Boundary Set-Point Regulation of a Linear 2×2 Hyperbolic PDE with Uncertain Bilinear Boundary Condition

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***Abstract*— We design an adaptive controller for a 2×2 linear hyperbolic PDE with uncertain boundary parameters where measurements are taken at both boundaries, while only one boundary is actuated. The uncertainty, which appears in the un-actuated boundary, is in a bilinear form, which allows us to use bilinear adaptive laws for which parameter estimate convergence is guaranteed under a simple, a priori verifiable, persistence of excitation criterion. The control objective is boundary set-point regulation where the set-point is dependent on one of the unknown parameters. Stability is proved in the L_2-sense and all signals in the closed loop are shown to be bounded.**

I. Introduction

Linear 2×2 hyperbolic PDEs can be used to describe a wide range of physical systems ranging from road traffic to transmission lines [1], [2]. Particularly important for the problem studied in this paper is an application in offshore oil drilling [3], [4] where the goal is to control the bottom-hole pressure to a set-point by actuating the top-side flow only. The problem is complicated by the fact that the pressure in the surrounding oil reservoir and the production index governing the reservoir flow as a function of down-hole pressure are unknown. To avoid any sudden in- or out-flow, the bottom-hole pressure must match the uncertain reservoir pressure. In this paper, we solve this problem by first estimating the unknown parameters and then designing a tracking controller that achieves set-point regulation to a set-point specified by the parameter estimates. Parameter estimate convergence then guarantees that the control objective is achieved.

We use the *backstepping approach* to derive a controller and prove closed loop stability. Backstepping for PDEs, in its current form using invertible Volterra integral transformations, was first introduced for parabolic PDEs [5], [6] and extended to a 2×2 system of first order hyperbolic PDEs in [7]. The first result on adaptive control for hyperbolic PDEs was in [8] with extension to the 2×2 case in [9]. Particularly important to the work in this paper is [10], where a reference model was used to design a closed loop controller tracking a time-varying signal in a 2×2 hyperbolic system with unknown harmonic disturbances. The method in the present paper uses a similar reference model together with the state and parameter scheme from [11] to design a closed loop controller. The main contribution of the paper is the stability analysis proving closed loop stability in the L_2-sense, parameter convergence and boundary set-point regulation. This analysis is novel and non-trivial due to the fact that the set-point is unknown a priori. While two parameters are uncertain, only one parameter needs to converge to achieve the regulation objective. Therefore, the bilinear form of the uncertainty in the un-actuated boundary is exploited, setting the stage for applying adaptive laws based on the bilinear parametric model.

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A. Notation

For a signal $z : [0,1] \times [0,\infty) \to \mathbb{R}$, the L_2-norm is denoted $||z|| := \sqrt{\int_0^1 z^2(x,t)dx}$. For $f : [0,\infty) \to \mathbb{R}$, we use the vector spaces $f \in \mathscr{L}_p \leftrightarrow (\int_0^\infty |f(t)|^p dt)^{\frac{1}{p}} < \infty$ for $p \geq 1$ with the particular case $f \in \mathscr{L}_\infty \leftrightarrow \sup_{t\geq 0} |f(t)| < \infty$. Let $\mathscr{B}([0,1]) = \{f(x) : \sup_{x\in[0,1]} f(x) < \infty\}$. The projection operator Proj_a is defined to be $\text{Proj}_a(\tau,\omega) = 0$ for $\omega = a$ and $\tau \leq 0$, and $\text{Proj}_a(\tau,\omega) = \tau$ otherwise.

B. Problem statement

Consider the linear 2×2 hyperbolic system

$$u_t(x,t) + \lambda u_x(x,t) = c_1(x)v(x,t) \tag{1a}$$
$$v_t(x,t) - \mu v_x(x,t) = c_2(x)u(x,t) \tag{1b}$$
$$u(0,t) = rv(0,t) + k(\theta - y_0(t)) \tag{1c}$$
$$v(1,t) = U(t) \tag{1d}$$

where $u, v : [0,1] \times [0,\infty) \to \mathbb{R}$ are the system states, the parameters $\lambda, \mu > 0$, $r \in \mathbb{R}$ and $c_1(x), c_2(x) \in C([0,1])$ are known, while $k \in [\underline{k},\infty)$ and $\theta \in \mathbb{R}$ are unknown boundary parameters where $\underline{k} = \max(0,-1/a_0) + k_0$ for some $k_0 > 0$ which is needed to ensure well-possessedness of (1c) and identifiability of θ.

Measurements are taken at both boundaries. At the actuated boundary, we have $y_1(t) = u(1,t)$, while the measurement at the un-actuated boundary is given as the linear combination $y_0(t) = a_0 u(0,t) + b_0 v(0,t)$ where a_0 and b_0 are known constants with $a_0 \neq 0$ and $a_0 r + b_0 \neq 0$. The objective is to design a control input $U(t)$ so that system (1) is adaptively stabilized (all signals are bounded) in the L_2-sense and such that the objective

$$\lim_{t\to\infty} \int_t^{t+t_\delta} |\theta - y_0(\tau)| d\tau = 0 \tag{2}$$

is achieved for any $t_\delta > 0$. Furthermore, the parameter estimate $\hat{\theta}(t)$ should converge to θ asymptotically. It is assumed that the initial conditions $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$ satisfy u_0, $v_0 \in \mathscr{B}([0,1])$, in which case it can be shown (for $U(t)$ in the form used here) that (1) has a unique solution in $\mathscr{B}([0,1])$ for all $t \geq 0$.

C. Backstepping operators

Two sets of operators will be used to derive the adaptive law and control law. First, the operators $\mathscr{P}_1, \mathscr{P}_2 : \mathscr{B}([0,1]) \times \mathscr{B}([0,1]) \to \mathscr{B}([0,1])$ are given by

$$\mathscr{P}_1[a,b](x) := a(x) + \int_x^1 P^{uu}(x,\xi)a(\xi)d\xi + \int_x^1 P^{uv}(x,\xi)b(\xi)d\xi \tag{3a}$$

$$\mathscr{P}_2[a,b](x) := b(x) + \int_x^1 P^{vu}(x,\xi)a(\xi)d\xi + \int_x^1 P^{vv}(x,\xi)b(\xi)d\xi \tag{3b}$$

where the kernel $(P^{uu}, P^{uv}, P^{vu}, P^{vv})$, defined over $\mathscr{T}_1 = \{(x,\xi)\,|\,0 \le x \le \xi \le 1\}$, is the unique, bounded and continuous solution to a set of 4×4 hyperbolic equations stated in [7, Eq. 67-74] and proven in [12, Appendix A]. Next, the operators $\mathscr{K}_1, \mathscr{K}_2 : \mathscr{B}([0,1]) \times \mathscr{B}([0,1]) \to \mathscr{B}([0,1])$ are given by

$$\mathscr{K}_1[a,b](x) := a(x) - \int_0^x K^{uu}(x,\xi)a(\xi)d\xi - \int_0^x K^{uv}(x,\xi)b(\xi)d\xi \tag{4a}$$

$$\mathscr{K}_2[a,b](x) := b(x) - \int_0^x K^{vu}(x,\xi)a(\xi)d\xi - \int_0^x K^{vv}(x,\xi)b(\xi)d\xi \tag{4b}$$

where the kernel $(K^{uu}, K^{uv}, K^{vu}, K^{vv})$, defined over $\mathscr{T}_2 = \{(x,\xi)\,|\,0 \le \xi \le x \le 1\}$, is the unique, bounded and continuous solution to a set of 4×4 hyperbolic equations stated in [12, Eq. 3.30-3.37] and also proven in [12, Appendix A]. Furthermore, it is shown in [12] that the operators (3) and (4) are invertible.

