Doctoral Theses at NTNU, 2008:65

Johannes Tjønnås

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Doctoral Thesis

ISBN 978-82-471-7250-6 (printed ver.)
ISBN 978-82-471-7264-3 (electronic ver.)
ISSN 1503-8181

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Printed by Tapir Uttrykk

Summary

This work addresses the control allocation problem for a nonlinear over-actuated time-varying system where parameters affine in the actuator dynamics and actuator force model may be assumed unknown. Instead of optimizing the control allocation at each time instant, a dynamic approach is considered by constructing update-laws that represent asymptotically optimal allocation search and adaptation. A previous result on uniform global asymptotic stability (UGAS) of the equilibrium of cascaded time-varying systems, is in the thesis shown to also hold for closed (not necessarily compact) sets composed by set-stable subsystems of a cascade. In view of this result, the optimal control allocation approach is studied by using Lyapunov analysis for cascaded set-stable systems, and uniform global/local asymptotic stability is guaranteed for the sets described by; the system dynamics, the optimizing allocation update-law and the adaptive update-law.

The performance of the proposed control allocation scheme is demonstrated throughout the thesis by simulations of a scaled-model ship manoeuvred at low-speed. Furthermore, the application of a yaw stabilization scheme for an automotive vehicle is presented. The stabilization strategy consists of; a high level module that deals with the vehicle motion control objective (yaw rate reference generation and tracking), a low level module that handles the braking control for each wheel (longitudinal slip control and maximal tyre road friction parameter estimation) and an intermediate level dynamic control allocation module. The control allocation module generates longitudinal slip reference for the low level brake controller and commands front wheel steering angle corrections, such that the actual torque about the yaw axis tends to the desired torque calculated by the high level module. The conditions for uniform asymptotic stability are given and the scheme has been implemented in a realistic nonlinear multi-body vehicle simulation environment. The simulation cases show that the control strategy stabilizes the vehicle in extreme manoeuvres where the nonlinear vehicle yaw dynamics otherwise become unstable in the sense of over- or understeering.

Preface

This thesis contains research for my doctoral studies from September 2004 to February 2008 at the department of Engineering Cybernetics (ITK) at the Norwegian University of Science and Technology. My supervisor has been Professor Tor Arne Johansen and the work has been sponsored by the Research Council of Norway through the Strategic University Program on Computational Method in Nonlinear Motion Control and the European Commission through the STREP project CEmACS, contract 004175.

The aim of the thesis is to present ideas around a dynamic adaptive control allocation scheme. Although the control allocation strategy is compared with existing approaches, no attempts of comparing the practical implementation results with existing methods are done.

The main chapters that present the theoretical results; Dynamic control allocation, Dynamic adaptive control allocation and Control allocation with actuator dynamic are built up around the same cascaded set stability approach. The idea is first to show stability of a perturbing system under certain assumptions, then in a second step, the combined perturbing and perturbed system cascade is analyzed. Furthermore, the main results are founded on a set of global assumptions and the main focus has been on the analysis in the global case. The reason for this is that the global proves enables a fairly straight forward way to verify the local results when the assumptions only hold locally.

Acknowledgements

I would most of all like to thank my supervisor, Tor Arne Johansen who has been inspiring, patient and very helpful in all aspects throughout my Ph.D. work. I would also like to thank Lars Imsland and Petter Tøndel at SINTEF and Brad Schofield at Lund University for interesting discussions on vehicle dynamics. The insightful comments from Jens C. Kalkkuhl and Avshalom Suissa, related to the DaimlerCrysler's proprietary multi-body simulation environment CASCaDE, have also been very helpful when dealing with the implementation and verification of the proposed algorithms. The collaboration with Antoine Chaillet and Elena Panteley at LSS - supélec, in the field of cascaded systems has been highly appreciated and the result achieved, serves as a theoretical foundation for some of the main results in this thesis.

Throughout my stay at ITK, I have shared office space with Jostein Bakkeheim. Many interesting topics have been discussed, spanning from mathematical technicalities to practical how-tos. Luca Pivano, Gullik Jensen and Frank Jakobsen have also often joined the discussions and I am happy to say that most problems may be solved on a white board. I am glad to have known Alexey Pavlov, for his existential views on life and mathematical insight, and all my colleagues at ITK who has interfered with this work through interesting debates, in particular; Aksel Transeth, Morten Breivik and Jørgen Spjøtvold.

The secretarial staff; Tove K. Johnsen, Eva Amdal and Unni Johansen, and the technical staff, in particular Stefano Bertelli, have handled all the administrative issues and provided an excellently environment for research.

Finally, I am very thankful to my loving and supportive family in Gransherad, and last but not least the support from my lovely fiancée, Maria Suong Le Thi.

Contents

Chapter 1

Introduction

1.1 Motivation

In the control allocation philosophy the actuators and effectors of a system are combined in order to produce some desired effect. For the (overactuated) control problem, where a unique desired effect on the system may be produced by different actuator input constellations and settings, a controller design involving the control allocation approach is beneficial. A key issue is the handling of redundancy in order to achieve dynamic reconfigurability and fault tolerance. Such designs offer a modular structure, decoupling the high level controller (defining the desired effect) from the allocation problem.

The control allocation problem is commonly formulated as an optimization problem systematically handling; redundant sets of actuators, actuator constraints and minimizing power consumption, wear/tear and other undesirable effects. For mechanical systems with fast dynamics, the optimization problem needs to be solved at a fast sampling rate which in general is a demanding and sometimes safety-critical task, even with the state of the art numerical optimization software available.

The main contribution of this thesis is to show that the instantaneous control allocation problem, for classes of nonlinear and uncertain systems, does not necessarily need to be solved exactly at each time instant. In order to ensure convergence and stability properties for the closed-loop system, a control allocation synthesis (with no iterative optimization loops) based on Lagrangian parameters is proposed, and the stability of a corresponding optimal set is pursued through the use of Lyapunov analysis. Furthermore, parameter uncertainty related to disturbances or actuator failure, is treated in an adaptive allocation solution of the closed-loop control problem.

1.1.1 System description and motivating examples

The main motivation for using the control allocation approach, in the solution of a control problem, is the simplification it offers in the design and control synthesis. Typically, the dynamic model of a mechanical system is based on Newton's three laws of motion, such that the forces and moments acting on the system occur affinely in the model. Generally, we have the plant motion dynamics (*the high level model*):

$$\dot{x} = f(t,x) + g(t,x)\tau, \tag{1.1}$$

where $f(t,x)$ and $g(t,x)$ defines the model dynamics, $x \in \mathbb{R}^n$ is a vector of size n, that denotes the plant motion states, t is the time and $\tau \in \mathbb{R}^d$ is a vector of size d, that denotes the *generalized forces and moments* acting as the system input. These forces and moments can not be generated directly, but may be manipulated through the actuators and effectors. The static actuator-force mapping model (*the static actuator/effector model*) takes the typical form

$$\begin{aligned} \tau &= \Phi(t,x,u), \\ \Phi(t,x,u,\theta) &:= \Phi_0(t,x,u) + \Phi_\tau(t,x,u)\theta_\tau + \Phi_u(t,x,u)\theta_u, \end{aligned} \tag{1.2}$$

where $u \in \mathbb{R}^r$ is the constrained input/control vector of size r, $\theta := (\theta_\tau^{\mathrm{T}}, \theta_u^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{m=m_\tau+m_u}$ is a vector of size m containing parameters that may be unknown. θ_u is related to the actuator model while θ_τ only appear in the static actuator-force mapping. By assuming that the actuator dynamics are much faster then the plant motion, u is commonly treated as the plant input. In some cases the actuator dynamics cannot be neglected and the dynamic actuator model (*the low level model*)

$$\dot{u} = f_{u0}(t,x,u,u_{cmd}) + f_{u\theta}(t,x,u,u_{cmd})\theta_u, \tag{1.3}$$

is considered, where u_{cmd} is the actuator input, and thus the only manipulated input available to the plant.

Structurally the plant is described by Figure 1.1.

The control allocation concept

Based on the actuator-force mapping (1.2), neglecting the actuator dynamics (1.3), the principle of the control allocation problem may be stated by

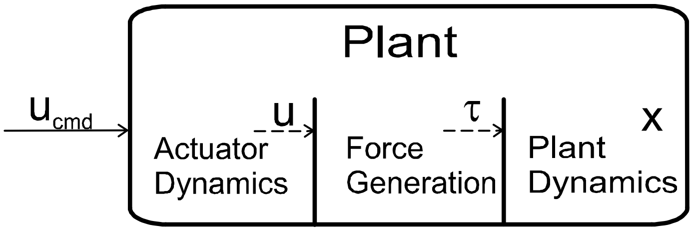

Figure 1.1: The plant

the selection of u that satisfy:

$$\Phi(t, x, u, \theta) = \tau_c \quad (1.4)$$
$$\underline{u} \leq u \leq \overline{u}, \quad (1.5)$$

where $\underline{u}$ and $\overline{u}$ are the actuator constraints, and τ_c is a desired generalized force generated by the high level control synthesis for (1.1). If a control satisfying (1.4) and (1.5) is not unique, some optimization criterion may be considered. When feasible controls does not exist, i.e. equation (1.4) is not satisfied for any u bounded by (1.5), the control allocation problem may be formulated as a two step sequential problem:

$$u = \arg\min_{u\in\Omega} J(u)$$
$$\Omega = \arg\min_{\underline{u}\leq u\leq\overline{u}} |\Phi(t, x, u, \theta) - \tau_c|,$$

by first minimizing the error between the desired and a feasible control, and further optimizing the actuator effort with respect to an optimizing criterion. By considering actuator dynamics and parameter uncertainty in the control/control allocation design, the closed loop structure will also consist of a low level controller and adaptive parameter estimators, Figure 1.2. In the following the control allocation strategy is presented and exemplified in the framework of; aircraft, automotive and ship control.

Aircraft moment/torque allocation

The orientation of an aircraft is in airplane terminology described by the three rotational degrees: yaw, pitch and roll. These rotational degrees are controlled by the aircraft control surfaces. The control surfaces are any object on the plane that can be manipulated in order to alter the airflow around the body and thus locally change the aerodynamic forces and create

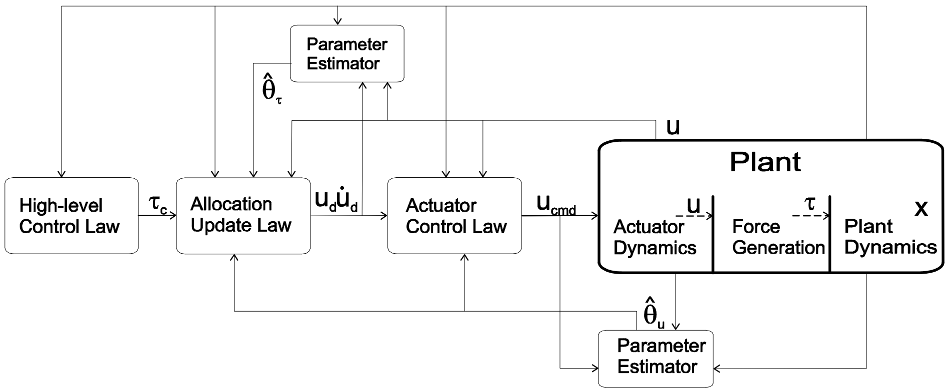

Figure 1.2: Block diagram of the cloosed loop control/adaptive optimizing control allocation scheme

moments around the rotational axes. For a conventional aircraft, the aileron surfaces are mainly designated to roll control, the elevators to pitch control and the rudder to yaw control. Flaps and slats are wing altering shapes that when deployed, increase lift. Spoilers which change lift, drag and roll are also common on conventional airplanes. In order to make air travel more safe, the concept of *Propulsion Controlled Aircraft* (PCA) has been introduced Jonckheere and Yu [1999]. If the control surfaces on the aircraft are disabled, the PCA system enables the pilot to control the aircraft pitch and yaw by solely manipulating the engine thrust.

A pilot has in general, only direct control of the plane attitude and the longitudinal motion. The task of an allocation algorithm is to determine which and how much the control surfaces should be deflected, based on some desired effect specified by the pilot. The control allocation problem becomes involved due to control surface redundancy and because a control surface deflection usually does not generate pure single axis moments (ailerons are mainly used for roll moment generation, but they also have an effect on the yaw moment). Further complexity arises as propulsion manipulation and additional control surfaces are added to the design in order to generate airplanes with greater manoeuvrability and redundancy.

Normally the plane dynamics can be described by (1.1) where x may be the translational positions (longitudinal, lateral and lift), the attitudes (pitch, roll and yaw) and the velocities of these states. In this case x is a twelve dimensional vector. u in (1.2) is a vector that contains the positions of all the airplane thrusters and control surfaces, constrained by propulsion

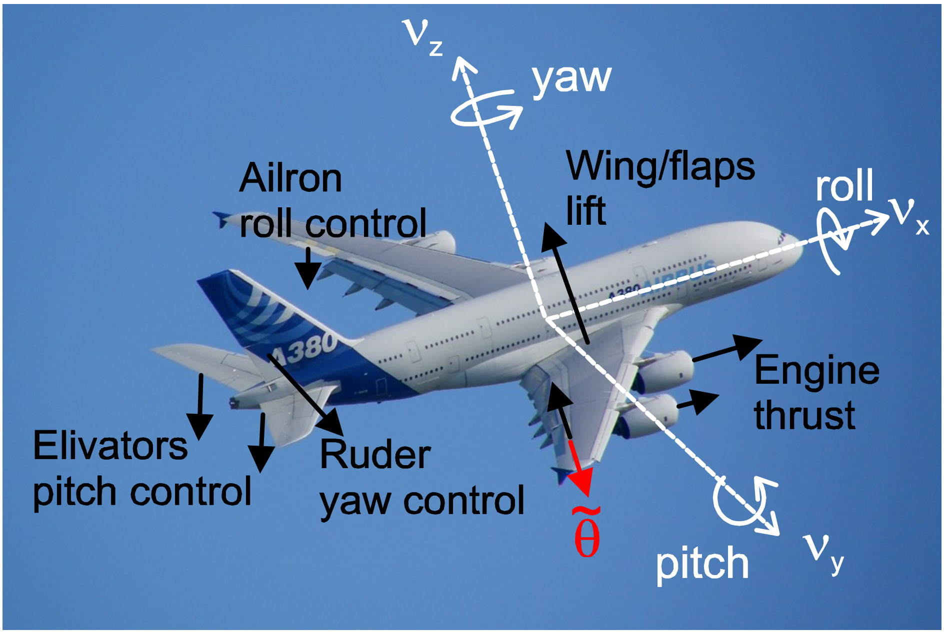

Figure 1.3: Airplane illustration. Photo courtesy: Axwel [2007]

saturation and maximal angle of deflection. The generalized force vector τ represents the forces in ν_x, ν_y and ν_z direction (see Figure 1.3) and the moments around the respective axes. Typically, the pilot specifies rate references for longitudinal speed, pitch, roll and yaw angular velocities. These references are fed to a high level controller and desired forces and moments (τ_c) are calculated. Moreover by the control allocation algorithm the desired controls (u_d) are manipulated in order to satisfy $\tau = \tau_c$ based on some actuator constraints and costs. If the actuator dynamics is not negligible, u_d may be considered as a reference for the low level actuator controls.

In case of in-flight actuator malfunction or damage, the allocation algorithm should adapt. This functionality can be included by the parametrization from (1.2) where θ_u can be treated as an effect parameter, or it can be handled by changing the actuator position constraints in order to model a stuck, slow or ineffective control surface.

The dynamic of a high performance aircraft is fast, thus the closed loop control system needs to be fast, and the speed and deterministic behavior of the control allocation algorithm is crucial. An explicit control allocation scheme usually has the specific property of being fast and deterministic.

See Bordignon [1996] and Beck [2002] for an in-depth discussion of the airplane control allocation problem.

Automotive safety control systems

In recent years, many new active driver assistance systems for increased safety and vehicle handling has been designed and applied in the automotive industry. The *anti-lock braking system* (ABS) that increases brake performance, the *brake assist* (BA) aiding the driver in critical braking situations and the *traction control system/antispin* (TCS) preventing loss of traction, are typical systems available in most modern cars. Active chassis vehicle control systems, like yaw motion control incorporated in for example; the *electronic stability program* (ESP), the *vehicle dynamic control* (VDC), the *electronic stability control* (ESC) etc. and the rollover prevention systems, may use ABS, BA and TCS as subsystem in order to increase the manoeuvrability of the vehicle.

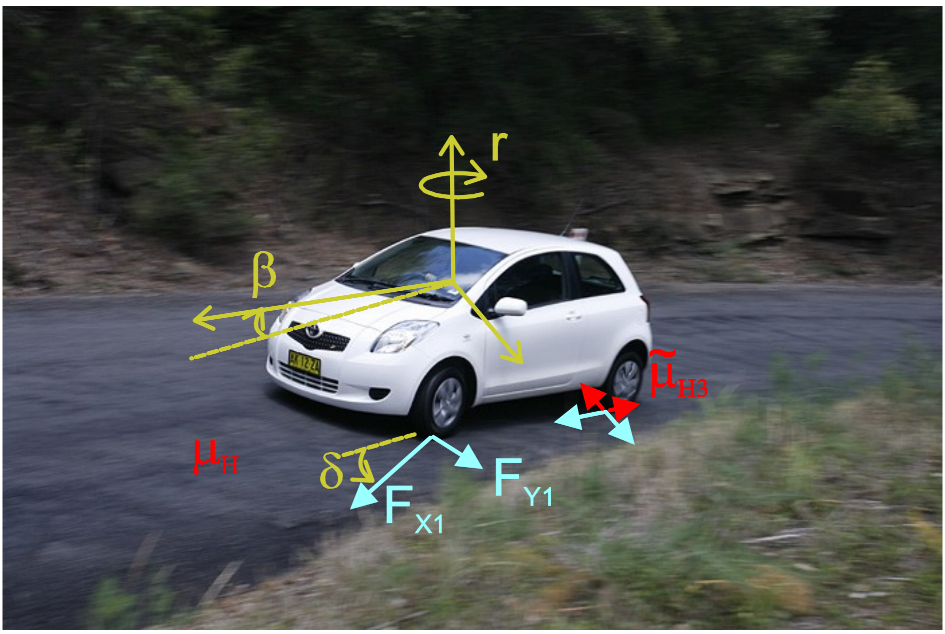

Figure 1.4: Vehicle illustration. Photo courtesy: Drive [2007]

In an automotive vehicle yaw stabilization scheme, the control problem consists of controlling the yaw rate (r) based on the driver steering angle input (δ) and the vehicle side slip (β), such that over- and understeering is prevented. A high level controller is assigned to stabilize the yaw rate error (which defines x in (1.1)), by calculating a desired torque (τ_c) around the yaw axis. The allocation problem is then to determine and achieve desired wheel-road forces that results in the desired moment. It is common to use brake actuators and steering angle actuators (constrained by maximal steering angle deflection and braking forces). But other approaches where active damping and torque biasing are used as controls may also be considered. In many ABS solutions dynamic wheel slip models are used in the control synthesis. Similarly these models (1.3) may be used in braking based yaw stabilization schemes. The tyre road friction model, incorporated in (1.2) and (1.3) is essential in the control allocation approach. This model is highly dependent on the tyre properties and the road conditions. The road condition uncertainty may be parameterized in the friction model by the maximal tyre road friction coefficient ($\theta_u = \mu_H$). Since this parameter change, corresponding to the road conditions, knowledge of this uncertain parameter is crucial for the performance of the allocation algorithm.

By solving the control allocation problem dynamically (not necessarily finding the optimal solution at each sampling instant), a real-time implementation can be realized without the use of any numeric optimization software. In general, this is an advantage since implementations on vehicles with low-cost hardware may be considered.

Background on vehicle control systems can be found in Kiencke and Nielsen [2000], and in Chapter 6 a detailed description of the yaw stabilization problem is presented as an application of the main result of this thesis.

Dynamic positioning of a scale model ship

Low-speed manoeuvring and station keeping of ships are commonly characterized as problems of dynamic positioning (DP). DP systems are classified by rules based on IMO [1994], issued by the International Maritime Organization (IMO), describing DP system properties for class 1, 2 and 3. Important elements in the classification are fault handling, system redundancy and fire/flood proof properties of the components in the system. DP systems are much used in the offshore industry and involves operations like; loading, pipe and cable laying, trenching, dredging, drilling and target following. Normally the vessels used in high accuracy and reliability operations like pipe laying and drilling are equipped with DP systems of class 2 and 3. Typically vessel actuators available for a DP system are; the main aft propellers in conjuncture with rudders, tunnel thrusters going through the hull of the vessel and azimuth thrusters that can rotate under the hull and produce forces in different directions. Other actuators are water jets and stabilizing fins, but they are more often used in higher speed regime control systems. Dependent on the vessels operational regime and

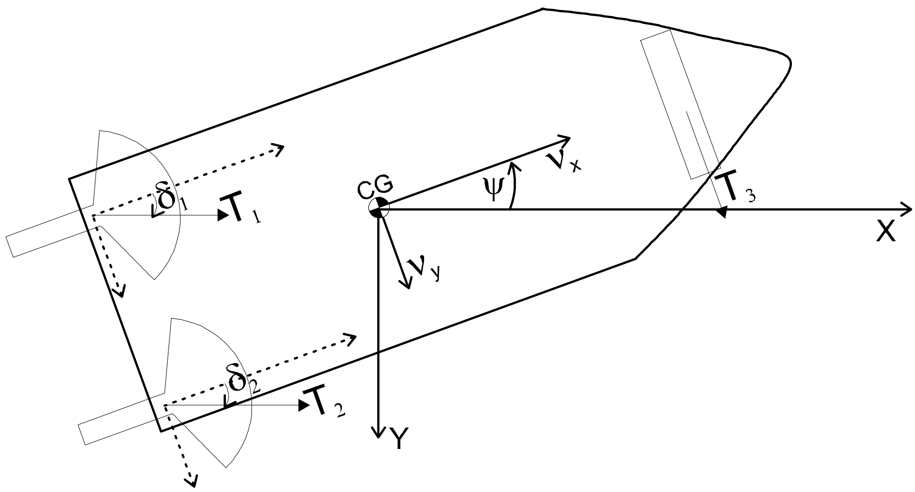

Figure 1.5: Model ship diagram

requirements for handling environmental disturbances, like thruster losses due to; *axial water inflow, cross coupling drag, thruster-hull interactions, propeller ventilation* and *thruster-thruster interaction* (see Sørensen et al. [1997] and Fossen and Blanke [2000] for details), the vessel thruster configurations vary. For example a typical supply vessel thruster configuration can consist of two main propellers, two tunnel thrusters and two azimuth thrusters which give 8 control variables (6 rpm and 2 directional controls) in total. In Figure 1.5 the actuator configuration for a model ship is shown (used in the following example). In safety critical and accuracy demanding operations, the actuator redundancy and actuator degeneration/failure detection, which motivate the adaptive structure, are of importance in the making of a fault tolerant reliable DP system.

In addition, it is desired to minimize the use of actuators in order to reduce maintenance costs due to wear, and to minimize fuel consumption.

Through out this thesis, the theoretical results will be exemplified on an overactuated scaled-model ship moving at low-speed. A detailed description of the scale-model is presented in the following.

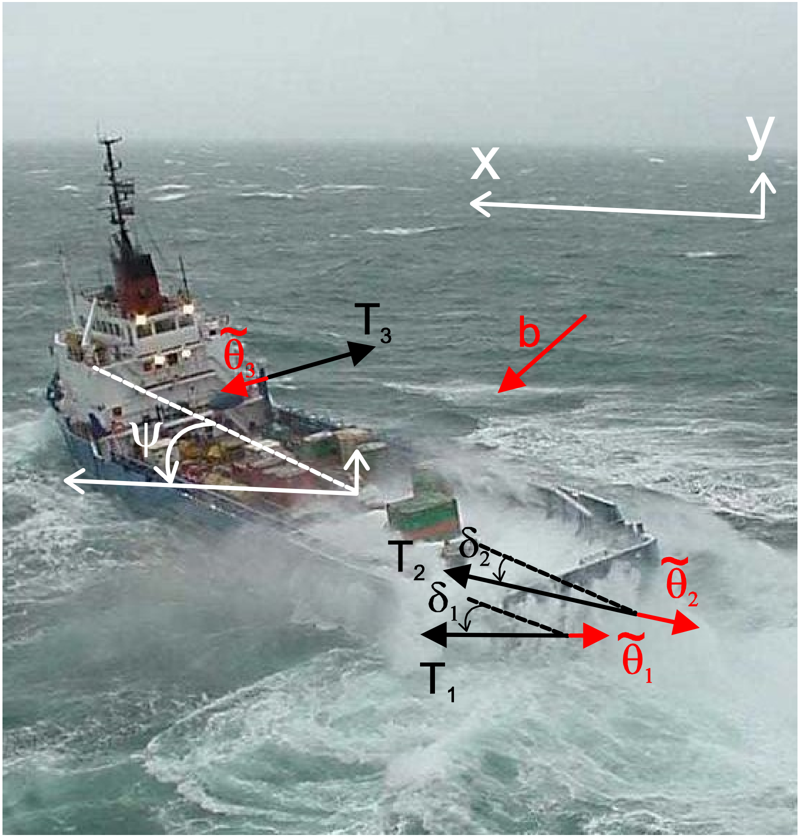

Figure 1.6: Ship illustration. Photo found in: Sørensen and Perez [2006]

The position of the scale model-ship is controlled while experiencing disturbances caused by wind and current, and propellers trust losses. The scenario is based on a 3 Degree Of Freedom (DOF) horizontal plane model:

$$\begin{aligned} \dot{\eta}_e &= R(\psi)\nu, \\ \dot{\nu} &= -M^{-1}D\nu + M^{-1}(\tau + b), \\ \tau &= \Phi(\nu, u, \theta), \end{aligned} \tag{1.6}$$

where $\eta_e := (x_e, y_e, \psi_e)^{\mathrm{T}} := (x_p - x_d, y_p - y_d, \psi_p - \psi_d)^{\mathrm{T}}$ is the north and east positions and compass heading deviation. Subscript p and d denotes the actual and desired states. $\nu := (\nu_x, \nu_y, r)^{\mathrm{T}}$ is the body-fixed velocities in, surge, sway and yaw, τ is the generalized force vector in surge, sway and yaw, $b := (b_1, b_2, b_3)^{\mathrm{T}}$ is a disturbance vector due to wind, waves and current and $R(\psi) := \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is the rotation matrix function between the body fixed and the earth fixed coordinate frame. M is the mass matrix of the ship and D is the damping matrix. The example presented is based on Lindegaard and Fossen [2003], and is also studied in Johansen [2004]. In the considered model there are five force producing devices; the two main propellers aft of the hull, in conjunction with two rudders, and one tunnel thruster going through the hull of the vessel at the bow. ω_i denotes the propeller angular velocity where $|\omega_i| \le \omega_{\max}$ and δ_i denotes the rudder deflection, where $|\delta_i| \le \delta_{\max}$. $i = 1, 2$ denotes the aft actuators, while $i = 3$ denotes the bow tunnel thruster. The following notation and models are introduced in order to rewrite the problem in the form of (1.1)-(1.3).

$$x := (\eta_e, \nu)^{\mathrm{T}}, \quad \theta_1 := (\theta_{u1}, \theta_{u2}, \theta_{u3})^{\mathrm{T}}, \quad \theta_\tau := (\theta_{\tau 1}, \theta_{\tau 2}, \theta_{\tau 3})^{\mathrm{T}}, \tag{1.7}$$

$$\tau := (\tau_1, \tau_2, \tau_3)^{\mathrm{T}}, \quad u := (\omega_1, \omega_2, \omega_3, \delta_1, \delta_2)^{\mathrm{T}}, \tag{1.8}$$

$$f := \begin{pmatrix} R(\psi_e + \psi_d)\nu \\ -M^{-1}D\nu \end{pmatrix}, \quad g := \begin{pmatrix} 0 \\ M^{-1} \end{pmatrix}, \tag{1.9}$$

$$\Phi(\nu, u, \theta) := G_u(u) \begin{pmatrix} T_1(\nu_x, \omega_1, \theta_{u1}) \\ T_2(\nu_x, \omega_2, \theta_{u2}) \\ T_3(\nu_x, \nu_y, \omega_3, \theta_{u3}) \end{pmatrix} + R(\psi_p)\theta_\tau, \tag{1.10}$$

$$G_u(u) := \begin{pmatrix} (1 - D_1) & (1 - D_2) & 0 \\ L_1 & L_2 & 1 \\ \Phi_{31} & \Phi_{32} & l_{3,x} \end{pmatrix}, \tag{1.11}$$

$$\Phi_{31}(u) := -l_{1,y}(1 - D_1(u) + l_{1,x}L_1(u)),$$
$$\Phi_{32}(u) := -l_{2,y}\left(1 - D_2(u) + l_{2,x}L_2(u)\right).$$

The thruster forces are defined by:

$$T_i(\upsilon_x, \omega_i, \theta_{ui}) := T_{ni}(\omega_i) - \phi_i(\omega_i, \upsilon_x)\theta_{ui}, \tag{1.12}$$

$$T_{ni}(\omega_i) := \begin{cases} k_{Tp_i}\omega_i^2 & \omega_i \ge 0 \\ k_{Tn_i}|\omega_i|\omega_i & \omega_i < 0 \end{cases}, \tag{1.13}$$

$$\phi_1(\omega_1, \nu_x) := \omega_1 \nu_x, \tag{1.14}$$
$$\phi_2(\omega_2, \nu_x) := \omega_2 \nu_x, \tag{1.15}$$
$$\phi_3(\omega_3, \nu) := \sqrt{\nu_x^2 + \nu_y^2}\, |\omega_3|\, \omega_3, \tag{1.16}$$
$$\theta_{u3} := k_{T\theta_3}, \tag{1.17}$$
$$\theta_{u1} := \begin{cases} k_{T\theta_1}(1-w) & \nu_x \geq 0 \\ k_{T\theta_1} & \nu_x < 0 \end{cases}, \tag{1.18}$$
$$\theta_{u2} := \begin{cases} k_{T\theta_2}(1-w) & \nu_x \geq 0 \\ k_{T\theta_2} & \nu_x < 0 \end{cases}, \tag{1.19}$$

where $0 < w < 1$ is the wake fraction number, $\phi_i(\omega_i, \upsilon_x)\theta_{ui}$ is the thrust loss due to changes in the advance speed, $\upsilon_a = (1-w)\upsilon_x$, and the unknown parameters θ_{ui} represents the thruster loss factors. The rudder lift and drag forces are:

$$L_i(u) := \begin{cases} (1 + k_{Ln_i}\omega_i)(k_{L\delta 1_i} + k_{L\delta 2_i}\,|\delta_i|)\delta_i & ,\omega_i \geq 0 \\ 0 & ,\omega_i < 0 \end{cases}, \tag{1.20}$$
$$D_i(u) := \begin{cases} (1 + k_{Dn_i}\omega_i)(k_{D\delta 1_i}\,|\delta_i| + k_{D\delta 2_i}\delta_i^2) & ,\omega_i \geq 0 \\ 0 & ,\omega_i < 0 \end{cases}. \tag{1.21}$$

Furthermore it is clear from (1.12) that
$\Phi(\nu, u, \theta) = G_u(u)Q(u) + G_u(u)\phi(\omega, \nu_x)\theta_u + R(\psi_e)\theta_\tau$, where $\phi(\omega, \nu_x) := diag(\phi_1, \phi_2, \phi_3)$ and $Q(u)$ represents the nominal propeller thrust and θ_τ represents unknown external disturbances, such as ocean current, that are considered constant or slowly time-varying in the earth fixed coordinate frame.

The *High level controller*

$$\tau_c := -K_i R^T(\psi)\xi - K_p R^T(\psi)\eta_e - K_d \nu, \tag{1.22}$$

proposed in Lindegaard and Fossen [2003], stabilizes the equilibrium of the system (1.6) augmented with the integrator

$$\dot{\xi} = \eta_e, \tag{1.23}$$

uniformly and exponentially, for some physically limited yaw rate.

The actuator dynamics for each propeller is based on the propeller model presented in Pivano et al. [2007] and given by

$$J_{mi}\dot{\omega}_i = -k_{fi}(\omega_i) - \frac{T_{ni}}{a_T}(\omega_i) + \frac{\phi_i(\omega_i, \upsilon_x)\theta_{1i}}{a_T} + u_{cmdi}, \tag{1.24}$$

where J_m is the shaft moment of inertia, k_f is a positive coefficient related to the viscous friction, a_T is a positive model constant Pivano et al. [2006] and u_{cmd} is the commanded motor torque. The rudder model is linearly time-variant and the dynamics are given by:

$$m_i\dot{\delta} = a_i\delta_i + b_i u_{cmd\delta i}, \tag{1.25}$$

where a_i, b_i are known scalar parameter bounded away from zero.

The parameters of the system are given in Table A.1 in Appendix A.5, and the simulation results generated by setting $\tau = \tau_c$ is presented in Figure A.1 and A.2, Appendix A.5. The control allocation problem, where the desired thruster and rudder deflection controls u are calculated based on the virtual control τ_c and the minimization of the actuator power consumption, is presented as examples in Chapter 3, 4 and 5.

1.2 Literature review

Since the first motorized aircraft were built at the beginning of the 20th century, the problem of control allocation has been an important field of research in the aviation society. Early flight control system implementations were entirely mechanical, and closely connected to the design and construction of the aircraft. In the 1950-60s *fly-by-wire* (electronically signaled control system) technology was developed and as the hydraulic circuits had replaced the mechanical transmissions, computerized control and allocation systems became available such that the control allocation designs was no longer directly connected to the airplane construction. Through the last decades, many different control allocation schemes have been proposed and developed for air vehicle control systems, Virnig and Bodden [1994], Enns [1998], Durham [1993], Buffington and Enns [1996], Luo et al. [2007], Härkegård [2002]. As the control allocation design advantages in general apply for overactuated mechanical control systems, a variety of control allocation schemes are proposed for automotive vehicle control systems, e.g. Ackermann et al. [1995], Alberti and Babbel [1996], Wang and Longoria [2006], Schofield et al. [2006], Piyabongkarn et al. [2006], and marine control systems due to requirements from redundancy and fault tolerance, e.g. Lindfors [1993], Lindegaard and Fossen [2003], Johansen et al. [2004], Fossen and Johansen [2006].

Mixing and blending systems are other processes where control allocation approaches could be applied. In Westerlund et al. [1979], a cement mixing plant is described, where the control objective is to maintain a desired composition (specified by three components, x) of the cement raw meal, despite disturbances from the raw material silos (five raw materials, u). In Johansen and Sbárbaro [2003] and Johansen and Sbarbaro [2005] an optimizing color blending process is considered where the resulting color mixture, measured in RGB values (vector of size 3), is governed by the input of three or more colorants. Although the controller designs from Westerlund et al. [1979] and Johansen and Sbárbaro [2003] do not have the control allocation structure (separation of the control and allocation), the redundancy of controls suggests that an optimizing scheme like the *model predictive control* (MPC) could be applied directly or through a problem reformulation incorporated with an observer scheme.

An overactuated system contains some available redundancy in the actuator design, such that given a desired virtual control τ_c, the control allocation problem is naturally formulated as an *nonlinear programming problem* (NLP) subject to some actuator limits,

$$\min_{\underline{u} \leq u \leq \overline{u}} J(t, x, u) \quad s.t. \quad \tau_c - \Phi(t, x, u, \theta) = 0. \tag{1.26}$$

The available degrees of freedom can be used to minimize the cost $J(t, x, u)$, which may contain terms that penalizes power consumption, wear/tear, effects related to actuator configuration (e.g. singularity avoidance) and safety critical effects, e.g. forbidden sectors in a marine vessel supporting diving operations or large side slip in an automotive vehicle. In the following the control allocation method is characterized as either being *implicit* (the solution of (1.26) is iterative) or *explicit*. If $J = c^{\mathrm{T}}u$ and $\Phi = Bu$, the problem can be formulated as a standard *linear programming* (LP) problem,

$$\min_u c^{\mathrm{T}}u \quad s.t. \quad Bu = \tau_c, u + z_1 = \overline{u}, u - z_2 = \underline{u}, \quad z_1 \geq 0, z_2 \geq 0.$$

Simplex (since the 1950s) and *interior point* (IP) methods (since 1980s), both implicit methods, are today the most commonly used algorithms for solving LP's, Nocedal and Wright [1999]. The simplex methods, where the search for optimality is done by visiting the vertices of the polytope described by the constraints of the problem, are usually the practically most efficient algorithms. Interior point methods, where the optimality search is done from the interior and/or exterior of the constraint polytope, has better theoretical convergence properties, Nocedal and Wright [1999], and are often preferred for large scale problem (> $10k$ variables) or when warm state initialization based on the previous time step solution is not stored or found. By approximating the NLP with a piecewise linear representation, a modified simplex, Reklaitis et al. [1983] or a mixed integer method, Wolsey [1998], may be used to solve the optimization problem. In Bolender and Doman [2004], two such algorithms are implemented, compared and tested for the closed loop performance of a re-entry vehicle. It was shown that solving the simplex based approach is significantly faster than solving the same problem using a mixed-integer formulation. Advantages of control allocation schemes that utilize online optimization are flexibility and ability for online re-configuration. The drawbacks with such algorithms are the complexity of the implementation code, the variable number of iterations and further complexity introduced by anti-cycling procedures. An in-depth discussion on online optimization methods, with the conclusion that constrained optimization methods can realistically be considered for real-time control allocation in flight control systems, can be found in Bodson [2002].

The direct control allocation method, first posed in Durham [1993], is a control allocation strategy based on geometric reasoning. The idea relies on finding the intersection between the boundary of the attainable controls and the desired virtual control τ_c. The actuator controls are then scaled down from this intersection. The problem can be formulated as an LP

$$\max_{u_1,\rho} \rho \quad s.t. \quad Bu_1 = \rho\tau_c \quad \underline{u} \leq u_1 \leq \overline{u},$$

proposed in Bodson [2002], and is implemented by; if $\rho > 1$, then $u = \frac{1}{\rho}u_1$ else $u = u_1$. The facet searching algorithm from Durham [1993] and null space interaction method in Bordignon and Durham [1995] are other methods suggested for solving the direct control allocation problem. Advantages of the direct control allocation method are the maximum control utilization and the existence of computationally fast implementations, Petersen and Bodson [2002]. Difficulties with the direct allocation scheme, are rate-limit implementation (the actuator constraint set should contain zero) and lack of axis prioritization.

If the allocation problem is unconstrained (no actuator bounds) and the cost function is quadratically dependent on the control input vector, $J = u^{\mathrm{T}}Wu$, (*unconstrained least square problem)* an explicit optimal solution based on *Lagrangian multipliers* (λ) is available. Based on the *Lagrangian function*

$$L = u^{\mathrm{T}}Wu - (\tau_c - Bu)^{\mathrm{T}}\lambda,$$

the explicit generalized inverse solution

$$u = W^{-1}B^{\mathrm{T}}\left(BW^{-1}B^{\mathrm{T}}\right)^{-1}\tau_c$$

can be derived by solving the algebraic equations for the optimal condition; $\frac{\partial L}{\partial u} = 0$, $\frac{\partial L}{\partial \lambda} = 0$. By assuming $u > 0$, the generalized inverse solution is, in Sørdalen [1997], used to solve the trust allocation problem for marine vessels. And in Wang and Longoria [2006] it is used to distribute brake forces in a automotive vehicle dynamic control scheme. A variety of explicit algorithms based on the generalized inverse are in the literature posed for the *constrained least square problem*. Daisy chain solutions; Durham [1993], redistributed pseudoinverse solutions; Virnig and Bodden [1994] and Eberhardt and Ward [1999], and cascaded generalized inverse solutions; Durham [1993] and Bordignon [1996], are examples of such algorithms.

In a Daisy chain control allocation strategy, the actuator controls (u) are partitioned into groups such that if the virtual control demands are not satisfied by the first group of actuators, the remaining demands are passed to the second group of actuators. If there still are virtual control demands that are not satisfied, those are passed to the third group etcetera. The cascaded generalized inverse and redistributed pseudoinverse solutions are multistep algorithms. If in the first step the control commands suggested are not satisfied and some actuator controls are saturated, then by multiple steps the saturating controls are set to the limits and the remaining control commands are allocated to the actuator controls which are not saturated. The algorithms end when no more control freedom is available. Daisy chain and multiple step methods are referred to as non-optimal or approximately optimal solutions, because the allocated actuator controls are not obtained from the entire attainable set, Beck [2002] and Petersen and Bodson [2006]. The advantages of such explicit algorithms are ease of implementation, fast computations and fixed number of iteration. On the downside the optimal solution is not necessarily found and large errors may occur.

In Petersen and Bodson [2006] an IP method, based on Vanderbei [1999], is implemented in order to solve a *quadratic program* (QP, linear constraints and second order polynomial cost function) exactly. The method is compared with a fixed point, Burken et al. [1999], and an active set method, Härkegård [2002]. The advantage of the IP method is uniform convergence and knowledge of the relative distance to the optimal solution. The active set method applied to a QP takes a similar form as the simplex method for the LP. The actuator controls are divided into a saturated (active) set and an unsaturated (free) set, and the updates of these sets are calculated based on the pseudoinverse solution of the free set and the Lagrangian parameters reflected by the active set (see Härkegård [2002] and Petersen and Bodson [2006] for details). The active set algorithm converges to the optimum in a finite number of steps and it is shown to be efficient for problems of small to medium size. But even though feasible suboptimal solutions are produced at each iteration step, the relative distance to the optimum is not known. An interesting application of the active set algorithm was shown in Schofield [2006], where the algorithm was used to allocate wheel braking commands in order to prevent vehicle rollover. The fixed point method is a recursive algorithm similar to a gradient search. The algorithm has a proven global convergence but may be slow when unattainable commands are given. An advantage of the QP over the LP formulation for the control allocation problem, is that a QP solution involve all the actuators that will effect the equality constraint in (1.26) since the cost function is quadratic, while the nature of a LP solution relies on a smaller number of actuators due to the linear cost function. This may reduce the degeneration of performance, related to a QP solution compared with a LP solution, after an actuator failure, since more actuators are used in a QP solution, Page and Steinberg [2002].

Receding horizon control (RHC) or MPC, that first appeared in the sixties, is a powerful control method with applications to large scale multi variable constrained systems of sufficiently slow dynamics, Qin and Badgewell [1997] or where fast computation is available for small scale systems, Luo et al. [2004]. The control action is calculated at each sample for a given horizon, based on an open loop optimization. Since the process is repeated at each sample a closed loop control is obtained. The RHC optimization problem may be formulated in discrete-time setup by:

$$
\begin{aligned}
\min_{u \in U} \quad & J_E(x(k+N)) + \sum_{i=k}^{k+N-1} J(x(i),u(i)), \\
x(i) \quad \in \quad & X \quad \forall i \in (k+1,...,k+N), \\
x(i+1) \quad = \quad & f(x(i),u(i)) \quad \forall i \in (k,...,k+N-1), \\
x(k) \quad = \quad & x_0,
\end{aligned}
$$

where x_0 is the initial state, X and U defines the constraints on x and u, J_E is terminal state cost, J is the stage cost and $f(x(i),u(i))$ represent the model difference equation. In Luo et al. [2005] and Luo et al. [2007] a MPC program (implicit) is suggested in a control allocation approach which accounts for actuator dynamics described by a linear time-varying actuator model. Simulation studies of a reentry vehicle show significant performance improvement compared to static control allocation algorithms in the presence of realistic actuator dynamics.

In Bemporad et al. [2002a]/Bemporad et al. [2002b] it was shown that the linear RHC ($f(x(i), u(i)) := Ax(i) + Bu(i)$) with quadratic/linear cost can be formulated as a *multiparametric QP* (mpQP)/*multiparametric LP* (mpLP) program. A multiparametric problem is generally formulated as

$$J^*(p) := \min_z J(p, z) \quad s.t. \quad g_J(z, p) \geq 0, \quad h_J(z, p) = 0,$$

where z is the optimization variable, p is a parameter vector, J is a cost function, g_J is the inequality constraints and h_J is the equality constraints. If $J(p, z) := c^{\mathrm{T}} z$, $g_J(z, p) := z$ and $h_J(z, p) := Az - Sp - b$ the problem has a mpLP form. And it is called a mpQP if, $J(p, z) = z^{\mathrm{T}} Hz + pFp + c^{\mathrm{T}} p$ and $g_J(z, p) = -Az + b + Sp$. H, F, A, S, c, b are matrices and vectors defined according to the sizes of the vectors z and p. In Tøndel and Johansen [2005] an explicit *piecewise linear* (PWL) control allocation law is constructed by a mpQP, based on a RHC formulation of a vehicle lateral stabilization problem. In Johansen et al. [2005] an explicit PWL control allocation law based on the off-line solution of mpQP, is verified by experimental results for a scale model ship. The main advantage of a PWL control is simple implementation, no need of real-time optimization software and fast online control computations, Tøndel et al. [2003]. The worst computation time can be stated a priori such that solutions are generated within hard real-time bounds. Due to rapid growth of solution complexity and memory requirements, as problem size increases, the explicit solutions of a mpLP/mpQP are not available for typical large scale problems. Reconfiguration of the optimization problem, due to for example sensor or actuator failure, may also be more complicated and demand considerable off-line computation time and real-time computer memory, compared with an on-line optimization solution. An excellent review of multiparametric programming and RHC can be found in Tøndel [2003].

In order for the allocation algorithm to perform in an optimal way, the knowledge of the actuator-force mapping model $\Phi(t, x, u, \theta)$ is of importance. This model is dependent on environmental conditions and actuator failures like; changing ground surfaces (environmental) and wheel lift off or brake power losses (actuator failures) for automotive vehicles, wind currents and air conditions (environmental) and control surface or engine malfunction (actuator failure) for an aircraft, and currents and waves disturbances (environmental) and thruster losses (actuator degradation) affecting a ship. In Davidson et al. [2001] a combined static (no actuator dynamic) nonlinear control allocation and failure detection scheme is developed and implemented in the simulation environment of a modified *Lockheed-Martin Innovative Control Effector* (LM-ICE) aircraft version. The aircraft's degraded manoeuvrability and survivability is improved through the re-configuration of the allocation problem when subject to failure. For automotive vehicles it is in Shim and Margolis [2001] shown that the knowledge of the friction coefficient offers significant improvement of the vehicle response during yaw rate control.

The foundation of this thesis relies on the control allocation scheme presented in Johansen [2004], where the minimization problem (2.5) is solved dynamically by defining update laws u and λ and considering convergence to the first order optimal solution described by the Lagrangian function

$$L(t, x, u, \theta) := J(t, x, u) + (\tau_c - \Phi(t, x, u, \theta))^{\mathrm{T}}\lambda.$$

This leads to a solution, opposed to for example the developments, Enns [1998], Buffington et al. [1998], Sørdalen [1997], Johansen et al. [2005], Bodson [2002] and Härkegård [2002], where the control allocation problem is viewed as a *static* or *quasi-dynamic* optimization problem that is solved independently of the dynamic control problem considering non-adaptive linear effector models of the form $\tau = Gu$. The main feature of the presented approach is that convergence to the optimal solution can be guaranteed without solving the minimization problem (2.5) exactly for each sample. This allows the construction of an efficient algorithm by relatively simple implementation, and the stability of the closed loop to be analyzed. The core object of this work is to analyze the dynamic solutions through set stability arguments and incorporate parameter uncertainty and actuator dynamic in the control allocation design.

1.3 Contribution and outline

The results of this thesis relates to the over-actuated nonlinear control allocation problem, with uncertainties in the actuator-force mapping (1.2) and/or actuator model (1.3). The work is mainly based on ideas from Johansen [2004], where the control allocation is formulated in the context of Lyapunov design, with the emphasis on extensions and assumption relaxation incorporating adaptation and actuator dynamics. Motivated by the control allocation strategy from Johansen [2004], the contribution of the thesis can be presented in three parts, where the Lyapunov design allow the control allocation problem to be combined and analyzed together with nonlinear control and parameter estimation; the theoretical results for dynamic control allocation, the development of a cascade analysis tool used in the analysis of the control allocation schemes, and the implementation of the control allocation algorithm for controlling the yaw motion of an automotive vehicle using brakes and active steering.

- A modular design approach is formalized by the construction of a cascade lemma for set stable systems, Tjønnås et al. [2006]. This result motivates a modular approach where the subsystems can be analyzed separately and conclusions about the properties of the cascaded system may be drawn based on the interconnection term instead of analyzing the cascade as one system, which in many cases induces complexity. Roughly speaking, it is shown that if a set is uniform global asymptotic stable (UGAS) with respect to a system, then the composed set generated by a cascade of two such systems is itself UGAS under the assumption that the solutions of the cascaded system are UGB with respect to the composed set. The cascade result is in the following presented in Subsection 1.4.3, together with some corollaries that are important for local and not necessarily converging cases.

- Dynamic nonlinear and adaptive control allocation designs are proposed as direct update laws in, Tjønnås and Johansen [2008a] and Tjønnås and Johansen [2005]. Actuator dynamics are considered in, Tjønnås and Johansen [2007a] and Tjønnås and Johansen [2007b]. The ideas from Johansen [2004] are extend by utilizing the set-stability result for cascaded systems established in Tjønnås et al. [2006] and the result enables us to relax the assumptions in Johansen [2004] where $f(t, x)$, $g(t, x)$ and $\Phi(t, x, u, \theta)$ from equation the system (1.1) and (1.2) are assumed to be globally Lipschitz in x. Further the virtual controller τ_c does only need to render equilibrium of (1.1) UGAS for $\tau = \tau_c$, not UGES as assumed in Johansen [2004]. The implementation of the adaptive law presented in Tjønnås and Johansen [2005] depends directly on the Lyapunov function used in the analysis. In Tjønnås and Johansen [2008a] the analysis and implementation are separated, and although the Lyapunov functions in the analysis need to satisfy certain requirements, the adaptive implementation does not assume knowledge of these Lyapunov functions. The chapters 3, 4 and 5 present the theoretical results related to the control allocation.

- The proposed designs are verified in an automotive case study by realistic simulations in Tjønnås and Johansen [2008b] and Tjønnås and Johansen [2006]. Based on the dynamic control allocation strategy, the stabilization of automotive vehicles using active steering and adaptive braking has been formulated as a control allocation problem. The stabilization strategy is based on two modules independent in design, a high level module that deals with the motion control objective and a low level module that deals with the actuator control allocation. The high level module consists of yaw-rate reference generator and high level controller that provides the low level control allocation module with a desired torque about the yaw axis. The task of the low level module is to command the individual brakes, the longitudinal clamping force, such that the actual torque about the yaw axis tends to the desired torque. These commands are generated by a dynamic control allocation algorithm that also takes actuator constraints and uncertainty in the tyre-road friction model into consideration. The scheme has been implemented in a realistic nonlinear multi body vehicle simulation environment. The control allocation scheme presented in the framework of yaw stabilization is given in Chapter 6.

In Chapter 2 the control structure, the problem statement and main assumptions are presented.

