Adaptive Control of Linear 2×2 Hyperbolic Systems

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Abstract

We design two closely related state feedback adaptive control laws for stabilization of a class of 2×2 linear hyperbolic system of partial differential equations (PDEs) with constant but uncertain in-domain and boundary parameters. One control law uses an identifier, while the other is based on swapping design. We establish boundedness of all signals in the closed loop system, pointwise in space and time, and convergence of the system states to zero pointwise in space. The theory is demonstrated in simulations.

Key words: Distributed systems; Backstepping. Adaptive control.

1 Introduction

1.1 Background

We will in this paper consider systems on the form of 2×2 linear hyperbolic partial differential equations (PDEs), which can be used to model for instance traffic flow [9] and pressure and flow profiles in oil wells [15]. Since equations of this type can be used to model a vast range of different physical systems, extensive research regarding control of this kind of systems have been performed, and we list control Lyapunov functions [7], Riemann invariants [10] and frequency approaches [16] to name a few.

The pioneering backstepping approach presented in [17] for stabilization of partial differential equations of the parabolic type, has in recent years shown to be quite useful and a general framework for analysis of PDEs. The key ingredient of this approach is the introduction of an invertible Volterra-like transformation that maps the system to be investigated into an auxiliary system designed to possess some desirable stability properties. Due to the invertibility of the transformation, the stability properties of the two systems are the same.

The first use of backstepping to hyperbolic systems was presented in [14], where among other applications, hyperbolic PDEs were used to model actuator and sensor delays in ordinary differential equations. Extensions of the backstepping technique to second order hyperbolic systems were presented in [18], and in [23] to 2×2 coupled linear hyperbolic PDEs. Explicit non-adaptive controllers for a subclass of the systems covered in [23] was also offered in [22].

Adaptive stabilization of PDEs with unknown system parameters is a field that is well-established in the case of parabolic PDEs, with contributions like [13], [19], [20] and [21]. Material regarding adaptive control of hyperbolic PDEs, however, is currently limited. The first result was presented in [6], where an adaptive output feedback control law was derived for a single hyperbolic partial-integro differential equation with non-local source terms, while a subproblem of this was presented in [24] offering a full-state feedback solution. Recently, state feedback stabilization of coupled 2×2 linear hyperbolic systems of PDEs with uncertain in-domain coefficients was solved in [2] and [3] using an identifier and swapping design, respectively. Boundedness and square integrability in the L_2-sense of the states were established, while the important result of convergence of the states to zero were not established. In the present paper, boundedness, square integrability and convergence to zero of system states pointwise in space are provided, thereby completing the missing aspects of [2], [3]. Another minor extension is provided by considering the boundary parameters unknown in addition to the in-domain coefficients considered in [2], [3]. A significant drawback of the result, limiting it's practical value, is the need for full state measurements. Full state measurements are rarely available in practice, however, for the particular problem motivating the present work, they can be considered available in an approximate sense. When drilling oil wells, it is im-

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portant to control pressure accurately in the well. The main obstacle to accurate modelling of the flow dynamics in the well is the uncertainty of friction parameters, which appear as coupling terms in the domain of the hyperbolic PDE. Emerging technology, referred to as wired pipe, allows for distributed sensing along the drill-pipe throughout the well. Sensors can be installed at every pipe connection, about 30 meters apart, thereby providing an approximate measurement of the distributed state of the PDE. That being said, solving the output feedback problem is the ultimate goal, and is a topic of our current research.

1.2 Paper structure

In Section 2, we formally pose the control problem to be investigated. An adaptive control law based on an identifier is presented in Section 3. The control law is formally stated as Theorem 4. Then, in Section 4, another control law based on swapping design is presented, and the control law is formally stated as Theorem 7. Boundedness and square integrability of all states in the closed loop system in the L_2-sense are proved for both controllers, and pointwise boundedness, square integrability and convergence to zero of the system states are also proved. The performance of the controllers is demonstrated in simulations in Section 5, while Section 6 offers some concluding remarks, and lists some pros and cons regarding the two proposed controllers.

1.3 Notation

For a time-varying, real signal $f(t)$, the following vector spaces are used

$$f \in \mathcal{L}_p \Leftrightarrow \left(\int_0^\infty |f(t)|^p dt \right)^{\frac{1}{p}} < \infty \tag{1}$$

for $p \geq 1$ with the particular case

$$f \in \mathcal{L}_\infty \Leftrightarrow \sup_{t \geq 0} |f(t)| < \infty. \tag{2}$$

For the (possibly time-varying) vector signal $u(x)$ defined for $0 \leq x \leq 1$, we introduce the following integral operator

$$I_a[u] = \int_0^1 e^{ax} u(x) dx \tag{3}$$

with the derived norm

$$||u||_a^2 = I_a\left[u^T u\right] = \int_0^1 e^{ax} u^T(x) u(x) dx. \tag{4}$$

The operator (3) is linear and has the property

$$2I_a[uu_x] = e^a u^2(1) - u^2(0) - a||u||_a^2. \tag{5}$$

The norm $||u||_a$ is equivalent to the standard L_2 norm, in the sense that there exist positive constants k_1, k_2 so that

$$k_1||u||_a \leq ||u|| \leq k_2||u||_a, \tag{6}$$

and also that $||u|| = ||u||_0$. Moreover, for the sum of norms of u and v the shorthand notation

$$||u, v|| = ||u|| + ||v|| \tag{7}$$

is used. Lastly, we will in subsequent sections often omit writing the argument in time, so that e.g. $u(x) = u(x, t)$ and $||z|| = ||z(t)||$.

2 Problem description

We consider systems on the form of 2×2 linear hyperbolic partial differential equations with constant in-domain coefficients. These type of systems were also investigated [22], and are on the form

$$u_t(x,t) + \lambda u_x(x,t) = c_1 u(x,t) + c_2 v(x,t) \tag{8a}$$

$$v_t(x,t) - \mu v_x(x,t) = c_3 u(x,t) + c_4 v(x,t) \tag{8b}$$

$$u(0,t) = qv(0,t) \tag{8c}$$

$$v(1,t) = U(t) \tag{8d}$$

defined for $0 \leq x \leq 1$, $t \geq 0$, where u, v are the system states, and

$$0 < \lambda \in \mathbb{R}, \quad 0 < \mu \in \mathbb{R} \tag{9}$$

are known transport speeds while the coefficients

$$c_1, c_2, c_3, c_4, q \in \mathbb{R} \tag{10}$$

are unknown. However, we assume we have some bounds on c_i, $i = 1 \dots 4$ and q. That is, we are in possession of some positive constants $\bar{c}_i$, $i = 1 \dots 4$ and $\bar{q}$ so that

$$|c_i| \leq \bar{c}_i, \ i = 1 \dots 4, \qquad |q| \leq \bar{q}. \tag{11}$$

These assumptions merely accommodate the use of the projection operator (see Appendix A for the definition and properties) to limit the parameter estimates, and do not restrict the class of systems (8) considered since the bounds are arbitrary. Finally, we assume the initial states $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$ satisfy

$$u_0, v_0 \in L_2. \tag{12}$$

The goal is to design a state feedback adaptive control law U that achieves regulation of the system states u and v to zero pointwise in space and time. Moreover, all additional signals should be bounded.

3 Adaptive control using an identifier

3.1 Introduction

In identifier-based design, a dynamical system - referred to as an identifier - is introduced. The identifier is usually a copy of the system dynamics with certain injection gains added for the purpose of making the adaptive laws integrable. Lyapunov theory is then used to derive adaptive laws, and also prove that the error between the system states and identifier states is bounded. The backstepping technique is used for controller design and to map the identifier into a target system for which stability analysis is easier. Boundedness of the identifier is then proved using the target system. Due to invertibility of the backstepping transform and the estimation error also being bounded, the original system states are bounded as well. An identifier is sometimes termed an observer, although it's purpose is parameter estimation and not state estimation.

3.2 Identification using an identifier

Consider the identifier

$$\partial_t \hat{u}_1(x) + \lambda \partial_x \hat{u}_1(x) = \varpi^T(x)\hat{b}_1 + \rho e_1(x)||\varpi||^2 \quad (13a)$$

$$\partial_t \hat{v}_1(x) - \mu \partial_x \hat{v}_1(x) = \varpi^T(x)\hat{b}_2 + \rho \epsilon_1(x)||\varpi||^2 \quad (13b)$$

$$\hat{u}_1(0) = \frac{\hat{q}v(0) + u(0)v^2(0)}{1 + v^2(0)} \quad (13c)$$

$$\hat{v}_1(1) = U \quad (13d)$$

for some design gain $\rho > 0$, and where

$$e_1(x) = u(x) - \hat{u}_1(x) \quad (14a)$$

$$\epsilon_1(x) = v(x) - \hat{v}_1(x). \quad (14b)$$

are errors between u and v and their estimates $\hat{u}$ and $\hat{v}$,

$$\varpi(x) = \begin{bmatrix} u(x) & v(x) \end{bmatrix}^T \quad (15a)$$

$$b_1 = \begin{bmatrix} c_1 & c_2 \end{bmatrix}^T, \quad b_2 = \begin{bmatrix} c_3 & c_4 \end{bmatrix}^T, \quad (15b)$$

and $\hat{b}_1$ and $\hat{b}_2$ are estimates of b_1 and b_2, respectively. The dynamics of (14) is

$$\partial_t e_1(x) + \lambda \partial_x e_1(x) = \varpi^T(x)\tilde{b}_1 - \rho e_1(x)||\varpi||^2 \quad (16a)$$

$$\partial_t \epsilon_1(x) - \mu \partial_x \epsilon_1(x) = \varpi^T(x)\tilde{b}_2 - \rho \epsilon_1(x)||\varpi||^2 \quad (16b)$$

$$e_1(0) = \frac{\tilde{q}v(0)}{1 + v^2(0)} \quad (16c)$$

$$\epsilon_1(1) = 0 \quad (16d)$$

where

$$\tilde{q} = q - \hat{q}, \quad \tilde{c}_i = c_i - \hat{c}_i, \quad \text{for } i = 1 \ldots 4. \quad (17)$$

Lemma 1 *Consider the system* (8)*, the identifier* (13) *and let* Γ_1, Γ_2 *be positive definite matrices,* $\gamma_5 > 0$*, and*

$$\bar{b}_1 = \begin{bmatrix} \bar{c}_1 & \bar{c}_2 \end{bmatrix}^T \qquad \bar{b}_2 = \begin{bmatrix} \bar{c}_3 & \bar{c}_4 \end{bmatrix}^T. \quad (18)$$

The following adaptive laws

$$\begin{aligned} \dot{\hat{b}}_1 &= \mathrm{proj}_{\bar{b}_1}\left\{\Gamma_1 I_{-\gamma}\left[e_1\varpi\right], \hat{b}_1\right\} \\ &= \mathrm{proj}_{\bar{b}_1}\left\{\Gamma_1 \int_0^1 e^{-\gamma x} e_1(x)\varpi(x)dx, \hat{b}_1\right\} \quad (19a) \\ \dot{\hat{b}}_2 &= \mathrm{proj}_{\bar{b}_2}\left\{\Gamma_2 I_{\gamma}\left[\epsilon_1\varpi\right], \hat{b}_2\right\} \\ &= \mathrm{proj}_{\bar{b}_2}\left\{\Gamma_2 \int_0^1 e^{\gamma x} \epsilon_1(x)\varpi(x)dx, \hat{b}_2\right\} \quad (19b) \\ \dot{\hat{q}} &= \mathrm{proj}_{\bar{q}}\left\{\gamma_5 e_1(0)v(0)\right\} \quad (19c) \end{aligned}$$

with initial conditions satisfying the bounds (11)*, guarantee the following properties*

$$|\hat{c}_i| \leq \bar{c}_i, \; i = 1 \ldots 4, \quad |\hat{q}| \leq \bar{q} \quad (20a)$$

$$||e_1||, ||\epsilon_1|| \in \mathcal{L}_\infty \cap \mathcal{L}_2 \quad (20b)$$

$$||e_1||||\varpi||, ||\epsilon_1||||\varpi|| \in \mathcal{L}_2 \quad (20c)$$

$$|e_1(0)|, |e_1(1)|, |\epsilon_1(0)|, |e_1(0)v(0)| \in \mathcal{L}_2 \quad (20d)$$

$$|\dot{\hat{b}}_1|, |\dot{\hat{b}}_2|, |\dot{\hat{q}}| \in \mathcal{L}_2 \quad (20e)$$

$$\frac{\tilde{q}v(0)}{\sqrt{1 + v^2(0)}} \in \mathcal{L}_2 \quad (20f)$$

PROOF. Consider the Lyapunov function candidate

$$V_1 = V_2 + \tilde{b}_1^T \Gamma_1^{-1} \tilde{b}_1 + \tilde{b}_2^T \Gamma_2^{-1} \tilde{b}_2 + \frac{\lambda}{2\gamma_5}\tilde{q}^2 \quad (21)$$

where

$$V_2 = ||e_1||_{-\gamma}^2 + ||\epsilon_1||_{\gamma}^2. \quad (22)$$

Differentiating (21) with respect to time and inserting the dynamics (16a)–(16b) we find

$$\begin{aligned} \dot{V}_1 &= 2I_{-\gamma}\left[e_1(-\lambda\partial_x e_1 + \varpi^T\tilde{b}_1 - \rho e_1||\varpi||^2)\right] \\ &+ 2I_{\gamma}\left[\epsilon_1(\mu\partial_x\epsilon_1 + \varpi^T\tilde{b}_2 - \rho\epsilon_1||\varpi||^2)\right] \\ &+ 2\tilde{b}_1^T\Gamma_1^{-1}\dot{\tilde{b}}_1 + 2\tilde{b}_2^T\Gamma_2^{-1}\dot{\tilde{b}}_2 + \lambda\gamma_5^{-1}\tilde{q}\dot{\tilde{q}}. \quad (23) \end{aligned}$$