II. State and parameter estimation

To generate state and parameter estimates, we will use a swapping-based design where a set of filters are used to find a parametric model relating the measured signals to the unknown parameters. The filters, parametric model and adaptive law are the same as in [11] and are only included here for completeness. The following parametric model is used

$$u(x,t) = a(x,t) + k\left(\theta m(x,t) + w(x,t)\right) + e(x,t) \tag{5a}$$

$$v(x,t) = b(x,t) + k\left(\theta n(x,t) + z(x,t)\right) + \varepsilon(x,t) \tag{5b}$$

where (a,b), (m,n) and (w,z) are filters given by

$$a_t(x,t) + \lambda a_x(x,t) = c_1(x)b(x,t) + P_1(x)(y_1(t) - a(1,t)) \tag{6a}$$

$$b_t(x,t) - \mu b_x(x,t) = c_2(x)a(x,t) + P_2(x)(y_1(t) - a(1,t)) \tag{6b}$$

$$a(0,t) = rb(0,t) \tag{6c}$$

$$b(1,t) = U(t), \tag{6d}$$

$$m_t(x,t) + \lambda m_x(x,t) = c_1(x)n(x,t) - P_1(x)m(1,t) \tag{7a}$$

$$n_t(x,t) - \mu n_x(x,t) = c_2(x)m(x,t) - P_2(x)m(1,t) \tag{7b}$$

$$m(0,t) = rn(0,t) + 1 \tag{7c}$$

$$n(1,t) = 0 \tag{7d}$$

and

$$w_t(x,t) + \lambda w_x(x,t) = c_1(x)z(x,t) - P_1(x)w(1,t) \tag{8a}$$

$$z_t(x,t) - \mu z_x(x,t) = c_2(x)w(x,t) - P_2(x)w(1,t) \tag{8b}$$

$$w(0,t) = rz(0,t) - y_0(t) \tag{8c}$$

$$z(1,t) = 0 \tag{8d}$$

with initial conditions in $\mathscr{B}([0,1])$. It can be shown, using the backstepping transformation $\alpha(x,t) = \mathscr{P}_1[e,\varepsilon](x,t)$ and $\beta(x,t) = \mathscr{P}_2[e,\varepsilon](x,t)$ and selecting the injection terms as $P_1(x) = \lambda P^{uu}(x,1)$ and $P_2(x) = \lambda P^{vu}(x,1)$, that the dynamics of (α,β) is given by simple cascaded transport equations with a zero boundary condition. From the invertibility of the backstepping operator, it follows that $e(x,t) = \varepsilon(x,t) = 0$ for all $t \ge t_F = \lambda^{-1} + \mu^{-1}$. Replacing θ, k in (5) with their respective estimates $\hat{\theta}, \hat{k}$ gives the adaptive state estimates

$$\hat{u}(x,t) = a(x,t) + \hat{k}(t)\left(\hat{\theta}(t)m(x,t) + w(x,t)\right) \tag{9a}$$

$$\hat{v}(x,t) = b(x,t) + \hat{k}(t)\left(\hat{\theta}(t)n(x,t) + z(x,t)\right) \tag{9b}$$

where the errors between the true states and the state estimates are denoted

$$\hat{e}(x,t) = u(x,t) - \hat{u}(x,t) \tag{10a}$$

$$\hat{\varepsilon}(x,t) = v(x,t) - \hat{v}(x,t). \tag{10b}$$

Since the uncertain parameters appear at $x = 0$, we would like to exploit the measurement $y_0(t)$ for parameter estimation. Towards that end, we define

$$\breve{a}(t) := a_0 a(0,t) + b_0 b(0,t) \tag{11a}$$

$$\breve{m}(t) := a_0 m(0,t) + b_0 n(0,t) \tag{11b}$$

$$\breve{w}(t) := a_0 w(0,t) + b_0 z(0,t) \tag{11c}$$

$$\breve{e}(t) := a_0 \hat{e}(0,t) + b_0 \hat{\varepsilon}(0,t) \tag{11d}$$

and substitute (9) evaluated at $x = 0$ into $y_0(t) = a_0 u(0,t) + b_0 v(0,t)$ to obtain the relationship

$$\breve{e}(t) = y_0(t) - \breve{a}(t) - \hat{k}(t)\left(\hat{\theta}(t)\breve{m}(t) + \breve{w}(t)\right). \tag{12}$$

In order to estimate the parameter θ, we will need a persistent excitation condition on $\breve{m}$. It turns out that such a condition can be derived in terms of known system parameters, which is important since it can then be checked a priori.

Lemma 1: If

$$(a_0 r + b_0)\int_0^1 P^{vu}(0,\xi)d\xi + a_0 \ne 0, \tag{13}$$

then $\breve{m}(t)$ converges to a nonzero constant in finite time t_F.

Proof: Using the backstepping transformations $m = \mathscr{P}_1[\bar{m},\bar{n}](x,t)$, $n = \mathscr{P}_2[\bar{m},\bar{n}](x,t)$ with injection terms $P_1(x) = \lambda P^{uu}(x,1)$ and $P_2(x) = \lambda P^{vu}(x,1)$, system (7) is transformed into a simple cascaded transport equation with boundary conditions $\bar{m}(0,t) = r\bar{n}(0,t) + 1$ and $\bar{n}(1,t) = 0$. Thus, for $t \ge t_F$, $\bar{n}(x,t) \equiv 0$ and $\bar{m}(x,t) \equiv 1$, so the backstepping transform gives $n(0,t) = \int_0^1 P^{vu}(0,\xi)d\xi$. Using (7c) and inserting into (11b) gives

$$\check{m}(t) = (a_0 r + b_0)\int_0^1 P^{vu}(0,\xi)d\xi + a_0, \tag{14}$$

which by assumption is nonzero. ■

The gradient method for bilinear parametric models in [13, Theorem 4.52] can be used to minimize a cost function based on the square error $\check{e}^2(t)$ and thereby forming an adaptive law for the parameter estimates $\hat{\theta}, \hat{k}$. Although a lower bound $\underline{k}$ for k is known and the parametric model could be written in a linear form, keeping the bilinear form has some desirable properties regarding parameter convergence.

Lemma 2: Consider the adaptive laws $\dot{\hat{\theta}} = \dot{\hat{k}} = 0$ for $t < t_F$ and

$$\dot{\hat{\theta}}(t) = \gamma_1 \frac{\check{e}(t)}{1+\check{w}^2(t)}\check{m}(t) \tag{15a}$$

$$\dot{\hat{k}}(t) = \mathrm{Proj}_{\underline{k}}\left(\gamma_2\left[\hat{\theta}(t)\check{m}(t) + \check{w}(t)\right]\frac{\check{e}(t)}{1+\check{w}^2(t)}, \hat{k}\right) \tag{15b}$$

for $t \geq t_F$, where $\gamma_1, \gamma_2 > 0$ are the adaptation gains, $\check{m}(t)$ and $\check{w}(t)$ are given in (11), and $\check{e}(t)$ is the adaptive estimation error (11d). Suppose system (1) has a unique solution u, v for all $t \geq 0$, and that the initial estimate satisfies $\hat{k}(0) \geq \underline{k}$. Then, the adaptive laws (15) have the following properties:

1) $\hat{\theta}, \hat{k} \in \mathscr{L}_\infty$.
2) $\hat{k}(t) \geq \underline{k}$ for all $t \geq 0$
3) $\dot{\hat{\theta}}, \dot{\hat{k}} \in \mathscr{L}_2 \cap \mathscr{L}_\infty$.
4) $\frac{\check{e}}{\sqrt{1+\check{w}^2}} \in \mathscr{L}_\infty \cap \mathscr{L}_2$
5) If $\check{w}$ is bounded for *almost all* $t \geq 0$, (13) holds and $\hat{\theta}\check{m} + \check{w} \in \mathscr{L}_2$, then $\hat{\theta}$ converges to θ and $\hat{k}$ converges to some constant.