1.4 Notation, definitions and preliminaries

1.4.1 Notation

$\mathbb{R}$ and $\mathbb{R}^n$ denote the set of real numbers and the n-dimensional *Euclidean space.* $\mathbb{R}_{\geq 0}$ represent the set of real numbers which has a value greater or equal to zero. ":=" and "=:" has the meaning of being defined by right and left. "iff" has the meaning: "if and only if". Let $x := (x_1, ..., x_n)^{\mathrm{T}}$ be a n-dimensional column vector such that $x \in \mathbb{R}^n$, then the *Euclidean norm* is defined by $|x| := \sqrt{x^{\mathrm{T}}x}$. Furthermore the *Euclidean metric* is defined by $d(x_a, x_b) := |x_a - x_b| := \sqrt{(x_a - x_b)^{\mathrm{T}}(x_a - x_b)}$, where $x_a, x_b \in \mathbb{R}^n$.

Let $(\mathbb{D}, d)$ be a metric space where $\mathbb{D} \subseteq \mathbb{R}^n$, if $x_a \in \mathbb{D}$ and $r > 0$ then the *open ball* of radius r about x_a is defined by

$$\mathcal{B}(x_a, r) := \{x_a \in \mathbb{D} : d(x_b, x_a) < r\}.$$

The set $\mathcal{O} \subseteq \mathbb{R}^n$ is *open* if for every $x \in \mathcal{O}$ there exists a scalar $r > 0$ such that $\mathcal{B}(x, r) \subseteq \mathcal{O}$. $\mathcal{O}$ is *closed* if its complement $(\mathbb{R}^n/\mathcal{O})$ is open. $|\cdot|_{\mathcal{O}} : \mathbb{R}^q \mapsto \mathbb{R}_{\geq 0}$ denotes the distance from a point $x_a \in \mathbb{R}^n$ to a set $\mathcal{O} \subset \mathbb{R}^n$, $|x_a|_{\mathcal{O}} := \inf\{|x_a - x_b| : x_b \in \mathcal{O}\}$. The induced matrix norm is defined by: $\|A\| := \{|Ax| : x \in \mathbb{R}^n, |x| \leq 1\}$.

If $f : X \to Y$ is a function, then X is the domain and $f(X)$ is the range. The function f is *injective* if $f(x_a) = f(x_b)$ only when $x_a = x_b$, *surjective* if $f(X) = Y$ and *bijective* if it is both surjective and injective. The function f is *continuous* if for every point $x_a \in X$, and $\epsilon > 0$ there exists a $\delta > 0$ such that $|x_a - x_b| < \delta \Rightarrow |f(x_a) - f(x_b)| < \epsilon$. If a function $f \in C^0$ the function is continuous, if $f \in C^1$, then f and its first derivative are continuous, etcetera. f is *locally Lipschitz* on the domain $\mathbb{D} \subseteq \mathbb{R}^n$ if each point $x_a \in \mathbb{D}$ has a neighborhood $\mathbb{D}_0$ there exists a constant $L \geq 0$, where $|f(x_a) - f(x_b)| \leq L\,|x_a - x_b|$ for $x_b \in \mathbb{D}_0$. If there exists a $L > 0$ such that, $|f(x_a) - f(x_b)| \leq L\,|x_a - x_b|$ is satisfied for $\mathbb{D}_0 = \mathbb{D} = \mathbb{R}^n$, then f is *globally Lipschitz.*

Let $\phi : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}$, then $\phi \in \mathcal{L}_p$ if $\left(\int_0^\infty |\phi(\tau)|^p \, d\tau\right)^{\frac{1}{p}}$ exists, and $\phi \in \mathcal{L}_\infty$ if $\sup_{\tau > 0} \phi(\tau)$ exists.

The function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a *class* $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. α is of *class* $\mathcal{K}_\infty$ if in addition $\alpha(s) \to \infty$ as $s \to \infty$. $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a *class* $\mathcal{KL}$ function if, for each fixed t, the mapping $\beta(\cdot, t)$ is of class $\mathcal{K}$ and for each fixed s the mapping $\beta(s, \cdot)$ is continuous, decreasing and tends to zero as its argument tends to $+\infty$. The function f is *uniformly bounded* by y, if there exist a function G_F of *class* $\mathcal{K}_\infty$ and a scalar $c > 0$ such that $|F(t, y, z)| < G_f\left(|y|\right) + c$ for all y, z and t.

The time derivatives of a signal $x(t)$ are denoted $\dot{x} := \frac{dx}{dt}$, $\ddot{x} := \frac{d^2x}{dt^2}$, ... If the partial derivative of a function $f(x_a, x_b)$ is written $\frac{\partial f}{\partial x_a}$, it is defined by $\frac{\partial f}{\partial x_a} := \left.\frac{\partial f(s, x_b)}{\partial s}\right|_{s = x_a}$. We say that a function $V : \mathbb{R}^q \to \mathbb{R}_{\geq 0}$ is *smooth* if it is infinitely differentiable.

A matrix is strictly *Hurwitz* if all eigenvalues has a real part less then zero.

1.4.2 Definitions

Definition 1.1 *The signal matrix* $\Phi(t)$ *is* Persistently Excited *(PE) if, there exist constants* T *and* $\gamma > 0$, *such that*

$$\int_t^{t+T} \Phi(\tau)^T \Phi(\tau) d\tau \geq \gamma I \ , \quad \forall t > t_0. \tag{1.27}$$

The definitions that follows are either motivated by, or can be found in Teel et al. [2002] and Lin et al. [1996]. They pertain to systems of the form

$$\dot{z} = F(z)\,, \tag{1.28}$$

where $z := (p, x^{\mathrm{T}})^{\mathrm{T}}$, p is the time-state

$$\dot{p} = 1, \quad p_0 = t_0. \tag{1.29}$$

and $F : \mathbb{D} \to \mathbb{R}^q$ is locally Lipschitz with $\mathbb{D} \subset \mathbb{R}^q$. In the following, if referred to a set, it has the properties of being *nonempty.*

The solution of an autonomous dynamic system is denoted by $z(t, x_0)$ where $z_0 = z(0, z_0)$ is the initial state.

Definition 1.2 *The system (1.28) is said to be* forward complete *if, for each $z_0 \in \mathbb{D}$, the solution $z(\cdot, z_0) \in \mathbb{D}$ is defined on $\mathbb{R}_{\geq t_0}$.*

Definition 1.3 *The system (1.28) is said to be* finite escape time detectable *through $|\cdot|_{\mathcal{A}}$, if any solution, $z(t, z_0) \in \mathbb{D}$, which is right maximally defined on a bounded interval $[t_0, T)$, satisfies $\lim_{t \nearrow T} |z(t, z_0)|_{\mathcal{A}} = \infty$.*

Definition 1.4 *If the system (1.28) is forward complete, then the closed set $\mathcal{A} \subset \mathbb{D}$ is:*

- Uniformly Stable *(US), if there exists a function, $\nu \in \mathcal{K}$, and a constant, $c > 0$, such that, $\forall \ |z_0|_{\mathcal{A}} < c$,*

$$|z(t, z_0)|_{\mathcal{A}} \leq \nu(|z_0|_{\mathcal{A}}), \quad \forall t \geq 0\,. \tag{1.30}$$

- Uniformly Globally Stable *(UGS), when $\mathbb{D} = \mathbb{R}^q$, if (1.30) is satisfied with, $\nu \in \mathcal{K}_\infty$, and for any $z_0 \in \mathbb{R}^q$.*

- Uniformly Attractive *(UA) if there exists a constant $c > 0$ such that for all $|z_0|_{\mathcal{A}} < c$ and any $\mu > 0$ there exists $T = T(\mu) > 0$, such that*

$$|z_0|_{\mathcal{A}} \leq c, \ t \geq T \quad \Rightarrow \quad |z(t, z_0)|_{\mathcal{A}} \leq \mu, \tag{1.31}$$

- Uniformly Globally Attractive *(UGA), when $\mathbb{D} = \mathbb{R}^q$, if for each pair of strictly positive numbers (c, μ) there exists $T = T(\mu) > 0$ such that for all $z_0 \in \mathbb{R}^q$, (1.31) holds.*

- Uniformly Asymptotically Stable *(UAS) if it is US and UA.*

- Uniformly Globally Asymptotically Stable *(UGAS), when $\mathbb{D} = \mathbb{R}^q$, if it is UGS and UGA.*

When (1.28) is forward complete, UGAS is well known to be equivalent to the following $\mathcal{KL}$ characterization (see e.g. Lin et al. [1996], Teel and Praly [2000]): There exists a class $\mathcal{KL}$ function β such that, for all $z_0 \in \mathbb{R}^q$, $|z(t, z_0)|_{\mathcal{A}} \leq \beta(|z_0|_{\mathcal{A}}, t)$ $\forall t \geq t_0$.

- Uniformly Globally Exponentially Stable *(UGES), when* $\mathbb{D} = \mathbb{R}^q$*, if it is UGS and if there exist numbers* $k, \lambda > 0$ *such that for all* $z_0 \in \mathbb{R}^q$, $|z(t, z_0)|_{\mathcal{A}} \leq k|z_0|_{\mathcal{A}} e^{\lambda t}$ $\forall t \geq t_0$.

From Khalil [1996], we adapt the definition of uniform boundedness of solutions to the case when $\mathcal{A}$ is not reduced to the origin $\{0\}$.

Definition 1.5 *With respect to the closed set* $\mathcal{A} \subset \mathbb{R}^q$*, the solutions of system (1.28) are said to be:*

- Uniformly Bounded (UB) *if there exist a positive constant* c*, such that for every positive constant* $r < c$ *there is a positive constant* $\mu = \mu(r)$*, such that*

$$|z_0|_{\mathcal{A}} \leq r \quad \Rightarrow \quad |z(t, z_0)|_{\mathcal{A}} \leq \mu \,, \qquad \forall t \geq 0 \,. \tag{1.32}$$

- Uniformly Globally Bounded *(UGB), if for every* $r \in \mathbb{R}_{\geq 0}$*, there is a positive constant* $\mu = \mu(r)$ *such that (1.32) is satisfied.*

Definition 1.6 *A smooth* Lyapunov function *for (1.28) with respect to a non-empty, closed forward invariant set* $\mathcal{A} \subset \mathbb{D}$ *is a function* $V : \mathbb{D} \to \mathbb{R}$ *that satisfies: i) there exists two* $\mathcal{K}$ *functions* α_1 *and* α_2 *such that for any* $z \in \mathbb{D}$, $\alpha_1(|z|_{\mathcal{A}}) \leq V(z) \leq \alpha_2(|z|_{\mathcal{A}})$*. ii) There exists a continuous and positive definite/semidefinite function* α_3 *such that for any* $z \in \mathbb{D} \backslash \mathcal{A}$*:* $\frac{\partial V}{\partial z}(z) F(z) \leq -\alpha_3(|z|_{\mathcal{A}})$.

1.4.3 Some set stability results

Theorem 1.1 *Assume that the system (1.28) is finite escape-time detectable through* $|z|_{\mathcal{A}}$*. If there exists a smooth Lyapunov function for the system (1.28) with respect to a nonempty, closed, forward invariant set* $\mathcal{O}$*, then* $\mathcal{O}$ *is UGS with respect to (1.28). Furthermore, if* α_3 *is a positive definite function, then* $\mathcal{O}$ *is UGAS with respect to (1.28). Moreover, if* $\alpha_i(|z|_{\mathcal{O}}) = k_i\, |z|_{\mathcal{O}}^r$ *for* $i = 1, 2, 3$ *where* k_i *and* r *are strictly positive values and* $r > 1$*, then* $\mathcal{O}$ *is UGES with respect to (1.28).*
Proof. *The proof of this result can be found in Skjetne [2005]* ■

Consider the cascaded system

$$\dot{x}_1 = f_{c1}(t, x_1) + g_c(t, x_1, x_2), \tag{1.33}$$

$$\dot{x}_2 = f_{c2}(t, x_2). \tag{1.34}$$

When the functions f_{c1}, f_{c2} and g_c are locally Lipschitz in all arguments, this class of nonlinear time-varying systems can be represented by the following autonomous systems

$$\dot{z}_1 = F_1(z_1) + G(z), \tag{1.35}$$

$$\dot{z}_2 = F_2(z_2), \tag{1.36}$$

where $z_1 := (p, x_1^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{q_1}$, $z_2 := (p, x_2^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{q_2}$, $z := (p, x^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{q}$, $x := (x_1^{\mathrm{T}}, x_2^{\mathrm{T}})^{\mathrm{T}}$, $F_1(z_1) := (1, f_{c1}(p, x_1)^{\mathrm{T}})^{\mathrm{T}}$, $G(z) := (0, g_c(p, x)^{\mathrm{T}})^{\mathrm{T}}$ and $F_2(z_2) := (1, f_{c2}(p, x_2)^{\mathrm{T}})^{\mathrm{T}}$. Based on the cascaded system formulation the main results from Tjønnås et al. [2006] are.

Lemma 1.1 *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be some closed subsets of $\mathbb{R}^{q_1}$ and $\mathbb{R}^{q_2}$ respectively. Then, under the following assumptions, the set $\mathcal{O} := \mathcal{O}_1 \times \mathcal{O}_2$ is UGAS with respect to the cascade (1.35)-(1.36).*

A 1.1 *The set $\mathcal{O}_2$ is UGAS with respect to the system (1.36) and that the solution of the system (1.35)-(1.36) is UGB with respect to $\mathcal{O}$.*

A 1.2 *The functions F_1, F_2 and G are locally Lipschitz.*

A 1.3 *The cascade (1.35)-(1.36) is forward complete.*

A 1.4 *There exist a continuous function $G_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and a class $\mathcal{K}$ function G_2 such that, for all $z \in \mathbb{R}^q$,*

$$|G(z)| \leq G_1(|z|_{\mathcal{O}}) G_2(|z_2|_{\mathcal{O}_2}). \tag{1.37}$$

A 1.5 *There exists a continuously differentiable function $\overline{V}_1 : \mathbb{R}^{q_1} \to \mathbb{R}_{\geq 0}$, class $\mathcal{K}_\infty$ functions $\overline{\alpha}_1$, $\overline{\alpha}_2$ and $\bar{\alpha}_3$, and a continuous function $\bar{\varsigma}_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for all $z_1 \in \mathbb{R}^{q_1}$,*

$$\overline{\alpha}_1(|z_1|_{\mathcal{O}_1}) \leq \overline{V}_1(z_1) \leq \overline{\alpha}_2(|z_1|_{\mathcal{O}_1}), \tag{1.38}$$

$$\frac{\partial \overline{V}_1}{\partial z_1}(z_1) F_1(z_1) \leq -\bar{\alpha}_3(|z_1|_{\mathcal{O}_1}), \tag{1.39}$$

$$\left| \frac{\partial \overline{V}_1}{\partial x_1}(z_1) \right| \leq \bar{\varsigma}\left(|z_1|_{\mathcal{O}_1}\right) . \tag{1.40}$$

Proof. *We start by introducing the following result, which borrows from [Praly and Wang, 1996, Proposition 13], originally presented in Lakshmikantham and Leela [1969]. We have that under Assumption 4, for any nonnegative constant c, there exists a continuously differentiable Lyapunov function $V_1 : \mathbb{R}^{q_1} \to \mathbb{R}_{\geq 0}$, class $\mathcal{K}_\infty$ functions α_1, α_2, and a continuous nondecreasing function $\varsigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, for all $z_1 \in \mathbb{R}^{q_1}$,*

$$\alpha_1(|z_1|_{\mathcal{O}_1}) \leq V_1(z_1) \leq \alpha_2(|z_1|_{\mathcal{O}_1}), \tag{1.41}$$

$$\frac{\partial V_1}{\partial z_1}(z_1)F_1(z_1) \leq -cV_1(z_1), \tag{1.42}$$

$$\left|\frac{\partial V_1}{\partial z_1}(z_1)\right| \leq \varsigma\left(|z_1|_{\mathcal{O}_1}\right) . \tag{1.43}$$

Let the function $\bar{V}_1$ of Assumption 4 generate a continuously differentiable function V_1 with $c = 1$. In view of Assumption 3, the derivative of V_1 along the solutions of (1.35) then yields

$$\dot{V}_1(z_1) \leq -V_1(z_1) + \varsigma(|z_1|_{\mathcal{O}_1})G_1(|z|_{\mathcal{O}})G_2(|z_2|_{\mathcal{O}_2}) .$$

From the UGB property, there exist $\mu \geq 0$ and $\eta \in \mathcal{K}_\infty$ such that, for all $z_0 \in \mathbb{R}^q$,

$$|z(t, z_0)|_{\mathcal{O}} \leq \eta(|z_0|_{\mathcal{O}}) + \mu\,, \quad \forall t \geq 0\,. \tag{1.44}$$

Defining $v(t, z_0) := V_1(z_1(t, z_0))$ and $v_0 := V_1(z_{10})$, we get that
[1]$\max\{|z_1|_{\mathcal{O}_1}, |z_2|_{\mathcal{O}_2}\} \leq |z|_{\mathcal{O}}$.

$$\dot{v}(t, z_0) \leq -v(t, z_0) + B(|z_0|_{\mathcal{O}})G_2(|z_2(t, z_{20})|_{\mathcal{O}_2}) ,$$

where $B(\cdot) := \max_{0 \leq s \leq \eta(\cdot)+\mu} \varsigma(s)G_1(\eta(\cdot) + \mu)$. From the UGAS of (1.36) with respect to $\mathcal{O}_2$, there exists $\beta_2 \in \mathcal{KL}$ such that, for all $z_{20} \in \mathbb{R}^{q_2}$,

$$|z_2(t, z_{20})|_{\mathcal{O}_2} \leq \beta_2(|z_{20}|_{\mathcal{O}_2}, t)\,, \quad \forall t \geq 0\,. \tag{1.45}$$

Accordingly, we obtain that

$$\dot{v}(t, z_0) \leq -v(t, z_0) + \tilde{\beta}(|z_0|_{\mathcal{O}}, t)\,, \tag{1.46}$$

where $\tilde{\beta}(r, t) := B(r)G_2(\beta_2(r, t))$. Notice that $\tilde{\beta}$ is a class $\mathcal{KL}$ function. Using that $\tilde{\beta}(|z_0|_{\mathcal{O}}, t-0) \leq \tilde{\beta}(|z_0|_{\mathcal{O}}, 0)$ and integrating (1.46) yields, through the comparison lemma,

$$v(t, z_0) \leq v_0 e^{-t} + \tilde{\beta}(|z_0|_{\mathcal{O}}, 0)\,.$$

[1] This is done by noticing that $\max(|z_1|_{\mathcal{O}_1}, |z_2|_{\mathcal{O}_2}) \leq |z|_{\mathcal{A}}$

It follows that, for all $t \geq 0$,

$$|z_1(t,z_0)|_{\mathcal{O}_1} \leq \alpha_1^{-1}\left(\alpha_2(|z_{10}|_{\mathcal{O}_1}) + \tilde{\beta}(|z_0|_{\mathcal{O}},0)\right),$$

which, with the UGAS of $\mathcal{O}_2$ *for (1.36), implies that*

$$|z(t,z_0)|_{\mathcal{O}} \leq \nu(|z_0|_{\mathcal{O}}), \quad \forall t \geq 0, \tag{1.47}$$

where $\nu(\cdot) := \sqrt{\alpha_1^{-1}(\alpha_2(\cdot) + \tilde{\beta}(\cdot,0))^2 + \beta_2(\cdot,0)^2}$ *is a class* $\mathcal{K}_\infty$ *function. UGS of* $\mathcal{O}$ *follows. To prove uniform global attractiveness, consider any positive constants* ε_1 *and* r *such that* $\varepsilon_1 < r$ *and let* $T_1(\varepsilon_1, r) \geq 0$ *be such that*[2] $\tilde{\beta}(r,T_1) = \frac{\varepsilon_1}{2}$, *then it follows from the integration of (1.46) from* T_1 *to any* $t \geq T_1$ *that, for any* $|z_0|_{\mathcal{O}} \leq r$,

$$\begin{aligned} v(t,z_0) &\leq v(T_1,z_0)e^{-(t-T_1)} + \int_{T_1}^{t} \tilde{\beta}(|z_{20}|_{\mathcal{O}_2},T_1)e^{-(t-s)}ds \\ &\leq v(T_1,z_0)e^{-(t-T_1)} + \tilde{\beta}(r,T_1)\left(1 - e^{-(t-T_1)}\right). \end{aligned}$$

Consequently, in view of (1.47),

$$v(t,z_0) \leq \alpha_2 \circ \nu(|z_0|_{\mathcal{O}})e^{-(t-T_1)} + \frac{\varepsilon_1}{2}.$$

Letting $T := T_1 + \ln\left(\frac{2}{\varepsilon_1}\left(\alpha_2 \circ \nu(r) + \tilde{\beta}(r,0)\right)\right)$ *gives* $v(t) \leq \varepsilon_1$ *for all* $t \geq T$. *If we define* $\varepsilon := \alpha_1^{-1}(\varepsilon_1)$, *it follows that* $|z_1(t,z_{10})|_{\mathcal{O}} \leq \varepsilon$ *for all* $t \geq T$. *Since* ε *is arbitrary and* $\mathcal{O}_2$ *is UGAS for (1.36), we conclude that* $\mathcal{O}$ *is UGA, and the conclusion follows.* ∎

Corollary 1.1 *Let* $\mathcal{O}_1$ *and* $\mathcal{O}_2 \subset \mathbb{D}$ *be closed subsets of* $\mathbb{R}^{q_1}$ *and* $\mathbb{R}^{q_2}$ *respectively, and the assumptions A 1.2 - A1.5 be satisfied. Then, with respect to the cascade (1.35)-(1.36), the set* $\mathcal{O}$ *is:*

- *UGS, when* $\mathbb{D} = \mathbb{R}^{q_2}$, *if* $\mathcal{O}_2$ *is UGS with respect to the system (1.36) and that the solution of system (1.35)-(1.36) is UGB with respect to* $\mathcal{O}$.
- *US, if* $\mathcal{O}_2$ *is US with respect to the system (1.36).*
- *UAS, if* $\mathcal{O}_2$ *is UAS with respect to the system (1.36).*

[2] If $\tilde{\beta}(r,0) \leq \frac{\varepsilon_1}{2}$, pick T_1 as 0

Proof. *Based on the proof of Lemma 1.1 the following can be stated:*

- *(UGS Proof) From the UGB and UGS property of $\mathcal{O}$ and $\mathcal{O}_2$, (1.47) is satisfied by noting that $\tilde{\beta}$ in (1.46) is a class $\mathcal{K}_\infty$ function.*

- *(US Proof) We first prove that the solutions of the system (1.35)-(1.36) are UB with respect to $\mathcal{O} := \mathcal{O}_1 \times \mathcal{O}_2$, then we use Lemma 1.1 to prove the stability result. From Mazenc and Praly [1996] Lemma B.1 there exist continuous functions $B_{z_1} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $B_{z_2} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, where $B_{z_2}(0) = 0$, such that $\varsigma(|z_1|_{\mathcal{O}_1})G_1(|z|_{\mathcal{O}})G_2(|z_2|_{\mathcal{O}_2}) \leq B_{z_1}(|z_1|_{\mathcal{O}_1})B_{z_2}(|z_2|_{\mathcal{O}_2})$. From $B_{z_1}(|z_1|_{\mathcal{O}_1})$ being continuous, for any $\epsilon_1 > 0$ there exists a $\delta_1 > 0$ such that $|z_1|_{\mathcal{O}_1} < \delta_1 \Rightarrow |B_{z_1}(|z_1|_{\mathcal{O}_1}) - B_{z_1}(0)| \leq \epsilon_1$. Fix ϵ_1 and choose ϵ_2 such that there exist a δ_2, by US of $\mathcal{O}_2$, that satisfy $\bar{\alpha}_3^{-1}\left((\epsilon_1 + B_{z_1}(0))\, B_{z_2}(\delta_2)\right) < \delta_1$ and $\delta_2 < \delta_1$. Then if $\bar{\alpha}_3(|z_{10}|_{\mathcal{O}_1}) < B_{z_1}(|z_{10}|_{\mathcal{O}_1})B_{z_1}(\delta_2)$:*

  $$\begin{aligned} \dot{V}_1 &\leq -\bar{\alpha}_3(|z_1|_{\mathcal{O}_1}) + B_{z_1}(|z_1|_{\mathcal{O}_1})B_{z_2}(|z_2|_{\mathcal{O}_2}) \\ &\leq -\bar{\alpha}_3(|z_1|_{\mathcal{O}_1}) + (\epsilon_1 + B_{z_1}(0))\, B_{z_2}(\delta_2), \end{aligned}$$

  such that

  $$|z_1(t)|_{\mathcal{O}_1} \leq \bar{\alpha}_3^{-1}\left((\epsilon_1 + B_{z_1}(0))\, B_{z_2}(\delta_2)\right),$$

  and $|z(t)|_{\mathcal{O}} \leq c_1$ where

  $$c_1 := 2\max(\bar{\alpha}_3^{-1}\left((\epsilon_1 + B_{z_1}(0))\, B_{z_2}(\delta_2)\right), \delta_2).$$

  Else for $\bar{\alpha}_3(|z_{10}|_{\mathcal{O}_1}) \geq B_{z_1}(|z_{10}|_{\mathcal{O}_1})B_{z_1}(\delta_2)$:

  $$|z_1(t)|_{\mathcal{O}_1} \leq \bar{\alpha}_1^{-1}(\bar{\alpha}_2(\delta_2)),$$

  and $|z(t)|_{\mathcal{O}} \leq c_2$ where $c_2 := 2\max(\bar{\alpha}_1^{-1}(\bar{\alpha}_2(\delta_2)), \delta_2)$. Thus for all $|z_0|_{\mathcal{O}} \leq \delta_2$, $|z(t)|_{\mathcal{O}} \leq c$, where $c(\delta_2) := \max(c_1, c_2)$, the solutions of system (1.35)-(1.36) are UB with respect to $\mathcal{O}$. From the UB and US property of $\mathcal{O}$ and $\mathcal{O}_2$ there exist positive constants c_z and c_{z_2} such that for all $|z_0|_{\mathcal{O}} \leq c_{z_2}$ and $|z_{20}|_{\mathcal{O}_2} \leq c_{z_2}$, (1.47) is satisfied by noticing that $\tilde{\beta}$ in (1.46) is, in this case, a class $\mathcal{K}$ function.

- *(UAS Proof) By the same arguments as in the proof of the US result, the solutions of system (1.35)-(1.36) are UB with respect to $\mathcal{O}$. UB of set $\mathcal{O}$ and UAS of set $\mathcal{O}_2$ imply that there exists some positive constants c_z and c_{z_2}, such that the steps in the proof of Lemma 1.1 can be followed for some initial condition $|z_0|_{\mathcal{O}} \leq c_z$ and $|z_{20}|_{\mathcal{O}_2} \leq c_{z_2}$.*

■

1.4.4 Important mathematical results

Theorem 4.6, Lyapunov equation, Khalil [1996] page 136

A matrix A is Hurwitz; that is $\operatorname{Re}\lambda_i < 0$ for all eigenvalues of A, if and only if for any given positive definite symmetric matrix Q there exists a positive definite symmetric matrix P that satisfies the *Lyapunov equation*

$$PA^{\mathrm{T}} + AP = -Q. \tag{1.48}$$

Moreover, if A is Hurwitz, then P is the unique solution of (1.48).

Barbalat's Lemma, Krstic et al. [1995] page 491

Consider the function $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}$. If ϕ is uniformly continuous and $\lim_{t\to\infty} \int_0^\infty \phi(\tau)d\tau$ exists and is finite, then

$$\lim_{t\to\infty} \phi(t) = 0.$$

Corollary 1.2 *Consider the function* $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}$. *If* $\phi, \dot{\phi} \in \mathcal{L}_\infty$ *and* $\phi \in \mathcal{L}_p$ *for some* $p \in [1, \infty)$, *then*

$$\lim_{t\to\infty} \phi(t) = 0.$$

Proposition 2.4.7, Abrahamson et al. [1988]

Let E and F be real Banach spaces, $f : U \subset E \to F$ a C^1 map, $x, y \in U$ and c a C^1 arc in U connecting x to y; i.e., c is a continuous map $c : [0,1] \to U$, which is C^1 on $[0,1]$, $c(0) = x$, and $c(1) = y$. Then

$$f(y) - f(x) = \int_0^1 \frac{df(c(t))}{dc}\frac{dc(t)}{dt}dt.$$

If U is convex and $c(t) = (1-t)x + ty$, then

$$f(y) - f(x) = \int_0^1 \frac{df((1-t)x + ty)}{dc}dt(y - x).$$

Mean value theorem, Khalil [1996] page 651

Assume that $f : \mathbb{R}^n \mapsto \mathbb{R}$ is continuously differentiable at each point x of an open set $S \subset \mathbb{R}^n$. Let x and y be two points of S such that the line segment $L(x,y) \subset S$. Then there exists a point z of $L(x,y)$ such that

$$f(y) - f(x) = \frac{df(z)}{dz}(y - x).$$

Implicit function theorem, Adams [2002] page 769

Consider a system of n equations in $n+m$ variables,

$$F_1(x_a, x_b) = 0$$
$$\vdots$$
$$F_n(x_a, x_b) = 0$$

where $x_a \in \mathbb{R}^m$, $x_b \in \mathbb{R}^n$, and a point $P_0 := (a, b)$, where $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$, that satisfy the system. Suppose for each of the functions F_i has continuous first partial derivatives with respect to each of the variables x_a and x_b, near P_0. Finally, suppose that $\left.\frac{\partial(F_1..F_n)}{\partial(x_b)}\right|_{P_0} \neq 0$. Then the system can be solved for x_b as a function of x_a, near P_0.

Chapter 2

Objective, problem statement and solution strategy

Consider the system description (1.1), (1.2) and (1.3):

$$\begin{aligned}\dot{x} &= f(t,x) + g(t,x)\tau, &(2.1)\\ \tau &= \Phi(t,x,u,\theta) := \Phi_0(t,x,u) + \Phi_\tau(t,x,u)\theta_\tau + \Phi_u(t,x,u)\theta_u, &(2.2)\\ \dot{u} &= f_{u0}(t,x,u,u_{cmd}) + f_{u\theta}(t,x,u,u_{cmd})\theta_u, &(2.3)\end{aligned}$$

where $t \geq 0$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^r$, $\tau \in \mathbb{R}^d$, $\theta := (\theta_u^{\mathrm{T}}, \theta_\tau^{\mathrm{T}})^{\mathrm{T}}$, $\theta_u \in \mathbb{R}^{m_n}$, $\theta_\tau \in \mathbb{R}^{m_\tau}$, $u_{cmd} \in \mathbb{R}^c$.

The objective of this thesis is to present a control strategy for systems of the form (2.1)-(2.3) with focus on the control allocation. This involves; constructing control allocation and parameter estimate update laws, defining a set of sufficient assumptions for closed loop uniform global asymptotic stability, and presenting an analytical evaluation of the resulting closed loop properties.

By rewriting equation (2.1) into

$$\begin{aligned}\dot{x} = \; & f(t,x) + g(t,x)k(t,x)\\ & + g(t,x)\left(\Phi(t,x,u_d,\hat{\theta}) - k(t,x)\right)\\ & + g(t,x)\left(\Phi(t,x,u,\hat{\theta}) - \Phi(t,x,u_d,\hat{\theta})\right)\\ & + g(t,x)\left(\Phi(t,x,u,\theta) - \Phi(t,x,u,\hat{\theta})\right), \qquad (2.4)\end{aligned}$$

the control problem can be modularized into a structure of the four following sub-problems, see also Figure 2.1:

1. **The high level control algorithm.** The vector of generalized forces τ, is treated as an available input to the system (2.1), and a virtual control law $\tau_c = k(t,x)$ is designed such that the origin of (2.1) is UGAS when $\tau = \tau_c$.

2. **The low level control algorithm.** Based on the actuator dynamic (2.3) a control law u_{cmd} is defined such that for any smooth reference u_d, u will track u_d asymptotically.

3. **The control allocation algorithm (connecting the high and low level control).** The main objective of the control allocation algorithm is to distribute a set of low level references (u_d) to the low level control, based on the desired virtual control τ_c. The static actuator-force mapping $\Phi(t,x,u,\theta)$ from (2.2), represents the connection between the output of the low level system (2.3) and the input to the high level system (2.1). And the control allocation problem is formulated as the static minimization problem

$$\min_{u_d} J(t,x,u_d) \quad s.t \quad \tau_c - \Phi(t,x,u_d,\hat{\theta}) = 0, \tag{2.5}$$

where J is a cost function that incorporates objectives such as minimum power consumption and actuator constraints (implemented as barrier functions). Based on this formulation, the Lagrangian function

$$L_{\hat{\theta}u}(t,x,u_d,\tilde{u},\lambda) := J(t,x,u_d) + (k(t,x) - \Phi(t,x,u_d,\hat{\theta}))^{\mathrm{T}}\lambda \tag{2.6}$$

is introduced, and update laws for the reference u_d and the Lagrangian parameter λ are then defined such that u_d and λ converge to a set defined by the time-varying optimality condition.

4. **The adaptive algorithm.** In order to cope with a possibly unknown parameter vector θ in the actuator and force-mapping models, adaptive laws are defined for the estimate $\hat{\theta}$. The parameter estimate is used in the control allocation scheme and a certainty equivalent adaptive optimal control allocation algorithm can be defined.

For each of these steps, Lyapunov based theory may be applied such that for each subsystem some stability and convergence properties are achieved.

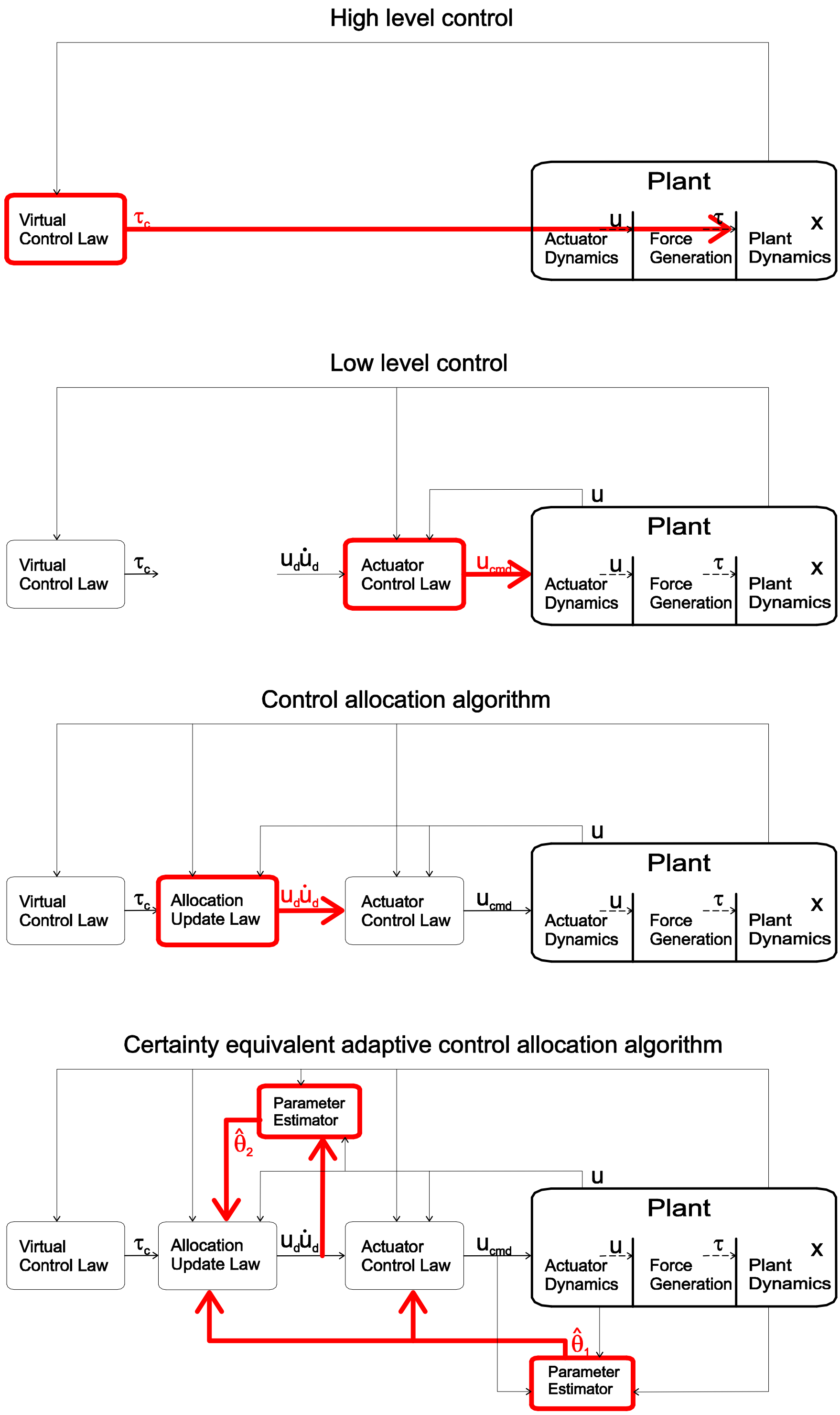

Figure 2.1: The control allocation philosophy

The closed loop system is analyzed by applying cascade theory, such that based on the subsystem properties, the qualitative attributes of the closed loop system can be specified if the solutions are bounded. The cascade partitioning through out this thesis is based on the perturbed system, (2.4), where the last three terms represent inputs from the perturbing system. Since in this setting the perturbing system may be dependent on the perturbed system through the state x, the idea is to treat x as a time-varying signal $x(t)$ and show that the stability attributes of the perturbing system holds as long as x exists, see Figure 2.2.

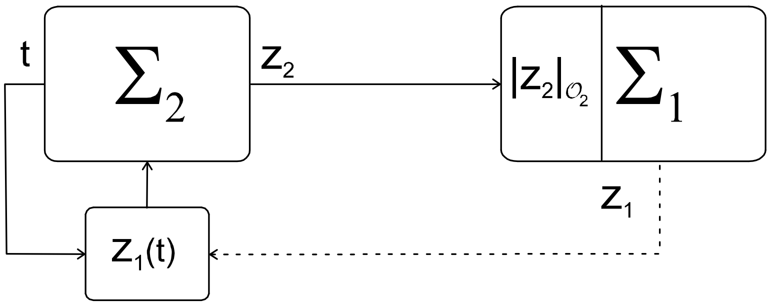

Figure 2.2: General representation of a time-variant cascade, where Σ_1 is the *perturbed system* that will be UGAS with respect to a set, $\mathcal{O}_1$, when $|z_2|_{\mathcal{O}_2} = 0$. Σ_2 is the *perturbing system.* Note that Σ_2 may be perturbed indirectly by Σ_1 since z_1 may be considered as a time-varying signal, $z_1(t)$, as long as this signal exists for all t.

2.1 Standing assumptions

In this section the standing assumptions used throughout the thesis are presented. Although further assumptions will be defined later, they will be specified with respect to the following assumptions.

If in the following a function F is said to be uniformly bounded by y, this means that there exist a function G_f of *class* $\mathcal{K}_\infty$ and a scalar $c > 0$ such that $|F(t, y, z)| < G_f(|y|) + c$ for all y, z and t.

Assumption 2.1 *(Plant assumptions)*

a) *The states x and u are known for all t.*

b) *The function f is uniformly locally Lipschitz in x and uniformly bounded by x. The function g is uniformly bounded and its partial derivatives are bounded by x.*

c) *The function Φ is twice differentiable and uniformly bounded by x and u. Moreover its partial derivatives are uniformly bounded by x.*

d) *There exist constants $\varrho_2 > \varrho_1 > 0$, such that $\forall t$, x, u and θ*

$$\varrho_1 I \leq \frac{\partial \Phi}{\partial u}(t,x,u,\theta)\left(\frac{\partial \Phi}{\partial u}(t,x,u,\theta)\right)^T \leq \varrho_2 I. \tag{2.7}$$

Remark 2.1 *Assumption 2.1 d) can be viewed as a controllability assumption in the sense that: i) the mapping $\Phi(t,x,\cdot,\theta) : \mathbb{R}^r \to \mathbb{R}^d$ is surjective for all t, x and θ and ii) for all t, x and θ there exists a continuous function $f_u(t,x,\theta)$ such that $\Phi(t,x,f_u(t,x,\theta),\theta) = k(t,x)$. The surjective property can be seen by the* Moore-Penrose pseudoinverse *solution, $u_1 = \frac{\partial \Phi}{\partial u}\big|_{u_c}\left(\frac{\partial \Phi}{\partial u}\big|_{u_c}\frac{\partial \Phi}{\partial u}\big|_{u_c}^T\right)^{-1} y$, of the equation, $y = \frac{\partial \Phi}{\partial u}\big|_{u_c} u_1$, which exists for any u_c. Thus for every y there exists a solution $\Phi(t,x,u_1,\theta) = \Phi(t,x,0,\theta) + y$ by the* Mean Value Theorem, *where $u_c \in (0, u_1)$. ii) follows from the* Implicit Function Theorem *by i) and Assumption 2.1 d).*

Assumption 2.2 *(Control algorithm assumption)*

a) *There exists a high level control $\tau_c := k(t,x)$, that render the equilibrium of (2.1) UGAS for $\tau = \tau_c$. The function k is uniformly bounded by x and differentiable. Its partial derivatives are uniformly bounded by x.*

Hence there exists a Lyapunov function $V_x : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and $\mathcal{K}_\infty$ functions α_{x1}, α_{x2}, α_{x3} and α_{x4} such that

$$\alpha_{x1}(|x|) \leq V_x(t,x) \leq \alpha_{x2}(|x|), \tag{2.8}$$

$$\frac{\partial V_x}{\partial t} + \frac{\partial V_x}{\partial x}\left(f(t,x) + g(t,x)k(t,x)\right) \leq -\alpha_{x3}(|x|), \tag{2.9}$$

$$\left|\frac{\partial V_x}{\partial x}\right| \leq \alpha_{x4}(|x|), \tag{2.10}$$

for the system $\dot{x} = f(t,x) + g(t,x)k(t,x)$ with respect to its origin.

b) *There exists a low-level control* $u_{cmd} := k_u(t, x, u, u_d, \dot{u}_d, \hat{\theta}_u)$ *that makes the equilibrium of*

$$\dot{\tilde{u}} = f_{\tilde{u}}(t, x, u, u_d, \hat{\theta}_u, \theta_u), \tag{2.11}$$

where $\tilde{u} := u - u_d$ *and*

$$f_{\tilde{u}}(t, x, u, u_d, \hat{\theta}_u, \theta_u) := f_{u0}(t, x, u, k_u(t, x, u, u_d, \dot{u}_d(t), \hat{\theta}_u)) + f_{u\theta}(t, x, u, k_u(t, x, u, u_d, \dot{u}_d(t), \hat{\theta}_u))\theta_u - \dot{u}_d(t),$$

UGAS if $\hat{\theta}_u = \theta_u$ *and* x, u_d, $\dot{u}_d$ *exist for all* $t > 0$.

c) *There exists a* $\mathcal{K}_\infty$ *function* $\alpha_k : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, *such that*

$$\alpha_{x4}(|x|)\alpha_k(|x|) \leq \alpha_{x3}(|x|). \tag{2.12}$$

Remark 2.2 *If the origin of* $\dot{x} = f(t, x) + g(t, x)k(t, x)$ *is UGES, then Assumption 2.2 is generally satisfied, with for example* α_{x3} *quadratic,* α_{x4} *linear and* α_k *sublinear, and* V_x *does not need to be known explicitly in order to verify Assumption 2.2 c).*

Classical control nonlinear design tools, used to solve the feedback control problem arising from Assumption 2.2, are; feedback linearization, sliding mode control, backstepping and passivity-based control, see Khalil [1996] and Krstic et al. [1995]. Analytically the choice of controller design tool is not of importance, but Lyapunov based control methods like for example backstepping, if the systems (2.1) and (2.3) are in *strict-feedback* from, may be preferred since the Lyapunov control function (clf) can be used directly in the verification of Assumption 2.2.

Assumption 2.3 *(Optimality assumptions)*

a) *The cost function* $J : \mathbb{R}_{\geq t_0} \times \mathbb{R}^{n \times r} \to \mathbb{R}$ *is twice differentiable and* $J(t, x, u_d) \to \infty$ *as* $|u_d| \to \infty$. *Furthermore* $\frac{\partial J}{\partial u_d}$, $\frac{\partial^2 J}{\partial t \partial u_d}$ *and* $\frac{\partial^2 J}{\partial x \partial u_d}$ *are uniformly bounded by* x *and* u_d.

b) *There exist constants* $k_2 > k_1 > 0$, *such that* $\forall$ t, x, $\hat{\theta}$, $\tilde{u}$ *and* $\left(u_d^T, \lambda^T\right)^T \notin \mathcal{O}_{u_d\lambda}(t, x, \tilde{u}, \hat{\theta})$, *where* $\mathcal{O}_{u_d\lambda}(t, x, \tilde{u}, \hat{\theta}) := \left\{u_d^T, \lambda^T \middle| \frac{\partial L_{\hat{\theta}u}}{\partial u_d} = 0, \frac{\partial L_{\hat{\theta}u}}{\partial \lambda} = 0\right\}$,

$$k_1 I \leq \frac{\partial^2 L_{\hat{\theta}u}}{\partial u_d^2}(t, x, u_d, \tilde{u}, \lambda, \hat{\theta}) \leq k_2 I. \tag{2.13}$$

If $\left(u_d^T, \lambda^T\right)^T \in \mathcal{O}_{u_d\lambda}(t, x, \tilde{u}, \hat{\theta})$ *the lower bound is replaced by* $\frac{\partial^2 L_{\hat{\theta}u}}{\partial u_d^2} \geq 0$

Remark 2.3 *The second order sufficient conditions in Theorem 12.6 in Nocedal and Wright [1999] are satisfied for all* t, x, u, λ *and* θ *by Assumption 2.1 and 2.3, thus the set* $\mathcal{O}_{u\lambda}(t,x)$ *describes global optimal solutions of the problem (2.5).*

2.2 Problem statement

From the above assumptions the problem statement is divided into three parts:

i) Consider the model (2.1)-(2.2), assume that θ is known and that there exist a high level controller satisfying Assumption 2.2 a), then define an update law $\dot{u} := f_u(t,x,u)$, such that the stability property of the perturbed system (2.1) is conserved for the closed loop, and $u(t)$ converges to an optimal solution with respect to the static minimization problem (2.5), where $u_d = u$. (Chapter 3)

ii) Consider the model (2.1)-(2.2), assume that θ is not known, then solve Problem i) with $\theta = \hat{\theta}$, $\dot{u} := f_u(t,x,u,\hat{\theta})$ and define an adaptive update-law $\dot{\hat{\theta}} := f_{\hat{\theta}}(t,x,u,\hat{\theta})$, where $\hat{\theta} \in \mathbb{R}^m$ is an estimate of θ. (Chapter 4)

iii) Consider the model (2.1)-(2.3), assume that θ is not known and that there exist high and low level controllers satisfying Assumptions 2.2 a) and b), then define update laws $\dot{u}_d := f_u(t,x,u,u_d,\hat{\theta})$ and $\dot{\hat{\theta}} := f_{\hat{\theta}}(t,x,u,u_d,\hat{\theta})$ such that some stability properties may be derived for closed loop system, and $u_d(t)$ converges to an optimal solution with respect to the static minimization problem (2.5). (Chapter 5)

Chapter 3

Dynamic Control Allocation

In this chapter the update laws for the control input u, and the Lagrangian multipliers λ, are established, such that stability and convergence to the time-varying first order optimal set defined by the optimization problem (2.5) can be concluded, and the stabilizing properties of the virtual controller from Assumption 2.2 a), are conserved for the closed loop. According to Remark 2.3, first order optimality is sufficient for global optimality due to the Assumptions 2.1 and 2.3.

The problem formulation in this chapter relies on the minimization problem from (2.5), which without considering actuator dynamic and parameter uncertainty takes the form

$$\min_{u} J(t,x,u) \quad s.t \quad \tau_c - \Phi(t,x,u,\theta) = 0. \tag{3.1}$$

Furthermore the Lagrangian function is given by

$$L(t,x,u,\lambda) := J(t,x,u) + (k(t,x) - \Phi(t,x,u,\theta))^{\mathrm{T}}\lambda, \tag{3.2}$$

and the dynamic equation (2.1), related to (2.4), may be rewritten by

$$\begin{aligned} \dot{x} &= f(t,x) + g(t,x)k(t,x) \\ &\quad + g(t,x)\left(\Phi(t,x,u,\theta) - k(t,x)\right). \end{aligned} \tag{3.3}$$

3.1 Stability of the control allocation

In what follows, it will be proved that the time and state-varying optimal set

$$\mathcal{O}_{u\lambda}(t,x) := \left\{ \left(u^{\mathrm{T}}, \lambda^{\mathrm{T}}\right)^{\mathrm{T}} \in \mathbb{R}^{r+d} \,\middle|\, \left(\left(\frac{\partial L}{\partial u}\right)^{\mathrm{T}}, \left(\frac{\partial L}{\partial \lambda}\right)^{\mathrm{T}} \right)^{\mathrm{T}} = 0 \right\},$$

in a certain sense is UGES with respect to the control allocation algorithm based on the update-laws,

$$\begin{pmatrix} \dot{u} \\ \dot{\lambda} \end{pmatrix} = -\Gamma \mathbb{H} \begin{pmatrix} \frac{\partial L}{\partial u} \\ \frac{\partial L}{\partial \lambda} \end{pmatrix} - u_{ff}, \tag{3.4}$$

where $\mathbb{H} := \begin{pmatrix} \frac{\partial^2 L}{\partial u^2} & \frac{\partial^2 L}{\partial \lambda \partial u} \\ \frac{\partial^2 L}{\partial u \partial \lambda} & 0 \end{pmatrix}$, Γ is a symmetric positive definite matrix and u_{ff} is a feed-forward like term:

$$\begin{aligned} u_{ff} := {} & \mathbb{H}^{-1} \begin{pmatrix} \frac{\partial^2 L}{\partial t \partial u} \\ \frac{\partial^2 L}{\partial t \partial \lambda} \end{pmatrix} + \mathbb{H}^{-1} \begin{pmatrix} \frac{\partial^2 L}{\partial x \partial u} \\ \frac{\partial^2 L}{\partial x \partial \lambda} \end{pmatrix} f(t,x) \\ & + \mathbb{H}^{-1} \begin{pmatrix} \frac{\partial^2 L}{\partial x \partial u} \\ \frac{\partial^2 L}{\partial x \partial \lambda} \end{pmatrix} g(t,x)\Phi(t,x,u,\theta), \end{aligned}$$

if $det(\frac{\partial^2 L}{\partial u^2}) \geq \epsilon$ and $u_{ff} = 0$ if $det(\frac{\partial^2 L}{\partial u^2}) < \epsilon$, where $k_1^r > \epsilon > 0$ and k_1 is defined in Assumption 2.3 b). By the Assumptions 2.1 and 2.3 the existence of the proposed update laws and the time-varying first order optimal solution are guaranteed, i.e. for all t and x, $\mathcal{O}_{u\lambda}(t,x) \neq \varnothing$.

