Extremum-seeking control for steady-state performance optimization of nonlinear plants with time-varying steady-state outputs*

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***Abstract*— Extremum-seeking control is a useful tool for the steady-state performance optimization of plants for which the dynamics and disturbance situation can be unknown. The case when the steady-state plant outputs are constant received a lot of attention, however, in many applications time-varying outputs characterize plant performance. As a result, no static parameter-to-steady-state performance map can be obtained. In this work, an extremum-seeking control method is proposed that uses a so-called dynamic cost function to cope with these time-varying outputs. We show that, under appropriate conditions and given arbitrarily large sets of initial conditions, the solutions of the extremum-seeking control scheme are uniformly ultimately bounded in view of bounded and time-varying external disturbances, and the region of convergence towards the optimal tunable plant parameters can be made arbitrarily small. Moreover, its working principle is illustrated by means of the performance optimal tuning of a variable-gain controller for a motion control application.**

I. INTRODUCTION

Extremum-seeking control, categorized as being an adaptive control approach, is a data-driven and, in essence, model-free control technique for optimizing the steady-state performance of a stable or stabilized plant in real-time, by automated adaptation of tunable plant parameters [1], [2], [3]. Due to its model-free character, extremum-seeking control is a particularly useful tool in applications where only little knowledge of the plant dynamics is available and, as such, has been applied in many different engineering domains such as internal combustion engines [4], antilock bracking systems [5], control of sawtooth instabilities in fusion tokamak plasmas [6], and many more, see, e.g., [1], [2], and references therein. Practical applications are usually subject to external disturbances which are in general not known a priori, which further emphasizes the power of extremum-seeking control as a model-free technique. As a result, and in addition the lack of plant knowledge, the (performance optimal) steady-state output of the plant is often not analytically known, and can only be assessed through output measurements. An extremum-seeking controller is able to exploit these measured plant outputs irrespectively of the availability of plant and disturbance knowledge, and subsequently uses the measured outputs to steer the tunable plant parameters to their performance optimal values, thereby achieving optimal steady-state performance.

In most of the work on extremum-seeking control, the general requirement for the plant to be optimized is the existence of a (unknown) static parameter-to-steady-state performance map, i.e., a *static* input-to-output map, referred to as the objective function, whose extremum corresponds to the optimal steady-state plant performance [2], [7], [3]. Using measured outputs of the plant, gradient-based extremum-seeking control approaches can be employed that estimate the gradient of the objective function and steer the plant parameters to their optimal values in real-time by means of a gradient-based update law. In many applications, such a static input-to-output map, where steady-state performance is characterized by an equilibrium solution, does not exist because performance is related to *time-varying* plant behavior. This time-varying plant behavior can originate for example from reference tracking or disturbance attenuation problems, which are encountered, for example, in industrial motion systems, such as, e.g., wafer scanning systems [13], [14], pick-and-place systems, electron microscopes, and printing applications [15].

In [9], an extremum-seeking controller is developed for general dynamical plants that do not exhibit equilibrium solutions but instead have limit cycle behavior, which can only be reduced in size by some tunable plant parameter but cannot be eliminated completely. The authors added a detector that captures the amplitude of the limit cycle, which is assumed to be sinusoidal. Considering the plant and the detector as one extended plant with the plant parameter as input and the amplitude of the limit cycle as output, a constant steady-state relation between the input and output is obtained. The work in [9] was most suitable for sinusoidal outputs, and has been applied, e.g., in the suppression of subsonic cavity flow resonances [16], and automatic mode matching in vibrating gyroscopes [17].

In [10], an extremum-seeking control scheme is designed for steady-state output optimization of a class of differentially flat periodic nonlinear systems. Using the flatness property of the dynamics, a period of the periodic steady-state output of the plant is computed. Extremum-seeking control is then used to optimize the computed steady-state output in real-time, based on a user-defined cost functional evaluated over that periodic steady-state output. In [8], a similar approach as in [10] is pursued for the steady-state

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output optimization of periodic Hamiltonian systems. We need to emphasize that the wide use of extremum-seeking control can be attributed to the feature of it being model-free, while the methods in [10] and [8] use explicit knowledge of the relation between the parameters and the steady-state output of the plant.

In [11], an extremum-seeking scheme is proposed for the optimization of general nonlinear plants with periodic steady-state outputs. Knowing the period time of the periodic steady-state output, a cost function is designed that links the periodic output of the plant to a performance measure, such as, e.g., an $\mathcal{L}^p$-norm of the error response computed over the periodic time interval. Again, considering the plant and the cost function as one extended plant with the plant parameter as input and the $\mathcal{L}^p$-norm as output, a constant steady-state relation between the input and output is obtained. In [18], this method was experimentally demonstrated for the adaptive design of variable-gain controllers for a motion control application.

In many (industrial) applications, the steady-state response characterizing system performance is time-varying, and periodicity of the steady-state response is not evident due to the fact that responses can be induced by complex time-varying disturbances and reference signals. In such generic cases, a static input-to-output performance map may not be readily defined as in the periodic cases in [18], [9], and [11].