Proof: The proof of Properties 1 and 3 for the adaptive laws without projection is given in [11, Theorem 3]. Since $\hat{k}(0) \geq \underline{k}$, Property 2 follows trivially from projection. Inserting (5), (9) and (11d) into the adaptive law (15a) yields

$$\dot{\tilde{\theta}}(t) = -f(t)\left(m^2\tilde{\theta}(t) + m\tilde{k}(t)\left(\hat{\theta}(t)\check{m}(t) + \check{w}(t)\right)\right) \tag{16}$$

where $f(t) = \gamma_1/(1+\check{w}^2(t)) > 0$ and $m = \check{m}(t) \neq 0$ for all $t > t_F$. The nonzero constant m exists by Lemma 1. Forming $V_0(t) = \frac{1}{2}\tilde{\theta}^2(t)$, time differentiating and applying Young's inequality to the cross term, we get

$$\begin{aligned}\dot{V}_0 =& -m^2 k f(t)\tilde{\theta}^2(t) - m\tilde{\theta}(t)f(t)\tilde{k}(t)\left(\hat{\theta}(t)\check{m}(t) + \check{w}(t)\right)\\ \leq& -\left(m^2 k - \rho m/2\right)f(t)\tilde{\theta}^2(t)\\ &+ \frac{f(t)m^2\tilde{k}^2(t)}{2\rho}\left(\hat{\theta}(t)\check{m}(t) + \check{w}(t)\right)^2\end{aligned} \tag{17}$$

for some $\rho > 0$. Selecting $\rho = |m|k > 0$ yields

$$\dot{V}_0(t) \leq -\frac{k}{2}f(t)\tilde{\theta}^2(t) + \frac{|m|}{2k}f(t)\tilde{k}^2(t)\left(\hat{\theta}\check{m}(t) + \check{w}(t)\right)^2. \tag{18}$$

Since by assumption $\check{w}$ is bounded for almost all $t \geq 0$, it follows that $\operatorname{ess\,inf}_{t\geq 0} f(t) > 0$, which along with Property 1 and boundedness of $f(t)$, provide the existence of constants b and $c > 0$ such that

$$\dot{V}_0(t) \leq -c\tilde{\theta}^2(t) + g(t)\tilde{\theta}^2(t) + b\left(\hat{\theta}\check{m}(t) + \check{w}(t)\right)^2, \tag{19}$$

where $g(t) = 0$ almost everywhere and therefore $g \in \mathscr{L}_1$. Since $(\hat{\theta}\check{m} + \check{w})^2 \in \mathscr{L}_1$ by assumption, it follows from [14, Lemma D.6] that $V_0 \in \mathscr{L}_1 \cap \mathscr{L}_\infty$ which together with [15, Lemma 2.7] imply $V_0, \tilde{\theta} \to 0$. Convergence in $\hat{k}$ to some constant can be shown similarly by inserting (5), (9) and (11d) into (15b). ■

III. Closed loop control

To motivate our choice of control law, assume for a moment that the states (u, v) and parameters (k, θ) are known. It is possible to show, through a suitable backstepping transformation, that system (1) is equivalent to a set of cascaded transport equations with boundary conditions

$$\omega(0,t) = r\zeta(0,t) + k(\theta - y_0(t)) \tag{20a}$$

$$\begin{aligned}\zeta(1,t) =& U(t) - \int_0^1 K^{vu}(1,\xi)u(\xi)d\xi\\ &- \int_0^1 K^{vv}(1,\xi)v(\xi)d\xi,\end{aligned} \tag{20b}$$

and $y_0(t) = a_0\omega(0,t) + b_0\zeta(0,t)$. Therefore, if $\zeta(0,t) = \theta/(a_0 r + b_0)$, we obtain $\theta - y_0(t) = 0$. Selecting

$$\begin{aligned}U(t) =& \int_0^1 K^{vu}(1,\xi)u(\xi,t)d\xi + \int_0^1 K^{vv}(1,\xi)v(\xi,t)d\xi\\ &+ \frac{1}{a_0 r + b_0}\theta,\end{aligned} \tag{21}$$

we have $\zeta(1,t) = \theta/(a_0 r + b_0)$. Since $\zeta(x,t)$ is the solution to a simple transport equation, $\zeta(0,t) = \theta/(a_0 r + b_0)$ in finite time and the control objective (2) is achieved. Going back to the case with unknown states and parameters, we now propose a control law by applying a *certainty equivalence principle*.

Theorem 1: Suppose (13) holds. Then the control law

$$\begin{aligned}U(t) =& \int_0^1 K^{vu}(1,\xi)\hat{u}(\xi,t)d\xi + \int_0^1 K^{vv}(1,\xi)\hat{v}(\xi,t)d\xi\\ &+ \frac{1}{a_0 r + b_0}\hat{\theta}(t).\end{aligned} \tag{22}$$

in closed loop with system (1), state estimates (9) and adaptive law (15), guarantees (2). Moreover, all signals in the closed loop are bounded, $\hat{\theta}$ converges to θ, and $\hat{k}$ converges to some constant.

The proof of Theorem 1 is the main contribution of this paper, with the formal stability analysis given in Section IV. The remainder of this section is used to further motivate our selection of the control law (22).

Since the states are unknown, we start with the state estimates (9), differentiate with respect to time and space, respectively, and insert the filter dynamics (6)–(8) to arrive at the state estimation dynamics

$$\begin{aligned}\hat{u}_t(x,t) + \lambda\hat{u}_x(x,t) =& c_1(x)\hat{v}(x,t) + \dot{\hat{k}}(t)\left(\hat{\theta}(t)m(x,t) + w(x,t)\right)\\ &+ \hat{k}(t)\dot{\hat{\theta}}(t)m(x,t) + P_1(x)\hat{e}(1,t)\end{aligned} \tag{23a}$$

$$\begin{aligned}\hat{v}_t(x,t) - \mu\hat{v}_x(x,t) =& c_2(x)\hat{u}(x,t) + \dot{\hat{k}}(t)\left(\hat{\theta}(t)n(x,t) + z(x,t)\right)\\ &+ \hat{k}(t)\dot{\hat{\theta}}(t)n(x,t) + P_2(x)\hat{e}(1,t)\end{aligned} \tag{23b}$$

$$\hat{u}(0,t) = r\hat{v}(0,t) + \hat{k}(t)\left(\hat{\theta}(t) - y_0(t)\right) \tag{23c}$$

$$\hat{v}(1,t) = U(t). \tag{23d}$$

Using the backstepping transformations $\omega(x,t) = \mathscr{K}_1[\hat{u},\hat{v}](x,t)$ and $\zeta(x,t) = \mathscr{K}_2[\hat{u},\hat{v}](x,t)$, we transform the system (23) into the equivalent target system

$$\begin{aligned}\omega_t(x,t) + \lambda\omega_x(x,t) =& \hat{\theta}(t)H_1(x,t)\dot{\hat{k}}(t) + G_1(x,t)\dot{\hat{k}}(t)\\ &+ \hat{k}(t)H_1(x,t)\dot{\hat{\theta}}(t) + \Omega_1(x)\hat{e}(1,t)\\ &+ \Psi_1(x)\hat{k}(t)\left(\hat{\theta}(t) - y_0(t)\right)\end{aligned} \tag{24a}$$

$$\begin{aligned}\zeta_t(x,t) - \mu\zeta_x(x,t) =& \hat{\theta}(t)H_2(x,t)\dot{\hat{k}}(t) + G_2(x,t)\dot{\hat{k}}(t)\\ &+ \hat{k}(t)H_2(x,t)\dot{\hat{\theta}}(t) + \Omega_2(x)\hat{e}(1,t)\\ &+ \Psi_2(x)\hat{k}(t)\left(\hat{\theta}(t) - y_0(t)\right)\end{aligned} \tag{24b}$$

$$\omega(0,t) = r\zeta(0,t) + \hat{k}(t)\left(\hat{\theta}(t) - y_0(t)\right) \tag{24c}$$

$$\begin{aligned}\zeta(1,t) =& U(t) - \int_0^1 K^{vu}(1,\xi)\hat{u}(\xi)d\xi\\ &- \int_0^1 K^{vv}(1,\xi)\hat{v}(\xi)d\xi\end{aligned} \tag{24d}$$

where

$$G_1(x,t) = \mathscr{K}_1[w,z](x,t), \qquad H_1(x,t) = \mathscr{K}_1[m,n](x,t) \tag{25a}$$

$$G_2(x,t) = \mathscr{K}_2[w,z](x,t), \qquad H_2(x,t) = \mathscr{K}_2[m,n](x,t) \tag{25b}$$

$$\Omega_1(x) = \mathscr{K}_1[P_1,P_2](x), \qquad \Psi_1(x) = -K^{uu}(x,0)\lambda \tag{25c}$$

$$\Omega_2(x) = \mathscr{K}_2[P_1,P_2](x), \qquad \Psi_2(x) = -K^{vu}(x,0)\lambda. \tag{25d}$$