Lemma 3.1 *If Assumptions 2.1, 2.2 a) and 2.3 are satisfied, then $\mathcal{O}_{u\lambda}$ is non-empty for all t, x and θ. Further, for all t, x, θ and $(u,\lambda) \in \mathcal{O}_{u\lambda}$, there exists a continuous function $\varsigma_{\mathcal{O}_{u\lambda}} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that:*

$$\left|(u^T, \lambda^T)^T\right| \leq \varsigma_{\mathcal{O}_{u\lambda}} \left(\left| \left(x^T, \theta^T\right)^T \right| \right)$$

Proof. *The boundary of the set $\mathcal{O}_{u\lambda}$ and the time-varying first order optimal solution is described by:*

$$\begin{aligned} \frac{\partial L}{\partial u}(t,x,u,\lambda,\theta) &= \frac{\partial J}{\partial u}(t,x,u) - \left(\frac{\partial \Phi}{\partial u}(t,x,u,\theta)\right)^T \lambda, \\ \frac{\partial L}{\partial \lambda}(t,x,u,\lambda,\theta) &= k(t,x) - \Phi(t,x,u,\theta), \end{aligned}$$

By using theorem 2.4.7 in Abrahamson et al. [1988], it can be shown that

$$\begin{aligned}
\frac{\partial L}{\partial u}(t,x,u,\lambda,\theta) - \frac{\partial L}{\partial u}(t,x,u_c,\lambda,\theta) &= \left.\frac{\partial^2 L}{\partial u^2}\right|_c (u-u_c),\\
\frac{\partial L}{\partial u}(t,x,u_c,\lambda,\theta) - \frac{\partial L}{\partial u}(t,x,u_c,\lambda^*,\theta) &= \left.\frac{\partial^2 L}{\partial \lambda \partial u}\right|_c (\lambda-\lambda^*),\\
\frac{\partial L}{\partial u}(t,x,u_c,\lambda^*,\theta) - \frac{\partial L}{\partial u}(t,x,u^*,\lambda^*,\theta) &= \left.\frac{\partial^2 L}{\partial u^2}\right|_* (u_c-u^*),\\
\frac{\partial L}{\partial \lambda}(t,x,u,\lambda,\theta) - \frac{\partial L}{\partial \lambda}(t,x,u^*,\lambda,\theta) &= \left.\frac{\partial^2 L}{\partial u \partial \lambda}\right|_* (u-u^*),\\
\frac{\partial L}{\partial \lambda}(t,x,u^*,\lambda,\theta) - \frac{\partial L}{\partial \lambda}(t,x,u^*,\lambda^*,\theta) &= 0,
\end{aligned}$$

where $\left.\frac{\partial^2 L}{\partial u^2}\right|_c := \int_0^1 \frac{\partial^2 L}{\partial u^2}(t,x,(1-s)u_c + su,\lambda,\theta)ds$,
$\left.\frac{\partial^2 L}{\partial u^2}\right|_* := \int_0^1 \frac{\partial^2 L}{\partial u^2}(t,x,(1-s)u^* + su_c,\lambda,\theta)ds$,
$\left.\frac{\partial^2 L}{\partial \lambda \partial u}\right|_c := \int_0^1 \frac{\partial^2 L}{\partial \lambda \partial u}(t,x,u_c,(1-s)\lambda^* + s\lambda,\theta)ds = -\frac{\partial \Phi}{\partial u}^T(t,x,u_c,\theta)$
and $\left.\frac{\partial^2 L}{\partial u \partial \lambda}\right|_* := \int_0^1 \frac{\partial^2 L}{\partial u \partial \lambda}(t,x,(1-s)u^* + su,\lambda,\theta)ds = -\frac{\partial \Phi}{\partial u}(t,x,u_c,\theta)$.
Since $\frac{\partial L}{\partial \lambda}(t,x,u^*,\lambda^*,\theta) = 0$ *and* $\frac{\partial L}{\partial u}(t,x,u^*,\lambda^*,\theta) = 0$, *we get*

$$\begin{aligned}
\frac{\partial L}{\partial u}(t,x,u,\lambda,\theta) &= \left.\frac{\partial^2 L}{\partial u^2}\right|_c (u-u_c) + \left.\frac{\partial^2 L}{\partial u^2}\right|_* (u_c-u^*)\\
&\quad -\frac{\partial \Phi}{\partial u}^T(t,x,u_c,\theta)(\lambda-\lambda^*), && (3.5)\\
\frac{\partial L}{\partial \lambda}(t,x,u,\lambda,\theta) &= -\frac{\partial \Phi}{\partial u}(t,x,u_c,\theta)(u-u^*), && (3.6)
\end{aligned}$$

and from $u_c := \vartheta(u-u^*) + u^*$ *where* $\vartheta := diag(\vartheta_i)$ *and* $0 < \vartheta_i < 1$:

$$\left.\frac{\partial^2 L}{\partial u^2}\right|_c (u-u_c) + \left.\frac{\partial^2 L}{\partial u^2}\right|_* (u_c-u^*) = (1-\vartheta)\left.\frac{\partial^2 L}{\partial u^2}\right|_c (u-u^*) + \vartheta \left.\frac{\partial^2 L}{\partial u^2}\right|_* (u-u^*).$$

From Assumption 2.3, $\left.\frac{\partial^2 L}{\partial u^2}\right|_c$ *and* $\left.\frac{\partial^2 L}{\partial u^2}\right|_*$ *are positive definite matrices, such that* $P_{*c} := (1-\vartheta)\left.\frac{\partial^2 L}{\partial u^2}\right|_c + \vartheta \left.\frac{\partial^2 L}{\partial u^2}\right|_*$ *is also a positive definite matrix. Further*

$$\left(\frac{\partial L}{\partial u}\right)^T \frac{\partial L}{\partial u} + \left(\frac{\partial L}{\partial \lambda}\right)^T \frac{\partial L}{\partial \lambda} = \begin{pmatrix} u-u^* \\ \lambda-\lambda^* \end{pmatrix}^T \mathbb{H}_*^T \mathbb{H}_* \begin{pmatrix} u-u^* \\ \lambda-\lambda^* \end{pmatrix}, \qquad (3.7)$$

where $\mathbb{H}_* = \begin{pmatrix} P_{*c} & -\frac{\partial \Phi}{\partial u}^T(t,x,u_c,\theta) \\ -\frac{\partial \Phi}{\partial u}(t,x,u_c,\theta) & 0 \end{pmatrix}$. *By Assumption 2.3 and 2.1 it can be seen that:*

$$\begin{aligned} |\det(\mathbb{H}_*)| &= \left|\det(P_{*c})\det\left(-\frac{\partial \Phi}{\partial u}(t,x,u_c,\theta)P_{*c}^{-1}\left(\frac{\partial \Phi}{\partial u}(t,x,u_c,\theta)\right)^T\right)\right| \\ &> 0, \end{aligned} \tag{3.8}$$

and $\mathbb{H}_*$ *is non-singular and there exist constants* $\bar{\rho}_2 \geq \bar{\rho}_1 > 0$ *such that* $\bar{\rho}_1 I \leq \mathbb{H}_*^T\mathbb{H}_* \leq \bar{\rho}_2 I$. *Furthermore choose* $u = 0$ *and* $\lambda = 0$, *such that* $\left|\frac{\partial L}{\partial u}(t,x,0,0,\theta)\right| \leq \varsigma_{\partial J}(|x|)$ *and* $\left|\frac{\partial L}{\partial \lambda}(t,x,0,0,\theta)\right| \leq G_\Phi(|x|)(1+|\theta|)+\varsigma_k(|x|)$ *from Assumption 2.1, 2.2 and 2.3.*
Thus from (3.7) $\left|\left(u^{*T},\lambda^{*T}\right)^T\right| \leq \varsigma_{\mathcal{O}u\lambda}\left(\left|\left(x^T,\theta^T\right)^T\right|\right)$
where $\varsigma_{\mathcal{O}u\lambda}(|s|) := \bar{\rho}_1^{-1}\left(G_\Phi(|s|)(1+|s|)+\varsigma_k(|s|)\right)^2 + \bar{\rho}_1^{-1}\varsigma_{\partial J}(|s|)^2$ ■

The idea of proving stability and convergence of the optimal set, $\mathcal{O}_{u\lambda}$, relies on the construction of the Lyapunov-like function:

$$V_{u\lambda}(t,x,u,\lambda) := \frac{1}{2}\left(\left(\frac{\partial L}{\partial u}\right)^{\mathrm{T}}\frac{\partial L}{\partial u} + \left(\frac{\partial L}{\partial \lambda}\right)^{\mathrm{T}}\frac{\partial L}{\partial \lambda}\right).$$

It follows that, along the trajectories of (3.3) and (3.4), the time-derivative of $V_{u\lambda}$ is given by:

$$\begin{aligned} \dot{V}_{u\lambda} &= \left(\left(\frac{\partial L}{\partial u}\right)^{\mathrm{T}}\frac{\partial^2 L}{\partial u^2} + \left(\frac{\partial L}{\partial \lambda}\right)^{\mathrm{T}}\frac{\partial^2 L}{\partial u \partial \lambda}\right)\dot{u} + \left(\frac{\partial L}{\partial u}\right)^{\mathrm{T}}\frac{\partial^2 L}{\partial \lambda \partial u}\dot{\lambda} \\ &\quad + \left(\left(\frac{\partial L}{\partial u}\right)^{\mathrm{T}}\frac{\partial^2 L}{\partial x \partial u} + \left(\frac{\partial L}{\partial \lambda}\right)^{\mathrm{T}}\frac{\partial^2 L}{\partial x \partial \lambda}\right)\dot{x} \\ &\quad + \left(\frac{\partial L}{\partial u}\right)^{\mathrm{T}}\frac{\partial^2 L}{\partial t \partial u} + \left(\frac{\partial L}{\partial \lambda}\right)^{\mathrm{T}}\frac{\partial^2 L}{\partial t \partial \lambda} \\ &= \left(\left(\frac{\partial L}{\partial u}\right)^{\mathrm{T}}, \left(\frac{\partial L}{\partial \lambda}\right)^{\mathrm{T}}\right)\mathbb{H}\left(\dot{u}^{\mathrm{T}}, \dot{\lambda}^{\mathrm{T}}\right)^{\mathrm{T}} + \left(\left(\frac{\partial L}{\partial u}\right)^{\mathrm{T}}, \left(\frac{\partial L}{\partial \lambda}\right)^{\mathrm{T}}\right)u_{ff} \\ &= -\left(\left(\frac{\partial L}{\partial u}\right)^{\mathrm{T}}, \left(\frac{\partial L}{\partial \lambda}\right)^{\mathrm{T}}\right)\mathbb{H}\Gamma\mathbb{H}\left(\left(\frac{\partial L}{\partial u}\right)^{\mathrm{T}}, \left(\frac{\partial L}{\partial \lambda}\right)^{\mathrm{T}}\right)^{\mathrm{T}} \\ &\leq -c\left(\left(\frac{\partial L}{\partial u}\right)^{\mathrm{T}}\frac{\partial L}{\partial u} + \left(\frac{\partial L}{\partial \lambda}\right)^{\mathrm{T}}\frac{\partial L}{\partial \lambda}\right), \\ &= -2cV_{u\lambda} \end{aligned} \tag{3.9}$$

where $c := \inf_t \lambda_{\min}(\mathbb{H}\Gamma\mathbb{H}) > 0$, according to assumptions 2.1 and 2.3, as long as the solutions of (3.4) are not in the interior of the set $\mathcal{O}_{u\lambda}(t, x(t))$. The Lyapunov analysis and results are formalized, based on set-stability.

Proposition 3.1 *Let Assumptions 2.1, 2.2 a) and 2.3 be satisfied. If $x(t)$ exists for all t, then the optimal set $\mathcal{O}_{u\lambda}(t, x(t))$ is UGES with respect to the system (3.4).*

Proof. *In order to prove this result, it will be shown that i) $\mathcal{O}_{u\lambda}(t, x(t))$ is a closed forward invariant set with respect to system (3.4), ii) the system (3.4) and (1.29) is finite escape time detectable through $|z_{u\lambda}|_{\mathcal{O}_{u\lambda}}$, and that iii) $\bar{V}_{u\lambda}(t, z_{u\lambda}) := V_{u\lambda}(t, x(t), u, \lambda)$, where $z_{u\lambda} := (u^T, \lambda^T)^T$, is a radially unbounded Lyapunov function. From (3.9) and Theorem 1.1 it then follows that $\mathcal{O}_{u\lambda}(t, x(t))$ is UGES with respect to the system (3.4)*

i) *Define $G(t, x, u, \lambda, \theta) := \left(\frac{\partial L}{\partial u}^T, \frac{\partial L}{\partial \lambda}^T\right)^T$. From Proposition 1.1.9 b) in Bertsekas et al. [2003], $G : \mathbb{R}^q \to \mathbb{R}^{q_G}$ is continuous iff $G^{-1}(U)$ is closed in $\mathbb{R}^q$ for every closed U in $\mathbb{R}^{q_G}$. From the definition of $\mathcal{O}_{u\lambda}$, $U = \{0\}$, and since G is continuous (by Assumption 2.1 - 2.3), $\mathcal{O}_{u\lambda}$ is a closed set. The set is forward invariant if at t_1, $G(t_1, x(t_1), u(t_1), \lambda(t_1), \theta) = 0$ and $\frac{d(G(t,x,u,\lambda))}{dt} = 0$ $\forall t \geq t_1$ with respect to (3.3) and (3.4). Since there exists a continuous solution of (3.4) as long as x exists, we only need to check this condition on the boundary of $\mathcal{O}_{u\lambda}(t, x(t))$ (Note that det($\mathbb{H}$) $\neq$0 on the boundary of $\mathcal{O}_{u\lambda}(t, x(t))$ by Assumption 1d) and 3b)). We get $\frac{d\partial L}{dt\partial\lambda} = \frac{\partial^2 L}{\partial t\partial\lambda} + \frac{\partial^2 L}{\partial x\partial\lambda}\dot{x} + \frac{\partial^2 L}{\partial u\partial\lambda}\dot{u}$ and $\frac{d\partial L}{dt\partial u} = \frac{\partial^2 L}{\partial t\partial u} + \frac{\partial^2 L}{\partial x\partial u}\dot{x} + \frac{\partial^2 L}{\partial u\partial u}\dot{u} + \frac{\partial^2 L}{\partial\lambda\partial u}\dot{\lambda}$, thus*

$$\begin{aligned}\begin{pmatrix} \frac{d\partial L}{dt\partial u} \\ \frac{d\partial L}{dt\partial\lambda} \end{pmatrix} &= \mathbb{H}\begin{pmatrix} \dot{u} \\ \dot{\lambda} \end{pmatrix} + \begin{pmatrix} \frac{\partial^2 L}{\partial x\partial u} \\ \frac{\partial^2 L}{\partial x\partial\lambda} \end{pmatrix}\dot{x} + \begin{pmatrix} \frac{\partial^2 L}{\partial t\partial u} \\ \frac{\partial^2 L}{\partial t\partial\lambda} \end{pmatrix} \\ &= -\mathbb{H}\Gamma\mathbb{H}\begin{pmatrix} \frac{\partial L}{\partial u} \\ \frac{\partial L}{\partial\lambda} \end{pmatrix} = 0.\end{aligned}$$

ii) *Since $x(t)$ is assumed to be forward complete, it follows from Lemma 3.1 that there always exist a pair $\left(u^{*T}, \lambda^{*T}\right)^T$ that satisfy the time-varying first order optimal conditions, and thus the system (3.4) and (1.29) is finite escape time detectable through $|z_{u\lambda}|_{\mathcal{O}_{u\lambda}}$.*

iii) *In the proof of Lemma 3.1, from (3.7)*

$$\left(\frac{\partial L}{\partial u}\right)^T\frac{\partial L}{\partial u} + \left(\frac{\partial L}{\partial\lambda}\right)^T\frac{\partial L}{\partial\lambda} = \begin{pmatrix} u - u^* \\ \lambda - \lambda^* \end{pmatrix}^T \mathbb{H}_*^T\mathbb{H}_*\begin{pmatrix} u - u^* \\ \lambda - \lambda^* \end{pmatrix},$$

it follows that there exists constants $\kappa_2 \geq \kappa_1 > 0$ *such that* $\bar{\kappa}_1 |z_{u\lambda}|^2_{\mathcal{O}_{u\lambda}} \leq \left(\left(\frac{\partial L}{\partial u}\right)^T \frac{\partial L}{\partial u} + \left(\frac{\partial L}{\partial \lambda}\right)^T \frac{\partial L}{\partial \lambda} \right) \leq \bar{\kappa}_2 |z_{u\lambda}|^2_{\mathcal{O}_{u\lambda}}$, *since the matrix* $\mathbb{H}_*^T \mathbb{H}_*$ *is positive definite and bounded by Assumption 2.1 and 2.3. By a similar argument, in (3.8) change* P_{*c} *with* $\frac{\partial^2 L}{\partial u^2}$, *and it follows that* $\mathbb{H}$ *is non-singular, hence the control allocation law (3.4) always exist and,* $\bar{\kappa}_1 |z_{u\lambda}|_{\mathcal{O}_{u\lambda}} \leq \bar{V}_{u\lambda}(t, z_{u\lambda}) \leq \bar{\kappa}_2 |z_{u\lambda}|_{\mathcal{O}_{u\lambda}}$, *such that* $\bar{V}_{u\lambda}(t, z_{u\lambda})$ *is a radially unbounded Lyapunov function.*

■

This result implies that the set $\mathcal{O}_{u\lambda}(t, x(t))$ is uniformly attractive and stable, such that optimality is achieved asymptotically.

Remark 3.1 *Provided that the gain matrix,* $\Gamma > 0$, *is bounded away from zero,* Γ *may be chosen time-varying. If for example* $\Gamma = \gamma (\mathbb{H}\mathbb{H})^{-1}$ *for some* $\gamma > 0$, *then*

$$\begin{pmatrix} \dot{u} \\ \dot{\lambda} \end{pmatrix} = -\gamma \mathbb{H}^{-1} \begin{pmatrix} \frac{\partial L}{\partial u} \\ \frac{\partial L}{\partial \lambda} \end{pmatrix} - \mathbb{H}^{-1} u_{ff},$$

where the first term is the Newton direction when L is considered the cost function to be minimized. In case $\mathbb{H}\mathbb{H}$ *is poorly conditioned, one may choose* $\Gamma = \gamma (\mathbb{H}\mathbb{H} + \rho I)^{-1}$ *for some* $\rho > 0$, *to avoid c in (3.9) from being small.*

Remark 3.2 *If in Assumption 2.3 the terms* k_2 *and* ϱ_2 *are relaxed to continuous positive functions* $\varsigma_k(|z_{u\lambda}|_{\mathcal{O}_{u\lambda}})$ *and* $\varsigma_\varrho(|z_{u\lambda}|_{\mathcal{O}_{u\lambda}})$, *then* $\kappa_1 |z_{u\lambda}|^2_{\mathcal{O}_{u\lambda}} \leq \left(\left(\frac{\partial L}{\partial u}\right)^T \frac{\partial L}{\partial u} + \left(\frac{\partial L}{\partial \lambda}\right)^T \frac{\partial L}{\partial \lambda} \right) \leq \alpha \left(|z_{u\lambda}|_{\mathcal{O}_{u\lambda}} \right)$ *where* $\alpha \in \mathcal{K}_\infty$, *and by the proof of Proposition 3.1, an UGAS result is obtained.*

Corollary 3.1 *Let Assumptions 2.1, 2.2 a) and 2.3 be satisfied and* $|z_{u\lambda 0}|_{\mathcal{O}_{u\lambda}} < r$ *where* $r > 0$. *If* $x(t)$ *exists for* $t \in [t_0, T)$ *where* $T \geq t_0$, *then there exists a positive constant* $B(r) > 0$ *such that for all* $t \in [t_0, T)$, $|z_{u\lambda}(t)|_{\mathcal{O}_{u\lambda}(t,x(t))} \leq B(r)$.

Proof. *Since* $\dot{V}_{u\lambda} \leq 0$, $V_{u\lambda}(t, x(t), u(t), \lambda(t)) \leq V_{u\lambda}(t_0, x(t_0), u(t_0), \lambda(t_0))$ *for all* $t \in [t_0, T)$. *From iii) of the proof of Proposition 1* $\kappa_1 |z_{u\lambda}|^2_{\mathcal{O}_{u\lambda}} \leq V_{u\lambda} \leq \kappa_2 |z_{u\lambda}|^2_{\mathcal{O}_{u\lambda}}$ *for all* $t \in [t_0, T)$, *it follows that* $|z_{u\lambda}(t)|_{\mathcal{O}_{u\lambda}(t,x(t))} \leq \sqrt{\frac{1}{\kappa_1} V_{u\lambda}(t, x(t), u(t), \lambda(t))} \leq \sqrt{\frac{1}{\kappa_1} V_{u\lambda}(t_0, x(t_0), u(t_0), \lambda(t_0))} \leq \sqrt{\frac{\kappa_2}{\kappa_1}} |z_{u\lambda 0}|_{\mathcal{O}_{u\lambda}} =: B(r)$ ■

The technical boundedness property of the solutions of system (3.4) with respect to the set $\mathcal{O}_{u\lambda}$, given in Corollary 3.1, is useful when analyzing the closed loop system, see the proof in the following section.

3.2 Stability of the combined control and control allocation

The optimal set for the combined control and control allocation problem is defined by:

$$\mathcal{O}_{xu\lambda}(t) := \mathcal{O}_x \times \mathcal{O}_{u\lambda}(t,0). \tag{3.10}$$

The stability properties of this combined set is analyzed by using Lemma 1.1. In Figure 2.2 it is illustrated how the two time-varying systems can interact with each other in a cascade. In this case the perturbed system Σ_1, is represented by (3.3), the perturbing system Σ_2, is represented by the update-laws for u and λ, (3.4). Loosely explained, Lemma 1.1 is used to conclude UGAS of the set $\mathcal{O}_{xu\lambda}$ if, $\mathcal{O}_x$ and $\mathcal{O}_{u\lambda}$ individually are UGAS (which is established by Assumption 2.2 and the previous section) and the combined system is UGB with respect to $\mathcal{O}_{xu\lambda}$.

The properties of the closed loop system, the plant, the virtual controller and the dynamic control allocation law, are analyzed next.

Proposition 3.2 *If Assumptions 2.1-2.3 are satisfied, then the set $\mathcal{O}_{xu\lambda}$ is UGAS with respect to the system (3.3) and (3.4).*

Proof. *We prove boundedness and completeness of the system (3.3)-(3.4), and use Lemma 1.1. For notational purpose, we define $z_x := x$, $z_{u\lambda} := \left(u^T, \lambda^T\right)^T$ and $z_{xu\lambda} := \left(x^T, u^T, \lambda^T\right)^T$. Let $|z_{xu\lambda 0}|_{\mathcal{O}_{xu\lambda}} \leq r$, where $r > 0$, and assume that $|z_x(t)|_{\mathcal{O}_x}$ escapes to infinity at T. Then for any constant $M(r)$ there exists a $t \in [t_0, T)$ such that $M(r) \leq |z_x(t)|_{\mathcal{O}_x}$. In what follows we show that $M(r)$ can not be chosen arbitrarily.*

Define $v(t,x) := V_x(t,x)$ such that

$$\begin{aligned}
\dot{v} &\leq -\alpha_{x3}(|z_x|_{\mathcal{O}_x}) - \frac{\partial V_x}{\partial x} g(t,x)(\Phi(t,x,u,\theta) - k(t,x)) \\
&\leq -\alpha_{x3}(|z_x|_{\mathcal{O}_x}) + \left|\frac{\partial V_x}{\partial x}\right| |g(t,x)| \left|\frac{\partial L}{\partial \lambda}(t,x,u)\right| \\
&\leq -\alpha_{x3}(|z_x|_{\mathcal{O}_x}) + \alpha_{x4}(|z_x|_{\mathcal{O}_x}) K\kappa_2 \left|z_{u\lambda}\right|_{\mathcal{O}_{u\lambda}}.
\end{aligned} \tag{3.11}$$

From Corollary 3.1 there exists a constant $B(r)$, such that for all $t \in [t_0, T)$, $B(r) \geq B(|z_{u\lambda 0}|_{\mathcal{O}_{u\lambda}}) \geq |z_{u\lambda}(t)|_{\mathcal{O}_{u\lambda}}$. Further by Assumption 2.1 and 2.2 it follows that

$$\begin{aligned}
\dot{v} &\leq -\alpha_{x3}(|z_x|_{\mathcal{O}_x}) + \alpha_{x4}(|z_x|_{\mathcal{O}_x}) K\kappa_2 B(r) \\
&\leq -\alpha_k(|z_x|_{\mathcal{O}_x})\alpha_{x4}(|z_x|_{\mathcal{O}_x}) + \alpha_{x4}(|z_x|_{\mathcal{O}_x}) K\kappa_2 B(r) \\
&\leq -\alpha_{x4}(|z_x|_{\mathcal{O}_x}) \left(\alpha_k(|z_x|_{\mathcal{O}_x}) - K\kappa_2 B(r)\right) \qquad (3.12) \\
&\leq -\alpha_{x4}(|z_x|_{\mathcal{O}_x}) \left(\alpha_k(\alpha_{x2}^{-1}((v))) - K\kappa_2 B(r)\right). \qquad (3.13)
\end{aligned}$$

If $|z_{x0}|_{\mathcal{O}_x} > \alpha_k^{-1}(K\kappa_2 B(r))$*, then from (3.12),* $v(t_0, z_{x0}) \geq v(t, z_x(t))$ *and* $|z_x(t)|_{\mathcal{O}_x} \leq \alpha_{x1}^{-1}(v(t_0, z_{x0})) \leq \alpha_{x1}^{-1}(\alpha_{x2}(r))$ *else,* $|z_{x0}|_{\mathcal{O}_x} \leq \alpha_k^{-1}(K\kappa_2 B(r))$ *and from (3.13),* $v(t, z_x(t)) \leq \alpha_{x2}\left(\alpha_k^{-1}(K\kappa_2 B(r))\right)$ *such that* $|z_x(t)|_{\mathcal{O}_x} \leq \alpha_{x1}^{-1}\left(\alpha_{x2}(\alpha_k^{-1}(K\kappa_2 B(r)))\right)$*. By choosing*

$$M(r) := \max\left(\alpha_{x1}^{-1}(\alpha_{x2}(r)), \alpha_{x1}^{-1}\left(\alpha_{x2}\left(\alpha_k^{-1}(K\kappa_2 B(r))\right)\right)\right),$$

the assumption of $|z_x(t)|_{\mathcal{O}_x} = x(t)$ *escaping to infinity is contradicted, since* $|z_x(t)|_{\mathcal{O}_x} \leq M(r)$*. By proof of contradiction* $|z_x(t)|_{\mathcal{O}_x}$ *is globally uniformly bounded. From Proposition 3.1 and the Assumptions 2.1-2.2, the assumptions of Lemma 1.1 are satisfied and the result is proved.* ■

From Proposition 3.2, the optimal set $\mathcal{O}_{xu\lambda}(t)$ is uniformly stable and attractive, such that optimality is achieved asymptotically for the closed loop.

Corollary 3.2 *If for* $\mathbb{U} \subset \mathbb{R}^r$ *there exists a constant* $c_x > 0$ *such that for* $|x| \leq c_x$ *the domain* $\mathbb{U}_z \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{U} \times \mathbb{R}^d$ *contain* $\mathcal{O}_{xu\lambda}$*, then if the Assumptions 2.1-2.3 are satisfied, the set* $\mathcal{O}_{xu\lambda}$ *is UAS with respect to the system (3.3) and (3.4).*

Proof. *Since* $\mathcal{O}_{xu\lambda} \subset \mathbb{U}_z$*, there exists a positive constant* $r \geq |z_{xu\lambda 0}|_{\mathcal{O}_{xu\lambda}}$ *such that* $|z_x(t)|_{\mathcal{O}_x} \leq c_{x0}$ *where* $0 < c_{x0} < c_x$*, hence the domain* $\mathbb{U}_{u\lambda} \subset \mathbb{R}_{\geq 0} \times \mathbb{U} \times \mathbb{R}^d$ *contain* $\mathcal{O}_{u\lambda}$*. UAS of* $\mathcal{O}_{u\lambda}$ *follows from the Lyapunov-like function* $V_{u\lambda}$*. UB follows from* $|z_{xu\lambda 0}|_{\mathcal{O}_{xu\lambda}} \leq r$*, and the UAS property of* $\mathcal{O}_{xu\lambda}$ *follows from Corollary 1.1* ■

3.3 Example: DP of a scale model ship

The model ship dynamics considered in this example are described in the introduction chapter, Section 1.1.1, and the model parameters can be found in Table A.1. The control allocation algorithm in this chapter relies on Assumption 2.1, 2.2 a) and c), and 2.3. Furthermore, no actuator dynamic is considered and all parameters are assumed to be known. This means that the generalized force vector τ is considered to be known, and the external disturbance b should be handled by the high level controller (integrator action), such that the model which the allocation algorithm is based upon, takes the form (rewriting (1.6))

$$\begin{aligned} \dot{\eta}_e &= R(\psi)\nu, \\ \dot{\nu} &= -M^{-1}D\nu + M^{-1}(\tau + b), \\ \tau &= \Phi(\nu, u), \end{aligned} \tag{3.14}$$

where, $\Phi(\nu, u) := G_u(u)\left(T_1(\omega_1), T_1(\omega_2), T_1(\omega_3)\right)^{\mathrm{T}}$, and the propulsion forces are defined by:

$$T_i(\omega_i) := \begin{cases} k_{Tp_i}\omega_i^2 & ,\omega_i \geq 0 \\ k_{Tn_i}\left|\omega_i\right|\omega_i & ,\omega_i < 0 \end{cases} . \tag{3.15}$$

Analytically the model does not satisfy Assumption 2.1 c) and d) for $u = 0$. This means that the control allocation algorithm should be based on an approximation of the model ship model. By replacing τ from (3.14) with $\tau_a = \Phi(\nu, u) + \Phi_L(u)$, where $\Phi_L(u) := \varsigma(\omega_1 + \omega_2, \omega_3, \omega_1 - \omega_2 + \omega_3)^{\mathrm{T}}$ in the control allocation algorithm, Assumption 2.1 d) is satisfied for all $\varsigma > 0$. Moreover, although Φ analytically is not twice differentiable (not unique at $u = 0$) and Assumption 2.1 c) is not satisfied, a numerical implementation of the partial derivatives will always yield unique solutions.

The cost function used in this simulation scenario is defined by:

$$J(u) := \sum_{i=1}^{3} k_i\left|\omega_i\right|\omega_i^2 + k_{i2}\omega_i^2 + \sum_{i=1}^{2} q_i\delta_i^2, \tag{3.16}$$

$$\begin{aligned} &\left|\omega_i\right| \leq 18Hz, \quad \left|\delta_i\right| \leq 35\deg, \\ &k_1 = k_2 = 0.01, \quad k_3 = 0.02, \quad q_1 = q_2 = 500, \\ &k_{i2} = 10^{-3}. \end{aligned}$$

Assumption 2.3 is satisfied locally since for bounded u and λ, k_{i2} ensures that $\frac{\partial^2 L}{\partial u^2} > 0$. Based on the high level controller (1.22) and previous discussion, the Assumptions 2.1-2.3 are satisfied locally and from Corollary 3.2 an UAS result is obtained for this model ship control problem.

The gain matrices are chosen as follows:
$K_p := M \cdot diag(3.13, 3.13, 12.5)10^{-2}$, $K_d := M \cdot diag(3.75, 3.75, 7.5)10^{-1}$, $K_I := M \cdot diag(0.2, 0.2, 4)10^{-3}$ and $\Gamma := \left(\mathbb{H}^{\mathrm{T}} W \mathbb{H} + \varepsilon I\right)^{-1}$ where $W := diag\left(1, 1, 1, 1, 1, 0.6, 0.6, 0.6\right)$ and $\varepsilon := 10^{-9}$. The weighting matrix W is chosen such that the deviation of $\left|\frac{\partial L}{\partial \lambda}\right| = \left|k(t, x) - \Phi(t, x, u, \theta)\right|$ from zero, is penalized more then the deviation of $\left|\frac{\partial L}{\partial u}\right|$ from zero, in the search direction.

Based on the wind and current disturbance vector $b := (0.06, 0.08, 0.04)^{\mathrm{T}}$, the simulation results are presented in the Figures 3.1-3.3. The control objective is satisfied and the commanded virtual controls τ_c, are tracked closely by the real, actuator generated forces, except for $t \approx 400$, where the control allocation is suboptimal due to actuator saturation: see Figure 3.3. The simulations are carried out in the MATLAB environment with a sampling rate of $10Hz$.

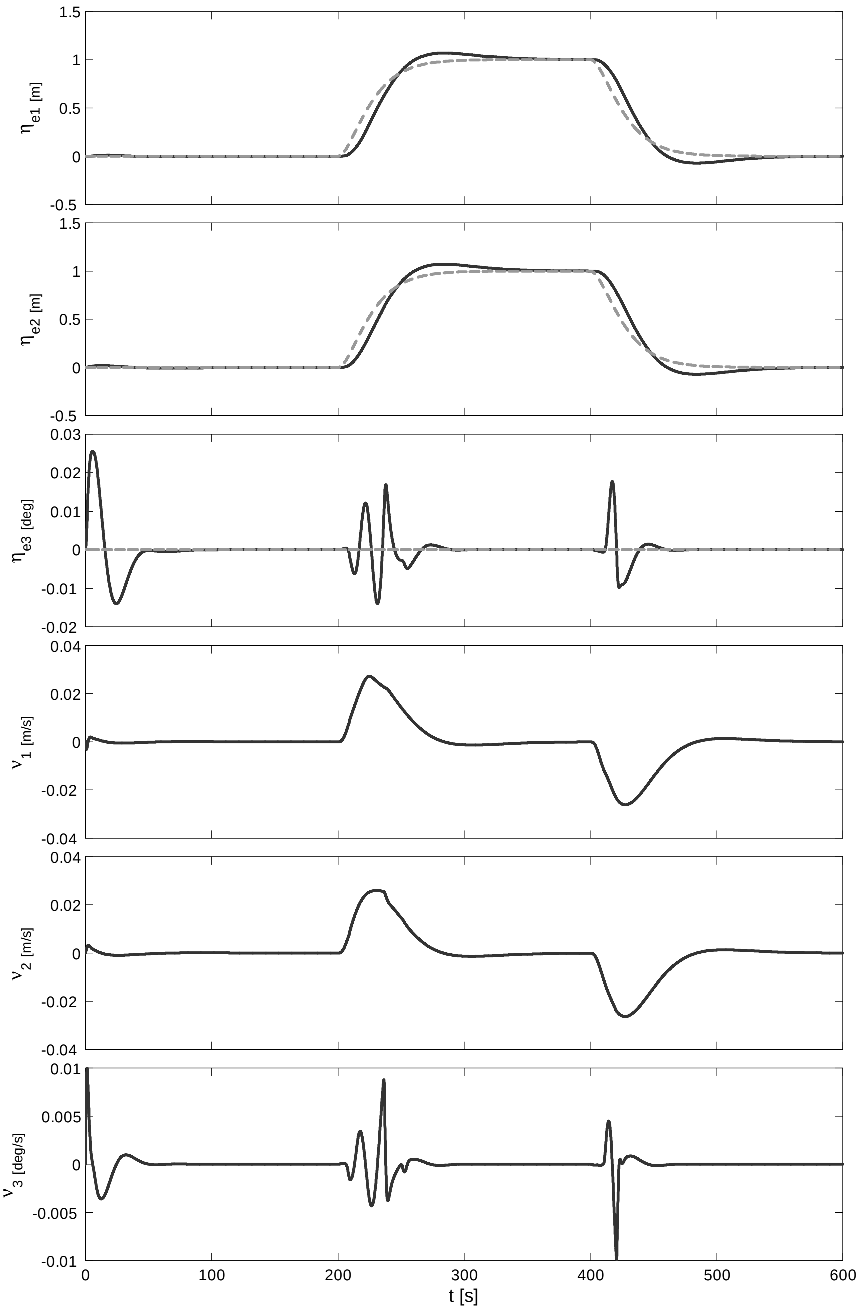

Figure 3.1: The ship; desired position (dashed), actual position (solid) and velocities.

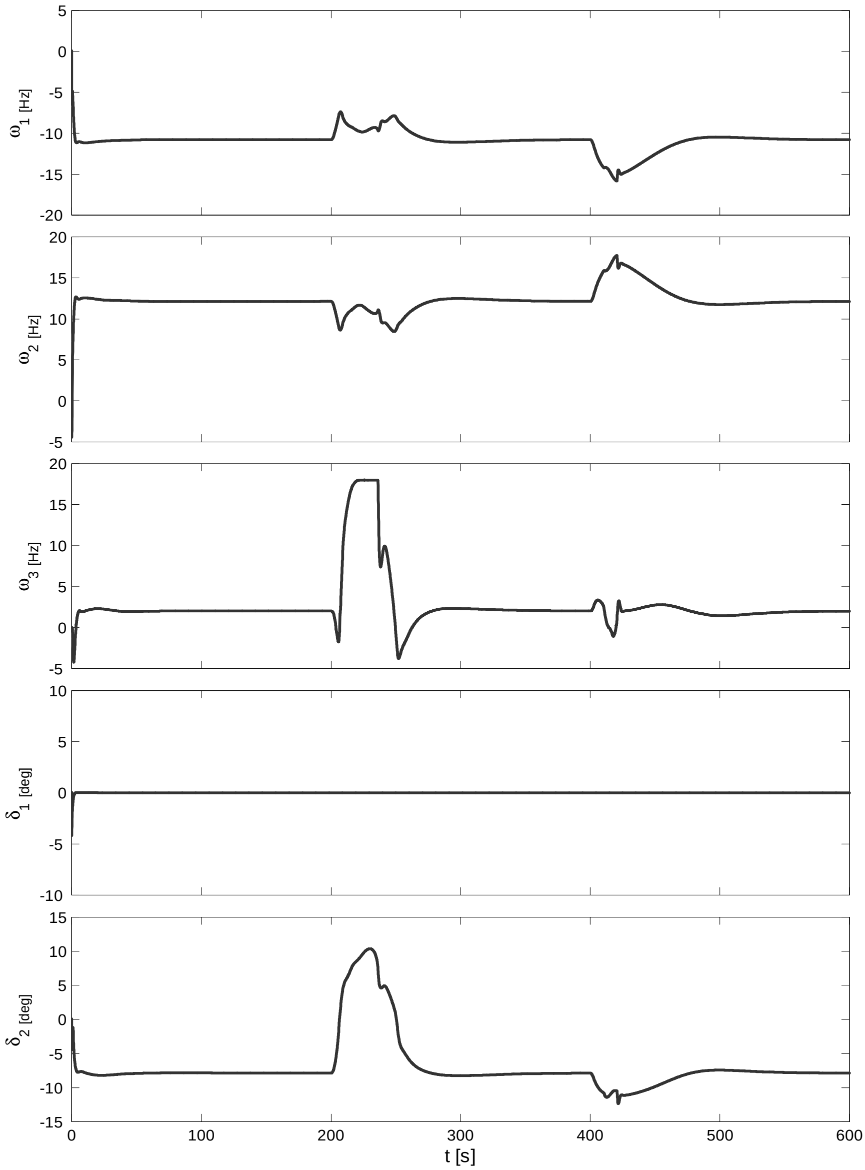

Figure 3.2: The acutal propeller velocities and rudder deflections.

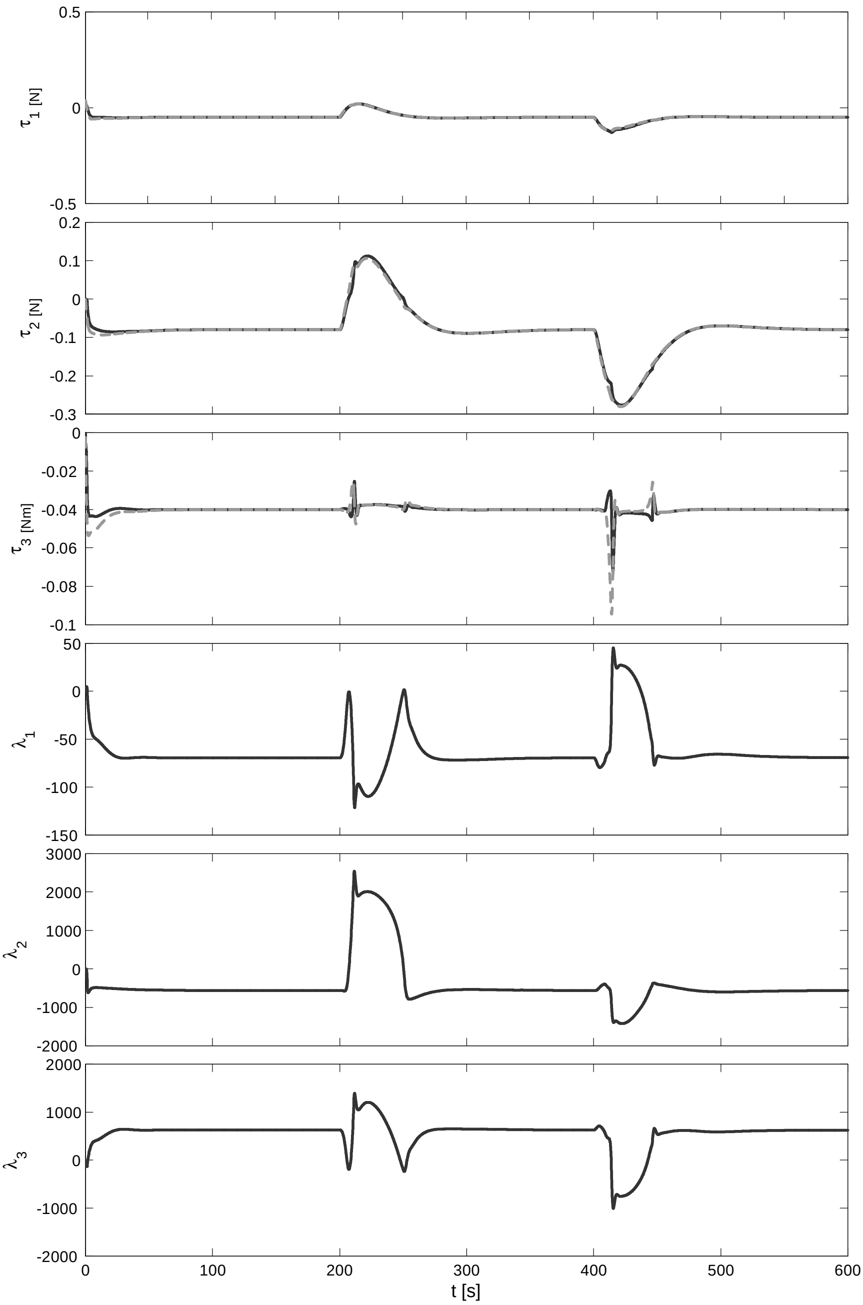

Figure 3.3: The desired virtual control forces (dashed) and the actual control forces (solid) generated by the actuators. The lagrangian parameters are also shown.

Chapter 4

Dynamic Adaptive Control Allocation

In this chapter an adaptive mechanism is include in the optimal control allocation design in order to account for an unknown, but bounded, parameter vector θ. A Lyapunov based indirect parameter estimation scheme (see Krstic et al. [1995] for a systematic Lyapunov based procedure) is utilized. And since the state space of (2.1) by Assumption 2.1 a) is assumed to be known, the estimation model

$$\dot{\hat{x}} = A(x - \hat{x}) + f(t, x) + g(t, x)\Phi(t, x, u, \hat{\theta}), \tag{4.1}$$

is considered in the construction of the adaptive law. This estimation model has the same structure as a *series parallel model* (SP), Ioannou and Sun [1996] and Landau [1979]. For analytical purpose, the filtered error estimate of the unknown parameter vector:

$$\dot{\epsilon} = -A\epsilon + g(t, x)\Phi_\theta(t, x, u)\tilde{\theta}, \tag{4.2}$$

where $\Phi_\theta(t, x, u) := \left(\Phi_\tau(t, x, u)^{\mathrm{T}}, \Phi_u(t, x, u)^{\mathrm{T}}\right)$, $\tilde{\theta} = \theta - \hat{\theta}$, $\epsilon = x - \hat{x}$ and $(-A)$ is Hurwitz, will be used.

Remark 4.1 *If the virtual control τ can be measured, e.g. by accelerometers and gyroscopes, the estimate (4.1) is not necessary and the construction and analysis of the adaptive law becomes much simpler.*

The analysis and design of the adaptive law can be carried out in several ways. We consider the approach, with reference to Figure 2.2, where the perturbing system (Σ_2) is expanded with an adaptive law. An advantage with this approach is that the adaptive law is independent of V_x.

Remark 4.2 *A different approach would be to expand the perturbed system (Σ_1) with an adaptive law. In this case the adaptive law will be dependent on the initial Lyapunov function (similar to Tjønnås and Johansen [2005]), but convergence results like, $x(t) \to 0$ as $t \to \infty$, may be concluded even if a persistence of excitation condition is not satisfied.*

4.1 Stability of the certainty equivalent control allocation

In order to see the cascaded coupling between the system (3.3) and the adaptive and optimal control allocation update-laws, equation (3.3) can be rewritten by:

$$\begin{aligned}\dot{x} &= f(t,x) + g(t,x)k(t,x) \\ &\quad + g(t,x)\left(k(t,x) - \Phi(t,x,u,\hat{\theta})\right) \\ &\quad + g(t,x)\Phi_\theta(t,x,u)\tilde{\theta}. \end{aligned} \tag{4.3}$$

Based on the perturbing system (Σ_2, expanded with estimation and adaptation dynamic, in Figure 2.2), we consider the estimated optimal solution set

$$\mathcal{O}_{u\lambda\tilde{\theta}}(t,x) := \left\{ (u^{\mathrm{T}}, \lambda^{\mathrm{T}}, \epsilon^{\mathrm{T}}, \tilde{\theta}^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{n_{u\lambda\tilde{\theta}}} \middle| f_{\mathcal{O}_{u\lambda\tilde{\theta}}}(t, z_{xu\lambda\tilde{\theta}}) = 0 \right\} \tag{4.4}$$

where $n_{u\lambda\tilde{\theta}} := r + d + n + m$, $z_{xu\lambda\tilde{\theta}} := \left(x^{\mathrm{T}}, u^{\mathrm{T}}, \lambda^{\mathrm{T}}, \epsilon^{\mathrm{T}}, \tilde{\theta}^{\mathrm{T}}\right)^{\mathrm{T}}$, $f_{\mathcal{O}_{u\lambda\tilde{\theta}}} := \left(\left(\frac{\partial L_{\hat{\theta}}}{\partial u}\right)^{\mathrm{T}}, \left(\frac{\partial L_{\hat{\theta}}}{\partial \lambda}\right)^{\mathrm{T}}, \epsilon^{\mathrm{T}}, \tilde{\theta}^{\mathrm{T}} \right)$ and

$$L_{\hat{\theta}}(t,x,u,\lambda,\hat{\theta}) := J(t,x,u) + (k(t,x) - \Phi(t,x,u,\hat{\theta}))^{\mathrm{T}}\lambda.$$

The task will be to show that the following certainty equivalent control allocation algorithm ensures stability, in a certain sense, of the set $\mathcal{O}_{u\lambda\tilde{\theta}}(t,x)$, with respect to (4.2), (4.5) and (4.6):

$$\begin{pmatrix} \dot{u} \\ \dot{\lambda} \end{pmatrix} = -\Gamma \mathbb{H}_{\hat{\theta}} \begin{pmatrix} \frac{\partial L_{\hat{\theta}}}{\partial u} \\ \frac{\partial L_{\hat{\theta}}}{\partial \lambda} \end{pmatrix} - \mathbb{H}_{\hat{\theta}}^{-1} u_{ff\hat{\theta}}, \tag{4.5}$$

$$\begin{aligned}\dot{\hat{\theta}} &= \Gamma_{\hat{\theta}}^{-1} \left(\Phi_\theta(t,x,u)\right)^{\mathrm{T}} \left(g(t,x)\right)^{\mathrm{T}} \left(\Gamma_\epsilon \epsilon + \left(\frac{\partial^2 L_{\hat{\theta}}}{\partial x \partial u}\right)^{\mathrm{T}} \frac{\partial L_{\hat{\theta}}}{\partial u} \right) \\ &\quad + \Gamma_{\hat{\theta}}^{-1} \left(\Phi_\theta(t,x,u)\right)^{\mathrm{T}} \left(g(t,x)\right)^{\mathrm{T}} \left(\frac{\partial^2 L_{\hat{\theta}}}{\partial x \partial \lambda}\right)^{\mathrm{T}} \frac{\partial L_{\hat{\theta}}}{\partial \lambda}, \end{aligned} \tag{4.6}$$

where $\mathbb{H}_{\hat{\theta}} := \begin{pmatrix} \frac{\partial^2 L_{\hat{\theta}}}{\partial u^2} & \frac{\partial^2 L_{\hat{\theta}}}{\partial \lambda \partial u} \\ \frac{\partial^2 L_{\hat{\theta}}}{\partial u \partial \lambda} & 0 \end{pmatrix}$, the matrices Γ, Γ_ϵ and $\Gamma_{\hat{\theta}}$, are symmetric positive definite and $u_{ff\hat{\theta}}$ is a feed-forward like term given by:

$$u_{ff\hat{\theta}} := \mathbb{H}_{\hat{\theta}}^{-1} \left(\begin{pmatrix} \frac{\partial^2 L}{\partial t \partial u} \\ \frac{\partial^2 L}{\partial t \partial \lambda} \end{pmatrix} + \begin{pmatrix} \frac{\partial^2 L}{\partial x \partial u} \\ \frac{\partial^2 L}{\partial x \partial \lambda} \end{pmatrix} f(t,x) + \begin{pmatrix} \frac{\partial^2 L_{\hat{\theta}}}{\partial \hat{\theta} \partial u} \\ \frac{\partial^2 L_{\hat{\theta}}}{\partial \hat{\theta} \partial \lambda} \end{pmatrix} \dot{\hat{\theta}} \right)$$
$$+ \mathbb{H}_{\hat{\theta}}^{-1} \begin{pmatrix} \frac{\partial^2 L}{\partial x \partial u} \\ \frac{\partial^2 L}{\partial x \partial \lambda} \end{pmatrix} g(t,x)\Phi(t,x,u,\hat{\theta}),$$

or $u_{ff\hat{\theta}} := 0$ if $det(\frac{\partial^2 L}{\partial u^2}) < \epsilon$, where $(k_1)^r > \epsilon > 0$. Γ_ϵ is uniquely defined by the solution of the Lyapunov equation $Q_\epsilon = \frac{1}{2}(A^{\mathrm{T}}\Gamma_\epsilon + \Gamma_\epsilon A)$, where Q_ϵ is a positive definite design matrix. As before a Lyapunov-like function is considered:

$$V_{u\lambda\tilde{\theta}}(t,x,u,\lambda,\epsilon,\tilde{\theta}) := \frac{1}{2}\left(\left(\frac{\partial L_{\hat{\theta}}}{\partial u}\right)^{\mathrm{T}} \frac{\partial L_{\hat{\theta}}}{\partial u} + \left(\frac{\partial L_{\hat{\theta}}}{\partial \lambda}\right)^{\mathrm{T}} \frac{\partial L_{\hat{\theta}}}{\partial \lambda} \right) + \frac{\epsilon^{\mathrm{T}}\Gamma_\epsilon \epsilon}{2} + \frac{\tilde{\theta}^{\mathrm{T}}\Gamma_{\hat{\theta}}\tilde{\theta}}{2}. \tag{4.7}$$

And its time derivative along the trajectories of the closed loop system is:

$$\begin{aligned} \dot{V}_{u\lambda\hat{\theta}} &= \left(\left(\frac{\partial L_{\hat{\theta}}}{\partial u}\right)^{\mathrm{T}} \frac{\partial^2 L_{\hat{\theta}}}{\partial u^2} + \left(\frac{\partial L_{\hat{\theta}}}{\partial \lambda}\right)^{\mathrm{T}} \frac{\partial^2 L_{\hat{\theta}}}{\partial u \partial \lambda} \right) \dot{u} \\ &\quad + \left(\frac{\partial L_{\hat{\theta}}}{\partial u}\right)^{\mathrm{T}} \frac{\partial^2 L_{\hat{\theta}}}{\partial \lambda \partial u} \dot{\lambda} + \left(\left(\frac{\partial L_{\hat{\theta}}}{\partial u}\right)^{\mathrm{T}} \frac{\partial^2 L_{\hat{\theta}}}{\partial x \partial u} + \left(\frac{\partial L_{\hat{\theta}}}{\partial \lambda}\right)^{\mathrm{T}} \frac{\partial^2 L_{\hat{\theta}}}{\partial x \partial \lambda} \right) \dot{x} \\ &\quad - \left(\left(\frac{\partial L_{\hat{\theta}}}{\partial u}\right)^{\mathrm{T}} \frac{\partial^2 L_{\hat{\theta}}}{\partial \hat{\theta} \partial u} + \left(\frac{\partial L_{\hat{\theta}}}{\partial \lambda}\right)^{\mathrm{T}} \frac{\partial^2 L_{\hat{\theta}}}{\partial \hat{\theta} \partial \lambda} \right) \dot{\tilde{\theta}} \\ &\quad + \left(\frac{\partial L_{\hat{\theta}}}{\partial u}\right)^{\mathrm{T}} \frac{\partial^2 L_{\hat{\theta}}}{\partial t \partial u} + \left(\frac{\partial L_{\hat{\theta}}}{\partial \lambda}\right)^{\mathrm{T}} \frac{\partial^2 L_{\hat{\theta}}}{\partial t \partial \lambda} + \epsilon^{\mathrm{T}}\dot{\epsilon} + \tilde{\theta}^{\mathrm{T}}\Gamma_{\hat{\theta}}\dot{\tilde{\theta}}. \end{aligned}$$

Further by inserting (4.2), (4.5) and (4.6)

$$\begin{aligned} \dot{V}_{u\lambda\hat{\theta}} &= -\left(\left(\frac{\partial L_{\hat{\theta}}}{\partial u}\right)^{\mathrm{T}}, \left(\frac{\partial L_{\hat{\theta}}}{\partial \lambda}\right)^{\mathrm{T}} \right) \mathbb{H}_{\hat{\theta}}\Gamma\mathbb{H}_{\hat{\theta}} \left(\left(\frac{\partial L_{\hat{\theta}}}{\partial u}\right)^{\mathrm{T}}, \left(\frac{\partial L_{\hat{\theta}}}{\partial \lambda}\right)^{\mathrm{T}} \right)^{\mathrm{T}} \\ &\quad -\epsilon^{\mathrm{T}}\frac{1}{2}(A^{\mathrm{T}}\Gamma_\epsilon + \Gamma_\epsilon A)\epsilon & (4.8) \\ &\le -c_{\hat{\theta}} \left(\left(\frac{\partial L_{\hat{\theta}}}{\partial u}\right)^{\mathrm{T}} \frac{\partial L_{\hat{\theta}}}{\partial u} + \left(\frac{\partial L_{\hat{\theta}}}{\partial \lambda}\right)^{\mathrm{T}} \frac{\partial L_{\hat{\theta}}}{\partial \lambda} \right) - \epsilon^{\mathrm{T}}Q_\epsilon\epsilon, & (4.9) \end{aligned}$$

where $c_{\hat\theta} := \inf_t \lambda_{\min}\left(\mathbb{H}_{\hat\theta}\Gamma\mathbb{H}_{\hat\theta}\right) > 0$.