Using the property (5), inserting the adaptive laws (19a) and (19b), and using property (A.12) of Lemma 9 in Appendix A give

$$\begin{aligned} \dot{V}_1 &\leq -\lambda e^{-\gamma} e_1^2(1) + \lambda e_1^2(0) - \lambda\gamma||e_1||_{-\gamma}^2 + \mu e^{\gamma}\epsilon_1^2(1) \\ &- \mu\epsilon_1^2(0) - \mu\gamma||\epsilon_1||_{\gamma}^2 - 2\rho I_{-\gamma}\left[e_1^2\right]||\varpi||^2 \\ &- 2\rho I_{\gamma}\left[\epsilon_1^2\right]||\varpi||^2 + \lambda\gamma_5^{-1}\tilde{q}\dot{\tilde{q}}. \quad (24) \end{aligned}$$

Substituting the boundary conditions (16c)–(16d) and the adaptive law (19c), using property (A.12) of Lemma 9 in Appendix A, and

$$e_1(0) = \tilde{q}v(0) - e_1(0)v^2(0), \quad (25)$$

we find

$$\begin{aligned} \dot{V}_1 &\leq -\lambda e^{-\gamma} e_1^2(1) - \mu\epsilon_1^2(0) - e_1^2(0)v^2(0) - \lambda\gamma||e_1||_{-\gamma}^2 \\ &- \mu\gamma||\epsilon_1||_{\gamma}^2 - 2\rho||e_1||_{-\gamma}^2||\varpi||^2 - 2\rho||\epsilon_1||_{\gamma}^2||\varpi||^2 \quad (26) \end{aligned}$$

which shows that V_1 is bounded and from the definition of V_1 and V_2 gives (20a) and $||e_1||, ||\epsilon_1|| \in \mathcal{L}_\infty$. Integrating (26) in time from zero to infinity gives $||e_1||, ||\epsilon_1|| \in \mathcal{L}_2$, (20c) and $|e_1(1)|, |\epsilon_1(0)|, |e_1(0)v(0)| \in \mathcal{L}_2$. From the properties (20c), $|e_1(0)v(0)| \in \mathcal{L}_2$ and the adaptive laws (19), (20e) follows. Using the following Lyapunov function candidate

$$V_3 = \frac{1}{2\gamma_5}\tilde{q}^2, \quad (27)$$

and the property (A.12), we find

$$\dot{V}_3 \leq -\tilde{q}e_1(0)v(0) \leq -\frac{\tilde{q}^2 v^2(0)}{1 + v^2(0)}. \quad (28)$$

This means that V_3 is bounded from above, and hence $V_3 \in \mathcal{L}_\infty$. Integrating (28) from zero to infinity gives $\dot{V}_3 \in \mathcal{L}_1$ and hence (20f). From (25) and (16c), we have

$$e_1^2(0) = e_1(0)(\tilde{q}v(0) - e_1(0)v^2(0)) = \frac{\tilde{q}^2 v^2(0)}{1 + v^2(0)} - e_1^2(0)v^2(0) \tag{29}$$

and from $|e_1(0)v(0)| \in \mathcal{L}_2$ and (20f), $|e_1(0)| \in \mathcal{L}_2$ follows. □

3.3 Adaptive control

In this section, a stabilizing adaptive controller is designed for system (8). The controller is derived using the backstepping technique where an invertible Volterra transformation is used to map the identifier (13) into a target system which is shown to be stable using Lyapunov analysis.

3.3.1 Kernel equations

Consider the following equation in $K = K(x, \xi, t)$, $L = L(x, \xi, t)$

$$\mu K_x - \lambda K_\xi = (\hat{c}_1 - \hat{c}_4)K + \hat{c}_3 L \tag{30a}$$
$$\mu L_x + \mu L_\xi = \hat{c}_2 K \tag{30b}$$
$$L(x, 0) = \hat{q}\frac{\lambda}{\mu}K(x, 0) \tag{30c}$$
$$K(x, x) = -\frac{\hat{c}_3}{\lambda + \mu} \tag{30d}$$

defined over $\mathcal{T}_1$, given as

$$\mathcal{T}_1 = \mathcal{T} \times \{t \geq 0\} \tag{31a}$$
$$\mathcal{T} = \{(x, \xi) \mid 0 \leq \xi \leq x \leq 1\}. \tag{31b}$$

The well-posedness of the equation is addressed in the following Lemma.

Lemma 2 *For every time $t \geq 0$, equation (30) has a unique, continuous solution (K, L) with the following properties*

$$|K(x, \xi, t)| \leq \bar{K}, \qquad \forall (x, \xi) \in \mathcal{T}, t \geq 0 \tag{32a}$$
$$|L(x, \xi, t)| \leq \bar{L}, \qquad \forall (x, \xi) \in \mathcal{T}, t \geq 0 \tag{32b}$$
$$|K_t(x, \xi, \cdot)| \in \mathcal{L}_2, \qquad \forall (x, \xi) \in \mathcal{T}. \tag{32c}$$
$$|L_t(x, \xi, \cdot)| \in \mathcal{L}_2, \qquad \forall (x, \xi) \in \mathcal{T}. \tag{32d}$$

for some positive constants $\bar{K}$, $\bar{L}$ depending on the parameter bounds (11).

The proof of this lemma is given in Appendix B.

Remark 3 *For every time $t \geq 0$, explicit solutions of (30) can (through a transformation) be found in [22].*

3.3.2 Main theorem

Theorem 4 *Consider the system (8) and the identifier (13), and consider the control law*

$$U = \int_0^1 K(1, \xi)\hat{u}_1(\xi)d\xi + \int_0^1 L(1, \xi)\hat{v}_1(\xi)d\xi \tag{33}$$

where K, L is the solution to (30). Then all signals in the closed loop system are bounded and integrable in the $\mathcal{L}_2$-sense. Moreover, $u(x, \cdot), v(x, \cdot) \in \mathcal{L}_\infty \cap \mathcal{L}_2$ for all $x \in [0, 1]$, and $u(x, \cdot), v(x, \cdot) \to 0$ for all $x \in [0, 1]$.

The proof of Theorem 4 is the subject of the next sections.

3.3.3 Backstepping transformation

For every time $t \geq 0$, consider the following adaptive backstepping transformation

$$w_1(x) = \hat{u}_1(x) \tag{34a}$$
$$z_1(x) = \hat{v}_1(x) - \int_0^x K(x, \xi)\hat{u}_1(\xi)d\xi - \int_0^x L(x, \xi)\hat{v}_1(\xi)d\xi =: T[\hat{u}_1, \hat{v}_1](x) \tag{34b}$$

where K, L is the solution to (30). As with all backstepping transformations with uniformly bounded kernels, transformation (34) is invertible with an inverse on the form

$$\hat{u}_1(x) = w_1(x) \tag{35a}$$
$$\hat{v}_1(x) = T^{-1}[w_1, z_1](x) \tag{35b}$$

where T^{-1} is an integral operator taking a similar form as (34).

Lemma 5 *The transformation (34) and the control law (33) with kernels satisfying (30) map the identifier (13) into the following target system*

$$\partial_t w_1(x) + \lambda \partial_x w_1(x) = \hat{c}_1 w_1(x) + \hat{c}_2 z_1(x) + \int_0^x \omega(x, \xi)w_1(\xi)d\xi + \int_0^x \kappa(x, \xi)z_1(\xi)d\xi + \hat{c}_1 e_1(x) + \hat{c}_2 \epsilon_1(x) + \rho e_1(x)||\varpi||^2 \tag{36a}$$
$$\partial_t z_1(x) - \mu \partial_x z_1(x) = \hat{c}_4 z_1(x) - \lambda K(x, 0)\hat{q}\epsilon_1(0) - \lambda K(x, 0)\tilde{q}v(0) + \lambda K(x, 0)e_1(0) - \int_0^x K_t(x, \xi)w_1(\xi)d\xi - \int_0^x L_t(x, \xi)T^{-1}[w_1, z_1](\xi)d\xi + T[\hat{c}_1 e_1 + \hat{c}_2 \epsilon_1, \hat{c}_3 e_1 + \hat{c}_4 \epsilon_1](x) + \rho T[e_1, \epsilon_1](x)||\varpi||^2 \tag{36b}$$
$$w_1(0) = qz_1(0) + q\epsilon_1(0) - e_1(0) \tag{36c}$$
$$z_1(1) = 0 \tag{36d}$$

with

$$\omega(x, \xi) = \hat{c}_2 K(x, \xi) + \int_\xi^x \kappa(x, s)K(s, \xi)ds \tag{37a}$$
$$\kappa(x, \xi) = \hat{c}_2 L(x, \xi) + \int_\xi^x \kappa(x, s)L(s, \xi)ds \tag{37b}$$

satisfying

$$|\omega(x, \xi, t)| \leq \bar{\omega}, \qquad \forall (x, \xi) \in \mathcal{T}, t \geq 0 \tag{38a}$$
$$|\kappa(x, \xi, t)| \leq \bar{\kappa}, \qquad \forall (x, \xi) \in \mathcal{T}, t \geq 0 \tag{38b}$$

for some positive constants $\bar{\omega}$, $\bar{\kappa}$ depending on the param-

eter bounds (11)*. Moreover, there exist positive constants* k_1 *and* k_2 *such that*

$$k_1||w_1, z_1|| \leq ||\hat{u}_1, \hat{v}_1|| \leq k_2||w_1, z_1|| \tag{39}$$

for all $t \geq 0$*, where the notation* (7) *has been used, and* k_1*,* k_2 *depend on the parameter bounds* (11)*.*

PROOF. Differentiating (34b) with respect to time, inserting the dynamics (13a)–(13b), integration by parts, and inserting the boundary condition (13c) we find

$$\begin{aligned}
\partial_t \hat{v}_1(x) &= \partial_t z_1(x) + \int_0^x K_t(x,\xi)\hat{u}_1(\xi)d\xi \\
&+ \int_0^x L_t(x,\xi)\hat{v}_1(\xi)d\xi - \lambda K(x,x)\hat{u}_1(x) \\
&+ \lambda \hat{q} K(x,0)\hat{v}_1(0) + \lambda \hat{q} K(x,0)\epsilon_1(0) \\
&+ \lambda K(x,0)\tilde{q}v(0) - \lambda K(x,0)e_1(0) \\
&+ \int_0^x K_\xi(x,\xi)\lambda\hat{u}_1(\xi)d\xi + \int_0^x K(x,\xi)\hat{c}_1\hat{u}_1(\xi)d\xi \\
&+ \int_0^x K(x,\xi)\hat{c}_1 e_1(\xi)d\xi + \int_0^x K(x,\xi)\hat{c}_2\hat{v}_1(\xi)d\xi \\
&+ \int_0^x K(x,\xi)\hat{c}_2\epsilon_1(\xi)d\xi + \rho\int_0^x K(x,\xi)e_1(\xi)d\xi||\varpi||^2 \\
&+ L(x,x)\mu\hat{v}_1(x) - L(x,0)\mu\hat{v}_1(0) \\
&- \int_0^x L_\xi(x,\xi)\mu\hat{v}_1(\xi)d\xi + \int_0^x L(x,\xi)\hat{c}_3\hat{u}_1(\xi)d\xi \\
&+ \int_0^x L(x,\xi)\hat{c}_3 e_1(\xi)d\xi + \int_0^x L(x,\xi)\hat{c}_4\hat{v}_1(\xi)d\xi \\
&+ \int_0^x L(x,\xi)\hat{c}_4\epsilon_1(\xi)d\xi \\
&+ \rho\int_0^x L(x,\xi)\epsilon_1(\xi)d\xi||\varpi||^2.
\end{aligned} \tag{40}$$

Equivalently, differentiating (34b) with respect to space, we obtain

$$\begin{aligned}
\partial_x \hat{v}_1(x) &= \partial_x z_1(x) + K(x,x)\hat{u}_1(x) + L(x,x)\hat{v}_1(x) \\
&+ \int_0^x K_x(x,\xi)\hat{u}_1(\xi)d\xi + \int_0^x L_x(x,\xi)\hat{v}_1(\xi)d\xi.
\end{aligned} \tag{41}$$

Inserting (40) and (41) into (13b), using the equations (30) and inserting the inverse transformation (35), one obtains (36b). Inserting (34) into (36a), changing the order of integration in the double integrals and using (37), we obtain (13a). The existence of a unique solution κ of (37b) is ensured by Volterra equation theory (see e.g. [4, Lemma 9] for a proof). The boundary condition (36c) follows from inserting (34) into (13c) and noting that

$$w_1(0) = qv(0) - e_1(0) \tag{42}$$

and

$$v(0) = z_1(0) + \epsilon_1(0). \tag{43}$$

From the properties (32a)–(32b) with bounds $\bar{K}$ and $\bar{L}$ depending on the parameter bounds (11), the bounds (38) immediately follow. A similar argument holds for the bounds (39), since the backstepping transformation (34) has uniformly bounded kernels and is thus invertible.