At this point, it is more convenient to use a linear form of the boundary condition. Defining

$$\hat{q} = \frac{r - b_0\hat{k}}{1 + a_0\hat{k}}, \quad \hat{d} = \frac{\hat{k}\hat{\theta}}{1 + a_0\hat{k}}, \quad \hat{\kappa} = \frac{-\hat{k}}{1 + a_0\hat{k}}, \tag{26}$$

gives the boundary condition

$$\omega(0,t) = \zeta(0,t)\hat{q}(t) + \hat{d}(t) + \hat{\kappa}(t)\check{e}(t). \tag{27}$$

We select the *reference model*

$$\varphi_t(x,t) + \lambda\varphi_x(x,t) = \Psi_1(x)\hat{k}(t)\left(\hat{\theta}(t) - y_0(t)\right) \tag{28a}$$

$$\phi_t(x,t) - \mu\phi_x(x,t) = \Psi_2(x)\hat{k}(t)\left(\hat{\theta}(t) - y_0(t)\right) \tag{28b}$$

$$\varphi(0,t) = \hat{q}(t)\phi(0,t) + \hat{d}(t) \tag{28c}$$

$$\phi(1,t) = \zeta^*(t). \tag{28d}$$

To achieve $\varphi(0,t) = r\phi(0,t)$, we must have $\phi(0,t) = \hat{d}(t)/(r - \hat{q}(t))$, we therefore select the *reference signal*

$$\zeta^*(t) = \hat{d}(t)/(r - \hat{q}(t)) = \hat{\theta}(t)/(a_0 r + b_0). \tag{29}$$

Notice that if $\hat{\theta}(t) \to y_0(t)$, the source terms on the right hand side of (28a) and (28b) will converge to zero, and the reference model is reduced to a set of cascaded transport equations with the solution $\phi(0,t) = \zeta^*(t - \mu^{-1})$. The idea is now to select a control law $U(t)$ such that the tracking error $\nu(x,t) = \omega(x,t) - \varphi(x,t)$ and $\eta(x,t) = \zeta(x,t) - \phi(x,t)$ converge to zero. Direct substitution of (22), (24) and (28) shows that the tracking errors have the dynamics

$$\begin{aligned}\nu_t(x,t) + \lambda\nu_x(x,t) =& \hat{\theta}(t)H_1(x,t)\dot{\hat{k}}(t) + G_1(x,t)\dot{\hat{k}}(t)\\ &+ \hat{k}(t)H_1(x,t)\dot{\hat{\theta}}(t) + \Omega_1(x)\hat{e}(1,t)\end{aligned} \tag{30a}$$

$$\begin{aligned}\eta_t(x,t) - \mu\eta_x(x,t) =& \hat{\theta}(t)H_2(x,t)\dot{\hat{k}}(t) + G_2(x,t)\dot{\hat{k}}(t)\\ &+ \hat{k}(t)H_2(x,t)\dot{\hat{\theta}}(t) + \Omega_2(x)\hat{e}(1,t)\end{aligned} \tag{30b}$$

$$\nu(0,t) = \eta(0,t)\hat{q}(t) + \hat{\kappa}(t)\check{e}(t) \tag{30c}$$

$$\eta(1,t) = 0. \tag{30d}$$

Notice in particular the zero boundary condition achieved when using the control law (22).

IV. Stability Analysis

Our strategy in proving Theorem 1 is to study both the stability of the state tracking error (ν,η) and the state estimation error $(\hat{e},\hat{\varepsilon})$ in (10). The dynamics of the state tracking error have already been derived in (30). The state estimation error dynamics will be derived in Section IV-A together with a backstepping transformation easing the stability analysis. Since the overall goal is stabilization of system (1), we need a relation between the (ω,ζ) dynamics in (24) and the state tracking error (ν,η). Such a relation is found in Section IV-B. Once this relation is found, the system state (u,v) is trivially related to (ν,η) through the invertible backstepping transformation $\omega(x,t) = \mathscr{K}_1[\hat{u},\hat{v}](x,t)$, $\zeta(x,t) = \mathscr{K}_2[\hat{u},\hat{v}](x,t)$ and the state estimation error $(\hat{e},\hat{\varepsilon})$. Finally, the proof of Theorem 1 is given in Section IV-C.

A. Backstepping of error dynamics and filters

Differentiating (10) with respect to time and space, respectively, inserting the dynamics (1) and (23) and using (11d) for the boundary condition, gives the estimation error dynamics

$$\begin{aligned}\hat{e}_t(x,t) + \lambda\hat{e}_x(x,t) =& c_1(x)\hat{\varepsilon}(x,t) - \hat{k}(t)\dot{\hat{\theta}}(t)m(x,t)\\ &- \dot{\hat{k}}(t)\left(\hat{\theta}(t)m(x,t) - w(x,t)\right) - P_1(x)\hat{e}(1,t)\end{aligned} \tag{31a}$$

$$\begin{aligned}\hat{\varepsilon}_t(x,t) - \mu\hat{\varepsilon}_x(x,t) =& c_2(x)\hat{e}(x,t) - \hat{k}(t)\dot{\hat{\theta}}(t)n(x,t)\\ &- \dot{\hat{k}}(t)\left(\hat{\theta}(t)n(x,t) - z(x,t)\right) - P_2(x)\hat{e}(1,t)\end{aligned} \tag{31b}$$

$$\hat{e}(0,t) = -\frac{b_0}{a_0}\hat{\varepsilon}(0,t) + \frac{1}{a_0}\check{e}(t) \tag{31c}$$

$$\hat{\varepsilon}(1,t) = 0 \tag{31d}$$

Using the backstepping transformations $\hat{e} = \mathscr{P}_1[\hat{\alpha},\hat{\beta}](x,t)$ and $\hat{\varepsilon} = \mathscr{P}_2[\hat{\alpha},\hat{\beta}](x,t)$ with injection terms $P_1(x) = \lambda P^{uu}(x,1)$ and $P_2(x) = \lambda P^{vu}(x,1)$ and a suitable kernel boundary conditions, maps system (31) into the target system

$$\hat{\alpha}_t(x,t) + \lambda\hat{\alpha}_x(x,t) = B_1(x,t) \tag{32a}$$

$$\hat{\beta}_t(x,t) - \mu\hat{\beta}_x(x,t) = B_2(x,t) \tag{32b}$$

$$\hat{\alpha}(0,t) = -\frac{b_0}{a_0}\hat{\beta}(0,t) + \frac{1}{a_0}\check{e}(t) \tag{32c}$$

$$\hat{\beta}(1,t) = 0, \tag{32d}$$

where (B_1,B_2) is given as the solution to the 2×2 Volterra equation

$$\begin{aligned}B_1(x,t) =& \int_x^1 P^{uu}(x,\xi)B_1(\xi,t)d\xi + \int_x^1 P^{uv}(x,\xi)B_2(\xi,t)d\xi\\ &+ \dot{\hat{k}}\left(\hat{\theta}(t)m(x,t) - w(x,t)\right) + \hat{k}\dot{\hat{\theta}}m(x,t)\end{aligned} \tag{33a}$$

$$B_2(x,t) = \int_x^1 P^{vu}(x,\xi)B_1(\xi,t)d\xi + \int_x^1 P^{vv}(x,\xi)B_2(\xi,t)d\xi + \dot{\hat{k}}\left(\hat{\theta}(t)n(x,t) - z(x,t)\right) + \hat{k}\dot{\hat{\theta}}n(x,t). \quad (33b)$$