Proposition 4.1 *Let Assumptions 2.1, 2.2 a) and 2.3 be satisfied. Then if $x(t)$ exists for all t, the set $\mathcal{O}_{u\lambda\tilde\theta}(t, x(t))$ is UGS with respect to the system (4.1), (4.2), (4.5) and (4.6), and $\left(\epsilon, \frac{\partial L_{\hat\theta}}{\partial u}, \frac{\partial L_{\hat\theta}}{\partial \lambda}\right)$ converges asymptotically to zero.*

Proof. *Define $G(t, x, u, \lambda, \tilde\theta) := \left(\epsilon, \tilde\theta, \frac{\partial L_{\hat\theta}}{\partial u}^T, \frac{\partial L_{\hat\theta}}{\partial \lambda}^T\right)^T$, $\bar V_{u\lambda\tilde\theta}(t, z_{u\lambda\tilde\theta}) := V_{u\lambda\tilde\theta}(t, x(t), u, \lambda, \epsilon, \tilde\theta)$ and change L with $L_{\hat\theta}$ such that (3.7) become*

$$\left(\frac{\partial L_{\hat\theta}}{\partial u}\right)^T\frac{\partial L_{\hat\theta}}{\partial u}+\left(\frac{\partial L_{\hat\theta}}{\partial \lambda}\right)^T\frac{\partial L_{\hat\theta}}{\partial \lambda}=\begin{pmatrix}u-u^*\\ \lambda-\lambda^*\end{pmatrix}^T\mathbb{H}_{\hat\theta*}^T\mathbb{H}_{\hat\theta*}\begin{pmatrix}u-u^*\\ \lambda-\lambda^*\end{pmatrix}, \qquad (4.10)$$

then the proof of this result follow with the same steps as in the proof of Proposition 3.1. I.e. there exist constants $\bar\kappa_2 \geq \bar\kappa_1 > 0$ such that $\bar\kappa_1 |z_{u\lambda}|^2_{\mathcal{O}_{u\lambda}} \leq \left(\left(\frac{\partial L_{\hat\theta}}{\partial u}\right)^T\frac{\partial L_{\hat\theta}}{\partial u}+\left(\frac{\partial L_{\hat\theta}}{\partial \lambda}\right)^T\frac{\partial L_{\hat\theta}}{\partial \lambda}\right) \leq \bar\kappa_2 |z_{u\lambda}|^2_{\mathcal{O}_{u\lambda}}$ since $\mathbb{H}_{\hat\theta*}^T\mathbb{H}_{\hat\theta*}$ is positive definite and bounded from Assumption 2.1 and 2.3. By a similar argument, in (3.8) change P_{*c} with $\frac{\partial^2 L_{\hat\theta}}{\partial u^2}$, $\mathbb{H}_{\hat\theta}$ is non-singular, hence the control allocation law (4.5) always exist and, $\kappa_1 \left|z_{u\lambda\tilde\theta}\right|_{\mathcal{O}_{u\lambda\tilde\theta}} \leq \bar V_{u\lambda\tilde\theta}(t, z_{u\lambda\tilde\theta}) \leq \kappa_2 \left|z_{u\lambda\tilde\theta}\right|_{\mathcal{O}_{u\lambda\tilde\theta}}$, where $\kappa_1 := \frac{1}{2}\min(\bar\kappa_1, \Gamma_\epsilon, \Gamma_{\hat\theta})$ $\kappa_2 := \frac{1}{2}\max(\bar\kappa_2, \Gamma_\epsilon, \Gamma_{\hat\theta})$, such that $\bar V_{u\lambda\tilde\theta}(t, z_{u\lambda\tilde\theta})$ is a radially unbounded Lyapunov function.

Since (4.9) is negative semidefinite, UGS of the set $\mathcal{O}_{u\lambda\tilde\theta}$ can be concluded by Theorem 1.1. The convergence result follows by Barbalat's lemma:

$$\begin{aligned}\min(c_{\hat\theta}, \lambda_{\min}(Q_\epsilon)) \lim_{t\to\infty}\int_{t_0}^t \left|\left(\epsilon(s), \frac{\partial \bar L_{\hat\theta}}{\partial u}(s), \frac{\partial \bar L_{\hat\theta}}{\partial \lambda}(s)\right)\right|^2 ds &\leq \\ \lim_{t\to\infty}\int_{t_0}^t \dot{\bar V}_{u\lambda\tilde\theta}(s, z_{u\lambda\tilde\theta}(s))ds &\leq \\ \bar V_{u\lambda\tilde\theta}(t_0, z_{u\lambda\tilde\theta}(t_0)) &< \infty\end{aligned}$$

where $\frac{\partial \bar L_{\hat\theta}}{\partial u}(t) := \frac{\partial L_{\hat\theta}}{\partial u}(t, x(t), u(t), \lambda(t), \hat\theta(t))$ and $\frac{\partial \bar L_{\hat\theta}}{\partial \lambda}(t) := \frac{\partial L_{\hat\theta}}{\partial \lambda}\left(t, x(t), u(t), \lambda(t), \hat\theta(t)\right)$ such that $\left(\epsilon(s), \frac{\partial \bar L_{\hat\theta}}{\partial u}(s), \frac{\partial \bar L_{\hat\theta}}{\partial \lambda}(s)\right) \in \mathcal{L}_2$ and $\mathcal{L}_\infty$ (due to the UGS result), and thus $\left(\bar\epsilon(s), \frac{\partial \bar L_{\hat\theta}}{\partial u}(s), \frac{\partial \bar L_{\hat\theta}}{\partial \lambda}(s)\right) \to 0$ as $s \to \infty$ ■

Assumption 4.1 *(Persistence of Excitation) The signal matrix*

$$\Phi_g(t) := g(t, x(t))\Phi_\theta(t, x(t), u(t)),$$

is persistently excited, which with respect to Definition 1.1 means that there exist constants T *and* $\gamma > 0$*, such that*

$$\int_t^{t+T} \Phi_g(\tau)^T \Phi_g(\tau) d\tau \geq \gamma I \ , \quad \forall t > t_0.$$

Remark 4.3 *The system trajectories* $x(t)$ *defined by (2.1) typically represent the tracking error i.e.* $x := x_s - x_d$*, where* x_s *is the state vector of the system, and* x_d *represents the desired reference for these states. This means that PE assumption on* $\Phi_g(t)$ *is dependent on the reference trajectory* x_d*, in addition to disturbances and noise, and imply that some "richness" properties of these signals are satisfied.*

Proposition 4.2 *Let* $x(t)$ *be UGB, then if Assumption 4.1 and the assumptions of Proposition 4.1 are satisfied, the set* $\mathcal{O}_{u\lambda\tilde{\theta}}$ *is UGAS with respect to system (4.1), (4.2), (4.5) and (4.6).*

Proof. See Appendix A.1 ■

Unless the PE condition is satisfied for Φ_g, only stability of the optimal set is guaranteed. Thus in the sense of achieving asymptotic optimality, parameter convergence is of importance.

4.2 Stability of the combined control and certainty equivalent control allocation

In this section the properties of the combined control and the certainty equivalent control allocation set

$$\mathcal{O}_{xu\lambda\tilde{\theta}}(t) := \mathcal{O}_x(t) \times \mathcal{O}_{u\lambda\tilde{\theta}}(t, 0) \tag{4.11}$$

is investigated. In the framework of cascaded systems, the system defined by (4.5), (4.6) together with (4.2) is considered to be the perturbing system Σ_2, while (3.3) represents the perturbed system Σ_1. Before establishing the main results of this section, an assumption on the interconnection term between the two systems is stated. We start by stating the following property:

Lemma 4.1 *By Assumption 2.1 and Lemma 3.1 there exist continuous functions* ς_x*,* ς_{xu}*,* $\varsigma_u : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$*, such that*

$$|\Phi_\theta(t, x, u)| \leq \varsigma_x(|x|)\varsigma_{xu}(|x|) + \varsigma_x(|x|)\varsigma_u(\left|z_{u\lambda\tilde{\theta}}\right|_{\mathcal{O}_{u\lambda\tilde{\theta}}}).$$

Proof. *From Mazenc and Praly [1996]'s lemma B.1: Since* $|\Phi_\theta(t, x, u)| \leq G_\Phi\left(\left|\left(x^T, u^T\right)^T\right|\right)$*, there exist continuous functions* $\varsigma_x, \bar{\varsigma}_u : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

such that $G_\Phi\left(\left|\left(x^T,u^T\right)^T\right|\right) \leq \varsigma_x(|x|)\bar{\varsigma}_u(|u|)$*. Further* $|u| \leq \left|\left(u^T,\lambda^T\right)^T\right| \leq \left|z_{u\lambda\tilde{\theta}}\right|_{\mathcal{O}_{u\lambda\tilde{\theta}}} + \varsigma_{\mathcal{O}_{u\lambda}}(|x|)$ *(triangular inequality) such that*

$$\begin{aligned}\bar{\varsigma}_u(|u|) &\leq \max_{0\leq s\leq\left|z_{u\lambda\tilde{\theta}}\right|_{\mathcal{O}_{u\lambda\tilde{\theta}}}+\varsigma_{\mathcal{O}_{u\lambda}}(|x|)} \bar{\varsigma}_u(s)\\ &\leq \varsigma_u(\left|z_{u\lambda\tilde{\theta}}\right|_{\mathcal{O}_{u\lambda\tilde{\theta}}}) + \varsigma_{xu}(|x|),\end{aligned}$$

where $\varsigma_u(\left|z_{u\lambda\tilde{\theta}}\right|_{\mathcal{O}_{u\lambda\tilde{\theta}}}) := \max_{0\leq s\leq 2\left|z_{u\lambda\tilde{\theta}}\right|_{\mathcal{O}_{u\lambda\tilde{\theta}}}} \bar{\varsigma}_u(s)$ *and* $\varsigma_{xu}(|x|) := \max_{0\leq s\leq 2\varsigma_{\mathcal{O}_{u\lambda}}(|x|)} \bar{\varsigma}_u(s)$

■

Assumption 2.2 *(Continued)*

d) *The function,* α_k*, from Assumption 2.2 c) has the following additional property:*

$$\alpha_k^{-1}(|x|)\alpha_{x3}(|x|) \geq \alpha_{x4}(|x|)\varsigma_{x\max}(|x|)\ , \tag{4.12}$$

where $\varsigma_{x\max}(|x|) := \max(1, \varsigma_x(|x|), \varsigma_x(|x|)\varsigma_{xu}(|x|))$.

Remark 4.4 *If there exists a constant* $K > 0$ *such that* $\varsigma_{x\max}(|x|) \leq K\ \forall x$*, which is common in a mechanical system, Assumption 2.2 d) is satisfied.*

Proposition 4.3 *If Assumptions 2.1-2.3 are satisfied, then the set* $\mathcal{O}_{xu\lambda\tilde{\theta}}$ *is UGS, with respect to system (4.2), (4.3), (4.5) and (4.6). If in addition Assumption 4.1 is satisfied, then* $\mathcal{O}_{xu\lambda\tilde{\theta}}$ *is UGAS with respect to (4.2), (4.3), (4.5), (4.6).*

Proof. *As before the main part of this proof is to prove boundedness and completeness and invoke Lemma 1.1. Let* $\left|z_{xu\lambda\tilde{\theta}0}\right|_{\mathcal{O}_{xu\lambda\tilde{\theta}}} \leq r$*, where* $r > 0$*, and assume that* $|z_x(t)|_{\mathcal{O}_x}$ *escapes to infinity at* T*. Then for any constant* $M(r)$ *there exists a* $t \in [t_0, T)$ *such that* $M(r) \leq |z_x(t)|_{\mathcal{O}_x}$*. In what follows we show that* $M(r)$ *can not be chosen arbitrarily. Define* $v(t,z_x) := V_x(t,x)$ *such that*

$$\begin{aligned}\dot{v} &\leq -\alpha_{x3}(|z_x|_{\mathcal{O}_x}) + \frac{\partial V_x}{\partial x}g(t,x)\left(k(t,x) - \Phi(t,x,u,\hat{\theta})\right)\\ &\quad+\frac{\partial V_x}{\partial x}g_x(t,x)\Phi_\theta(t,x,u)\tilde{\theta} \qquad (4.13)\\ &\leq -\alpha_{x3}(|z_x|_{\mathcal{O}_x}) + \left|\frac{\partial V_x}{\partial x}\right||g(t,x)|\left|\frac{\partial L_{\hat{\theta}}}{\partial \lambda}(t,x,u,\hat{\theta})\right|\\ &\quad+\left|\frac{\partial V_x}{\partial x}\right||g_x(t,x)|\left|\Phi_\theta(t,x,u)\tilde{\theta}\right|\\ &\leq -\alpha_{x3}(|z_x|_{\mathcal{O}_x}) \qquad (4.14)\\ &\quad+\alpha_{x4}(|z_x|_{\mathcal{O}_x})K(\kappa_2+|\Phi_\theta(t,x,u)|)\left|z_{u\lambda\tilde{\theta}}\right|_{\mathcal{O}_{u\lambda\tilde{\theta}}}. \qquad (4.15)\end{aligned}$$

By the same arguments as in the proof of Proposition 3.2 and the use of Corollary 3.1 there exists a positive constant $B(r) \geq 0$*, for all* $t \in [t_0, T)$*, such that for* $\left|z_{u\lambda\tilde{\theta}0}\right|_{\mathcal{O}_{u\lambda\tilde{\theta}}} \leq r$*,* $\left|z_{u\lambda\tilde{\theta}}(t)\right|_{\mathcal{O}_{u\lambda\tilde{\theta}}} \leq B(\left|z_{u\lambda\tilde{\theta}0}\right|_{\mathcal{O}_{u\lambda\tilde{\theta}}}) \leq B(r)$*. From Assumption 2.1 and 2.2,*

$$\begin{aligned}
\dot{v} &\leq -\alpha_{x3}(|z_x|_{\mathcal{O}_x}) \\
&\quad +\alpha_{x4}(|z_x|_{\mathcal{O}_x})K\left(\kappa_2 + |\Phi_\theta(t,x,u)|\right)B(r) \\
&\leq -\alpha_k(|z_x|_{\mathcal{O}_x})\alpha_{x4}(|z_x|_{\mathcal{O}_x})\varsigma_{x\max}((|z_x|_{\mathcal{O}_x}) \\
&\quad +\alpha_{x4}(|z_x|_{\mathcal{O}_x})K\left(\kappa_2 + |\Phi_\theta(t,x,u)|\right)B(r) \\
&\leq -\alpha_{x4}(|z_x|_{\mathcal{O}_x})\alpha_k(|z_x|_{\mathcal{O}_x})\varsigma_{x\max}(|z_x|_{\mathcal{O}_x}) \\
&\quad +\alpha_{x4}(|z_x|_{\mathcal{O}_x})K\kappa_2 B(r) \\
&\quad +\alpha_{x4}(|z_x|_{\mathcal{O}_x})K\varsigma_x(|z_x|_{\mathcal{O}_x})\varsigma_{xu}(|z_x|_{\mathcal{O}_x})B(r) \\
&\quad +\alpha_{x4}(|z_x|_{\mathcal{O}_x})K\varsigma_x(|z_x|_{\mathcal{O}_x})\varsigma_u(B(r))B(r) \qquad (4.16) \\
&\leq -\alpha_{x4}(|z_x|_{\mathcal{O}_x})\alpha_k(\alpha_{x2}^{-1}(v))\varsigma_{x\max}(|z_x|_{\mathcal{O}_x}) \\
&\quad +\alpha_{x4}(|z_x|_{\mathcal{O}_x})K\kappa_2 B(r) \\
&\quad +\alpha_{x4}(|z_x|_{\mathcal{O}_x})K\varsigma_x(|z_x|_{\mathcal{O}_x})\varsigma_{xu}(|z_x|_{\mathcal{O}_x})B(r) \\
&\quad +\alpha_{x4}(|z_x|_{\mathcal{O}_x})K\varsigma_x(|z_x|_{\mathcal{O}_x})\varsigma_u(B(r))B(r). \qquad (4.17)
\end{aligned}$$

Thus, if

$$|z_{x0}|_{\mathcal{O}_x} > \alpha_k^{-1}(K\left(\kappa_2 + 1 + \varsigma_u(B(r))\right)B(r))$$

then from (5.19),
$v(t_0, z_{x0}) \geq v(t, z_x(t))$ *and* $|z_x(t)|_{\mathcal{O}_x} \leq \alpha_{x1}^{-1}(v(z_{x0})) \leq \alpha_{x1}^{-1}(\alpha_{x2}(r))$*, else,*

$$|z_{x0}|_{\mathcal{O}_x} \leq \alpha_k^{-1}(K\left(\kappa_2 + 1 + \varsigma_u(B(r))\right)B(r))$$

and from (5.20), $v(t, z_x(t)) \leq \alpha_{x2}(\alpha_k^{-1}(K\left(\kappa_2 + 1 + \varsigma_u(B(r))\right)B(r)))$ *such that*

$$|z_x(t)|_{\mathcal{O}_x} \leq \alpha_{x1}^{-1}\big(\alpha_{x2}\big(\alpha_k^{-1}(K\left(\kappa_2 + 1 + \varsigma_u(B(r))\right)B(r))\big)\big).$$

By choosing

$$M(r) := \max\left(\alpha_{x1}^{-1}(\alpha_{x2}(r)), \alpha_{x1}^{-1}\big(\alpha_{x2}\big(\alpha_k^{-1}\left(K\left(\kappa_2 + 1 + \varsigma_u(B(r))\right)B(r)\right)\big)\big)\right),$$

the assumption of $|z_x(t)|_{\mathcal{O}_x}$ *escaping to infinity is contradicted, since* $M(r) > |z_x(t)|_{\mathcal{O}_x}$ *and* $|\cdot|_{\mathcal{O}_x}$ *is finite escape time detectable. Furthermore* $\mathcal{O}_x$ *is UGB. From Propositions 4.1 and 4.2 and the assumptions of these propositions, the assumptions of Lemma 1.1 and Corollary 1.1 are satisfied and the result is proved* ■

Proposition 4.3 implies that the time-varying optimal set $\mathcal{O}_{xu\lambda\tilde{\theta}}(t)$ is uniformly stable, and in addition uniformly attractive if Assumption 4.1 is satisfied. Thus optimal control allocation is achieved asymptotically for the closed loop. A local version of this result can be proven using Corollary 1.1 and arguments similar to the proof of Corollary 3.2.

Corollary 4.1 *If for $\mathbb{U} \subset \mathbb{R}^r$ there exist constant $c_x > 0$ such that for $|x| \leq c_x$ the domain $\mathbb{U}_z \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{U} \times \mathbb{R}^{d+n+m}$ contain $\mathcal{O}_{xu\lambda\tilde{\theta}}$, then if the Assumptions 2.1-2.3 are satisfied, the set $\mathcal{O}_{xu\lambda\tilde{\theta}}$ is US with respect to the system (4.2), (4.3), (4.5) and (4.6). If in addition Assumption 4.1 is satisfied, $\mathcal{O}_{xu\lambda\tilde{\theta}}$ is UAS with respect to the system (4.2), (4.3), (4.5) and (4.6).*
Proof. *The result follows by Corollary 1.1 and arguments similar to the ones given in the proof of Corollary 3.2* ■

Since the set with respect to system Σ_2 may only be US, due to actuator/effector constraints and parameter uncertainty, only US of the cascade may be concluded. But if the PE property is satisfied on Φ_g, a UAS result may be achieved with both optimal and adaptive convergence.

4.3 Example: DP of a scale model ship continued

In this section the example from Chapter 3 is continued with the assumption of uncertainty in the force generation model. This uncertainties can be viewed as thrust losses and are in the following denoted by $\theta := (\theta_{u1}, \theta_{u2})^{\mathrm{T}}$. The high level dynamics are given by (3.14) and based on the model presented in 3, the rewritten parameterized actuator-force mapping model takes the following form:

$$\Phi(\nu, u, \theta) := \begin{pmatrix} (1-D_1)T_1(\omega_1) + (1-D_2)T_2(\omega_2) & 0 \\ L_1 T_1(\omega_1) + L_2 T_2(\omega_2) & T_3(\omega_3) \\ \Phi_{31} T_1(\omega_1) + \Phi_{32} T_2(\omega_2) & l_{3,x} T_3(\omega_3) \end{pmatrix} \begin{pmatrix} \theta_{u1} \\ \theta_{u2} \end{pmatrix}, \tag{4.18}$$

where the thruster forces are defined by (3.15).

Note that θ_{u2} is also related to the parameters $k_{T_{p3}}$ and $k_{T_{n3}}$ in a multiplicative way, which suggests that the estimate of θ_{u2} gives a direct estimate of the tunnel thruster loss factor. In order to keep $\hat{\theta}$ from being zero, due to a physical consideration, a projection algorithm can be used. I.e. from

(4.5)

$$\dot{\hat{\theta}} = \begin{cases} \Gamma_{\hat{\theta}}^{-1}\xi & \text{(i)} \\ \Gamma_{\hat{\theta}}^{-1}\xi + \Gamma_{\hat{\theta}}^{-1}\frac{(1,1)^{\mathrm{T}}(1,1)}{(1,1)\Gamma_{\hat{\theta}}^{-1}(1,1)^{\mathrm{T}}}\Gamma_{\hat{\theta}}^{-1}\xi & \text{(ii)} \end{cases},$$

where (i) is used if $\hat{\theta} \in S^0$ or $\theta \in \delta(S)$ and $\left(\Gamma_{\hat{\theta}}^{-1}\xi\right)^{\mathrm{T}}(1,1)^{\mathrm{T}} \leq 0$, and (ii) is used otherwise. Further

$$\xi := \Phi_\theta(u)^{\mathrm{T}} {g_x}^{\mathrm{T}} \left(\Gamma_\epsilon \epsilon + \frac{\partial^2 L_{\hat{\theta}}}{\partial x \partial u}^{\mathrm{T}} \frac{\partial L_{\hat{\theta}}}{\partial u} + \frac{\partial^2 L_{\hat{\theta}}}{\partial x \partial \lambda}^{\mathrm{T}} \frac{\partial L_{\hat{\theta}}}{\partial \lambda} \right),$$

$S := \left\{ \hat{\theta} \in \mathbb{R}^2 \,\middle|\, -\min_i\{\hat{\theta}_i\} \leq \hat{\theta}_{\min} \right\}$, S^0 is the interior of S and $\delta(S)$ is the boundary of S and $\hat{\theta}_{\min} > 0$.

The high level controller and allocation algorithm parameters are given in Chapter 3, and the adaptive parameters are defined by: $A_\epsilon := I_{9\times 9}$, $Q_\theta := diag(1,1)$, $Q_\epsilon := diag(a, 150\cdot(100,100,1)^{\mathrm{T}})$, and $a := (1,1,1,1,1,1)^{\mathrm{T}}$. The loss related parameters are given by $\theta := (0.8, 0.9)^{\mathrm{T}}$ and the simulation results are presented in the Figures 4.1-4.4. Due to the disturbance and reference change, the parameter update law for θ_{u1} is exited on the whole time scale of the simulation since the aft propellers are counteracting the external forces acting the ship. But the update law for estimating θ_{u2} is only exited at $t \approx 200$ and $t \approx 400$ since the tunnel thruster is only used to counteract initial transients due to position reference changes. As before, the control objective is satisfied with the exceptions at $t \approx 200$ and $t \approx 400$ where the control allocation is suboptimal due to actuator saturation.

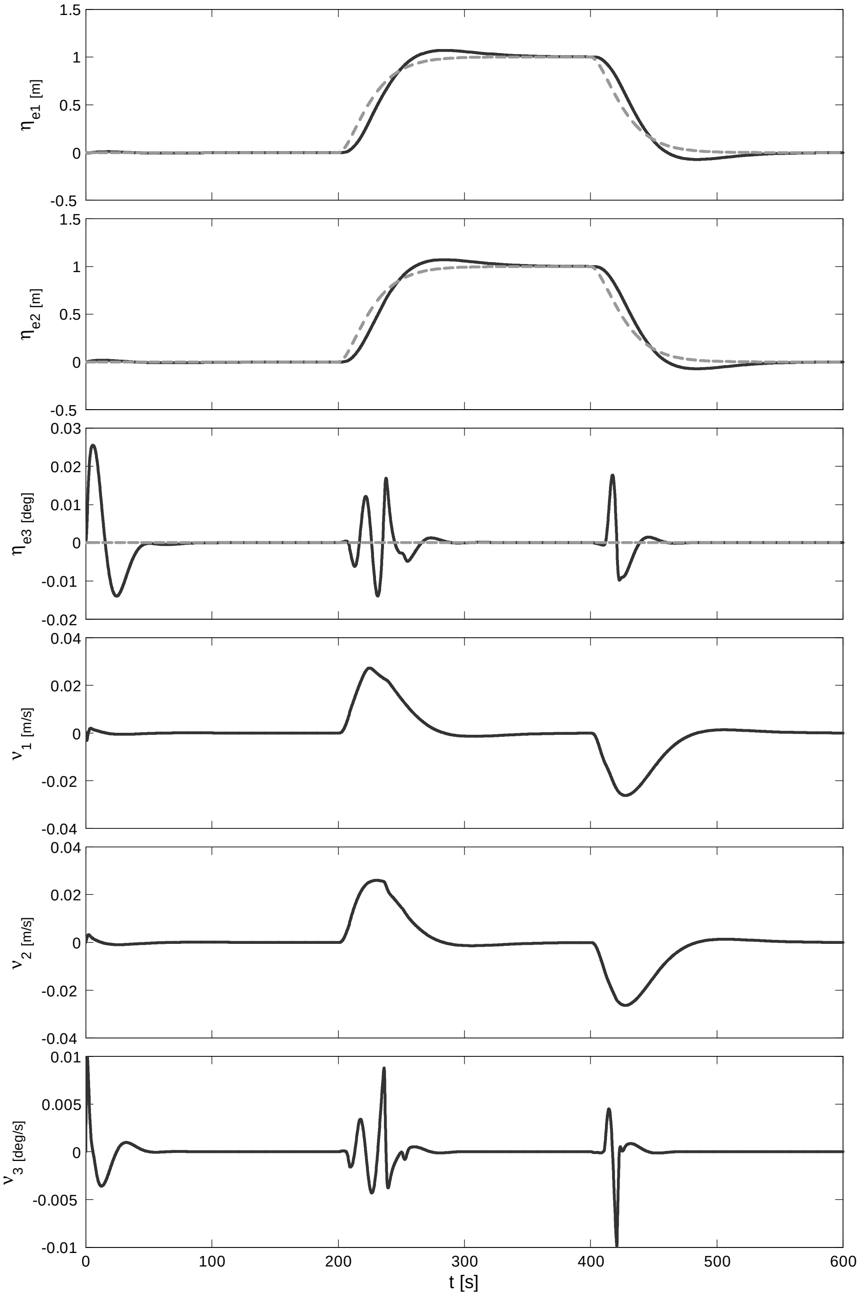

Figure 4.1: The ship; desired position (dashed), actual position (solid) and velocities.

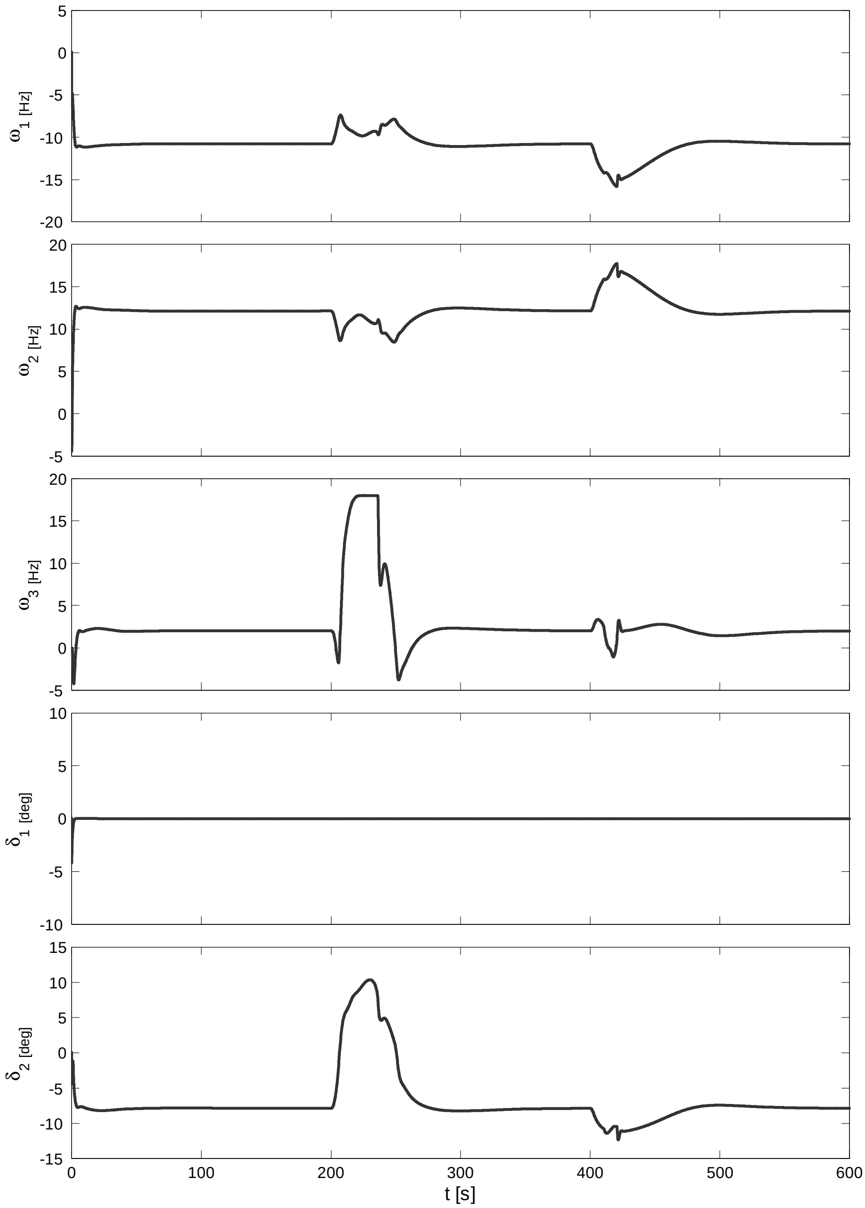

Figure 4.2: The actual propeller velocities and rudder deflections.

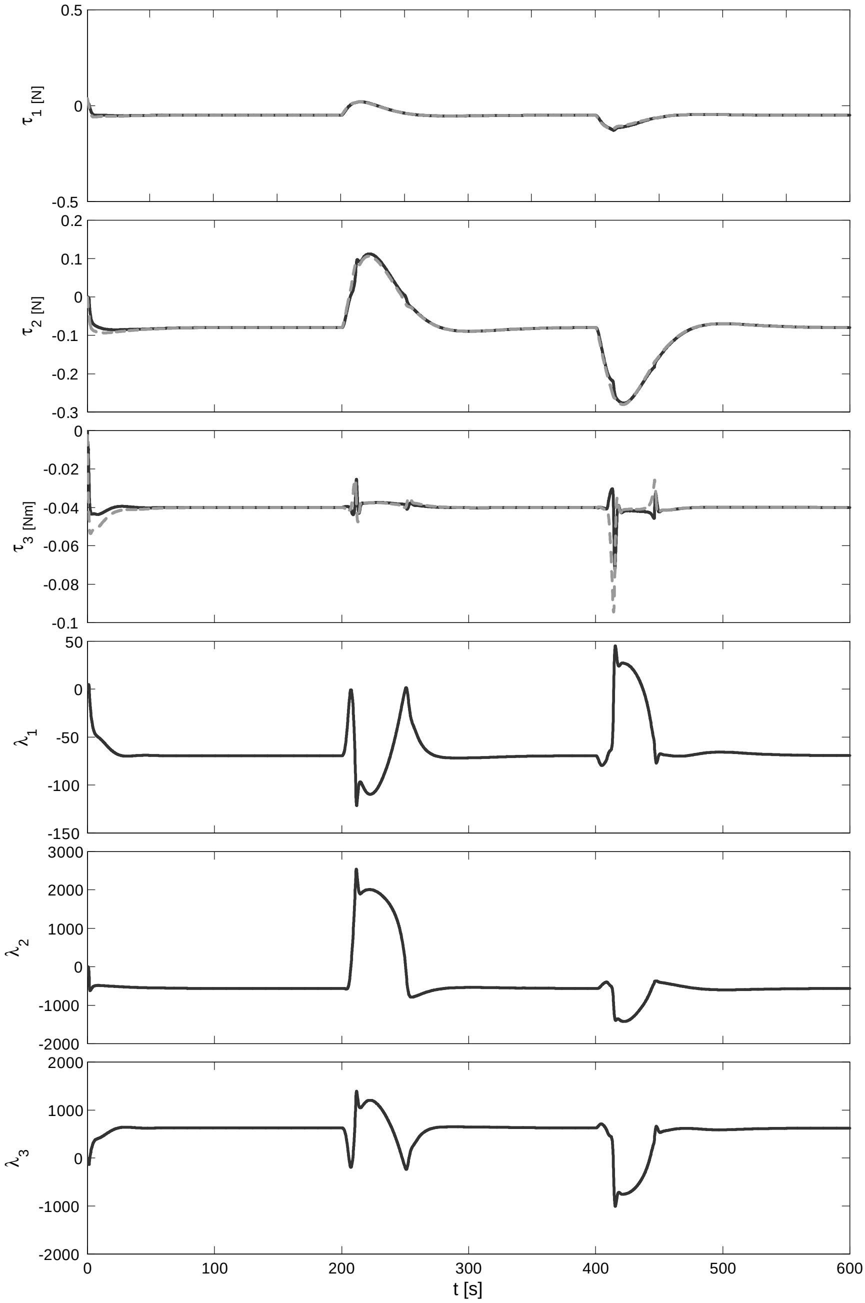

Figure 4.3: The desired virtual control forces (dashed) and the actual control forces (solid) generated by the actuators. The lagrangian parameters are also shown.

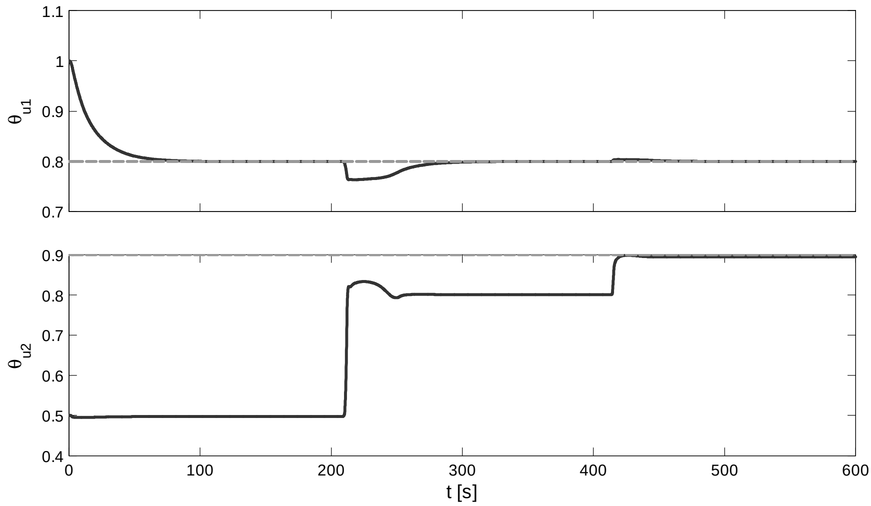

Figure 4.4: The loss parameters and estimates.

Chapter 5

Control Allocation with Actuator Dynamics

When actuator dynamics is considered, the task of the dynamic control allocation algorithm is to connect the high and low level controls by taking the desired virtual control τ_c as an input and computing the desired actuator reference u_d as an output. Based on the minimization problem (2.5) the Lagrangian function from (5.1)

$$L_{\hat{\theta}u}(t,x,u_d,\lambda) := J(t,x,u_d) + (k(t,x) - \Phi(t,x,u_d,\hat{\theta}))^{\mathrm{T}}\lambda, \tag{5.1}$$

is introduced. The idea is then to define update laws for the actuator reference u_d and the Lagrangian parameter λ, based on a Lyapunov approach, such that u_d and λ converges to a set defined by the first-order optimal condition for $L_{\hat{\theta}u}$.

Since the parameter vector θ from the actuator and the actuator-force mapping models (2.3) and (2.2) are unknown, an adaptive update law for $\hat{\theta}$ is defined. The parameter estimates are used in the Lagrangian function (5.1) and a certainty equivalent adaptive optimal control allocation can be defined. The following observers, similar to the one given in Chapter 4, are used in order to support estimation of the parameters:

$$\begin{aligned}\dot{\hat{u}} =& A_x(u-\hat{u}) + f_{u0}(t,x,u,u_{cmd}) + f_{u\theta}(t,x,u,u_{cmd})\theta_u,\\ \dot{\hat{x}} =& A_x(x-\hat{x}) + f(t,x) + g(t,x)\Phi(t,x,u,\hat{\theta}).\end{aligned}$$

where $(-A_u)$ and $(-A_x)$ are Hurwitz matrices. For analytical purpose, the error dynamics are given by:

$$\dot{\eta}_u = -A_u\eta_u + \bar{f}_{u\theta}(t,x,u_d,\tilde{u},\hat{\theta})\tilde{\theta}_u, \tag{5.2}$$
$$\dot{\eta}_x = -A_x\eta_x + \Phi_{\theta_\tau}(t,x,u)\tilde{\theta}_\tau + \Phi_{\theta_u}(t,x,u)\tilde{\theta}_u. \tag{5.3}$$

5.1 Stability of the control allocation and actuator dynamics

Based on equation (2.4) the considered high level system can be represented by:

$$\begin{aligned}\dot{x} \;=\; & f(t,x) + g(t,x)k(t,x)\\ & +g(t,x)\left(\Phi(t,x,u_d,\hat{\theta}) - k(t,x)\right)\\ & +g(t,x)\left(\Phi(t,x,u,\hat{\theta}) - \Phi(t,x,u_d,\hat{\theta})\right)\\ & +g(t,x)\Phi_\theta(t,x,u,\theta)\tilde{\theta}.\end{aligned} \tag{5.4}$$

In the following, the system (5.4) is, with reference to Figure 2.2, the perturbed system denoted Σ_1, and the idea is to construct update laws for u_d, λ and $\hat{\theta}$ such that the last three parts of the equation converges to zero in a stabile sense. As from the presentations in Chapter 3 and 4 the idea is then: i) by defining a Lyapunov like function $V_{u_d\lambda\tilde{u}\eta\tilde{\theta}}$, show that there exist appropriate update-laws for u_d, λ and $\hat{\theta}$, the perturbing part, such that $\dot{V}_{u_d\lambda\tilde{u}\eta\tilde{\theta}} \leq 0$. And ii) use the cascade Lemma 1.1 to prove convergence and boundedness of the closed loop system Σ_1 and Σ_2, where Σ_2 is defined by the allocation update laws, the parameter estimate errors and the low level dynamics.

For Σ_2 the stability and convergence arguments are related to the set:

$$\mathcal{O}_{u_d\lambda\tilde{\theta}}(t,x(t)) := \left\{ z_{u_d\lambda\tilde{\theta}} \in \mathbb{R}^{3r+d+n+m} \,\middle|\, f_{\mathcal{O}_{u\lambda\tilde{\theta}}}(t,x,z_{u_d\lambda\tilde{\theta}}) = 0 \right\}, \tag{5.5}$$

where $z_{u_d\lambda\tilde{\theta}} := \left(u_d^{\mathrm{T}}, \lambda^{\mathrm{T}}, \tilde{u}^{\mathrm{T}}, \eta_u^{\mathrm{T}}, \eta_x^{\mathrm{T}}, \tilde{\theta}^{\mathrm{T}}\right)^{\mathrm{T}}$ and $f_{\mathcal{O}_{u\lambda\tilde{\theta}}}(t,x,z_{u_d\lambda\tilde{\theta}}) := \left(\left(\frac{\partial L_{\hat{\theta}u}}{\partial u}\right)^{\mathrm{T}}, \left(\frac{\partial L_{\hat{\theta}u}}{\partial \lambda}\right)^{\mathrm{T}}, \tilde{u}^{\mathrm{T}}, \eta_u^{\mathrm{T}}, \eta_x^{\mathrm{T}}, \tilde{\theta}^{\mathrm{T}}\right)$. Consider the Lyapunov like function

$$V_{u_d\lambda\tilde{u}\eta\tilde{\theta}}(t,x,u_d,\lambda,\tilde{u},\eta_u,\eta_x) := V_{\tilde{u}}(t,\tilde{u}) + \frac{1}{2}\eta_u^{\mathrm{T}}\Gamma_u\eta_u + \frac{1}{2}\eta_x^{\mathrm{T}}\Gamma_x\eta_x + \frac{1}{2}\left(\frac{\partial L_{\hat{\theta}u}^{\mathrm{T}}}{\partial u_d}\frac{\partial L_{\hat{\theta}u}}{\partial u_d} + \frac{\partial L_{\hat{\theta}u}^{\mathrm{T}}}{\partial \lambda}\frac{\partial L_{\hat{\theta}u}}{\partial \lambda}\right) + \frac{1}{2}\tilde{\theta}_u^{\mathrm{T}}\Gamma_{\theta_u}\tilde{\theta}_u + \frac{1}{2}\tilde{\theta}_\tau^{\mathrm{T}}\Gamma_{\theta_\tau}\tilde{\theta}_\tau \tag{5.6}$$

The time derivative of $V_{u_d\lambda\tilde{u}\eta\tilde{\theta}}$ along the trajectories of (5.2), (5.3) and (5.4) is:

$$\begin{aligned}
\dot{V}_{u_d\lambda\tilde{u}\eta\tilde{\theta}} &= \dot{V}_{\tilde{u}} - \frac{\partial V_{\tilde{u}}}{\partial \tilde{u}} f_{u\theta}(t,x,u,u_d,u_{cmd})\tilde{\theta}_u - \eta_u^{\mathrm{T}}\frac{1}{2}\left(A_u^{\mathrm{T}}\Gamma_u + \Gamma_u A_u\right)\eta_u \\
&+ \eta_u^{\mathrm{T}}\Gamma_u f_{u\theta}(t,x,u,u_d,u_{cmd})\tilde{\theta}_u + \eta_x^{\mathrm{T}}\Gamma_x\Phi_{\theta_\tau}(t,x,u)\tilde{\theta}_\tau \\
&+ \eta_x^{\mathrm{T}}\Gamma_x\Phi_{\theta_u}(t,x,u)\tilde{\theta}_u + \tilde{\theta}_\tau^{\mathrm{T}}\Gamma_{\tilde{\theta}_\tau}\dot{\tilde{\theta}}_\tau + \tilde{\theta}_u^{\mathrm{T}}\Gamma_{\tilde{\theta}_u}\dot{\tilde{\theta}}_u - \eta_x^{\mathrm{T}}\frac{1}{2}\left(A_x^{\mathrm{T}}\Gamma_x + \Gamma_x A_x\right)\eta_x \\
&+ \left(\frac{\partial L_{\hat{\theta}u}^{\mathrm{T}}}{\partial u_d}\frac{\partial^2 L_{\hat{\theta}u}}{\partial u_d^2} + \frac{\partial L_{\hat{\theta}u}^{\mathrm{T}}}{\partial \lambda}\frac{\partial^2 L_{\hat{\theta}u}}{\partial u_d\partial\lambda}\right)\dot{u}_d + \frac{\partial L_{\hat{\theta}u}^{\mathrm{T}}}{\partial u_d}\frac{\partial^2 L_{\hat{\theta}u}}{\partial\lambda\partial u_d}\dot{\lambda} \\
&+ \frac{\partial L_{\hat{\theta}u}^{\mathrm{T}}}{\partial u_d}\frac{\partial^2 L_{\hat{\theta}u}}{\partial x\partial u_d}\left(f(t,x) + g(t,x)\Phi(t,x,u,\hat{\theta})\right) \\
&+ \frac{\partial L_{\hat{\theta}u}^{\mathrm{T}}}{\partial \lambda}\frac{\partial^2 L_{\hat{\theta}u}}{\partial x\partial\lambda}\left(f(t,x) + g(t,x)\Phi(t,x,u,\hat{\theta})\right) \\
&+ \frac{\partial L_{\hat{\theta}u}^{\mathrm{T}}}{\partial u_d}\frac{\partial^2 L_{\hat{\theta}u}}{\partial x\partial u_d}g(t,x)\left(\Phi_{\theta_\tau}(t,x,u)\tilde{\theta}_\tau + \Phi_{\theta_u}(t,x,u)\tilde{\theta}_u\right) \\
&+ \frac{\partial L_{\hat{\theta}u}^{\mathrm{T}}}{\partial \lambda}\frac{\partial^2 L_{\hat{\theta}u}}{\partial x\partial\lambda}g(t,x)\left(\Phi_{\theta_\tau}(t,x,u)\tilde{\theta}_\tau + \Phi_{\theta_u}(t,x,u)\tilde{\theta}_u\right) \\
&- \left(\frac{\partial L_{\hat{\theta}u}^{\mathrm{T}}}{\partial u_d}\frac{\partial^2 L_{\hat{\theta}u}}{\partial\hat{\theta}\partial u_d} + \frac{\partial L_{\hat{\theta}u}^{\mathrm{T}}}{\partial \lambda}\frac{\partial^2 L_{\hat{\theta}u}}{\partial\hat{\theta}\partial\lambda}\right)\dot{\hat{\theta}} + \frac{\partial L_{\hat{\theta}u}^{\mathrm{T}}}{\partial u_d}\frac{\partial^2 L_{\hat{\theta}u}}{\partial t\partial u_d} + \frac{\partial L_{\hat{\theta}u}^{\mathrm{T}}}{\partial \lambda}\frac{\partial^2 L_{\hat{\theta}u}}{\partial t\partial\lambda},
\end{aligned} \tag{5.7}$$