□

3.3.4 Stability

We are ready to prove the Theorem 4.

PROOF. [Proof of Theorem 4] Consider the Lyapunov function candidate

$$V_4 = V_5 + aV_6, \tag{44}$$

with

$$V_5 = ||w_1||^2_{-\delta}, \quad V_6 = ||z_1||^2_k \tag{45}$$

and where $a > 0$, k, δ are constants to be decided. It can be shown using the properties of Lemmas 1 and 5, and assuming $\delta, k \geq 1$ (see Appendix C for details), that the derivatives of (45) satisfy

$$\dot{V}_5 \leq h_1 z_1^2(0) - \left[\lambda\delta - h_2\right]V_5 + h_3 V_6 + l_1 V_5 + l_2 \tag{46a}$$

$$\begin{aligned}
\dot{V}_6 \leq &- \left[\mu - h_6\tilde{q}^2 e^k\right] z_1^2(0) + h_5 V_5 - \left[k\mu - h_4\right] V_6 \\
&+ l_3 V_5 + l_4 V_6 + l_5
\end{aligned} \tag{46b}$$

where l_i, $i = 1, \ldots, 5$ are integrable functions (i.e. in $\mathcal{L}_1$), and $h_1 \ldots h_6$ are positive constants. Letting

$$a = \frac{h_1 + 1}{\mu} \tag{47}$$

gives

$$\begin{aligned}
\dot{V}_4 \leq &-\left[\lambda\delta - h_2 - ah_5\right]V_5 - \left[ak\mu - ah_4 - h_3\right] V_6 \\
&- \left[1 - ah_6\tilde{q}^2 e^k\right] z_1^2(0) \\
&+ l_1 V_5 + al_3 V_5 + al_4 V_6 + l_2 + al_5
\end{aligned} \tag{48}$$

and selecting

$$\delta > \frac{h_2 + ah_5}{\lambda}, \qquad k > \frac{h_3 + ah_4}{a\mu} \tag{49}$$

give

$$\dot{V}_4 \leq -cV_4 - \left[1 - ah_6\tilde{q}^2 e^k\right] z_1^2(0) + l_6 V_4 + l_7 \tag{50}$$

for some positive constant c and integrable functions l_6 and l_7. From the relationship

$$z_1(0) = \hat{v}_1(0) = v(0) + \epsilon_1(0) \tag{51}$$

we find

$$\begin{aligned}
z_1^2(0) &\leq 2v^2(0) + 2\epsilon_1^2(0) \\
&\leq 2\frac{v^2(0)}{1 + v^2(0)}[1 + v^2(0)] + 2\epsilon_1^2(0) \\
&\leq 2\frac{v^2(0)}{1 + v^2(0)}\left[1 + 2z_1^2(0) + 2\epsilon_1^2(0)\right] + 2\epsilon_1^2(0)
\end{aligned}$$

$$\leq 2\frac{v^2(0)}{1+v^2(0)}\left[1+2z_1^2(0)\right]+6\epsilon_1^2(0) \tag{52}$$

Inserting this into (50), we obtain

$$\dot{V}_4 \leq -cV_4 - \left[1-b\tilde{q}^2 g\right] z_1^2(0) + l_6 V_4 + l_8 \tag{53}$$

where

$$g = \frac{v^2(0)}{1+v^2(0)} \tag{54}$$

has the property

$$0 \leq g < 1, \quad \forall t \geq 0, \tag{55}$$

the function

$$l_8 = l_7 + 2ah_6 e^k \frac{\tilde{q}^2 v^2(0)}{1+v^2(0)} + 6ah_6\tilde{q}^2 e^k \epsilon_1^2(0) \tag{56}$$

is integrable, and

$$b = 4ah_6 e^k \tag{57}$$

is a positive constant. From (28) and the property (A.12), we have

$$\dot{V}_3 \leq -\tilde{q}^2 g \tag{58}$$

It then follows from Lemma 10 in Appendix E that

$$V_4 \in \mathcal{L}_1 \cap \mathcal{L}_\infty \tag{59}$$

and hence $||w_1||$ and $||z_2||$ are bounded and integrable. Due to the equivalence of norms (39) we then have $||\hat{u}_1||, ||\hat{v}_1|| \in \mathcal{L}_\infty \cap \mathcal{L}_2$ and

$$||u||, ||v|| \in \mathcal{L}_\infty \cap \mathcal{L}_2. \tag{60}$$

We proceed by showing pointwise boundedness and square integrability. It was in [23] shown that system (8) is through an invertible backstepping transformation equivalent to the system

$$\alpha_t(x) + \lambda\alpha_x(x) = h(x)\beta(0) \tag{61a}$$
$$\beta_t(x) - \mu\beta_x(x) = 0 \tag{61b}$$
$$\alpha(0) = q\beta(0) \tag{61c}$$
$$\beta(1) = U - \int_0^1 G_1(\xi)u(\xi)d\xi - \int_0^1 G_2(\xi)v(\xi)d\xi, \tag{61d}$$

for some bounded functions $h, G_1, G_2 \in \mathcal{C}$ of the unknown parameters $c_1 \dots c_4$ and q. Equation (61) can explicitly be solved for $t > \lambda^{-1} + \mu^{-1}$ to yield (see e.g. [1, Lemma 1] for details)

$$\alpha(x,t) = q\beta(1, t-\mu^{-1}-\lambda^{-1}x) + \lambda^{-1}\int_0^x h(\tau)\beta(1, t-\mu^{-1}-\lambda^{-1}(x-\tau))d\tau \tag{62a}$$
$$\beta(x,t) = \beta(1, t-\mu^{-1}(1-x)). \tag{62b}$$

From (61d), the control law (33) and the properties $||u||, ||v||, ||\hat{u}_1||, ||\hat{v}_1|| \in \mathcal{L}_\infty \cap \mathcal{L}_2$, it follows that $\beta(1,\cdot) \in \mathcal{L}_\infty \cap \mathcal{L}_2$, and from (62), that $\alpha(x,\cdot), \beta(x,\cdot) \in \mathcal{L}_\infty \cap \mathcal{L}_2$ for all $x \in [0,1]$. From the invertibility of the transform, we will therefore also have $u(x,\cdot), v(x,\cdot) \in \mathcal{L}_\infty \cap \mathcal{L}_2$ for all $x \in [0,1]$. From the structure of the identifier (13), we will also have $\hat{u}_1(x,\cdot), \hat{v}_1(x,\cdot) \in \mathcal{L}_\infty \cap \mathcal{L}_2$, and hence $e_1(x,\cdot), \epsilon_1(x,\cdot) \in \mathcal{L}_\infty \cap \mathcal{L}_2$.

We proceed by proving convergence to zero. We will use Barbalat's Lemma (see e.g. [12, Corollary A.7]), which can be applied if $V_4 \in \mathcal{L}_1 \cap \mathcal{L}_\infty$, and $\dot{V}_4 \in \mathcal{L}_\infty$. We already know that $V_4 \in \mathcal{L}_1 \cap \mathcal{L}_\infty$, and it can be seen from the derivations of (46a) and (46b) stated in Appendix C, that $\dot{V}_4 \in \mathcal{L}_\infty$ if $e_1(0), \epsilon_1(0), z_1(0) \in \mathcal{L}_\infty$. The first one follows trivially from (16c), while the two latter follow from $\epsilon_1(x,\cdot), \hat{v}(x,\cdot), \in \mathcal{L}_\infty$, since $z_1(0) = \hat{v}_1(0)$. Thus, $\dot{V}_4 \in \mathcal{L}_\infty$ and Barbalat's Lemma gives $V_4 \to 0$ and hence $||w_1||, ||z_1|| \to 0$ and $||\hat{u}_1||, ||\hat{v}_1|| \to 0$. We note from (22) that $\dot{V}_2 \in \mathcal{L}_\infty$, and hence $||e_1||, ||\epsilon_2|| \to 0$, which from (14) implies

$$||u||, ||v|| \to 0. \tag{63}$$

The same argument as above using (62) then yields $\alpha(x,\cdot), \beta(x,\cdot) \to 0$ and thus

$$u(x,\cdot), v(x,\cdot) \to 0 \tag{64}$$

for all $x \in [0,1]$. □

4 Swapping design

4.1 Introduction

When using swapping design, one designs filters created so that they can be used to express the system states as linear, static combinations of the filters, the unknown parameters and some error terms. The error terms are then shown to go to zero. From the static parameterization of the system states, standard parameter identification laws can be used to estimate the unknown parameters. The number of filters required when using this method equals the number of unknown parameters plus one.

4.2 Filter equations

We introduce the boundary parameter filter

$$\eta_t(x) + \lambda\eta_x(x) = 0, \qquad \eta(0) = v(0) \tag{65a}$$
$$\phi_t(x) - \mu\phi_x(x) = 0, \qquad \phi(1) = U \tag{65b}$$
$$p_t(x) + \lambda p_x(x) = \varpi(x), \qquad p(0) = 0 \tag{65c}$$
$$r_t(x) - \mu r_x(x) = \varpi(x), \qquad r(1) = 0 \tag{65d}$$

where

$$p(x) = \left[p_1(x) \;\; p_2(x)\right]^T \tag{66a}$$
$$r(x) = \left[r_1(x) \;\; r_2(x)\right]^T \tag{66b}$$

and ϖ is defined in (15a). All filters are defined for $x \in [0,1]$, $t \geq 0$, and the initial conditions $\eta(x,0) = \eta_0(x)$, $\phi(x,0) = \phi_0(x)$, $p(x,0) = p_0(x)$ and $r(x,0) = r_0(x,0)$

are assumed to satisfy $\eta_0, \phi_0, p_0, r_0 \in L_2([0,1])$. We can then construct non-adaptive estimates of the system states as

$$\bar{u}(x) = p^T(x)b_1 + \eta(x)q \tag{67a}$$
$$\bar{v}(x) = r^T(x)b_2 + \phi(x) \tag{67b}$$

where b_1 and b_2 are defined in (15b). The corresponding estimation errors

$$e_2(x) = u(x) - \bar{u}(x) \tag{68a}$$
$$\epsilon_2(x) = v(x) - \bar{v}(x) \tag{68b}$$

can straightforwardly be shown to satisfy

$$\partial_t e_2(x) + \lambda \partial_x e_2(x) = 0 \tag{69a}$$
$$\partial_t \epsilon_2(x) - \mu \partial_x \epsilon_2(x) = 0 \tag{69b}$$
$$e_2(0) = 0 \tag{69c}$$
$$\epsilon_2(1) = 0, \tag{69d}$$

which will be identically zero for $t > \max\{\lambda^{-1}, \mu^{-1}\}$.

4.3 Adaptive laws

Motivated by the static relationships (67) and (71), we propose the following adaptive state estimates $\hat{u}_2$ and $\hat{v}_2$

$$\hat{u}_2(x) = p^T(x)\hat{b}_1 + \eta(x)\hat{q} \tag{70a}$$
$$\hat{v}_2(x) = r^T(x)\hat{b}_2 + \phi(x). \tag{70b}$$

with associated "prediction errors"

$$\hat{e}_2(x) = u(x) - \hat{u}_2(x) \tag{71a}$$
$$\hat{\epsilon}_2(x) = v(x) - \hat{v}_2(x), \tag{71b}$$

and the following adaptive laws

$$\dot{\hat{b}}_1 = \mathrm{proj}_{\bar{b}_1}\left\{\Gamma_1 \frac{\int_0^1 p(x)\hat{e}_2(x)dx}{1 + ||p||^2 + ||\eta||^2}\right\} \tag{72a}$$
$$\dot{\hat{b}}_2 = \mathrm{proj}_{\bar{b}_2}\left\{\Gamma_2\left[\frac{\int_0^1 r(x)\hat{\epsilon}_2(x)dx}{1 + ||r||^2} + \frac{\hat{\epsilon}_2(0)r(0)}{1 + |r(0)|^2}\right]\right\} \tag{72b}$$
$$\dot{\hat{q}} = \mathrm{proj}_{\bar{q}}\left\{\gamma_5 \frac{\int_0^1 \eta(x)\hat{e}_2(x)dx}{1 + ||p||^2 + ||\eta||^2}\right\}, \tag{72c}$$

with initial conditions chosen inside the bounds (11), and

$$\Gamma_1 = \mathrm{diag}\{\gamma_1, \gamma_2\}, \qquad \Gamma_2 = \mathrm{diag}\{\gamma_3, \gamma_4\} \tag{73}$$

with γ_i, $i = 1 \dots 5$ as positive design gains.

