w_{\alpha_2}(x) = w_{\lambda_1}(x) = W_2(x)$ ,  $w_{x_1}(x) = w_{x_2}(x) = x$  and  $w_{u_1}(u) = w_{u_2}(u) = u$  for all  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ .

Then, applying Theorem 6.3.2 and defining the new SP-ISS Lyapunov function  $V_T$  as in (6.14), we obtain that the system (6.1) is SP-ISS with measuring functions  $w_{\underline{\alpha}}(x) = w_{\bar{\alpha}}(x) = w_{\alpha}(x) = W_1(x) + W_2(x)$ ,  $w_x(x) = x$ ,  $w_{\sigma}(u) = |u|$  and  $w_u(u) = u$ . The conclusion follows from the fact that  $W_1(x) + W_2(x)$  is a positive definite and radially unbounded and hence there exist  $\tilde{\alpha}_1, \tilde{\alpha}_2 \in \mathcal{K}_{\infty}$  such that

$$\tilde{\alpha}_1(|x|) \leq W_1(x) + W_2(x) \leq \tilde{\alpha}_2(|x|)$$

for all  $x \in \mathbb{R}^n$ .

■

#### 6.4.4 Observer-based input-to-state stabilization

Observer-based stabilization of discrete-time nonlinear systems that was considered in [65, 66] uses a very similar construction to the ones considered in this paper. It was shown in [65, 66] that if a discrete-time plant can be robustly stabilized with full state feedback (in an ISS sense) and there exists an observer for the system satisfying

appropriate Lyapunov conditions (that is, the system is weakly detectable), then the plant is also stabilized using the controller/observer pair where the controller uses the state estimate obtained from the observer. Both local and global results were considered in [65, 66].

In this section, we show that our results, particularly Theorem 6.3.2, can be used to generalize results of [65, 66] in two directions: (i) we present results on observer based input-to-state stabilization of discrete-time systems (in [65, 66] only stabilization was considered); (ii) results on semiglobal practical ISS of parameterized systems (6.1) are presented (in [65, 66] only global and local stabilization of non-parameterized discrete-time systems were considered).

In this section we consider the parameterized family of plants:

$$\begin{aligned} x(k+1) &= F_T(x(k), u(k), v(k)) \\ y(k) &= h(x(k)) , \end{aligned} \tag{6.29}$$

where  $u$  and  $v$  are respectively the control and exogenous inputs, with the following observer

$$z(k+1) = G_T(z(k), h(x(k)), u(k), v(k)) , \tag{6.30}$$

and controller

$$u(k) = \phi_T(z(k)) , \tag{6.31}$$

that are defined for sufficiently small  $T$ . Let  $\tilde{x} := (x^T \ z^T)^T$ , and we assume the following:

**Assumption 6.4.3** *For any strictly positive real numbers  $\Delta_{\tilde{x}}, \Delta_u, \Delta_v$  there exist strictly positive real numbers  $M$  and  $T^*$  such that*

$$\begin{aligned} |\tilde{x}| \leq \Delta_{\tilde{x}}, |u| \leq \Delta_u, |v| \leq \Delta_v, T \in (0, T^*) \\ \implies \max\{|F_T(x, u, v)|, |G_T(x, z, u, v)|, |\phi_T(z)|\} \leq M . \end{aligned} \tag{6.32}$$

■

Then, we can state the following result:

**Corollary 6.4.4** *Consider the family of systems (6.29), (6.30) and (6.31) and suppose that Assumption 6.4.3 holds. Suppose that there exist,  $\underline{\alpha}_1, \bar{\alpha}_1, \alpha_1, \underline{\alpha}_2, \bar{\alpha}_2, \alpha_2 \in \mathcal{K}_\infty$ ,  $\sigma_1, \lambda_1, \sigma_2 \in \mathcal{G}$ , such that for any triple of strictly positive real numbers  $(\Delta_{\tilde{x}}, \Delta_v, \nu)$  there exists  $T^* > 0$  and for any  $T \in (0, T^*)$  there exist  $V_{1T} : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  and  $V_{2T} : \mathbb{R}^{2n} \rightarrow \mathbb{R}_{\geq 0}$  such that for all  $|\tilde{x}| \leq \Delta_{\tilde{x}}, |v| \leq \Delta_v, T \in (0, T^*)$  we have the following:*

$$\begin{aligned} \underline{\alpha}_1(|x|) &\leq V_{1T}(x) \leq \bar{\alpha}_1(|x|) \\ V_{1T}(F_T(x, \phi_T(z), v)) - V_{1T}(x) &\leq T \left( -\alpha_1(|x|) + \lambda_1(|x - z|) + \sigma_1(|v|) + \nu \right) \end{aligned} \quad (6.33)$$

$$\begin{aligned} \underline{\alpha}_2(|x - z|) &\leq V_{2T}(x, z) \leq \bar{\alpha}_2(|x - z|) \\ V_{2T}(F_T(x, \phi_T(z), v), G_T(z, h(x), \phi_T(z), v)) - V_{2T}(x, z) &\leq T \left( -\alpha_2(|x - z|) + \sigma_2(|v|) + \nu \right). \end{aligned} \quad (6.34)$$

Then, there exist  $\underline{\alpha}, \bar{\alpha}, \alpha \in \mathcal{K}_\infty$  and  $\sigma \in \mathcal{G}$  such that for any triple of strictly positive real numbers  $(\tilde{\Delta}_{\tilde{x}}, \tilde{\Delta}_v, \tilde{\nu})$  there exists  $\tilde{T} > 0$  and for any  $T \in (0, \tilde{T})$  there exist  $V_T : \mathbb{R}^{2n} \rightarrow \mathbb{R}_{\geq 0}$  such that for all  $|\tilde{x}| \leq \tilde{\Delta}_{\tilde{x}}, |v| \leq \tilde{\Delta}_v, T \in (0, \tilde{T})$  we have:

$$SP - ISS \left\{ \begin{array}{l} \underline{\alpha}(|\tilde{x}|) \leq V_T(x, z) \leq \bar{\alpha}(|\tilde{x}|) \\ V_T(F_T(x, \phi_T(z), v), G_T(z, h(x), \phi_T(z), v)) - V_T(x, z) \\ \leq T \left( -\alpha(|\tilde{x}|) + \sigma(|v|) + \tilde{\nu} \right). \end{array} \right. \quad (6.35)$$

■

**Proof of Corollary 6.4.4:** It can be seen immediately that all conditions of Theorem 6.3.2 hold, by noting that: (i) the systems (6.29), (6.31) is  $(V_{1T}, \underline{\alpha}_1, \bar{\alpha}_1, \alpha_1, \lambda_1, \sigma_1)$ -SP-IOSS with measuring functions  $w_{\underline{\alpha}_1}(\tilde{x}) = w_{\bar{\alpha}_1}(\tilde{x}) = w_{\alpha_1}(\tilde{x}) = x$ ,  $w_{\lambda_1}(\tilde{x}) = x - z$ ,  $w_{\sigma_1}(v) = w_{u_1}(v) = v$ , so that  $\underline{\kappa}_1, \bar{\kappa}_1$  exist; moreover from Assumption 6.4.3 and Remark 6.3.1 we have that the third condition of Lemma 6.3.1 holds; hence, condition 2 of Theorem 6.3.2 holds; (ii) the systems (6.29), (6.30) and (6.31) is  $(V_{2T}, \underline{\alpha}_2, \bar{\alpha}_2, \alpha_2, \sigma_2)$ -SP-ISS with measuring functions  $w_{\underline{\alpha}_2}(\tilde{x}) = w_{\bar{\alpha}_2}(\tilde{x}) = w_{\alpha_2}(\tilde{x}) = x - z$ ,  $w_{\sigma_2}(\tilde{x}) = \tilde{x}$  and  $w_{\sigma_2}(v) = w_{u_2}(v) = v$ , so that  $\underline{\kappa}_2, \bar{\kappa}_2$  exist; moreover, from Assumption 6.4.3 and Remark 6.3.1 we have that the third condition of Lemma 6.3.1 holds; hence, condition 1 of Theorem 6.3.2 holds; (iii) the third condition of Theorem 6.3.2 holds since  $|w_{\alpha_2}(\tilde{x})| = |w_{\lambda_1}(\tilde{x})| = |x - z|$ ,  $|w_{x_1}(\tilde{x})| = |w_{x_2}(\tilde{x})| = |\tilde{x}|$  and  $|w_{u_1}(v)| = |w_{u_2}(v)| = |v|$  for all  $x \in \mathbb{R}^n, v \in \mathbb{R}^m$ .

Then, applying Theorem 6.3.2 and defining the new SP-ISS Lyapunov function  $V_T$  as in (6.14), we obtain that the system (6.29), (6.30) and (6.31) is SP-ISS with measuring functions  $w_{\underline{\alpha}}(\tilde{x}) = w_{\bar{\alpha}}(\tilde{x}) = w_{\alpha}(\tilde{x}) = |x| + |x - z|$ ,  $w_x(\tilde{x}) = \tilde{x}$ ,  $w_{\sigma}(v) = |v|$  and  $w_u(v) = v$ . The proof follows from the fact that  $\frac{1}{2}|\tilde{x}| \leq |x| + |x - z| \leq 2|\tilde{x}|$ . ■

**Remark 6.4.1** *There are many variations of conditions in (6.33) that could be used to state similar results (for more details see [65, 66]). Also, there is a small discrepancy between the way we write conditions (6.33) and conditions used in [65, 66]. However, it is not hard to show that these conditions are equivalent. For example, we note that instead of the second inequality in (6.33) we could use:*

$$V_{1T}(F_T(x, \phi_T(x + d), v)) - V_{1T}(x) \leq T \left( -\alpha_1(|x|) + \lambda_1(|d|) + \sigma_1(|v|) + \nu \right), \quad (6.36)$$

where  $d$  is a “new disturbance” (similar conditions were used in [65, 66]). This condition states that the full state feedback controller  $u = \phi_T(x)$  robustly stabilizes the plant (6.29) in an ISS sense. Since for the controller that uses the estimates state we can write  $\phi_T(z) = \phi_T(x + (z - x))$  and let  $d = x - z$ , we can see that this is the same condition as the one we used in (6.33). ■

## 6.5 Proofs of main results

In this section, we provide proofs of our main results. The proofs of corollaries are omitted since they follow directly from the main results. We use a proof technique, which is similar to the one used in [3] and [111]. The proof of Lemma 6.3.1 is the main difference between the continuous-time results in [3] and our discrete-time results.

### Proof of Lemma 6.3.1

We denote  $V_T(F_T) := V_T(F_T(x, u))$  and  $V_T := V_T(x)$ . Suppose that all conditions in Lemma 6.3.1 are satisfied. Fix an arbitrary  $q \in \mathcal{SN}$  and let  $\rho$  be defined using (6.8). We prove next that  $\rho(V_T)$  is a SP-IOSS Lyapunov function for the system with appropriate bounding and measuring functions stated in the lemma.

First, note that from the Mean Value Theorem and the fact that  $q(\cdot) = \frac{d\rho}{ds}(\cdot)$  is

## 6. Changing Supply Rates for Input-Output to State Stable Systems 159

nondecreasing, it follows that

$$\rho(a) - \rho(b) \leq q(a)[a - b] \quad \forall a \geq 0, b \geq 0. \quad (6.37)$$

Let arbitrary strictly positive real numbers  $(\Delta'_x, \Delta'_u, \nu')$  be given. Let the numbers  $\Delta'_x, \Delta'_u$  generate numbers  $M, T_1^*$  via the third condition of the lemma, so that (6.9) holds. Let  $\nu_1$  be such that

$$\max\{\lambda(M), \sigma(M)\}[q(s + \nu_1) - q(s)] \leq \frac{\nu'}{2}, \forall s \in [0, \bar{\alpha}(M) + 2 \max\{\lambda(M), \sigma(M)\}].$$

Such  $\nu_1$  always exists since  $q(\cdot)$  is continuous. We define:

$$\Delta_x := \Delta'_x, \quad \Delta_u := \Delta'_u, \quad \nu := \min \left\{ \frac{\nu'}{2q \circ \bar{\alpha}(M)}, \nu_1 \right\}.$$

Let  $(\Delta_x, \Delta_u, \nu)$  determine  $T_2^* > 0$  and  $V_T$  using the first condition of the lemma, such that for all  $T \in (0, T_2^*)$  and all  $|w_x(x)| \leq \Delta_x, |w_u(u)| \leq \Delta_u$  the inequalities (6.2) and (6.3) hold. Fix  $T^* := \min\{T_1^*, T_2^*, 1\}$ . In the rest of the proof we always consider arbitrary  $T \in (0, T^*), |w_x(x)| \leq \Delta_x$  and  $|w_u(u)| \leq \Delta_u$ .

A direct consequence of condition 1 of the lemma and the fact that  $T^* \leq 1$  is that:

$$V_T \geq \max\{\underline{\alpha}(|w_{\underline{\alpha}}(x)|), T\alpha(|w_{\alpha}(x)|) - T\lambda(|w_{\lambda}(x)|) - T\sigma(|w_{\sigma}(u)|) - T\nu\} \quad (6.38)$$

$$V_T(F_T) \leq \bar{\alpha}(|w_{\bar{\alpha}}(x)|) + \lambda(|w_{\lambda}(x)|) + \sigma(|w_{\sigma}(u)|) + \nu. \quad (6.39)$$

Note first that

$$\rho \circ \underline{\alpha}(|w_{\underline{\alpha}}(x)|) \leq \rho(V_T) \leq \rho \circ \bar{\alpha}(|w_{\bar{\alpha}}(x)|),$$

which shows that (6.2) holds with the new bounding functions  $\underline{\alpha}'(s) = \rho \circ \underline{\alpha}(s)$  and  $\bar{\alpha}'(s) = \rho \circ \bar{\alpha}(s)$  and the same measuring functions. Now we prove that (6.3) holds for  $\rho(V_T)$  with the new bounding functions and the same measuring functions. Let use the notation  $\Delta\rho(V_T) := \rho(V_T(F_T)) - \rho(V_T)$ . The following two preliminary cases are first considered:

1.  $V_T(F_T) \leq \frac{1}{2}V_T$

Using the inequalities (6.37) and (6.38) and the definition of  $M$  and  $\nu$  we obtain

$$\begin{aligned}
 \Delta\rho(V_T) &:= \rho(V_T(F_T)) - \rho(V_T) \\
 &\leq \rho\left(\frac{1}{2}V_T\right) - \rho(V_T) \\
 &\leq q\left(\frac{1}{2}V_T\right) \left[-\frac{1}{2}V_T\right] \\
 &\leq \frac{T}{2}q\left(\frac{1}{2}V_T\right) \cdot (-\alpha(|w_\alpha(x)|) + \lambda(|w_\lambda(x)|) + \sigma(|w_\sigma(u)|) + \nu) \\
 &\leq \frac{T}{2}q\left(\frac{1}{2}V_T\right) \cdot (-\alpha(|w_\alpha(x)|) + \lambda(|w_\lambda(x)|)) + \sigma(|w_\sigma(u)|) \\
 &\quad + T\frac{q \circ \bar{\alpha}(M)}{2}\nu \\
 &\leq \frac{T}{2}q\left(\frac{1}{2}V_T\right) \cdot (-\alpha(|w_\alpha(x)|) + \lambda(|w_\lambda(x)|)) + \sigma(|w_\sigma(u)|) + T\frac{\nu'}{4}
 \end{aligned} \tag{6.40}$$

2.  $V_T(F_T) > \frac{1}{2}V_T$

Using the inequalities (6.37) and (6.3) and the definition of  $M$  and  $\nu$  we obtain

$$\begin{aligned}
 \Delta\rho(V_T) &:= \rho(V_T(F_T)) - \rho(V_T) \\
 &\leq q(V_T(F_T)) [V_T(F_T) - V_T] \\
 &\leq Tq(V_T(F_T)) \cdot (-\alpha(|w_\alpha(x)|) + \lambda(|w_\lambda(x)|) + \sigma(|w_\sigma(u)|) + \nu) \\
 &\leq Tq(V_T(F_T)) \cdot (-\alpha(|w_\alpha(x)|) + \lambda(|w_\lambda(x)|)) + \sigma(|w_\sigma(u)|) \\
 &\quad + Tq \circ \bar{\alpha}(M)\nu \\
 &\leq Tq(V_T(F_T)) \cdot (-\alpha(|w_\alpha(x)|) + \lambda(|w_\lambda(x)|)) + \sigma(|w_\sigma(u)|) + T\frac{\nu'}{2}.
 \end{aligned} \tag{6.41}$$

The proof is completed by considering the following three cases:

Case 1:  $\lambda(|w_\lambda(x)|) + \sigma(|w_\sigma(u)|) \leq \frac{1}{2}\alpha(|w_\alpha(x)|)$

•  $V_T(F_T) \leq \frac{1}{2}V_T$

We use (6.40) to write:

$$\begin{aligned}
 \Delta\rho(V_T) &:= \rho(V_T(F_T)) - \rho(V_T) \leq \frac{T}{2}q\left(\frac{1}{2}V_T\right) \cdot \left(-\frac{1}{2}\alpha(|w_\alpha(x)|)\right) + T\frac{\nu'}{4} \\
 &\leq -\frac{T}{4}q\left(\frac{1}{2}V_T\right) \cdot \alpha(|w_\alpha(x)|) + T\frac{\nu'}{4}
 \end{aligned} \tag{6.42}$$



- $V_T(F_T) > \frac{1}{2}V_T$

We use (6.41) and the fact that  $q$  is nondecreasing to write:

$$\begin{aligned} \Delta\rho(V_T) &:= \rho(V_T(F_T)) - \rho(V_T) \leq Tq(V_T(F_T)) \cdot \left(-\frac{1}{2}\alpha(|w_\alpha(x)|)\right) + T\frac{\nu'}{2} \\ &\leq -\frac{T}{2}q\left(\frac{1}{2}V_T\right) \cdot \alpha(|w_\alpha(x)|) + T\frac{\nu'}{2} \\ &\leq -\frac{T}{4}q\left(\frac{1}{2}V_T\right) \cdot \alpha(|w_\alpha(x)|) + T\frac{\nu'}{2} \end{aligned} \quad (6.43)$$

Since  $q$  is nondecreasing, using (6.38) and the second condition of the lemma, the following always holds for Case 1:

$$\rho(V_T(F_T)) - \rho(V_T) \leq -\frac{T}{4}q\left(\frac{1}{2}\underline{\alpha} \circ \underline{\kappa}(|w_\alpha(x)|)\right) \cdot \alpha(|w_\alpha(x)|) + T\frac{\nu'}{2} \quad (6.44)$$

Case 2:  $\lambda(|w_\lambda(x)|) + \sigma(|w_\sigma(u)|) > \frac{1}{2}\alpha(|w_\alpha(x)|)$ ,  $\lambda(|w_\lambda(x)|) \geq \sigma(|w_\sigma(u)|)$

- $V_T(F_T) \leq \frac{1}{2}V_T$

We use (6.40), (6.2), the fact that  $q$  is nondecreasing,  $T^* \leq 1$  and the choice of  $\nu_1$  to write:

$$\begin{aligned} \Delta\rho(V_T) &:= \rho(V_T(F_T)) - \rho(V_T) \\ &\leq \frac{T}{2}q\left(\frac{1}{2}V_T\right) \cdot (-\alpha(|w_\alpha(x)|) + 2\lambda(|w_\lambda(x)|)) + T\frac{\nu'}{4} \\ &\leq -\frac{T}{2}q\left(\frac{1}{2}V_T\right) \cdot \alpha(|w_\alpha(x)|) + Tq\left(\frac{1}{2}\bar{\alpha}(|w_{\bar{\alpha}}(x)|)\right) \cdot \lambda(|w_\lambda(x)|) + T\frac{\nu'}{4} \\ &\leq -\frac{T}{2}q\left(\frac{1}{2}V_T\right) \cdot \alpha(|w_\alpha(x)|) \\ &\quad + Tq(\bar{\alpha}(|w_{\bar{\alpha}}(x)|) + 2\lambda(|w_\lambda(x)|) + \nu_1) \cdot \lambda(|w_\lambda(x)|) + T\frac{\nu'}{4} \\ &\leq -\frac{T}{2}q\left(\frac{1}{2}V_T\right) \cdot \alpha(|w_\alpha(x)|) \\ &\quad + Tq(\bar{\alpha}(|w_{\bar{\alpha}}(x)|) + 2\lambda(|w_\lambda(x)|)) \cdot \lambda(|w_\lambda(x)|) + T\frac{\nu'}{2} + T\frac{\nu'}{4} \end{aligned} \quad (6.45)$$

- $V_T(F_T) > \frac{1}{2}V_T$

We use (6.41), (6.2), the fact that  $q$  is nondecreasing,  $T^* \leq 1$  and the choice of  $\nu_1$  to

write:

$$\begin{aligned}
 \Delta\rho(V_T) &:= \rho(V_T(F_T)) - \rho(V_T) \\
 &\leq Tq(V_T(F_T)) \cdot (-\alpha(|w_\alpha(x)|) + 2\lambda(|w_\lambda(x)|)) + T\frac{\nu'}{2} \\
 &\leq -Tq\left(\frac{1}{2}V_T\right) \cdot \alpha(|w_\alpha(x)|) \\
 &\quad + 2Tq(\bar{\alpha}(|w_{\bar{\alpha}}(x)|) + 2\lambda(|w_\lambda(x)|) + \nu_1) \cdot \lambda(|w_\lambda(x)|) + T\frac{\nu'}{2} \\
 &\leq -Tq\left(\frac{1}{2}V_T\right) \cdot \alpha(|w_\alpha(x)|) \\
 &\quad + 2Tq(\bar{\alpha}(|w_{\bar{\alpha}}(x)|) + 2\lambda(|w_\lambda(x)|)) \cdot \lambda(|w_\lambda(x)|) + T\frac{\nu'}{2} + T\frac{\nu'}{2}
 \end{aligned} \tag{6.46}$$