Equation (5.7) gives rise to the following allocation algorithm:

$$\begin{pmatrix} \dot{u}_d \\ \dot{\lambda} \end{pmatrix} = -\Gamma\mathbb{H}_{\hat{\theta}u}\begin{pmatrix} \frac{\partial L_{\hat{\theta}u}}{\partial u_d} \\ \frac{\partial L_{\hat{\theta}u}}{\partial \lambda} \end{pmatrix} - u_{\hat{\theta}uff}, \tag{5.8}$$

where $\mathbb{H}_{\hat{\theta}u} := \begin{pmatrix} \frac{\partial^2 L_{\hat{\theta}u}}{\partial u_d^2} & \frac{\partial^2 L_{\hat{\theta}u}}{\partial\lambda\partial u_d} \\ \frac{\partial^2 L_{\hat{\theta}u}}{\partial u_d\partial\lambda} & 0 \end{pmatrix}$, the matrices Γ, Γ_{θ_u}, Γ_{θ_τ}, Γ_x, and Γ_u are symmetric positive definite weighting matrices and u_{ff} is a feed-forward like term:

$$u_{\hat{\theta}uff} := \mathbb{H}_{\hat{\theta}u}^{-1}\begin{pmatrix}\frac{\partial^2 L_{\hat{\theta}u}}{\partial t \partial u_d} \\ \frac{\partial^2 L_{\hat{\theta}u}}{\partial t \partial \lambda}\end{pmatrix} + \mathbb{H}_{\hat{\theta}u}^{-1}\begin{pmatrix}\frac{\partial^2 L_{\hat{\theta}u}}{\partial x \partial u_d} \\ \frac{\partial^2 L_{\hat{\theta}u}}{\partial x \partial \lambda}\end{pmatrix} f(t,x)$$
$$+ \mathbb{H}_{\hat{\theta}u}^{-1}\begin{pmatrix}\frac{\partial^2 L_{\hat{\theta}u}}{\partial x \partial u_d} \\ \frac{\partial^2 L_{\hat{\theta}u}}{\partial x \partial \lambda}\end{pmatrix} g(t,x)\Phi(t,x,u,\hat{\theta}) + \mathbb{H}_{\hat{\theta}u}^{-1}\begin{pmatrix}\frac{\partial^2 L_{\hat{\theta}u}}{\partial \hat{\theta} \partial u_d} \\ \frac{\partial^2 L_{\hat{\theta}u}}{\partial \hat{\theta} \partial \lambda}\end{pmatrix}\dot{\hat{\theta}},$$

if $\det(\frac{\partial^2 L_{\hat{\theta}u}}{\partial u_d^2}) \geq \epsilon$ and $u_{\hat{\theta}_u ff} := 0$ if $\det(\frac{\partial^2 L_{\hat{\theta}u}}{\partial u_d^2}) < \epsilon$, where $(k_1)^r > \epsilon > 0$. Γ_u and Γ_x are uniquely defined by the Lyapunov equations; $Q_u :=: \frac{1}{2}\left(A_u^{\mathrm{T}}\Gamma_u + \Gamma_u A_u\right)$ and $Q_x :=: \frac{1}{2}\left(A_x^{\mathrm{T}}\Gamma_x + \Gamma_x A_x\right)$, where Q_u and Q_x are positive definite design matrices. By inserting the trajectories from (5.8) in (5.7), $\dot{V}_{u_d\lambda\tilde{u}\eta\tilde{\theta}}$ is given by:

$$\begin{aligned}
\dot{V}_{u_d\lambda\tilde{u}\eta\tilde{\theta}} &= -\eta_u^{\mathrm{T}} Q_u \eta_u - \alpha_{\tilde{u}3}(|\tilde{u}|) - \eta_x^{\mathrm{T}} Q_x \eta_x \\
&- \left(\frac{\partial L_{\hat{\theta}u}}{\partial u_d}^{\mathrm{T}}, \frac{\partial L_{\hat{\theta}u}}{\partial \lambda}^{\mathrm{T}}\right)\mathbb{H}_{\hat{\theta}u}\Gamma\mathbb{H}_{\hat{\theta}u}\left(\frac{\partial L_{\hat{\theta}u}}{\partial u_d}^{\mathrm{T}}, \frac{\partial L_{\hat{\theta}u}}{\partial \lambda}^{\mathrm{T}}\right)^{\mathrm{T}} + \dot{\tilde{\theta}}_u^{\mathrm{T}}\Gamma_{\tilde{\theta}_u}\tilde{\theta}_u \\
&+ \left(\frac{\partial V_{\tilde{u}}}{\partial \tilde{u}} + \eta_u^{\mathrm{T}}\Gamma_\eta\right) f_{u\theta}(t,x,u,u_d,u_{cmd})\tilde{\theta}_u + \dot{\tilde{\theta}}_\tau^{\mathrm{T}}\Gamma_{\tilde{\theta}_\tau}\tilde{\theta}_\tau \\
&+ \left(\eta_x^{\mathrm{T}}\Gamma_x + \frac{\partial L_{\hat{\theta}u}^{\mathrm{T}}}{\partial u_d}\frac{\partial^2 L_{\hat{\theta}u}}{\partial x \partial u_d} + \frac{\partial L_{\hat{\theta}u}^{\mathrm{T}}}{\partial \lambda}\frac{\partial^2 L_{\hat{\theta}u}}{\partial x \partial \lambda}\right) g(t,x)\Phi_{\theta_u}(t,x,u)\tilde{\theta}_u \\
&+ \left(\eta_x^{\mathrm{T}}\Gamma_x + \frac{\partial L_{\hat{\theta}u}^{\mathrm{T}}}{\partial u_d}\frac{\partial^2 L_{\hat{\theta}u}}{\partial x \partial u_d} + \frac{\partial L_{\hat{\theta}u}^{\mathrm{T}}}{\partial \lambda}\frac{\partial^2 L_{\hat{\theta}u}}{\partial x \partial \lambda}\right) g(t,x)\Phi_{\theta_\tau}(t,x,u)\tilde{\theta}_\tau,
\end{aligned} \tag{5.9}$$

which motivate the following adaptive laws:

$$\begin{aligned}
\dot{\hat{\theta}}_u^{\mathrm{T}} &= \left(\frac{\partial V_{\tilde{u}}}{\partial \tilde{u}} + \eta_u^{\mathrm{T}}\Gamma_u\right) f_{u\theta}(t,x,u,u_d,u_{cmd})\Gamma_{\theta_u}^{-1} \\
&+ \left(\eta_x^{\mathrm{T}}\Gamma_x + \frac{\partial L_{\hat{\theta}u}^{\mathrm{T}}}{\partial u_d}\frac{\partial^2 L_{\hat{\theta}u}}{\partial x \partial u_d} + \frac{\partial L_{\hat{\theta}u}^{\mathrm{T}}}{\partial \lambda}\frac{\partial^2 L_{\hat{\theta}u}}{\partial x \partial \lambda}\right) g(t,x)\Phi_{\theta_u}(t,x,u)\Gamma_{\theta_u}^{-1}
\end{aligned} \tag{5.10}$$

and

$$\dot{\hat{\theta}}_\tau^{\mathrm{T}} = \left(\eta_x^{\mathrm{T}}\Gamma_x + \frac{\partial L_{\hat{\theta}u}^{\mathrm{T}}}{\partial u_d}\frac{\partial^2 L_{\hat{\theta}u}}{\partial x \partial u_d} + \frac{\partial L_{\hat{\theta}u}^{\mathrm{T}}}{\partial \lambda}\frac{\partial^2 L_{\hat{\theta}u}}{\partial x \partial \lambda}\right) g(t,x)\Phi_{\theta_\tau}(t,x,u)\Gamma_{\theta_\tau}^{-1}, \tag{5.11}$$

such that

$$\begin{aligned}\dot{V}_{u_d\lambda\tilde{u}\eta\tilde{\theta}} &= -\eta_u^{\mathrm{T}} Q_u \eta_u - \alpha_{\tilde{u}3}(|\tilde{u}|) - \eta_x^{\mathrm{T}} Q_x \eta_x \\ &\quad - \left(\frac{\partial L_{\hat{\theta}u}}{\partial u_d}^{\mathrm{T}}, \frac{\partial L_{\hat{\theta}u}}{\partial \lambda}^{\mathrm{T}}\right) \mathbb{H}_{\hat{\theta}u} \Gamma \mathbb{H}_{\hat{\theta}u} \left(\frac{\partial L_{\hat{\theta}u}}{\partial u_d}^{\mathrm{T}}, \frac{\partial L_{\hat{\theta}u}}{\partial \lambda}^{\mathrm{T}}\right)^{\mathrm{T}} \\ &\leq -\eta_u^{\mathrm{T}} Q_u \eta_u - \alpha_{\tilde{u}3}(|\tilde{u}|) - \eta_x^{\mathrm{T}} Q_x \eta_x \\ &\quad - c_{\hat{\theta}} \left(\frac{\partial L_{\hat{\theta}u}}{\partial u_d}^{\mathrm{T}} \frac{\partial L_{\hat{\theta}u}}{\partial u_d}^{\mathrm{T}} + \frac{\partial L_{\hat{\theta}u}}{\partial \lambda}^{\mathrm{T}} \frac{\partial L_{\hat{\theta}u}}{\partial \lambda}^{\mathrm{T}}\right), \end{aligned} \tag{5.12}$$

where $c_{\hat{\theta}} := \inf_t \lambda_{\min}\left(\mathbb{H}_{\hat{\theta}u}\Gamma\mathbb{H}_{\hat{\theta}u}\right) > 0$.

Proposition 5.1 *If the assumptions 2.1, 2.2 and 2.3 are satisfied, and $x(t)$ exists for all t, then the set $\mathcal{O}_{u_d\lambda\tilde{\theta}}(t, x(t))$ is UGS with respect to system (5.2), (5.3), (5.4), (5.8), (5.10) and (5.11).*
Furthermore $\left(\frac{\partial L_{\hat{\theta}u}}{\partial u_d}, \frac{\partial L_{\hat{\theta}u}}{\partial \lambda}, \eta_u, \eta_x, \tilde{u}\right)$ will converge to zero as $t \to \infty$. If in addition $f_p(t) := f_{u\theta}(t, x(t), u(t), u_d(t), u_{cmd}(t))$ and
$\Phi_g(t) := g(t, x(t))\Phi_{\theta\tau}(t, x(t), u(t))$ are PE, then the set $\mathcal{O}_{u_d\lambda\tilde{\theta}}(t, x(t))$ is UGAS with respect to the system (5.2), (5.3), (5.4), (5.8), (5.10) and (5.11).
Proof. *Existence of solutions, forward invariance of the set $\mathcal{O}_{u_d\lambda\tilde{\theta}}$ and finite escape time detectability $\left|z_{u_d\lambda\tilde{\theta}}\right|_{\mathcal{O}_{u_d\lambda\tilde{\theta}}}$ follows by the same arguments as given in Proposition 3.1. Furthermore it follows that there exist class $\mathcal{K}_\infty$ functions $\bar{\alpha}_1$ and $\bar{\alpha}_2$ such that $\bar{\alpha}_1(\left|z_{u_d\lambda\tilde{\theta}}\right|_{\mathcal{O}_{u_d\lambda\tilde{\theta}}}) \leq \bar{V}_{u_d\lambda\tilde{u}\eta\tilde{\theta}}(t, z_{u_d\lambda\tilde{\theta}}) \leq \bar{\alpha}_2(\left|z_{u_d\lambda\tilde{\theta}}\right|_{\mathcal{O}_{u_d\lambda\tilde{\theta}}})$ where,*

$$\bar{\alpha}_1(\left|z_{u_d\lambda\tilde{\theta}}\right|_{\mathcal{O}_{u_d\lambda\tilde{\theta}}}) := \min\left(\frac{\kappa_1}{2}\left|z_{u_d\lambda\tilde{\theta}}\right|^2_{\mathcal{O}_{u_d\lambda\tilde{\theta}}}, \frac{1}{2}\alpha_{\tilde{u}1}(\left|z_{u_d\lambda\tilde{\theta}}\right|_{\mathcal{O}_{u_d\lambda\tilde{\theta}}})\right), \tag{5.13}$$

$$\bar{\alpha}_2(\left|z_{u_d\lambda\tilde{\theta}}\right|_{\mathcal{O}_{u_d\lambda\tilde{\theta}}}) := k_2\left|z_{u_d\lambda\tilde{\theta}}\right|^2_{\mathcal{O}_{u_d\lambda\tilde{\theta}}} + \alpha_{\tilde{u}2}(\left|z_{u_d\lambda\tilde{\theta}}\right|_{\mathcal{O}_{u_d\lambda\tilde{\theta}}}), \tag{5.14}$$

and $\kappa_1 := \frac{1}{2}\min(\lambda_{\min}\Gamma_u, \lambda_{\min}\Gamma_x, \lambda_{\min}\Gamma_{\theta_u}, \lambda_{\min}\Gamma_{\theta_\tau}, \bar{\kappa}_1)$,
$\kappa_2 := \max(\lambda_{\max}\Gamma_u, \lambda_{\max}\Gamma_x, \lambda_{\max}\Gamma_{\theta_u}, \lambda_{\max}\Gamma_{\theta_\tau}, \bar{\kappa}_2)$. $\kappa_2 \geq \kappa_1 > 0$ since there exist positive constants $\bar{\kappa}_2 \geq \bar{\kappa}_1$ such that
$\bar{\kappa}_1\left|\left(u^T, \lambda^T\right)^T\right|^2_{\mathcal{O}_{u\lambda}} \leq \left(\left(\frac{\partial L_{\hat{\theta}u}}{\partial u}\right)^T \frac{\partial L_{\hat{\theta}u}}{\partial u} + \left(\frac{\partial L_{\hat{\theta}u}}{\partial \lambda}\right)^T \frac{\partial L_{\hat{\theta}u}}{\partial \lambda}\right) \leq \bar{\kappa}_2\left|\left(u^T, \lambda^T\right)^T\right|^2_{\mathcal{O}_{u\lambda}}$

from the equation:

$$\left(\frac{\partial L_{\hat{\theta}u}}{\partial u}\right)^T \frac{\partial L_{\hat{\theta}u}}{\partial u} + \left(\frac{\partial L_{\hat{\theta}u}}{\partial \lambda}\right)^T \frac{\partial L_{\hat{\theta}u}}{\partial \lambda} = \begin{pmatrix} u - u^* \\ \lambda - \lambda^* \end{pmatrix}^T \mathbb{H}^T_{\hat{\theta}u*} \mathbb{H}_{\hat{\theta}u*} \begin{pmatrix} u - u^* \\ \lambda - \lambda^* \end{pmatrix}, \quad (5.15)$$

where $\mathbb{H}^T_{\hat{\theta}u*}\mathbb{H}_{\hat{\theta}u*}$ *can be shown to be a bounded positive definite matrix by following the same steps as in the proof of Proposition 3.1.* $\bar{\alpha}_1$ *is of class* $\mathcal{K}_\infty$ *since in general: Let* $s_1 \geq s_2 \geq 0$, *then* $s_1^2 + \alpha_{\tilde{u}1}(s_2) \geq s_1^2 \geq \left(\frac{s_1+s_2}{2}\right)^2 \geq \min\left(\left(\frac{s_1+s_2}{2}\right)^2, \alpha_{\tilde{u}1}\left(\frac{s_1+s_2}{2}\right)\right)$ *and the same argument can be made for* $s_2 > s_1$. $\bar{\alpha}_1$ *is obviously a* $\mathcal{K}_\infty$ *since it is a sum of two* $\mathcal{K}_\infty$ *functions.*

Moreover the arguments from the proof of Proposition 4.1 applies to the case UGS and convergence of $\left(\frac{\partial L_{\hat{\theta}u}}{\partial u_d}, \frac{\partial L_{\hat{\theta}u}}{\partial \lambda}, \eta_u, \eta_x, \tilde{u}\right)$. *The UGAS proof follows by the same steps as given in the proof of Proposition 4.2* ■

Proposition 5.1 implies that the time-varying first order optimal set $\mathcal{O}_{u_d\lambda\tilde{\theta}}(t)$ is uniformly stable, and in addition uniformly attractive if a PE assumption is satisfied.

5.2 Stability of the closed loop control allocation

In the following the solutions of the closed loop certainty equivalent control allocation procedure are defined with respect to the set:

$$\mathcal{O}_{xu_d\lambda\tilde{\theta}} := \mathcal{O}_x(t) \times \mathcal{O}_{u_d\lambda\tilde{\theta}}(t, 0).$$

Formally the theoretical achievements are given in the following propositions.

Proposition 5.2 *If Assumptions 2.1-2.3 are satisfied, then the set* $\mathcal{O}_{xu_d\lambda\tilde{\theta}}$ *is UGS, with respect to system (5.2), (5.3), (5.4), (5.8), (5.10) and (5.11). If in addition* $f_p(t) := f_{u\theta}(t, x(t), u(t), u_{cmd}(t))$ *and* $\Phi_g(t) := g(t, x(t))\Phi_{\theta\tau}(t, x(t), u(t))$ *are PE, then* $\mathcal{O}_{xu_d\lambda\tilde{\theta}}$ *is UGAS with respect to (5.2), (5.3), (5.4), (5.8), (5.10), (5.11).*

Proof. *As before the main part of this proof is to prove boundedness and completeness and invoke Lemma 1.1. Let* $\left|z_{xu\lambda\tilde{\theta}0}\right|_{\mathcal{O}_{xu\lambda\tilde{\theta}}} \leq r$, *where* $r > 0$, *and assume that* $|z_x(t)|_{\mathcal{O}_x}$ *escapes to infinity at* T. *Then for any constant* $M(r)$ *there exists a* $t \in [t_0, T)$ *such that* $M(r) \leq |z_x(t)|_{\mathcal{O}_x}$. *In what follows we show that* $M(r)$ *can not be chosen arbitrarily. Define* $v(t, z_x) := V_x(t, x)$

such that

$$\begin{aligned}
\dot{v} &\leq -\alpha_{x3}(|z_x|_{\mathcal{O}_x}) + \frac{\partial V_x}{\partial x} g(t,x) \left(\Phi(t,x,u_d,\hat{\theta}) - k(t,x) \right) \\
&\quad + \frac{\partial V_x}{\partial x} g(t,x) \left(\Phi(t,x,u,\hat{\theta}) - \Phi(t,x,u_d,\hat{\theta}) \right) \\
&\quad + \frac{\partial V_x}{\partial x} g(t,x) \Phi_\theta(t,x,u)\tilde{\theta} \qquad (5.16)\\
&\leq -\alpha_{x3}(|z_x|_{\mathcal{O}_x}) + \left| \frac{\partial V_x}{\partial x} \right| |g(t,x)| \left| \frac{\partial L_{\hat{\theta}u}}{\partial \lambda}(t,x,u_d,\hat{\theta}) \right| \\
&\quad + \left| \frac{\partial V_x}{\partial x} \right| |g(t,x)| \left| \left(\Phi(t,x,u,\hat{\theta}) - \Phi(t,x,u_d,\hat{\theta}) \right) \right| |\tilde{u}| \qquad (5.17)\\
&\quad + \left| \frac{\partial V_x}{\partial x} \right| |g(t,x)| \left| \Phi_\theta(t,x,u)\tilde{\theta} \right| \\
&\leq -\alpha_{x3}(|z_x|_{\mathcal{O}_x}) \\
&\quad + \alpha_{x4}(|z_x|_{\mathcal{O}_x}) K \left(\varrho_2 + \kappa_2 + |\Phi_\theta(t,x,u)| \right) \left| z_{u_d\lambda\tilde{\theta}} \right|_{\mathcal{O}_{u_d\lambda\tilde{\theta}}}. \qquad (5.18)
\end{aligned}$$

There exists a positive constant $B(r) \geq 0$, for all $t \in [t_0, T)$, such that for $\left|z_{u\lambda\tilde{\theta}0}\right|_{\mathcal{O}_{u\lambda\tilde{\theta}}} \leq r$, $\left|z_{u\lambda\tilde{\theta}}(t)\right|_{\mathcal{O}_{u\lambda\tilde{\theta}}} \leq B(\left|z_{u\lambda\tilde{\theta}0}\right|_{\mathcal{O}_{u\lambda\tilde{\theta}}}) \leq B(r)$ (similar argument as given in Proposition 3.2 and 4.3). From Assumption 2.1 and 2.2,

$$\begin{aligned}
\dot{v} &\leq -\alpha_{x3}(|z_x|_{\mathcal{O}_x}) \\
&\quad + \alpha_{x4}(|z_x|_{\mathcal{O}_x}) K \left(\varrho_2 + \kappa_2 + |\Phi_\theta(t,x,u)| \right) B(r) \\
&\leq -\alpha_k(|z_x|_{\mathcal{O}_x}) \alpha_{x4}(|z_x|_{\mathcal{O}_x}) \varsigma_{x\max}((|z_x|_{\mathcal{O}_x}) \\
&\quad + \alpha_{x4}(|z_x|_{\mathcal{O}_x}) K \left(\varrho_2 + \kappa_2 + |\Phi_\theta(t,x,u)| \right) B(r) \\
&\leq -\alpha_{x4}(|z_x|_{\mathcal{O}_x}) \alpha_k(|z_x|_{\mathcal{O}_x}) \varsigma_{x\max}(|z_x|_{\mathcal{O}_x}) \\
&\quad + \alpha_{x4}(|z_x|_{\mathcal{O}_x}) K \left(\varrho_2 + \kappa_2 \right) B(r) \\
&\quad + \alpha_{x4}(|z_x|_{\mathcal{O}_x}) K \varsigma_x(|z_x|_{\mathcal{O}_x}) \varsigma_{xu}(|z_x|_{\mathcal{O}_x}) B(r) \\
&\quad + \alpha_{x4}(|z_x|_{\mathcal{O}_x}) K \varsigma_x(|z_x|_{\mathcal{O}_x}) \varsigma_u(B(r)) B(r) \qquad (5.19)\\
&\leq -\alpha_{x4}(|z_x|_{\mathcal{O}_x}) \alpha_k(\alpha_{x2}^{-1}(v)) \varsigma_{x\max}(|z_x|_{\mathcal{O}_x}) \\
&\quad + \alpha_{x4}(|z_x|_{\mathcal{O}_x}) K \left(\varrho_2 + \kappa_2 \right) B(r) \\
&\quad + \alpha_{x4}(|z_x|_{\mathcal{O}_x}) K \varsigma_x(|z_x|_{\mathcal{O}_x}) \varsigma_{xu}(|z_x|_{\mathcal{O}_x}) B(r) \\
&\quad + \alpha_{x4}(|z_x|_{\mathcal{O}_x}) K \varsigma_x(|z_x|_{\mathcal{O}_x}) \varsigma_u(B(r)) B(r). \qquad (5.20)
\end{aligned}$$

Thus, if

$$|z_{x0}|_{\mathcal{O}_x} > \alpha_k^{-1}(K \left(\varrho_2 + \kappa_2 + 1 + \varsigma_u(B(r)) \right) B(r))$$

then from (5.19),
$v(t_0, z_{x0}) \geq v(t, z_x(t))$ *and* $|z_x(t)|_{\mathcal{O}_x} \leq \alpha_{x1}^{-1}(v(z_{x0})) \leq \alpha_{x1}^{-1}(\alpha_{x2}(r))$ *, else,*

$$|z_{x0}|_{\mathcal{O}_x} \leq \alpha_k^{-1}(K(\varrho_2 + \kappa_2 + 1 + \varsigma_u(B(r)))\, B(r))$$

and from (5.20),
$v(t, z_x(t)) \leq \alpha_{x2}(\alpha_k^{-1}(K(\varrho_2 + \kappa_2 + 1 + \varsigma_u(B(r)))\, B(r)))$ *such that*

$$|z_x(t)|_{\mathcal{O}_x} \leq \alpha_{x1}^{-1}\Big(\alpha_{x2}\big(\alpha_k^{-1}(K(\varrho_2 + \kappa_2 + 1 + \varsigma_u(B(r)))\, B(r))\big)\Big).$$

By choosing

$$M(r) := \max\left(\alpha_{x1}^{-1}(\alpha_{x2}(r))\, \alpha_{x1}^{-1}\big(\alpha_{x2}\big(\alpha_k^{-1}(K(\varrho_2 + \kappa_2 + 1 + \varsigma_u(B(r)))\, B(r))\big)\big)\right),$$

the assumption of $|z_x(t)|_{\mathcal{O}_x}$ *escaping to infinity is contradicted, since* $M(r) > |z_x(t)|_{\mathcal{O}_x}$ *and* $|\cdot|_{\mathcal{O}_x}$ *is finite escape time detectable. Furthermore,* $\mathcal{O}_x$ *is UGB. From Propositions 4.1 and 4.2 and the assumptions of these propositions, the assumptions of Lemma 1.1 and Corollary 1.1 are satisfied and the result is proved.* ∎

Proposition 5.2 implies that the time-varying optimal set $\mathcal{O}_{xu_d\lambda\tilde{\theta}}(t)$ is uniformly stable, and in addition uniformly attractive if the signals $f_p(t)$ and $\Phi_g(t)$ are PE. Thus optimal control allocation is achieved asymptotically for the closed loop. A local version of this result can be proved by using Corollary 1.1.

Corollary 5.1 *If for* $\mathbb{U} \subset \mathbb{R}^r$ *there exists a constant* $c_x > 0$ *such that for* $|x| \leq c_x$ *the domain* $\mathbb{U}_z \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times \mathbb{U} \times \mathbb{R}^{d+n+m}$ *contain* $\mathcal{O}_{xu_d\lambda\tilde{\theta}}$*, then if the Assumptions 2.1-2.3 are satisfied, the set* $\mathcal{O}_{xu_d\lambda\tilde{\theta}}$ *is US with respect to the system (5.2), (5.3), (5.4), (5.8), (5.10) and (5.11). If in addition Assumption 4.1 is satisfied,* $\mathcal{O}_{xu_d\lambda\tilde{\theta}}$ *is UAS with respect to the system (5.2), (5.3), (5.4), (5.8), (5.10) and (5.11).*
Proof. *The result follows by Corollary 1.1 and arguments similar to the ones given in Corollary 3.2.* ∎

Remark 5.1 *The structure of the perturbed system from equation (2.4), can also be written in the form:*

$$\begin{aligned} \dot{x} &= f(t,x) + g(t,x)k(t,x) \\ &\quad + g(t,x)\left(\Phi(t,x,u_d+\tilde{u},\hat{\theta}) - k(t,x)\right) \\ &\quad + g(t,x)\Phi_\theta(t,x,u,\theta)\tilde{\theta}. \end{aligned} \tag{5.21}$$

This formulation motivates an adaptive control allocation algorithm that provides desired actuator control references, based on the actual given generalized forces. The minimization problem can be expressed as:

$$\min_{u_d} J(t,x,u_d) \quad s.t \quad \tau_c - \Phi(t,x,u_d+\tilde{u},\hat{\theta}) = 0, \tag{5.22}$$

and the Lagrangian function becomes:

$$L_{\hat{\theta}\tilde{u}}(t,x,u_d,\tilde{u},\lambda,\hat{\theta}) := J(t,x,u_d) + (\tau_c - \Phi(t,x,u_d+\tilde{u},\hat{\theta}))^T\lambda. \tag{5.23}$$

The allocation algorithm is obtained by replacing $L_{\hat{\theta}u}$ with $L_{\hat{\theta}\tilde{u}}$ in the Lyapunov function (5.6) and the update laws (5.8) and (5.11). In addition the estimate of $\hat{\theta}_u$ is given by:

$$\begin{aligned}\dot{\hat{\theta}}_u^T &= \left(\frac{\partial V_{\tilde{u}}}{\partial \tilde{u}} + \eta_u^T\Gamma_u\right) f_{u\theta}(t,x,u,u_{cmd})\Gamma_{\theta_u}^{-1} \\ &+ \left(\frac{\partial L_{\hat{\theta}\tilde{u}}^T}{\partial u_d}\frac{\partial^2 L_{\hat{\theta}\tilde{u}}}{\partial \tilde{u}\partial u_d} + \frac{\partial L_{\hat{\theta}\tilde{u}}^T}{\partial \lambda}\frac{\partial^2 L_{\hat{\theta}\tilde{u}}}{\partial \tilde{u}\partial \lambda}\right) f_{u\theta}(t,x,u,u_d,u_{cmd})\Gamma_{\theta_u}^{-1} \\ &+ \left(\eta_x^T\Gamma_x + \frac{\partial L_{\hat{\theta}\tilde{u}}^T}{\partial u_d}\frac{\partial^2 L_{\hat{\theta}\tilde{u}}}{\partial x\partial u_d} + \frac{\partial L_{\hat{\theta}\tilde{u}}^T}{\partial \lambda}\frac{\partial^2 L_{\hat{\theta}\tilde{u}}}{\partial x\partial \lambda}\right) g(t,x)\Phi_{\theta_u}(t,x,u)\Gamma_{\theta_u}^{-1},\end{aligned} \tag{5.24}$$

and the feed forward term $u_{\hat{\theta}\tilde{u}ff}$ takes the following form:

$$\begin{aligned}u_{\hat{\theta}\tilde{u}ff} &:= \mathbb{H}_{\hat{\theta}u}^{-1}\begin{pmatrix}\frac{\partial^2 L_{\hat{\theta}\tilde{u}}}{\partial t\partial u_d} \\ \frac{\partial^2 L_{\hat{\theta}\tilde{u}}}{\partial t\partial \lambda}\end{pmatrix} + \mathbb{H}_{\hat{\theta}u}^{-1}\begin{pmatrix}\frac{\partial^2 L_{\hat{\theta}\tilde{u}}}{\partial x\partial u_d} \\ \frac{\partial^2 L_{\hat{\theta}\tilde{u}}}{\partial x\partial \lambda}\end{pmatrix} f(t,x) \\ &+ \mathbb{H}_{\hat{\theta}u}^{-1}\begin{pmatrix}\frac{\partial^2 L_{\hat{\theta}\tilde{u}}}{\partial x\partial u_d} \\ \frac{\partial^2 L_{\hat{\theta}\tilde{u}}}{\partial x\partial \lambda}\end{pmatrix} g(t,x)(\Phi(t,x,u_d+\tilde{u},\hat{\theta}) \\ &+ \mathbb{H}_{\hat{\theta}u}^{-1}\begin{pmatrix}\frac{\partial^2 L_{\hat{\theta}\tilde{u}}}{\partial \tilde{u}\partial u_d} \\ \frac{\partial^2 L_{\hat{\theta}\tilde{u}}}{\partial \tilde{u}\partial \lambda}\end{pmatrix} f_{\tilde{u}}(t,x,u,u_d,\hat{\theta}_u,\hat{\theta}_u) + \mathbb{H}_{\hat{\theta}\tilde{u}}^{-1}\begin{pmatrix}\frac{\partial^2 L_{\hat{\theta}\tilde{u}}}{\partial \hat{\theta}\partial u_d} \\ \frac{\partial^2 L_{\hat{\theta}\tilde{u}}}{\partial \hat{\theta}\partial \lambda}\end{pmatrix}\dot{\hat{\theta}}.\end{aligned}$$

A disadvantage of this algorithm is that more complexity is introduced, but the transient performance, introduced to the closed loop by the actuator dynamics, may be improved.

5.3 Example: DP of a scale model ship continued

The detailed description of a ship model including the actuator dynamics and the actuator-force mapping model, presented in the introducing chapter, is here considered. Furthermore, the high level controller defined in the introduction, and implemented in the examples of Chapter 3 and 4, is together with the cost function from Chapter 3 used in the following.

The actuator error dynamics for each propeller are based on the propeller model from (5.25) and the error dynamic is given by:

$$\begin{aligned} J_{mi}\dot{\tilde{\omega}}_i = & - k_{fi}\left(\tilde{\omega}_i + \omega_{di}\right) - \frac{T_{ni}}{a_T}(\tilde{\omega}_i + \omega_{di}) \\ & + \frac{\phi_i(\omega_i, \upsilon_x)\theta_{1i}}{a_T} + u_{cmdi} - J_{mi}\dot{\omega}_{di} \end{aligned} \tag{5.25}$$

where $\tilde{\omega}_i := (\omega_i - \omega_{id})$, J_m is the shaft moment of inertia, k_f is a positive coefficient related to the viscous friction, a_T is a positive model constant, Pivano et al. [2006], and u_{cmd} is the commanded motor torque. By the quadratic Lyapunov function $V_\omega := \frac{\tilde{\omega}_i^2}{2}$, it can be seen that the control law

$$\begin{aligned} u_{cmdi} := & - K_{\omega p}(\tilde{\omega}_i) - \frac{\phi_i(\omega_i, \upsilon_x)\hat{\theta}_{1i}}{a_T} + J_{mi}\dot{\omega}_{di} \\ & + \frac{T_{ni}(\omega_{di})}{a_T} + k_{fi}\omega_{di} \end{aligned} \tag{5.26}$$

makes the origin of (5.25) UGES when $\hat{\theta}_{1i} = \theta_{1i}$, since $\dot{V}_\omega = -\frac{1}{J_{mi}}\left(k_{fi} + \frac{T_{ni}}{a_T} + K_{\omega p}\right)\tilde{\omega}_i^2$. The rudder model is linearly time-variant and the error dynamics, based on (5.27), are given by:

$$m_i\dot{\tilde{\delta}} = a_i(t)\left(\tilde{\delta} + \delta_{di}\right) + b_i u_{cmd\delta i} - m_i\dot{\delta}_{di}, \tag{5.27}$$

where $\tilde{\delta} := \delta_i - \delta_{di}$ and a_i, b_i are known scalar parameters bounded away from zero. Furthermore the controller

$$b_i u_{cmd\delta i} := -K_\delta\tilde{\delta} - a_i(t)\left(\tilde{\delta} + \delta_{di}\right) + m_i\dot{\delta}_{di} \tag{5.28}$$

makes the origin of (5.27) UGES. The parameters for the actuator model and controllers are: $a_T = 1$, $J_{mi} = 10^{-2}$, $k_{fi} = 10^{-4}$, $a_i = -10^{-4}$, $b_i = 10^{-5}$, $m_i = 10^{-2}$, $K_{\omega p} = 5 \cdot 10^{-3}$ and $K_\delta = 10^{-3}$. The following simulation results are obtained with the basis in the model and high level controller parameterization presented in Chapter 3 and Table A.1 in Appendix A.5. As before, the control objective is satisfied and the commanded virtual controls are tracked closely by the forces generated by the adaptive control allocation law: see Figure 5.4. Note that there are some deviations since ω saturates at ca. $220s$ and $420s$. Also, note that the parameter estimates only converge to the true values when the ship is moving (the thrust loss is not zero).

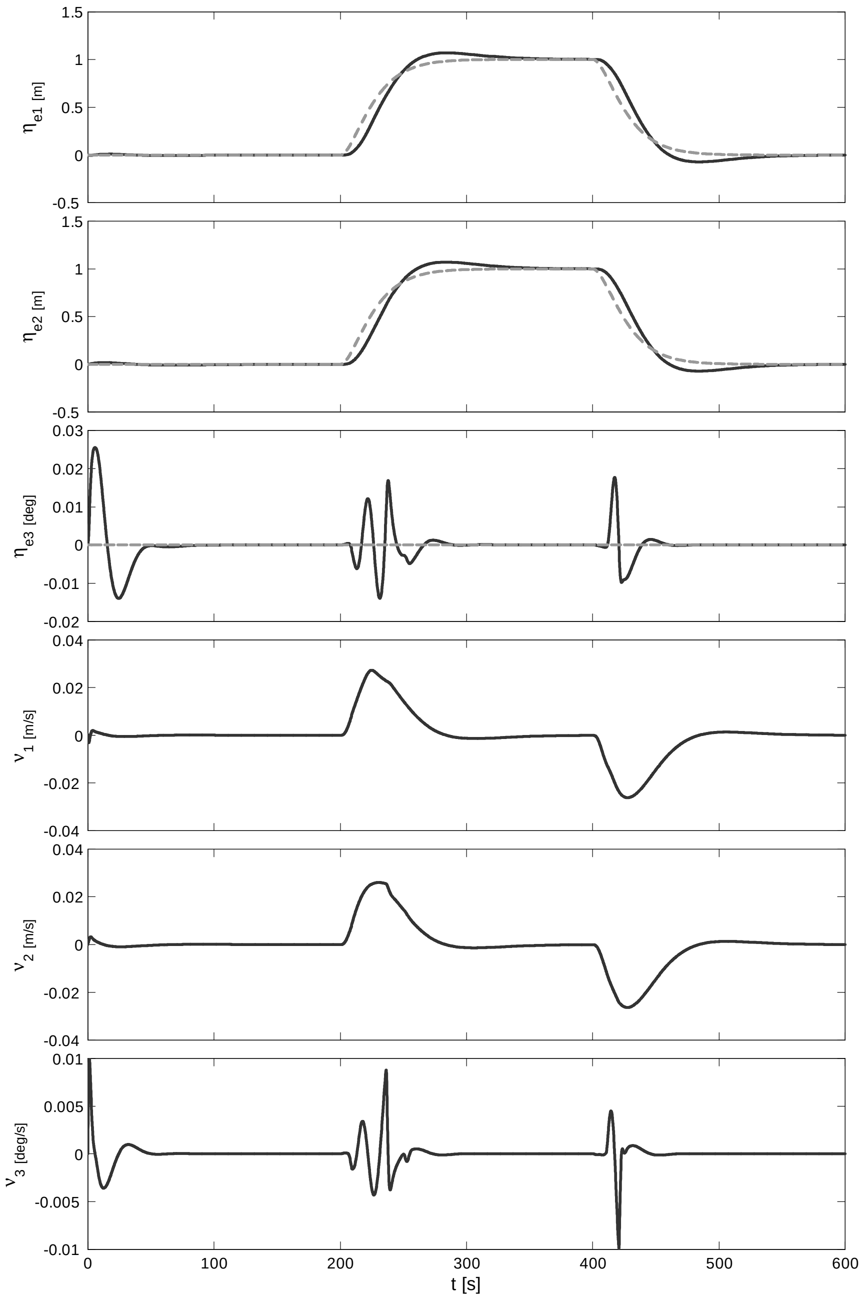

Figure 5.1: The ship; desired position (dashed), actual position (solid) and velocities.

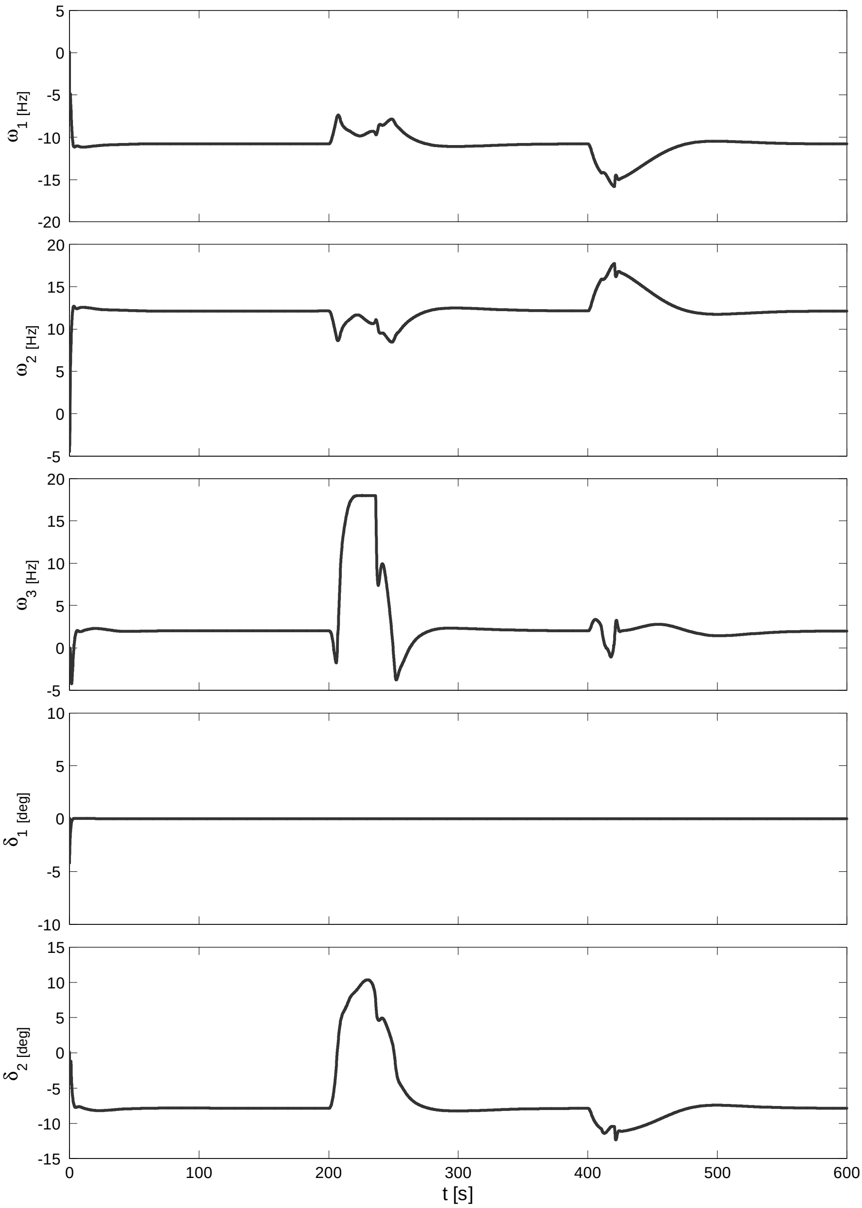

Figure 5.2: The actual propeller velocities and rudder deflections.

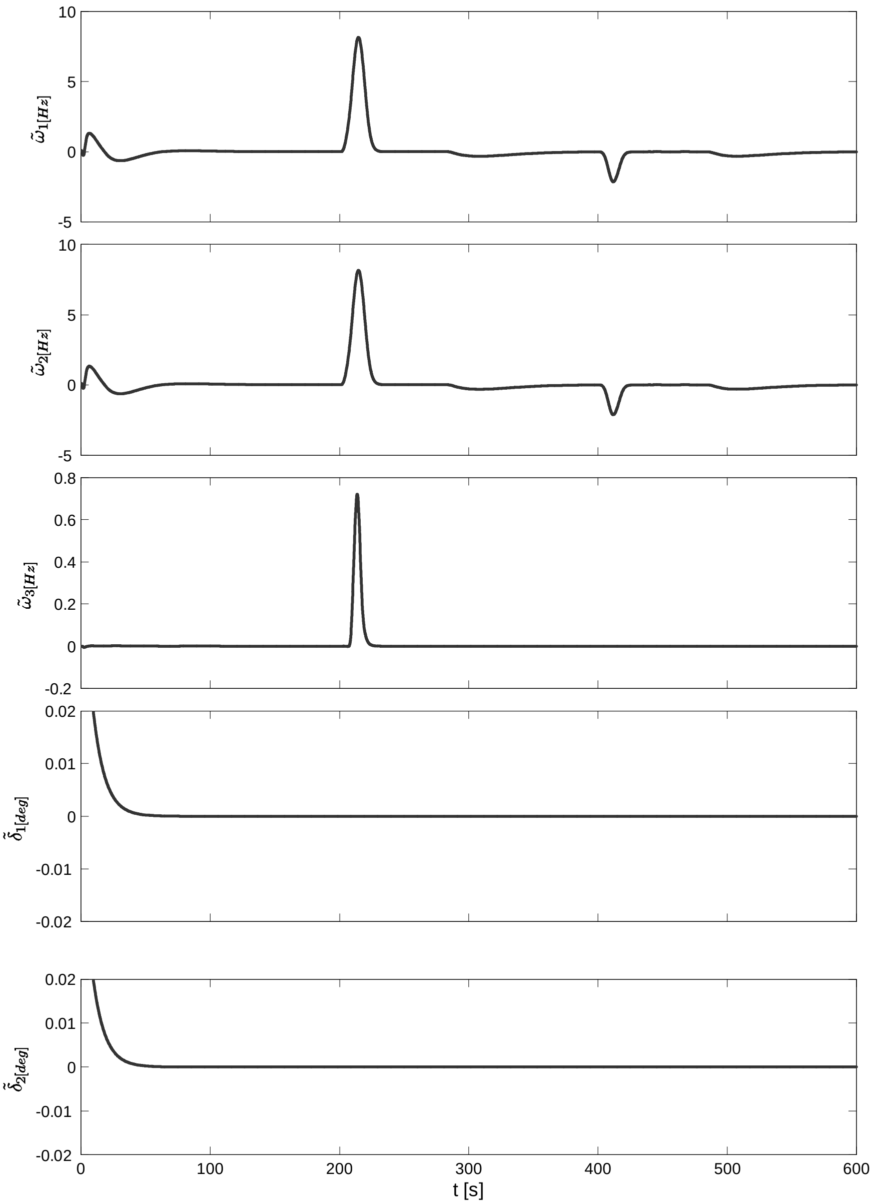

Figure 5.3: The difference between the actual and desired actuator values.

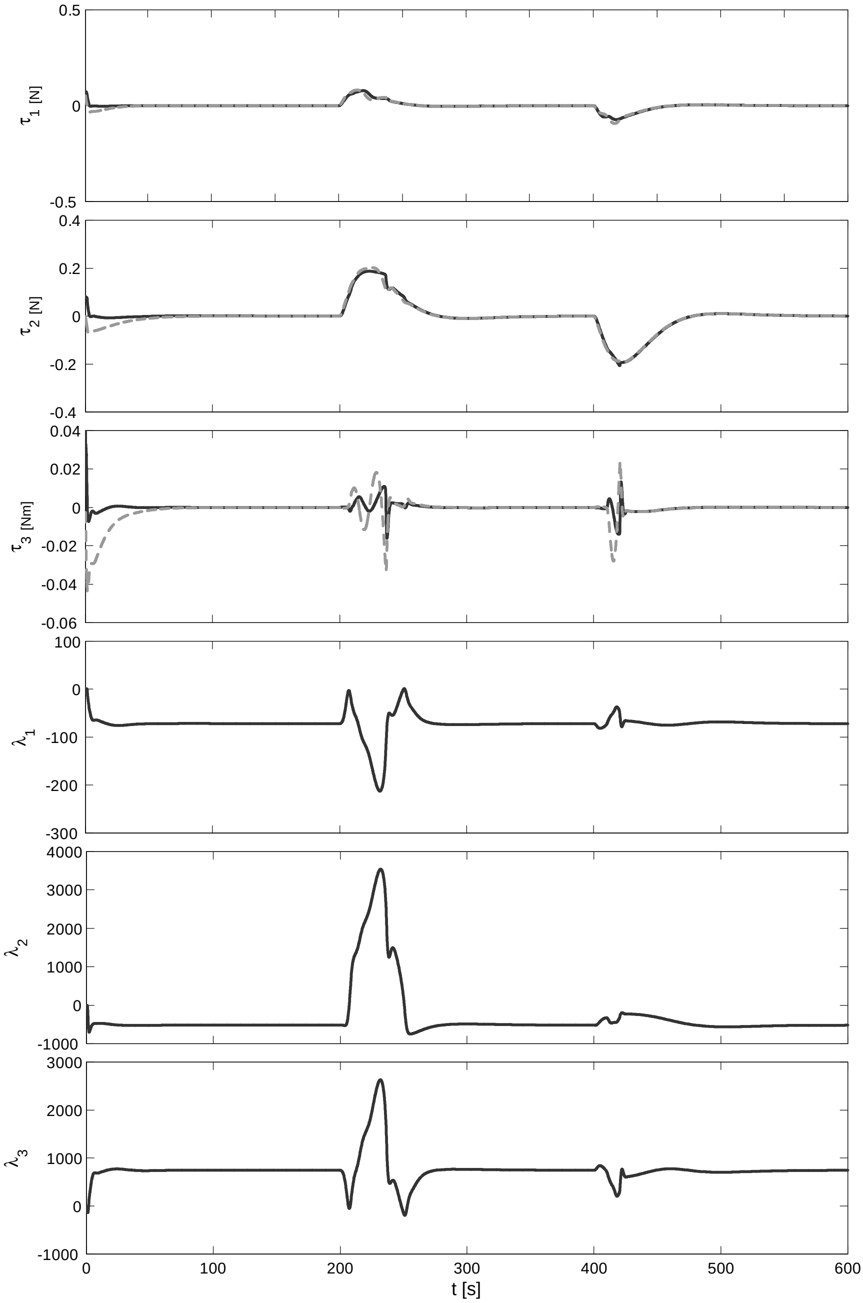

Figure 5.4: The desired virtual control forces (dashed) and the actual control forces (solid) generated by the actuators. The lagrangian parameters are also shown.

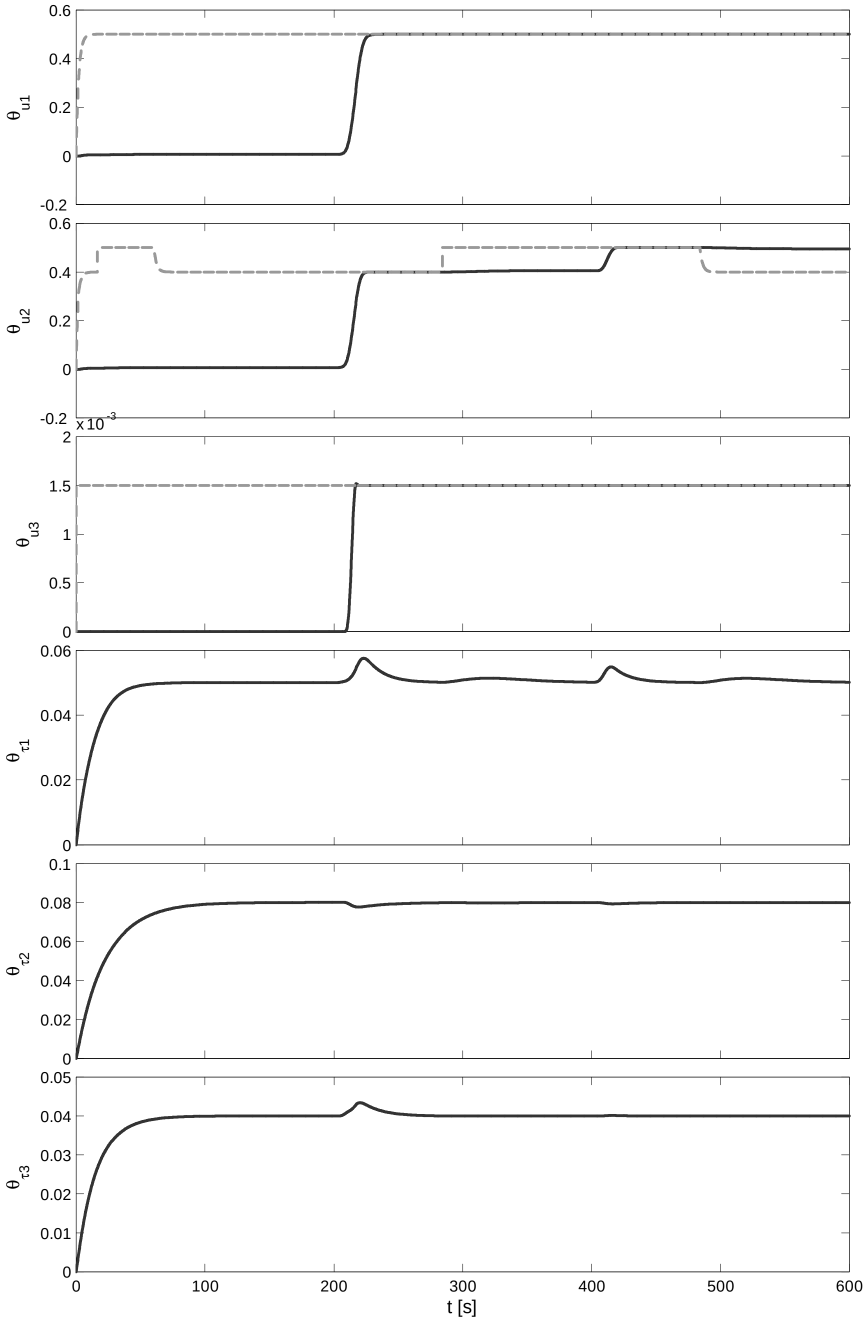

Figure 5.5: The actual (dashed) and estimated (solid) loss parameters, and external disturbance estimates.

Chapter 6

Stabilization of automotive vehicles using active steering and adaptive brake control allocation

6.1 Introduction

Some of the major advances in automotive technology in the past decade have been in the area of safety, and most modern passenger vehicles are now equipped with an active safety system. Important components in these systems are Anti-lock Braking Systems (ABS), Traction Control Systems/Antispin (TCS) and recently Electronic Stability Program (ESP). ABS and TCS are systems designed to maximize the contact-force between the tyres and the road during braking and acceleration, while ESP is introduced in order to control the yaw motion and to prevent over- and/or understeering. There are several ways of controlling the yaw dynamic of a vehicle, for example the torque-biasing system on a four-wheel drive vehicle, where the motor torque may be transferred from the front to the rear axle and between the left and right wheels through an electronic-controlled limited slip differential (ELSD), Piyabongkarn et al. [2006]. But most vehicle stability systems are controlled by active steering; Ackermann et al. [1995], Ackermann et al. [1999], Ackermann and Bünte [1997], Mammer and Koenig [2002] and Ackermann and Bünte [1998], active braking; van Zanten et al. [1995], Alberti and Babbel [1996] or by a combination of active steering and braking; Wang and Longoria [2006], Guvenc et al. [2003]

and Yu and Moskwa [1994].