B. Relationship between state estimates and tracking error

Lemma 3: Let ϖ be given by the auxiliary filter

$$\varpi_t(x,t) - \mu\varpi_x(x,t) = 0 \quad (34a)$$

$$\varpi(1,t) = \nu(0,t) - r\eta(0,t) =: \varpi_1(t) \quad (34b)$$

with initial condition $\varpi(\cdot,0) \in \mathscr{B}(0,1)$. The signal $\rho(t) = \omega(0,t) - r\zeta(0,t)$ is related to $\varpi_1(t)$ by

$$\varpi_1(t) = \rho(t) - \int_{t-1/\mu}^t M(\tau,t)\rho(\tau)d\tau + R(t) \quad (35)$$

for all $t > \mu^{-1}$ where

$$M(s,t) = (\hat{q}(t) - r)\Psi_2((t-s)\mu) \quad (36)$$

$$R(t) = \hat{\kappa}(t)\left(\hat{\theta}(t-1/\mu) - \hat{\theta}(t)\right). \quad (37)$$

Furthermore, the relation (35) is invertible with inverse

$$\rho(t) = \varpi_1(t) + \int_{t-\mu^{-1}}^t N(\tau,t)\varpi_1(\tau)d\tau + S(t) \quad (38)$$

for all $t > \mu^{-1}$ where N is related to M by the integral equation

$$N(\tau,t) = M(\tau,t) + \int_\tau^{\tau+\mu^{-1}} M(s,t)N(\tau,s)ds \quad (39)$$

and S given by

$$S(t) = -R(t) - \int_{t-\mu^{-1}}^t N(\tau,t)R(\tau)d\tau. \quad (40)$$

Proof: The reference model (28) can be solved explicitly as

$$\varphi(x,t) = \int_{t-x/\lambda}^t \Psi_1(x+\lambda(\tau-t))\hat{k}(\tau)\left(\hat{\theta}(\tau) - y_0(\tau)\right)d\tau + \varphi(0,t-\frac{x}{\lambda}) \quad (41a)$$

$$\phi(x,t) = \int_{t-(1-x)/\mu}^t \Psi_2(x-\mu(\tau-t))\hat{k}(\tau)\left(\hat{\theta}(\tau) - y_0(\tau)\right)d\tau + \phi(1,t-\frac{1-x}{\mu}). \quad (41b)$$

Evaluating (41) at $x = 0$ and using boundary conditions (28c) and (28d) yield

$$\phi(0,t) = \zeta^*(t-\frac{1}{\mu}) + Q(t) \quad (42a)$$

$$\varphi(0,t) = \hat{d}(t) + \hat{q}(t)\left(\zeta^*(t-\frac{1}{\mu}) + Q(t)\right) \quad (42b)$$

where

$$Q(t) = \int_{t-1/\mu}^t \Psi_2(\mu(t-\tau))\hat{k}(\tau)\left(\hat{\theta}(\tau) - y_0(\tau)\right)d\tau. \quad (43)$$

We have $\varpi_1(t) = \rho(t) - \varphi(0,t) + r\phi(0,t)$. Direct substitution and using (42) then gives the relation

$$\varpi_1(t) = \rho(t) - \left(\hat{d}(t) + \hat{q}(t)\left(\zeta^*(t-1/\mu) + Q(t)\right)\right) + r\left(\zeta^*(t-1/\mu) + Q(t)\right) \quad (44)$$

and after some lengthy but straightforward algebraic manipulation, we obtain (35). For the inverse relation, let

$$a(t) = b(t) - \int_{t-\mu^{-1}}^t M(\tau,t)b(\tau)d\tau =: \mathscr{M}[b](t) \quad (45)$$

for some signals $a,b : [0,\infty) \to \mathbb{R}$, and assume the transformation has an inverse in the form

$$\mathscr{M}^{-1}[a](t) = a(t) + \int_{t-d}^t N(s,t)a(s)ds = b(t). \quad (46)$$

Since the lower integration limit $t - \mu^{-1}$ is bounded below by zero, the above integral equation can be written as a Volterra equation by defining

$$M_0(\tau,t) := M(\tau,t)\chi[\tau > t - \mu^{-1}] \quad (47a)$$

$$N_0(\tau,t) := N(\tau,t)\chi[\tau > t - \mu^{-1}], \quad (47b)$$

where $\chi[condition]$ is the indicator function where $\chi = 1$ whenever *condition* is satisfied and 0 else, yielding

$$a(t) = b(t) - \int_0^t M_0(\tau,t)b(\tau)d\tau \quad (48)$$

$$b(t) = a(t) + \int_0^t N_0(\tau,t)a(\tau)d\tau. \quad (49)$$

M_0 and N_0 are then related by

$$N_0(\tau,t) = M_0(\tau,t) + \int_\tau^t M_0(s,t)N_0(\tau,s)ds \quad (50)$$

Since M_0 is bounded, there exist a unique solution N_0 to (50) (see [16, Lemma 9]). Using (47) this can be written

$$\begin{aligned} N_0(\tau,t) =& M(\tau,t)\chi[\tau > t-\mu^{-1}] \\ &+ \int_\tau^t M(s,t)\chi[s > t-\mu^{-1}]N_0(\tau,s)ds \\ =& \begin{cases} M + \int_\tau^t M(s,t)N_0(\tau,s)ds, & \text{for } \tau > t-\mu^{-1} \\ \int_{t-\mu^{-1}}^t M(\tau,t)N_0(\tau,\tau)ds, & \text{for } t-\mu^{-1} \geq \tau \geq 0 \end{cases} \end{aligned}$$

Since this should hold for all $t > \mu^{-1}$, the only solution to N_0 when $t - \mu^{-1} \geq \tau \geq 0$ is the trivial solution $N_0(\tau,t) = 0$, and we are left with

$$N_0(\tau,t) = \begin{cases} M(\tau,t) + \int_\tau^t M(\tau,t)N_0(\tau,s)ds, & \tau > t-\mu^{-1} \\ 0, & \text{otherwise} \end{cases}$$

and (39) follows. Using the $\mathscr{M}$ operator on (35) gives $\varpi_1(t) = \mathscr{M}[\rho](t) + R(t)$, and since M is invertible

$$\rho(t) = \mathscr{M}^{-1}[\varpi_1](t) - \mathscr{M}^{-1}[R](t), \quad (51)$$

which shows that (35) is invertible with inverse (38) and $S(t)$ given by (40). ■

C. Proof of Theorem 1

Proof: Consider the Lyapunov function candidates

$$V_1(t) = \lambda^{-1}\int_0^1 e^{-\delta_1 x}\nu^2(x,t)dx \quad (52a)$$

$$V_2(t) = \mu^{-1}\int_0^1 e^{\sigma_1 x}\eta^2(x,t)dx \quad (52b)$$

$$V_3(t) = \lambda^{-1}\int_0^1 e^{-\delta_2 x}\hat{\alpha}^2(x,t)dx \quad (52c)$$

$$V_4(t) = \mu^{-1}\int_0^1 e^{\sigma_2 x}\hat{\beta}^2(x,t)dx, \quad (52d)$$

$$V_5(t) = \lambda^{-1}\int_0^1 e^{-\delta_3 x}\bar{w}^2(x,t)dx \quad (52e)$$

$$V_6(t) = \mu^{-1}\int_0^1 e^{\sigma_3 x}\varpi^2(x,t)dx, \quad (52f)$$

where (ν,η) is given by (30), $(\hat{\alpha},\hat{\beta})$ by (32) and ϖ by (34). The signal $\bar{w}$ is obtained from the backstepping transformations $w = \mathscr{P}_1[\bar{w},\bar{z}](x,t), z = \mathscr{P}_2[\bar{w},\bar{z}](x,t)$ with injection terms $P_1(x) = \lambda P^{uu}(x,1)$ and $P_2(x) = \lambda P^{vu}(x,1)$, mapping the filter system (8) into

$$\bar{w}_t(x,t) + \lambda\bar{w}_x(x,t) = 0 \quad (53a)$$

$$\bar{z}_t(x,t) - \mu\bar{z}_x(x,t) = 0 \quad (53b)$$

$$\bar{w}(0,t) = r\bar{z}(0,t) - y_0(t) \quad (53c)$$

$$\bar{z}(1,t) = 0. \quad (53d)$$

It is possible to show that the time derivatives satisfy (see Section VI-B)