Since  $q$  is nondecreasing, using (6.38), (6.45), (6.46), the second condition of the lemma, the condition that  $\lambda(|w_\lambda(x)|) > \frac{1}{4}\alpha(|w_\alpha(x)|)$  and the definition of  $\theta_\lambda$  given by (6.11), the following always holds for Case 2:

$$\begin{aligned}
 \rho(V_T(F_T)) - \rho(V_T) &\leq -\frac{T}{2}q\left(\frac{1}{2}\underline{\alpha} \circ \underline{\kappa}(|w_\alpha(x)|)\right) \cdot \alpha(|w_\alpha(x)|) \\
 &\quad + 2Tq \circ \theta_\lambda(|w_\lambda(x)|) \cdot \lambda(|w_\lambda(x)|) + T\nu'
 \end{aligned} \tag{6.47}$$

Case 3:  $\lambda(|w_\lambda(x)|) + \sigma(|w_\sigma(u)|) > \frac{1}{2}\alpha(|w_\alpha(x)|)$ ,  $\lambda(|w_\lambda(x)|) < \sigma(|w_\sigma(u)|)$

•  $V_T(F_T) \leq \frac{1}{2}V_T$

We use (6.40), (6.2), the fact that  $q$  is nondecreasing,  $T^* \leq 1$  and the choice of  $\nu_1$  to write:

$$\begin{aligned}
 \Delta\rho(V_T) &:= \rho(V_T(F_T)) - \rho(V_T) \\
 &\leq -\frac{T}{2}q\left(\frac{1}{2}V_T\right) \cdot \alpha(|w_\alpha(x)|) + Tq\left(\frac{1}{2}V_T\right) \cdot \sigma(|w_\sigma(u)|) + T\frac{\nu'}{4} \\
 &\leq -\frac{T}{2}q\left(\frac{1}{2}V_T\right) \cdot \alpha(|w_\alpha(x)|) + Tq\left(\frac{1}{2}\bar{\alpha}(|w_{\bar{\alpha}}(x)|)\right) \cdot \sigma(|w_\sigma(u)|) + T\frac{\nu'}{4} \\
 &\leq -\frac{T}{2}q\left(\frac{1}{2}V_T\right) \cdot \alpha(|w_\alpha(x)|) \\
 &\quad + Tq(\bar{\alpha}(|w_{\bar{\alpha}}(x)|) + 2\sigma(|w_\sigma(u)|) + \nu_1) \cdot \sigma(|w_\sigma(u)|) + T\frac{\nu'}{4} \\
 &\leq -\frac{T}{2}q\left(\frac{1}{2}V_T\right) \cdot \alpha(|w_\alpha(x)|) \\
 &\quad + Tq(\bar{\alpha}(|w_{\bar{\alpha}}(x)|) + 2\sigma(|w_\sigma(u)|)) \cdot \sigma(|w_\sigma(u)|) + T\frac{\nu'}{2} + T\frac{\nu'}{4}
 \end{aligned} \tag{6.48}$$

•  $V_T(F_T) > \frac{1}{2}V_T$

We use (6.41), (6.2), the fact that  $q$  is nondecreasing,  $T^* \leq 1$  and the choice of  $\nu_1$  to

write:

$$\begin{aligned}
 \Delta\rho(V_T) &:= \rho(V_T(F_T)) - \rho(V_T) \\
 &\leq Tq(V_T(F_T)) \cdot (-\alpha(|w_\alpha(x)|) + 2\sigma(|w_\sigma(u)|)) + T\frac{\nu'}{2} \\
 &\leq -Tq(V_T(F_T)) \cdot \alpha(|w_\alpha(x)|) + 2Tq(V_T(F_T)) \cdot \sigma(|w_\sigma(u)|) + T\frac{\nu'}{2} \\
 &\leq -Tq\left(\frac{1}{2}V_T\right) \cdot \alpha(|w_\alpha(x)|) \\
 &\quad + 2Tq(\bar{\alpha}(|w_{\bar{\alpha}}(x)|) + 2\sigma(|w_\sigma(u)|) + \nu_1) \cdot \sigma(|w_\sigma(u)|) + T\frac{\nu'}{2} \\
 &\leq -Tq\left(\frac{1}{2}V_T\right) \cdot \alpha(|w_\alpha(x)|) \\
 &\quad + 2Tq(\bar{\alpha}(|w_{\bar{\alpha}}(x)|) + 2\sigma(|w_\sigma(u)|)) \cdot \sigma(|w_\sigma(u)|) + T\frac{\nu'}{2} + T\frac{\nu'}{2}
 \end{aligned} \tag{6.49}$$

Since  $q$  is nondecreasing, using (6.38), (6.48), (6.49), the second condition of the lemma, the condition that  $\sigma(|w_\sigma(u)|) > \frac{1}{4}\alpha(|w_\alpha(x)|)$  and the definition of  $\theta_\sigma$  given by (6.10), the following always holds for Case 3:

$$\begin{aligned}
 \rho(V_T(F_T)) - \rho(V_T) &\leq -\frac{T}{2}q\left(\frac{1}{2}\underline{\alpha} \circ \underline{\kappa}(|w_\alpha(x)|)\right) \cdot \alpha(|w_\alpha(x)|) \\
 &\quad + 2Tq \circ \theta_\sigma(|w_\sigma(u)|) \cdot \sigma(|w_\sigma(u)|) + T\nu' \tag{6.50}
 \end{aligned}$$

We have shown through these three cases that the following holds:

$$\begin{aligned}
 \rho(V_T(F_T(x, u))) - \rho(V_T(x)) &\leq T\left[-\frac{1}{4}q \circ \frac{1}{2}\underline{\alpha} \circ \underline{\kappa}(|w_\alpha(x)|) \cdot \alpha(|w_\alpha(x)|) \right. \\
 &\quad \left. + 2q \circ \theta_\lambda(|w_\lambda(x)|) \cdot \lambda(|w_\lambda(x)|) + 2q \circ \theta_\sigma(|w_\sigma(u)|) \cdot \sigma(|w_\sigma(u)|) + \nu'\right], \tag{6.51}
 \end{aligned}$$

which completes the proof of Lemma 6.3.1. ■

**Proof of Theorem 6.3.1** Suppose that all conditions of the theorem be satisfied. Let  $\underline{\alpha}_1, \bar{\alpha}_1, \alpha_1, \sigma_1$  come from the condition 1 and  $\underline{\alpha}_2, \bar{\alpha}_2, \alpha_2, \lambda_2, \sigma_2$  come from the condition 2. Define  $\tilde{q}$  as:

$$\tilde{q}(r) := \inf_{r \leq s} \frac{\alpha_1 \circ \gamma_1^{-1}(s)}{2(1 + \lambda_2(s))}, \tag{6.52}$$

where  $\gamma_1$  comes from the third condition of the theorem. Notice that  $\tilde{q}$  is by definition a nondecreasing function. Condition 4 of the theorem implies  $\tilde{q}(r) > 0$  for all  $r > 0$ . Let  $q(s) := \tilde{q} \circ \theta_{\lambda_2}^{-1}(s)$ , where  $\theta_{\lambda_2}$  is defined in (6.11). Using  $q(\cdot)$  we define  $\rho(\cdot)$  via (6.8). Let  $\rho$  generate via Lemma 5.4.1 the new bounding functions  $\underline{\alpha}'_2, \bar{\alpha}'_2, \alpha'_2, \lambda'_2, \sigma'_2$ .

Let arbitrary strictly positive real numbers  $(\Delta_x, \Delta_u, \nu)$  be given. Let  $(\Delta_x, \Delta_u, \frac{\nu}{2})$  generate via condition 1 the number  $T_1^*$  and  $V_{1T}$ . Let  $(\gamma_2(\Delta_x), \gamma_3(\Delta_u), \frac{\nu}{2})$  generate via condition 2 and Lemma 5.4.1 the number  $T_2^*$  and  $\rho(V_{2T})$ . Let  $T^* = \min\{T_1^*, T_2^*\}$  and define now  $V_T$  as (6.12). Let  $w_x(x) := w_{x_1}(x)$  and  $w_u(u) := w_{u_1}(u)$ . We consider now arbitrary  $|w_x(x)| \leq \Delta_x$ ,  $|w_u(u)| \leq \Delta_u$  and  $T \in (0, T^*)$ . Note that this implies via condition 3 of the theorem that  $w_{x_2}(x) \leq \gamma_2(\Delta_x)$  and  $w_{u_2}(x) \leq \gamma_3(\Delta_u)$ .

First, it follows from the definition of  $V_T$  that

$$\underline{\alpha}_1(|w_{\underline{\alpha}_1}(x)|) + \rho \circ \underline{\alpha}_2(|w_{\underline{\alpha}_2}(x)|) \leq V_T(x) \leq \bar{\alpha}_1(|w_{\bar{\alpha}_1}(x)|) + \rho \circ \bar{\alpha}_2(|w_{\bar{\alpha}_2}(x)|) . \quad (6.53)$$

Then by Remark 6.2.1, there exist  $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$  such that

$$\underline{\alpha}(|w_{\underline{\alpha}_1}(x)| + |w_{\underline{\alpha}_2}(x)|) \leq V_T(x) \leq \bar{\alpha}(|w_{\bar{\alpha}_1}(x)| + |w_{\bar{\alpha}_2}(x)|) . \quad (6.54)$$

Using condition 4 of the theorem, the dissipation inequality for  $V_T$  can be written as:

$$\begin{aligned} V_T(F_T(x, u)) - V_T(x) &= V_{1T}(F_T) - V_{1T} + \rho(V_{2T}(F_T)) - \rho(V_{2T}) \\ &\leq T \left[ \sigma_1(|w_{\sigma_1}(u)|) + \sigma'_2(|w_{\sigma_2}(u)|) + \frac{\nu}{2} - \alpha_1(|w_{\alpha_1}(x)|) \right. \\ &\quad \left. + \lambda'_2 \circ \gamma_1(|w_{\alpha_1}(x)|) - \alpha'_2(|w_{\alpha_2}(x)|) + \frac{\nu}{2} \right] \\ &\leq T \left[ \sigma_1(|w_{\sigma_1}(u)|) + \sigma'_2(|w_{\sigma_2}(u)|) + \frac{\nu}{2} - \alpha_1(|w_{\alpha_1}(x)|) \right. \\ &\quad \left. + \frac{\alpha_1(|w_{\alpha_1}(x)|) \lambda_2 \circ \gamma_1(|w_{\alpha_1}(x)|)}{2(1 + \lambda_2 \circ \gamma_1(|w_{\alpha_1}(x)|))} - \alpha'_2(|w_{\alpha_2}(x)|) + \frac{\nu}{2} \right] . \end{aligned} \quad (6.55)$$

Since

$$\frac{\lambda_2(s)}{1 + \lambda_2(s)} \leq 1 , \quad \forall s \geq 0 ,$$

by monotonicity of  $q(\cdot)$  and using Remark 6.2.1, there exist  $\alpha \in \mathcal{K}_\infty$  and  $\sigma \in \mathcal{K}$  so that we can write

$$V_T(F_T(x, u)) - V_T(x) \leq -T\alpha(|w_{\alpha_1}(x)| + |w_{\alpha_2}(x)|) + T\sigma(|w_{\sigma_1}(u)| + |w_{\sigma_2}(u)|) + T\nu .$$

This completes the proof of Theorem 6.3.1. ■

**Proof of Theorem 6.3.2** Suppose that all conditions of the theorem are satisfied.

Let  $\underline{\alpha}_1, \bar{\alpha}_1, \alpha_1, \sigma_1$  come from the condition 1 and  $\underline{\alpha}_2, \bar{\alpha}_2, \alpha_2, \lambda_2, \sigma_2$  come from the condition 2. Define a function  $\alpha'_1 \in \mathcal{K}_\infty$  as follows

$$\alpha'_1(s) := \begin{cases} \alpha_1(s) & \text{for small } s, \\ \lambda_2 \circ \gamma_1(s) & \text{for large } s , \end{cases} \quad (6.56)$$

where  $\gamma_1$  comes from the third condition of the theorem. It is clear that  $\alpha'_1(s) = O[\alpha_1(s)]$  for  $s \rightarrow 0^+$ . Hence, by Lemma 6.2.1 there exists  $\tilde{q}_1 \in \mathcal{SN}$  such that  $\tilde{q}_1(s) \cdot \alpha_1(s) \geq \alpha'_1(s)$ . Further, define a function  $\lambda'_2(s) := \frac{1}{2}\alpha'_1 \circ \gamma_1^{-1}(s)$  and note that  $\lambda'_2 \in \mathcal{K}$  and it is clear that  $\lambda_2(s) = O[\lambda'_2(s)]$  for  $s \rightarrow +\infty$ . Then by Lemma 6.2.2, there exists  $\tilde{q}_2 \in \mathcal{SN}$  such that  $\tilde{q}_2(s) \cdot \lambda_2(s) \leq \lambda'_2(s)$ . Let  $q_1(s) := 4\tilde{q}_1 \circ \underline{\kappa}_1^{-1} \circ \underline{\alpha}_1^{-1}(2s)$  and  $q_2(s) := \frac{1}{2}\tilde{q}_2 \circ \theta_{\lambda_2}^{-1}(s)$ , where  $\theta_{\lambda_2}$  is given in (6.11). We use  $q_1$  and  $q_2$  respectively to define  $\rho_1$  and  $\rho_2$ , and then let  $(q_1, \rho_1)$  and  $(q_2, \rho_2)$  respectively generate via Lemma 5.4.1 new bounding functions  $\underline{\alpha}'_1, \overline{\alpha}'_1, \alpha'_1, \sigma'_1$  and  $\underline{\alpha}'_2, \overline{\alpha}'_2, \alpha'_2, \sigma'_2$ .

Let arbitrary strictly positive real numbers  $(\Delta_x, \Delta_u, \nu)$  be given. Let the numbers  $(\Delta_x, \Delta_u, \frac{\nu}{2})$  generate  $T_1^*$  and  $\rho_1(V_{1T})$  via item 1 of the theorem and Lemma 5.4.1, and let  $(\gamma_2(\Delta_x), \gamma_3(\Delta_u), \frac{\nu}{2})$  generate  $T_2^*$  and  $\rho_2(V_{2T})$  via item 2 of the theorem and Lemma 5.4.1. Let  $T^* := \min\{T_1^*, T_2^*\}$ . We define  $V_T$  as (6.14). Let  $w_x(x) := w_{x_1}(x)$  and  $w_u(u) := w_{u_1}(u)$ . In all calculations below we consider arbitrary  $|w_x(x)| \leq \Delta_x$ ,  $|w_u(u)| \leq \Delta_u$  and  $T \in (0, T^*)$ . Note that this implies  $|w_{x_2}(x)| \leq \gamma_2(\Delta_x)$  and  $|w_{u_2}(u)| \leq \gamma_3(\Delta_u)$ .

It follows from the definition of  $V_T$  that

$$\begin{aligned} \rho_1 \circ \underline{\alpha}_1(|w_{\alpha_1}(x)|) + \rho_2 \circ \underline{\alpha}_2(|w_{\alpha_2}(x)|) \\ \leq V_T(x) \leq \rho_1 \circ \overline{\alpha}_1(|w_{\alpha_1}(x)|) + \rho_2 \circ \overline{\alpha}_2(|w_{\alpha_2}(x)|) . \end{aligned}$$

Then by Remark 6.2.1, there exist  $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$  such that (6.54) holds. Using condition 3 of the theorem and the definition of  $\lambda'_2$ , we have:

$$\begin{aligned} V_T(F_T(x, u)) - V_T(x) &= \rho_1(V_{1T}(F_T)) - \rho_1(V_{1T}) + \rho_2(V_{2T}(F_T)) - \rho_2(V_{2T}) \\ &\leq T \left[ -\alpha'_1(|w_{\alpha_1}(x)|) + \sigma'_1(|w_{\sigma_1}(u)|) + \frac{\nu}{2} - \alpha'_2(|w_{\alpha_2}(x)|) \right. \\ &\quad \left. + \lambda'_2 \circ \gamma_1(|w_{\alpha_1}(x)|) + \sigma'_2(|w_{\sigma_2}(u)|) + \frac{\nu}{2} \right] \\ &\leq T \left[ -\alpha'_2(|w_{\alpha_2}(x)|) - \frac{1}{2}\alpha'_1(|w_{\alpha_1}(x)|) + \sigma'_1(|w_{\sigma_1}(u)|) \right. \\ &\quad \left. + \sigma'_2(|w_{\sigma_2}(u)|) + \nu \right] . \end{aligned} \tag{6.57}$$

Finally, using Remark 6.2.1, there exist  $\sigma \in \mathcal{K}$  and  $\alpha \in \mathcal{K}_\infty$  that

$$V_T(F_T(x, u)) - V_T(x) \leq T \left[ \sigma(|w_{\sigma_1}(u)| + |w_{\sigma_2}(u)|) - \alpha(|w_{\alpha_1}(x)| + |w_{\alpha_2}(x)|) + \nu \right] .$$

This completes the proof of Theorem 6.3.2. ■

## 6.6 Conclusion

We have presented results on changing supply rates for discrete-time SP-IOSS systems that allow for a partial construction of Lyapunov functions. Our results apply to investigation of different semiglobal practical stability properties of discrete-time parameterized systems that arise when an approximate discrete-time model is used for controller design of a sampled-data nonlinear system. We have applied our results to several problems, such as the LaSalle criterion for SP-ISS of discrete-time systems. We emphasize that there is a great potential for further applications of our results. A case study implementing the tools presented in this chapter to a stabilization problem of a robotic manipulator is presented in Chapter 8, where a discrete-time SP-ISS controller was designed for a manipulator based on its Euler approximate model. This strongly motivates a development of systematic controller design procedures for sampled-data nonlinear systems based on their approximate discrete-time models where results of this chapter could play an important role.

In the next chapter, another Lyapunov-based design tool is developed. Results on small gain theorem for discrete-time nonlinear ISS interconnected systems based on the Lyapunov characterization of the ISS property of the subsystems are presented.

## Chapter 7

# Lyapunov Based Small-Gain Theorem for Input-to-State Stability

### 7.1 Introduction

In this chapter, another Lyapunov-based tool for controller design is developed to facilitate the design framework presented in Chapter 5. While the changing of supply rates technique has been used in Chapter 6, in this chapter the well known small gain theorem is explored to develop the results. We propose a version of small gain theorem based on Lyapunov characterization for input-to-state stable (ISS) discrete-time interconnected systems. In addition, we also provide a method for constructing a Lyapunov function for the interconnected systems.

Small gain theorem is one of the most important tools in robustness analysis and controller design for nonlinear control systems. A particularly useful version of the small gain theorem for nonlinear continuous-time systems was proved in [60] by Jiang et al. and it is based on the input-to-state stability (ISS) property introduced by Sontag in [139] (see also [141]). A range of related results for continuous-time systems can be found in [54, 127, 154, 155] and for nonlinear discrete-time systems in [61]. All of the above results rely on trajectory based proofs of the small gain theorem and

they do not construct a Lyapunov function for the overall interconnected system. The first partial construction of a Lyapunov function for the feedback connection of two continuous-time ISS systems satisfying a small-gain condition that we are aware of was proposed in [59].

The material presented in this chapter is based on the results from [79, 81]. It is the main purpose of the work in this chapter to present a discrete-time version of the results in [59]. In addition, similar to the results of Chapter 6, we consider families of discrete-time systems parameterized by the sampling period, and we propose a construction of a family of Lyapunov functions for these families of systems. Indeed, we present a partial construction of a family of ISS Lyapunov functions from the families of ISS Lyapunov functions of two interconnected discrete-time ISS systems satisfying a small-gain condition. While the constructed family of Lyapunov function in the discrete-time case has the same form as the one constructed in [59] for continuous-time systems, the proofs of the two results are significantly different.

The main result of this chapter is a useful tool for a range of nonlinear discrete-time control problems. In particular, the constructed family of Lyapunov function can be used together with results presented in Chapter 5 to design ISS controllers for nonlinear sampled-data systems via their approximate discrete-time plant models. In order for our result to be compatible with the results of Chapter 5, we consider families of discrete-time interconnected systems that are parameterized with the sampling period  $T$ . If we combine our result with those of Chapter 5, we can guarantee that whenever an approximate discrete-time model satisfies our small gain condition, this guarantees that the exact discrete-time model is ISS for all small values of  $T$ .

A counter-example is presented to show that if another small gain condition (different from the one that we use) is satisfied for an approximate discrete-time model for all small values of the parameter  $T$ , this may not imply stability of the exact model for small  $T$ . Indeed, in the counter-example the approximate model is stable for all small values of  $T$  but the exact discrete-time model of the system is unstable for all small values of  $T$ . However, if a particular small gain condition that we use is satisfied for an approximate discrete-time model, then we can construct a Lyapunov function for the



overall approximate discrete-time model, which together with results from Chapter 5 guarantees that the exact discrete-time model is ISS for all small values of  $T$ .

We also remark that our main result is closely related to results on changes of supply rates for ISS discrete-time systems investigated in [111] and IOSS discrete-time systems investigated in Chapter 6 and it can be regarded as an appropriate generalization of the results in [111].

This chapter is organized as follows. In Section 7.2 we introduce notation and definitions needed to present the results. We present the main result in Section 7.3 and provide the proof in Section 7.4. A counter example is presented in Section 7.5, and we close the chapter with conclusion in Section 7.6.

## 7.2 Preliminaries

Consider a family of parameterized discrete-time systems

$$x(k+1) = F_T(x(k), u(k)) . \quad (7.1)$$

where  $x \in \mathbb{R}^n$  and  $u \in \mathbb{R}^m$  are respectively the state and input of the system. It is assumed that  $F_T$  is well defined on arbitrarily large compact sets for sufficiently small  $T$ , where  $T > 0$  is the sampling period, which parameterizes the system and can be arbitrarily assigned. Parameterized discrete-time systems (7.1) commonly arise when an approximate discrete-time model is used for designing a digital controller for a nonlinear sampled-data system (see [106, 116]). For instance, if we use the Euler model of  $\dot{x} = f(x, u)$  for controller design then we have  $F_T(x, u) := x + Tf(x, u)$ . Non-parameterized discrete-time systems are a special case of (7.1) when  $T$  is constant (for instance  $T = 1$ ). We use the following definition.