The solution to the yaw motion control problem is in Piyabongkarn et al. [2006] and Wang and Longoria [2006] presented in the structure of a typical control allocation architecture, where a high level controller calculates a desired yaw moment. This yaw moment is then passed to an allocation/distribution block and mapped to desired references for the low level actuator controllers. The advantage of this structure is the modularity of separating the high and low level controls through the introduction of the allocation/distribution algorithm. It is in Shim and Margolis [2001], shown that the knowledge of the friction coefficient offers significant improvement of the vehicle response during yaw rate control. By utilizing vehicle motion information, typical friction coefficient estimation approaches consider either longitudinal or lateral motion measurements, see Rajamani et al. [2006] and references therein. Moreover, depending on sensors, the algorithms in Rajamani et al. [2006] provide estimates of the tyre-road friction parameters individually for each wheel, which is useful information for the yaw stabilizing schemes when the road conditions are unknown and non uniform.

In Tøndel and Johansen [2005] an explicit piecewise linear approximate solution is created by using multiparametric programming and solving the optimization problem off-line, while in Tjønnås and Johansen [2006] a yaw stabilization scheme for an automotive vehicle using brakes, based on the dynamic optimizing control allocation approach from Johansen [2004] and Tjønnås and Johansen [2005] was presented by the authors. This strategy offers the benefits of a modular approach combining convergence and stability properties for yaw rate tracking (high level control), optimality of the allocation problem and adaptation of the averaged maximal tyre-road friction parameter.

The following work is based on the control strategy from Tjønnås and Johansen [2006] and rely on the dynamic control allocation law from Chapter 5, which extends the results from Tjønnås and Johansen [2005] by considering actuator dynamics in the control architecture. Furthermore, the yaw stabilizing algorithm presented here include active front wheel steering in combination with low level control of the longitudinal wheel slip and an adaptive law that estimates the maximal tyre-road friction parameter for each wheel. In addition to the measurements (yaw rate, steering wheel angle, absolute velocity of the vehicle and wheel side slip) that was necessary in the algorithm from Tjønnås and Johansen [2006], the algorithm presented in this work require measurements (estimates) of the velocity and angular velocity from each wheel.

In Hattori et al. [2002] and Plumlee et al. [2004] the quasi-static control allocation problem is solved by a real-time optimizing program. The following approach solves the control allocation problem dynamically (by not necessarily finding the optimal solution at each sampling instant), such that a real-time implementation can be realized without the use of numeric optimization software. In general, this is an advantage since implementations on vehicles with low-cost hardware may be considered. The main benefits of this approach are the low memory requirement, the low computational complexity and stability conservation due to the asymptotic optimality approach. By including adaptive laws for estimating the maximal tyre-road friction parameter for each wheel, the yaw stabilizing scheme takes variable road conditions into account, and ultimately the control allocation algorithm will perform better.

The chapter is composed as follows: In Section 6.2 the control structure is presented. The high level model and controller are derived in section 6.3, while the low level model and controller, along with the qualitative behavior of the tyre-road friction model and the main design assumptions are given in section 6.4. The dynamic control allocation strategy is presented in Section 6.5 and in Section 6.6 the simulation scenarios are presented and discussed.

6.2 Control structure

In this section the control scheme and its intrinsic modular structure as a solution to the yaw stabilization problem is shown. The main result demonstrates how the computationally efficient dynamic control allocation algorithm, from Chapter 3 and 5, may be applied as a part of this solution.

The variables and parameters used in the following are described in Figure 6.2 and Table 6.1

The control inputs are in addition to controlling the brake pressure for each wheel, Tjønnås and Johansen [2006], an allied correction of the steering angles of the front wheels ($\Delta\delta_1 := \Delta\delta_2 := \Delta\delta_u$). This means that in total five actuators are available for the control allocation algorithm to manipulate. No steering on the rear wheels is assumed ($\delta_3 := \delta_4 := 0$) .

The control allocation approach structurally consists of the following modules (also shown in Figure 6.1):

1. **The high level yaw rate motion control algorithm.** Based on the vehicle motion model and the driver input $\delta := (\delta_1, \delta_2, 0, 0)^{\mathrm{T}}$, a yaw rate reference r_{ref} is defined and the high level dynamics of the

yaw rate error $\tilde{r} := r - r_{ref}$ is described. By treating the virtual control torque M as an available input to the yaw rate system, a virtual control law M_c is designed such that the origin of the error system is Uniformly Globally Asymptotically Stable (UGAS) when $M = M_c$.

2. **The low level braking control algorithm.** From the dynamics describing the longitudinal wheel slip $\lambda_x := (\lambda_{x1}, \lambda_{x2}, \lambda_{x3}, \lambda_{x4})^{\mathrm{T}}$, a control law for the clamping force $T_b := (T_{b1}, T_{b2}, T_{b3}, T_{b4})^{\mathrm{T}}$ is defined such that for any smooth reference $\lambda_{xd} := (\lambda_{x1d}, \lambda_{x2d}, \lambda_{x3d}, \lambda_{x4d})^{\mathrm{T}}$, then λ_x will track λ_{xd} asymptotically. The steering dynamics are assumed to be neglectable and therefore no low level steering control law is considered. In order to cope with the possibly unknown maximal tyre-road friction parameter vector $\mu_H := (\mu_{H1}, \mu_{H2}, \mu_{H3}, \mu_{H4})^{\mathrm{T}}$, appearing in the effector and actuator models, an adaptive law for online estimation of μ_H is defined.

3. **The dynamic control allocation algorithm (connecting the high and low level controls).** The main objective of the control allocation algorithm is to distribute the desired steering angle correction $\Delta\delta := (\Delta\delta_u, \Delta\delta_u, 0, 0)^{\mathrm{T}}$ and the desired low level reference (λ_{xd}) to the low level control, based on the desired virtual control (M_c). The static torque mapping $M = \Phi_M(\delta, \lambda_x, \alpha, \mu_H)$ (see equation (6.28) for definition) represents the connection between the output of low level system and the input to the high level system. The control allocation problem can be formulated as the static minimization problem

$$\min_{u_d} J(t, u_d) \quad s.t. \quad M_c - \bar{M}(t, u_d, \hat{\mu}_H) = 0 \;, \tag{6.1}$$

where $\tilde{u} := u - u_d$, $u_d := (\lambda_{xd}, \Delta\delta_u)$, $\bar{M}(t, u_d, \hat{\mu}_H) := M(\delta(t) + \Delta\delta, \lambda_{xd}, \alpha(t), \hat{\mu}_H)$ and $\alpha := (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^{\mathrm{T}}$. The cost function J is defined such that minimum braking and actuator constraints are incorporated. Based on this formulation, the Lagrangian function

$$L := J(t, u_d) + (M_c - \bar{M}(t, u_d, \hat{\mu}_H))\pi, \tag{6.2}$$

is introduced and update-laws for, the steering actuator $\Delta\delta_u$, the longitudinal wheel slip reference λ_{xd} and the Lagrangian parameter π are defined according to the dynamic control allocation algorithm in Chapter 5.

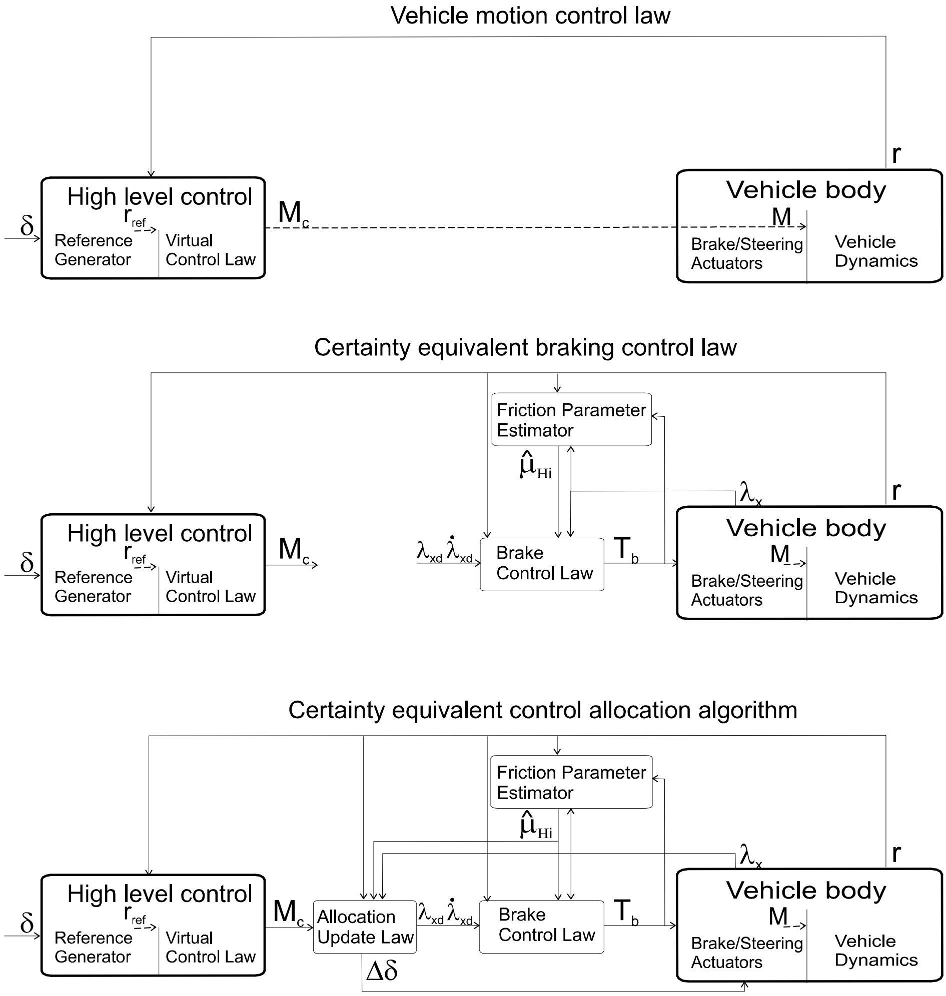

Figure 6.1: Adaptive control allocation design philosophy

The benefits of this modular control structure are independence in design and tuning, and support of fault tolerant control (e.g. if one actuator fails, the allocation module handle the re-configuration and changed constraints by modifying cost function parameters, without a direct effect on the high level implementation).

Except from the procedure of estimating the tyre road friction parameter, no observer dynamics are included in the analysis or synthesis presented in this paper. This means that all states and variables used in the algorithm are viewed as sampled sensor measurements. However, due to cost and reliability issues, vehicle velocity and side slip angle are rarely measured directly and observers based on yaw rate, wheel speed, acceleration and steering angle measurements are needed. Simulations of the controlled system using the nonlinear observers presented in Imsland et al. [2006] and Imsland et al. [2007] have been carried out and the results are promising, but left for further work and analysis.

6.3 High level vehicle model and control design

The dynamic control allocation approach involves modeling of the vehicle over three stages: the high level vehicle motion dynamics, the tyre force model and the low level longitudinal wheel slip dynamics.

The high level vehicle motion dynamics

$$\begin{pmatrix} \dot{\nu} \\ \dot{\beta} \\ \dot{r} \end{pmatrix} = - \begin{pmatrix} 0 \\ r \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{m}\cos(\beta) & \frac{1}{m}\sin(\beta) & 0 \\ -\frac{1}{m\nu}\sin(\beta) & \frac{1}{m\nu}\cos(\beta) & 0 \\ 0 & 0 & \frac{1}{J_z} \end{pmatrix} \begin{pmatrix} f_x \\ f_y \\ M \end{pmatrix}, \quad (6.3)$$

are based on a horizontal plane two-track model that can be found in Kiencke and Nielsen [2000], and serves as the basis for the high level controller design. The parameters are described in Figure 6.2 and Table 6.1.

6.3.1 Vehicle yaw stabilization

The control objective is to prevent the vehicle from over- and/or understeering, i.e. the yaw rate r, of the vehicle should be close to some desired yaw rate r_{ref}, defined by the commanded steering angle. In order to generate this reference the steady-state of the side slip dynamics are considered. Thus from the model (6.3)

$$\dot{\beta} = -r + \frac{-\sin\beta f_x + \cos\beta f_y}{m\nu},$$

i	Subscript for each wheel $i \in [1, 2, 3, 4]$
ν	Absolute speed at the vehicle CG
ν_i	Absolute speed at the wheel CG
ν_{xi}	Wheel velocity in the vertical wheel plane
ν_{yi}	Wheel velocity perpendicular to the vertical wheel plane
β	Vehicle side slip angle
ψ	Yaw angle
r	Yaw rate ($\dot{\psi}$)
r_{ref}	Desired Yaw rate
F_{xi}	Friction force on wheel in longitudinal wheel direction
F_{yi}	Friction force on wheel in lateral wheel direction
F_{zi}	Vertical force on ground from each wheel
δ_i	Steering angle ($\delta_3 := \delta_4 := 0$)
$\Delta\delta_u$	Steering angle actuator
$\Delta\delta_i$	Steering angle correction ($\Delta\delta_1 := \Delta\delta_2 := \Delta\delta_u, \Delta\delta_3 := \Delta\delta_4 := 0$)
f_x	Force at vehicle CG in longitudinal direction
f_y	Force at vehicle CG in lateral direction
M	Torque about the yaw axis
M_c	Desired torque about the yaw axis
m	Vehicle mass
m_{wi}	Vehicle mass distributed on each wheel i
J_z	Vehicle moment of inertia about the yaw axis
J_ω	Wheel moment of inertial
μ_{Hi}	Maximum tyre road friction coefficient
μ_{yi}	Lateral tyre road friction coefficient
μ_{xi}	Longitudinal tyre road friction coefficient
α_i	Wheel side slip angle
λ_{xi}	Wheel slip in longitudinal wheel direction
λ_{xid}	Desired longitudinal wheel slip
ω_i	Angular velocity of wheel i
T_{bi}	Braking torque on each wheel i
R	Radius of the wheels
ρ_i	Vehicle construction parameter
h_i	Wheel distance from the CG

Table 6.1: Model variables

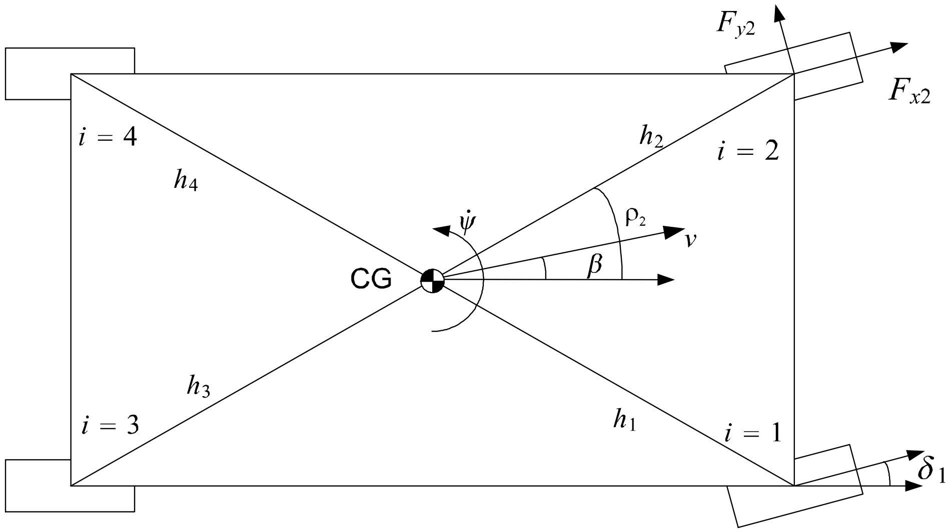

Figure 6.2: Vehicle schematic

the desired reference is generated by letting $\dot{\beta} = \beta = 0$ and $\lambda_x = 0$, such that:

$$r_{ref} := \frac{f_y(\alpha, \lambda_x = 0, \delta, \mu_{nH})}{m\nu}, \tag{6.4}$$

where μ_{nH} is a tuning parameter for the virtual maximal tyre-road friction, i.e. if the vehicle follows this yaw rate reference, the driver will experience a yaw motion related to a virtual surface described by μ_{nH}.

Remark 6.1 *Note that at $\nu = 0$ the yaw rate reference generation has a singularity, which means that (6.4) is not suitable for low speeds. This problem is avoided by introducing a threshold $\nu_T > 0$ such that the algorithm is only active at $\nu > \nu_T$. This is formally stated in Assumption 6.2 and a common limitation of wheel slip control, see e.g. Johansen et al. [2003a].*

For safe driving such as rollover prevention and vehicle controllability conservation, Kiencke and Nielsen [2000], the side-slip angle should be limited by

$$|\beta| \leq 10^\circ - 7^\circ \frac{\nu^2}{(40[m/s])^2} \ . \tag{6.5}$$

Although this bound is not explicitly enforced in this scheme, simulations show that due to the yaw reference tracking, the side slip angle satisfies (6.5).

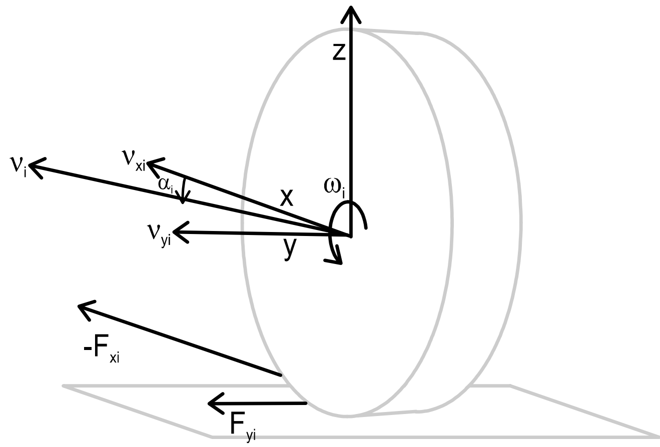

Figure 6.3: Quarter car model

The high level control design is based on the reduced model from (6.3):

$$\dot{r} = \frac{M}{J_z}. \tag{6.6}$$

Let $\tilde{r} := r - r_{ref}$ denote the yaw rate error such that a virtual tracking PI controller can be described by

$$M_c(t, \tilde{r}, e_r) := -K_P\tilde{r} - K_I e_r + J_z \dot{r}_{ref} \tag{6.7}$$

where

$$\dot{e}_r = \tilde{r} \tag{6.8}$$

$$\begin{aligned}\dot{\tilde{r}} &= \frac{M}{J_z} - \dot{r}_{ref} \\ &= -\frac{K_P}{J_z}\tilde{r} - \frac{K_I}{J_z}e_r + \frac{M - M_c}{J_z}.\end{aligned} \tag{6.9}$$

With $M = M_c$ the linear tracking error dynamics become:

$$\dot{e}_r = \tilde{r} \tag{6.10}$$

$$\dot{\tilde{r}} = -\frac{K_P}{J_z}\tilde{r} - \frac{K_I}{J_z}e_r, \tag{6.11}$$

and clearly the equilibrium $(\tilde{r}, e_r) = 0$ of (6.10)-(6.11) is Uniformly Globally Exponentially Stable (UGES) for $K_P, K_I > 0$.

6.4 Low level model and controller

In the literature the wheel slip definitions vary. For example, in Burckhardt [1993] the longitudinal wheel slip is defined in the wheel absolute velocity direction, while in Reimpell and Sponagel [1995] it is defined in the wheel vertical plane. The definitions also vary based on drive or braking modes. The following longitudinal and lateral wheel slip definitions are used here:

$$\lambda_{xi} := \frac{\nu_{xi} - \omega_i R}{\nu_i}, \ \ \lambda_{yi} := \sin(\alpha_i), \tag{6.12}$$

$$\nu_i = \sqrt{\nu_{xi}^2 + \nu_{yi}^2}. \tag{6.13}$$

Furthermore the longitudinal wheel slip is given in the wheel vertical plane. Based on the wheel slip definitions and the horizontal quarter car model:

$$m_{wi}\dot{\nu}_{yi} = F_{yi}, \tag{6.14}$$

$$m_{wi}\dot{\nu}_{xi} = F_{xi}, \tag{6.15}$$

$$J_\omega \dot{\omega}_i = -RF_{xi} - T_{bi} sign(\omega_i), \tag{6.16}$$

the low level longitudinal wheel slip dynamics is derived, see Appendix A.2:

$$\dot{\lambda}_{xi} = \frac{R}{\nu_i J_\omega}\left(sign(\omega_i)T_{bi} - \phi_t(\lambda_{xi}, \alpha_i, \mu_{Hi}, \nu_{xi}, \nu_i)\right), \tag{6.17}$$

where

$$\begin{aligned}\phi_t(\lambda_{xi},\alpha_i,\mu_{Hi},\nu_{xi},\nu_i) :=& -\frac{J_\omega}{Rm_{wi}}\sin(\alpha_i)\lambda_{xi}F_{yi}(\lambda_{xi},\alpha_i,\mu_{Hi}) \\ & - \left(\frac{J_\omega\left(1 - \lambda_{xi}\frac{\nu_{xi}}{\nu_i}\right)}{Rm_{wi}} + R\right)F_{xi}(\lambda_{xi},\alpha_i,\mu_{Hi}).\end{aligned} \tag{6.18}$$

The parameters are described in Figures 6.2 and 6.3, and Table 6.1. Note that the low level wheel slip model also has a singularity at $\nu_i = 0$, but since in principal a yaw stabilizing algorithm is not needed in a region around this singularity, a threshold will be introduced in order to switch off the algorithm, see Assumption 2. The model of the forces generated at the contact point between the tyre and the road, represented by F_{xi} and F_{yi}, are considered in the following.

6.4.1 Tyre-road friction model

The longitudinal and lateral friction forces acting on the tyres are described by the product between the normal forces and the friction coefficients:

$$F_{xi} := -F_{zi}\mu_{xi}(\lambda_{xi}, \alpha_i, \mu_{Hi})$$
$$F_{yi} := -F_{zi}\mu_{yi}(\lambda_{xi}, \alpha_i, \mu_{Hi}).$$

The presented control allocation approach, do not rely strongly on the detailed structure of the tyre road friction model, but certain qualitative properties are required. These properties are summarized in the following assumptions and ensure constraint definitions, tuning possibilities and convergence (via Persistence of Excitation (PE) arguments) of the allocation algorithm.

Assumption 6.1 *(Friction model assumptions)*

a) *There exist a limit $\lambda_{Tri}(\alpha_i, \mu_{H_i}) > 0$ such that for any fixed $|\alpha_i| < \frac{\pi}{2}$, $\mu_{Hi} \in [0.1, 1]$ and $|s| \leq |\lambda_{Dri}| < \lambda_{Tri}(\alpha_i, \mu_{H_i})$, then $\frac{\partial \mu_{xi}(s, \alpha_i, \mu_{Hi})}{\partial s} > 0$, where λ_{Dri} is a design parameter.*

b) *There exist a limit $\lambda_{Txi} > 0$ such that for any fixed s_1 and $s_2 \in [0.1, 1]$, $|\alpha_i| < \frac{\pi}{2}$ and $\lambda_{xi} \geq \lambda_{Axi} \geq \lambda_{Txi}$, then $\frac{\partial \mu_{xi}(\lambda_{xi}, \alpha_i, \mu_{Hi})}{\partial \mu_{Hi}} > 0$, $\frac{\partial \mu_{yi}(\lambda_{xi}, \alpha_i, \mu_{Hi})}{\partial \mu_{Hi}} > 0$, $\left|\frac{\partial^2 \mu_{xi}(\lambda_{xi}, \alpha_i, s_2)}{\partial s_2^2}\right| \ll \left|\frac{\partial \mu_{xi}(\lambda_{xi}, \alpha_i, s_1)}{\partial s_1}\right|$ and $\left|\frac{\partial^2 \mu_{yi}(\lambda_{xi}, \alpha_i, s_2)}{\partial s_2^2}\right| \ll \left|\frac{\partial \mu_{yi}(\lambda_{xi}, \alpha_i, s_1)}{\partial s_1}\right|$, where λ_{Axi} is a design parameter (adaptive switch). The same is true for $\frac{\partial \mu_{yi}(\lambda_{xi}, \alpha_i, \mu_{Hi})}{\partial \mu_{Hi}}$.*

c) *Let the map $(\lambda_{xi}, \alpha_i, \mu_{Hi}) \mapsto \mu_{xi}(\lambda_{xi}, \alpha_i, \mu_{Hi})$ be such that all of its first and second partial derivatives are uniformly bounded by λ_{xi} and μ_{Hi}, and let the same be true for the map $(\lambda_{xi}, \alpha_i, \mu_{Hi}) \mapsto \mu_{yi}(\lambda_{xi}, \alpha_i, \mu_{Hi})$.*

In Figure 6.4 and 6.5 typical friction coefficients are shown with dependence on the longitudinal wheel slip, the maximal tyre-road friction parameter and the wheel side slip angle.

6.4.2 Wheel slip dynamics

In this section the main model assumptions and properties of the wheel slip dynamics are stated. These assumptions are either directly related to the parameters and the qualitative behavior of the model presented, or deals with the limitations of this yaw stabilization approach (domain/state restriction).

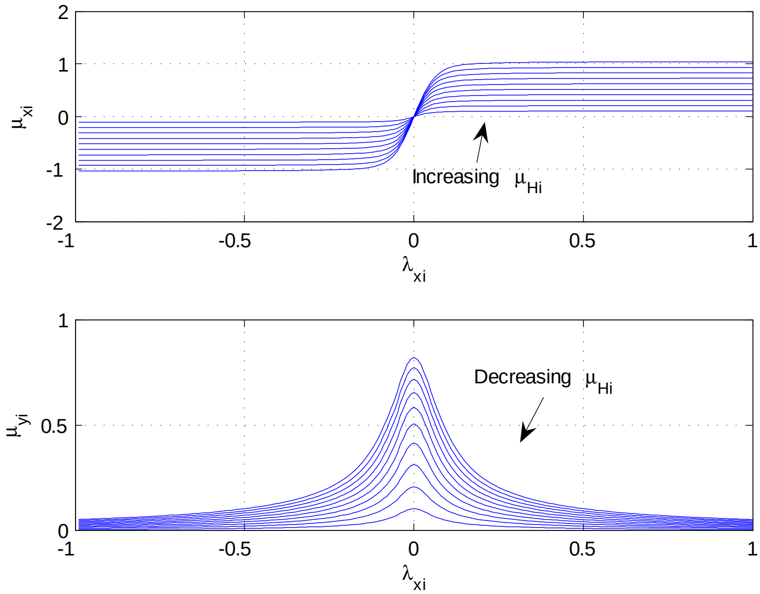

Figure 6.4: The effect of changing the maximal tyre road friction coefficient μ_{Hi}, in the mapping from the longitudinal wheel slip λ_{xi} to the forces in longitudinal μ_{xi} and lateral μ_{yi} direction. $\alpha = 3$[deg]

Assumption 6.2 *(Model and domain assumptions)*

a) m_{wi}, F_{zi}, R, *and* $J_\omega > 0$. *Furthermore* $J_\omega \ll m_{wi}R^2$.

b) $\nu_{xi}(t) > \epsilon > 0 \ \forall t > t_0$, *i.e.* $|\alpha_i(t)| < \frac{\pi}{2} \ \forall t > t_0$.

c) $\omega_i(t) > 0 \ \forall t > t_0$, *i.e.* $\lambda_{xi} \in (-\infty, 1)$. *Moreover the longitudinal wheel slip is assumed to have the lower bound:* $\lambda_{xi} \geq -1$.

d) $\Delta\delta_u \in [-\Delta\delta_{\max}, \Delta\delta_{\max}]$.

e) $\mu_{Hi} \in [0.1, 1]$.

By Assumption 6.2 the analysis and control design are limited to the cases where; the vehicle always has forward speed b), the wheels are never rotating in reverse and are always fixed on the ground a) and c), the steering angle correction are constrained around the nominal angle set by the driver

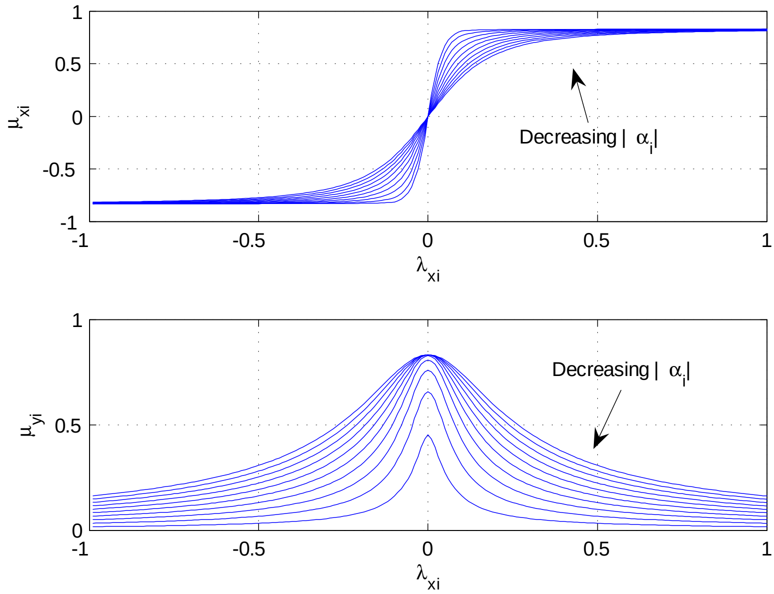

Figure 6.5: The effect of changing the wheel side slip coefficient α_i, in the mapping from longitudinal wheel slip λ_{xi} to forces in longitudinal μ_{xi} and lateral μ_{yi} direction. $\mu_{Hi} = 0.8$

d), the road conditions may vary from icy surface to dry asphalt. Furthermore, Assumptions 6.1 and 6.2 lead to the following approximations, which simplify the control design, the allocation algorithm and the analysis:

Approximation 6.1 *(Longitudinal wheel slip dynamic simplification) From Assumption 6.2;*

$$\frac{J_\omega\left(1-\lambda_{xi}\frac{\nu_{xi}}{\nu_i}\right)}{Rm_{wi}} \ll R.$$

Furthermore, $\lambda_{xi}\mu_{yi}(\lambda_{xi}, \alpha_i, \mu_{Hi})$ *and* $\mu_{xi}(\lambda_{xi}, \alpha_i, \mu_{Hi})$ *are in the same order of magnitude and approach zero as* $\lambda_{xi} \to 0$*, such that:*

$$\phi(\lambda_{xi}, \alpha_i, \mu_{Hi}) := RF_{zi}\mu_{xi}(\lambda_{xi}, \alpha_i, \mu_{Hi})$$

imply

$$\phi(\lambda_{xi}, \alpha_i, \mu_{Hi}) \approx \phi_t(\lambda_{xi}, \alpha_i, \mu_{Hi}, \nu_{xi}, \nu_i).$$

From Approximation 6.1 the low level longitudinal wheel slip dynamic may be simplified such that it is no longer dependent on the lateral friction coefficient.

Approximation 6.2 *(Affine maximal tyre-road friction parameter) From the truncated Taylor series expansion*

$$\phi(\lambda_{xi},\alpha_i,\mu_{Hi}) = \phi(\lambda_{xi},\alpha_i,\hat{\mu}_{Hi}) + \left.\frac{\partial\phi(\lambda_{xi},\alpha_i,s_i)}{\partial s_i}\right|_{s_i=\hat{\mu}_{Hi}} \tilde{\mu}_{Hi} + \left.\frac{\partial^2\phi(\lambda_{xi},\alpha_i,s_i)}{\partial s_i^2}\right|_{s_i=\hat{\mu}_{Hi}+\varsigma\tilde{\mu}_{Hi}} \tilde{\mu}_{Hi}^2,$$

where $\tilde{\mu}_{Hi} := \mu_{Hi} - \hat{\mu}_{Hi}$ *and* $1 > \varsigma > 0$, *and the function*

$$\bar{\phi}(\lambda_{xi},\alpha_i,\hat{\mu}_{Hi}) := \left.\frac{\partial\phi(\lambda_{xi},\alpha_i,s_i)}{\partial s_i}\right|_{s_i=\hat{\mu}_{Hi}},$$

it follows by Assumption 6.1 b) and 6.2, that

$$\bar{\phi}(\lambda_{xi},\alpha_i,\hat{\mu}_{Hi})\tilde{\mu}_{Hi} \approx \phi(\lambda_{xi},\alpha_i,\mu_{Hi}) - \phi(\lambda_{xi},\alpha_i,\hat{\mu}_{Hi}).$$

This approximation allows a control strategy based on the adaptive control allocation design presented in Chapter 5, but it is not necessary in order to conclude robust stability of the low level controller, see Remark 6.2.

Moreover, the following properties are incorporated in the longitudinal wheel slip dynamics:

Property 6.1 *(Properties of the wheel slip dynamics)*

a) *Let* ν_{xi}, ν_{yi} *and* ω_i *be treated as time-varying signals* $\nu_{xi}(t)$, $\nu_{yi}(t)$ *and* $\omega_i(t)$, *then if Assumption 6.2 is satisfied, the equilibrium* $\lambda_{xi} = 0$ *of the unforced system (6.17) is asymptotically stable. The same is true for the approximated system*

$$\dot{\lambda}_{xi} = -\frac{R}{\nu_i J_\omega}\left(\phi(\lambda_{xi},\alpha_i,\hat{\mu}_{Hi}) + \bar{\phi}(\lambda_{xi},\alpha_i,\hat{\mu}_{Hi})\tilde{\mu}_{Hi} - T_{bi}\right) \tag{6.19}$$

b) *For any* $|\lambda_{xri}| \leq \lambda_{Dri}$ *(defined in Assumption 6.1) and any fixed* ω_i, α_i, ν_i *and* ν_{xi} *satisfying Assumption 6.2, there exists a* T_{bi} *such that* $\lambda_{xi} = \lambda_{xri}$ *is an unique equilibrium point of the system (6.17) and the system (6.19).*

Property 1 b) is useful in defining the constraints on the references in the dynamic control allocation problem.

6.4.3 Low level wheel slip control

In the control synthesis, the low level system is described by the longitudinal wheel slip dynamic such that slip control laws developed for anti-lock brake systems (ABS) may be used. When the ABS is active, its main purpose is to maximize the friction force between the road and the tyres. For any given μ_{Hi} and α_i these maximal forces are uniquely defined by the fixed desired longitudinal wheel slip. In a yaw stabilizing scheme the desired longitudinal wheel slip is not fixed and a reference tracking controller is needed.

The low level control objective is based on the longitudinal wheel slip error dynamics derived from the approximation (6.19):

$$\dot{\tilde{\lambda}}_{xi} = -\frac{R}{\nu_i J_z}\left(\phi(\lambda_{xi}, \alpha_i, \hat{\mu}_{Hi}) + \bar{\phi}(\lambda_{xi}, \alpha_i, \hat{\mu}_{Hi})\tilde{\mu}_{Hi} - T_{bi}\right) - \dot{\lambda}_{xid}, \quad (6.20)$$

where $\tilde{\lambda}_{xi} := \lambda_{xi} - \lambda_{xid}$. Many different wheel slip control strategies can be found in the literature; see for example Lyapunov based solutions Freeman [1995], Yu [1997] and Johansen et al. [2003b], sliding-mode based controllers Choi and Cho [1998], Wu and Shih [2001] and Schinkel and Hunt [2002] and PID approaches Taheri and Law [1991], Jiang [2000] and Solyom and Rantzer [2002]. Here we consider a standard Lyapunov approach with adaptation and feed forward which allows a straight forward use of the dynamic control allocation result from Tjønnås and Johansen [2007a].

When μ_{Hi} is not known, the following adaptive law and certainty equivalent control law are suggested:

$$\dot{\hat{\mu}}_{Hi} = -\gamma_{\mu i}^{-1}\frac{R}{\nu_i J_z}\bar{\phi}(\lambda_{xi}, \alpha_i, \hat{\mu}_{Hi})\left(\tilde{\lambda}_{xi} + \gamma_{\bar{\lambda}i}\bar{\lambda}_{xi}\right) \quad (6.21)$$

$$T_{bi} := \phi(\lambda_{xi}, \alpha_i, \hat{\mu}_{Hi}) + \frac{\nu_i J_z}{R}\dot{\lambda}_{xid} - \frac{\nu_i J_z}{R}\Gamma_P\tilde{\lambda}_{xi}, \quad (6.22)$$

where $\bar{\lambda}_{xi} := \lambda_{xi} - \hat{\lambda}_{xi}$ and

$$\dot{\hat{\lambda}}_{xi} = A\left(\lambda_{xi} - \hat{\lambda}_{xi}\right) + \frac{R}{\nu J_z}\left(T_{bi} - \phi(\lambda_{xi}, \alpha_i, \hat{\mu}_{Hi})\right). \quad (6.23)$$

Proposition 6.1 *Let Assumption 6.1 and 6.2 be satisfied, then the equilibrium* $(\tilde{\lambda}_{xi}, \bar{\lambda}_{xi}, \tilde{\mu}_{Hi}) = 0$ *for the system (6.20), (6.21) and (6.23) and the controller (6.22) is US. If in addition* $\bar{\Phi}_i(t) := \frac{R}{\nu_i(t)J_z}\bar{\phi}(\lambda_{xi}(t), \alpha_i(t), \hat{\mu}_{Hi}(t))$ *is persistently excited, i.e. there exist constants* T *and* $\gamma > 0$, *such that*

$$\int_t^{t+T}\bar{\Phi}_i(\tau)^T\bar{\Phi}_i(\tau)d\tau \geq \gamma I \ , \quad \forall t > t_0,$$

then the equilibrium $(\tilde{\lambda}_{xi}, \bar{\lambda}_{xi}, \tilde{\mu}_{Hi}) = 0$ *is UAS with respect to the system (6.20), (6.21) and (6.23) and the controller (6.22).*
Proof. *Consider the quadratic Lyapunov function*

$$V_{\lambda i}(\tilde{\lambda}_{xi}, \bar{\lambda}_{xi}, \tilde{\mu}_{Hi}) := \frac{1}{2}\left(\tilde{\lambda}_{xi}^2 + \gamma_{\bar{\lambda}i}\bar{\lambda}_{xi}^2 + \gamma_{\mu i}\tilde{\mu}_{Hi}^2\right),$$

then along the trajectories of (6.20), (6.21) and (6.23):

$$\begin{aligned}
\dot{V}_{\lambda i} =& -\tilde{\lambda}_{xi}\Gamma_P\tilde{\lambda}_{xi} - \tilde{\lambda}_{xi}\frac{R}{\nu_i J_z}\bar{\phi}(\lambda_{xi}, \alpha_i, \hat{\mu}_{Hi})\tilde{\mu}_H \\
& - \gamma_{\bar{\lambda}i}\bar{\lambda}_{xi}\frac{R}{\nu_i J_z}\bar{\phi}(\lambda_{xi}, \alpha_i, \hat{\mu}_{Hi})\tilde{\mu}_H - \gamma_{\mu i}\tilde{\mu}_H\dot{\hat{\mu}}_{Hi} \\
& - \gamma_{\bar{\lambda}i}\bar{\lambda}_{xi}A\bar{\lambda}_{xi} \\
=& -\tilde{\lambda}_{xi}\Gamma_P\tilde{\lambda}_{xi} - \gamma_{\bar{\lambda}i}\bar{\lambda}_{xi}A\bar{\lambda}_{xi}.
\end{aligned}$$

and uniform stability is achieved. The asymptotic stability proof relies on an extension of Matrosov's theorem, Theorem 1 in Loria et al. [2005] given in Appendix A.4. The result is used to prove that if $\bar{\Phi}(\tau)$ *is PE then the origin* $(\tilde{\lambda}_{xi}, \bar{\lambda}_{xi}, \tilde{\mu}_{Hi}) = 0$ *is UAS for system (6.20), (6.23) and 6.21. As US is already confirmed, we only need to find some bounded auxiliary functions that will satisfy the Assumptions A.2-A.4. Choose:*

$$V_{aux1} := -\bar{\lambda}_{xi}\bar{\Phi}_i(t)\tilde{\mu}_{Hi}$$

then

$$\begin{aligned}
\dot{V}_{aux1} &= -\dot{\bar{\lambda}}_{xi}\bar{\Phi}_g(t)\tilde{\mu}_{Hi} - \bar{\lambda}_{xi}\frac{d\left(\bar{\Phi}_i(t)\tilde{\mu}_{Hi}\right)}{dt} \\
&= -\left(-A\bar{\lambda}_{xi} + \bar{\Phi}_i(t)\tilde{\mu}_{Hi}\right)^T\bar{\Phi}_i(t)\tilde{\mu}_{Hi} - \bar{\lambda}_{xi}\frac{d\left(\bar{\Phi}_i(t)\tilde{\mu}_{Hi}\right)}{dt} \\
&= -\tilde{\mu}_{Hi}\bar{\Phi}_i(t)^T\bar{\Phi}_i(t)\tilde{\mu}_{Hi} + \bar{\lambda}_{xi}A^T\bar{\Phi}_i(t)\tilde{\mu}_{Hi} - \bar{\lambda}_{xi}\frac{d\left(\bar{\Phi}_i(t)\tilde{\mu}_{Hi}\right)}{dt} \\
&\leq -\tilde{\mu}_{Hi}^T\bar{\Phi}_i(t)^T\bar{\Phi}_i(t)\tilde{\mu}_{Hi} + \bar{\lambda}_{xi}\gamma_{aux1}(t, \bar{\lambda}_{xi}, \tilde{\mu}_{Hi})
\end{aligned} \quad (6.24)$$

where $\gamma_{aux1}(t, \bar{\lambda}_{xi}, \tilde{\mu}_{Hi}) := \bar{\lambda}_{xi}A^T\bar{\Phi}_i(t)\tilde{\mu}_{Hi} - \bar{\lambda}_{xi}\frac{d\left(\bar{\Phi}_i(t)\tilde{\mu}_{Hi}\right)}{dt}$*. Furthermore, choose*

$$V_{aux2} := -\int_t^\infty \tilde{\mu}_{Hi}\bar{\Phi}_i(\tau)^T\bar{\Phi}_i(\tau)\tilde{\mu}_{Hi}e^{-(\tau-t)}d\tau$$

then

$$\begin{aligned}\dot{V}_{aux2} =& -\frac{\partial}{\partial t}\left(e^t\int_t^\infty \tilde{\mu}_{Hi}\bar{\Phi}_i(\tau)^T\bar{\Phi}_i(\tau)\tilde{\mu}_{Hi}e^{-\tau}d\tau\right)+\frac{\partial V_{aux}}{\partial\tilde{\mu}_{Hi}}\dot{\tilde{\mu}}_{Hi}\\ =& -e^t\int_t^\infty \tilde{\mu}_{Hi}\bar{\Phi}_i(\tau)^T\bar{\Phi}_i(\tau)\tilde{\mu}_{Hi}e^{-\tau}d\tau+\frac{\partial V_{aux}}{\partial\tilde{\mu}_{Hi}}\dot{\tilde{\mu}}_{Hi}\\ &-e^t\frac{\partial}{\partial t}\left(\int_t^\infty \tilde{\mu}_{Hi}\bar{\Phi}_i(\tau)^T\bar{\Phi}_i(\tau)\tilde{\mu}_{Hi}e^{-\tau}d\tau\right)\\ =&V_{aux2}+\tilde{\mu}_{Hi}^T\bar{\Phi}_i(t)^T\bar{\Phi}_i(t)\tilde{\mu}_{Hi}-\frac{\partial V_{aux}}{\partial\tilde{\mu}_{Hi}}\dot{\tilde{\mu}}_{Hi}. \end{aligned} \tag{6.25}$$

Define

$$\begin{aligned} Y_1(\tilde{\lambda}_{xi},\bar{\lambda}_{xi}) &:= \Gamma_P\tilde{\lambda}_{xi}^2 + A\bar{\lambda}_{xi}^2\\ Y_2(\tilde{\lambda}_{xi},\xi(t,\tilde{\lambda}_{xi},\tilde{\mu}_{Hi})) &:= -\xi_1(t,\tilde{\lambda}_{xi},\tilde{\mu}_{Hi})+\bar{\lambda}_{xi}\xi_2(t,\bar{\lambda}_{xi},\tilde{\mu}_{Hi})\\ Y_3(\tilde{\lambda}_{xi},\bar{\lambda}_{xi},\xi(t,\tilde{\lambda}_{xi},\tilde{\mu}_{Hi})) &:= V_{aux2}+\xi_1(t,\tilde{\lambda}_{xi},\tilde{\mu}_{Hi})-\xi_1(t,\tilde{\lambda}_{xi},\tilde{\mu}_{Hi}) \end{aligned}$$

where

$$\xi(t,\tilde{\lambda}_{xi},\bar{\lambda}_{xi},\tilde{\mu}_{Hi}) := \left[\left(\tilde{\mu}_{Hi}\bar{\Phi}_g(t)^T\bar{\Phi}_g(t)\tilde{\mu}_{Hi}\right)^T, \gamma_{aux1}(t,\bar{\lambda}_{xi},\tilde{\mu}_{Hi}), \frac{\partial V_{aux}}{\partial\tilde{\mu}_{Hi}}\dot{\tilde{\mu}}_{Hi}\right].$$

With $V_{\lambda i}$,V_{aux1}, V_{aux2}, Y_1, Y_2, Y_3 and $\xi(t,\tilde{\lambda}_{xi},\bar{\lambda}_{xi},\tilde{\mu}_{Hi})$ the Assumptions A.3 and A.4 are satisfied. Furthermore by Assumption 6.1 and 6.2, the Assumption A.2 is also satisfied, and the result is proved ■

Remark 6.2 *Since the equilibrium of system (6.20), (6.21) and (6.23), controlled by (6.22) is UAS, arguments for local robust stability can be made, see Loria et al. [2002] and similar arguments in Grip et al. [2006] and Imsland et al. [2006]. Moreover, without considering Approximation 2, it can be shown that the controller (6.22) render the equilibrium $(\tilde{\lambda}_{xi},\bar{\lambda}_{xi},\tilde{\mu}_{Hi}) = 0$, Uniformly Exponentially Stable (UES) with respect to system (6.20), (6.21) and (6.23) where (6.20) is based on (6.17), see Appendix A.3. This suggests that Approximation 2 may not be necessary for the closed loop-system (the high and low level systems connected through the allocation algorithm). But although arguments similar to the ones used in the analysis of the low level control, excluding Approximation 2, may apply to the analysis of the allocation algorithm, they are more involved and at the moment not theoretically founded.*

Remark 6.3 *As long as* $\frac{\partial \phi(\lambda_{xi},\alpha_i,s_i)}{\partial s_i} > 0$, *a more general class of adaptive update laws*

$$\dot{\hat{\mu}}_{Hi} = \gamma_{\tilde{\lambda}i}(t)\tilde{\lambda}_{xi} + \gamma_{\bar{\lambda}i}(t)\bar{\lambda}_{xi},$$

where $\gamma_{\tilde{\lambda}i}(t)$, $\gamma_{\bar{\lambda}i}(t) > 0 \forall t$, *may be suggested. This can be seen by applying* $\phi(\lambda_{xi},\alpha_i,\mu_{Hi}) - \phi(\lambda_{xi},\alpha_i,\hat{\mu}_{Hi}) = \left.\frac{\partial \phi(\lambda_{xi},\alpha_i,s_i)}{\partial s_i}\right|_{s_i=\hat{\mu}_{Hi}+\varsigma\tilde{\mu}_{Hi}} \tilde{\mu}_{Hi}$, *from the* mean value theorem, *in the analysis. With this algorithm the Approximation 6.2 is not necessary, but one may need to require technical bounds on* $\dot{\gamma}_{\tilde{\lambda}i}(t)$, $\dot{\gamma}_{\bar{\lambda}i}(t)$ *and* $\frac{d}{dt}\left.\frac{\partial \phi(\lambda_{xi},\alpha_i,s_i)}{\partial s_i}\right|_{s_i=\hat{\mu}_{Hi}+\varsigma\tilde{\mu}_{Hi}}$, *which again leads to enforcing bounds on* T_{bi}. *A way to deal with this may be to introduce a more sophisticated approach with an anti-windup algorithm.*

6.5 The control allocation algorithm

We have chosen a fairly straight forward Lyapunov based design both for the high level yaw stabilizing controller and the low level longitudinal wheel slip controller. Other controllers may be applied as long as uniform asymptotic stability of the respective systems equilibrium is achieved. The performance of the high and low level controllers are crucial in order to solve the yaw stabilizing problem, but the main contribution of this work is to show how the dynamic control allocation algorithm may be used for coordinating the actuators without the use of additional optimizing software.

The static mapping from the tyre force (low level system) to the yaw moment (high level system) can be found in Kiencke and Nielsen [2000] and is given by:

$$\begin{pmatrix} f_x \\ f_y \end{pmatrix} := \sum_{i=1}^{4} \begin{pmatrix} \cos(\delta_i) & -\sin(\delta_i) \\ \sin(\delta_i) & \cos(\delta_i) \end{pmatrix} \begin{pmatrix} F_{xi} \\ F_{yi} \end{pmatrix} \tag{6.26}$$

$$M = \Phi_M(\delta, \lambda_x, \alpha, \mu_H) \tag{6.27}$$

$$\Phi_M := \sum_{i=1}^{4} \begin{pmatrix} -\sin(\rho_i) \\ \cos(\rho_i) \end{pmatrix}^{\mathrm{T}} h_i \begin{pmatrix} \cos(\delta_i) & -\sin(\delta_i) \\ \sin(\delta_i) & \cos(\delta_i) \end{pmatrix} \begin{pmatrix} F_{xi} \\ F_{yi} \end{pmatrix}, \tag{6.28}$$

where the parameters are described in Table 6.1.

The basis of the control allocation algorithm lies in finding update laws for the longitudinal wheel slip (λ_{xid}) and the steering angle correction ($\Delta\delta_u$) that asymptotically solves the optimization problem (6.1).

The instantaneous cost function is divided into two parts, $J(u_d) := J_1(u_d) + J_2(u_d)$, where the function $J_1(u_d)$ represents the actuator penalty and the $J_2(u_d)$ is a barrier function representation of the actuator constraints.

$$J_1(u_d) := u_d^{\mathrm{T}} W_u u_d \tag{6.29}$$

$$\begin{aligned} J_2(u_d) := & -w_\lambda \sum_{i=1}^{4} \ln(\lambda_{xid} - \lambda_{x\min}) - w_\lambda \sum_{i=1}^{4} \ln(-\lambda_{xid} + \lambda_{x\max}) \\ & -w_\delta \sum_{i=1}^{2} \ln(\Delta\delta_i - \Delta\delta_{\min}) - w_\delta \sum_{i=1}^{2} \ln(-\Delta\delta_i + \Delta\delta_{\max}) \end{aligned} \tag{6.30}$$

where $\lambda_{x\max} = \lambda_{Dri}$, $\lambda_{x\min}$, $\Delta\delta_{\max}$, and $\Delta\delta_{\min}$ are wheel-slip and steering angle manipulation constraints. W_u is a positive definite weighting matrix, while w_λ and w_δ are positive parameters.