Lemma 6 *Consider the system* (8) *with filters* (65a)–(65d)*. The adaptive laws* (72) *guarantee the following properties*

$$|\hat{c}_i| \leq \bar{c}_i, \; i = 1 \dots 4, \quad |\hat{q}| \leq \bar{q} \tag{74a}$$
$$||e_2||, ||\epsilon_2|| \in \mathcal{L}_\infty \cap \mathcal{L}_2 \tag{74b}$$
$$\frac{||\hat{e}_2||}{\sqrt{1 + ||\eta||^2 + ||p||^2}}, \frac{||\hat{\epsilon}_2||}{\sqrt{1 + ||r||^2}} \in \mathcal{L}_2 \tag{74c}$$
$$|e_2(1)|, |\epsilon_2(0)| \in \mathcal{L}_2 \tag{74d}$$
$$|\dot{\hat{b}}_1|, |\dot{\hat{b}}_2|, |\dot{\hat{q}}| \in \mathcal{L}_\infty \cap \mathcal{L}_2 \tag{74e}$$
$$\frac{\hat{\epsilon}_2(0)}{\sqrt{1 + |r(0)|^2}} \in \mathcal{L}_\infty \cap \mathcal{L}_2 \tag{74f}$$

PROOF. Consider the Lyapunov function candidate

$$V_7 = \lambda^{-1}\int_0^1 (2 - x)e_2^2(x)dx + \mu^{-1}\int_0^1 (1 + x)\epsilon_2^2(x)dx + \frac{1}{2}\tilde{b}_1^2\Gamma_1^{-1}\tilde{b}_1 + \frac{1}{2}\tilde{b}_2^2\Gamma_2^{-1}\tilde{b}_2 + \frac{1}{2}\gamma_5^{-1}\tilde{q}^2. \tag{75}$$

Differentiating with respect to time, inserting the adaptive laws (72), using property (A.12) of Lemma 9 in Appendix A, and integration by parts, we obtain

$$\dot{V}_7 \leq -e_2^2(1) - \int_0^1 e_2^2(x)dx - \epsilon_2^2(0) - \int_0^1 \epsilon_2^2(x)dx - \frac{\int_0^1 (p^T(x)\tilde{b}_1 + \eta(x)\tilde{q})\hat{e}_2(x)dx}{1 + ||p||^2 + ||\eta||^2} - \frac{\int_0^1 r^T(x)\tilde{b}_2\hat{\epsilon}_2(x)dx}{1 + ||r||^2} - \frac{r^T(0)\tilde{b}_2\hat{\epsilon}_2(0)}{1 + |r(0)|^2}. \tag{76}$$

From (67), (68), (71) and (70), we note that

$$\hat{e}_2(x) - e_2(x) = p^T(x)\tilde{b}_1 + \eta(x)\tilde{q} \tag{77a}$$
$$\hat{\epsilon}_2(x) - \epsilon_2(x) = r^T(x)\tilde{b}_2 \tag{77b}$$

and find

$$\dot{V}_7 \leq -e_2^2(1) - \int_0^1 e_2^2(x)dx - \epsilon_2^2(0) - \int_0^1 \epsilon_2^2(x)dx - \frac{\int_0^1 \hat{e}_2^2(x)dx}{1 + ||\eta||^2 + ||p||^2} + \frac{\int_0^1 \hat{e}_2(x)e_2(x)dx}{1 + ||\eta||^2 + ||p||^2} - \frac{\int_0^1 \hat{\epsilon}_2^2(x)dx}{1 + ||r||^2} + \frac{\int_0^1 \hat{\epsilon}_2(x)\epsilon_2(x)dx}{1 + ||r||^2} - \frac{\hat{\epsilon}_2^2(0)}{1 + |r(0)|^2} + \frac{\hat{\epsilon}_2(0)\epsilon_2(0)}{1 + |r(0)|^2}. \tag{78}$$

Using Cauchy-Schwarz' inequality on the cross terms, one finds

$$\dot{V}_7 \leq -e_2^2(1) - \frac{1}{2}\epsilon_2^2(0) - \frac{1}{2}||e_2||^2 - \frac{1}{2}||\epsilon_2||^2 - \frac{1}{2}\frac{||\hat{e}_2||^2}{1 + ||\eta||^2 + ||r||^2} - \frac{1}{2}\frac{||\hat{\epsilon}_2||^2}{1 + ||r||^2} - \frac{1}{2}\frac{\hat{\epsilon}_2^2(0)}{1 + |r(0)|^2}. \tag{79}$$

Thus V_7 is bounded, and (74a) and $||e_2||, ||\epsilon_2|| \in \mathcal{L}_\infty$ follow. Integrating (79) gives $||e_2||, ||\epsilon_2|| \in \mathcal{L}_2$, (74c)–(74d) and $\frac{\hat{\epsilon}_2(0)}{\sqrt{1+|r(0)|^2}} \in \mathcal{L}_2$. Inserting

$$\hat{\epsilon}_2(0) = \epsilon_2(0) + r^T(0)\tilde{b}_2 \tag{80}$$

into $\frac{\hat{\epsilon}_2(0)}{\sqrt{1+|r(0)|^2}}$, we obtain

$$\frac{|\hat{\epsilon}_2(0)|}{\sqrt{1+|r(0)|^2}} \leq \frac{|\epsilon_2(0)|}{\sqrt{1+|r(0)|^2}} + |\tilde{b}_2|. \tag{81}$$

Since $\epsilon_2(0) = 0$ for $t > \mu^{-1}$, we obtain $\frac{\hat{\epsilon}_2(0)}{\sqrt{1+|r(0)|^2}} \in \mathcal{L}_\infty$. Lastly, consider for instance the first adaptive law of (72a). Using the Cauchy-Schwarz inequality

$$\begin{aligned}
|\dot{\hat{c}}_1| &\leq \gamma_1 \frac{\int_0^1 |p_1(x)\hat{e}_2(x)|dx}{1+||\eta||^2+||p||^2} \\
&\leq \gamma_1 \frac{||p_1||}{\sqrt{1+||\eta||^2+||p||^2}} \frac{||\hat{e}_2||}{\sqrt{1+||\eta||^2+||p||^2}} \\
&\leq \gamma_1 \frac{||\hat{e}_2||}{\sqrt{1+||\eta||^2+||p||^2}}
\end{aligned} \tag{82}$$

which from (74c) shows that $|\dot{\hat{c}}_1|$ is square integrable. Using (77a) and the fact that $e_2 \equiv 0$ for $t > \lambda^{-1}$, we get

$$\begin{aligned}
|\dot{\hat{c}}_1| &\leq \gamma_1 \frac{||p_1||||\hat{e}_2||}{1+||\eta||^2+||p||^2} \leq \gamma_1 \frac{||\hat{e}_2||}{\sqrt{1+||\eta||^2+||p||^2}} \\
&\leq \gamma_1 \frac{||p^T\tilde{b}_1|| + ||\eta\tilde{q}||}{\sqrt{1+||\eta||^2+||p||^2}} \\
&\leq \gamma_1(|\tilde{b}_1| + |\tilde{q}|)
\end{aligned} \tag{83}$$

which from (74a) shows that $|\dot{\hat{c}}_1|$ is bounded. A similar argument holds for the additional adaptive laws (72), and hence (74e) follows. □

4.4 Adaptive control

4.4.1 Main theorem

Theorem 7 *Consider the system* (8) *and the state estimates $\hat{u}$, $\hat{v}$ generated from* (70) *using the filters* (65a)–(65d) *and the adaptive laws* (72). *Consider also the control law*

$$U = \int_0^1 K(1,\xi)\hat{u}_2(\xi)d\xi + \int_0^1 L(1,\xi)\hat{v}_2(\xi)d\xi \tag{84}$$

where (K, L) is the solution to (30). *Then all signals in the closed loop system are bounded and integrable in the L_2-sense. Moreover, $u(x,\cdot), v(x,\cdot) \in \mathcal{L}_\infty \cap \mathcal{L}_2$ for all $x \in [0,1]$, and $u(x,\cdot), v(x,\cdot) \to 0$ for all $x \in [0,1]$.*

Again, we split the proof over the next subsections.

4.4.2 State estimate dynamics

First off, we state the dynamics for the estimates $\hat{u}_2$ and $\hat{v}_2$ generated from (70). It can straight forwardly be shown to be

$$\begin{aligned}
\partial_t\hat{u}_2(x) + \lambda\partial_x\hat{u}_2(x) &= \hat{c}_1u(x) + \hat{c}_2v(x) \\
&\quad + p^T(x)\dot{\hat{b}}_1 + \eta(x)\dot{\hat{q}} &&(85a)\\
\partial_t\hat{v}_2(x) - \mu\partial_x\hat{v}_2(x) &= \hat{c}_3u(x) + \hat{c}_4v(x) \\
&\quad + r^T(x)\dot{\hat{b}}_2 &&(85b)\\
\hat{u}_2(0) &= \hat{q}v(0) &&(85c)\\
\hat{v}_2(1) &= U. &&(85d)
\end{aligned}$$

4.4.3 Backstepping

Consider the same backstepping transformation as in the previous section, namely (34)

$$\begin{aligned}
w_2(x) &= \hat{u}_2(x) &&(86a)\\
z_2(x) &= T[\hat{u}_2, \hat{v}_2](x) &&(86b)
\end{aligned}$$

with inverse (35), namely

$$\begin{aligned}
\hat{u}_2(x) &= w_2(x) &&(87a)\\
\hat{v}_2(x) &= T^{-1}[w_2, z_2](x). &&(87b)
\end{aligned}$$

Lemma 8 *The backstepping transformation* (86) *and the controller* (84) *with the backstepping kernels satisfying* (30) *maps the dynamics* (85) *into the target system*

$$\begin{aligned}
\partial_tw_2(x) + \lambda\partial_xw_2(x) &= \hat{c}_1w_2(x) + \hat{c}_2z_2(x) \\
&\quad + \hat{c}_1\hat{e}_2(x) + \hat{c}_2\hat{\epsilon}_2(x) \\
&\quad + \int_0^x \omega(x,\xi)w_2(\xi)d\xi + \int_0^x \kappa(x,\xi)z_2(\xi)d\xi \\
&\quad + p^T(x)\dot{\hat{b}}_1 + \eta(x)\dot{\hat{q}} &&(88a)\\
\partial_tz_2(x) - \mu\partial_xz_2(x) &= \hat{c}_4z_2(x) - \int_0^x K_t(x,\xi)w_2(\xi)d\xi \\
&\quad - \int_0^x L_t(x,\xi)T^{-1}[w_2,z_2](\xi)d\xi \\
&\quad - \lambda K(x,0)\hat{q}\hat{\epsilon}_2(0) \\
&\quad + T[\hat{c}_2\hat{\epsilon}_2 + \hat{c}_1\hat{e}_2, \hat{c}_3\hat{e}_2 + \hat{c}_4\hat{\epsilon}_2](x) \\
&\quad - \int_0^x K(x,\xi)\eta(\xi)d\xi\dot{\hat{q}} \\
&\quad + T[p^T\dot{\hat{b}}_1, r^T\dot{\hat{b}}_2](x) &&(88b)\\
w_2(0) &= \hat{q}z_2(0) + \hat{q}\hat{\epsilon}_2(0) &&(88c)\\
z_2(1) &= 0, &&(88d)
\end{aligned}$$

where ω, κ satisfy (38). *Moreover, there exists constants k_3 and k_4 such that*

$$k_3||w_2, z_2|| \leq ||\hat{u}_2, \hat{v}_2|| \leq k_4||w_2, z_2|| \tag{89}$$

for all $t \geq 0$.

PROOF. The proof follows the same steps as the proof of Lemma 5 and is therefore omitted. □

4.4.4 Boundedness

PROOF. [Proof of Theorem 7] Consider the Lyapunov function candidate

$$V_8 = \sum_{i=9}^{13} a_iV_i \tag{90}$$

where

$$\begin{aligned}
&V_9 = ||p||^2_{-\delta} \quad V_{10} = ||r||^2_k \quad V_{11} = ||\eta||^2_{-\delta} &&(91a)\\
&V_{12} = ||w_2||^2_{-\delta} \quad V_{13} = ||z_2||^2_k &&(91b)
\end{aligned}$$

and the coefficients a_i, $i = 9 \dots 13$ are some positive constants.