**Definition 7.2.1** *The system (7.1) is semiglobally practically input-to-state stable (SP-ISS) w.r.t. input  $u$  if there exist functions  $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$ , a positive definite function  $\alpha$  and  $\gamma \in \mathcal{K}$ , and for any strictly positive real numbers  $\Delta_x, \Delta_u, \nu$  and  $\tilde{\nu}$  there exists  $T^* > 0$  such that for all  $T \in (0, T^*)$  there exists a continuous function  $V_T : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$*

such that for all  $|x| \leq \Delta_x$ ,  $|u| \leq \Delta_u$  and  $T \in (0, T^*)$  the following holds:

$$\underline{\alpha}(|x|) \leq V_T(x) \leq \bar{\alpha}(|x|) , \quad (7.2)$$

$$V_T(x) \geq \gamma(|u|) + \nu \Rightarrow V_T(F_T) - V_T(x) \leq -T\alpha(V_T(x)) , \quad (7.3)$$

$$V_T(F_T) \leq V_T(x) + \tilde{\nu} . \quad (7.4)$$

The function  $V_T$  is called a SP-ISS Lyapunov function for the system (7.1). ■

**Definition 7.2.2 (Lipschitz uniform in small T)** A family of functions  $V_T : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  is Lipschitz uniformly in small  $T$  if given any  $\Delta_x > 0$  there exists  $T^* > 0$  such that for all  $T \in (0, T^*)$  and  $\max\{|x|, |y|\} \leq \Delta_x$  the following holds:

$$|V_T(x) - V_T(y)| \leq L|x - y| . \quad (7.5)$$

■

**Remark 7.2.1** We note that for continuous-time systems, if  $\Delta_x > \underline{\alpha}^{-1}(\gamma(\Delta_u) + \nu)$  then the condition (7.4) is not needed in the definition of SP-ISS and a condition that corresponds to (7.3) is enough to guarantee an appropriate ISS bound on the trajectories of the system. However, for discrete-time systems, the condition (7.3) alone is not enough to guarantee even the boundedness of the trajectories of the system no matter how large  $\Delta_x$  is compared to  $\Delta_u$  and  $\nu$ . This is illustrated by the system  $x(k+1) = F_T(x(k))$  where  $F_T(\cdot)$  is any continuous function satisfying (the example is taken from [114])

$$F_T(x) = \begin{cases} 2\Delta_x & |x| \leq \nu/2 \\ 2|x| & |x| \geq 2\Delta_x \\ 0 & \nu \leq |x| \leq \Delta_x , \end{cases} \quad (7.6)$$

and  $\Delta_x$  and  $\nu$  are arbitrarily positive real numbers. With, for example,  $V_T(x) = |x|$ , we have for all  $|x| \leq \Delta_x$  that

$$|x| \geq \nu \Rightarrow \Delta V_T := V_T(F_T(x)) - V_T(x) = -V_T(x). \quad (7.7)$$

Yet, every trajectory grows without bound. Note that the condition (7.7) gives the right bound on  $\Delta V_T$  for all  $\Delta_x \geq |x| \geq \nu$ . However, this example shows that some

information about  $V_T(F_T(x))$  is required even for values of  $x$  such that  $|x| \leq \nu$  in order to assert a bound on trajectories of the system. Consequently, we have included the condition (7.4) as a part of the SP-ISS characterization in Definition 7.2.1. We note that the condition (7.4) is not restrictive and is satisfied in most situations of interest. Example 7.2.1 illustrates a particular case of this condition. ■

**Example 7.2.1** Consider a continuous-time nonlinear system  $\dot{x} = f(x, u)$  where  $f$  is bounded on compact sets. Suppose we use the Euler discrete-time model of the system  $x(k+1) = F_T(x(k), u(k)) := x(k) + Tf(x(k), u(k))$  to analyze its properties. Consider also a Lyapunov function  $V_T$  that is uniformly (locally) Lipschitz in small  $T$ . Then, we can write on compact sets:

$$V_T(F_T) = V_T(x) + V_T(x + Tf(x, u)) - V_T(x) \leq V_T(x) + LT |f(x, u)| . \quad (7.8)$$

From the boundedness of  $f$ , there exists  $M > 0$  so that  $|f(x, u)| \leq M$ . Then given any  $\tilde{\nu} > 0$  there exists  $T^* > 0$  (we can take  $T^* = \frac{\tilde{\nu}}{LM}$ ) so that for all  $T \in (0, T^*)$  we have that (7.4) holds. ■

**Remark 7.2.2** If instead of (7.3), we used the following Lyapunov condition in Definition 7.2.1

$$\Delta V_T \leq -Ta(|x|) + T\gamma(|u|) + T\nu , \quad a, \gamma \in \mathcal{K}_\infty , \quad (7.9)$$

then we would not need (7.4). However, the above given formulation leads to a more complicated statement and proof of our main result and hence we have opted to use the conditions as stated in Definition 7.2.1. We emphasize that (7.3) and (7.4) are equivalent to (7.9) if an appropriate condition holds. Indeed, it is trivial to see that (7.9) implies both (7.3) and (7.4). The opposite holds if there exists  $\sigma \in \mathcal{K}_\infty$  such that for any strictly positive  $r, \nu$  there exists  $T^* > 0$  such that the following holds

$$\max_{T \in (0, T^*), |x| \leq \gamma(r), |u| \leq r} \left| \frac{\Delta V_T}{T} \right| \leq \sigma(r) + \nu , \quad (7.10)$$

and then we can write that for any  $(\Delta_x, \Delta_u, \nu)$  there exists  $T^* > 0$  such that the following holds:

$$\Delta V_T \leq -T\alpha(V_T) + T\sigma(|u|) + T\nu .$$

The condition (7.10) is slightly stronger than (7.4) but it often holds.

We note that the condition (7.9) was used in [106] to provide a framework for design of input-to-state stabilizing controllers for sampled-data systems via their approximate discrete-time models. Hence, results of this chapter in cases when the condition (7.10) holds provide a tool for ISS controller design within the framework of [106]. In Section 7.5 we present an example which illustrates the importance of the particular definition of SP-ISS that we use when the controller design is based on an approximate discrete-time plant model. ■

### 7.3 Main result

In this section we state and prove Theorem 7.3.1, which is the main result of this chapter. Theorem 7.3.1 is a discrete-time version of the continuous-time result [59]. The statements of both results are similar but the proofs are notably different and the differences are commented on below (see Remark 7.4.1).

The focus of this chapter is a family of parameterized discrete-time interconnected systems

$$\begin{aligned}\Sigma_1 : x_1(k+1) &= F_{1T}(x_1(k), x_2(k), u(k)) , \\ \Sigma_2 : x_2(k+1) &= F_{2T}(x_1(k), x_2(k), u(k)) .\end{aligned}\tag{7.11}$$

In the sequel we will assume that the subsystem  $\Sigma_1$  is SP-ISS with respect to inputs  $x_2$  and  $u$  and the subsystem  $\Sigma_2$  is SP-ISS with respect to inputs  $x_1$  and  $u$ . More precisely, we suppose that for  $i, j \in \{1, 2\}, i \neq j$ , there exist functions  $\underline{\alpha}_i, \bar{\alpha}_i \in \mathcal{K}_\infty$ , positive definite functions  $\alpha_i$ , functions  $\gamma_{x_i}, \gamma_{u_i} \in \mathcal{K}$ , and for any strictly positive real numbers  $(\Delta_{x_i}, \Delta_{x_j}, \Delta_{u_i}, \nu_i, \tilde{\nu}_i)$  there exist  $T_i^* > 0$  and for any  $T \in (0, T_i^*)$  there exist  $V_{iT} : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  such that the following hold for all  $T \in (0, T_i^*)$ ,  $|x_i| \leq \Delta_{x_i}$ ,  $|x_j| \leq \Delta_{x_j}$  and  $|u| \leq \Delta_{u_i}$ :

$$\underline{\alpha}_i(|x_i|) \leq V_{iT}(x_i) \leq \bar{\alpha}_i(|x_i|)\tag{7.12}$$

$$\begin{aligned}V_{iT}(x_i) &\geq \max\{\gamma_{x_i}(V_{jT}(x_j)), \gamma_{u_i}(|u|) + \nu_i\} \\ &\Rightarrow V_{iT}(F_{iT}) - V_{iT}(x_i) \leq -T\alpha_i(V_{iT}(x_i))\end{aligned}\tag{7.13}$$

$$V_{iT}(F_{iT}) \leq V_{iT}(x_i) + \tilde{\nu}_i .\tag{7.14}$$

Under the above given conditions and an appropriate small gain condition, we show that the overall system (7.11) is SP-ISS with respect to the input  $u$ . Moreover, we construct a SP-ISS Lyapunov function  $V_T$  for the overall system (7.11) using the SP-ISS Lyapunov functions  $V_{1T}$  and  $V_{2T}$  of the subsystems  $\Sigma_1$  and  $\Sigma_2$ . More precisely, we can state the following result.

**Theorem 7.3.1** *Consider the parameterized discrete-time interconnected system (7.11). Suppose that the following conditions hold:*

- C1. The subsystem  $\Sigma_1$  is SP-ISS with inputs  $x_2$  and  $u$ , and a SP-ISS Lyapunov function  $V_{1T}$ .*
- C2. The subsystem  $\Sigma_2$  is SP-ISS with inputs  $x_1$  and  $u$ , and a SP-ISS Lyapunov function  $V_{2T}$ .*
- C3. There exist  $\tau_1, \tau_2 \in \mathcal{K}_\infty$  such that  $(\text{Id} + \tau_1) \circ \gamma_{x_1} \circ (\text{Id} + \tau_2) \circ \gamma_{x_2}(s) < s, \quad \forall s > 0$ .*

*Then, the system (7.11) is SP-ISS w.r.t. the input  $u$ ; moreover, there exists  $\rho \in \mathcal{K}_\infty$  such that the function*

$$V_T(x_1, x_2) := \max\{V_{1T}(x_1), \rho(V_{2T}(x_2))\}, \quad (7.15)$$

*is SP-ISS Lyapunov function for the system (7.11). Moreover, if  $V_{1T}, V_{2T}$  are locally Lipschitz uniformly in small  $T$ , then  $V_T$  is locally Lipschitz uniformly in small  $T$ . ■*

The following technical lemmas are used to prove the main result.

**Lemma 7.3.1** [59] *Let  $\sigma_1 \in \mathcal{K}$  and  $\sigma_2 \in \mathcal{K}_\infty$  satisfy  $\sigma_1(r) < \sigma_2(r)$  for all  $r > 0$ . Then there exists a  $\mathcal{K}_\infty$  function  $\sigma$  such that*

- $\sigma_1(r) < \sigma(r) < \sigma_2(r)$  for all  $r > 0$ ;
- $\sigma(r)$  is  $C^1$  on  $(0, \infty)$  and  $\sigma'(r) =: \tilde{q}(r)$  is a positive function. ■

Note that the above given function  $\tilde{q}$  is positive but it is not positive definite in general.

The following lemma is a simple consequence of [3, Lemma IV.1]

**Lemma 7.3.2** [3] *Let  $\tilde{q} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{> 0}$  be a positive function. Then there exist a positive definite function  $q$  and functions  $q_1 \in \mathcal{K}_\infty$  and  $q_2 \in \mathcal{L}$  such that  $\tilde{q}(r) \geq q(r) \geq q_1(r) \cdot q_2(r), \forall r \geq 0$ . ■*

Note that the existence of  $q$  is trivial to show, whereas the existence of  $q_1$  and  $q_2$  was proved in [3].

**Lemma 7.3.3** *Suppose that we are given a function  $\rho \in \mathcal{K}_\infty$  where  $q(r) := \rho'(r)$  is a positive function, a positive definite function  $\alpha$ , such that  $(\text{Id} - T\alpha)$  is positive definite and  $T \in [0, 1)$ . Then, we can write:*

$$\max_{0 \leq s \leq \rho(r)} (\text{Id} - T\alpha)(s) - \rho(r) \leq -Ta_1 \circ \rho(r) \quad (7.16)$$

$$\max_{0 \leq \rho(s) \leq r} \rho \circ (\text{Id} - T\alpha)(s) - r \leq -Ta_2(r) , \quad (7.17)$$

for some positive definite functions  $a_1$  and  $a_2$ . ■

## 7.4 Proof of the main result

**Proof of Lemma 7.3.3:** The inequality (7.16) follows easily from considering two cases  $0 \leq s \leq \frac{\rho(r)}{2}$  and  $\frac{\rho(r)}{2} \leq s \leq \rho(r)$ . In particular, we obtain  $a_1(r) := \max\{\frac{1}{2}r, \alpha(r)\}$ . We now prove (7.17) in more detail. First, note that if  $0 \leq \rho(s) \leq \frac{r}{2}$ , then

$$\rho \circ (\text{Id} - T\alpha)(s) - r \leq \rho(s) - r \leq -\frac{r}{2} . \quad (7.18)$$

Consider now  $\frac{r}{2} \leq \rho(s) \leq r$ . First, we use the Mean Value Theorem to write:

$$\rho \circ (\text{Id} - T\alpha)(s) - r \leq \max_{\frac{r}{2} \leq \rho(s) \leq r} \rho \circ (\text{Id} - T\alpha)(s) - \rho(s) = \tilde{q}(s^*)[-T\alpha(s)] , \quad (7.19)$$

where  $s^* \in [(\text{Id} - \alpha)(s), s]$  since  $T < 1$ . Using Lemma 7.3.2 we can find two functions  $q_1 \in \mathcal{K}_\infty$  and  $q_2 \in \mathcal{L}$  such that

$$\begin{aligned} -T\tilde{q}(s^*)\alpha(s) &\leq -Tq(s^*)\alpha(s) \leq -Tq_1(s^*) \cdot q_2(s^*) \cdot \alpha(s) \\ &\leq -Tq_1 \circ (\text{Id} - \alpha)(s) \cdot q_2(s) \cdot \alpha(s) =: -T\alpha^*(s) , \end{aligned}$$

where  $\alpha^*(\cdot)$  is a positive definite function. Applying Lemma 7.3.2 again we obtain  $q_1^* \in \mathcal{K}_\infty$ ,  $q_2^* \in \mathcal{L}$  and then using the fact that  $s \in [\rho^{-1}(r/2), \rho^{-1}(r)]$  we can write:

$$-T\alpha^*(s) \leq -Tq_1^*(s) \cdot q_2^*(s) \leq -Tq_1^* \circ \rho^{-1}(r/2) \cdot q_2^* \circ \rho^{-1}(r) =: -T\alpha_1(r) .$$

This completes the proof of (7.17) with  $a_2(r) := \min\{\frac{1}{2}r, \alpha_1(r)\}$ . ■

**Proof of Theorem 7.3.1:** Suppose that all conditions of Theorem 7.3.1 are satisfied. Let  $\underline{\alpha}_1, \bar{\alpha}_1, \alpha_1, \gamma_{x_1}, \gamma_{u_1}$  come from conditions C1, and let  $\underline{\alpha}_2, \bar{\alpha}_2, \alpha_2, \gamma_{x_2}, \gamma_{u_2}$  come from condition C2. Let  $\tau_1, \tau_2 \in \mathcal{K}_\infty$  come from condition C3. Note that without loss of generality we can assume that  $(\text{Id} - \alpha_i)$ ,  $i = 1, 2$  are positive definite. For simplicity of notation we introduce  $\tilde{\gamma}_{x_1}(s) := (\text{Id} + \tau_1) \circ \gamma_{x_1}$ ,  $\tilde{\gamma}_{x_2}(s) := (\text{Id} + \tau_2) \circ \gamma_{x_2}$ . Similar to [59] we denote  $b := \lim_{r \rightarrow \infty} \tilde{\gamma}_{x_2}(r)$  and since  $\tilde{\gamma}_{x_2} \in \mathcal{K}$ , then  $\tilde{\gamma}_{x_2}^{-1}$  is defined on  $[0, b)$ ,  $\tilde{\gamma}_{x_2}^{-1}(r) \rightarrow \infty$  as  $r \rightarrow b^-$  and from C3 we have that

$$\tilde{\gamma}_{x_1}(r) < \tilde{\gamma}_{x_2}^{-1}(r), \quad \forall r \in (0, b). \quad (7.20)$$

Let  $\hat{\gamma}_x \in \mathcal{K}_\infty$  be such that

- $\hat{\gamma}_x(r) \leq \tilde{\gamma}_{x_2}^{-1}(r)$  for all  $r \in [0, b)$ ;
- $\tilde{\gamma}_{x_1}(r) < \hat{\gamma}_x(r)$  for all  $r > 0$ .

(if  $\tilde{\gamma}_{x_2} \in \mathcal{K}_\infty$ , then we can take  $\hat{\gamma}_x(r) = \tilde{\gamma}_{x_2}^{-1}(r)$ ). Let  $\rho \in \mathcal{K}_\infty$  come from Lemma 7.3.1 such that

$$\tilde{\gamma}_{x_1}(r) < \rho(r) < \hat{\gamma}_x(r), \quad \forall r > 0. \quad (7.21)$$

Denote  $\tilde{q}(r) := \frac{d\rho}{dr}(r)$ , where  $\tilde{q}$  is a positive function. Let  $V_T$  be defined as:

$$V_T(x_1, x_2) := \max\{V_{1T}(x_1), \rho(V_{2T}(x_2))\}. \quad (7.22)$$

We use the notation  $x := (x_1^T \ x_2^T)^T$ ,  $F_T := (F_{1T}^T \ F_{2T}^T)^T$  and the norm  $|x| := |x_1| + |x_2|$ . We show that the interconnected system (7.11) is SP-ISS with input  $u$  by proving that  $V_T$  is a SP-ISS Lyapunov function for the system.

Let arbitrary strictly positive real numbers  $(\Delta_x, \Delta_u, \nu, \tilde{\nu})$  be given. Let  $\Delta_{x_1} = \Delta_{x_2} = \Delta_x$  and  $\Delta_{u_1} = \Delta_{u_2} = \Delta_u$ . Let  $\varepsilon_1, \varepsilon_2 \in \mathcal{K}_\infty$  be arbitrary functions such that  $(\text{Id} - \varepsilon_i)$  are positive definite functions for  $i = 1, 2$ . Let  $\nu_1$  be such that

$$\max \left\{ \nu_1, \max_{s \in [0, \Delta_u]} [\varepsilon_1^{-1}(\gamma_{u_1}(s) + \nu_1) - \varepsilon_1^{-1} \circ \gamma_{u_1}(s)] \right\} \leq \nu \quad (7.23)$$

and let  $\nu_2$  be such that

$$\max \left\{ \max_{s \in [0, \Delta_u]} [\rho(\gamma_{u_2}(s) + \nu_2) - \rho \circ \gamma_{u_2}(s)], \right. \\ \left. \max_{s \in [0, \Delta_u]} [\varepsilon_2^{-1} \circ \rho(\gamma_{u_2}(s) + \nu_2) - \varepsilon_2^{-1} \circ \rho \circ \gamma_{u_2}(s)] \right\} \leq \nu. \quad (7.24)$$

Let  $\tilde{\nu}_1 > 0$  and  $\tilde{\nu}_2 > 0$  be such that

$$\max\{\tilde{\nu}_1, \max_{s \in [0, \bar{\alpha}_2(\Delta_{x_2})]} [\rho(s + \tilde{\nu}_2) - \rho(s)]\} \leq \tilde{\nu} , \quad (7.25)$$

$$(\text{Id} + \tau_1^{-1})(\tilde{\nu}_1) \leq \frac{\nu_1}{2} , \quad (\text{Id} + \tau_2^{-1})(\tilde{\nu}_2) \leq \frac{\nu_2}{2} . \quad (7.26)$$

Let the strictly positive real numbers  $(\Delta_{x_1}, \Delta_{x_2}, \Delta_{u_1}, \frac{\nu_1}{2}, \tilde{\nu}_1)$  determine  $T_1^* > 0$  via the condition C1. Let  $(\Delta_{x_1}, \Delta_{x_2}, \Delta_{u_2}, \frac{\nu_2}{2}, \tilde{\nu}_2)$  determine  $T_2^* > 0$  via the condition C2. Let  $T^* := \min\{1, T_1^*, T_2^*\}$ . In the rest of the proof we assume that  $|x| \leq \Delta_x$ ,  $|u| \leq \Delta_u$  and  $T \in (0, T^*)$ .