$$\dot{V}_1 \leq -V_1 + l_1(t) + l_2(t)V_5(t) + l_3(t)V_6 + 2\bar{q}^2\eta^2(0,t) + \frac{\bar{\Omega}_1}{\lambda\delta_1}\left(1-e^{-\delta_1}\right)|\hat{\alpha}(1,t)|^2 \quad (54a)$$

$$\dot{V}_2 \leq -V_2 + l_4(t) + l_5(t)V_5 - \eta^2(0,t) + \frac{\bar{\Omega}_2}{\mu\sigma}\left(e^{\sigma_1}-1\right)|\hat{\alpha}(1,t)|^2 \quad (54b)$$

$$\dot{V}_3 \leq -V_3 + l_6(t) + l_7(t)V_5 + l_8(t)V_6 - e^{-\delta_2}\hat{\alpha}^2(1,t) + 2\frac{b_0^2}{a_0^2}\hat{\beta}^2(0,t) \quad (54c)$$

$$\dot{V}_4 \leq -V_4 + l_9(t) + l_{10}(t)V_5 - \hat{\beta}^2(0,t) \quad (54d)$$

$$\dot{V}_5 \leq -\delta_3\lambda V_5 + l_{11}(t) + l_{12}(t)V_5 + l_{13}(t)V_6 + \frac{4}{\underline{k}^2}\bar{S}^2 + 4\bar{\theta}^2 + \frac{4}{\underline{k}^2}\bar{N}^2V_6 + \frac{4}{\underline{k}^2}\bar{q}_r\eta^2(0,t) \quad (54e)$$

$$\dot{V}_6 \leq -\sigma_3\mu V_6 + l_{14}(t) + l_{15}(t)V_5 + l_{16}(t)V_6 + \bar{q}_r\eta^2(0,t) \quad (54f)$$

Forming $V_7(t) = \sum_{i=1}^6 a_iV_i$ where

$$\sigma_3 = \mu^{-1}\left(\frac{4}{\underline{k}^2}\bar{N}^2 + 1\right) \quad (55a)$$

$$a_1 = a_5 = a_6 = 1 \quad (55b)$$

$$a_2 = 2\bar{q}^2 + \frac{4}{\underline{k}^2}\bar{q}_r + \bar{q}_r \quad (55c)$$

$$a_3 = e^{\delta_2}\frac{\bar{\Omega}_1}{\lambda\delta_1}\left(1-e^{-\delta_1}\right) + a_2e^{\delta_2}\frac{\bar{\Omega}_2}{\mu\sigma}\left(e^{\sigma_1}-1\right) \quad (55d)$$

$$a_4 = a_3 2\frac{b_0^2}{a_0^2}, \quad (55e)$$

yields $\dot{V}_7 \leq -V_7 + l_{17}(t) + l_{18}(t)V_7 + \frac{4}{\underline{k}^2}\bar{S}^2 + 4\bar{\theta}^2$ for some integrable functions l_{17} and l_{18}. This shows that $V_7 \in \mathscr{L}_\infty$ and in turn $||\nu||, ||\eta||, ||\hat{\alpha}||, ||\hat{\beta}||, ||\bar{w}||, ||\varpi|| \in \mathscr{L}_\infty$. Since $||\bar{w}||$ and $||\varpi||$ are bounded, we have that the products $(l_2(t)+l_5(t)+l_7(t)+l_{10}+l_{12}+l_{15})V_5$ and $(l_3(t)+l_8(t)+l_{13}+l_{16})V_6$ are integrable. Forming $V_8(t) = \sum_{i=1}^4 a_iV_i + a_6V_6$ then yields $\dot{V}_8 \leq -V_8 + l_{19}(t) + l_{20}(t)V_8$ for some integrable functions l_{19} and l_{20}. It follows from [14, Lemma B.6] that $V_8 \in \mathscr{L}_1$, and hence

$$||\nu||, ||\eta||, ||\hat{\alpha}||, ||\hat{\beta}||, ||\varpi|| \in \mathscr{L}_2 \cap \mathscr{L}_\infty. \quad (56)$$

And from [15, Lemma 2.17] that $V_8 \to 0$ and hence

$$||\nu||, ||\eta||, ||\hat{\alpha}||, ||\hat{\beta}||, ||\varpi|| \to 0. \quad (57)$$

The prove parameter convergence and the control objective (2), we first prove that $\left(\hat{\theta} - y_0\right) \in \mathscr{L}_2$. We have $||\varpi|| \in \mathscr{L}_2$. I.e.

$$\lim_{T\to\infty}\int_0^T\int_0^1 \varpi^2(x,t)dxdt < \infty. \quad (58)$$

Inserting the explicit solution of (34) for $t > \mu^{-1}$ yields

$$\lim_{T\to\infty}\int_{\mu^{-1}}^T\int_0^1 \varpi_1^2(t-\mu^{-1}(1-x))dxdt < \infty. \quad (59)$$

Substituting $\tau = t - \mu^{-1}(1-x)$ and changing the order of integration yields

$$\lim_{T\to\infty}\left[\int_0^{\mu^{-1}}\int_{\mu^{-1}}^{\tau+\mu^{-1}} + \int_{\mu^{-1}}^{T-\mu^{-1}}\int_\tau^{\tau+\mu^{-1}} + \int_{T-\mu^{-1}}^T\int_\tau^T\right] \times dt\mu\varpi_1^2(\tau)d\tau < \infty. \quad (60)$$

All the inner integrals evaluate to μ^{-1} or less, and we have

$$\lim_{T\to\infty}\int_0^T \varpi_1^2(\tau)d\tau < \infty. \quad (61)$$

That is $(\nu(0,\cdot) - r\eta(0,\cdot)) \in \mathscr{L}_2$.

Using Cauchy-Schwarz' inequality and changing the order of integration similarly, we have

$$\int_{t_0}^T\left|\hat{\theta}(t) - \hat{\theta}(t-t_0)\right|^2dt \leq \int_{t_0}^T t_0\int_{t-t_0}^t\left|\dot{\hat{\theta}}(\tau)\right|^2d\tau dt \leq t_0^2\int_0^T\left|\dot{\hat{\theta}}(\tau)\right|^2d\tau, \quad (62)$$

for some $t_0 \leq t$. From the definition of R in (37), we have

$$\int_{\mu^{-1}}^T R^2(t)dt \leq \frac{\bar{k}^2}{\mu^2}\int_0^T\left|\dot{\hat{\theta}}(\tau)\right|^2d\tau. \quad (63)$$

Taking the limit as $T\to\infty$ and from $\dot{\hat{\theta}} \in \mathscr{L}_2$, it follows that $R \in \mathscr{L}_2$. And from the invertible transformation (40) that $S \in \mathscr{L}_2$. Now, since $\varpi_1, ||\varpi||, S \in \mathscr{L}_2$, we have from (38) that $\rho = (\omega(0,\cdot) - r\zeta(0,\cdot)) \in \mathscr{L}_2$. Finally, using boundary condition (24c) gives $\left(\hat{\theta} - y_0\right) \in \mathscr{L}_2$.

From the definition (11) and the backstepping transformation leading to the target systems $(\bar{w},\bar{z})$ in (53) and $(\bar{m},\bar{n})$ in the proof of Lemma 1, we have for $t \geq t_F$

$$\hat{\theta}(t)\check{m}(t) + \check{w}(t) = a_0\left[\hat{\theta}(t) - y_0(t)\right] + (a_0r + b_0)\int_0^1 P^{vu}(0,\xi)\left[\hat{\theta}(t) - y_0(t-\xi\lambda^{-1})\right]d\xi, \quad (64)$$

where we have used that $\bar{z}(x,t) = \bar{n}(x,t) = 0$ for all $t \geq \mu^{-1}$, and $\bar{m}(x,t) = 1$ and $\bar{w}(x,t) = -y_0(t-x\lambda^{-1})$ for all $t \geq \lambda^{-1}$. Changing the variable of integration and adding and subtracting $\hat{\theta}(\tau)$ inside the integral yields

$$\hat{\theta}(t)\check{m}(t) + \check{w}(t) \leq a_0\left[\hat{\theta}(t) - y_0(t)\right]$$

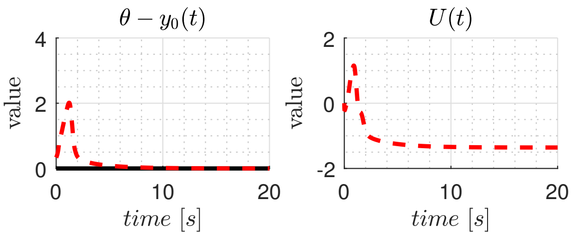

Fig. 1. Control objective (left) and control signal (right).