It can be shown using the properties of Lemmas 6 and 8, and assuming $\delta, k \geq 1$ (see Appendix D for details), that

$$\begin{aligned}\dot{V}_9 &\leq -(\lambda\delta - 1)V_9 + h_7 e^{\delta} V_{12} + h_8 V_{13} \\ &\quad + l_9 V_{11} + l_9 V_9 + l_{10} V_{10} + l_{11} \end{aligned} \tag{92a}$$

$$\begin{aligned}\dot{V}_{10} &\leq -\mu |r(0)|^2 - (\mu k - 1)V_{10} + h_7 e^{\delta + k} V_{12} + h_8 V_{13} \\ &\quad + l_{12} V_9 + l_{13} V_{10} + l_{12} V_{11} + l_{14} \end{aligned} \tag{92b}$$

$$\dot{V}_{11} \leq h_9 z_2^2(0) + h_9 \hat{\epsilon}_2^2(0) - \lambda\delta V_{11} \tag{92c}$$

$$\begin{aligned}\dot{V}_{12} &\leq h_{10} z_2^2(0) + h_{10} \hat{\epsilon}_2^2(0) - [\lambda\delta - h_{11}] V_{12} \\ &\quad + 2V_{13} + l_{15} V_9 + l_{16} V_{10} + l_{16} V_{11} + l_{18} \end{aligned} \tag{92d}$$

$$\begin{aligned}\dot{V}_{13} &\leq -\mu z_2^2(0) + e^k \epsilon_2^2(0) - [k\mu - h_{12}] V_{13} \\ &\quad + l_{19} V_9 + l_{20} V_{10} + l_{21} V_{11} \\ &\quad + l_{22} V_{12} + l_{23} V_{13} + l_{24} \end{aligned} \tag{92e}$$

where $h_7 \dots h_{12}$ are positive constants and $l_8 \dots l_{24}$ are integrable functions. From (90), we find

$$\begin{aligned}\dot{V}_8 &\leq -a_9(\lambda\delta - 1)V_9 - a_{10}(\mu k - 1)V_{10} - a_{11}\lambda\delta V_{11} \\ &\quad - \left[a_{12}\lambda\delta - a_{12}h_{11} - a_9 h_7 e^{\delta} - a_{10} h_7 e^{\delta+k}\right] V_{12} \\ &\quad - [a_{13}k\mu - a_{13}h_{12} - a_9 h_8 - a_{10} h_8 - 2a_{12}] V_{13} \\ &\quad - a_{10}\mu r^2(0) - [a_{13}\mu - a_{11}h_9 - a_{12}h_{10}] z_2^2(0) \\ &\quad + \left[a_{11}h_9 + a_{12}h_{10} + a_{13}e^k\right] \hat{\epsilon}_2^2(0) \\ &\quad + [a_9 l_9 + a_{10} l_{12} + a_{12} l_{15} + a_{13} l_{19}] V_9 \\ &\quad + [a_9 l_{10} + a_{10} l_{13} + a_{12} l_{16} + a_{13} l_{20}] V_{10} \\ &\quad + [a_9 l_9 + a_{10} l_{12}\ a_{12} l_{16} + a_{13} l_{21}] V_{11} + a_{10} l_{14} \\ &\quad + a_{13} l_{22} V_{12} + a_{13} l_{23} V_{13} + a_9 l_{11} \\ &\quad + a_{12} l_{18} + a_{13} l_{24} \end{aligned} \tag{93}$$

Let

$$a_9 = e^{-\delta} \qquad a_{10} = e^{-k-\delta} \qquad a_{11} = 1 \tag{94a}$$

$$a_{12} = 1 \qquad a_{13} = \frac{h_9 + h_{10}}{\mu} \tag{94b}$$

and then choose

$$\delta > \max\left\{1, \frac{1}{\lambda}, \frac{h_{11} + 2h_7}{\lambda}\right\} \tag{95}$$

$$k > \max\left\{1, \frac{1}{\mu}, \frac{a_{13}h_{12} + 2h_8 + 2}{a_{13}\mu}\right\} \tag{96}$$

then

$$\dot{V}_8 \leq -cV_8 - a_{10}\mu r^2(0) + b\hat{\epsilon}_2^2(0) + l_{25}V_8 + l_{26} \tag{97}$$

for some integrable functions l_{25} and l_{26} and positive constants c and

$$b = h_9 + h_{10} + a_{13}e^k. \tag{98}$$

We now rewrite $\hat{\epsilon}_2^2(0)$ as

$$\hat{\epsilon}_2^2(0) = \frac{\hat{\epsilon}_2^2(0)}{1 + |r(0)|^2} + \frac{\hat{\epsilon}_2^2(0)}{1 + |r(0)|^2}|r(0)|^2, \tag{99}$$

where the first term on the right hand side is integrable. Inserting this, we find

$$\begin{aligned}\dot{V}_8 &\leq -cV_8 - \left[a_{10}\mu - b\frac{\hat{\epsilon}_2^2(0)}{1 + |r(0)|^2}\right] |r(0)|^2 \\ &\quad + l_{25}V_8 + l_{27} \end{aligned} \tag{100}$$

where

$$l_{27} = l_{26} + b\frac{\hat{\epsilon}_2^2(0)}{1 + |r(0)|^2} \tag{101}$$

is an integrable function. Using

$$\hat{\epsilon}_2(0) = \epsilon_2(0) + r^T(0)\tilde{b}_2 \tag{102}$$

and the fact that $\epsilon_2(0) = 0$ for $t > \mu^{-1}$, we obtain

$$\begin{aligned}\dot{V}_8 &\leq -cV_8 - \left[a_{10}\mu - b\tilde{b}_2^T G\tilde{b}_2\right] |r(0)|^2 \\ &\quad + l_{25}V_8 + l_{27} \end{aligned} \tag{103}$$

where

$$G = \frac{r(0)r^T(0)}{1 + |r(0)|^2} \tag{104}$$

satisfies $0 \leq G = G^T < 0$. Furthermore, from the adaptive law (72b) and the property (A.12), we have that

$$V_{14} = \frac{1}{2}\tilde{b}_2^T \Gamma_2^{-1} \tilde{b}_2 \tag{105}$$

satisfies

$$\dot{V}_{14} \leq -\tilde{b}_2^T G \tilde{b}_2. \tag{106}$$

It then follows from Lemma 10 in Appendix E that

$$V_8 \in \mathcal{L}_1 \cap \mathcal{L}_\infty \tag{107}$$

and $||p||, ||r||, ||\eta||, ||w||, ||z|| \in \mathcal{L}_\infty \cap \mathcal{L}_2$ are established. From the control law (84) and (70), it follows that U is bounded and integrable, and from (65b), that $\phi(x)$ is bounded and integrable for all x. From (68) and (67), we have

$$||u|| \leq ||p|||b_1| + ||\eta|||q| + ||e|| \tag{108a}$$

$$||v|| \leq ||r|||b_2| + ||\phi|| + ||\epsilon||, \tag{108b}$$

and hence, from (74a)–(74b), we get

$$||u||, ||v|| \in \mathcal{L}_\infty \cap \mathcal{L}_2. \tag{109}$$

From (61) and (62), we get

$$u(x, \cdot), v(x, \cdot) \in \mathcal{L}_\infty \cap \mathcal{L}_2 \tag{110}$$

for all $x \in [0, 1]$, and from the structure of the filters, it immediately follows that

$$p_1(x, \cdot), p_2(x, \cdot), r_1(x, \cdot), r_2(x, \cdot) \in \mathcal{L}_\infty \cap \mathcal{L}_2 \tag{111a}$$

$$\eta(x, \cdot), \phi(x, \cdot) \in \mathcal{L}_\infty \cap \mathcal{L}_2 \tag{111b}$$

for all $x \in [0, 1]$. Then the same line of reasoning using

Barbalat's Lemma as in the proof of Theorem 4 gives

$$u(x, \cdot), v(x, \cdot) \to 0. \tag{112}$$

for all $x \in [0, 1]$. □

5 Simulation

The system (8) and the control laws of Theorem 4 and 7 were implemented in MATLAB. The system parameters were set to

$$c_1 = -0.1,\ c_2 = 1,\ c_3 = 0.4,\ c_4 = 0.2,\ q = 4 \tag{113}$$

while the a priori upper bounds were set to

$$\bar{c}_1 = 10,\ \bar{c}_2 = 10,\ \bar{c}_3 = 10,\ \bar{c}_4 = 10,\ \bar{q} = 10. \tag{114}$$

This system is open loop ($U \equiv 0$) unstable. The design gains were for both controllers set to

$$\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = 10, \tag{115}$$

while for the controller of Theorem 4, we set

$$\gamma = \rho = 0.01. \tag{116}$$

The initial conditions for the system were in both cases the following

$$u(x, 0) = \sin(2\pi x), \qquad v(x, 0) = x, \tag{117}$$

while the initial conditions of the identifier, filters and estimated parameters were all set to zero. The system states for the case of open loop is clearly shown to diverge in Figure 1. In both the adaptive control cases, the system states are seen to be bounded and converge to zero in Figures 2 and 5, respectively. The actuation signals U are also seen to converge to zero in Figures 3 and 6. All estimated parameters are seen to be bounded in Figures 4 and 7. Convergence of the estimated parameters is not guaranteed by any of the derived control laws, however. The identifier-based method is slightly more aggressive in terms of control, as can be seen from comparing Figure 3 with Figure 6. However, this does not have an impact on the state norms observed in Figures 4 and 7.

Simulations were also performed to test for robustness with respect to errors in the transport speeds λ and μ, and indicate robustness to small errors in these parameters. For the specific case (113)–(117), stability was preserved for errors up to plus/minus 10%.

6 Conclusions

We have derived two adaptive control laws for stabilization of 2×2 linear hyperbolic PDEs with constant in-domain coefficients. One control law is based on an identifier, while the other is based on swapping design. Proof of pointwise boundedness, square integrability and convergence to zero are given and the theory is verified in simulations.

The identifier based method requires the introduction of an auxiliary system - the identifier - which has a dynamical order equal to the system itself, regardless of the number of unknowns. However, for swapping design, one requires the introduction of a number of filters equal to the number of unknown parameters plus one, and hence, the dynamic order is higher. One advantage of the swapping based controller, is that the static formulation opens for a larger family of adaptive laws to be used, as well as the possibility to create adaptive laws that are not only square integrable, but also bounded.

Simulations did not show any particular advantage of choosing one controller over the other one, with the two performing very similarly and both achieving convergence of the system states to zero after approximately the same amount of time. The identifier-based method is slightly more aggressive in terms of control effort, but not enough to have significant effect on the state norms. The transient performance can be slightly altered by tuning the adaptation gains used.

An obvious drawback of the proposed methods, is the need to have distributed measurements of the system states, a property which is usually unrealistic in practice. Ultimately, stabilization using output feedback should therefore be investigated.

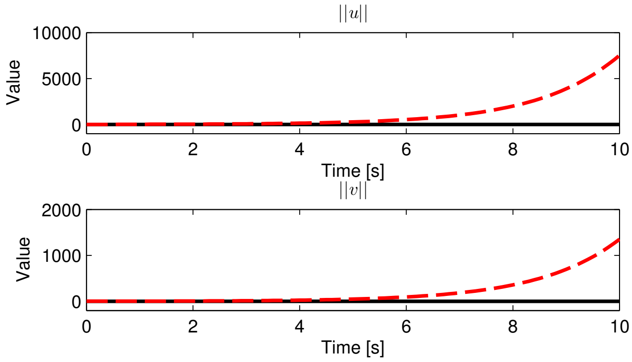

Fig. 1. Norm of the system states in the open loop case.

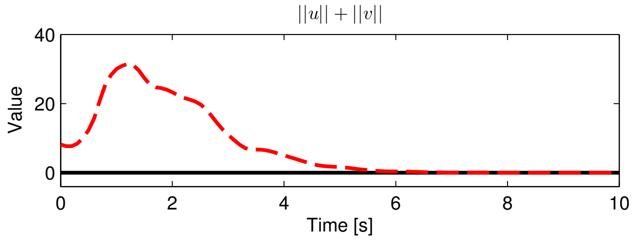

Fig. 2. Norm of the system states for the controller of Theorem 4.

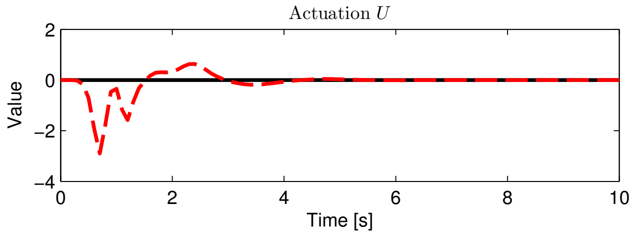

Fig. 3. Actuation signal for the controller of Theorem 4.

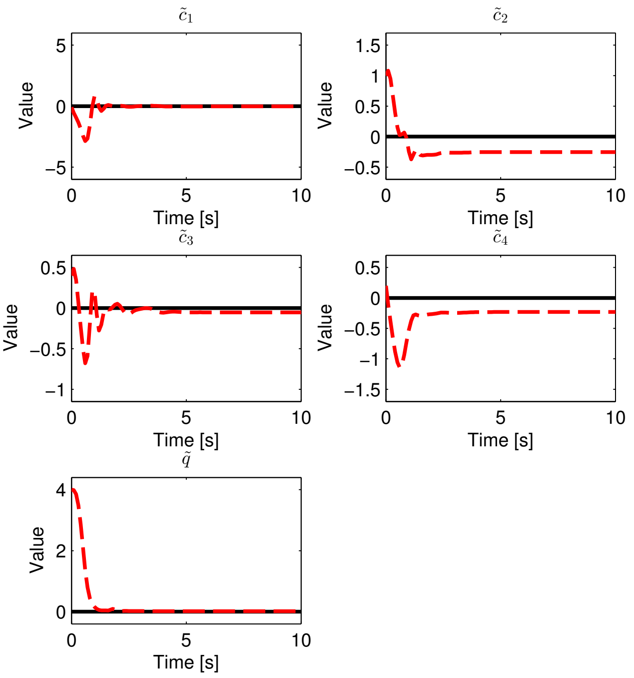

Fig. 4. Parameter estimation errors for the controller of Theorem 4.

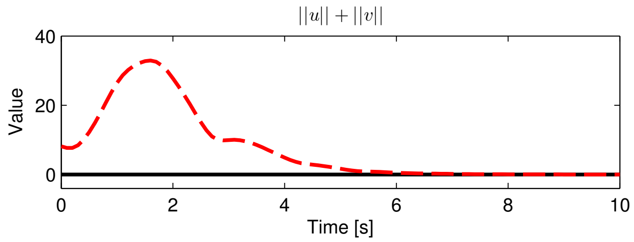

Fig. 5. Norm of the system states for the controller of Theorem 7.