First note that  $\tilde{\gamma}_{x_1}(s) \geq \gamma_{x_1}(s)$ ,  $\tilde{\gamma}_{x_2}(s) \geq \gamma_{x_2}(s)$  for all  $s \geq 0$ . Conditions C1 and C2 imply that:

$$\begin{aligned} V_{1T}(x_1) &\geq \max\{\gamma_{x_1}(V_{2T}(x_2)), \gamma_{u_1}(|u|) + \nu_1/2\} \\ &\Rightarrow V_{1T}(F_{1T}) - V_{1T}(x_1) \leq -T\alpha_1(V_{1T}(x_1)) , \end{aligned} \quad (7.27)$$

$$\begin{aligned} V_{2T}(x_2) &\geq \max\{\gamma_{x_2}(V_{1T}(x_1)), \gamma_{u_2}(|u|) + \nu_2/2\} \\ &\Rightarrow V_{2T}(F_{2T}) - V_{2T}(x_2) \leq -T\alpha_2(V_{2T}(x_2)) , \end{aligned} \quad (7.28)$$

$$V_{1T}(F_{1T}) \leq V_{1T}(x_1) + \tilde{\nu}_1 , \quad V_{2T}(F_{2T}) \leq V_{2T}(x_2) + \tilde{\nu}_2 . \quad (7.29)$$

Moreover, using respectively C1 and C2 and our choice of  $\nu_i, \tilde{\nu}_i$ ,  $i = 1, 2$  we can write respectively

$$V_{1T}(F_{1T}) \leq \max\{(\text{Id} - T\alpha_1)(V_{1T}(x_1)), \tilde{\gamma}_{x_1}(V_{2T}(x_2)), \gamma_{u_1}(|u|) + \nu_1\} , \quad (7.30)$$

$$V_{2T}(F_{2T}) \leq \max\{(\text{Id} - T\alpha_2)(V_{2T}(x_2)), \tilde{\gamma}_{x_2}(V_{1T}(x_1)), \gamma_{u_2}(|u|) + \nu_2\} . \quad (7.31)$$

We only prove (7.30) and the proof of (7.31) is omitted since it follows the same steps. Note that if  $V_{1T}(x_1) \geq \max\{\gamma_{x_1}(V_{2T}(x_2)), \gamma_{u_1}(|u|) + \nu_1/2\}$ , then from (7.27) we can write that:

$$V_{1T}(F_{1T}) \leq (\text{Id} - T\alpha_1)(V_{1T}(x_1)) . \quad (7.32)$$

On the other hand, if  $V_{1T}(x_1) \leq \max\{\gamma_{x_1}(V_{2T}(x_2)), \gamma_{u_1}(|u|) + \nu_1/2\}$ , then from (7.29) we have

$$V_{1T}(F_{1T}) \leq V_{1T}(x_1) + \tilde{\nu}_1 \leq \max\{\gamma_{x_1}(V_{2T}(x_2)) + \tilde{\nu}_1, \gamma_{u_1}(|u|) + \nu_1/2 + \tilde{\nu}_1\} .$$



## 7. Lyapunov Based Small-Gain Theorem for Input-to-State Stability 177

By considering two sub-cases  $\tau_1 \circ \gamma_{x_1}(V_{2T}(x_2)) \geq \tilde{\nu}_1$  and  $\tau_1 \circ \gamma_{x_1}(V_{2T}(x_2)) \leq \tilde{\nu}_1$ , and from definition of  $\nu_1$  and  $\tilde{\nu}_1$  we can write that

$$\begin{aligned} V_{1T}(F_{1T}) &\leq \max\{(\text{Id} + \tau_1) \circ \gamma_{x_1}(V_{2T}(x_2)), (\text{Id} + \tau_1^{-1})(\tilde{\nu}_1), \gamma_{u_1}(|u|) + \nu_1/2 + \tilde{\nu}_1\} \\ &\leq \max\{\tilde{\gamma}_{x_1}(V_{2T}(x_2)), (\text{Id} + \tau_1^{-1})(\tilde{\nu}_1), \gamma_{u_1}(|u|) + \nu_1/2 + \nu_1/2\} \quad (7.33) \\ &\leq \max\{\tilde{\gamma}_{x_1}(V_{2T}(x_2)), \nu_1/2, \gamma_{u_1}(|u|) + \nu_1\} \\ &= \max\{\tilde{\gamma}_{x_1}(V_{2T}(x_2)), \gamma_{u_1}(|u|) + \nu_1\} , \end{aligned}$$

and (7.32), (7.33) complete the proof of (7.30). We assume in the sequel that (7.27)-(7.31) hold.

We have that for any  $\rho \in \mathcal{K}_\infty$ ,  $r_1 \geq 0$ ,  $r_2 \geq 0$ , then  $\frac{1}{2}r_1 + \frac{1}{2}\rho(r_2) \leq \max\{r_1, \rho(r_2)\} \leq r_1 + \rho(r_2)$ ; and that for any  $\alpha_1, \alpha_2 \in \mathcal{K}$ , there exist  $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}$  such that  $\underline{\alpha}(s_1 + s_2) \leq \alpha_1(s_1) + \alpha_2(s_2) \leq \bar{\alpha}(s_1 + s_2)$ ,  $\forall s_1 \geq 0, s_2 \geq 0$ , where we can take  $\underline{\alpha}(s) := \min\{\alpha_1(\frac{s}{2}), \alpha_2(\frac{s}{2})\}$  and  $\bar{\alpha}(s) := 2 \max\{\alpha_1(s), \alpha_2(s)\}$ . Hence, using the definition of  $V_T$ , we have that  $V_T$  satisfies (7.2) with  $\underline{\alpha}(s) := \min\{\frac{1}{2}\underline{\alpha}_1(s/2), \frac{1}{2}\rho \circ \underline{\alpha}_2(s/2)\}$  and  $\bar{\alpha}(|x|) := 2 \max\{\bar{\alpha}_1(s), \rho \circ \bar{\alpha}_2(s)\}$ . Moreover, from the definition of  $V_T$  and (7.25) we have that

$$\begin{aligned} V_T(F_T) &= \max\{V_{1T}(F_{1T}), \rho(V_{2T}(F_{2T}))\} \\ &\leq \max\{V_{1T}(x_1) + \tilde{\nu}_1, \rho(V_{2T}(x_2) + \tilde{\nu}_2)\} \quad (7.34) \\ &\leq \max\{V_{1T}(x_1), \rho(V_{2T}(x_2))\} + \tilde{\nu} = V_T(x) + \tilde{\nu} , \end{aligned}$$

which proves that (7.4) holds.

To show that  $V_T$  satisfies (7.3), we consider the following four cases:

Case 1:  $V_{1T}(x_1) \geq \rho(V_{2T}(x_2))$  and  $V_{1T}(F_{1T}) \geq \rho(V_{2T}(F_{2T}))$ . It holds that

$$\Delta V_T := V_T(F_T) - V_T(x) = V_{1T}(F_{1T}) - V_{1T}(x_1) .$$

Conditions  $V_{1T}(x_1) \geq \rho(V_{2T}(x_2))$  and  $\gamma_{x_1}(r) \leq \tilde{\gamma}_{x_1}(r) < \rho(r)$  for all  $r > 0$  imply  $V_{1T}(x_1) > \gamma_{x_1}(V_{2T}(x_2))$ . Hence, from (7.27) it holds that if  $V_{1T}(x_1) \geq \gamma_{u_1}(|u|) + \nu_1$  then we have

$$V_{1T}(F_{1T}) - V_{1T}(x_1) \leq -T\alpha_1(V_{1T}(x_1)) .$$

Since  $V_T(x) = V_{1T}(x_1)$ ,  $\nu \geq \nu_1$  and  $\varepsilon_1^{-1}(r) > r$  we have

$$V_T(x) \geq \varepsilon_1^{-1} \circ \gamma_{u_1}(|u|) + \nu \geq \gamma_{u_1}(|u|) + \nu \Rightarrow \Delta V_T \leq -T\alpha_1(V_T(x)) . \quad (7.35)$$

Case 2:  $V_{1T}(x_1) < \rho(V_{2T}(x_2))$  and  $V_{1T}(F_{1T}) < \rho(V_{2T}(F_{2T}))$ . It holds that

$$\Delta V_T = \rho(V_{2T}(F_{2T})) - \rho(V_{2T}(x_2)) . \quad (7.36)$$

Conditions  $V_{1T}(x_1) < \rho(V_{2T}(x_2))$  and  $\rho^{-1}(r) > \tilde{\gamma}_{x_2}(r) \geq \gamma_{x_2}(r), \forall r > 0$  imply  $V_{2T}(x_2) > \gamma_{x_2}(V_{1T}(x_1))$ . Hence, from (7.28) it holds that if  $V_{2T}(x_2) \geq \gamma_{u_2}(|u|) + \nu_2$ , we have that

$$\begin{aligned} \Delta V_{2T} &= V_{2T}(F_{2T}) - V_{2T}(x_2) \leq -T\alpha_2(V_{2T}(x_2)) \\ &\Rightarrow V_{2T}(F_{2T}) \leq (\text{Id} - T\alpha_2)(V_{2T}(x_2)) . \end{aligned} \quad (7.37)$$

Then using the Mean Value Theorem and the construction of  $\rho$  via Lemma 7.3.1, we have that

$$\begin{aligned} \Delta V_T &= \rho(V_{2T}(F_{2T})) - \rho(V_{2T}(x_2)) \\ &\leq \rho \circ (\text{Id} - T\alpha_2)(V_{2T}(x_2)) - \rho(V_{2T}(x_2)) = -T\tilde{q}(V_{2T}^*) \cdot \alpha_2(V_{2T}(x_2)) , \end{aligned} \quad (7.38)$$

with  $V_{2T}^* \in [(\text{Id} - \alpha_2)(V_{2T}(x_2)), V_{2T}(x_2)]$  (since  $T < 1$ ) and  $\tilde{q}$  is a positive function. Let  $\tilde{q}$  generate via Lemma 7.3.2 the functions  $q_1 \in \mathcal{K}_\infty$  and  $q_2 \in \mathcal{L}$ . We use the fact that  $V_T(x) = \rho(V_{2T}(x_2))$  to write that

$$\begin{aligned} \Delta V_T &\leq -T\tilde{q}(V_{2T}^*) \cdot \alpha_2(V_{2T}(x_2)) \\ &\leq -Tq_1(V_{2T}^*) \cdot q_2(V_{2T}^*) \cdot \alpha_2(V_{2T}(x_2)) \\ &\leq -Tq_1 \circ (\text{Id} - \alpha_2)(V_{2T}(x_2)) \cdot q_2(V_{2T}(x_2)) \cdot \alpha_2(V_{2T}(x_2)) \\ &=: -Ta_{2a}(V_{2T}(x_2)) = -Ta_{2a} \circ \rho^{-1}(V_T(x)) = -Ta_2(V_T(x)) . \end{aligned} \quad (7.39)$$

Since  $V_T(x) = \rho(V_{2T}(x_2))$ ,  $\varepsilon_2^{-1}(r) > r$  and by (7.24), we have

$$V_T(x) \geq \varepsilon_2^{-1} \circ \rho \circ \gamma_{u_2}(|u|) + \nu \geq \rho \circ \gamma_{u_2}(|u|) + \nu \Rightarrow \Delta V_T \leq -Ta_2(V_T(x)) . \quad (7.40)$$

Case 3:  $V_{1T}(x_1) < \rho(V_{2T}(x_2))$  and  $V_{1T}(F_{1T}) \geq \rho(V_{2T}(F_{2T}))$ . Using (7.30) it holds that

$$\begin{aligned} \Delta V_T &= V_{1T}(F_{1T}) - \rho(V_{2T}(x_2)) \\ &\leq \max\{(\text{Id} - T\alpha_1)(V_{1T}(x_1)), \tilde{\gamma}_{x_1}(V_{2T}(x_2)), \gamma_{u_1}(|u|) + \nu_1\} - \rho(V_{2T}(x_2)) . \end{aligned} \quad (7.41)$$

## 7. Lyapunov Based Small-Gain Theorem for Input-to-State Stability 179

We now overbound each of the terms in (7.41). First, by using (7.16) from Lemma 7.3.3 with  $V_{1T}(x_1) = s$  and  $V_{2T}(x_2) = r$ , we obtain

$$(\text{Id} - T\alpha_1)(V_{1T}(x_1)) \leq -Ta_{3a} \circ \rho(V_{2T}(x_2)) = -Ta_{3a}(V_T(x)) . \quad (7.42)$$

Next, since  $\tilde{\gamma}_{x_1}(r) < \rho(r), \forall r > 0$ , the function  $a_{3b}(r) := \rho(r) - \tilde{\gamma}_{x_1}(r)$  is positive definite and since  $T < 1$  we have that

$$\begin{aligned} \tilde{\gamma}_{x_1}(V_{2T}(x_2)) - \rho(V_{2T}(x_2)) &= -a_{3b}(V_{2T}(x_2)) \\ &= -a_{3b} \circ \rho^{-1}(V_T(x)) \leq -Ta_{3b} \circ \rho^{-1}(V_T(x)). \end{aligned} \quad (7.43)$$

Finally, we consider the third term. Since  $\varepsilon_1 \in \mathcal{K}_\infty$  is such that  $(\text{Id} - \varepsilon_1)$  is a positive definite function, if  $\rho(V_{2T}(x_2)) > \varepsilon_1^{-1} \circ (\gamma_{u_1}(|u|) + \nu_1)$  then it holds that

$$-\rho(V_{2T}(x_2)) + \gamma_{u_1}(|u|) + \nu_1 \leq -(\text{Id} - \varepsilon_1) \circ \rho(V_{2T}(x_2)) . \quad (7.44)$$

Since  $V_T(x) = \rho(V_{2T}(x_2))$  and using the definition of  $\nu_1$  and  $T < 1$ , we can write:

$$\begin{aligned} V_T(x) > \varepsilon_1^{-1} \circ \gamma_{u_1}(|u|) + \nu &\Rightarrow \rho(V_{2T}(x_2)) > \varepsilon_1^{-1} \circ (\gamma_{u_1}(|u|) + \nu_1) \\ &\Rightarrow -(\text{Id} - \varepsilon_1) \circ \rho(V_{2T}(x_2)) = -(\text{Id} - \varepsilon_1)(V_T(x)) \\ &=: -a_{3c}(V_T(x)) \leq -Ta_{3c}(V_T(x)) . \end{aligned} \quad (7.45)$$

Combining (7.42), (7.43) and (7.45), with  $a_3(r) := \min\{a_{3a}(r), a_{3b} \circ \rho^{-1}(r), a_{3c}(r)\}$ , we have that

$$V_T(x) > \varepsilon_1^{-1} \circ \gamma_{u_1}(|u|) + \nu \Rightarrow \Delta V_T \leq -Ta_3(V_T(x)) . \quad (7.46)$$

Case 4:  $V_{1T}(x_1) \geq \rho(V_{2T}(x_2))$  and  $V_{1T}(F_{1T}) < \rho(V_{2T}(F_{2T}))$ . Using condition (7.31) we have

$$\begin{aligned} \Delta V_T &= \rho(V_{2T}(F_{2T})) - V_{1T}(x_1) \\ &\leq \max\{\rho \circ (\text{Id} - T\alpha_2)(V_{2T}(x_2)), \rho \circ \tilde{\gamma}_{x_2}(V_{1T}(x_1)), \rho(\gamma_{u_2}(|u|) + \nu_2)\} \\ &\quad - V_{1T}(x_1) . \end{aligned} \quad (7.47)$$

Now we bound the terms on the right hand side of (7.47). First, using (7.17) of Lemma 7.3.3 with  $s = V_{2T}(x_2)$  and  $r = V_{1T}(x_1)$  we can write

$$\rho \circ (\text{Id} - T\alpha_2)(V_{2T}(x_2)) - V_{1T}(x_1) \leq -Ta_{4a}(V_{1T}(x_1)) = -Ta_{4a}(V_T(x)) . \quad (7.48)$$

## 180 7. Lyapunov Based Small-Gain Theorem for Input-to-State Stability

Since  $a_{4b}(r) := (\text{Id} - \rho \circ \tilde{\gamma}_{x_2})(r)$  is positive definite,  $T < 1$  and  $V_T(x) = V_{1T}(x_1)$ , we have that

$$\begin{aligned} \rho \circ \tilde{\gamma}_{x_2}(V_{1T}(x_1)) - V_{1T}(x_1) &\leq (\rho \circ \tilde{\gamma}_{x_2} - \text{Id})(V_{1T}(x_1)) =: -a_{4b}(V_{1T}(x_1)) \\ &= -a_{4b}(V_T(x)) \leq -Ta_{4b}(V_T(x)) . \end{aligned} \quad (7.49)$$

Finally, we consider the third term. Since  $\varepsilon_2 \in \mathcal{K}_\infty$  is such that  $(\text{Id} - \varepsilon_2)$  is a positive definite function, if  $V_{1T}(x_1) \geq \varepsilon_2^{-1} \circ \rho(\gamma_{u_2}(|u|) + \nu_2)$  then it holds that

$$\rho(\gamma_{u_2}(|u|) + \nu_2) - V_{1T}(x_1) \leq \varepsilon_2(V_{1T}(x_1)) - V_{1T}(x_1) = -(\text{Id} - \varepsilon_2)(V_{1T}(x_1)) \quad (7.50)$$

Since  $V_T(x) = V_{1T}(x_1)$  and using the definition of  $\nu_2$  and  $T < 1$ , we can write that

$$\begin{aligned} V_T(x) \geq \varepsilon_2^{-1} \circ \rho \circ \gamma_{u_2}(|u|) + \nu &\Rightarrow V_{1T}(x_1) \geq \varepsilon_2^{-1} \circ \rho(\gamma_{u_2}(|u|) + \nu_2) \\ &\Rightarrow -(\text{Id} - \varepsilon_2)(V_{1T}(x_1)) =: -a_{4c}(V_T(x)) \\ &\leq -Ta_{4c}(V_T(x)) . \end{aligned} \quad (7.51)$$

Combining (7.48), (7.49) and (7.51), with  $a_4(r) := \min\{a_{4a}(r), a_{4b}(r), a_{4c}(r)\}$ , we can write that

$$V_T(x) \geq \varepsilon_2^{-1} \circ \rho \circ \gamma_{u_2}(|u|) + \nu \Rightarrow \Delta V_T \leq -Ta_4(V_T(x)) . \quad (7.52)$$

By combining (7.35), (7.40), (7.46) and (7.52) and the fact that  $\varepsilon_i^{-1}(r) > r, \forall r > 0$ ,  $i = 1, 2$ , we have shown that (7.3) holds with

$$\alpha(r) := \min\{\alpha_1(r), a_2(r), a_3(r), a_4(r)\} \quad (7.53)$$

$$\gamma(r) := \max\{\varepsilon_1^{-1} \circ \gamma_{u_1}(r), \varepsilon_2^{-1} \circ \rho \circ \gamma_{u_2}(r)\} \quad (7.54)$$

where  $\alpha$  is a positive definite function and  $\gamma \in \mathcal{K}$ . Hence, the system (7.11) is SP-ISS.

The last thing left to prove is that if  $V_{1T}$  and  $V_{2T}$  are Lipschitz, uniformly in small  $T$  then  $V_T$  is Lipschitz, uniformly in small  $T$ . Let  $\Delta_x > 0$  be given. Let  $L_1, T_1^*$  and  $L_2, T_2^*$  come respectively from the Lipschitz properties of  $V_{1T}$  and  $V_{2T}$  for the set  $|x_i| \leq \Delta_x, i = 1, 2$ . Note also that since  $\rho \in C^1$ , it is locally Lipschitz and let  $L_\rho$  be its Lipschitz constant for the set  $V_{2T}(x_2) \leq \bar{\alpha}_2(\Delta_x)$ . Denote  $x := (x_1^T \ x_2^T)^T$  and  $y := (y_1^T \ y_2^T)^T$ . Let  $T^* := \min\{1, T_1^*, T_2^*\}$  and consider arbitrary  $T \in (0, T^*)$  and

$\max\{|x|, |y|\} \leq \Delta_x$ . Introduce the sets:  $A := \{x : V_{1T}(x_1) > \rho(V_{2T}(x_2))\}$ ;  $B := \{x : V_{1T}(x_1) = \rho(V_{2T}(x_2))\}$ ;  $C := \{x : V_{1T}(x_1) < \rho(V_{2T}(x_2))\}$ . We consider the following cases, to prove our claim:

Case 1:  $(x, y \in A)$  or  $(x \in A \text{ and } y \in B)$  or  $(x \in B \text{ and } y \in A)$  or  $(x, y \in B)$

$$|V_T(x) - V_T(y)| = |V_{1T}(x_1) - V_{1T}(y_1)| \leq L_1 |x_1 - y_1| . \quad (7.55)$$

Case 2:  $(x, y \in C)$  or  $(x \in C \text{ and } y \in B)$  or  $(x \in B \text{ and } y \in C)$ .

$$|V_T(x) - V_T(y)| = |\rho(V_{2T}(x_2)) - \rho(V_{2T}(y_2))| \leq L_\rho L_2 |x_2 - y_2| . \quad (7.56)$$

Case 3:  $x \in A$  and  $y \in C$

$$|V_T(x) - V_T(y)| = |V_{1T}(x_1) - \rho(V_{2T}(y_2))| . \quad (7.57)$$

Since  $x \in A$  implies  $V_{1T}(x_1) > \rho(V_{2T}(x_2))$  and  $y \in C$  implies  $V_{1T}(y_1) < \rho(V_{2T}(y_2))$ , we have that:

1. If  $V_{1T}(x_1) > \rho(V_{2T}(y_2))$  then

$$\begin{aligned} |V_{1T}(x_1) - \rho(V_{2T}(y_2))| &= V_{1T}(x_1) - \rho(V_{2T}(y_2)) \\ &\leq V_{1T}(x_1) - V_{1T}(y_1) \\ &\leq L_1 |x_1 - y_1| . \end{aligned} \quad (7.58)$$

2. If  $V_{1T}(x_1) \leq \rho(V_{2T}(y_2))$  then

$$\begin{aligned} |V_{1T}(x_1) - \rho(V_{2T}(y_2))| &= -V_{1T}(x_1) + \rho(V_{2T}(y_2)) \\ &\leq \rho(V_{2T}(y_2)) - \rho(V_{2T}(x_2)) \\ &\leq L_\rho L_2 |x_2 - y_2| . \end{aligned} \quad (7.59)$$

Case 4:  $x \in C$  and  $y \in A$ . This case follows by symmetry from Case 3.

Hence, we can conclude that

$$|V_T(x) - V_T(y)| \leq L(|x_1 - y_1| + |x_2 - y_2|) , \quad (7.60)$$

where  $L := \max\{L_1, L_\rho L_2\}$ . Therefore,  $V_T$  is Lipschitz uniformly in small  $T$ . ■

**Remark 7.4.1** *We note that the proofs of the continuous-time result in [59] and the discrete-time result in Theorem 7.3.1 are notably different although the constructed*

function  $V_T$  has the same form. In particular, while the result in [59] was proved by considering three different cases, we need to consider four cases in discrete-time, some of which contained up to three different sub-cases. Moreover, in the proof of the discrete-time result we needed to use The Mean Value Theorem and Lemma 7.3.2, which were not needed in the proof of the continuous-time result in [59]. ■

## 7.5 Counter example

The following example illustrates that it may happen that an approximate discrete-time model satisfies a small gain condition but if the gains depend on  $T$  (hence, the subsystems are not SP-ISS in the sense of our Definition 7.2.1), then the approximate discrete-time model may be stable for all small values of  $T$  but the exact discrete-time model is unstable for all small values of  $T$ . This example motivates our approach and in particular the consideration of families of parameterized discrete-time systems and the SP-ISS property that we use.