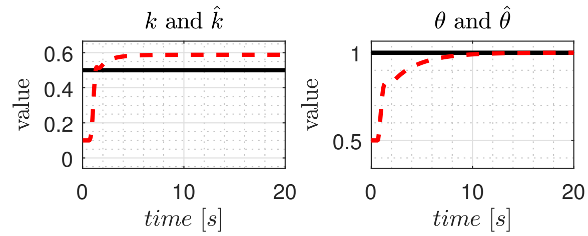

Fig. 2. Parameter estimates (dashed red) and actual parameters (solid black).

$$+(a_0r+b_0)\lambda\int_{t-\lambda^{-1}}^{t} P^{vu}(0,\lambda(t-\tau))\left[\hat{\theta}(\tau)-y_0(\tau)\right]d\tau$$
$$+(a_0r+b_0)\lambda\int_{t-\lambda^{-1}}^{t} P^{vu}(0,\lambda(t-\tau))\left[\hat{\theta}(t)-\hat{\theta}(\tau)\right]d\tau. \tag{65}$$

Squaring both sides, applying Cauchy-Schwarz' inequality and integrating from $t=\lambda^{-1}$ to T yields after changing the order of integration

$$\int_{\lambda^{-1}}^{T}\left(\hat{\theta}(t)\check{m}(t)+\check{w}(t)\right)^2dt \leq \int_{\lambda^{-1}}^{T} 2a_0^2\left[\hat{\theta}(t)-y_0(t)\right]^2dt$$
$$+4(a_0r+b_0)^2\bar{P}^2\lambda^{-1}\int_0^T\left[\hat{\theta}(\tau)-y_0(\tau)\right]^2d\tau$$
$$+4(a_0r+b_0)^2\bar{P}^2\lambda^{-3}\int_0^T\left|\dot{\hat{\theta}}(\tau)\right|^2d\tau. \tag{66}$$

Since $\dot{\hat{\theta}}\in\mathscr{L}_2$ and $(\hat{\theta}-y_0)\in\mathscr{L}_2$, it follows that $\left(\hat{\theta}\check{m}+\check{w}\right)\in\mathscr{L}_2$. From Property 5 in Lemma 2 and $||\bar{w}||\in\mathscr{L}_\infty$, implying $\check{w}$ bounded for almost all $t\geq 0$, we then have $\hat{\theta}\to\theta$ and $\hat{k}\to k_\infty$ for some constant k_∞. Furthermore,

$$\int_t^{t+t_\delta}|\theta-y_0(\tau)|d\tau \leq \int_t^{t+t_\delta}\left|\hat{\theta}(t)-y_0(\tau)\right|d\tau + \int_t^{t+t_\delta}\left|\theta-\hat{\theta}(t)\right|d\tau \tag{67}$$

and by the squeeze theorem, since the right hand side converges, the control objective (2) follows. Lastly, since $||\varpi||,\zeta^*,\hat{q},\hat{d}\in\mathscr{L}_\infty$, we have from the explicit solution (41) that $||\varphi||,||\phi||\in\mathscr{L}_\infty$. From the tracking errors $\omega(x,t)=\nu(x,t)+\varphi(x,t)$ and $\zeta(x,t)=\eta(x,t)+\phi(x,t)$ and the invertibility of the transformations $\omega(x,t)=\mathscr{K}_1[\hat{u},\hat{v}](x,t)$ and $\zeta(x,t)=\mathscr{K}_2[\hat{u},\hat{v}](x,t)$, we then have $||u||,||v||\in\mathscr{L}_\infty$. ■

V. Simulation and Concluding Remarks

The swapping-based estimation scheme consisting of the swapping filters (6)-(8), state estimates (9) and the adaptive law of Lemma 2, and the control law (22) with reference signal (29) was implemented in MATLAB using a 2nd order upwind scheme for the spatial discretization and MATLAB's `ode(45)` solver for the temporal discretization. The system parameters were chosen as $\lambda=\mu=3$, $c_1(x)=3e^{-2x}$, $c_2(x)=3e^{2x}$, $\theta=1$, $k=0.5$ and $r=-1$ with initial estimates $\hat{k}(0)=0.1$ and $\hat{\theta}(0)=0.5$, and initial condition $u_0(x)=1/3$ and $v_0(x)=x$. The adaptation gain was chosen as $\gamma_1=\gamma_2=10$. The system is open loop ($U(t)\equiv 0$) unstable.

Figure 1 shows that the control objective (2) is achieved and that the control input converges to some non-zero value. Furthermore, Figure 2 shows that $\hat{\theta}$ correctly estimates the unknown parameter θ and that $\hat{k}$ converges to some constant $k_\infty\neq k$. This was achieved with no other requirement than the a priori verifiable PE condition (13). Moreover, exploiting the measurement at the non-actuated boundary y_0 in the filter systems (6)–(8) and relationship (12) yields fast parameter adaptation with no propagation delay. This can be seen from Figure 2 where, after an initial $t_F=2/3$ s, the parameter estimates are directly updated without any artificial propagation delay through the filters.

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VI. APPENDIX

A. Additional properties

Lemma 4: Consider G_1, G_2, H_1, H_2, Ω_1 and Ω_2 given in (25), $\bar{w}$ by (53), B_i in (33). The following properties hold for all $t \geq t_f$

1) $H_1(x,\cdot), H_2(x,\cdot), \Omega_1(x,\cdot), \Omega_2(x,\cdot) \in \mathscr{L}_\infty$.
2) $||G_i(t)||^2 \leq h_i||\bar{w}(t)||^2$ for $i \in \{1,2\}$.
3) $||B_i(t)||^2 \leq h_3 \left|\dot{\hat{k}}(t)\right|^2 + h_4 \left|\dot{\hat{\theta}}(t)\right|^2 + h_5 \left|\dot{\hat{k}}(t)\right|^2 ||\bar{w}||^2$.
4) $y_0^2(t) \leq \frac{4}{\underline{k}^2} |\varpi_1(t)|^2 + \frac{4}{\underline{k}^2}\bar{S}^2 + 4\bar{\theta}^2 + \frac{4}{\underline{k}^2}\bar{N}^2||\varpi||^2$
5) $\varpi_1^2(t) \leq \bar{q}_r|\eta(0,t)|^2 + 2\bar{\kappa}^2\varepsilon^2(t)(1+||\varpi||^2+||\bar{w}||^2)$
6) $\varepsilon^2 := \frac{\hat{e}^2}{1+||\bar{w}||^2+||\varpi||^2} \in \mathscr{L}_2$

for some constants $h_i > 0$, $i = 1...5$, and where we have defined $\bar{S} = \sup_{t\geq 0} S(t)$, $\bar{\theta} = \sup_{t\geq 0}\hat{\theta}(t)$, $\bar{k} = \sup_{t\geq 0}\hat{k}(t)$ and $\bar{q}_r = \sup_{t\geq 0} 2(\hat{q}(t)-r)^2 = \sup_{t\geq 0} 2(a_0 r + b_0)^2\hat{\kappa}(t)$.