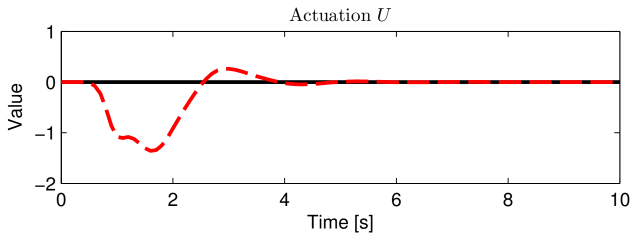

Fig. 6. Actuation signal for the controller of Theorem 7.

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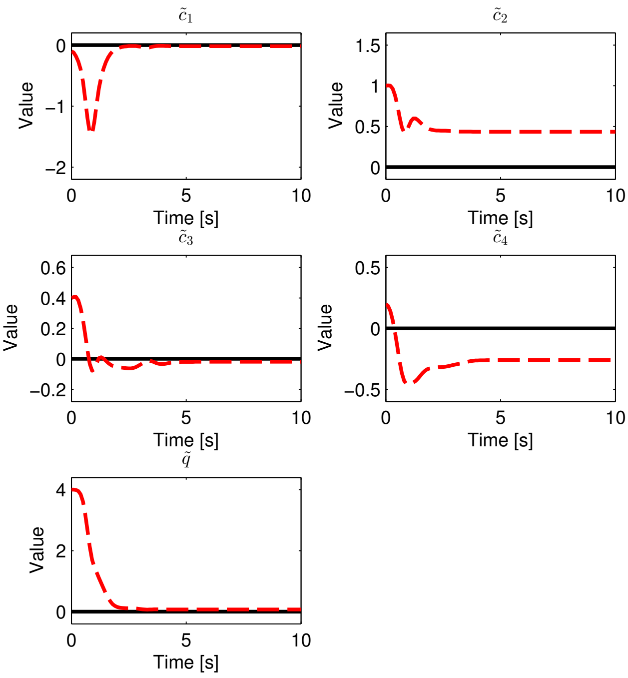

Fig. 7. Parameter estimation errors for the controller of Theorem 7.

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A Projection operator

Let

$$\theta = \begin{bmatrix}\theta_1 & \dots & \theta_n\end{bmatrix}^T \tag{A.1}$$

be a vector of unknowns. Assume a vector of bounds

$$\bar{\theta} = \begin{bmatrix}\bar{\theta}_1 & \dots & \bar{\theta}_n\end{bmatrix}^T \tag{A.2}$$

is known, so that for all $i = 1 \dots n$

$$|\theta_i| \leq \bar{\theta}_i. \tag{A.3}$$

Assume

$$\hat{\theta} = \begin{bmatrix}\hat{\theta}_1 & \dots & \hat{\theta}_n\end{bmatrix}^T \tag{A.4}$$

is an estimate of θ, which is generated using the following adaptive law

$$\dot{\hat{\theta}} = \mathrm{proj}_{\bar{\theta}}\{\tau, \hat{\theta}\} \tag{A.5}$$

for a vector of adaptive laws

$$\tau = \begin{bmatrix}\tau_1 & \dots & \tau_n\end{bmatrix}^T \tag{A.6}$$

and initial conditions

$$\hat{\theta}(0) = \begin{bmatrix}\hat{\theta}_1(0) & \dots & \hat{\theta}_n(0)\end{bmatrix}^T \tag{A.7}$$

satisfying, for all $i = 1 \dots n$

$$|\hat{\theta}_i(0)| \leq \bar{\theta}_i, \tag{A.8}$$

while the projection operator acts element-wise, and is for every element given as

$$\mathrm{proj}_{\bar{\theta}_i}(\tau_i, \hat{\theta}_i) = \begin{cases} 0 & \text{if } |\hat{\theta}_i| \geq \bar{\theta}_i \text{ and } \hat{\theta}_i\tau_i \geq 0 \\ \tau_i & \text{otherwise.} \end{cases} \tag{A.9}$$

Lemma 9 *The adaptive law* (A.5) *using the projection operator* (A.9) *has the following properties for all* $t \geq 0$*:*
(1) For $i = 1 \dots n$ *have*

$$|\hat{\theta}_i| \leq \bar{\theta}_i. \tag{A.10}$$

(2) The following inequalities hold

$$\left(\mathrm{proj}_{\bar{\theta}}(\tau, \hat{\theta})\right)^2 \leq \tau^2 \tag{A.11}$$

and

$$-\tilde{\theta}^T \mathrm{proj}_{\bar{\theta}}(\tau, \hat{\theta}) \leq -\tilde{\theta}^T \tau \tag{A.12}$$

where

$$\tilde{\theta} = \theta - \hat{\theta}. \tag{A.13}$$

PROOF. Proof of (A.11) and (A.12) follow similar steps as in [11, Lemma E.1], while for (A.10), we first note that $\hat{\theta}_i$ is continuous. Assume $\hat{\theta}_i(t) > \bar{\theta}_i$ for some t. Since $\hat{\theta}_i$ is continuous, there exist t^*, T such that $\hat{\theta}_i(t) > \bar{\theta}_i$ for all $t \in (t^*, t^* + T]$, and $\dot{\hat{\theta}}(\bar{t}) > 0$ for some $\bar{t} \in (t^*, t^* + T]$. Then $\dot{\hat{\theta}}_i(\bar{t}) > 0$ implies that $\tau_i(\bar{t}) > 0$, which gives $\hat{\theta}_i(\bar{t})\tau_i(\bar{t}) > 0$, contradicting (A.9).

B Well-posedness of kernel equations

PROOF. [Proof of Lemma 2] For every time t, the existence of a unique, pointwise bounded solution of (30) is ensured by [8, Theorem A.1], which also states pointwise bounds on the form

$$|K(x,\xi,t)| \le f_1(\hat{c}_1(t), \dots \hat{c}_4(t), \hat{q}(t)) \quad \text{(B.1a)}$$
$$|K(x,\xi,t)| \le f_2(\hat{c}_1(t), \dots \hat{c}_4(t), \hat{q}(t)) \quad \text{(B.1b)}$$

for all $(x,\xi) \in \mathcal{T}$ and $t \ge 0$. The set $\mathcal{A}$ of admissible $\hat{c}_1, \dots \hat{c}_4, \hat{q}$, is compact due to projection. f_1 and f_2 are continuous functions of the estimates $\hat{c}_1, \dots \hat{c}_4, \hat{q}$, and hence f_1 and f_2 attain maximum (and minimum) values on $\mathcal{A}$. Let $\bar{K}$ be the maximum value of f_1, and $\bar{L}$ the maximum value of f_2 over $\mathcal{A}$ give the bounds (32a)–(32b). Now differentiating (30) with respect to time, we find

$$\begin{aligned}\mu K_{tx} - \lambda K_{t\xi} &= (\hat{c}_1 - \hat{c}_4)K_t + \hat{c}_3 L_t \\ &\quad + (\dot{\hat{c}}_1 - \dot{\hat{c}}_4)K + \dot{\hat{c}}_3 L \quad \text{(B.2a)}\end{aligned}$$
$$\mu L_{tx} + \mu L_{t\xi} = \hat{c}_2 K_t + \dot{\hat{c}}_2 K \quad \text{(B.2b)}$$
$$L_t(x,0) = \hat{q}\frac{\lambda}{\mu}K_t(x,0) + \dot{\hat{q}}\frac{\lambda}{\mu}K(x,0) \quad \text{(B.2c)}$$
$$K_t(x,x) = -\frac{\dot{\hat{c}}_3}{\lambda+\mu}. \quad \text{(B.2d)}$$

Again using [8, Theorem A.1] on Equation (B.2) in (K_t, L_t), we find that the equations have a unique solution (K_t, L_t) which is bounded as follows

$$\begin{aligned}|K_t(x,\xi)| &\le M_1|\dot{\hat{c}}_1| + M_2|\dot{\hat{c}}_2| + M_3|\dot{\hat{c}}_3| \\ &\quad + M_4|\dot{\hat{c}}_4| + M_5|\dot{\hat{q}}| \quad \text{(B.3a)}\\ |L_t(x,\xi)| &\le M_6|\dot{\hat{c}}_1| + M_7|\dot{\hat{c}}_2| + M_8|\dot{\hat{c}}_3| \\ &\quad + M_9|\dot{\hat{c}}_4| + M_{10}|\dot{\hat{q}}| \quad \text{(B.3b)}\end{aligned}$$

for some positive constants M_i, $i = 1 \dots 10$. From the property (20e), we find (32c)–(32d). □

C Details regarding Theorem 4

We will in this section frequently use Cauchy-Schwarz' inequality

$$I_a[guv] \le \frac{k_1||g||}{2}||u||_a^2 + \frac{||g||}{2k_1}||v||_a^2 \quad \text{(C.1a)}$$

which holds for any a and arbitrary positive constant k_1, and the following inequalities which hold provided $a \ge 1$

$$\begin{aligned}I_{-a}&\left[u(x)\int_0^x f(x,\xi)v(\xi)d\xi\right] \\ &\le \frac{\bar{f}k_2}{2}||u||_{-a}^2 + \frac{\bar{f}}{2k_2}||v||_{-a}^2 \quad \text{(C.1b)}\\ I_a&\left[u(x)\int_0^x f(x,\xi)v(\xi)d\xi\right] \\ &\le \frac{||f||k_3}{2}||u||_a^2 + \frac{||f||}{2k_3}e^a||v||_0^2 \quad \text{(C.1c)}\end{aligned}$$

for arbitrary positive constants k_2 and k_3, and where $\bar{f}$ bounds $|f(x,\xi)|$ for all $(x,\xi) \in \mathcal{T}$.

Also, we state an important property regarding the transform (34) and its inverse (35). Consider three signals a, b, u defined for $x \in [0,1]$, $t \ge 0$, given from

$$u(x) = T[a,b](x) \quad \text{(C.2a)}$$
$$b(x) = T^{-1}[a,u](x). \quad \text{(C.2b)}$$

Since the kernels (K, L) are uniformly, pointwise bounded for every t, the following inequalities hold

$$||u|| \le A_1||a|| + A_2||b|| \quad \text{(C.3a)}$$
$$||b|| \le B_1||a|| + B_2||u|| \quad \text{(C.3b)}$$

for some positive constants A_1, A_2, B_1, B_2 depending on the parameter bounds (11). From (14), (34) and (35), we then have

$$\begin{aligned}||v|| &\le ||\hat{v}_1|| + ||\epsilon_1|| = ||T^{-1}[w_1, z_1]|| + ||\epsilon_1|| \\ &\le B_1||w_1|| + B_2||z_1|| + ||\epsilon_1|| \quad \text{(C.4a)}\\ ||u|| &\le ||\hat{u}_1|| + ||e_1|| \\ &\le ||w_1|| + ||e_1||. \quad \text{(C.4b)}\end{aligned}$$

C.1 Bounds on V_5

Differentiating V_5 in (45), inserting the dynamics (36a), we find

$$\begin{aligned}\dot{V}_5 &= -2\lambda I_{-\delta}[w_1\partial_x w_1] + 2\hat{c}_1 I_{-\delta}\left[w_1^2\right] + 2\hat{c}_2 I_{-\delta}[w_1 z_1] \\ &\quad + 2I_{-\delta}\left[w_1\int_0^x \omega(x,\xi)w_1(\xi)d\xi\right] \\ &\quad + 2I_{-\delta}\left[w_1\int_0^x \kappa(x,\xi)z_1(\xi)d\xi\right] + 2\hat{c}_1 I_{-\delta}[w_1 e_1] \\ &\quad + 2\hat{c}_2 I_{-\delta}[w_1\epsilon_1] + 2\rho I_{-\delta}[w_1 e_1]||\varpi||^2. \quad \text{(C.5)}\end{aligned}$$

Using the properties (5) and (C.1) and the bounds (20a) of Lemma 1 and (38) of Lemma 5, we bound the above term as follows

$$\begin{aligned}\dot{V}_5 &\le \lambda w_1^2(0) - \lambda\delta||w_1||_{-\delta}^2 + 2\hat{c}_1||w_1||_{-\delta}^2 \\ &+ \bar{c}_2^2||w_1||_{-\delta}^2 + ||z_1||_{-\delta}^2 + 2\bar{\omega}||w_1||_{-\delta}^2 \\ &+ \bar{\kappa}^2||w_1||_{-\delta}^2 + ||z_1||_{-\delta}^2 + \bar{c}_1^2||w_1||_{-\delta}^2 + ||e_1||_{-\delta}^2 \\ &+ \bar{c}_2^2||w_1||_{-\delta}^2 + ||\epsilon_1||_{-\delta}^2 + 2\rho I_{-\delta}[w_1 e_1]||\varpi||^2. \quad \text{(C.6)}\end{aligned}$$