Consider a continuous-time plant  $\dot{x}_1 = x_1 + u$ , which is between a sampler and zero order hold. Suppose that we want to carry out the controller design using the Euler discrete-time approximate model of the plant

$$x_1(k+1) = (1+T)x_1(k) + Tu(k) . \quad (7.61)$$

Suppose that we use the following family of dynamic controllers

$$x_2(k+1) = -0.5x_2(k) - T^2x_1(k) , \quad (7.62)$$

$$u(k) = -\frac{1}{T}x_2(k) - \frac{2}{T}x_1(k) . \quad (7.63)$$

Note that the approximate closed-loop system (7.61), (7.62), (7.63) can be regarded as a feedback interconnection of two scalar systems (7.61) with (7.63) and (7.62). Moreover, using Lyapunov functions  $V_{1T}(x_1) = |x_1|$  and  $V_{2T}(x_2) = |x_2|$  and suppose that  $T < 1$ , we can write the following:

$$|x_1| \geq \frac{2}{T}|x_2| \Rightarrow \Delta V_{1T} \leq -\frac{T}{2}|x_1| , \quad (7.64)$$

$$|x_2| \geq 4T^2|x_1| \Rightarrow \Delta V_{2T} \leq -\frac{T}{4}|x_2| . \quad (7.65)$$

In this case the gains are  $\gamma_{x_1}(s) = \frac{2}{T}s$  and  $\gamma_{x_2}(s) = 4T^2s$ . Note that for any  $M \in (0, 1/8)$  there exist sufficiently small  $\tau_1, \tau_2 \in \mathcal{K}_\infty$  so that our small gain condition holds for all  $T \in (0, M]$ . We have computed the eigenvalues of the approximate closed-loop system matrix and obtained that  $\lambda_1^a = -\frac{1}{2} + 2T^2 + O(T^3)$  and  $\lambda_2^a = -1 + T - 2T^2 + O(T^3)$ , which indicates that indeed the approximate closed-loop model is stable for sufficiently small  $T$ . However, if we consider the exact closed-loop system consisting of the exact discrete-time plant model  $x_1(k+1) = e^T x_1(k) + (e^T - 1)u(k)$  and (7.62), (7.63), we obtain that the eigenvalues of the system matrix are  $\lambda_1^e = -\frac{1}{2} + 2T^2 + O(T^3)$  and  $\lambda_2^e = -1 - \frac{11}{6}T^2 + O(T^3)$  and obviously we have that  $|\lambda_2^e| > 1$  for all sufficiently small  $T$ . In this case, since  $\gamma_{x_1}$  and  $\gamma_{x_2}$  depend on  $T$ , it is not possible to construct a Lyapunov function  $V_T$  via (7.15) that satisfies appropriate bounds in Definition 7.2.1 uniformly in small  $T$ .

## 7.6 Conclusion

We have presented in this chapter a Lyapunov based small gain theorem for parameterized discrete-time SP-ISS systems. The result is a discrete-time counterpart of the continuous-time results in [59]. We have also presented a method for constructing a family of SP-ISS Lyapunov functions for a family of parameterized discrete-time interconnected systems, which can be considered in one direction as a generalization of constructions presented in Chapter 6. A counter example that motivates our results in the case when a discrete-time controller for a continuous-time plant in sampled-data system is designed based on the approximate discrete-time model of the plant has also been presented, which has shown the importance of our result in controller design based on approximate discrete-time models.





## Part III

# Case Study



## Chapter 8

# Stabilization Problem for A Two-Link Manipulator

### 8.1 Introduction

In Part I and Part II, various theoretical results for Lyapunov based design and analysis of nonlinear sampled-data systems were developed. We have focused on emulation design in Part I, and on approximate based direct discrete-time design in Part II. Different frameworks for the two design approaches were proposed in Chapter 3 and Chapter 5, while various design tools were developed in Chapter 6 and Chapter 7. In this chapter, we present a case study to implement results obtained in those earlier chapters.

In Chapter 1, an autonomous vehicle, which belongs to a class of robotic systems, was used to illustrate a nonlinear sampled-data system. In this chapter, we consider another class of robotic systems, the robotic manipulators. A robotic manipulator is one of the benchmark systems to study nonlinear control. It is a familiar example of trajectory-controllable mechanical systems and is widely used in automation processes. For example, it is used as pick and place tools in an automated manufacturing, as surgical equipment in a robotic surgery, as automatic sprayer in an automated agriculture, and as experimental equipment to learn control systems at school laboratories. Relating to this thesis, the most important issue when implementing robotic

systems is the use of digital controllers. Robotic systems are generally controlled using computers. Since a robot itself is a continuous-time plant, then robotic systems are generally sampled-data systems.

A robotic manipulator represents an important class of nonlinear, time-varying, multi input multi output (MIMO) dynamic systems. This class of systems is quite complex and not easy to deal with. For a while, the difficulty was mitigated by the fact that robotic manipulators were highly geared, thereby strongly reducing the interactive dynamic effects between links. In addition, their nonlinear dynamics is not negligible. Hence, the systems present a challenging control problem, since traditional linear control approaches do not easily apply.

Despite all the structural complexities of a robotic manipulator, its dynamic belongs to a special class of dissipative systems, which are called Lagrangian control systems. Lagrangian control system is an outcome of a powerful modeling technique whose starting point is the definition of the energy functions in terms of sets of generalized variables, which leads to the definition of the Lagrangian function [123]. Therefore, it is logical to expect that the Lyapunov based design and the energy based design are suitable to tackle the control design problem of this kind of system.

In this case study, the control problem for a two-link manipulator that has the same structure as the one considered in [3] is revisited, and results from Part I and Part II are applied to solve the stabilization problem of the manipulator with respect to some inputs. In particular, the framework for controller emulation design of Chapter 3 and the framework for direct discrete-time design of Chapter 5 are applied for designing two controllers that input-to-state stabilize the manipulator system. The first controller is designed using emulation, while the second is designed using direct discrete-time design based on plant approximate model. One of the methods for constructing Lyapunov functions presented in Chapter 6 is utilized as a tool to design the controllers and to analyze the property of the closed-loop manipulator system.

As have been presented in Chapter 2, various algorithms of numerical integration are available for the purpose of discrete-time controller design (see [151]). In this case study, the Euler approximation is chosen to build the model. The advantages of using

the Euler are the following:

1. It provides a very simple form of approximate model.
2. It preserves various important structures of the continuous-time plant that often allows the use of design tools which are similar to the tools used in a continuous-time design.
3. It satisfies the consistency property which is a necessary condition for guaranteeing that the stability of the approximate model will also result in the stability of the exact model.

After performing the design, we then implement the controllers to control the two-link manipulator. Intensive simulations using Matlab Simulink are carried out to study the performance of the closed-loop system when each of the two controllers are implemented. Based on these simulation results, we further provide thorough analysis by comparing several different configurations of the system.

This chapter is organized as follows. In Section 8.2 we provide the preliminaries. The two-link robotic manipulator model is derived in Section 8.3, and the controller design is presented in Section 8.4. Simulation results are provided, and deep analysis and discussion of the results are presented in Section 8.5. Finally, a conclusion is presented in Section 8.6.

## 8.2 Preliminaries

A robotic manipulator is an example of Lagrangian systems, which belong to a class of dissipative systems. Lagrangian systems arose from variational calculus and gave a first general analytical definition of physical systems in analytical mechanics. However, they also make it possible to describe the dynamics of various engineering systems such as electromechanical systems or electrical circuits. Lagrangian systems have also motivated a lot of intensive research in control, in order to derive various control laws by taking into account the structure of the system's dynamics derived from energy based modeling. Intensive literature about the design and analysis of Lagrangian systems are for instance [91, 123].

A formal definition of Lagrangian control systems is stated in the following.

**Definition 8.2.1** [91, 123] *A Lagrangian control system is defined by a real function  $L(q, \dot{q}, u) : \mathbb{R}^{2n} \times \mathbb{R}^m \rightarrow \mathbb{R}$  and the equations:*

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}}(q, \dot{q}, u) \right) - \frac{\partial L}{\partial q}(q, \dot{q}, u) = 0 , \quad (8.1)$$

where  $q \in \mathbb{R}^n$  and  $u \in \mathbb{R}^m$ , and  $L(q, \dot{q}, u)$  is called the *Lagrangian function*. ■

When the Lagrangian function is chosen to be

$$L(q, \dot{q}, F_e) = L(q, \dot{q}) + q^T F_e , \quad (8.2)$$

it defines a *Lagrangian system with external forces*, in which the Lagrangian equation satisfies

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}}(q, \dot{q}) \right) - \frac{\partial L}{\partial q}(q, \dot{q}) = F_e , \quad (8.3)$$

where  $F_e \in \mathbb{R}^n$  is the vector of generalized external forces acting on the system.

An important subclass of the Lagrangian control system with external forces is the so called simple mechanical systems, where the Lagrangian function takes a particular form, i.e:

$$L(q, \dot{q}) = K(q, \dot{q}) - P(q) , \quad (8.4)$$

where  $P$  is the potential energy and  $K$  is the kinetic energy defined by

$$K(q, \dot{q}) = \frac{1}{2} \dot{q}^T M(q) \dot{q} \quad (8.5)$$

and the matrix  $M(q) \in \mathbb{R}^{n \times n}$  is positive definite inertia matrix. The following lemma provides a formal formulation of simple mechanical systems.

**Lemma 8.2.1** [91] *The Lagrangian equations for a simple mechanical system may be written as:*

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = F_e , \quad (8.6)$$

where  $g(q) = \frac{du}{dq}(q) \in \mathbb{R}^n$ ,

$$C(q, \dot{q}) = \sum_{k=1}^n \Gamma_{ijk} \dot{q}_k \quad (8.7)$$

and  $\Gamma_{ijk}$  are called the Christoffel's symbols associated with the inertia matrix  $M(q)$  and are defined by:

$$\Gamma_{ijk} = \frac{1}{2} \left( \frac{\partial M_{ij}}{\partial q_k} + \frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{kj}}{\partial q_i} \right). \quad (8.8)$$

■

A class of systems which typically may be represented in this formulation is the dynamics of multibody systems, for which systematic derivation procedures were obtained. A robotic manipulator considered in this chapter is an example of this class of systems.

### 8.3 Manipulator system modeling

Consider a two-link robotic manipulator shown in Figure 8.1, with mass of the arm  $M$  and length  $L$ , and the gripper with mass  $m$ . The physical parameters of the manipulator that we use for the model are respectively  $m = 1 \text{ kg}$  and  $ML^2 = 3 \text{ kg} \cdot \text{m}^2$ . From the definition the Euler-Lagrange system, the dynamical model of the manipulator can be derived from the knowledge of its energy functions. Let  $K$  and  $P$  be the kinetic and potential energy of the system.

$$K(q, \dot{q}) = \frac{(ML^2/3 + mq_2^2)\dot{q}_1^2}{2} + \frac{m}{2}\dot{q}_2^2 \quad (8.9)$$

$$P(q) = \frac{ML^2}{3}q_1^2 + \frac{m}{2}q_2^2 + \frac{m}{4}q_2^4. \quad (8.10)$$

The Lagrangian function is then

$$L(q, \dot{q}) = \frac{(ML^2/3 + mq_2^2)\dot{q}_1^2}{2} + \frac{m}{2}\dot{q}_2^2 - \frac{ML^2}{3}q_1^2 - \frac{m}{2}q_2^2 - \frac{m}{4}q_2^4. \quad (8.11)$$

Applying (8.3), we have that the Lagrangian equation for the manipulator is the following:

$$\begin{bmatrix} (ML^2/3 + mq_2^2)\ddot{q}_1 + 2mq_2\dot{q}_1\dot{q}_2 \\ m\ddot{q}_2 \end{bmatrix} - \begin{bmatrix} -2\frac{ML^2}{3}q_1 \\ mq_2\dot{q}_1^2 - mq_2 - mq_2^3 \end{bmatrix} = \begin{bmatrix} F_{e1} \\ F_{e2} \end{bmatrix}. \quad (8.12)$$

Following Lemma 8.2.1, the robot equation can be formulated as

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = F_e. \quad (8.13)$$

In this example, we assume that the effect of gravitational torque is not taken into account, since they can be cancelled with a suitable choice of external force,  $\tau$ . As a result, the Lagrangian equation takes form

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} = \tau . \quad (8.14)$$

For the model of Figure 8.1 the matrices  $M(q)$  and  $C(q, \dot{q})$  satisfy

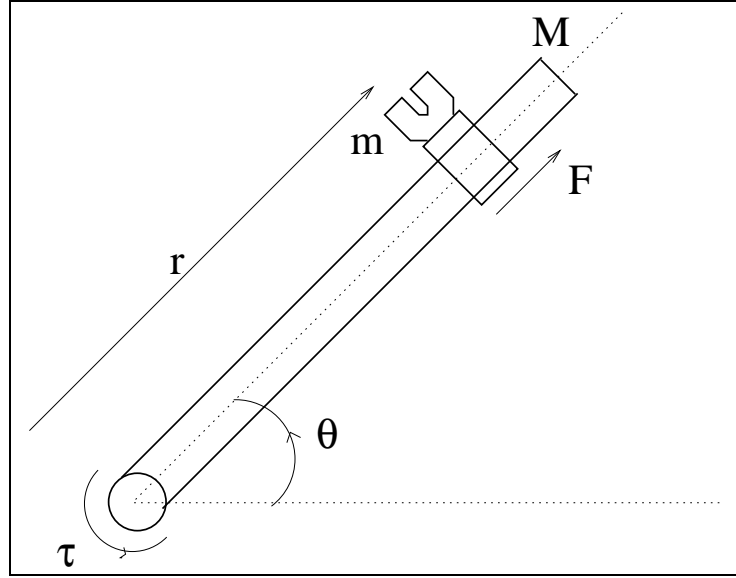


Figure 8.1: A two-link manipulator

$$M(q) = \begin{bmatrix} mr^2 + ML^2/3 & 0 \\ 0 & m \end{bmatrix}, \quad C(q, \dot{q}) = \begin{bmatrix} mr\dot{r} & mr\dot{\theta} \\ -mr\dot{\theta} & 0 \end{bmatrix} \quad (8.15)$$

Substituting  $M(q)$  and  $C(q, \dot{q})$  into (8.14), and denoting the angle of the link and the position of the gripper respectively as  $\theta$  and  $r$ , the continuous time dynamic equations of the manipulator are [5]:

$$\begin{aligned} (mr^2 + ML^2/3)\ddot{\theta} + 2mr\dot{r}\dot{\theta} &= \tau \\ m\ddot{r} - mr\dot{\theta}^2 &= F . \end{aligned} \quad (8.16)$$



We denote the state vector  $(\theta \ r \ \dot{\theta} \ \dot{r})^T$  as  $x := (q_1 \ q_2 \ z_1 \ z_2)^T$  and then write the model in a state space representation:

$$\begin{aligned}\dot{q}_1 &= z_1 \\ \dot{q}_2 &= z_2 \\ \dot{z}_1 &= -\frac{2mq_2z_1z_2}{mq_2^2 + ML^2/3} + \frac{\tau}{mq_2^2 + ML^2/3} \\ \dot{z}_2 &= q_2z_1^2 + \frac{F}{m},\end{aligned}\tag{8.17}$$

and the output equations:

$$\begin{aligned}y_1 &= z_1 \\ y_2 &= z_2.\end{aligned}\tag{8.18}$$

The external inputs  $\tau$  and  $F$  are respectively the external torque and external translational force to each link.

The manipulator is assigned a task to track a reference path. The input references are given by  $\theta_d$  and  $r_d$ , which are respectively the desired angular position and the translational position of the gripper. The control objective is to make the manipulator links to follow the input references with as soon as possible as smooth as possible transient response. For this, external forces are needed to drive the links of the manipulator.

## 8.4 Sampled-data controller design

In the previous section, we have obtained the complete dynamical model of the manipulator. In this section we present the controller design to input-to-state stabilize the manipulator. We will design two controllers for the system, one is designed using emulation and the other is designed using direct discrete-time design based on the Euler approximate model of the manipulator.

For the emulation controller, we design a continuous-time PD controller plus nonlinear terms in the same way as presented in [3]. We then discretize the controller for digital implementation. Further, for the approximate based design, the PD plus nonlinear terms controller is modified to obtain another controller which is redesigned using

a Lyapunov function obtained using changing of supply rates technique. Implementing both discrete-time controllers renders the input-to-state stability in a semiglobal practical sense to the closed loop sampled-data configuration of the robotic manipulator system. The design procedures are presented in the following subsections.

#### 8.4.1 PD plus nonlinear terms controller design

The simplest controller that stabilizes a manipulator system in some sense is a PD controller. This controller follows the form:

$$\tau = -k_d \dot{q} - k_p(q - q_d) . \quad (8.19)$$

However, as shown in [3], a PD controller designed for the manipulator (8.17) has failed to input-to-state stabilize the closed-loop system for any  $k_d, k_p > 0$ . Nonlinear resonance phenomenon occurs with the PD control for some non-constant input reference applied to the system.

A continuous-time nonlinear controller, which is an extension of the PD controller, was designed for the system (8.17) in [3]. In a continuous-time setting, it has been shown in [3] that the following controller

$$\begin{aligned} \tau_c &= -k_{d_1} z_1 - k_{p_1}(q_1 - q_{1d}) \\ F_c &= -k_{d_2} z_2 - k_{p_2}(q_2 - q_{2d}) - k_{nl}(q_2^3 - q_{2d}^3) , \end{aligned} \quad (8.20)$$

renders ISS for the closed-loop system (8.17), (8.20) with respect to the external inputs  $q_{1d}$  and  $q_{2d}$ .

Suppose now that the manipulator is to be controlled using computer. The control algorithm (8.20) is to be implemented using sample and zero order hold devices. It means that  $F$  and  $\tau$  are constant during each sampling interval and the state  $x$  is measured at sampling instants  $kT$ , where  $k \in \mathbb{N}$  and  $T$  is the sampling period.

In this case one may simply discretize the controller (8.20) in the following way, where we denote  $w := (q_{1d} \ q_{2d})^T$ :

$$\begin{aligned} \tau_c(x(k), w(k)) &= -k_{d_1} z_1(k) - k_{p_1}(q_1(k) - q_{1d}(k)) \\ F_c(x(k), w(k)) &= -k_{d_2} z_2(k) - k_{p_2}(q_2(k) - q_{2d}(k)) - k_{nl}(q_2^3(k) - q_{2d}^3(k)) , \end{aligned} \quad (8.21)$$

and implement the emulated controller digitally.

Finally, it is a straight away implementation of Corollary 3.5.1 from Chapter 3 to show that the sampled-data closed-loop system consisting of the continuous-time manipulator (8.17) and the emulation controller (8.21) achieves semiglobal practical ISS for some chosen sampling periods  $T > 0$ .

#### 8.4.2 Lyapunov direct digital redesign

As presented in the previous section, a PD plus nonlinear terms controller designed using emulation can input-to-state stabilize the manipulator system in a semiglobal practical sense. However, as shown in Chapter 4, emulation controller may not be the best choice to control the system. It is potentially better if one takes the sampling into account when designing a controller by using a discrete-time model of the plant. In this subsection, the framework of direct discrete-time design provided in Chapter 5 is applied to design a discrete-time controller for the manipulator.

Since it is impossible to obtain the exact discrete-time model of the manipulator, we use instead the Euler approximate discrete-time model for the controller design. The Euler approximate model of the manipulator when we substitute values of its physical parameters is the following:

$$\left. \begin{aligned} q_1(k+1) &= q_1(k) + Tz_1(k) \\ q_2(k+1) &= q_2(k) + Tz_2(k) \\ z_1(k+1) &= z_1(k) + T \left[ -\frac{2q_2(k)z_1(k)z_2(k)}{q_2(k)^2+1} + \frac{\tau(k)}{q_2(k)^2+1} \right] \\ z_2(k+1) &= z_2(k) + T [q_2(k)z_1(k)^2 + F(k)] \end{aligned} \right\} =: \tilde{F}_T^a, \quad (8.22)$$

where  $\tilde{F}_T^a := \tilde{F}_T^a(x(k), \tau(k), F(k))$  and  $T$  is the sampling period parameterizing the model.

In order to guarantee that the controller that achieves ISS for system (8.22) would also achieve SP-ISS of the sampled-data system, we need to use the results Theorem 5.4.1. By this theorem, there are three points need to satisfy to guarantee that the controller that achieves ISS for system (8.22) would also achieve SP-ISS of the sampled-data system. In particular, it is directly true that the consistency condition of Theorem 5.4.1 holds since we are using the Euler approximate model. Hence, the

second condition of Theorem 5.4.1 is satisfied.

We have supposed that the controller has the following form

$$\begin{aligned}\tau_T^{Euler} &= \tau_c + Tu_1(x) \\ F_T^{Euler} &= F_c + Tu_2(x) ,\end{aligned}\tag{8.23}$$

where  $\tau_c$  and  $F_c$  are given in (8.21), and  $u_1$  and  $u_2$  are functions that need to be designed based on (8.22) for the Euler approximate closed-loop system, so that the system has a Lyapunov characterization (6.19). We formally let the control input to be  $u := (u_1 \ u_2)^T$  and using (8.20), (8.22) and (8.23) we can write the approximate model as follows:

$$\begin{aligned}x(k+1) &= \tilde{F}_T^a(x(k), \tau(x(k), w(k)) + Tu_1(k), F(x(k), w(k)) + Tu_2(k)) \\ &=: F_T^a(x(k), u(k), w(k)) ,\end{aligned}\tag{8.24}$$

which has the desirable form given by (5.20). If  $u_1, u_2$  are bounded on compact sets we can conclude that the controller (8.23) is locally uniformly bounded and hence the third condition of Theorem 5.4.1 holds. Although other controller structures are possible, our choice is guided by the fact that we want to have that the continuous time and the Euler-based controllers coincide for  $T = 0$ , so that it makes sense to compare their performance. Systematic controller design procedure based on these ideas is an interesting topic for further research.