Proof: The details of the proof are technical and omitted due to page limitation. Property 1 follow from boundedness of the backstepping operator. The bound in Property 3 can be found by successive approximations. Properties 4 and 5 are trivial. Lastly, Property 6 follow from Property 4 in Lemma 2 and applying the upper bounds for y_0^2 and ϖ_1^2. ■

B. Details in proof of Theorem 1

1) Bounds on V_1 and V_2: From (52a) and (52b) and inserting the dynamics (30a) and (30b), we get

$$\begin{aligned}\dot{V}_1 =& 2\lambda^{-1}\int_0^1 e^{-\delta_1 x}\nu(x,t)\Big(-\lambda\nu_x(x,t) + G_1(x,t)\dot{\hat{k}}(t) \\ &+\hat{\theta}(t)H_1(x,t)\dot{\hat{k}}(t) + \hat{k}(t)H_1(x,t)\dot{\hat{\theta}}(t) \\ &+\Omega_1(x)\hat{\alpha}(1,t)\Big)dx, \end{aligned} \tag{68}$$

$$\begin{aligned}\dot{V}_2 =& 2\mu^{-1}\int_0^1 e^{\sigma_1 x}\eta(x,t)\Big(\mu\eta_x(x,t) + G_2(x,t)\dot{\hat{k}}(t) \\ &+\hat{\theta}(t)H_2(x,t)\dot{\hat{k}}(t) + \hat{k}(t)H_2(x,t)\dot{\hat{\theta}}(t) \\ &+\Omega_2(x)\hat{\alpha}(1,t)\Big)dx \end{aligned} \tag{69}$$

respectively, where the relation $\hat{e}(1,t) = \hat{\alpha}(1,t)$ has been used. Integration by parts, separating the cross terms using Young's inequality, using Properties 1, 2 and 6 of Lemma 4 and boundedness of $\hat{\theta}, \hat{k}, \dot{\hat{\theta}}, \dot{\hat{k}}$ from Lemma 2, defining the integrable functions

$$l_1(t) := 2\bar{\kappa}^2\varepsilon^2(t) + \frac{\bar{H}_1}{\lambda\delta_1}(1-e^{-\delta_1})(\bar{\theta}|\dot{\hat{k}}|^2 + \bar{k}|\dot{\hat{\theta}}|^2) \tag{70a}$$

$$l_2(t) := h_1 e^{\delta_3}|\dot{\hat{k}}(t)|^2 + 2\bar{\kappa}^2\varepsilon^2(t)\lambda e^{\delta_3} \tag{70b}$$

$$l_3(t) := 2\bar{\kappa}^2\varepsilon^2(t)\mu \tag{70c}$$

$$l_4(t) := \frac{\bar{H}_2}{\mu\sigma}(e^{\sigma_1}-1)(\bar{\theta}|\dot{\hat{k}}(t)|^2 + \bar{k}|\dot{\hat{\theta}}(t)|^2) \tag{70d}$$

$$l_5(t) := h_2 e^{\delta_3}|\dot{\hat{k}}(t)|^2, \tag{70e}$$

and selecting $\delta_1 = \lambda^{-1}\left(\bar{H}_1\bar{\theta} + \bar{H}_1\bar{k} + \bar{\Omega}_1 + 2\right)$ and $\sigma_1 = \mu^{-1}\left(\bar{H}_2\bar{\theta} + \bar{H}_2\bar{k} + \bar{\Omega}_2 + 2\right)$ yield

$$\begin{aligned}\dot{V}_1 \leq& -V_1 + l_1(t) + l_2(t)V_5(t) + l_3(t)V_6 \\ &+ 2\bar{q}^2\eta^2(0,t) + \frac{\bar{\Omega}_1}{\lambda\delta_1}\left(1-e^{-\delta_1}\right)|\hat{\alpha}(1,t)|^2, \end{aligned} \tag{71}$$

$$\begin{aligned}\dot{V}_2 \leq& -V_2 + l_4(t) + l_5(t)V_5 \\ &-\eta^2(0,t) + \frac{\bar{\Omega}_2}{\mu\sigma}(e^{\sigma_1}-1)|\hat{\alpha}(1,t)|^2. \end{aligned} \tag{72}$$

2) Bounds on V_3 and V_4: From (52c) and (52d) and inserting the dynamics (32a) and (32b), we get

$$\begin{aligned}\dot{V}_3 =& -2\int_0^1 e^{-\delta_2 x}\hat{\alpha}(x,t)\hat{\alpha}_x(x,t)dx \\ &+2\lambda^{-1}\int_0^1 e^{-\delta_2 x}\hat{\alpha}(x,t)B_1(x,t)dx, \end{aligned} \tag{73}$$

$$\begin{aligned}\dot{V}_4 =& 2\int_0^1 e^{\sigma_2 x}\hat{\beta}(x,t)\hat{\beta}_x(x,t)dx \\ &+2\mu^{-1}\int_0^1 e^{\sigma_2 x}\hat{\beta}(x,t)B_2(x,t)dx \end{aligned} \tag{74}$$

respectively. Integration by parts, separating the cross terms using Young's inequality, using Properties 3 and 6 of Lemma 4, defining

$$l_6(t) := (2/a_0^2)\varepsilon(t)^2 + \lambda^{-1}h_3|\dot{\hat{k}}(t)|^2 + \lambda^{-1}h_4|\dot{\hat{\theta}}(t)|^2 \tag{75a}$$

$$l_7(t) := e^{\delta_3}|\dot{\hat{k}}(t)|^2 + (2/a_0^2)\lambda e^{\delta_3}\varepsilon(t)^2 \tag{75b}$$

$$l_8(t) := (2/a_0^2)\mu\varepsilon(t)^2 \tag{75c}$$

$$l_9(t) := \mu^{-1}e^{\sigma_2}h_3|\dot{\hat{k}}(t)|^2 + \mu^{-1}e^{\sigma_2}h_4|\dot{\hat{\theta}}(t)|^2 \tag{75d}$$

$$l_{10}(t) := e^{\delta_3}|\dot{\hat{k}}(t)|^2, \tag{75e}$$

and selecting $\delta_1 = 2\lambda^{-1}$ and $\sigma_2 = 2\mu^{-1}$ yield

$$\begin{aligned}\dot{V}_3 \leq& -V_3 + l_6(t) + l_7(t)V_5 + l_8(t)V_6 \\ &-e^{-\delta_2}\hat{\alpha}^2(1,t) + 2(b_0^2/a_0^2)\hat{\beta}^2(0,t), \end{aligned} \tag{76}$$

$$\dot{V}_4 \leq -V_4 + l_9(t) + l_{10}(t)V_5 - \hat{\beta}^2(0,t). \tag{77}$$

3) Bounds on V_5 and V_6: Define

$$l_{11}(t) := \frac{8}{\underline{k}^2}\bar{\kappa}^2\varepsilon^2(t) =: (\underline{k}^2/4)l_{14}(t) \tag{78a}$$

$$l_{12}(t) := \frac{8}{\underline{k}^2}\bar{\kappa}^2\varepsilon^2(t)e^{\delta_3}\lambda =: (\underline{k}^2/4)l_{15}(t) \tag{78b}$$

$$l_{13}(t) := \frac{8}{\underline{k}^2}\bar{\kappa}^2\varepsilon^2(t)\mu =: (\underline{k}^2/4)l_{16}(t). \tag{78c}$$

Differentiating (52e), inserting the dynamics (53), integrating by parts, and using Properties 4 and 6 in Lemma 4 yield

$$\begin{aligned}\dot{V}_4 \leq& -\delta_3\lambda V_5 + l_{11}(t) + l_{12}(t)V_5 + l_{13}(t)V_6 \\ &+\frac{4}{\underline{k}^2}\bar{S}^2 + 4\bar{\theta}^2 + \frac{4}{\underline{k}^2}\bar{N}^2V_6 + \frac{4}{\underline{k}^2}\bar{q}_r\eta^2(0,t). \end{aligned} \tag{79}$$

From (52f), inserting the dynamics (34) and using Properties 5 and 6 in Lemma 4, we get

$$\begin{aligned}\dot{V}_6 \leq& -\sigma_3\mu V_6 + l_{14}(t) + l_{15}(t)V_5 + l_{16}(t)V_6 \\ &+\bar{q}_r\eta^2(0,t). \end{aligned} \tag{80}$$