Consider the latter term. We find

$$\begin{aligned}&2\rho I_{-\delta}[w_1 e_1]||\varpi||^2 = 2\rho I_{-\delta}[w_1 e_1]||\varpi||(||u|| + ||v||) \\ &\le \rho^2 e^{\delta}(I_{-\delta}[w_1 e_1])^2||\varpi||^2 + e^{-\delta}(||u|| + ||v||)^2 \\ &\le \rho^2 e^{\delta}(I_{-\delta}[w_1 e_1])^2||\varpi||^2 \\ &\quad + e^{-\delta}((1+B_1)||w_1|| + B_2||z_1|| + ||e_1|| + ||\epsilon_1||)^2 \\ &\le \rho^2 e^{\delta}||w_1||_{-\delta}^2||e_1||_{-\delta}^2||\varpi||^2 + 4e^{-\delta}(1+B_1)^2||w_1||^2 \\ &\quad + 4e^{-\delta}B_2^2||z_1||^2 + 4e^{-\delta}||e_1||^2 + 4e^{-\delta}||\epsilon_1||^2. \quad \text{(C.7)}\end{aligned}$$

Inserting this and the boundary condition (36c), we obtain

$$\begin{aligned}\dot{V}_5 &\le 3\lambda q^2 z_1^2(0) + 3q^2\epsilon_1^2(0) + 3e_1^2(0) \\ &\quad - \left[\lambda\delta - 2\bar{c}_1 - \bar{c}_2^2 - 2\bar{\omega} - \bar{\kappa}^2 - \bar{c}_1^2\right. \\ &\quad \left. - \bar{c}_2^2 - 4(1+B_1)^2\right]||w_1||_{-\delta}^2 \\ &\quad + (2 + 4B_2^2)||z_1||_k^2 + 5||e_1||_{-\delta}^2 + 5||\epsilon_1||_{-\delta}^2\end{aligned}$$

$$+ \rho^2 e^{\delta}||e_1||^2_{-\delta}||\varpi||^2||w_1||^2_{-\delta} \tag{C.8}$$

Which can be written as

$$\dot{V}_5 \leq h_1 z_1^2(0) - \left[\lambda\delta - h_2\right]||w_1||^2_{-\delta} + h_3||z_1||^2_k + l_1||w_1||^2_{-\delta} + l_2 \tag{C.9}$$

since

$$||z_1||_{-\delta} \leq ||z_1||_k \tag{C.10}$$

for the positive constants

$$h_1 = 3\lambda q^2 \tag{C.11a}$$
$$h_2 = 2\bar{c}_1 + \bar{c}_2^2 + 2\bar{\omega} + \bar{\kappa}^2 + \bar{c}_1^2 + \bar{c}_2^2 + 4(1+B_1)^2 \tag{C.11b}$$
$$h_3 = 2 + 4B_2^2 \tag{C.11c}$$

and integrable functions

$$l_1 = \rho^2 e^{\delta}||e_1||^2_{-\delta}||\varpi||^2 \tag{C.12a}$$
$$l_2 = 3e_1^2(0) + 3q^2\epsilon_1^2(0) + 5||e_1||^2_{-\delta} + 5||\epsilon_1||^2_{-\delta} \tag{C.12b}$$

C.2 Bounds on V_6

Similarly, differentiating V_6 in (45), inserting the dynamics (36b), we obtain

$$\begin{aligned}\dot{V}_6 &= 2\mu I_k\left[z_1\partial_x z_1\right] + 2\hat{c}_4 I_k\left[z_1^2\right] - 2\lambda I_k\left[z_1 K(x,0)\hat{q}\epsilon_1(0)\right] \\ &\quad - 2\lambda I_k\left[z_1 K(x,0)\tilde{q}v(0)\right] + 2\lambda I_k\left[z_1 K(x,0)e_1(0)\right] \\ &\quad - 2I_k\left[z_1\int_0^x K_t(x,\xi)w_1(\xi)d\xi\right] \\ &\quad - 2I_k\left[z_1\int_0^x L_t(x,\xi)T^{-1}[w_1,z_1](\xi)d\xi\right] \\ &\quad + 2I_k\left[z_1 T[\hat{c}_1e_1+\hat{c}_2\epsilon_1, \hat{c}_2e_1+\hat{c}_4\epsilon_1](x)\right] \\ &\quad + 2\rho I_k\left[z_1 T[e_1,\epsilon_1](x)\right]||\varpi||^2.\end{aligned} \tag{C.13}$$

Using the properties (5) and (C.1), and inserting the boundary condition, we obtain

$$\begin{aligned}\dot{V}_6 &\leq -\mu z_1^2(0) - \left[k\mu - 2\bar{c}_4 - 5\right]||z_1||^2_k + \lambda^2\bar{K}^2\bar{q}^2\epsilon_1^2(0)e^k \\ &\quad + \lambda^2\bar{K}^2\tilde{q}^2v^2(0)e^k + \lambda^2\bar{K}^2e_1^2(0)e^k + ||K_t||^2e^k||w_1||^2_0 \\ &\quad + 2||L_t||^2e^k(B_1^2||w_1||^2 + B_2^2||z_1||^2) \\ &\quad + ||T[\hat{c}_1e_1+\hat{c}_2\epsilon_1, \hat{c}_2e_1+\hat{c}_4\epsilon_1](x)||^2_k \\ &\quad + 2\rho I_k\left[z_1T[e_1,\epsilon_1](x)\right]||\varpi||^2\end{aligned} \tag{C.14}$$

where we have used that

$$||1||_k = \frac{1}{k}(e^k - 1) \leq e^k \tag{C.15}$$

for $k \geq 1$. Now using

$$v(0) = z_1(0) + \epsilon_1(0) \tag{C.16}$$

and

$$\begin{aligned}&2\rho I_k\left[z_1T[e_1,\epsilon_1](x)\right]||\varpi||^2 \\ &\leq e^{\delta}I_k\left[z_1T[e_1,\epsilon_1](x)\right]^2||\varpi||^2 + e^{-\delta}(||u||+||v||)^2 \\ &\leq e^{\delta+k}I_k\left[z_1^2T^2[e_1,\epsilon_1](x)\right]||\varpi||^2 + e^{-\delta}(||u||+||v||)^2 \\ &\leq 2e^{\delta+k}(A_1^2||e_1|| + A_2^2||\epsilon_1||)||\varpi||^2||z_1||^2_k \\ &\quad + e^{-\delta}((1+B_1)||w_1|| + B_2||z_1|| + ||e_1|| + ||\epsilon_1||)^2 \\ &\leq 2e^{\delta+k}(A_1^2||e_1|| + A_2^2||\epsilon_1||)||\varpi||^2||z_1||^2_k \\ &\quad + 4e^{-\delta}(1+B_1)^2||w_1||^2 + 4e^{-\delta}B_2^2||z_1||^2 \\ &\quad + 4e^{-\delta}||e_1||^2 + 4e^{-\delta}||\epsilon_1||^2\end{aligned} \tag{C.17}$$

we obtain

$$\begin{aligned}\dot{V}_6 &\leq -\mu z_1^2(0) - \left[k\mu - 2\bar{c}_4 - 5\right]||z_1||^2_k + \lambda^2\bar{K}^2\bar{q}^2\epsilon_1^2(0)e^k \\ &\quad + 2\lambda^2\bar{K}^2\tilde{q}^2z_1^2(0)e^k + 8\lambda^2\bar{K}^2\bar{q}^2\epsilon_1^2(0)e^k \\ &\quad + \lambda^2\bar{K}^2e_1^2(0)e^k + ||K_t||^2e^k||w_1||^2_0 \\ &\quad + 2||L_t||^2e^k(B_1^2||w_1||^2 + B_2^2||z_1||^2) \\ &\quad + ||T[\hat{c}_1e_1+\hat{c}_2\epsilon_1, \hat{c}_2e_1+\hat{c}_4\epsilon_1](x)||^2_k \\ &\quad + 2e^{\delta+k}(A_1^2||e_1|| + A_2^2||\epsilon_1||)||\varpi||^2||z_1||^2_k \\ &\quad + 4(1+B_1)^2||w_1||^2_{-\delta} + 4e^{-\delta}B_2^2||z_1||^2 \\ &\quad + 4e^{-\delta}||e_1||^2 + 4e^{-\delta}||\epsilon_1||^2.\end{aligned} \tag{C.18}$$

Defining the positive constants

$$h_4 = 2\lambda^2\bar{K}^2 \qquad h_5 = 4(1+B_1)^2 \tag{C.19a}$$
$$h_6 = 2\bar{c}_4 - 4B_2^2 - 5 \tag{C.19b}$$

and the integrable functions

$$l_3 = ||K_t||^2e^{\delta+k} + 2B_1^2||L_t||^2e^{\delta+k} \tag{C.20a}$$
$$l_4 = 2B_2^2||L_t||^2e^k + 2e^{\delta+k}(A_1^2||e_1|| + A_2^2||\epsilon_1||)||\varpi||^2 \tag{C.20b}$$
$$\begin{aligned}l_5 &= \lambda^2\bar{K}^2\bar{q}^2\epsilon_1^2(0)e^k + 8\lambda^2\bar{K}^2\bar{q}^2\epsilon_1^2(0)e^k \\ &\quad + 4e^{-\delta}||e_1||^2 + 4e^{-\delta}||\epsilon_1||^2 + \lambda^2\bar{K}^2e_1^2(0)e^k \\ &\quad + ||T[\hat{c}_1e_1+\hat{c}_2\epsilon_1, \hat{c}_2e_1+\hat{c}_4\epsilon_1](x)||^2_k,\end{aligned} \tag{C.20c}$$

(C.18) can be written

$$\begin{aligned}\dot{V}_6 &\leq -\left[\mu - h_6\tilde{q}^2e^k\right]z_1^2(0) + h_5||w_1||^2_{-\delta} \\ &\quad - \left[k\mu - h_4\right]||z_1||^2_k \\ &\quad + l_3||w_1||^2_{-\delta} + l_4||z_1||^2_k + l_5.\end{aligned} \tag{C.21}$$

D Details regarding Theorem 7

Since the backstepping transformation of Lemma 8 is the same as in Lemma 5, the bounds (C.3) hold.

D.1 Bounds on V_9

Using the dynamics (65c), we find

$$\dot{V}_9 = -2\lambda I_{-\delta}\left[p^Tp_x\right] + 2I_{-\delta}\left[p^T\varpi\right]. \tag{D.1}$$

Using the properties (5) and (C.1), we obtain

$$\dot{V}_9 \leq -\lambda e^{-\delta}p^T(1)p(1) + \lambda p^T(0)p(0) - \lambda\delta||p||^2_{-\delta} + ||p||^2_{-\delta} + ||\varpi||^2_{-\delta} \tag{D.2}$$

Inserting the boundary condition (65c), we obtain

$$\dot{V}_9 \leq -(\lambda\delta - 1)||p||^2_{-\delta} + ||\varpi||^2_{-\delta}. \tag{D.3}$$

For the latter term, we have, using the bounds (C.3)

$$\begin{aligned}||\varpi||_{-\delta}^2 &\leq ||\varpi||^2 = ||u||^2 + ||v||^2 \\ &\leq (||w_1|| + ||\hat{e}_1||)^2 + (||z_1|| + ||\hat{\epsilon}_1||)^2 \\ &\leq (2 + 3B_1^2)||w_2||^2 + 3B_2^2||z_2||^2 \\ &\quad + 2||\hat{e}_2||^2 + 3||\hat{\epsilon}_2||^2 \\ &\leq (2 + 3B_1^2)||w_2||^2 + 3B_2^2||z_2||^2 \\ &\quad + 2\frac{||\hat{e}_2||^2}{1 + ||\eta||^2 + ||p||^2}(1 + ||\eta||^2 + ||p||^2) \\ &\quad + 3\frac{||\hat{\epsilon}_2||^2}{1 + ||r||^2}(1 + ||r||^2) \end{aligned} \quad \text{(D.4)}$$

Inserting this, we obtain

$$\begin{aligned}\dot{V}_9 &\leq -(\lambda\delta - 1)||p||_{-\delta}^2 + h_7 e^{\delta}||w_2||_{-\delta}^2 + h_8||z_2||_k^2 \\ &\quad + l_9||\eta||_{-\delta}^2 + l_9||p||_{-\delta}^2 + l_{10}||r||_k^2 + l_{11} \end{aligned} \quad \text{(D.5)}$$

where

$$l_9 = 2\frac{||\hat{e}_2||^2}{1 + ||\eta||^2 + ||p||^2}e^{\delta} \quad \text{(D.6a)}$$

$$l_{10} = 3\frac{||\hat{\epsilon}_2||^2}{1 + ||r||^2}, \qquad l_{11} = e^{-\delta}l_9 + l_{10} \quad \text{(D.6b)}$$

are integrable functions and

$$h_7 = 2 + 3B_1^2, \quad h_8 = 3B_2^2 \quad \text{(D.7)}$$

are positive constants.

D.2 Bounds on V_{10}

Following the same steps as for V_9 in the previous subsection, using the dynamics (65d) and the boundary condition (65d), we find

$$\begin{aligned}\dot{V}_{10} &\leq -\mu r^2(0) - (\mu k - 1)||r||_k^2 + h_7 e^{\delta + k}||w_2||_{-\delta}^2 \\ &\quad + h_8||z_2||_k^2 + l_{12}||\eta||_{-\delta}^2 + l_{12}||p||_{-\delta}^2 \\ &\quad + l_{13}||r||_k^2 + l_{14} \end{aligned} \quad \text{(D.8)}$$

where

$$l_{12} = e^k l_9, \quad l_{13} = e^k l_{10}, \quad l_{14} = e^{-\delta}l_{12} + l_{13} \quad \text{(D.9)}$$

and where h_7 and h_8 are the same as in (D.7).