It remains to design  $u_1$  and  $u_2$  so that the SP-ISS Lyapunov conditions for approximate model, which is the first condition of Theorem 5.4.1, holds. In order to do this, we apply Corollary 6.4.1 and Remark 6.3.2. Recall the kinetic and potential energy of the system  $K$  and  $P$ , where

$$K = \frac{(1 + q_2^2)z_1^2}{2} + \frac{1}{2}z_2^2 ,\tag{8.25}$$

$$P = q_1^2 + \frac{1}{2}q_2^2 + \frac{1}{4}q_2^4 .\tag{8.26}$$

In the same way as in [3], we let the Lyapunov function  $V_{1T}$  be the energy function of the system:

$$V_{1T} = K + P = \frac{(1 + q_2^2)z_1^2}{2} + \frac{1}{2}z_2^2 + q_1^2 + \frac{1}{2}q_2^2 + \frac{1}{4}q_2^4 .\tag{8.27}$$

We next consider the first difference for  $V_{1T}$  to compute  $u_1$  and  $u_2$ , and we write

$$\begin{aligned}\Delta V_{1T} &= V_{1T}(F_T) - V_{1T}(x) \\ &= T(-2z_1^2 - z_2^2 + 2z_1q_{1d} + z_2q_{2d} + z_2q_{2d}^3) + T^2\left(z_1\left(u_1 + 3\frac{z_1}{q_2^2 + 1}\right.\right. \\ &\quad \left.\left.+ 0.5z_1^3q_2^2\right) + z_2\left(u_2 + 0.5\frac{z_2z_1^2}{q_2^2 + 1} + z_2 + 1.5z_2q_2^2\right) + f(q, z, q_d)\right) \\ &\quad + O(T^3),\end{aligned}\tag{8.28}$$

where  $z := (z_1 \ z_2)^T$ ,  $q := (q_1 \ q_2)^T$ .  $u_1$  and  $u_2$  are designed to reduce the positiveness of the  $O(T^2)$  term on the right-hand side of (8.28) and we choose the following:

$$\begin{aligned}u_1(x) &= -k_{e1} \left(3\frac{z_1}{q_2^2 + 1} + 0.5z_1^3q_2^2\right) \\ u_2(x) &= -k_{e2} \left(0.5\frac{z_2z_1^2}{q_2^2 + 1} + z_2 + 1.5z_2q_2^2\right),\end{aligned}\tag{8.29}$$

where the values of  $k_{e1}$ ,  $k_{e2}$  are listed in Table 8.1. It is obvious from (8.23) and (8.29) that the control  $\tau_T^{Euler}$  and  $F_T^{Euler}$  satisfy the third condition of Theorem 5.4.1. Moreover, substitution of (8.29) to (8.28) results in the dissipation inequality:

$$\begin{aligned}\Delta V_{1T} &\leq T(-2z_1^2 - z_2^2 + 2z_1q_{1d} + z_2q_{2d} + z_2q_{2d}^3) + T^2\left(-3\frac{z_1^2}{q_2^2 + 1} - 0.5z_1^4q_2^2\right. \\ &\quad \left.- 0.5\frac{z_2^2z_1^2}{q_2^2 + 1} - z_2^2 - 1.5z_2^2q_2^2\right) + T^2f(q, z, q_d) + O(T^3) \\ &\leq T\left(-\frac{1}{2}|z|^2 + a_1|q_{1d}|^2 + a_2|q_{2d}|^6\right) + T^2\left(-3\frac{z_1^2}{q_2^2 + 1} - 0.5z_1^4q_2^2\right. \\ &\quad \left.- 0.5\frac{z_2^2z_1^2}{q_2^2 + 1} - z_2^2 - 1.5z_2^2q_2^2\right) + T^2f(q, z, q_d) + O(T^3),\end{aligned}\tag{8.30}$$

where  $a_1$  and  $a_2$  are sufficiently large positive numbers. The system is SP-qISS and hence the first part of condition 1 of Corollary 6.4.1 holds.

Define another Lyapunov function  $V_{2T}$  in the following form:

$$V_{2T} = K + P + \varepsilon \frac{q_2z_2 + q_1(1 + q_2^2)z_1}{(1 + q_2^4 + q_1^2)^{3/4}},\tag{8.31}$$

where  $\varepsilon > 0$  is a sufficiently small constant (to guarantee that  $V_{2T}$  positive definite).

We can write

$$\begin{aligned}
\Delta V_{2T} &= V_{2T}(F_T) - V_{2T}(x) \\
&= T \left[ -2z_1^2 - z_1^2 + 2z_1q_{1d} + z_2q_{2d} + z_2q_2d^3 \right. \\
&\quad \left. + \varepsilon \frac{z_2^2 + 2q_2^2z_2^2 + z_1^2 + q_2(F_c + Tu_2) + q_1(\tau_c + Tu_1)}{(1 + q_2^4 + q_1^2)^{3/4}} \right. \\
&\quad \left. + \frac{3}{4}\varepsilon \frac{4q_2^3z_2 + 2q_1z_1}{(1 + q_2^4 + q_1^2)^{7/4}}(q_2z_2 + q_1(1 + q_2^2)z_1) \right] + O(T^2) \\
&\leq T \left[ M_1(q_{1d}^2 + q_{2d}^2 + q_{2d}^6) - M_2|z|^2 + M_3|z|^2 + \varepsilon \frac{q_2F_c + q_1\tau_c}{(1 + q_2^4 + q_1^2)^{3/4}} \right] + O(T^2)
\end{aligned} \tag{8.32}$$

for sufficiently small  $T$ ,  $\varepsilon$  and  $M_2$  and sufficiently large  $M_1$  and  $M_3$ . Substituting the controller  $\tau_T^{Euler}$  and  $F_T^{Euler}$ , we can write the dissipation inequality as

$$\begin{aligned}
\Delta V_{2T} &\leq T \{ \tilde{M}_1(q_{1d}^2 + q_{2d}^2 + q_{2d}^6) + \tilde{M}_3|z|^2 \\
&\quad - \tilde{M}_2|z|^2 - \tilde{\varepsilon} \frac{q_2^4 + q_1^2}{(1 + q_2^4 + q_1^2)^{3/4}} \} + O(T^2)
\end{aligned} \tag{8.33}$$

for sufficiently small  $T$ ,  $\tilde{\varepsilon}$  and  $\tilde{M}_2$  and sufficiently large  $\tilde{M}_1$  and  $\tilde{M}_3$ . The system is SP-IOSS and hence the second part of condition 1 of Corollary 6.4.1 holds. Finally, since  $\alpha_1(s) = \frac{s^2}{2}$  and  $\lambda_2(s) = \tilde{M}_3s^2$ , it is obvious that condition 2 of Corollary 6.4.1 holds. Moreover, from Corollary 6.4.1 and Remark 6.3.2 we have that the closed-loop approximate model (8.22), (8.23) is SP-ISS and from the choice of  $V_{1T}$  and  $V_{2T}$  the SP-ISS Lyapunov function of the system follows the form

$$V_T = V_{1T} + \rho(V_{2T}),$$

with appropriate  $\rho \in \mathcal{K}_\infty$ . With this, the first condition of Theorem 5.4.1 holds. Therefore, we have that all conditions of Theorem 5.4.1 are satisfied, then it follows from the conclusion of the theorem that the exact discrete-time closed-loop system is SP-ISS. The local Lipschitz property of the system guarantee that the solutions of the discrete-time closed-loop model is uniformly bounded over  $T$  for arbitrarily fast sampling and arbitrarily large sets of states and inputs. Finally, using [117, Theorem 5], we can conclude that the closed-loop sampled-data system (8.17), (8.23) is SP-ISS.

## 8.5 Simulation results

In the earlier section, we have designed an emulation controller and an Euler based controller and have shown that both controllers input-to-state stabilize the closed-loop sampled-data configuration of the manipulator system in a semiglobal practical sense. In this section, our design results are implemented via a set of intensive simulations using Matlab Simulink package. The parameters of the controller used for the simulations, following [5], are listed in Table 8.1.

Table 8.1: Manipulator's and controller's parameters

Parameter	Value	Parameter	Value
$k_{p1}$	2	$k_{p2}$	1
$k_{d1}$	2	$k_{d2}$	1
$k_{e1}$	2	$k_{e2}$	2
-	-	$k_{nl}$	1

The simulation results are presented to illustrate performance of the system when we apply the Euler-based controller (8.23) and the emulation controller (8.21). The parameters for Simulation 8.1 are given in Table 8.2. A square wave input of amplitude 3 and period 0.5 was used in the simulations and the results are presented in Fig. 8.2.

Table 8.2: Parameters for Simulation 8.1

Parameter	Value
Sampling period (T)	0.25s
Initial state	$(0.1 \ 0.1 \ 0.1 \ 0.1)^T$
$\theta_d$	3 square(0.5t)
$r_d$	0

Fig. 8.2(a) shows the reference signal  $\theta_d$  and the actual angular position of the arm  $\theta$ , while Fig. 8.2(b) shows the desired position of the gripper  $r_d$  and the actual position  $r$  obtained when applying the Euler-based controller (8.23). Fig. 8.2(c) and Fig. 8.2(d) are respectively showing the response of the corresponding variables with emulation controller (8.21).

The simulation is carried out with a relatively large sampling period, to observe

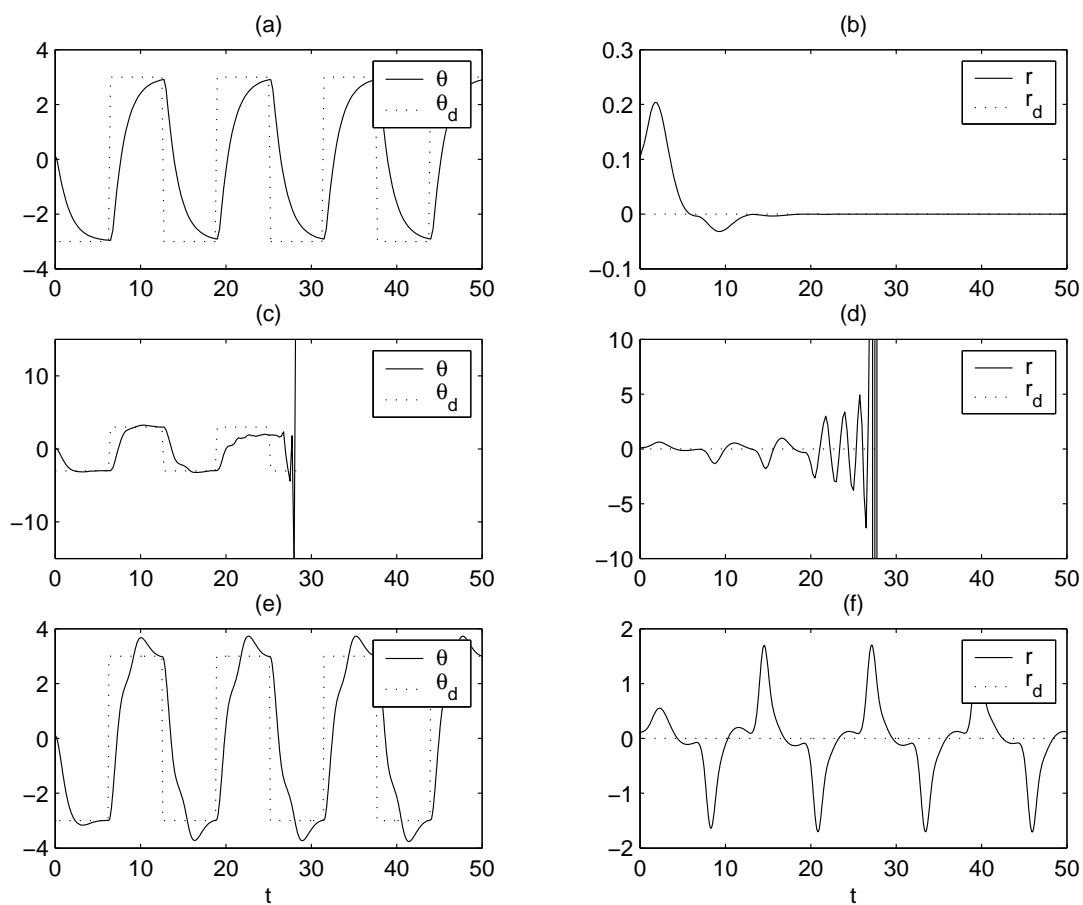


Figure 8.2: Simulation 8.1, responses with Euler-based controller (a,b) and emulation controller (c,d,e,f) when applying a square wave input.



the robustness of each controller to a square wave input. It is shown that with the given simulation set-up, Euler-based controller can still show a good performance with  $T = 0.25s$ . On the other hand, the trajectories of the system with the emulation controller (8.21), exhibit finite escape times for the same simulation parameters.

By computation it has been shown in [3] that the controller (8.20) rendered ISS to the closed-loop system. It follows from the result of [82] that ISS property is preserved in a semiglobal practical sense under fast sampling. Hence, applying the emulation controller (8.21) would render SP-ISS property to the closed-loop system. However, it is shown by simulation results that the Euler-based controller (8.23) performs significantly better than the emulation controller (8.20) for the corresponding simulation setup.

Moreover, it is shown in Fig. 8.2(d) and Fig. 8.2(e) that by reducing the time sampling into  $T = 0.1s$ , the emulation controller results in a bounded response, although the overshoot that occurs on the state  $r$  exceeds the feasible range of the physical parameters of the manipulator. Reducing further the sampling period results in performance that is closer to the continuous-time controller.

Table 8.3: Parameters for Simulation 8.2

Parameter	Value
Sampling period (T)	$0.3s$
Initial state	$(3\pi \ 0 \ \frac{1}{2}\pi \ 0.3)^T$
$\theta_d$	$\pi$
$r_d$	$0.3$

Another set of simulations were done in Simulation 8.2 using both controllers, applying a constant input signal to the system. Simulation 8.2 is carried out using parameter given in Table 8.3, where the initial condition of the states is chosen to be relatively large. Responses resulted from these simulations are shown in Fig. 8.3.

Fig. 8.3(a) and Fig. 8.3(b) show the responses obtained from the simulation with Euler-based controller, while Fig. 8.3(c) and Fig. 8.3(d) are obtained from the emulation controller. It is again shown that the Euler-based controller outperforms the emulation controller.

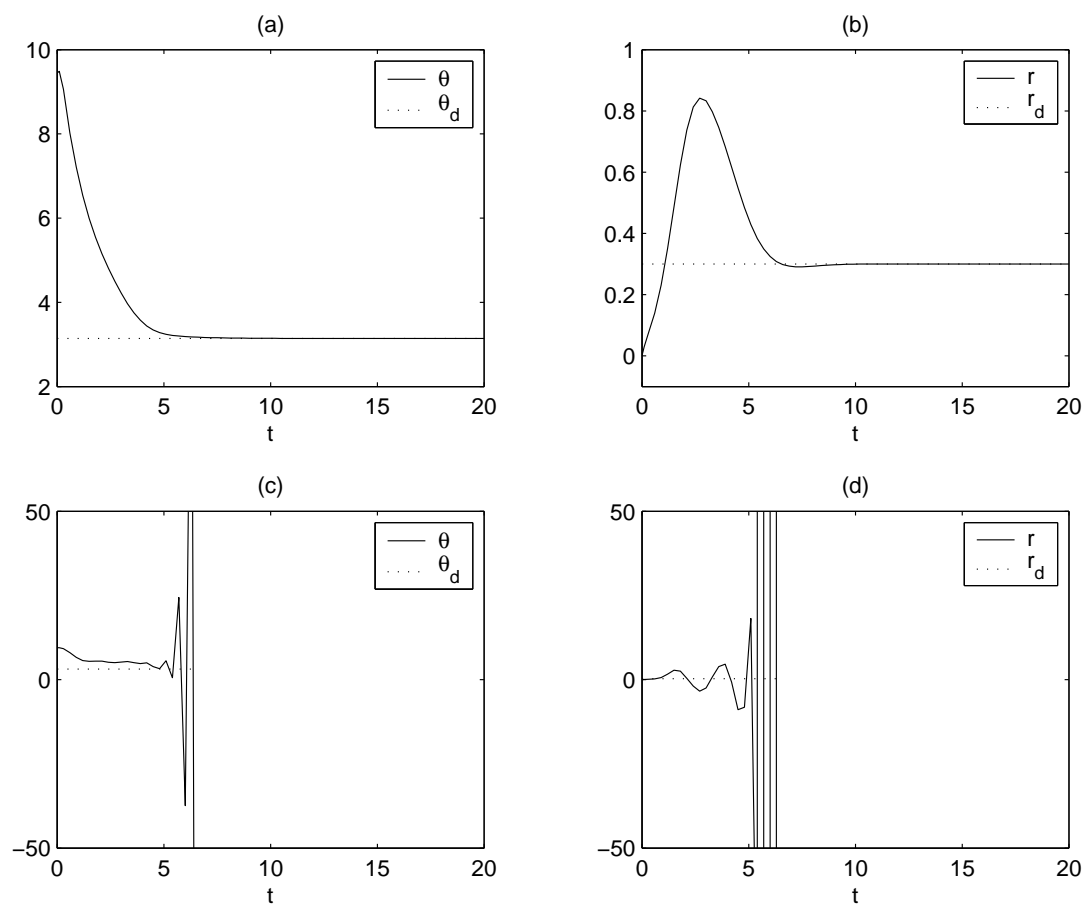


Figure 8.3: Simulation 8.2, responses with Euler-based controller (a,b) and emulation controller (c,d), when applying a constant input.

## 8.6 Conclusion

In this chapter, we have applied some results presented in this thesis to a model of a particular example from engineering practice. The case study presented in this chapter has illustrated the usefulness of our design frameworks. We have demonstrated the design of an emulation controller and a discrete-time SP-ISS controller for a manipulator based on its Euler approximate model. We have also shown that our composite Lyapunov function construction is a useful tool to overcome the difficulties of finding a Lyapunov function for nonlinear systems.

Using simulations, we have compared the performance of the Euler-based discrete-time controller with the performance of the discretized continuous-time controller obtained in [3] for the same problem. It was shown that the Euler-based discrete-time controller yielded better performance. This strongly motivates further development of systematic controller design procedures for sampled-data nonlinear systems based on their approximate discrete-time models, where results in Part II of this thesis could play an important role.



## Chapter 9

# Conclusions and Further Research

Advances in the digital computer technology have led to the era of sampled-data control systems. Sampled-data control configuration has now become prevalent in engineering applications. In addition, the requirement for more sophisticated control and the increase of system complexity, together with the fact that most plants are nonlinear in nature, have also motivated the utilization of nonlinear control techniques in design and analysis of control systems. The two issues together have raised problems of the design and analysis of nonlinear sampled-data control systems.

In this thesis, important issues in design and analysis for nonlinear sampled-data control systems have been investigated. Two main approaches to sampled-data design - the emulation design and the direct discrete-time design, particularly that based on approximate plant models - have been the focus of our research. Dissipativity theory, together with the Lyapunov stability method for nonlinear systems and its extensions have been used to obtain our results. In Chapter 1, the main research problems covered in this thesis have been formulated. Solutions for those problems, have been presented as the main contributions of this thesis.

In this chapter, the main contributions of the thesis will be summarized and potential expansion and extension is proposed for further research.

## 9.1 Conclusions

We have considered a general class of nonlinear systems with disturbances, controlled by a discrete-time controller. A discretization process is necessarily involved in the procedure for design and analysis of this class of systems. Throughout the thesis we have considered a discretization scheme, where we let the time sampling  $T$  be a parameter that can be assigned later for implementation. Approximation techniques have also been used to obtain a discrete-time controller model in Part I and a plant model in Part II, since obtaining the exact discrete-time model for nonlinear systems is almost impossible in general. For both cases, parameterized family of approximate discrete-time plant models and parameterized family of discrete-time controllers are used to obtain our results. This set up is more general and more realistic from an application point of view, than assuming the availability of the exact discrete-time model from the beginning.

In this thesis, we have focused on various important properties of nonlinear systems, particularly the dissipativity property and stability with the presence of inputs. We consider a semiglobal practical version of the properties for stating our results, since in most cases the properties are achieved in a semiglobal practical rather than global sense. A semiglobal practical property occurs naturally in sampled-data system as an effect of discretization.

The conclusions from each part of the thesis are the following. In Part I, we have focused on emulation design. In Chapter 3, results on controller emulation design have been presented. Dissipativity properties of sampled-data systems were the topic of concern, and the preservation of dissipation inequalities under sampling were investigated. The results were stated for a general class of nonlinear systems with inputs. It has been shown in this work that dissipation inequalities are preserved in a semiglobal practical sense under fast sampling. An explicit formula for computing the upper-bound of the permitted sampling periods was presented. Applications of our results to ISS and passivity were also presented, to show the generality of our main results.

In Chapter 4, theoretical results of Chapter 3 were applied to a jet engine control problem. Stabilization problems for a jet engine rotating stall and surge was studied.

In this example, a backstepping controller of [74] has been designed to asymptotically stabilize the closed-loop system in continuous-time, and then it was emulated for a discrete-time implementation to control the same continuous-time plant. Intensive simulations have been done to study the performance of the sampled-data closed-loop system. It has been shown that the results of Chapter 3 are applicable to these problem. The discrete-time emulated controller still asymptotically stabilizes the continuous-time plant, with some degradation of performance compared with the original continuous-time set-up. It has been proven in Chapter 3 that the asymptotic stability property of the sampled-data system is achieved in a semiglobal practical sense. Further step taken was to find another way to design a discrete-time controller for sampled-data implementation to improve for the achieved performance. An Euler based discrete-time backstepping controller similar to [110] was then designed for the jet engine system. Adopting the technique similar to [110], it was shown that with the Euler based controller, the closed-loop sampled-data system also achieves a semiglobal practical asymptotic stability property. The performance of the closed-loop sampled-data system with this controller was then compared with the one controlled using the emulated controller that has been designed earlier. It has been shown by this example that potential improvement can be achieved with the approximate based discrete-time design.