The main contribution of this work is as follows. First, we propose an extremum-seeking control method for steady-state performance optimization of general nonlinear plants with time-varying steady-state outputs. The proposed extremum-seeking control method includes a so-called *dynamic cost function* in terms of the time-varying output response. The dynamic cost function allows for the characterization of a static input-to-output performance map for general nonlinear plants with time-varying steady-state outputs. Second, under appropriate conditions and given arbitrarily large sets of initial conditions, we prove that the solutions of the closed-loop extremum-seeking control scheme are uniformly ultimately bounded in view of bounded and time-varying disturbances. Moreover, we show that the region of convergence towards the optimal tunable plant parameters can be made arbitrarily small. Third, an illustrative simulation example is presented in which performance is optimized of a variable-gain controlled motion system exhibiting generically time-varying outputs.

The paper is organized as follows. Section II presents the problem formulation. Section III gives the extremum-seeking controller. In Section IV the stability result is stated, and in Section V an illustrative example is provided.

II. EXTREMUM-SEEKING CONTROL PROBLEM FOR TIME-VARYING OUTPUTS

Consider the following multi-input-multi-output nonlinear plant:

$$\Sigma_p : \begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{w}(t)) \\ \boldsymbol{e}(t) = \boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{w}(t)), \end{cases} \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^{n_x}$ is the state of the plant, $\boldsymbol{u} \in \mathbb{R}^{n_u}$ is the input of the plant, $\boldsymbol{e} \in \mathbb{R}^{n_e}$ is the output of the plant, $\boldsymbol{w} \in \mathbb{R}^{n_w}$ are disturbances, and $t \in \mathbb{R}$ is time. In the context of extremum-seeking control, the input $\boldsymbol{u}$ is a vector of tunable plant parameters, the output $\boldsymbol{e}$ is a vector of measured performance variables, and $\boldsymbol{w}$ are (time-varying) disturbances, for which we adopt the following assumption.

Assumption 1: The disturbances $\boldsymbol{w}(t)$ are piecewise continuous, defined and bounded on $t \in \mathbb{R}$. Moreover, there exists a constant $\rho_{\boldsymbol{w}} \in \mathbb{R}_{>0}$ such that $\boldsymbol{w}(t) \in \mathcal{W}$ for all $t \in \mathbb{R}$, with $\mathcal{W} = \{\boldsymbol{w} \in \mathbb{R}^{n_w} : \|\boldsymbol{w}\| \leq \rho_{\boldsymbol{w}}\}$.

Although the functions $\boldsymbol{f}$ and $\boldsymbol{g}$ in (1) are considered unknown, we adopt the following assumption.

Assumption 2: The functions $\boldsymbol{f} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_x}$ and $\boldsymbol{g} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \times \mathbb{R}^{n_w} \to \mathbb{R}^{n_e}$ are twice continuously differentiable in $\boldsymbol{x}$ and $\boldsymbol{u}$ and continuous in $\boldsymbol{w}$. Moreover, there exist constants $L_{\boldsymbol{fx}}$, $L_{\boldsymbol{fu}}$, $L_{\boldsymbol{gx}}$, $L_{\boldsymbol{gu}} \in \mathbb{R}_{>0}$ such that

$$\begin{aligned} \left\|\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{w})\right\| &\leq L_{\boldsymbol{fx}}, & \left\|\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{w})\right\| &\leq L_{\boldsymbol{fu}}, \\ \left\|\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{w})\right\| &\leq L_{\boldsymbol{gx}}, & \left\|\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{u}}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{w})\right\| &\leq L_{\boldsymbol{gu}}, \end{aligned} \tag{2}$$

for all $\boldsymbol{x} \in \mathbb{R}^{n_x}$, all $\boldsymbol{u} \in \mathcal{U}$, and all $\boldsymbol{w} \in \mathcal{W}$, where $\mathcal{U} \subset \mathbb{R}^{n_u}$ is a compact set.

In addition to Assumptions 1 and 2, we adopt the following assumption on the plant Σ_p.

Assumption 3: The plant Σ_p in (1) is globally exponentially convergent[1] for all constant inputs $\boldsymbol{u} \in \mathcal{U}$.

Remark 1: Assumption 3 guarantees that, for any constant $\boldsymbol{u} \in \mathcal{U}$ and any $\boldsymbol{w}(t) \in \mathcal{W}$, there exists a unique globally exponentially stable (time-varying) steady-state solution. This assumption is the equivalent of the common assumption in extremum-seeking literature on the plant exhibiting globally asymptotically stable equilibria. In many (nonlinear) control problems, for example tracking, synchronization, observer design and output regulation problems, the convergent system property that all solutions of a closed-loop system converges to some steady-state solution and thus ”forget” their initial condition plays an important role. Moreover, this property is immediate for asymptotically stable linear time-invariant systems with inputs.