D.3 Bounds on V_{11}

Using the dynamics (65a) and the boundary condition (65a), we find

$$\begin{aligned}\dot{V}_{11} &= -2\lambda I_{-\delta}[\eta\eta_x] \\ &= -\lambda e^{-\delta}\eta^2(1) + \lambda\eta^2(0) - \lambda\delta||\eta||_{-\delta}^2 \\ &\leq \lambda v^2(0) - \lambda\delta||\eta||_{-\delta}^2 \\ &\leq h_9 z_2^2(0) + h_9\hat{\epsilon}_2^2(0) - \lambda\delta||\eta||_{-\delta}^2 \end{aligned} \quad \text{(D.10)}$$

where

$$h_9 = 2\lambda \quad \text{(D.11)}$$

is a positive constant.

D.4 Bounds on V_{12}

Using the dynamics (88a), we find

$$\begin{aligned}\dot{V}_{12} &= -\lambda e^{-\delta}w_2^2(1) + \lambda w_2^2(0) - \lambda\delta I_{-\delta}\left[w_2^2\right] \\ &\quad + 2\hat{c}_1 I_{-\delta}\left[w_2^2\right] + 2\hat{c}_2 I_{-\delta}[w_2 z_2] + 2\hat{c}_1 I_{-\delta}[w_2\hat{e}_2] \\ &\quad + 2\hat{c}_2 I_{-\delta}[w_2\hat{\epsilon}_2] + 2I_{-\delta}\left[w_2\int_0^x \omega(x,\xi)w_2(\xi)d\xi\right] \\ &\quad + 2I_{-\delta}\left[w_2\int_0^x \kappa(x,\xi)z_2(\xi)d\xi\right] \\ &\quad + 2I_{-\delta}\left[w_2 p^T(x)\dot{\hat{b}}_1\right] + 2I_{-\delta}\left[w_2\eta(x)\dot{\hat{q}}\right]. \end{aligned} \quad \text{(D.12)}$$

Using the properties (5) and (C.1), the boundary condition (88c) and the bounds (38), we obtain the upper bounds

$$\begin{aligned}\dot{V}_{12} &\leq 2\lambda\bar{q}^2 z_2^2(0) + 2\lambda\bar{q}^2\hat{\epsilon}_2^2(0) \\ &\quad - \left[\lambda\delta - 2\bar{c}_1 - \bar{c}_1^2 - 2\bar{c}_2^2 - 2\bar{\omega} - \bar{\kappa}^2 - 2\right]||w_2||_{-\delta}^2 \\ &\quad + 2||z_2||_{-\delta}^2 + ||\hat{e}_2||_{-\delta}^2 + ||\hat{\epsilon}_2||_{-\delta}^2 \\ &\quad + ||p^T\dot{\hat{b}}_1||_{-\delta}^2 + ||\eta\dot{\hat{q}}||_{-\delta}^2 \end{aligned} \quad \text{(D.13)}$$

which can be rewritten as

$$\begin{aligned}\dot{V}_{12} &\leq h_{10}z_2^2(0) + h_{10}\hat{\epsilon}_2^2(0) - [\lambda\delta - h_{11}]||w_2||_{-\delta}^2 \\ &\quad + 2||z_2||_{-\delta}^2 + l_{15}||p||_{-\delta}^2 + l_{16}||r||_k^2 \\ &\quad + l_{16}||\eta||_k^2 + l_{18} \end{aligned} \quad \text{(D.14)}$$

where

$$h_{10} = 2\lambda\bar{q}^2 \quad \text{(D.15a)}$$

$$h_{11} = 2\bar{c}_1 + \bar{c}_1^2 + 2\bar{c}_2^2 + 2\bar{\omega} + \bar{\kappa}^2 + 2 \quad \text{(D.15b)}$$

are positive constants, and

$$l_{15} = \frac{||\hat{e}_2||^2}{1 + ||\eta||^2 + ||p||^2}e^{\delta} + \dot{\hat{b}}_1^T\dot{\hat{b}}_1 \quad \text{(D.16a)}$$

$$l_{16} = \frac{||\hat{\epsilon}_2||^2}{1 + ||r||^2} \quad \text{(D.16b)}$$

$$l_{17} = \frac{||\hat{e}_2||^2}{1 + ||\eta||^2 + ||p||^2} + \dot{\hat{q}}^2 \quad \text{(D.16c)}$$

$$l_{18} = \frac{||\hat{e}_2||^2}{1 + ||\eta||^2 + ||p||^2} + \frac{||\hat{\epsilon}_2||^2}{1 + ||r||^2} \quad \text{(D.16d)}$$

are integrable functions.

D.5 Bounds on V_{13}

Lastly, using the dynamics (88b), we find

$$\begin{aligned}\dot{V}_{13} &= \mu z_2^2(1) - \mu z_2^2(0) - k\mu I_k\left[z_2^2\right] + 2\hat{c}_4 I_k\left[z_2^2\right] \\ &\quad - 2I_k\left[z_2\int_0^x K_t(x,\xi)w_2(\xi)d\xi\right] \\ &\quad - 2I_k\left[z_2\int_0^x L_t(x,\xi)T^{-1}[w_2,z_2](\xi)d\xi\right] \\ &\quad - 2\lambda\hat{q}I_k[z_2 K(x,0)\hat{\epsilon}_2(0)] + 2I_k[z_2 T[\hat{c}_2\hat{e}_2, \hat{c}_3\hat{e}_2]](x) \\ &\quad + 2I_k[z_2 T[\hat{c}_1\hat{e}_2, \hat{c}_4\hat{\epsilon}_2]](x)\end{aligned}$$

$$- 2I_k \left[z_2 \int_0^x K(x,\xi)\eta(\xi) d\xi \dot{\hat{q}} \right] + 2I_k \left[z_2 T[p^T \dot{\hat{b}}_1, r^T \dot{\hat{b}}_2] \right]. \quad \text{(D.17)}$$

Using the different properties, we bound this as follows

$$\begin{aligned} \dot{V}_{13} \leq & -\mu z_2^2(0) - \left[k\mu - 2\bar{c}_4 - 5 - \lambda^2 \bar{q}^2 \bar{K}^2 - \bar{K}^2 \right] ||z_2||_k^2 \\ & + ||K_t||^2 e^k ||w_2||_0^2 + 2||L_t||^2 (B_1^2 ||w_2||^2 + B_2^2 ||z_2||^2) \\ & + ||1||_k^2 \epsilon_2^2(0) + ||T[\hat{c}_2 \hat{e}_2, \hat{c}_3 \hat{e}_2]||_k^2 + ||T[\hat{c}_1 \hat{e}_2, \hat{c}_4 \hat{\epsilon}_2]||_k^2 \\ & + \dot{\hat{q}}^2 ||\eta||_0^2 + ||T[p^T \dot{\hat{b}}_1, r^T \dot{\hat{b}}_2]||_k^2. \end{aligned} \quad \text{(D.18)}$$

We investigate the four latter terms.

$$\begin{aligned} & ||T[\hat{c}_2 \hat{\epsilon}_2, \hat{c}_3 \hat{e}_2]||_k^2 + ||T[\hat{c}_1 \hat{e}_2, \hat{c}_4 \hat{\epsilon}_2]||_k^2 \\ & + \dot{\hat{q}}^2 ||\eta||_0^2 + ||T[p^T \dot{\hat{b}}_1, r^T \dot{\hat{b}}_2]||_k^2 \\ & \leq e^k ||T[\hat{c}_2 \hat{e}_2, \hat{c}_3 \hat{e}_2]||^2 + e^k ||T[\hat{c}_1 \hat{e}_2, \hat{c}_4 \hat{\epsilon}_2]||^2 \\ & \quad + \dot{\hat{q}}^2 ||\eta||^2 + e^k ||T[p^T \dot{\hat{b}}_1, r^T \dot{\hat{b}}_2]||^2 \\ & \leq 2e^k (A_1^2 \bar{c}_1^2 + A_2^2 \bar{c}_3^2) ||\hat{e}_2||^2 \\ & \quad + 2e^k (A_1^2 \bar{c}_2^2 + A_2^2 \bar{c}_4^2) ||\hat{\epsilon}_2||^2 \\ & \quad + \dot{\hat{q}}^2 ||\eta||^2 + 2e^k A_1^2 |\dot{\hat{b}}_1|^2 ||p||^2 + 2e^k A_2^2 |\dot{\hat{b}}_2|^2 ||r||^2 \\ & \leq \frac{2e^k (A_1^2 \bar{c}_1^2 + A_2^2 \bar{c}_3^2) ||\hat{e}_2||^2}{1 + ||\eta||^2 + ||p||^2} (1 + ||\eta||^2 + ||p||^2) \\ & \quad + \frac{e^k (A_1^2 \bar{c}_2^2 + A_2^2 \bar{c}_4^2 ||\hat{\epsilon}_2||^2}{1 + ||r||^2} (1 + ||r||^2) + \dot{\hat{q}}^2 ||\eta||^2 \\ & \quad + 2e^k A_1^2 |\dot{\hat{b}}_1|^2 ||p||^2 + 2e^k A_2^2 |\dot{\hat{b}}_2|^2 ||r||^2. \end{aligned} \quad \text{(D.19)}$$

Inserting this, and using the property (C.15), for $k \geq 1$, we obtain

$$\begin{aligned} \dot{V}_{13} \leq & -\mu z_2^2(0) + e^k \epsilon_2^2(0) - [k\mu - h_{12}] ||z_2||_k^2 \\ & + l_{19} ||p||_{-\delta}^2 + l_{20} ||r||_k^2 + l_{21} ||\eta||_{-\delta}^2 \\ & + l_{22} ||w_2||_{-\delta}^2 + l_{23} ||z_2||_k^2 + l_{24} \end{aligned} \quad \text{(D.20)}$$

for the positive constant

$$h_{12} = 2\bar{c}_4 + 5 + \lambda^2 \bar{q}^2 \bar{K}^2 + \bar{K}^2 \quad \text{(D.21)}$$

and the integrable functions

$$l_{19} = \frac{2e^{\delta+k} (A_1^2 \bar{c}_1^2 + A_2^2 \bar{c}_3^2) ||\hat{e}_2||^2}{1 + ||\eta||^2 + ||p||^2} + 2e^{\delta+k} A_1^2 |\dot{\hat{b}}_1|^2 \quad \text{(D.22a)}$$

$$l_{20} = \frac{e^k (A_1^2 \bar{c}_2^2 + A_2^2 \bar{c}_4^2) ||\hat{\epsilon}_2||^2}{1 + ||r||^2} + 2e^k A_2^2 |\dot{\hat{b}}_2|^2 \quad \text{(D.22b)}$$

$$l_{21} = \dot{\hat{q}}^2 e^\delta \quad \text{(D.22c)}$$

$$l_{22} = ||K_t||^2 e^{\delta+k} + 2B_1^2 ||L_t||^2 e^\delta \quad \text{(D.22d)}$$

$$l_{23} = 2B_2^2 ||L_t||^2 \quad \text{(D.22e)}$$

$$l_{24} = \frac{2e^k (A_1^2 \bar{c}_1^2 + A_2^2 \bar{c}_3^2) ||\hat{e}_2||^2}{1 + ||\eta||^2 + ||p||^2} + \frac{e^k (A_1^2 \bar{c}_2^2 + A_2^2 \bar{c}_4^2) ||\hat{\epsilon}_2||^2}{1 + ||r||^2}. \quad \text{(D.22f)}$$

E Stability lemma

Lemma 10 *Let $v_1(t)$, $v_2(t)$, $l_1(t)$, $l_2(t)$, $f(t)$, be real-valued functions, and $G(t)$ a real-valued matrix of dimensions $n \times n$ defined for $t \geq 0$, with*

$$v_1(t) = \frac{1}{2} \nu^T(t) \Gamma^{-1} \nu(t) \quad \text{(E.1)}$$

for a signal vector ν of length n and some matrix $\Gamma > 0$. Suppose

$$0 \leq v_1(t), v_2(t) \quad \forall t \geq 0 \quad \text{(E.2a)}$$

$$0 \leq l_1(t), l_2(t), f(t) \quad \forall t \geq 0 \quad \text{(E.2b)}$$

$$l_1, l_2 \in \mathcal{L}_1 \quad \text{(E.2c)}$$

$$|\nu| \in \mathcal{L}_\infty \quad \text{(E.2d)}$$

$$0 \leq G(t) = G^T(t) < I_{n \times n} \quad \text{(E.2e)}$$

$$\int_0^t f(s) ds \leq A e^{Bt} \quad \text{(E.2f)}$$

$$\dot{v}_1(t) \leq -\nu^T(t) G(t) \nu(t) \quad \text{(E.2g)}$$

$$\begin{aligned} \dot{v}_2(t) \leq & -cv_2(t) + l_1(t) v_2(t) + l_2(t) \\ & - a(1 - b\nu^T(t) G(t) \nu(t)) f(t) \end{aligned} \quad \text{(E.2h)}$$

for some positive constants A, B, a, b and c. Then $v_2 \in \mathcal{L}_1 \cap \mathcal{L}_\infty$.

PROOF. See [5].