The simulation study and observation done in Chapter 4 have motivated the rest of the research in this thesis. In Part II, a direct discrete-time design via approximate plant models was the main concern. Results in this part provide a framework and tools for approximate based design for sampled-data systems with inputs, using direct discrete-time technique. Our results enrich the collection of tools that control engineers can use to carry out complex design, which is especially important since this area of research is still very new. Although we believe there is still a lot of work to do in order to have systematic tools for controller design, a case study done at the end of this thesis have strengthened the hypothesis we had from Chapter 4 that direct discrete-time design may complement the emulation technique to carry out better design in order to obtain more satisfying performance.

In Chapter 5, direct discrete-time design based on an approximate model of a continuous-time plant was investigated. Input-to-state stability property was the main focus. A framework for input-to-state stabilization design for sampled-data systems via an approximate discrete-time model of the plant was proposed. It was proved that under certain conditions for the continuous-time plant model, the controller and the approximate discrete-time plant model, a discrete-time controller that is designed to input-to-state stabilize an approximate discrete-time model of a nonlinear plant with disturbances, would also input-to-state stabilize in a semiglobal practical sense the exact discrete-time plant model for a sufficiently small  $T$ . The framework provides a guideline for a Lyapunov-based direct discrete-time design for sampled-data systems, especially with the presence of disturbances.

In Chapter 6, results on changing supply rates for discrete-time SP-IOSS systems were presented. This allows for a partial construction of Lyapunov functions. Two formulae of the composite Lyapunov function have been constructed. The results apply to investigation of different semiglobal practical stability properties of discrete-time parameterized systems that commonly arise when doing a discrete-time design using an approximate discrete-time model. The results provide a tool to carry out the Lyapunov stability checking as a part of framework proposed in Chapter 5. In addition, we have also shown that SP-IOSS is quite a general setting of nonlinear system stability properties that covers various other nonlinear stability notions as its special cases. Applications of the main results were presented to a LaSalle criterion for SP-ISS, SP-ISS for parameterized time-varying discrete-time cascaded systems, SP-ISS via positive semidefinite Lyapunov functions for parameterized discrete-time systems, and observer-based ISS controller design for parameterized discrete-time systems. An engineering application was also presented as a case study in Chapter 8.

In Chapter 7, a Lyapunov based small gain theorem for discrete-time interconnected ISS system was proposed. It has been proved that if two SP-ISS subsystems are interconnected and the interconnection satisfies a certain small gain condition, then the interconnected system is also SP-ISS. The result is also very useful for a partial Lyapunov function construction, which is another method in addition to the methods



provided in Chapter 6.

In Chapter 8, study of the stabilization problem of a two link robotic manipulator was carried out. In this chapter, we have applied results of Chapter 3 to Chapter 7 to the manipulator system. This has shown the practicability of both results on emulation and discrete-time design which are presented in this thesis. Controller design for the manipulator was done using the two approaches. A comparison study based on simulation results has been done, showing the capability of using the discrete-time redesign to improve performance of the controller designed using emulation. This has shown that a trade off and selection of methods to use when carrying out a design is crucial to determine the quality of the design. Hence, availability of various design techniques is very important to increase freedom in performing a design.

## 9.2 Further research

Although this thesis has thoroughly discussed important issues arising in design and analysis of nonlinear sampled-data systems, there is still much scope for further research. We do not attempt to be exhaustive in listing these possible topics since it is our opinion that this area is still in its infancy and it would take a lot of work to develop a comprehensive set of tools that control engineers can use directly for sampled-data controller design. Consequently, the topics we propose as further research are more immediately related to the results presented in this thesis and can be regarded, therefore as a natural next step.

The results on emulation design in Part I of this thesis provide a general framework for emulation design. As shown in Chapter 3, the dissipativity property of a nonlinear system is preserved in a semiglobal practical sense, under sampling and controller emulation. In this chapter, an explicit formula to compute the bound  $T^*$  on the required sampling period that guarantees the preservation of the properties is provided. However, the time sampling obtained using this formula is rather conservative and often cannot be used for implementation purposes. The required time sampling is in fact not as small as what is obtained using the proposed formula. A research on how to compute a less conservative bound on  $T^*$  will be very useful to facilitate the

implementation of our framework. This can be achieved, for example, by using sharper bounds on solutions of the system than the conservative bounds based on Lipschitz conditions that we used. An example of how this can be done is to use the so called *one sided Lipschitz condition* to estimate bounds on trajectories (see [151]), which is very related to the stability properties of the system.

In Chapter 3, we limit our discussion to the case when a sampler and zero order hold is used. Developing more advanced sampling techniques will be very useful to decrease the required sampling rate and also to minimize discrepancies between the continuous-time controller and its emulation. Results presented in [15, 16] that apply to linear systems can motivate further exploration of emulation design for nonlinear sampled-data systems. Moreover, the consistency properties that we used are a minimum sufficient requirement for an emulated controller to work well. However, among the consistent discretizations of the continuous-time controllers, one may find a controller that performs better in some sense than other consistent controllers. Obtaining constructive methods of finding such optimal discretizations is an important avenue of research. In particular, since it is likely that this problem would need some type of nonlinear optimization to be carried out, it is important to identify classes of systems for which computation of optimal discretizations is feasible.

Another interesting issue related to the results of this chapter is to study the usefulness of the passivity results plus a type of detectability property to conclude semiglobal practical uniform asymptotic stability property of nonlinear systems.

The results of Part II of this thesis can be extended in several directions. Chapter 5 has provided an important framework for a direct discrete-time design based on an approximate model of the plant for input-to-state stabilization problem. The first possible extension is to cover other stability properties of nonlinear systems with disturbances. Important results relevant to this direction are given in [114] where a framework for stability with respect to compact sets of sampled-data differential inclusions was considered and in [104] where iISS and iLiSS stability of systems with disturbances was the interest. Results of [114] provide an alternative framework for design of ISS controllers via approximate models of an auxiliary differential inclusion.

Moreover, results in the same paper cover the case of fixed sampling periods where the family of approximate models is obtained by reducing the integration period of the numerical scheme. Hence, in general one may investigate approximate models of the form

$$x(k+1) = F_{T,h}(x(k), u(k), w[k]) \quad (9.1)$$

where  $T$  and  $h$  are respectively the sampling period of the sampled-data system and the integration period of the numerical method used to generate the approximate model. We only considered the situation when  $T = h$ , but the more general case when  $T \neq h$  can be treated in a similar manner. Considering the later case, more general approximate models and other stability properties such as IOS, IOSS and others are important areas for further research.

In Chapter 8, the framework presented in Chapter 5 has shown to be useful for real applications. The use of approximation to construct discrete-time models of the plant, equipped with the consistency properties, has provided a solution to the problem of the difficulty of modeling. In addition, checkable conditions are very important for the applicability of the framework. Yet, the result is prescriptive. When applying the framework for a design problem, direct discrete-time techniques that are suitable for the framework are rare. Digital redesign used in Chapter 8, discrete-time backstepping presented in [110, 113], which has been used in Chapter 4 and optimization based technique [44] are the only techniques available for this framework. Therefore, besides development of generic tools for parameterized discrete-time systems, it is important to develop recipes for controller design for classes of plants based on their approximate models. Further development of controller design recipes for other classes of systems is extremely important for practical engineers and this was the main motivation for research presented in this thesis.

The discrete-time backstepping presented in [110, 113] is one of the only few recipes available for direct discrete-time design. It was shown in Example 5.2.4 that compared with a controller that is obtained by emulation, the Euler based controller designed using [110, 113] rendered consistently larger DOA for the closed-loop systems. As the technique given in [113] considers systems without input, and the fact that enlargement

of DOA is an important issue, it will be a good motivation to study the case when inputs are taken into account.

In addition, besides the techniques applied for the controller design, the choice of discrete-time model used determines the quality of the design. Hence, exploration of discretization techniques for the plant model is an important direction for further research. Discretization using higher order approximation and functional approximation will be two potential directions to follow. Results of [42] could be the starting point of this further research. Moreover, the issue of control with saturation and control with bounded signals, which is very important from practical point of view (see [88, 90, 91, 123] and references therein), is also an interesting topic to pursue for further research.

Although results in Chapter 5 suggest a semiglobal practical stability of the system, designing a globally stabilizing controller for the plant is possible to achieve for a certain class of nonlinear systems. It is still an open question what is the largest class of continuous-time plant that can be globally stabilized with sampled feedback. An important result for this problem was presented in [39] for homogeneous systems.

It has been shown how partial construction of Lyapunov functions is very useful to overcome the difficulty of finding a Lyapunov function for nonlinear systems. Results, in particular, of Chapter 6 have shown to be useful in various applications. For instance, the application on the LaSalle criterion has given a solution to deal with LaSalle type stability in a semiglobal practical sense, where LaSalle's theorem does not apply. This motivates further research to develop a Matrosov's type result to deal with similar problem for time-varying systems. Certain steps in this direction have been reported in [89, 108] and [115], where the authors respectively considered stabilization of parameterized cascaded discrete-time systems and the Matrosov Theorems for time-varying parameterized discrete-time systems. Further development of such results for parameterized discrete-time systems could enrich the control designers' toolbox and lead to constructive methods for sampled-data controller design for classes of nonlinear systems via their approximate discrete-time models. For example, it would be interesting to investigate whether one can use passivity based ideas in

investigation of parameterized discrete-time systems in order to obtain new tools for controller design via approximate discrete-time models.

A small gain theorem is very powerful and there is still ever growing research on this topic. Most people are still looking at trajectory based results for design purpose. Indeed, learning from the usefulness of Lyapunov approach in many design tools, the Lyapunov based small gain theorem results seems promising to be a more constructive design tool than the trajectory based result. Hence providing results based on Lyapunov characterization is one possible direction to follow. Moreover, design based on the small gain theorem for discrete-time and sampled-data system is also a potential direction for further research.

Last but not the least, since most of the results in this thesis give generic recipes for design and analysis, breaking down and implementing the results to solve more specific applications within the framework is also a possible path to follow. This also opens a possibility for improving and combining results of this thesis with other results. Finally, it is extremely important to apply these results to real-world engineering problems for several reasons. First, it will be useful for the evaluation of the power of the newly developed algorithms. Second, practical applications normally generate new theoretical questions that are more related to implementation of nonlinear sampled-data controllers, such as quantization effects and signal conditioning that have been overlooked in this thesis. Also, having more case studies where performance of different controllers is compared would be very useful from a practical point of view, especially if it can be shown that certain methods turn out to work consistently better than others. At the moment no comprehensive studies seem to exist, whereas this type of work is important for evaluation of different approaches that can be considered in this thesis. We have partly addressed this questions in Chapter 4 and Chapter 8 where 2 different controllers designed using emulation and direct discrete-time techniques, respectively to stabilize a jet engine and a robotic manipulator, were compared.

To sum up, a range of further technical developments is still needed to equip the control engineers with a set of comprehensive and effective toolbox for digital nonlinear controller design and implementation. Evaluation and comparison of different methods

needs to be carried out via case studies to real engineering examples in order to better test and examine the obtained theory. This thesis has provided contributions in all of these directions, but a lot of work remains to be done in order to see these results used by practical control engineers, which is the original motivation and also the ultimate goal of our research.

# Bibliography

- [1] G. L. Amicucci, S. Monaco, and D Normand-Cyrot. Control Lyapunov stabilization of affine discrete-time systems. In *Proc. IEEE Conf. Decis. Contr.*, pages 923–924, San Diego, CA, 1997.
- [2] B. D. O. Anderson. Controller design: Moving from theory to practice. *IEEE Control Systems Magazine*, 13:16–25, 1993.
- [3] D. Angeli. Input-to-state stability of PD-controlled robotic systems. *Automatica*, 35:1285–1290, 1999.
- [4] D. Angeli. A Lyapunov approach to incremental stability properties. *IEEE Trans. Auto. Contr.*, 47:410–422, 2002.
- [5] D. Angeli, E. D. Sontag, and Y. Wang. A characterization of integral input to state stability. *IEEE Trans. Auto. Contr.*, 45:1082–1097, 2000.
- [6] A. Arapostathis, B. Jacubczyk, H. G. Lee, S. I. Marcus, and E. D. Sontag. The effect of sampling on linear equivalence and feedback linearization. *Systems and Control Letters*, 13:373–381, 1989.
- [7] M. Arcak, D. Angeli, and E. D. Sontag. A unifying integral ISS framework for stability of nonlinear cascades. *SIAM Journal on Control and Optimization*, 40:1888–1904, 2002.
- [8] K. J. Aström and B. Wittenmark. *Computer-Controlled System, Theory and Design*. PHI, 1997.

- 
- [9] P. Astuti, M. Corless, and D. Williamson. On the convergence of sampled-data nonlinear systems. *Differential Equations - Theory, Numerics and Applications*, pages 201–210, 1997.
  - [10] J. P. Barbot, M. Djemai, S. Monaco, and D. Normand-Cyrot. Analysis and control of nonlinear singularly perturbed systems under sampling. In C. T. Leondes, editor, *Control and Dynamic Systems: Advances in Theory and Application*, volume 79, pages 203–246. Academic Press, San Diego, 1996.
  - [11] J. P. Barbot, S. Monaco, and D. Normand-Cyrot. A sampled normal form for feedback linearization. *Math. of Control, Signals and Systems*, 9:162–188, 1996.
  - [12] S. Battilotti. Robust output feedback stabilization via a small gain theorem. *Internat. Jour. of Robust and Nonlin. Control*, 8:211–229, 1998.
  - [13] M. Bensoubaya, A. Ferfera, and A. Iggidr. Stabilization of nonlinear systems by use of semidefinite Lyapunov functions. *Appl. Math. Letters*, 12:11–17, 1999.
  - [14] M. M. Bridges, D. M. Dowson, and C. T. Abdallah. Control of rigid-link, flexible-joint robots: a survey of backstepping approach. *Journal of Robotic Systems*, 12:199–216, 1995.
  - [15] M. Cantoni. Necessity of sampled-data approximation result. In *Proc. IEEE Conf. Decis. Contr.*, pages 310–315, Orlando, Florida, 2001.
  - [16] M. Cantoni and G. Vinnicombe. Discretization of feedback controllers in a point-wise gap metric. In *Proc. IEEE Conf. Decis. Contr.*, pages 1918–1923, Sydney, Australia, 2000.
  - [17] B. Castillo, S. Di Gennaro, S. Monaco, and D. Normand-Cyrot. On regulation under sampling. *IEEE Trans. Automat. Contr.*, 42:864–868, 1997.
  - [18] T. Chen and B. A. Francis. Input-output stability of sampled-data systems. *IEEE Trans. Automat. Contr.*, 36:50–58, 1991.
  - [19] T. Chen and B. A. Francis. *Optimal Sampled-Data Control Systems*. Springer-Verlag, London, 1995.



- 
- [20] F. H. Clarke, Y. S. Ledyaev, E. D. Sontag, and A. I. Subbotin. Asymptotic controllability implies feedback stabilization. *IEEE Trans. Automat. Contr.*, 42:1394–1407, 1997.
- [21] A. M. Dabroom and H. K. Khalil. Discrete-time implementation of high-gain observers for numerical differentiation. *Int. Journal of Control*, 17:1523–1537, 1999.
- [22] A. M. de Paor and M. O'Malley. A describing function technique for sampling period selection and controller discretization. *Trans. Inst. Meas. Control*, 15:207–212, 1993.
- [23] D. Dochain and G. Bastin. Adaptive identification and control algorithms for nonlinear bacterial growth systems. *Automatica*, 20:621–634, 1984.
- [24] F. Esfandiari and H. K. Khalil. On the robustness of sampled-data control to unmodelled high frequency dynamics. *IEEE Trans. Auto. Contr.*, 34:900–903, 1989.
- [25] I. Fantoni and R. Lozano. Control of nonlinear mechanical systems. *European Journal of Control*, 7:328–348, 2001.
- [26] M. Farza, S. Othman, H. Hammouri, and M. Fick. Discrete-time nonlinear observer-based estimators for the on-line estimation of the kinetic rates in bioreactors. *Bioprocess Engineering*, 17:247–255, 1997.
- [27] R. Ferretti. Higher order approximations of linear control systems via Runge-Kutta schemes. *Computing*, 58:351–364, 1997.
- [28] T. Fliegner. *Contributions to The Control of Nonlinear Discrete-Time Systems*. PhD Thesis, Department of Applied Mathematics, University of Twente, 1995.
- [29] B. A. Francis and T. T. Georgiou. Stability theory for linear time-invariant plants with periodic digital controller. *IEEE Trans. Auto. Contr.*, 33:820–832, 1988.

- 
- [30] G. F. Franklin, J. D. Powel, and M. Workman. *Digital Control of Dynamic Systems, 3rd Ed.* Addison-Wesley, 1997.
  - [31] R. A. Freeman and P. V. Kokotović. *Robust Nonlinear Control Design, State-Space and Lyapunov Techniques.* Birkhauser, Boston, 1996.
  - [32] S. T. Glad. Output dead-beat control for nonlinear systems with one zero at infinity. *Syst. Contr. Letter*, 9:249–255, 1987.
  - [33] G. C. Goodwin, B. McInnis, and R. S. Long. Adaptive control algorithm for waste water treatment and pH neutralization. *Optimal Contr. Applic. Meth.*, 3:443–459, 1982.
  - [34] T. J. Gordon, C. Marsh, and M. G. Milsted. A comparison of adaptive LQG and nonlinear controllers for vehicle suspension systems. *Vehicle System Dynamics*, 20:321–340, 1991.
  - [35] J. W. Grizzle and J. M. Kang. Discrete-time control design with positive semidefinite Lyapunov functions. *Syst. Contr. Lett.*, 43:287–292, 2001.
  - [36] L. Grüne. Discrete feedback stabilization of semilinear control systems. *ESAIM: Control, Optim. Calc. Variations*, 1:207–224, 1996.
  - [37] L. Grüne. Asymptotic controllability and exponential stabilization of nonlinear control systems at singular points. *SIAM J. Contr. Optimiz.*, 36:1485–1503, 1998.
  - [38] L. Grüne. Stabilization by sampled and discrete feedback with positive sampling rate. In F. Lamnabhi-Lagarigue, D. Aeyels, and A. van der Schaft, editors, *Stability and Stabilization of Nonlinear Systems (NCN), Lecture Notes in Control and Information Sciences*, volume 246, pages 165–182. Springer Verlag, London, 1999.
  - [39] L. Grüne. Homogeneous state feedback stabilization of homogeneous systems. *SIAM Journal on Control and Optimization*, 38:1288–1314, 2000.

- 
- [40] L. Grüne. *Asymptotic Behavior of Dynamical and Control Systems under Perturbation and Discretization, Lecture Notes in Mathematics, Vol. 1783*. Springer Verlag, 2002.
  - [41] L. Grüne. Input-to-state dynamical stability and its Lyapunov function characterization. *IEEE Trans. Auto. Contr.*, 47:1499–1504, 2002.
  - [42] L. Grüne and P. E. Kloeden. Higher order numerical schemes for affinely controlled nonlinear systems. *Numerische Mathematik*, 89:669–690, 2001.
  - [43] L. Grüne and P. E. Kloeden. Numerical schemes of higher order for a class of nonlinear control systems. In *Proceedings of the Fifth International Conference on Numerical Methods and Applications - NM&A 02*, Borovets, Bulgaria, 2002.
  - [44] L. Grüne and D. Nešić. Optimization based stabilization of sampled-data nonlinear systems via their approximate discrete-time models. *SIAM Jour. Control and Optim.*, 42:98–122, 2003.
  - [45] A. M. Guillaume, G. Bastin, and G. Campion. Sampled-data adaptive control of a class of continuous nonlinear systems. *Int. Journal of Contr.*, 60:569–594, 1994.
  - [46] W. Hahn. *Stability of Motion*. Springer, 1967.
  - [47] H. Hanselmann. Implementation of digital controllers - A survey. *Automatica*, 23:7–32, 1987.
  - [48] H. Hermes. Homogeneous feedback control for homogeneous systems. *Syst. and Control Letters*, 24:7–11, 1995.
  - [49] D. Hill and P. Moylan. The stability of nonlinear dissipative systems. *IEEE Trans. Automat. Contr.*, 21:708–711, 1976.
  - [50] L. Hou, A. N. Michel, and H. Ye. Some qualitative properties of sampled-data control systems. *IEEE Trans. Automat. Contr.*, 42:1721–1725, 1997.

- [51] B. Hu and A. N. Michel. Some qualitative properties of multirate digital control systems. *IEEE Trans. Automat. Contr.*, 44:765–770, 1999.
- [52] A. R. Humphries and A. M. Stuart. Runge-Kutta methods for dissipative and gradient dynamical systems. *SIAM Jour. of Num. Analysis*, 31:1452–1485, 1994.
- [53] A. Iggidr, B. Kalitine, and R. Outbib. Semidefinite Lyapunov functions stability and stabilization. *Math. Contr. Sig. Syst.*, 9:95–106, 1996.
- [54] B. Ingalls and E. D. Sontag. A small-gain theorem with applications to input/output systems, incremental stability, detectability and interconnections. *Jour. Franklin Inst.*, 339:211–229, 2002.
- [55] B. P. Ingalls. *Comparisons of Notions of Stability for Nonlinear Control Systems with Outputs*. PhD Thesis, Rutgers, The State University of New Jersey, 2001.
- [56] A. Isidori. *Nonlinear Control Systems*. Springer, 1995.
- [57] A. Isidori. *Nonlinear Control Systems II*. Springer, 1999.
- [58] Z. P. Jiang and I. M. Y. Mareels. A small-gain control method for cascaded nonlinear systems with dynamic uncertainties. *IEEE Trans. Automat. Contr.*, 42:292–308, 1997.
- [59] Z. P. Jiang, I. M. Y. Mareels, and Y. Wang. A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems. *Automatica*, 32:1211–1215, 1996.
- [60] Z. P. Jiang, A. R. Teel, and L. Praly. Small-gain theorem for ISS systems and applications. *Math. of Control, Signals and Systems*, 7:95–120, 1994.
- [61] Z. P. Jiang and Y. Wang. Input-to-state stability for discrete-time nonlinear systems. *Automatica*, 37:857–869, 2001.
- [62] M. Jovanović. Nonlinear control of an electrohydraulic velocity servosystem. In *Proc. 26th IEEE American Control Conference*, pages 588–593, 2002.