Let the high level dynamics be defined by $x := (e_r, \tilde{r})^{\mathrm{T}}$, $g := (0, \frac{1}{J_z})^{\mathrm{T}}$ and $f(t,x) := (\tilde{r}, -K_p\tilde{r} - K_i e_r)^{\mathrm{T}}$. Then based on; the Lagrangian function (6.2), the high level control law (6.7), the adaptive law (6.21) and the certainty equivalent low level control law (6.22), the following certainty equivalent dynamic control allocation algorithm is constructed with reference to Chapter 5:

$$\left(\dot{u}_d^{\mathrm{T}}, \dot{\pi}^{\mathrm{T}}\right)^{\mathrm{T}} = -\gamma(t)\left(\mathbb{H}^{\mathrm{T}} W_\pi \mathbb{H}\right)^{-1} \mathbb{H} \left(\frac{\partial L}{\partial u_d}^{\mathrm{T}}, \frac{\partial L}{\partial \pi}^{\mathrm{T}}\right)^{\mathrm{T}} - \mathbb{H}^{-1} u_{dff} \tag{6.31}$$

$$\dot{\hat{\mu}}_H = -\Gamma_\mu^{-1} \bar{\Phi}^{\mathrm{T}}(\lambda_x, \alpha, \mu_H, \nu)\left(\tilde{\lambda}_{xi} + \Gamma_{\bar{\lambda}}^{-1} \bar{\lambda}_{xi}\right) + \Gamma_\mu^{-1} \bar{\Phi}_M^{\mathrm{T}}(\delta, \lambda_x, \alpha, \hat{\mu}_H) \frac{\partial^2 L^{\mathrm{T}}}{\partial x \partial \pi} \frac{\partial L}{\partial \pi}, \tag{6.32}$$

where the feed-forward like term is given by:

$$\begin{aligned} u_{dff} := & \left(\frac{\partial^2 L}{\partial t \partial u_d}^{\mathrm{T}}, \frac{\partial^2 L}{\partial t \partial \pi}^{\mathrm{T}}\right)^{\mathrm{T}} + \left(\frac{\partial^2 L}{\partial x \partial u_d}^{\mathrm{T}}, \frac{\partial^2 L}{\partial x \partial \pi}^{\mathrm{T}}\right)^{\mathrm{T}} f(t,x) \\ & + \left(\frac{\partial^2 L^{\mathrm{T}}}{\partial x \partial u_d}, \frac{\partial^2 L^{\mathrm{T}}}{\partial x \partial \pi}\right)^{\mathrm{T}} g \frac{1}{J_z} \bar{M} + \left(\frac{\partial^2 L^{\mathrm{T}}}{\partial \tilde{u} \partial u_d}, \frac{\partial^2 L^{\mathrm{T}}}{\partial \tilde{u} \partial \pi}\right)^{\mathrm{T}} \Gamma_P \tilde{\lambda}_{xi} \\ & + \left(\frac{\partial^2 L^{\mathrm{T}}}{\partial \hat{\mu}_H \partial u_d}, \frac{\partial^2 L^{\mathrm{T}}}{\partial \hat{\mu}_H \partial \pi}\right)^{\mathrm{T}} \Gamma_\mu^{-1}\left(\tilde{\lambda}_{xi} + \Gamma_{\bar{\lambda}}^{-1} \bar{\lambda}_{xi}\right) \bar{\Phi}(\lambda_x, \alpha, \mu_H). \end{aligned} \tag{6.33}$$

Furthermore, $\bar{\Phi}(\lambda_x, \alpha, \hat{\mu}_H) := \frac{R}{\nu_i J_z} diag(\bar{\phi}(\lambda_{xi}, \alpha_i, \hat{\mu}_{Hi}))$, $\bar{\Phi}_M(\delta, \lambda_x, \alpha, \hat{\mu}_H) := \frac{\partial \Phi_M(\delta, \lambda_x, \alpha, \hat{\mu}_H)}{\partial \hat{\mu}_H}$ and $\mathbb{H} := \begin{pmatrix} \frac{\partial^2 L}{\partial u_d^2} & \frac{\partial^2 L}{\partial \pi \partial u_d} \\ \frac{\partial^2 L}{\partial u_d \partial \pi} & 0 \end{pmatrix}$. The gain matrices are defined as $\Gamma_{\mu}^{-1} := diag(\gamma_{\mu i}^{-1})$, $\Gamma_{\bar{\lambda}}^{-1} := diag(\gamma_{\bar{\lambda} i}^{-1})$, $W_{\pi} > 0$ and $\gamma(t) = \gamma^{\mathrm{T}}(t)$. $\gamma(t)$ is a positive definite time-varying weighing matrix that ensures numerical feasibility. Based on the cost function choice, (6.29) and (6.30), singularity of $\mathbb{H}$ can be avoided for bounded π and appropriate choices of W_u and W_{π}.

Remark 6.4 *The first part of the algorithm (6.32) represents a typical term arriving from the Lyapunov design. And the remaining part of the update law (u_{dff}) is a feed-forward like term that is designed to track the time-varying equilibrium.*

Let the optimal equilibrium set of the closed loop (6.8), (6.9), (6.20), (6.23), (6.21) and (6.31) be defined by:

$$\mathcal{O}_{xu_d\pi\mu_H}(t) := \{x = 0\} \times \mathcal{O}_{u_d\pi\mu_H}(t),$$

where

$$\mathcal{O}_{u_d\pi\mu_H}(t) := \left\{ \left(u_d^{\mathrm{T}}, \tilde{\lambda}_x^{\mathrm{T}}, \pi, \bar{\lambda}_x^{\mathrm{T}}, \tilde{\mu}_H^{\mathrm{T}} \right) \in \mathbb{R}^{18} \middle| \left(\frac{\partial L^{\mathrm{T}}}{\partial u_d}, \frac{\partial L^{\mathrm{T}}}{\partial \pi}, \tilde{\lambda}_x^{\mathrm{T}} \bar{\lambda}_x^{\mathrm{T}} \tilde{\mu}_H^{\mathrm{T}} \right) = 0 \right\}.$$

Proposition 6.2 *Consider the Assumptions 6.1 and 6.2, the system (6.3), (6.19) and (6.27), and the control and allocation laws (6.7), (6.22) and (6.31), then the set $\mathcal{O}_{xu_d\pi\mu_H}(t)$ is UAS with respect to the system (6.8), (6.9), (6.20), (6.23), (6.21) and (6.31).*

Proof. *The proof follows by noticing that with; the control laws (6.7), (6.22), the control allocation law (6.31), the estimator (6.32), the cost function design (6.29)-(6.30) and the Assumptions 6.1-6.2, then the Assumptions 2.1, 2.2 and 2.3 from Chapter 2 are satisfied locally, and the Corollary 5.1 in Chapter 5 can be used* ■

From Proposition 6.2 we have a local solution where u converges asymptotically to u_d, u_d converges asymptotically to the optimal solution of problem (6.1) and $\hat{\mu}_H$ converges asymptotically to μ_H, such that M converges asymptotically to M_c and finally r converges to r_{ref} in a stable sense, which is the main goal of this work. Furthermore, this result applies for the special cases where the number of controlled actuators is reduced.

Consider the following sets:

$$\mathcal{O}_{x\lambda\pi\mu_H}(t) := \{x = 0\} \times \mathcal{O}_{\lambda\pi\mu_H}(t),$$

where

$$\mathcal{O}_{\lambda\pi\mu_H}(t) := \left\{ \left(\lambda_{xd}^{\mathrm{T}}, \tilde{\lambda}_x^{\mathrm{T}}, \pi, \bar{\lambda}_x^{\mathrm{T}}, \tilde{\mu}_H^{\mathrm{T}} \right) \in \mathbb{R}^{17} \middle| \left(\frac{\partial L^{\mathrm{T}}}{\partial \lambda_{xd}}, \frac{\partial L^{\mathrm{T}}}{\partial \pi}, \tilde{u}^{\mathrm{T}}, \bar{\lambda}_x^{\mathrm{T}}, \tilde{\mu}_H^{\mathrm{T}} \right) = 0 \right\},$$

and

$$\mathcal{O}_{x\delta\pi}(t) := \{x = 0\} \times \left\{ (\Delta\delta_u, \pi) \in \mathbb{R}^2 \,\middle|\, \left(\frac{\partial L^{\mathrm{T}}}{\partial \Delta\delta_u}, \frac{\partial L^{\mathrm{T}}}{\partial \pi} \right) = 0 \right\},$$

then the following results hold.

Corollary 6.1 *With the same argument as in Proposition 6.2 if:*

- $u := \lambda_x$ *(only braking actuators available), the set* $\mathcal{O}_{x\lambda\pi\mu_H}(t)$ *is UAS with respect to the system (6.8), (6.9), (6.20), (6.23), (6.21) and (6.31).*
- $u := \Delta\delta_u$ *(only steering actuator available), the set* $\mathcal{O}_{x\delta\pi}(t)$ *is UAS with respect to the system (6.8), (6.9), and (6.31).*

6.6 Simulation results

In order to validate the yaw stabilization scheme, a test-bench based on DaimlerChrysler's proprietary multi-body simulation environment CASCaDE (Computer Aided Simulation of Car, Driver and Environment) for MATLAB, is considered. Three simulation scenarios are investigated:

- **Understeering:** The understeered motion is a "stable" but not desired yaw motion of the vehicle. It is defined by the yaw rate being less then the desired yaw rate i.e. $|r| < |r_{ref}|$. Typically such behavior appear on low friction surfaces during relatively fast but limited steering maneuvers. In Figure 6.6 the initial conditions and the behavior of the uncontrolled periodically understeered vehicle is presented undergoing a sinusoidal steering manoeuvre on a surface with the maximal tyre-road friction $\mu_{Hi} = 0.3$ for $i \in \{1, 2, 3, 4\}$.
- **Oversteering:** The oversteered motion is an "unstable" vehicle motion, where the yaw rate is greater then the desired yaw rate i.e. $|r| > |r_{ref}|$. In Figure 6.7 an uncontrolled oversteered manoeuvre is created by a slowly increasing left-going steering wheel motion. The maximal tyre-road friction is $\mu_{Hi} = 0.7$ for $i \in \{1, 2, 3, 4\}$.
- **Fishhook:** The Fishhook manoeuvre is motivated by a vehicle changing lanes. In this scenario, the maximal tyre-road friction is initially $\mu_{Hi} = 0.9$ for $i \in \{1, 2, 3, 4\}$ (dry asphalt), but at ca. $1.6s$ the maximal tyre-road friction parameter changes to $\mu_{Hi} = 0.3$ for $i \in \{1, 2, 3, 4\}$

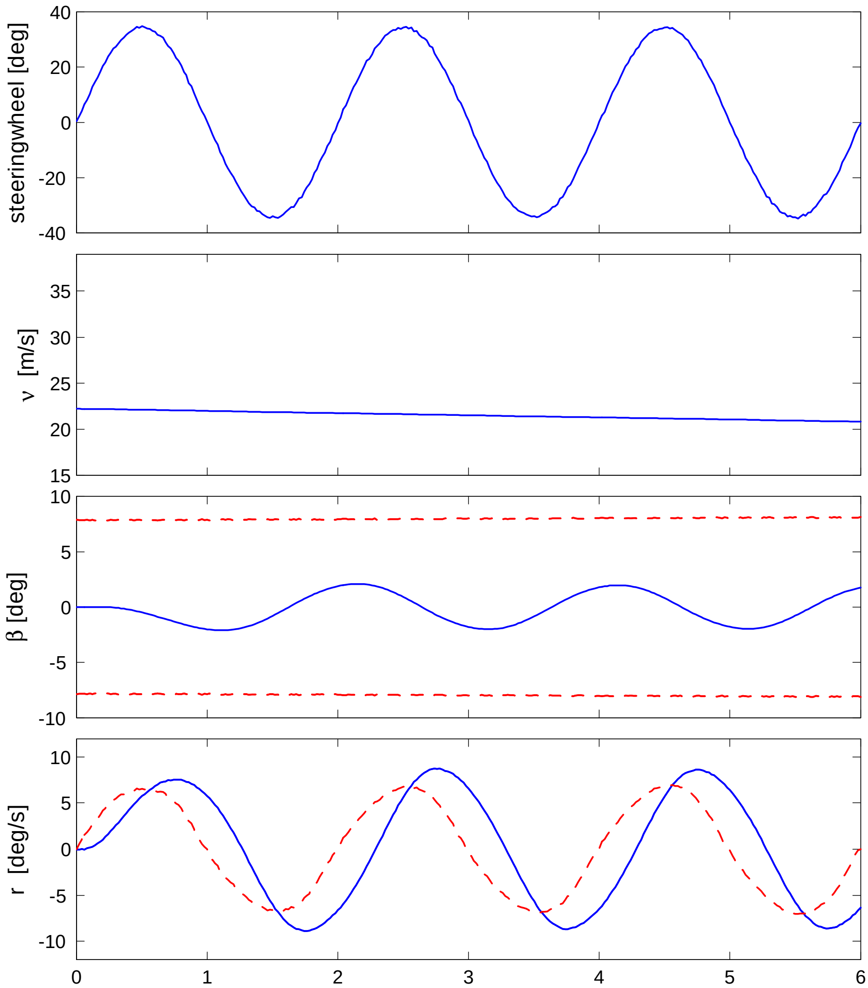

Figure 6.6: The steering manoeuvre, and states of a periodically understeered vehicle (no active yaw stabilization). The dashed lines are given by r_{ref} (6.4) and $\beta_{\max}$(6.5).

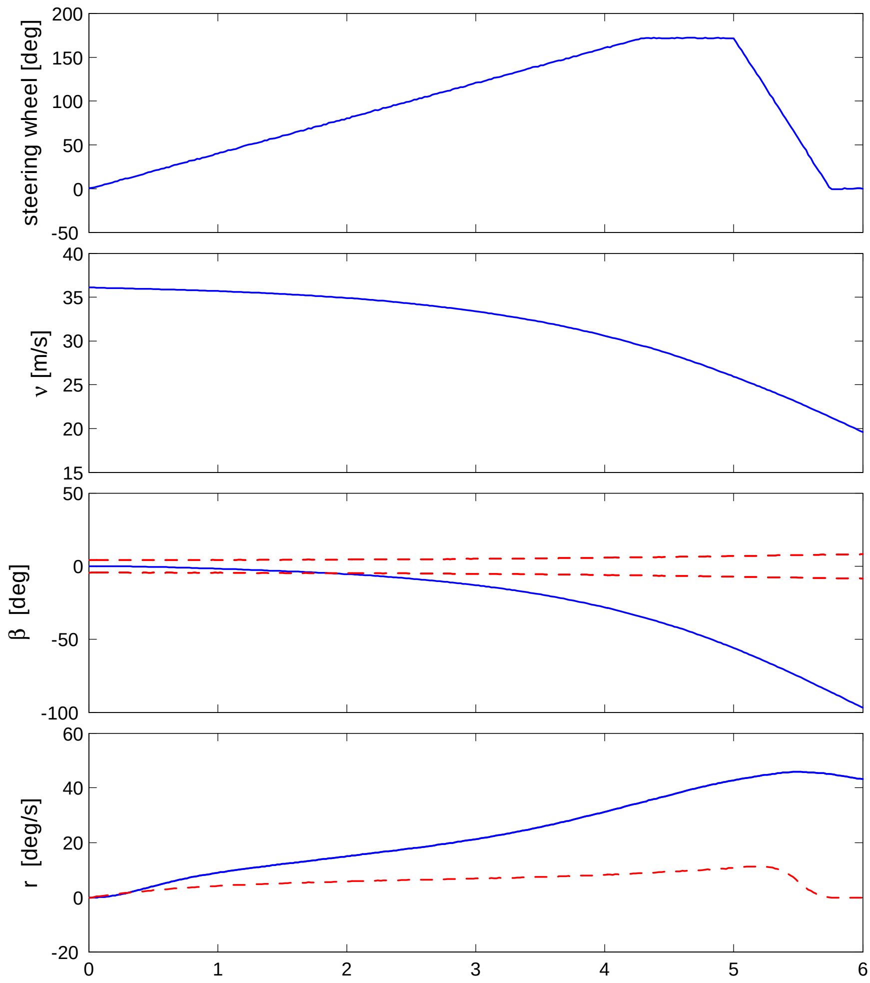

Figure 6.7: The steering manoeuvre, and states of an oversteered vehicle (no active yaw stabilization). The dashed lines are given by r_{ref} (6.4) and $\beta_{\max}$(6.5).

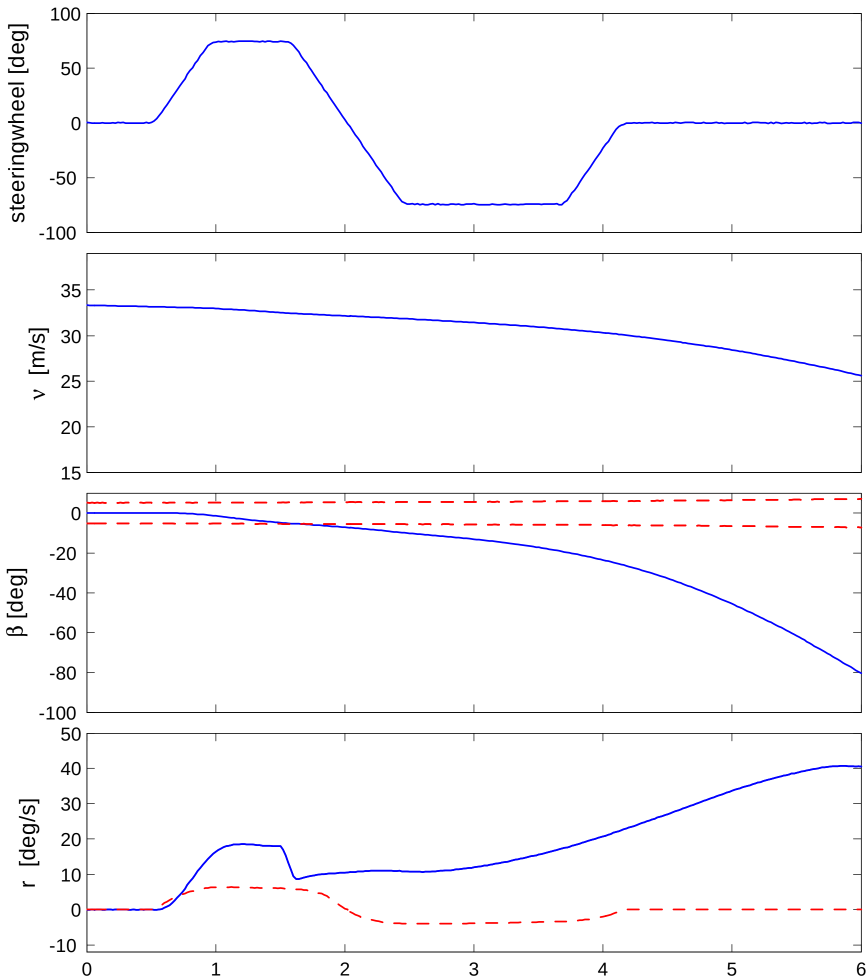

Figure 6.8: The fishhook steering manoeuvre and related vehicle states (no active yaw stabilization). The dashed lines are given by r_{ref} (6.4) and $\beta_{\max}$(6.5).

(wet/snowy surface). The initial conditions are given in the Figure 6.8.

The vehicle measurements and the yaw stabilization algorithm are operated with a $50Hz$ sampling rate. Moreover, the brake pressure commands are delayed with 0.02 seconds in order to simulate the effect of communication and computer delays, and dynamics of the actuators. The algorithm parameters used in the simulations are given by Table 6.2, where ε ensures singularity avoidance in (6.31), W_π is a weighting matrix penalizing the error of $\left|M_c - \bar{M}\right|$ against $\left|\frac{\partial L}{\partial \lambda_{xd}}\right|$, W_u defines the quadratic cost function of problem (6.1) and w_λ and w_δ defines the steepness of the barrier function (6.30). The control and adaptive algorithm parameters are tuned based on simulations. The longitudinal wheel slip constraints are chosen with respect to the system requirements (no wheel acceleration available) and Assumption 6.1. $\Delta\delta_{\max}$ and the fifth element of W_u may be driver style dependent parameters. A high value $\Delta\delta_{\max}$ and a low value of the fifth element of W_u compared with the remaining elements of W_u, indicate that most of the control effort will rely on the steering angle manipulation and result in a yaw stabilizing steer-by-wire algorithm. The steering wheel deflection is proportional to δ with a proportionality constant of approximately 20 (i.e if the steering wheel turns $20[deg]$, then $\delta = 1[deg]$).

The initial estimate of the maximal tyre-road friction parameter is $\hat{\mu}_{Hi} = 0.5$ for $i \in \{1, 2, 3, 4\}$ so that the adaptive mechanism of the algorithm is shown both for initial under and over estimation. The nominal maximal tyre road friction coefficient that describes the virtual reference surface (6.4) is given by $\mu_{nH} = 0.7$. In order to prevent the allocation update law (6.31) from generating infeasible actuator commands, due to the discretization of the system, γ is chosen to be a diagonal matrix where each element represents a positive step length that is made as small as necessary to achieve feasibility.

The CASCaDE model (see Figure 6.9) control inputs are defined through the steering angle (manipulated with $\Delta\delta$) and the desired brake force ($\frac{T_{bi}}{R}$), provided by the low level wheel slip controller. The input to the control algorithm is specified through the measurement block, where the vehicle states and variables are corrupted and delayed according to realistic vehicle sensor characteristics. In the figures and plots presented, only the real actual states are given.

The results from simulation of the three scenarios mentioned above, are shown in the Figures 6.10-6.12, 6.13-6.15 and 6.16-6.18, respectively.

Allocation tuning parameters	
W_π	$diag(1, 1, 1, 1, 5)$
W_u	$diag(3, 3, 3, 3, 7)10^3$
w_λ	0.1
w_δ	0.1
High and low level control parameters	
K_P	$15J_z$
K_I	$50J_z$
Γ_P	500
Actuator constraints	
$\lambda_{x\,\min}$	-10^{-4}
$\lambda_{x\,\max}$ (λ_{Dri})	$0.10 + 10^{-4}$
$\Delta\delta_{\max}$	2[deg]
Adaptive algorithm	
$\gamma_{\mu i}$	1
$\gamma_{\bar{\lambda} i}$	0.2
A	30
λ_{Axi}	0.006

Table 6.2: Algorithm parameters

From the plots it can be seen that the control objectives are satisfied, over- and understeering prevented and steerability conserved. It should be noted that since the steering angle is used actively, less control forces are commanded to the wheel brakes, which means that a controlled manoeuvre will have less impact on the absolute velocity, but at the same time, slow down the estimation of the maximal tyre-road friction parameter. Furthermore, the maximal tyre road friction parameters are adapted only when the measured longitudinal wheel slips are high enough. Moreover, the independence of the parameters estimates in the adaptive algorithm is shown from the maximal tyre road friction parameter estimation plot in the fish hook scenario (6.16), where the maximal tyre road friction parameters associated to the right side of the vehicle are estimated related to a surface with high friction, while the ones on the left side are related to a low friction surface. Also especifically note the transients at ca. $1.6s$ and $4.2s$ in yaw rate and the longitudinal wheel slip plots from Figure 6.16 and 6.17. The first transient is due to surface changes (see the scenario description), and the second occurs because the yaw stabilizing algorithm is switched off when there is no steering action.

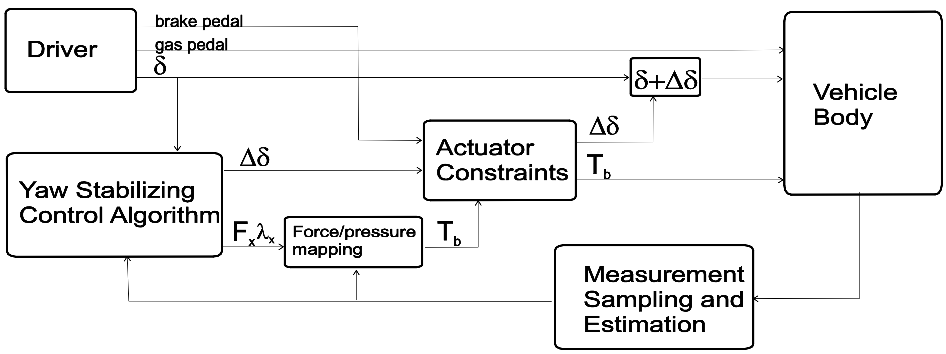

Figure 6.9: Block diagram of the simulation setup

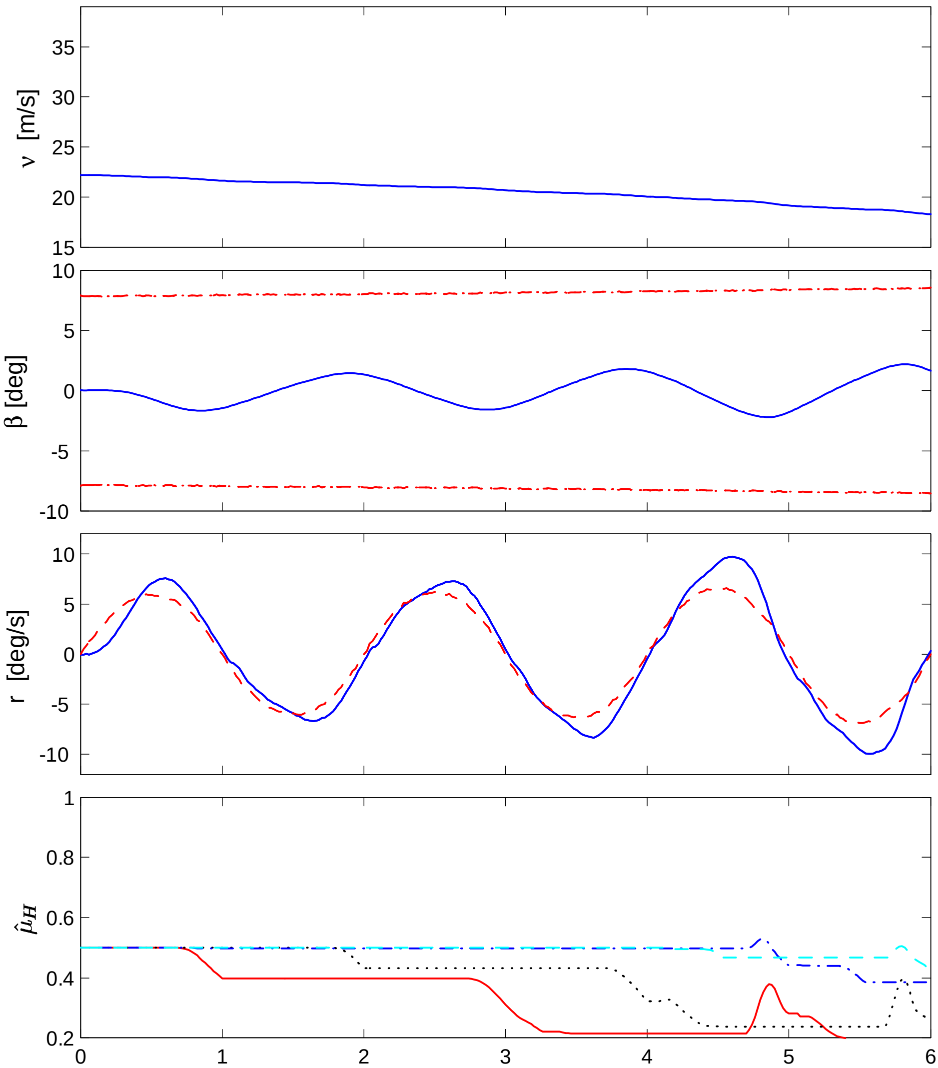

Figure 6.10: The understeering scenario with active yaw stabilization. System states: The dashed lines in plot 2 and 3 are given by $\beta_{\max}$ (6.4) and r_{ref} (6.4). In plot 4 the lines represent the following: solid-$\hat{\mu}_{H1}$, dotted-$\hat{\mu}_{H2}$, dashdot-$\hat{\mu}_{H3}$, dashed-$\hat{\mu}_{H4}$

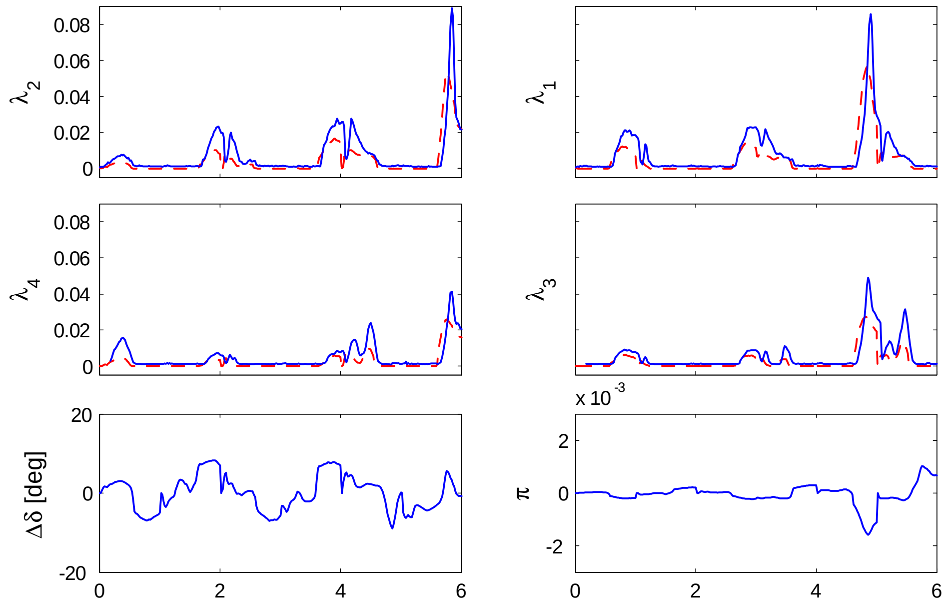

Figure 6.11: The understeering scenario with active yaw stabilization. The longitudinal wheel slip, the steering angle manipulation parameter and the Lagrangian parameter. The dotted lines denote the desired longitudinal wheel slip specified by the control allocator

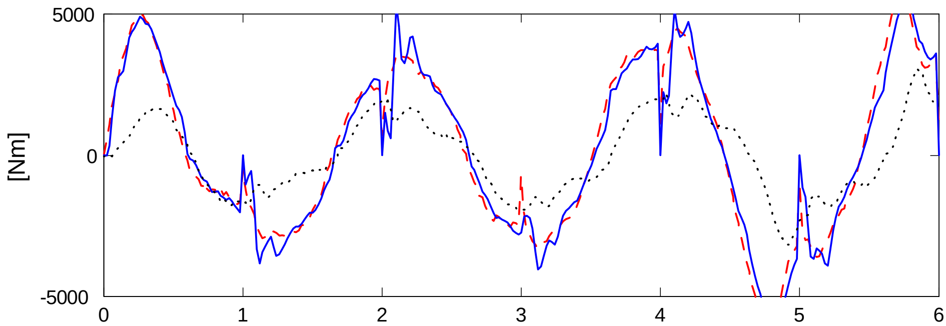

Figure 6.12: The understeering scenario with active yaw stabilization. Yaw torque: M_c-dashed, $M_{estimated}$-solid, M_{real}-dotted.

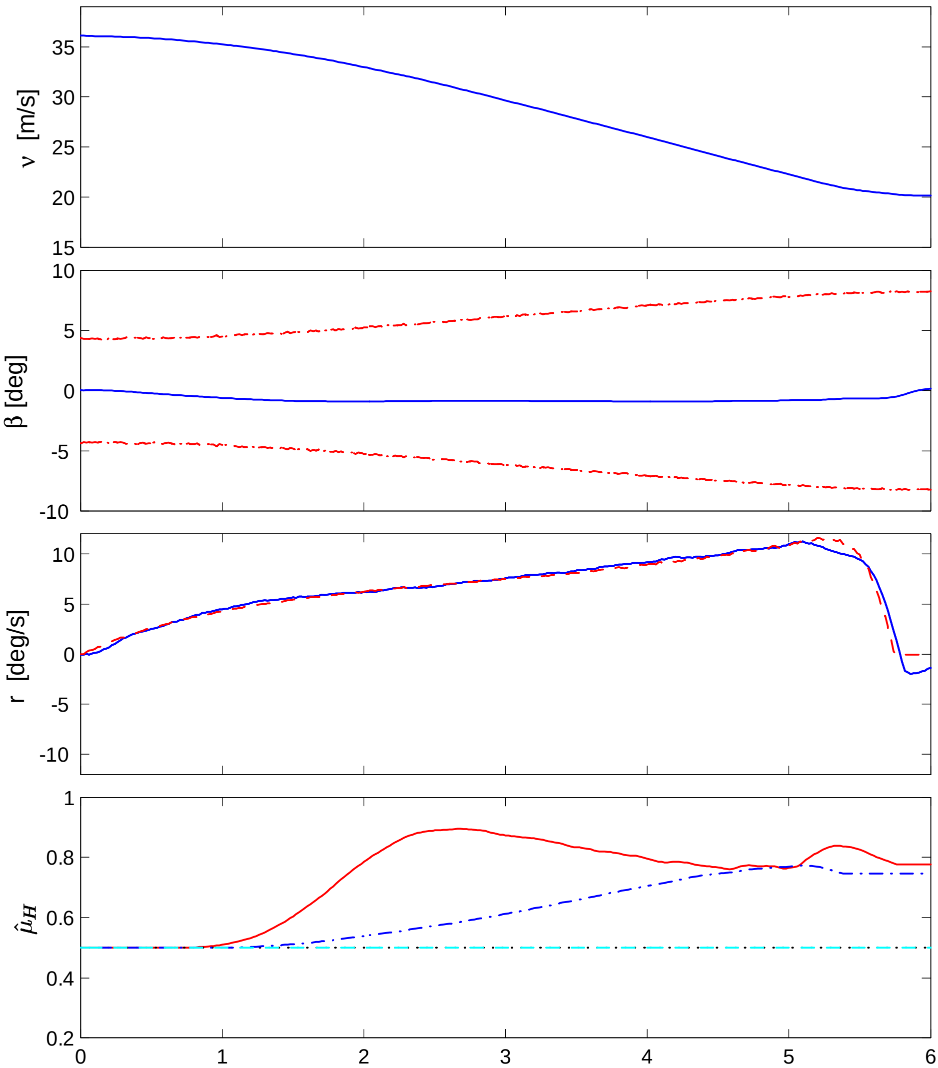

Figure 6.13: The oversteering scenario with active yaw stabilization. The dashed lines in plot 2 and 3 are given by $\beta_{\max}$ (6.4) and r_{ref} (6.4). In plot 4 the lines represent the following: solid-$\hat{\mu}_{H1}$, dotted-$\hat{\mu}_{H2}$, dashdot-$\hat{\mu}_{H3}$, dashed-$\hat{\mu}_{H4}$

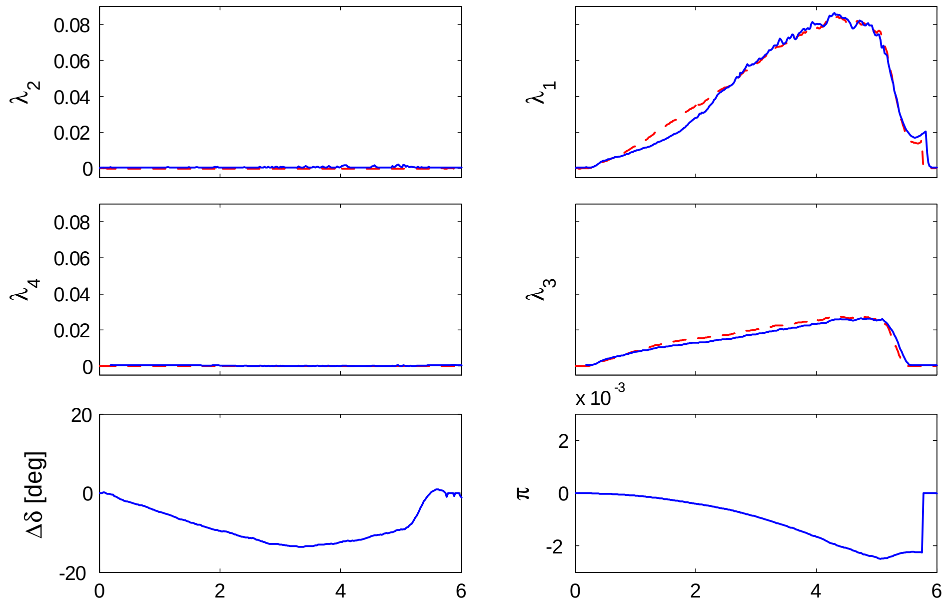

Figure 6.14: The oversteering scenario with active yaw stabilization. The longitudinal wheel slip, the steering angle manipulation parameter and Lagrangian parameter. The dotted lines denote the desired longitudinal wheel slip specified by the control allocator

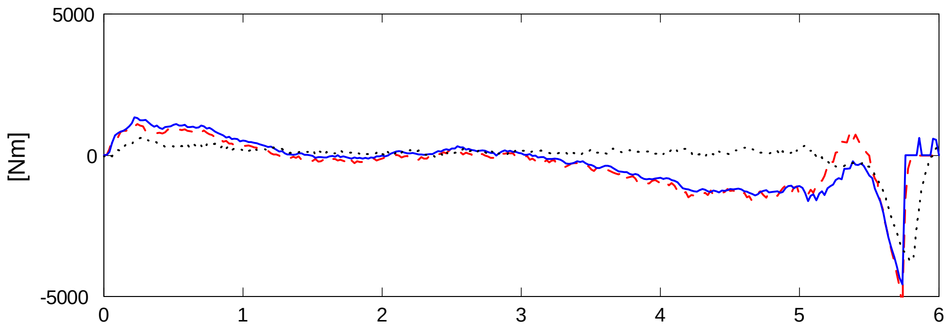

Figure 6.15: The oversteering scenario with active yaw stabilization. Yaw torque: M_c-dashed, $M_{estimated}$-solid, M_{real}-dotted.

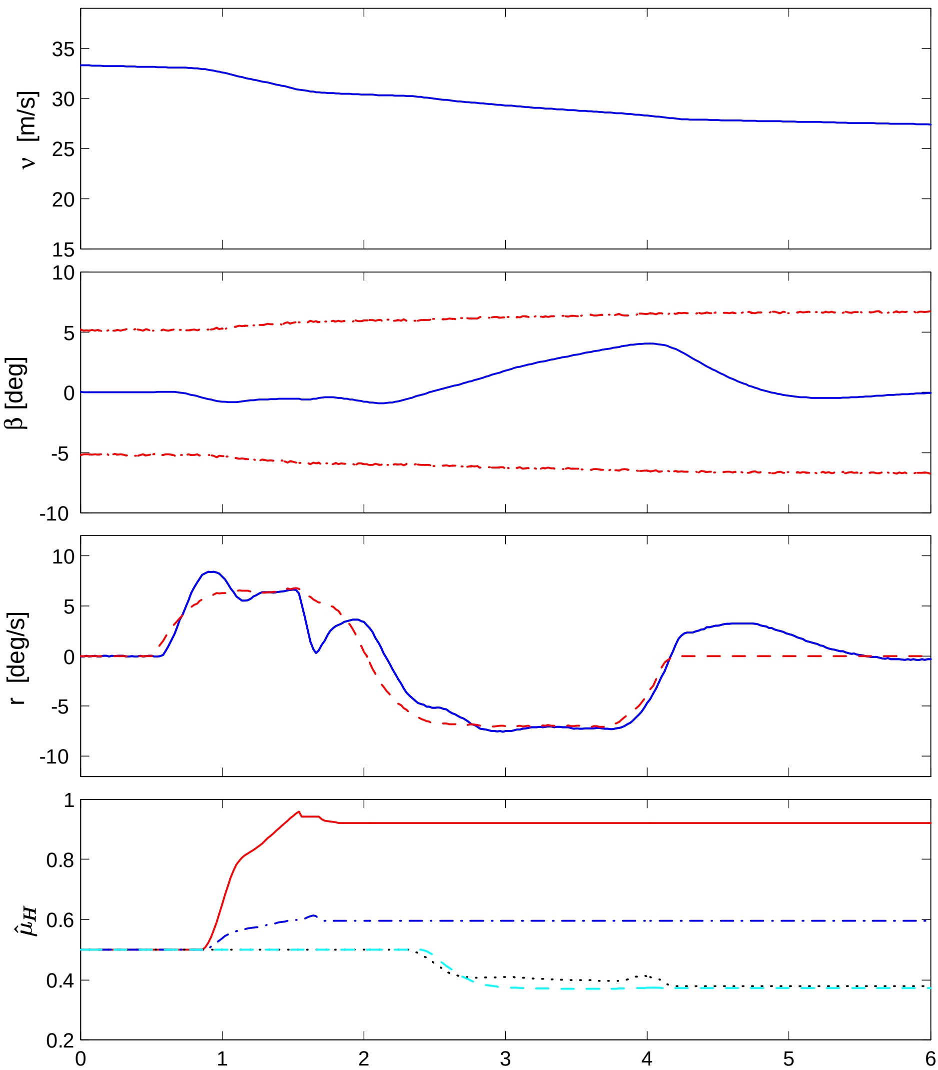

Figure 6.16: The fishhook test with active yaw stabilization. The dashed lines in plot 2 and 3 are given by $\beta_{\max}$ (6.4) and r_{ref} (6.4). In plot 4 the lines represent the following: solid-$\hat{\mu}_{H1}$, dotted-$\hat{\mu}_{H2}$, dashdot-$\hat{\mu}_{H3}$, dashed-$\hat{\mu}_{H4}$

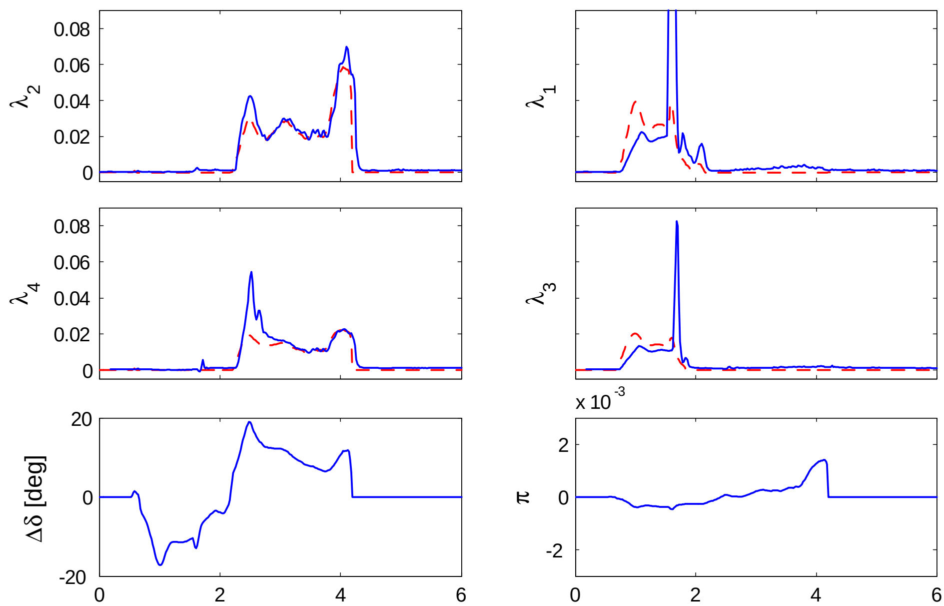

Figure 6.17: The fishhook test with active yaw stabilization. The longitudinal wheel slip, the steering angle manipulation parameter and Lagrangian parameter. The dotted lines denote the desired longitudinal wheel slip specified by the control allocator

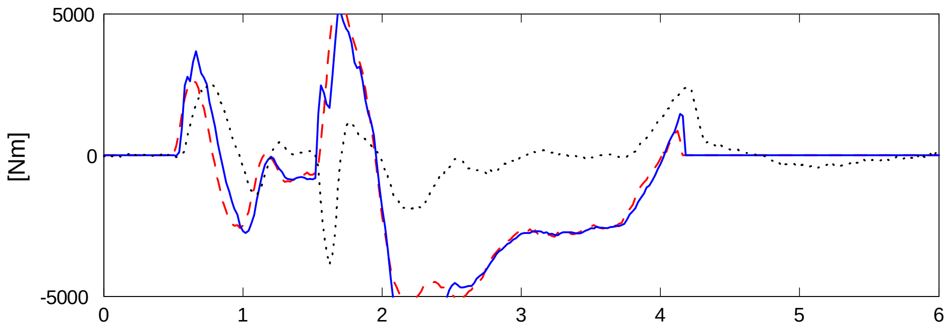

Figure 6.18: The fishhook test with active yaw stabilization. Yaw torque: M_c-dashed, $M_{estimated}$-solid, M_{real}-dotted.

Chapter 7

Conclusions

Based on a previous result about uniform global asymptotic stability (UGAS) of the equilibrium of cascaded time-varying systems, a similar result for a set-stable cascaded system is established (Lemma 1.1). The cascade result is applied in the analysis of a dynamic optimizing control allocation approach, first presented in Johansen [2004], and more general nonlinear systems may be considered. Moreover, founded on a control-Lyapunov design approach, adaptive optimizing nonlinear dynamic control allocation algorithms are derived. Under certain assumptions on the system (actuator and actuator-force model) and on the control design (growth rate conditions on the Lyapunov function), the cascade result is used to prove closed-loop stability and attractivity of a set representing the optimal actuator configuration. Typical applications for the control allocation algorithm are over-actuated mechanical systems, especially systems that exhibit fast dynamics since the algorithm is explicit and in general computational efficient.

The most important advantages of the control allocation approach presented in this thesis are based on the cascaded structure that appears from the optimizing problem formulation and the system model:

- On the theoretical level, as long as closed loop solution boundedness can be proved, the cascade result enables a modular analysis approach where UGAS of the closed loop equilibrium set can be concluded, if the equilibrium sets of the sub systems are also UGAS. This means that the sub systems can be analyzed separately. Since in most practical applications the actuators of a system are bounded, a local version of the result is presented.

- The control allocation strategy is based on a Lyapunov design, in which the first order optimal solution is described by the Lyapunov function when it is equal to zero. Through this approach an optimizing control allocation algorithm, which does not solve the optimization problem exactly at each time step, can be defined. Also the Lyapunov approach offers an analytical tool for; considering the stability of the optimal solution directly, and incorporating Lyapunov based estimation strategies and controller designs.

- The control allocation algorithms are not iterative and do not depend on any optimizing software, such that implementation on inexpensive hardware for fast (e.g. mechanical) systems can be considered.

The drawbacks and limitations of the algorithm are related to the assumptions given in Chapter 2 and commented in the section of possible further work.

- All states are assumed to be known. This means that measurements and fast observers (in the case where no measurements are available) are necessary in an implementation.

- Only the first order optimal solutions are incorporated in the presented Lyapunov design, this means that in the case of non-convex problems with more local maxima and minima, the algorithm does not distinguish between the extremum points.

- The control allocation algorithms contain matrix inversion, and the constraints of the optimization problem are implemented with barrier functions. This may introduce numerical problems that result in suboptimal solutions and degeneration of the algorithm performance, such that handling of the numerical aspects (e.g. singularity avoidance) is of importance.

The cascade result presented by Lemma 1.1 in Chapter 1, may simplify the stability analysis of the cascade by treating the subsystems separately under the assumption of uniform global boundedness of the closed-loop solution with respect to its equilibrium set. In most cases, when applying Lemma 1.1, the hardest requirement to satisfy is the boundedness assumption. In Chapter 3, 4 and 5 it is shown that the boundedness requirement is satisfied for the cascade consisting of the high level dynamics and the control allocation dynamics. Also by utilizing the cascade analysis tool it is shown in Chapter 3 that the system and high level controller assumptions presented in Johansen [2004] may be relaxed (see Section 1.3). In Chapter 4, parameter estimation is incorporated in the control allocation design, such that uncertainties and losses in the actuator force model can be estimated. In order for the control allocation algorithm to generate an actuator constellation that converges to the optimal solution in a stable sense, it is important that these force uncertainties and losses are accounted for. Regressor signal persistency of excitation in this setting is of importance for the optimal control allocation. By considering low level actuator dynamics and parameter uncertainties both in the actuator force mapping model and the actuator model, the algorithms in Chapter 5 extend the results from Chapter 3 and 4, and compensate for that transients introduced by the actuator dynamics.

The control allocation algorithms are verified by simulating a model ship undergoing DP control, and by presenting the yaw stabilizing problem in the framework of a modular efficient dynamic control allocation scheme. The modularity of the approach was shown by de-coupling the high level vehicle motion from the low level dynamics of the wheel slip with a control allocation scheme commanding desired longitudinal wheel slip and steering angle corrections. Furthermore, the scheme was tested in a highly realistic proprietary simulation environment by DaimlerChrysler, and over- and understeering of the vehicle was efficiently prevented.

7.1 Some ideas for further work on the topics of the thesis

- The main focus of further work related to the cascade Lemma 1.1, may be to provide and formalize ways of guaranteeing UGB, as in Theorem 1, Tjønnås et al. [2006], of the closed loop solutions with respect to the cascaded set, possibly in the framework of Pantely and Loria [2001]. Furthermore, work related to a generalized cascade lemma where the cascade consists of more then two systems (possibly represented by differential inclusions) is of interest.

- An obvious drawback with the presented control allocation algorithms is the assumption of a known state. In many applications, state measurements are not available, such that observers or state estimators are necessary. This means that the observer dynamic has to be accounted for in the control allocation, and work related to defining requirements on the observer properties, in order to conclude closed loop stability, is therefore of interest. A possibility may be to include the observers in the existing Lyapunov design framework.

- Consider anti-windup strategies and possibly use slack variables in the design of constrained control allocation (handling; infeasibility, physical actuator constraints to enable faster convergence, and actuator constraints related to problem objectives where undesired regions are prevented).

- Interesting further work related to the automotive application, involves strategies with increased functionality (augmentation of rollover prevention) and robustness (by including additional actuators, for example through the pressure control of an active hydropneumatic (AHP) suspension system). In the case of higher actuator redundancy, both improved performance and increased robustness of this yaw stabilizing system are expected.

- A comparison of the presented algorithm to the state of art available control allocation schemes is of interest, together with an experimental implementation and an in-depth investigation of numerical aspects, in order to evaluate and verify the practical significance of the algorithm.

- Implementation of the control allocation strategy on different process plants which are not necessarily mechanical systems, such as blending and mixing systems.

- Handling of non-convex problems.

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Appendix A

Proofs and simulation data

A.1 Proof of Proposition 4.2

The proof is divided into two parts: **i)** show that the integral of $\bar{f}_{\mathcal{O}_{u\lambda\tilde{\theta}}}(t)^{\mathrm{T}}\bar{f}_{\mathcal{O}_{u\lambda\tilde{\theta}}}(t)$, where

$$\bar{f}_{\mathcal{O}_{u\lambda\tilde{\theta}}}(t) := f_{\mathcal{O}_{u\lambda\tilde{\theta}}}(t, x(t), u(t), \lambda(t), \epsilon(t), \tilde{\theta}(t)),$$

is bounded, by using the PE property; and **ii)** use the integral bound to prove UGA by contradiction. The proof is based on ideas from Pantely et al. [2001] and Teel et al. [2002].

i) First we establish a bound on $\dot{\Phi}_g(t)$. We have

$$\begin{aligned}
\dot{\Phi}_g(t) \quad = \quad & \frac{\partial g_x(t,x)}{\partial t}\Phi_\theta(t,x,u) + g_x(t,x)\frac{\partial \Phi_\theta(t,x,u)}{\partial t} \\
& + \left(\frac{\partial g_x(t,x)}{\partial x}\Phi_\theta(t,x,u) + g_x(t,x)\frac{\partial \Phi_\theta(t,x,u)}{\partial x}\right)\dot{x} \\
& + g_x(t,x)\frac{\partial \Phi_\theta(t,x,u)}{\partial u}\dot{u},
\end{aligned}$$

thus from Assumption 2.1, system (4.3) and update-law (4.5), there exists a bound $\left|\dot{\Phi}_g(t)\right| \leq \varsigma_{1x\dot{\Phi}_g}(|x|) + \varsigma_{1x\dot{\Phi}_g}(|x|)\varsigma_{u\dot{\Phi}_g}(\left|z_{u\lambda\tilde{\theta}}\right|_{\mathcal{O}_{u\lambda\tilde{\theta}}})$ where $\varsigma_{1x\dot{\Phi}_g}, \varsigma_{1x\dot{\Phi}_g}, \varsigma_{1x\dot{\Phi}_g} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ are continuous functions. This can be seen by following the same approach as in Lemma 4.1. From the assumption that x is uniformly bounded, we use $|x| \leq B_x$. The integrability of $\tilde{\theta}^{\mathrm{T}}\tilde{\theta}$ is investigated by considering the auxiliary function:

$$V_{\theta aux1} := -\epsilon^{\mathrm{T}}\Phi_g(t)\tilde{\theta}, \tag{A.1}$$

bounded by $B_{V\theta aux1}(B_x, r) \geq V_{\theta aux1}$ where $r \geq \left|z_{u\lambda\tilde{\theta}0}\right|_{\mathcal{O}_{u\lambda\tilde{\theta}}}$ and $r > 0$. Its derivative along the solutions of (4.2) and (4.6) is given by

$$\begin{aligned} \dot{V}_{\theta aux1} &= -\dot{\epsilon}^{\mathrm{T}}\Phi_g(t)\tilde{\theta} - \epsilon^{\mathrm{T}}\dot{\Phi}_g(t)\tilde{\theta} - \epsilon^{\mathrm{T}}\Phi_g(t)\dot{\tilde{\theta}} \\ &= -\tilde{\theta}^{\mathrm{T}}\Phi_g(t)^{\mathrm{T}}\Phi_g(t)\tilde{\theta} + \epsilon^{\mathrm{T}}\left(A^{\mathrm{T}}\Phi_g(t) - \dot{\Phi}_g(t)\right)\tilde{\theta} \\ &\quad +\epsilon^{\mathrm{T}}\Phi_g(t)\Gamma_{\hat{\theta}}^{-1}\Phi_g(t)^{\mathrm{T}}\epsilon \qquad \text{(A.2)} \\ &\quad +\epsilon^{\mathrm{T}}\Phi_g(t)\Gamma_{\hat{\theta}}^{-1}\Phi_g(t)^{\mathrm{T}}\frac{\partial_x^2\bar{L}_{\hat{\theta}}}{\partial u\lambda}(t)\frac{\partial\bar{L}_{\hat{\theta}}}{\partial u\lambda}(t), \qquad \text{(A.3)} \end{aligned}$$

where $\frac{\partial_x^2\bar{L}_{\hat{\theta}}}{\partial u\lambda}(t) := \left(\left(\frac{\partial^2 L_{\hat{\theta}}}{\partial x\partial u}\right)^{\mathrm{T}}, \left(\frac{\partial^2 L_{\hat{\theta}}}{\partial x\partial \lambda}\right)^{\mathrm{T}}\right)(t, x(t), u(t), \lambda(t), \hat{\theta}(t))$ and $\frac{\partial\bar{L}_{\hat{\theta}}}{\partial u\lambda}(t) := \left(\left(\frac{\partial L_{\hat{\theta}}}{\partial u}\right)^{\mathrm{T}}, \left(\frac{\partial L_{\hat{\theta}}}{\partial \lambda}\right)^{\mathrm{T}}\right)^{\mathrm{T}}(t, x(t), u(t), \lambda(t), \hat{\theta}(t))$. From Young's inequality, $na^{\mathrm{T}}a + \frac{1}{n}b^{\mathrm{T}}b \geq 2\left|b^{\mathrm{T}}a\right|$, we have

$$\begin{aligned} & \epsilon^{\mathrm{T}}\left(A^{\mathrm{T}}\Phi_g(t) - \dot{\Phi}_g(t)\right)\tilde{\theta} \\ \leq\; & \frac{T}{\mu}\epsilon^{\mathrm{T}}\left(A^{\mathrm{T}}\Phi_g(t) - \dot{\Phi}_g(t)\right)\left(A^{\mathrm{T}}\Phi_g(t) - \dot{\Phi}_g(t)\right)^{\mathrm{T}}\epsilon \\ & +\frac{\mu}{4T}\tilde{\theta}^{\mathrm{T}}\tilde{\theta}, \qquad \text{(A.4)} \end{aligned}$$

where $a = \tilde{\theta}$, $b = \epsilon^{\mathrm{T}}\left(A^{\mathrm{T}}\Phi_g(t) - \dot{\Phi}_g(t)\right)$ and $n = \frac{\mu}{2T}$ such that μ and T are positive scalars of desirable choice. Next

$$\begin{aligned} & \epsilon^{\mathrm{T}}\Phi_g(t)\Gamma_{\hat{\theta}}^{-1}\Phi_g(t)^{\mathrm{T}}\frac{\partial_x^2\bar{L}_{\hat{\theta}}}{\partial u\lambda}(t)\frac{\partial\bar{L}_{\hat{\theta}}}{\partial u\lambda}(t) \\ \leq\; & \epsilon^{\mathrm{T}}\Phi_g(t)\Phi_g^{\mathrm{T}}(t)\epsilon + \left(\frac{\partial\bar{L}_{\hat{\theta}}}{\partial u\lambda}(t)\right)^{\mathrm{T}} K_{L\Phi}\frac{\partial\bar{L}_{\hat{\theta}}}{\partial u\lambda}(t) \end{aligned}$$

for $n = 1$, $a = \epsilon^{\mathrm{T}}\Phi_g(t)$ and $b = \Gamma_{\hat{\theta}}^{-1}\Phi_g(t)^{\mathrm{T}}\frac{\partial_x^2\bar{L}_{\hat{\theta}}}{\partial u\lambda}(t)\frac{\partial\bar{L}_{\hat{\theta}}}{\partial u\lambda}(t)$, where $K_{L\Phi} :=$

$\frac{\partial_x^2 \bar{L}_{\hat{\theta}}}{\partial u\lambda}(t)\Phi_g(t)\Gamma_{\hat{\theta}}^{-\mathrm{T}}\Gamma_{\hat{\theta}}^{-1}\Phi_g^{\mathrm{T}}(t)\frac{\partial_x^2 \bar{L}_{\hat{\theta}}^{\mathrm{T}}}{\partial u\lambda}(t)$. Hence

$$
\begin{aligned}
& \dot{V}_{\theta aux1} \\
\leq\ & \frac{T}{\mu}\epsilon^{\mathrm{T}}\left(A^{\mathrm{T}}\Phi_g(t)-\dot{\Phi}_g(t)\right)\left(A^{\mathrm{T}}\Phi_g(t)-\dot{\Phi}_g(t)\right)^{\mathrm{T}}\epsilon \\
& +\epsilon^{\mathrm{T}}\Phi_g(t)\Gamma_{\hat{\theta}}^{-1}\Phi_g(t)^{\mathrm{T}}\epsilon-\tilde{\theta}^{\mathrm{T}}\Phi_g(t)^{\mathrm{T}}\Phi_g(t)\tilde{\theta} \\
& +\frac{\mu}{4T}\tilde{\theta}^{\mathrm{T}}\tilde{\theta}+\epsilon^{\mathrm{T}}\Phi_g(t)\Phi_g^{\mathrm{T}}(t)\epsilon \\
& +\frac{\partial\bar{L}_{\hat{\theta}}^{\mathrm{T}}}{\partial u\lambda}(t)K_{L\Phi}\frac{\partial\bar{L}_{\hat{\theta}}}{\partial u\lambda}(t),
\end{aligned}
\tag{A.5}
$$

and thus

$$
\begin{aligned}
& \tilde{\theta}(t)^{\mathrm{T}}\left(\Phi_g(t)^{\mathrm{T}}\Phi_g(t)-\frac{\mu}{4T}I\right)\tilde{\theta}(t) \\
\leq\ & \frac{T}{\mu}\epsilon^{\mathrm{T}}\left(A^{\mathrm{T}}\Phi_g(t)-\dot{\Phi}_g(t)\right)\left(A^{\mathrm{T}}\Phi_g(t)-\dot{\Phi}_g(t)\right)^{\mathrm{T}}\epsilon \\
& +\epsilon^{\mathrm{T}}\Phi_g(t)\left(\Gamma_{\hat{\theta}}^{-1}+I\right)\Phi_g^{\mathrm{T}}(t)\epsilon \\
& -\dot{V}_{\theta aux1}+\frac{\partial\bar{L}_{\hat{\theta}}^{\mathrm{T}}}{\partial u\lambda}(t)K_{L\Phi}\frac{\partial\bar{L}_{\hat{\theta}}}{\partial u\lambda}(t)\ .
\end{aligned}
\tag{A.6}
$$

The solution of (4.5) can be represented by

$$
\tilde{\theta}(\tau)=\tilde{\theta}(t)-R_\Phi(t,\tau)\ , \tag{A.7}
$$

where $R_\Phi(t,\tau):=\Gamma_{\hat{\theta}}^{-1}\int_t^\tau\Phi_g^{\mathrm{T}}(s)\left(\epsilon(s)+\frac{\partial_x^2\bar{L}_{\hat{\theta}}}{\partial u\lambda}(s)\frac{\partial\bar{L}_{\hat{\theta}}}{\partial u\lambda}(s)\right)ds$, such that

$$
\begin{aligned}
& \tilde{\theta}(\tau)^{\mathrm{T}}\left(\Phi_g(\tau)^{\mathrm{T}}\Phi_g(\tau)-\frac{\mu}{4T}I\right)\tilde{\theta}(\tau) \\
=\ & \tilde{\theta}(t)^{\mathrm{T}}\left(\Phi_g(\tau)^{\mathrm{T}}\Phi_g(\tau)-\frac{\mu}{4T}I\right)\tilde{\theta}(t) \\
& +R_\Phi(t,\tau)^{\mathrm{T}}\left(\Phi_g(\tau)^{\mathrm{T}}\Phi_g(\tau)-\frac{\mu}{4T}I\right)R_\Phi(t,\tau) \\
& -2\tilde{\theta}(t)^{\mathrm{T}}\left(\Phi_g(\tau)^{\mathrm{T}}\Phi_g(\tau)-\frac{\mu}{4T}I\right)R_\Phi(t,\tau).
\end{aligned}
\tag{A.8}
$$

Again by Young's inequality, $-n_1a_1^{\mathrm{T}}a_1-\frac{1}{n_1}b_1^{\mathrm{T}}b_1\leq-2\left|b_1^{\mathrm{T}}a_1\right|$ and $-n_2a_2^{\mathrm{T}}a_2-\frac{1}{n_2}b_2^{\mathrm{T}}b_2\leq 2\left|b_2^{\mathrm{T}}a_2\right|$ with $a_1=\Phi_g(\tau)\tilde{\theta}$, $a_2=\sqrt{\frac{\mu}{4T}}\tilde{\theta}$, $b_1=\Phi_g(\tau)R_\Phi(t,\tau)$,

$b_2 = \sqrt{\frac{\mu}{4T}} R_\Phi(t,\tau)$ and $n_1 = n_2 = \frac{1}{\sigma}$, such that

$$\begin{aligned} & -2\tilde{\theta}(t)^{\mathrm{T}} \Phi_g(\tau)^{\mathrm{T}} \Phi_g(\tau) R_\Phi(t,\tau) \\ \geq \ & -\frac{1}{\sigma} \tilde{\theta}(t)^{\mathrm{T}} \Phi_g(\tau)^{\mathrm{T}} \Phi_g(\tau) \tilde{\theta}(t) \\ & -\sigma R_\Phi(t,\tau)^{\mathrm{T}} \Phi_g(\tau)^{\mathrm{T}} \Phi_g(\tau) R_\Phi(t,\tau) \end{aligned}$$

and

$$\begin{aligned} & 2\frac{\mu}{4T} \tilde{\theta}(t)^{\mathrm{T}} R_\Phi(t,\tau) \\ \geq \ & -\frac{1}{\sigma} \frac{\mu}{4T} \tilde{\theta}(t)^{\mathrm{T}} \Phi_g(\tau)^{\mathrm{T}} \Phi_g(\tau) \tilde{\theta}(t) \\ & -\sigma \frac{\mu}{4T} R_\Phi(t,\tau)^{\mathrm{T}} \Phi_g(\tau)^{\mathrm{T}} \Phi_g(\tau) R_\Phi(t,\tau) \end{aligned}$$

which gives

$$\begin{aligned} & \tilde{\theta}(\tau)^{\mathrm{T}} \left(\Phi_g(\tau)^{\mathrm{T}} \Phi_g(\tau) - \frac{\mu}{4T} I \right) \tilde{\theta}(\tau) \\ \geq \ & (1-\sigma) R_\Phi(t,\tau)^{\mathrm{T}} \left(\Phi_g(\tau)^{\mathrm{T}} \Phi_g(\tau) - \frac{(1+\sigma)}{(1-\sigma)} \frac{\mu}{4T} I \right) R_\Phi(t,\tau) \\ & + \left(1 - \frac{1}{\sigma}\right) \tilde{\theta}(t)^{\mathrm{T}} \left(\Phi_g(\tau)^{\mathrm{T}} \Phi_g(\tau) - \frac{(1+\frac{1}{\sigma})}{(1-\frac{1}{\sigma})} \frac{\mu}{4T} I \right) \tilde{\theta}(t) \end{aligned} \tag{A.9}$$

for $\sigma > 1$.

From Holder's inequality, $\int |fg| \leq \left(\int |f|^2\right)^{\frac{1}{2}} \left(\int |g|^2\right)^{\frac{1}{2}}$, with $g = 1$ and $f = \Gamma_{\hat{\theta}}^{-1} \Phi_g^{\mathrm{T}}(s) \epsilon(s)$ we get

$$\begin{aligned} & \left(\int_t^\tau \left| \Gamma_{\hat{\theta}}^{-1} \Phi_g^{\mathrm{T}}(s) \epsilon(s) \right| ds \right)^2 \\ \leq \ & \int_t^\tau \left| \Gamma_{\hat{\theta}}^{-1} \Phi_g^{\mathrm{T}}(s) \epsilon(s) \right|^2 ds \, (\tau - t) \end{aligned}$$

and by similar arguments

$$\begin{aligned} & \left(\int_t^\tau \left| \Gamma_{\hat{\theta}}^{-1} \Phi_g^{\mathrm{T}}(s) \frac{\partial_x^2 \bar{L}_{\hat{\theta}}}{\partial u \lambda}(s) \frac{\partial \bar{L}_{\hat{\theta}}}{\partial u \lambda}(s) \right| ds \right)^2 \\ \leq \ & \int_t^\tau \left| \Gamma_{\hat{\theta}}^{-1} \Phi_g^{\mathrm{T}}(s) \frac{\partial_x^2 \bar{L}_{\hat{\theta}}}{\partial u \lambda}(s) \frac{\partial \bar{L}_{\hat{\theta}}}{\partial u \lambda}(s) \right|^2 ds \, (\tau - t) \,, \end{aligned}$$

such that

$$
\begin{aligned}
& R_\Phi(t,\tau)^{\mathrm{T}} R_\Phi(t,\tau) \\
\leq\ & 2\left(\int_t^\tau \left|\Gamma_{\hat\theta}^{-1}\Phi_g^{\mathrm{T}}(s)\epsilon(s)\right| ds\right)^2 \\
& +2\left(\int_t^\tau \left|\Gamma_{\hat\theta}^{-1}\Phi_g^{\mathrm{T}}(s)\frac{\partial_x^2 \bar L_{\hat\theta}}{\partial u\lambda}(s)\frac{\partial \bar L_{\hat\theta}}{\partial u\lambda}(s)\right| ds\right)^2 \\
\leq\ & 2\int_t^\tau \left|\Gamma_{\hat\theta}^{-1}\Phi_g^{\mathrm{T}}(s)\epsilon(s)\right|^2 ds\,(\tau-t) \\
& +2\int_t^\tau \left|\Gamma_{\hat\theta}^{-1}\Phi_g^{\mathrm{T}}(s)\frac{\partial_x^2 \bar L_{\hat\theta}}{\partial u\lambda}(s)\frac{\partial \bar L_{\hat\theta}}{\partial u\lambda}(s)\right|^2 ds\,(\tau-t) \\
=\ & 2\int_t^\tau \epsilon(s)^{\mathrm{T}}\Phi_g(s)\Gamma_{\hat\theta}^{-\mathrm{T}}\Gamma_{\hat\theta}^{-1}\Phi_g^{\mathrm{T}}(s)\epsilon(s)ds\,(\tau-t) \\
& +2\int_t^\tau \frac{\partial \bar L_{\hat\theta}^{\mathrm{T}}}{\partial u\lambda}(s)K_{L\Phi}\frac{\partial \bar L_{\hat\theta}}{\partial u\lambda}(s)ds(\tau-t).
\end{aligned}
$$

Let $B_{\Phi T}(B_x,r,T) := \lambda_{\max}\left(\Phi_g(t)^{\mathrm{T}}\Phi_g(t) - \frac{(1+\sigma)}{(1-\sigma)}\frac{\mu}{4T}I\right)$ be the maximal eigenvalue of $\left(\Phi_g(\tau)^{\mathrm{T}}\Phi_g(\tau) - \frac{(1+\sigma)}{(1-\sigma)}\frac{\mu}{4T}I\right)$ and $\lambda_{\max}K_{L\Phi}$ be the maximal eigenvalue of $K_{L\Phi}$, then from combining (A.6) and (A.9) and investigating the integral over τ,

$$
\begin{aligned}
& \left(1-\frac{1}{\sigma}\right)\tilde\theta(t)^{\mathrm{T}}\int_t^{t+T}\left(\Phi_g(\tau)^{\mathrm{T}}\Phi_g(\tau)-\frac{\left(1+\frac{1}{\sigma}\right)}{\left(1-\frac{1}{\sigma}\right)}\frac{\mu}{4T}I\right)d\tau\tilde\theta(t) \\
\leq\ & B_{\Phi T}\int_t^{t+T}\left(\int_t^\tau \epsilon(s)^{\mathrm{T}}\Phi_g(s)\Phi_g^{\mathrm{T}}(s)\epsilon(s)ds\right)(\tau-t)d\tau \\
& +B_{\Phi T}\lambda_{\max}K_{L\Phi}\int_t^{t+T}\left(\int_t^\tau\left(\frac{\partial \bar L_{\hat\theta}}{\partial u\lambda}(s)\right)^{\mathrm{T}}\frac{\partial \bar L_{\hat\theta}}{\partial u\lambda}(s)ds\right)(\tau-t)\,d\tau \\
& +V_{\theta aux1}(t)-V_{\theta aux1}(t+T) \\
& +\lambda_{\max}K_{L\Phi}\int_t^{t+T}\left(\left(\frac{\partial \bar L_{\hat\theta}}{\partial u\lambda}(s)\right)^{\mathrm{T}}\frac{\partial \bar L_{\hat\theta}}{\partial u\lambda}(s)\right)ds \\
& +\int_t^{t+T}\epsilon(\tau)^{\mathrm{T}}\Phi_g(\tau)\left(\Gamma_{\hat\theta}^{-1}+I\right)\Phi_g^{\mathrm{T}}(\tau)\epsilon(\tau)d\tau \\
& +\frac{T}{\mu}\int_t^{t+T}\epsilon(\tau)^{\mathrm{T}}\left(A^{\mathrm{T}}\Phi_g(\tau)-\dot\Phi_g(\tau)\right)\left(A^{\mathrm{T}}\Phi_g(\tau)-\dot\Phi_g(\tau)\right)^{\mathrm{T}}\epsilon(\tau)d\tau.
\end{aligned}
$$

Further

$$\begin{aligned}
&\int_t^{t+T}\left(\int_t^{\tau}\epsilon(s)^{\mathrm{T}}\Phi_g(s)\Gamma_{\hat{\theta}}^{-\mathrm{T}}\Gamma_{\hat{\theta}}^{-1}\Phi_g^{\mathrm{T}}(s)\epsilon(s)ds\right)(\tau-t)d\tau\\
= \;& T\int_t^{t+T}\left(\int_s^{t+T}\epsilon(s)^{\mathrm{T}}\Phi_g(s)\Gamma_{\hat{\theta}}^{-\mathrm{T}}\Gamma_{\hat{\theta}}^{-1}\Phi_g^{\mathrm{T}}(s)\epsilon(s)d\tau\right)ds\\
\leq \;& T\int_t^{t+T}\left(\int_t^{t+T}\epsilon(s)^{\mathrm{T}}\Phi_g(s)\Gamma_{\hat{\theta}}^{-\mathrm{T}}\Gamma_{\hat{\theta}}^{-1}\Phi_g^{\mathrm{T}}(s)\epsilon(s)d\tau\right)ds\\
= \;& T^2\int_t^{t+T}\epsilon(s)^{\mathrm{T}}\Phi_g(s)\Gamma_{\hat{\theta}}^{-\mathrm{T}}\Gamma_{\hat{\theta}}^{-1}\Phi_g^{\mathrm{T}}(s)\epsilon(s)ds,
\end{aligned}$$

such that for

$$\begin{aligned}
&B_\epsilon(B_x,r,T)\\
:\;& =\lambda_{\max}\left(\left(A^{\mathrm{T}}\Phi_g(t)-\dot{\Phi}_g(t)\right)\left(A^{\mathrm{T}}\Phi_g(t)-\dot{\Phi}_g(t)\right)^{\mathrm{T}}+\right.\\
&\qquad\left.\Phi_g(t)\left(\left(B_{\Phi T}T^2+1\right)\Gamma_{\hat{\theta}}^{-1}+I\right)\Phi_g(t)^{\mathrm{T}}\right)
\end{aligned}$$

it follows that

$$\begin{aligned}
&\left(1-\frac{1}{\sigma}\right)\tilde{\theta}(t)^{\mathrm{T}}\int_t^{t+T}\left(\Phi_g(\tau)^{\mathrm{T}}\Phi_g(\tau)-\frac{\left(1+\frac{1}{\sigma}\right)}{\left(1-\frac{1}{\sigma}\right)}\frac{\mu}{4T}I\right)d\tau\tilde{\theta}(t)\\
\leq \;& \int_t^{t+T}\epsilon(s)^{\mathrm{T}}B_\epsilon\epsilon(s)ds+V_{\theta aux1}(t)-V_{\theta aux1}(t+T)\\
&+\lambda_{\max}K_{L\Phi}\left(B_{\Phi T}T^2+1\right)\int_t^{t+T}\left(\frac{\partial\bar{L}_{\hat{\theta}}}{\partial u\lambda}(\tau)\right)^{\mathrm{T}}\frac{\partial\bar{L}_{\hat{\theta}}}{\partial u\lambda}(\tau)d\tau\\
\leq \;& V_{\theta aux1}(t)-V_{\theta aux1}(t+T)\\
&+B_V\int_t^{t+T}\left(\left(\frac{\partial\bar{L}_{\hat{\theta}}}{\partial u\lambda}(\tau)\right)^{\mathrm{T}}\mathbb{H}_{\hat{\theta}}^{\mathrm{T}}(\tau)\Gamma\mathbb{H}_{\hat{\theta}}(\tau)\frac{\partial\bar{L}_{\hat{\theta}}}{\partial u\lambda}(\tau)+\epsilon(\tau)^{\mathrm{T}}A\epsilon(\tau)\right)d\tau
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{\theta}(t)^{\mathrm{T}}\int_t^{t+T}\left(\Phi_g(\tau)^{\mathrm{T}}\Phi_g(\tau)-\frac{\left(1+\frac{1}{\sigma}\right)}{\left(1-\frac{1}{\sigma}\right)}\frac{\mu}{4T}I\right)d\tau\tilde{\theta}(t)\\
\leq \;& \frac{1}{\left(1-\frac{1}{\sigma}\right)}\left(V_{\theta aux1}(t)-V_{\theta aux1}(t+T)\right)\\
&+B_V\left(V_{u\lambda\hat{\theta}}(T+t)-V_{u\lambda\hat{\theta}}(t)\right),
\end{aligned}$$

where $B_V(B_x, r, T) := \frac{\lambda_{\max} B_\epsilon(t,T)+\lambda_{\max} K_{L\Phi}\left(B_{\Phi T}T^2+1\right)}{\left(1-\frac{1}{\sigma}\right)\lambda_{\min} A}$. By the PE assumption we can choose $T, \gamma, \mu > 0$ and $\sigma > 1$ such that, $B_\gamma < \gamma - \frac{\left(1+\frac{1}{\sigma}\right)}{\left(1-\frac{1}{\sigma}\right)}\frac{\mu}{4}$, where $B_\gamma > 0$ i.e.

$$\begin{aligned}
& B_\gamma \tilde{\theta}(t)^{\mathrm{T}}\tilde{\theta}(t) \\
\leq\ & \tilde{\theta}(t)^{\mathrm{T}} \int_t^{t+T} \left(\Phi_g(\tau)^{\mathrm{T}}\Phi_g(\tau) - \frac{\left(1+\frac{1}{\sigma}\right)}{\left(1-\frac{1}{\sigma}\right)}\frac{\mu}{4T} I\right) d\tau \tilde{\theta}(t) \\
\leq\ & \frac{1}{\left(1-\frac{1}{\sigma}\right)} V_{\theta aux1}(t) - V_{\theta aux1}(t+T) \\
+\ & B_V \left(V_{u\lambda\hat{\theta}}(T+t) - V_{u\lambda\hat{\theta}}(t)\right),
\end{aligned}$$

$$\begin{aligned}
& \int_{t_0}^{t_0+T_T} \left(V_{\theta aux1}(\tau) - V_{\theta aux1}(\tau+T)\right) d\tau \\
=\ & \int_{t_0}^{t_0+T_T} V_{\theta aux1}(\tau) d\tau - \int_{t_0}^{t_0+T_T} V_{\theta aux1}(\tau+T) d\tau \\
=\ & \int_{t_0}^{t_0+T_T} V_{\theta aux1}(\tau) d\tau - \int_{t_0+T}^{t_0+T_T+T} V_{\theta aux1}(s) ds \\
=\ & \int_{t_0}^{t_0+T} V_{\theta aux1}(\tau) d\tau - \int_{t_0+T_T}^{t_0+T_T+T} V_{\theta aux1}(\tau) d\tau \\
\leq\ & 2T B_{V\theta aux1}(B_x, r)
\end{aligned}$$

and

$$\begin{aligned}
& \int_{t_0}^{t_0+T_T} \tilde{\theta}(\tau)^{\mathrm{T}}\tilde{\theta}(\tau) d\tau \\
\leq\ & \frac{1}{B_\gamma\left(1-\frac{1}{\sigma}\right)} \int_{t_0}^{t_0+T_T} \left(V_{\theta aux1}(\tau) - V_{\theta aux1}(\tau+T)\right) d\tau \\
+\ & \frac{B_V}{B_\gamma} \int_{t_0}^{t_0+T_T} \left(V_{u\lambda\hat{\theta}}(\tau) - V_{u\lambda\hat{\theta}}(\tau+T)\right) d\tau \\
\leq\ & \frac{2T}{B_\gamma}\left(\frac{B_{V\theta aux1}(B_x, r)}{\left(1-\frac{1}{\sigma}\right)} + B_V V_{u\lambda\hat{\theta}}(t_0)\right).
\end{aligned}$$

Thus $\tilde{\theta}(\tau)^{\mathrm{T}}\tilde{\theta}(\tau)$ is integrable. Let $\lambda_{\min} V$ be the minimum eigenvalue of

$(1, \mathbb{H}_{\hat{\theta}}^{\mathrm{T}}\Gamma\mathbb{H}_{\hat{\theta}}, A)$, then there exists a constant $B_{f_{\mathcal{O}_{u\lambda\tilde{\theta}}}}(B_x, r, T) > 0$, such that

$$\begin{aligned}
&\int_{t_0}^{t_0+T_T} \bar{f}_{\mathcal{O}_{u\lambda\tilde{\theta}}}(\tau)^{\mathrm{T}}\bar{f}_{\mathcal{O}_{u\lambda\tilde{\theta}}}(\tau)d\tau \\
\leq\ & \lim_{t\to\infty}\int_{t_0}^{t} \bar{f}_{\mathcal{O}_{u\lambda\tilde{\theta}}}(\tau)^{\mathrm{T}}\bar{f}_{\mathcal{O}_{u\lambda\tilde{\theta}}}(\tau)d\tau \\
\leq\ & \frac{1}{\lambda_{\min}V}\left(V_{u\lambda\hat{\theta}}(t_0) - V_{u\lambda\hat{\theta}}(\infty)\right) \\
& + \frac{2T}{B_\gamma}\left(\frac{B_{V\theta aux1}(B_x, r)}{(1-\frac{1}{\sigma})} + B_V V_{u\lambda\hat{\theta}}(t_0)\right) \\
\leq\ & B_{f_{\mathcal{O}_{u\lambda\tilde{\theta}}}},
\end{aligned} \tag{A.10}$$

and the integral of $\bar{f}_{\mathcal{O}_{u\lambda\tilde{\theta}}}(t)^{\mathrm{T}}\bar{f}_{\mathcal{O}_{u\lambda\tilde{\theta}}}(t)$ is bounded.

ii) From The UGS property we have

$$\left|z_{u\lambda\tilde{\theta}}(t)\right|_{\mathcal{O}_{u\lambda\tilde{\theta}}} \leq \rho(\left|z_{u\lambda\tilde{\theta}0}\right|_{\mathcal{O}_{u\lambda\tilde{\theta}}}), \qquad \forall t \geq t_0$$

where $\rho \in \mathcal{K}_\infty$. Fix $r > 0$, $\varepsilon > 0$. Define $\Omega := \rho(r)$ and $\omega := \min\left\{\Omega, \rho^{-1}(\varepsilon)\right\}$ and note that $B_{f_{\mathcal{O}_{u\lambda\tilde{\theta}}}}$ is given by ω and Ω. Define $T_T = \frac{2B_{f_{\mathcal{O}_{u\lambda\tilde{\theta}}}}}{\omega_1}$, where ω_1 is specified later, and assume that for all $\left|z_{u\lambda\tilde{\theta}0}\right|_{\mathcal{O}_{u\lambda\tilde{\theta}}} \leq r$, there exists $t' \in [t_0, T_T]$ such that $\left|z_{u\lambda\tilde{\theta}}(t', z_{u\lambda\tilde{\theta}0})\right|_{\mathcal{A}} \leq \rho^{-1}(\varepsilon)$. Thus $\left|z_{u\lambda\tilde{\theta}}(t, z_{u\lambda\tilde{\theta}0})\right|_{\mathcal{O}_{u\lambda\tilde{\theta}}} \leq \rho\left(\left|z_{u\lambda\tilde{\theta}}(t', z_{u\lambda\tilde{\theta}0})\right|_{\mathcal{O}_{u\lambda\tilde{\theta}}}\right) \leq \rho(\rho^{-1}(\varepsilon)) = \varepsilon$ for $\left|z_{u\lambda\tilde{\theta}0}\right|_{\mathcal{O}_{u\lambda\tilde{\theta}}} \leq r$ and $t \geq T_T + t_0$, which satisfies definition 1.4. Suppose the assumption is not true, i.e., there exists $\left|z_{u\lambda\tilde{\theta}0}\right|_{\mathcal{O}_{u\lambda\tilde{\theta}}} \leq r$ such that $\left|z_{u\lambda\tilde{\theta}}(t', z_{u\lambda\tilde{\theta}0})\right|_{\mathcal{O}_{u\lambda\tilde{\theta}}} > \rho^{-1}(\varepsilon)\ \forall t' \in [t_0, T_T]$. Thus

$$\omega \leq \left|z_{u\lambda\tilde{\theta}}(t', z_{u\lambda\tilde{\theta}0})\right|_{\mathcal{O}_{u\lambda\tilde{\theta}}} \leq \Omega, \qquad \forall t' \in [t_0, T_T]$$

which from radial unboundedness of $V_{u\lambda\hat{\theta}}$, related to (4.7), imply that there exist positive constants ω_1, and Ω_1 such that

$$\omega_1 \leq \bar{f}_{\mathcal{O}_{u\lambda\tilde{\theta}}}(t')^{\mathrm{T}}\bar{f}_{\mathcal{O}_{u\lambda\tilde{\theta}}}(t') \leq \Omega_1, \qquad \forall t' \in [t_0, T_T]$$

Then

$$\begin{aligned}
&\int_{t_0}^{t_0+T_T} \bar{f}_{\mathcal{O}_{u\lambda\tilde{\theta}}}(\tau)^{\mathrm{T}}\bar{f}_{\mathcal{O}_{u\lambda\tilde{\theta}}}(\tau)d\tau \geq \int_{t_0}^{t_0+T_T}\omega_1 d\tau \\
=\ & \omega_1 T_T \\
=\ & 2B_{f_{\mathcal{O}_{u\lambda\tilde{\theta}}}},
\end{aligned}$$

which contradicts (A.10), and the proposition is proved.

A.2 Derivation of the longitudinal slip model

$$
\begin{aligned}
\dot{\lambda}_{xi} &= \frac{\dot{\nu}_{xi}}{\nu_i} - \frac{\nu_{xi}}{\nu_i^2}\dot{\nu}_i - \frac{\dot{\omega}_i R}{\nu_i} + \frac{\omega_i R}{\nu_i^2}\dot{\nu}_i \qquad (A.11)\\
&= \frac{1}{\nu_i}\left(\dot{\nu}_{xi} - \frac{\nu_{xi}}{\nu_i}\dot{\nu}_i - \dot{\omega}_i R + \left(\frac{v_{xi}}{\nu_i} - \lambda_{xi}\right)\dot{\nu}_i\right)\\
&= \frac{1}{\nu_i}\left(\dot{\nu}_{xi} - \dot{\omega}_i R - \lambda_{xi}\frac{\nu_{xi}\dot{\nu}_{xi} + \nu_{yi}\dot{\nu}_{yi}}{\nu_i}\right)\\
&= \frac{1}{\nu_i}\left(\left(1 - \lambda_{xi}\frac{\nu_{xi}}{\nu_i}\right)\dot{\nu}_{xi} - \dot{\omega}_i R - \lambda_{xi}\frac{\nu_{yi}}{\nu_i}\dot{\nu}_{yi}\right)\\
&= \frac{1}{\nu_i}\left(\left(1 - \lambda_{xi}\frac{\nu_{xi}}{\nu_i}\right)\frac{F_{xi}}{m_{wi}} - \left(-RF_{xi} - T_{bi} sign(\omega_i)\right)\frac{R}{J_\omega}\right)\\
&+ \frac{1}{\nu_i}\lambda_{xi}\sin(\alpha_i)\frac{F_{yi}}{m_{wi}}\\
&= \frac{1}{\nu_i}\left(\left(\frac{\left(1 - \lambda_{xi}\frac{\nu_{xi}}{\nu_i}\right)}{m_{wi}} + \frac{R^2}{J_\omega}\right)F_{x_i} + \frac{R}{J_\omega} sign(\omega_i) T_{bi}\right)\\
&+ \frac{1}{\nu_i}\lambda_{xi}\sin(\alpha_i)\frac{F_{yi}}{m_{wi}}\\
&= \frac{R}{\nu_i J_\omega}\left(\left(\frac{J_\omega\left(1 - \lambda_{xi}\frac{\nu_{xi}}{\nu_i}\right)}{R m_{wi}} + R\right)F_{xi} + sign(\omega_i) T_{bi}\right)\\
&+ \frac{1}{\nu_i}\lambda_{xi}\sin(\alpha_i)\frac{F_{yi}}{m_{wi}}\\
&= \frac{R}{\nu_i J_\omega}\left(sign(\omega_i) T_{bi} - \phi_t(\lambda_{xi}, \alpha_i, \mu_{Hi}, \nu_{xi}, \nu_i)\right) \qquad (A.12)
\end{aligned}
$$

where

$$
\begin{aligned}
\phi_t(\lambda_{xi}, \alpha_i, \mu_{Hi}, \nu_{xi}, \nu_i) := &-\left(\frac{J\left(1 - \lambda_{xi}\frac{\nu_{xi}}{\nu_i}\right)}{R m_{wi}} + R\right)F_{xi}\\
&- \frac{J}{R m_{wi}}\sin(\alpha_i)\lambda_{xi}F_{yi}
\end{aligned}
$$

A.3 Proof of UES comment in Remark 6.2

Consider the Lyapunov function candidate

$$V_x = (\epsilon_1 + c_1\eta_1)\frac{x_1^2}{2} + (\epsilon_1 + \frac{c_1}{\eta_1})\frac{x_3^2}{2} + c_1x_1x_3 + (\epsilon_2 + c_2\eta_2)\frac{x_2^2}{2} + (\epsilon_2 + \frac{c_2}{\eta_2})\frac{x_3^2}{2} + c_2x_2x_3 \tag{A.13}$$

where $\epsilon_1, \epsilon_2, c_1, c_2, \eta_1, \eta_2 > 0$ and $x := (x_1, x_2, x_3)^{\mathrm{T}}$ is given by the transformation $(x_1, x_2, x_3)^{\mathrm{T}} = diag(\gamma_{\mu i}, \gamma_{\mu i}\gamma_{\bar{\lambda} i}^{-1}, 1)(\tilde{\lambda}_{xi}, \bar{\lambda}_{xi}, \tilde{\mu}_{Hi})^{\mathrm{T}}$, such that

$$\dot{x} = \begin{pmatrix} -a_1 & 0 & -b(t) \\ 0 & -a_2 & -b(t) \\ \bar{b}(t) & \bar{b}(t) & 0 \end{pmatrix} x \tag{A.14}$$

where $a_1 := \Gamma_p\gamma_{\mu i}$, $a_2 := A\gamma_{\mu i}\gamma_{\bar{\lambda} i}^{-1}$, $\bar{b}(t) := \frac{R}{\nu_i(t)J_z}\bar{\phi}(\lambda_{xi}(t), \alpha_i(t), \hat{\mu}_{Hi}(t))$, $b(t) := \frac{R}{\nu_i J_z}\bar{\phi}_n(\lambda_{xi}(t), \alpha_i(t), \hat{\mu}_{Hi}(t), \tilde{\mu}_{Hi}(t))$ and $\bar{\phi}_n(\lambda_{xi}, \alpha_i, \hat{\mu}_{Hi}, \tilde{\mu}_{Hi}) := \left.\frac{\partial\phi(\lambda_{xi}, \alpha_i, s_i)}{\partial s_i}\right|_{s_i = \hat{\mu}_{Hi} + \varsigma\tilde{\mu}_{Hi}}$ (from the *mean value theorem*). The derivative of (A.13) along the trajectories of (A.14) is given by:

$$\begin{aligned}\dot{V}_x = &-\left((\epsilon_1 + c_1\eta_1)a_1 - c_1\bar{b}(t)\right)x_1^2 - c_1b(t)x_3^2 \\ &+ \left(-(\epsilon_1 + c_1\eta_1)b(t) + (\epsilon_1 + \frac{c_1}{\eta_1})\bar{b}(t) - c_1a_1\right)x_1x_3 \\ &- \left((\epsilon_2 + c_2\eta_2)a_2 - c_2\bar{b}(t)\right)x_2^2 - c_2b(t)x_3^2 \\ &+ \left(-(\epsilon_2 + c_2\eta_2)b(t) + (\epsilon_2 + \frac{c_2}{\eta_2})\bar{b}(t) - c_2a_2\right)x_2x_3.\end{aligned} \tag{A.15}$$

By defining the variables $\Delta b(t) := b(t) - \bar{b}(t)$, $k_1(t) := \left|(\frac{1}{\eta_1} - \eta_1)\bar{b}(t) - a_1\right|$ and $k_2(t) := \left|(\frac{1}{\eta_2} - \eta_2)\bar{b}(t) - c_2a_2\right|$, and applying Young's inequality ($\sigma_1x_1^2 + \frac{1}{\sigma_1}x_3^2 \geq 2\,|x_1x_3|$):

$$\begin{aligned}\dot{V}_x \leq &-\left((\epsilon_1 + c_1\eta_1)a_1 - c_1\left(\bar{b}(t) + k_1(t)\sigma_1\right)\right)x_1^2 \\ &- (\epsilon_1 + c_1\eta_1)\Delta b(t)x_1x_3 - c_1\left(b(t) - k_1(t)\frac{1}{\sigma_1}\right)x_3^2 \\ &- \left((\epsilon_2 + c_2\eta_2)a_2 - c_2\left(\bar{b}(t) + k_2(t)\sigma_2\right)\right)x_2^2 \\ &- (\epsilon_2 + c_2\eta_2)\Delta b(t)x_2x_3 - c_2\left(b(t) - k_2(t)\frac{1}{\sigma_2}\right)x_3^2.\end{aligned}$$

From Assumption 6.1 b) and c) there exist bounds $B_l, B_h > 0$, where $B_l < b(t) < B_h$ and $B_l < \bar{b}(t) < B_h$, such that for $\sigma_1 > \frac{\left(\left|(\frac{1}{\eta_1}-\eta_1)\right|B_h+a_1\right)+\kappa_1}{B_l}$, $\sigma_2 > \frac{\left(\left|(\frac{1}{\eta_2}-\eta_2)\right|B_h+a_2\right)+\kappa_2}{B_l}$, $\epsilon_1 > \frac{\xi_1+c_1\left(B_h+\left(\left|(\frac{1}{\eta_1}-\eta_1)\right|B_h+a_1\right)\sigma_1\right)}{a_1} - c_1\eta_1$ and $\epsilon_2 > \frac{\xi_2+c_2\left(B_h+\left(\left|(\frac{1}{\eta_2}-\eta_2)\right|B_h+a_2\right)\sigma_2\right)}{a_2} - c_2\eta_2$,

$$\begin{aligned}\dot{V}_x \leq & -\xi_1 x_1^2 - (\epsilon_1 + c_1\eta_1)\Delta b(t)x_1x_3 - c_1\kappa_1 x_3^2 \\ & - \xi_2 x_2^2 - (\epsilon_2 + c_2\eta_2)\Delta b(t)x_2x_3 - c_2\kappa_2 x_3^2\end{aligned}$$

which is negative definite if $|\Delta b(t)|$ is significantly small and $\kappa_1, \kappa_2, \xi_1, \xi_2 > 0$. This leads to a Local UES result since for bounded $b(t)$ and $\bar{b}(t)$, $x_3 \to 0$ imply $\Delta b(t) \to 0$ uniformly.

A.4 The Nested Matrosov Theorem, Loria et al. [2005]

Under assumptions A.1-A.4 the origin of

$$\dot{x} = f(t, x) \tag{A.16}$$

is UGAS.

Assumption A.1 *The origin of (A.16) is UGS*

Assumption A.2 *There exist integers $j, m > 0$ and for each $\Delta > 0$ there exists*

- *a number $\mu > 0$;*
- *locally Lipschitz continuous functions $V_i : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}$, $i \in \{1, .., j\}$*
- *a function $\phi : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^m$;*
- *continuous function $Y_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $i \in \{1, .., j\}$;*
- *such that, for almost all $(t, x) \in \mathbb{R}_{\geq 0} \times \mathcal{B}(\cdot)$, and all $i \in \{1, .., j\}$*

$$\max\{|V_i(t, x)|, |\phi(t, x)|\} \leq \mu \tag{A.17}$$

$$\dot{V}_i(t, x) \leq Y_i(x, \phi(t, x)). \tag{A.18}$$

Assumption A.3 *For each integer $k \in \{1, .., j\}$, we have that*

A) $\{Y_i(z,\psi) = 0 \forall i \in \{1,..,k-1\}$, *and all* $(z,\psi) \in \mathcal{B}(\cdot) \times \mathcal{B}(\mu)\}$ *implies that*

B) $\{Y_k(z,\psi) \leq 0$, *for all* $(z,\psi) \in \mathcal{B}(\cdot) \times \mathcal{B}(\mu)\}$.

Assumption A.4 *We have that the statement*

A) $\{Y_i(z,\psi) = 0 \forall i \in \{1,..,j\}$, *and all* $(z,\psi) \in \mathcal{B}(\cdot) \times \mathcal{B}(\mu)\}$ *implies that*

B) $\{z = 0\}$.

A.5 Model ship parameters and plots

Variable	**Value**
M	$diag(25.8, 33.8, 2.76)$
D	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 7 & 0.1 \\ 0 & 0.1 & 0.5 \end{pmatrix}$
$k1Tp, k2Tp$	$3.74e^{-3}$
$k1Tn, k2Tn$	$5.05e^{-3}$
$k3Tp$	$1.84e^{-4}$
$k3Tn$	$1.88e^{-4}$
$k1Ln, k2Ln$	$2.10e^{-2}$
$k1Ld1, k2Ld1$	0.927
$k1Ld2, k2Ld2$	-0.557
$k1Dn, k2Dn$	$9.64e^{-3}$
$k1Dd1, k2Dd1$	0.079
$k1Dd2, k2Dd2$	0.615
a_T	1
J_{mi}	$1e^{-2}$
k_{fi}	$1e^{-4}$
m_i	$1e^{-2}$
a_i	$-1e^{-4}$
b_i	$1e^{-5}$

Table A.1: Model ship parameters

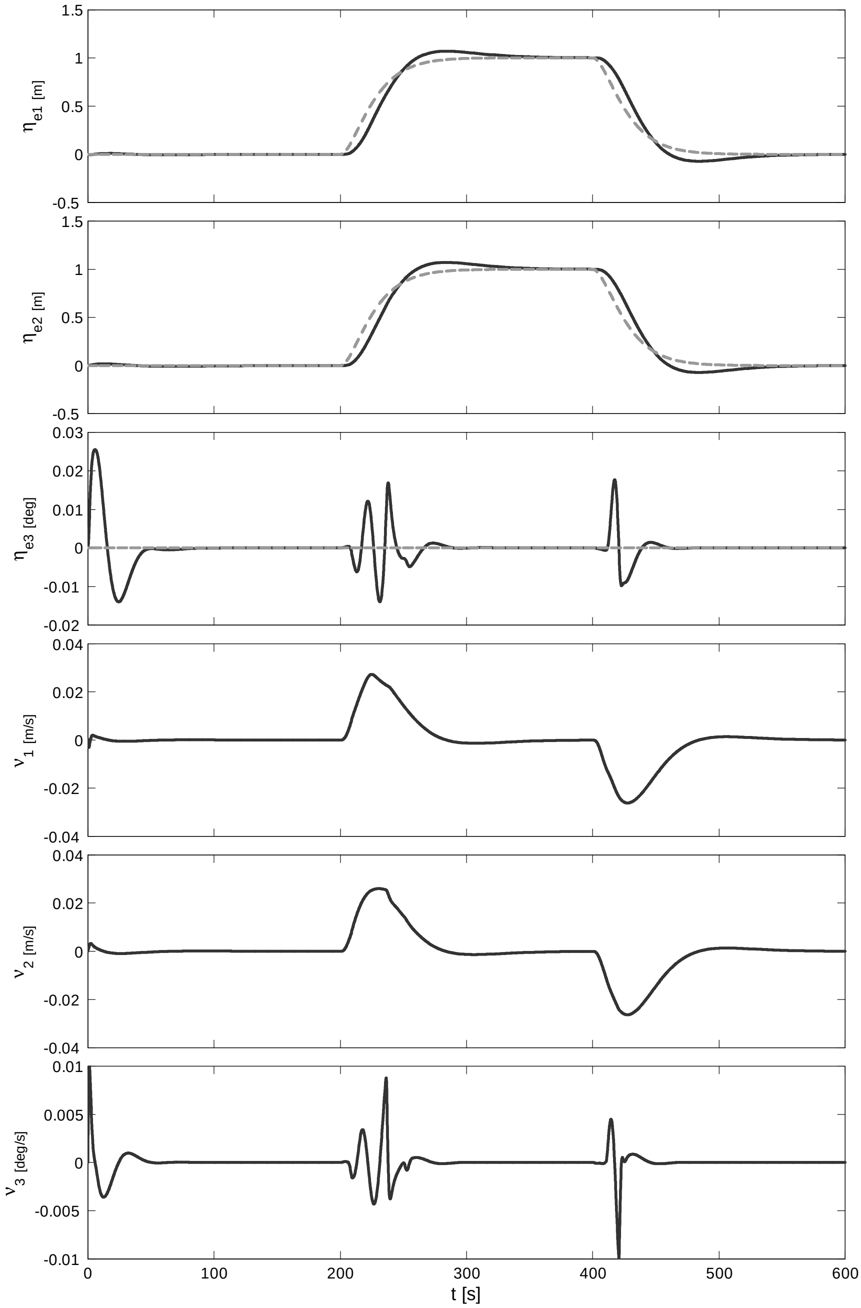

Figure A.1: The ship; desired position (dashed), actual position (solid) and velocities.

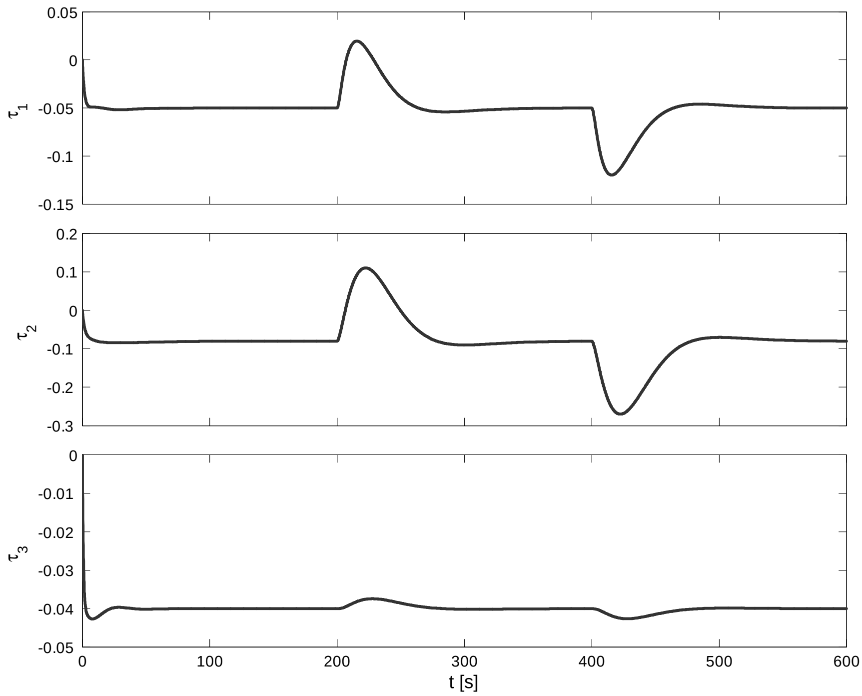

Figure A.2: The control forces.