- 
- [63] N. Karlsson, A. Teel, and D. Hrovat. A backstepping approach to control of active suspensions. In *Proc. IEEE Conf. Decis. Contr.*, pages 4170–4175, Orlando, Florida, 2001.
- [64] P. Katz. *Digital Control using Microprocessors*. Prentice Hall, 1981.
- [65] D. Kazakos and J. Tsinias. Stabilization of nonlinear discrete-time systems using state detection. *IEEE Trans. Auto. Contr.*, 38:1398–1400, 1993.
- [66] D. Kazakos and J. Tsinias. The input to state stability condition and global stabilization of discrete-time systems. *IEEE Trans. Auto. Contr.*, 39:2111–2113, 1994.
- [67] S. S. Keerthi and E. G. Gilbert. Optimal infinite-horizon feedback laws for a general class of constrained discrete-time systems: Stability and moving-horizon approximations. *Journal of Optimization Theory and Its Applications*, 57:265–293, 1988.
- [68] J. P. Keller and B. D. O. Anderson. A new approach to the discretization of continuous-time controller. *IEEE Trans. Automat. Contr.*, 37:214–223, 1992.
- [69] C. M. Kellet. *Advances in Converse and Control Lyapunov Function*, PhD Thesis. The University of California, Santa Barbara, 2002.
- [70] H. K. Khalil. *Nonlinear Control Systems 2nd Ed.* Prentice Hall, 1996.
- [71] P. E. Kloeden and E. Platen. *Numerical Solution of Stochastic Differential Equations*. Springer Verlag, 1992.
- [72] P. V. Kokotović and M. Arcak. Constructive nonlinear control: a historical perspective. *Automatica*, 37:637–662, 2001.
- [73] G. Kreisselmeier and T. Birkhölzer. Nonlinear numerical regulator design. *IEEE Trans. Automat. Contr.*, 39:33–46, 1994.
- [74] M. Krstić, I. Kanellakopoulos, and P. V. Kokotović. *Nonlinear and Adaptive Control Design*. Wiley, 1995.

- [75] M. Krstić and P. V. Kokotović. Lean backstepping design for a jet engine compressor model. In *Proc. 4th IEEE Conf. on Control Appl.*, pages 1047–1052, 1995.
- [76] D. S. Laila. Integrated design of discrete-time controller for an active suspension system. In *Proc. 42th IEEE Conference on Decision and Control*, Maui, Hawaii, 2003.
- [77] D. S. Laila and D. Nešić. On preservation of dissipation inequalities under sampling: the dynamic feedback case. In *Proc. Amer. Control Conf.*, volume 4, pages 2822–2827, Arlington, Virginia, 2001.
- [78] D. S. Laila and D. Nešić. Changing supply rates for input-output to state stable discrete-time systems. In *Proc. IFAC World Congress*, volume 1, pages 480–481, Barcelona, Spain, 2002.
- [79] D. S. Laila and D. Nešić. Lyapunov based small-gain theorem for parameterized discrete-time interconnected ISS systems. In *Proc. 41th IEEE Conference on Decision and Control*, volume 2, pages 2292–2297, Nevada, 2002.
- [80] D. S. Laila and D. Nešić. Changing supply rates for input-output to state stable discrete-time nonlinear systems with applications. *Automatica*, 39:821–835, 2003.
- [81] D. S. Laila and D. Nešić. Lyapunov based small-gain theorem for parameterized discrete-time interconnected ISS systems. *IEEE Trans. on Automatic Control*, 48:1783–1788, 2003.
- [82] D. S. Laila, D. Nešić, and A. R. Teel. Open and closed loop dissipation inequalities under sampling and controller emulation. *European Journal of Control*, 8:109–125, 2002.
- [83] D. A. Lawrence. Stability analysis of nonlinear sampled-data systems. In *Proc. 36th IEEE Conference on Decision and Control*, pages 365–366, San Diego, California, 1997.

- 
- [84] D. A. Lawrence. Stability property of nonlinear sampled-data systems with slowly varying inputs. *IEEE Trans. on Automatic Control*, 45:592–596, 2000.
  - [85] H. G. Lee, A. Araposthatis, and S. I. Marcus. On the digital control of nonlinear systems. In *Proc. IEEE Conf. Decis. Contr.*, volume 1, pages 480–481, 1988.
  - [86] J. S. Lin and I. Kanellakopoulos. Nonlinear design of active suspensions. *IEEE Control Systems Magazine*, June:45–59, 1997.
  - [87] Y. Lin, E. D. Sontag, and Y. Wang. A smooth converse Lyapunov theorem for robust stability. *SIAM J. Contr. Opt.*, 34:124–160, 1996.
  - [88] A. Loria, R. Kelly, R. Ortega, and V. Santibaez. On output feedback control of euler-lagrange systems with bounded inputs. *IEEE Trans. on Automatic Control*, 42:1138–1142, 1997.
  - [89] A. Loria and D. Nešić. On uniform boundedness of parameterized discrete-time cascades with decaying inputs: applications to cascades. *Systems and Control Letters*, 49:163–174, 2003.
  - [90] A. Loria and H. Nijmeijer. Output feedback tracking control of euler-lagrange systems via bounded controls. *Systems and Control Letters*, 33:151–161, 1997.
  - [91] R. Lozano, B. Brogliato, O. Egeland, and B. Maschke. *Dissipative Systems Analysis and Control*. Springer Verlag, London, 2000.
  - [92] I. M. Y. Mareels, H. B. Penfold, and R. J. Evans. Controlling nonlinear time-varying systems via Euler approximations. *Automatica*, 28:681–696, 1992.
  - [93] D. Q. Mayne and H. Michalska. Receding horizon control of nonlinear systems. *IEEE Trans. on Automatic Control*, 35:814–824, 1990.
  - [94] F. E. McCaughan. Bifurcation analysis of axial flow compressor stability. *SIAM Journal on Applied Mathematics*, 50:1232–1253, 1990.
  - [95] S. Monaco and D. Normand-Cyrot. On the sampling of a linear analytic control system. In *Proc. IEEE Conf. Decis. Contr.*, pages 1457–1462, Fort Lauderdale, 1985.

- 
- [96] S. Monaco and D. Normand-Cyrot. Nonlinear systems in discrete-time. In M. Fliess and M. Hazewinkel, editors, *Algebraic and Geometric Methods in Nonlinear Control Theory*, pages 411–430. D. Reidel Publishing Company, 1986.
  - [97] S. Monaco and D. Normand-Cyrot. Minimum phase nonlinear discrete-time systems and feedback stabilization. In *Proc. 26th IEEE Conf. Decis. Contr.*, pages 979–986, 1987.
  - [98] S. Monaco and D. Normand-Cyrot. On nonlinear digital control. In A.J. Fossard and D. Normand-Cyrot, editors, *Nonlinear Systems*, volume 3. Chapman & Hall, 1997.
  - [99] S. Monaco and D. Normand-Cyrot. On the conditions of passivity and losslessness in discrete-time. In *Proc. European Control Conference*, Brussels, 1997.
  - [100] S. Monaco and D. Normand-Cyrot. Issue on nonlinear digital control. *European Journal of Control*, 7:160–177, 2001.
  - [101] A. P. Morgan and K. S. Narendra. On the stability of nonautonomous differential equations  $\dot{x} = [a + b(t)]x$  with skew-symmetric matrix  $b(t)$ . *SIAM J. on Contr. and Opt.*, 15:163–176, 1977.
  - [102] M. S. Mousa, R. K. Miller, and A. N. Michel. Stability analysis of hybrid composite dynamical systems: descriptions involving operators and difference equations. *IEEE Trans. Automat. Contr.*, 31:603–615, 1986.
  - [103] D. S. Naidu and A. K. Rao. *Singular Perturbation Analysis of Discrete Control Systems*. Springer Verlag, New York, 1985.
  - [104] D. Nešić and D. Angeli. Integral versions of ISS for sampled-data nonlinear systems via their approximate discrete-time models. *IEEE Trans. Auto. Contr.*, 47:2033–2037, 2002.
  - [105] D. Nešić and D. S. Laila. Input-to-state stabilization for nonlinear sample-data systems via approximate discrete-time plant models. In *Proc. 40th IEEE Conf. Decis. Contr.*, pages 887–892, Orlando, FL, 2001.



- 
- [106] D. Nešić and D. S. Laila. A note on input-to-state stabilization for nonlinear sampled-data systems. *IEEE Trans. Auto. Contr.*, 47:1153–1158, 2002.
  - [107] D. Nešić, D. S. Laila, and A. R. Teel. On preservation of dissipation inequalities under sampling. In *Proc. 39th IEEE Conf. Decis. Contr.*, pages 2472–2477, Sydney, Australia, 2000.
  - [108] D. Nešić and A. Loria. On uniform asymptotic stability of time-varying parameterized discrete-time cascades. *IEEE Trans. Auto. Contr.*, submitted, 2002.
  - [109] D. Nešić and A. R. Teel. Set stabilization of sampled-data nonlinear differential inclusions via their approximate discrete-time models. In *Proc. 39th IEEE Conf. Decis. Contr.*, pages 2112–2117, Sydney, Australia, 2000.
  - [110] D. Nešić and A. R. Teel. Backstepping on the Euler approximate model for stabilization of sampled-data nonlinear systems. In *Proc. IEEE Conf. Decis. Contr.*, pages 1737–1742, Orlando, FL, 2001.
  - [111] D. Nešić and A. R. Teel. Changing supply functions in input to state stable systems: The discrete-time case. *IEEE Trans. Auto. Contr.*, 46:960–962, 2001.
  - [112] D. Nešić and A. R. Teel. Input-to-state stability for nonlinear time-varying systems via averaging. *Math. Control, Signals and Systems*, 14:257–280, 2001.
  - [113] D. Nešić and A. R. Teel. Stabilization of sampled-data nonlinear systems via backstepping on their Euler approximate model. *Automatica*, submitted, 2002.
  - [114] D. Nešić and A. R. Teel. A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models. *IEEE Trans. Auto. Contr.*, accepted, 2003.
  - [115] D. Nešić and A. R. Teel. Matrosov theorem for parameterized families of discrete-time systems. *Automatica*, accepted, 2003.
  - [116] D. Nešić, A. R. Teel, and P. Kokotović. Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations. *Syst. Contr. Lett.*, 38:259–270, 1999.

- [117] D. Nešić, A. R. Teel, and E. Sontag. Formulas relating  $\mathcal{KL}$  stability estimates of discrete-time and sampled-data nonlinear systems. *Syst. Contr. Lett.*, 38:49–60, 1999.
- [118] Semiconductor Business News. Intel rools out world'd fastest 2-GHz processor for notebooks. *Silicon Strategies*, 06/23/02 03:13 PM EST, taken 06/01/03 08:23 PM MEL:<http://www.siliconstrategies.com/story/OEG20020623S0001>, 2002.
- [119] K. Ogata. *Discrete-Time Control Systems, 1st Ed.* PHI, New Jersey, 1987.
- [120] K. Ogata. *Modern Control Engineering, 4th Ed.* PHI, New Jersey, 2001.
- [121] T. Ohtsuka and H. A. Fujii. Receding-horizon control of a space vehicle model using established continuation method. In *Proc. 3rd ISAC Workshop on Astrodynamics and Flight Mechanics*, pages 180–187, Institute of Space and Astronautical Science, Sagamihara, Japan, 1993.
- [122] T. Ohtsuka and H. A. Fujii. Real-time optimization algorithm for nonlinear receding-horizon control. *Automatica*, 33:1147–1154, 1997.
- [123] R. Ortega, A. Loria, P. J. Nicklasson, and H. Sira-Ramirez. *Passivity Based Control of Euler-Lagrange Systems*. Springer, London, 1998.
- [124] D. H. Owens, Y. Zheng, and S. A. Billings. Fast sampling and stability of nonlinear sampled-data systems: Part 1. Existence theorems. *IMA J. Math. Contr. Informat.*, 7:1–11, 1990.
- [125] J. Peuteman, D. Aeyels, and J. Soenen. On the investigation of dissipativity by a discrete observation of the storage function. In *Proc. 37th IEEE Conf. Decis. Contr*, pages 4150–4155, Tampa, FL, 1998.
- [126] D. A. Pierre and J. W. Pierre. Digital controller design - Alternative emulation approaches. *ISA Transaction*, 34:219–228, 1995.
- [127] L. Praly and Y. Wang. Stabilization in spite of matched unmodeled dynamics and an equivalent definition of input to state stability. *Math. of Control, Signals and Systems*, 9:1–33, 1996.

- 
- [128] Z. Qu. *Robust control of nonlinear uncertain systems*. John Wiley & Sons, New York, 1998.
- [129] N. Rouche, P. Habets, and M. Laloy. *Stability Theory by Lyapunov's Direct Method*. Springer, London, 1977.
- [130] J. La Salle. *The Stability and Control of Discrete Processes, Applied Mathematical Sciences vol. 62*. Springer Verlag, 1986.
- [131] J. La Salle and S. Lefschetz. *Stability by Lyapunov's Direct Method with Applications, Mathematics in Science and Engineering vol. 4*. Academic Press, 1961.
- [132] S. Sastry. *Nonlinear Systems. Analysis, Stability and Control*. Springer, 1999.
- [133] R. Sepulchre, M. Jancović, and P. V. Kokotović. *Constructive Nonlinear Control*. Springer, London, 1997.
- [134] K. Shillcutt. Patterned search planning and testing for the robotic antarctic meteorite search. In *Proc. International Topical Meeting on Robotics and Remote Systems for the Nuclear Industry, American Nuclear Society*, Pittsburgh, PA, 1999.
- [135] C. Simoes. *On Stabilization of Discrete-Time Nonlinear Systems, PhD Thesis*. Department of Applied Mathematics, University of Twente, 1996.
- [136] E. Skafidas, A. Fradkov, R. J. Evans, and I. M. Y. Mareels. Trajectory based adaptive control for nonlinear systems under matching conditions. *Automatica*, 34:287–299, 1998.
- [137] J. J. E. Slotine and W. Li. Adaptive manipulator control: A case study. *IEEE Trans. Auto. Contr.*, 33:995–1003, 1988.
- [138] J. J. E. Slotine and W. Li. *Applied Nonlinear Control*. Prentice-Hall, New York, 1991.
- [139] E. D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Trans. Auto. Contr.*, 34:435–443, 1989.

- 
- [140] E. D. Sontag. *Mathematical Control Theory*. Springer, 1998.
- [141] E. D. Sontag. The ISS philosophy as a unifying framework for stability-like behaviour. In A. Isidori, F. Lamnabhi-Lagarigue, and W. Respondek, editors, *Nonlinear Control in the Year 2000, Lecture Notes in Control and Information Sciences*, volume 2, pages 443–468. Springer, Berlin, 2000.
- [142] E. D. Sontag and A. R. Teel. Changing supply functions in input to state stable systems. *IEEE Trans. Auto. Contr.*, 40:1476–1478, 1995.
- [143] E. D. Sontag and Y. Wang. On characterizations of the input-to-state stability property. *System and Control Letters*, 24:351–359, 1995.
- [144] E. D. Sontag and Y. Wang. On characterizations of the input-to-state stability with respect to compact sets. In *Proceedings of IFAC Non-Linear Control Systems Design Symposium, (NOLCOS '95)*, pages 226–231, Tahoe City, CA, 1995.
- [145] E. D. Sontag and Y. Wang. Detectability of nonlinear systems. In *Proc. Conf. on Information Science and Systems (CISS'96)*, pages 1031–1036, Princeton, NJ, 1996.
- [146] E. D. Sontag and Y. Wang. Output to state stability and detectability of nonlinear systems. *System and Control Letters*, 29:279–290, 1997.
- [147] E. D. Sontag and Y. Wang. Lyapunov characterization of input to output stability. *SIAM J. Control and Optim.*, 39:226–249, 2001.
- [148] M. W. Spong, P. Corke, and R. Lozano. Nonlinear control of the reaction wheel pendulum. *Automatica*, 37:1845–1851, 2001.
- [149] A. M. Stuart. Numerical analysis of dynamical systems. *Acta Numerica*, pages 467–572, 1994.
- [150] A. M. Stuart and A. R. Humphries. Model problems in numerical stability theory for initial value problems. *SIAM Review*, 36:226–257, 1994.

- 
- [151] A. M. Stuart and A. R. Humphries. *Dynamical Systems and Numerical Analysis*. Cambridge Univ. Press, New York, 1996.
- [152] A. R. Teel. A nonlinear small gain theorem for the analysis of control systems with saturation. *IEEE Trans. Auto. Contr.*, 41:1256–1270, 1996.
- [153] A. R. Teel, L. Moreau, and D. Nešić. A unified framework for input-to-state stability in systems with two time scales. *IEEE Trans. Auto. Contr.*, 48:1526–1544, 2003.
- [154] A. R. Teel, D. Nešić, and P. Kokotović. A note on input-to-state stability of nonlinear sampled-data systems. In *Proc. IEEE Conf. Decis. Contr.*, pages 2473–2478, Tampa, FL, 1998.
- [155] M. P. Tzamtzi and S. G. Tzafestas. A small gain theorem for locally input to state stable interconnected systems. *Journal of The Franklin Institute*, 336:893–901, 1999.
- [156] V. Veliov. On the time-discretization of control systems. *SIAM Journal of Control and Optimization*, 35:1470–1486, 1997.
- [157] G. D. Warshaw and H. M. Schwartz. Compensation of sampled-data robot adaptive controllers for improved stability. In *Proc. 1993 Canadian Conference on Electrical and Computer Engineering*, pages 845–850, Vancouver, British Columbia, 1993.
- [158] G. D. Warshaw and H. M. Schwartz. Sampled-data robot adaptive control with stabilizing compensation. *The International Journal of Robotic Research*, 15:78–91, 1996.
- [159] G. D. Warshaw, H. M. Schwartz, and H. Asmer. Stability and performance of sampled-data robot adaptive controllers. In *Proc. 1992 IEEE/RSJ International Conference on Intelligent Robots and Systems*, pages 95–104, Raleigh, NC, 1992.
- [160] J. C. Willems. Dissipative dynamical systems part I, part II. *Arch. Ration. Mech. Anal.*, 45:325–393, 1972.

- [161] T. H. Yang and E. Polak. Moving horizon control of nonlinear systems with input saturation, disturbances and plant uncertainty. *Int. Jour. of Control*, 58:875–903, 1993.
- [162] L. Zaccarian, A. R. Teel, and D. Nešić. On finite gain  $L_p$  stability of nonlinear sampled-data systems. *Systems and Control Letters*, 49:201–212, 2003.
- [163] Y. Zheng, D. H. Owens, and S. A. Billings. Fast sampling and stability of nonlinear sampled-data systems: Part 2. Sampling rate estimation. *IMA J. Math. Contr. Informat.*, 7:13–33, 1990.
- [164] Y. Zheng, D. H. Owens, and S. A. Billings. Slow sampling and stability of nonlinear sampled-data systems. *Int. J. Control*, 51:251–265, 1990.

# Index

- approximation
  - higher order, 49
  - Euler, 48, 91, 188
- backstepping, 89
  - continuous-time, 90
  - discrete-time, 91
- bounding function, 141
- cascade-connected systems, 151
- consistency, 47, 48, 59
  - one-step strong, 60, 123
  - one-step weak, 59, 123
- control
  - digital, 11
- control Lyapunov function (clf), 91
- controllers
  - dynamic feedback, 121
- difference equation
  - functional, 120
  - ordinary, 120
- direct discrete-time design, 12, 18
- discrete-time model
  - approximate, 120
  - exact, 120
- dissipation inequality
  - strong form, 64
  - weak form, 64
  - differential, 57
  - integral, 57
- disturbance
  - measurable, 54, 119
  - uniformly Lipschitz, 54, 60
- domain of attraction, 44, 119
- emulation, 57
- emulation design, 12
- energy
  - kinetic, 190, 191
  - potential, 190, 191
- finite escape time, 59, 121
- Gronwall-Bellman Inequality, 37
- inertia matrix, 190
- initial value problem, 119
- input
  - Lebesgue measurable, 47
  - piecewise constant, 46
- input-output to state stability, 139
  - semiglobal practical, 140
- input-to-state stability, 80
  - quasi, 149

- semiglobal practical, 122, 169
- semiglobal practical derivative, 122
- continuous-time, 39
- discrete-time
  - nonparameterized systems, 41
  - parameterized systems, 42
- Lagrangian control, 188
- LaSalle's invariance principle, 142
- Lebesgue measurable, 56
- Lipschitz, 55
- Lyapunov
  - nonlinear system, 40, 42
  - SP-ISS, 44
  - partial construction, 139
- Mean Value Theorem, 37
- measuring function, 141
- MIMO, 188
- multibody systems, 191
- numerical integration, 131
- observer, 155
- passivity, 84
- PD controller, 194
- sampled-data control system, 2
- sampled-data design, 23
- sampled-data system, 1
- semiglobal practical
  - asymptotically stable, 89
- simple mechanical systems, 190
- small gain, 167
  - theorem, 173
- stability
  - global, 43
  - semiglobal practical, 43
- strictly proper stable filter, 66
- supply function, 141
- supply rate, 139
- uniformly locally bounded, 122
- zero-order-hold, 58