From Assumptions 1-3, it follows that for all constant inputs $\boldsymbol{u} \in \mathcal{U}$ and all disturbances $\boldsymbol{w}(t) \in \mathcal{W}$ there exists a unique steady-state solution of the plant Σ_p, which is defined and bounded on $t \in \mathbb{R}$ and globally exponentially stable (GES). The steady-state solution is denoted by $\bar{\boldsymbol{x}}_{\boldsymbol{w}}(t, \boldsymbol{u})$, emphasizing the dependency on time-varying disturbances $\boldsymbol{w}(t)$ and constant inputs $\boldsymbol{u}$, and satisfies

$$\dot{\bar{\boldsymbol{x}}}_{\boldsymbol{w}}(t, \boldsymbol{u}) = \boldsymbol{f}(\bar{\boldsymbol{x}}_{\boldsymbol{w}}(t, \boldsymbol{u}), \boldsymbol{u}, \boldsymbol{w}(t)). \tag{3}$$

For the steady-state solution $\bar{\boldsymbol{x}}_{\boldsymbol{w}}(t, \boldsymbol{u})$ we adopt the following assumption.

Assumption 4: The steady-state solution $\bar{\boldsymbol{x}}_{\boldsymbol{w}}(t, \boldsymbol{u})$ is twice continuously differentiable in $\boldsymbol{u}$ and satisfies

$$\left\|\frac{\partial \bar{\boldsymbol{x}}_{\boldsymbol{w}}}{\partial \boldsymbol{u}}(t, \boldsymbol{u})\right\| \leq L_{\boldsymbol{x}}, \tag{4}$$

for all $t \in \mathbb{R}$, all $\boldsymbol{u} \in \mathcal{U}$, and some constant $L_{\boldsymbol{x}} \in \mathbb{R}_{>0}$.

[1]For definitions of convergent systems the reader is referred to [19].

Furthermore, it follows from Assumption 3 that, for constant inputs $\boldsymbol{u} \in \mathcal{U}$ and (time-varying) disturbances $\boldsymbol{w}(t) \in \mathcal{W}$, there exists a unique steady-state output of the plant Σ_p in (1), denoted by $\bar{\mathbf{e}}_{\boldsymbol{w}}(t,\boldsymbol{u})$, which is given by

$$\bar{\boldsymbol{e}}_{\boldsymbol{w}}(t,\boldsymbol{u}) = \boldsymbol{g}(\bar{\boldsymbol{x}}_{\boldsymbol{w}}(t,\boldsymbol{u}),\boldsymbol{u},\boldsymbol{w}(t)). \tag{5}$$

It is the task of the designer to define a bounded cost function, denoted by Z, that quantifies the performance of interest for the plant under study. Then, the corresponding measured plant performance is given by

$$y(t) = Z(\boldsymbol{e}(t),\boldsymbol{u}(t)), \tag{6}$$

where $y \in \mathbb{R}$. For the function Z, we adopt the following assumption, which can be satisfied by design.

Assumption 5: The function $Z : \mathbb{R}^{n_e} \times \mathbb{R}^{n_{\boldsymbol{u}}} \to \mathbb{R}$ is twice continuously differentiable with respect to both arguments. Moreover, there exist constants $L_{Z\boldsymbol{e}}$, $L_{Z\boldsymbol{u}} \in \mathbb{R}_{>0}$ such that

$$\left\| \frac{\partial^2 Z}{\partial \boldsymbol{e} \partial \boldsymbol{e}^T}(\boldsymbol{e},\boldsymbol{u}) \right\| \leq L_{Z\boldsymbol{e}} \quad \text{and} \quad \left\| \frac{\partial^2 Z}{\partial \boldsymbol{e} \partial \boldsymbol{u}^T}(\boldsymbol{e},\boldsymbol{u}) \right\| \leq L_{Z\boldsymbol{u}}, \tag{7}$$

for all $\boldsymbol{e} \in \mathbb{R}^{n_e}$, and all $\boldsymbol{u} \in \mathcal{U}$.

Remark 2: For example in the context of motion control systems, typically the measured performance variable $\boldsymbol{e}$ is the tracking error. To quantify performance of motion control systems based on the tracking error, a useful cost function Z can be the $\mathcal{L}^2$-norm, i.e., $Z(\boldsymbol{e},\boldsymbol{u}) = \|\boldsymbol{e}\|^2 \in \mathbb{R}_{\geq 0}$.

For all constant inputs $\boldsymbol{u} \in \mathcal{U}$ and all (time-varying) disturbances $\boldsymbol{w}(t) \in \mathcal{W}$, the steady-state plant performance $\bar{y}_{\boldsymbol{w}}(t,\boldsymbol{u})$ is given by

$$\bar{y}_{\boldsymbol{w}}(t,\boldsymbol{u}) = Z(\boldsymbol{g}(\bar{\boldsymbol{x}}_{\boldsymbol{w}}(t,\boldsymbol{u}),\boldsymbol{u},\boldsymbol{w}(t)),\boldsymbol{u}). \tag{8}$$

Our aim is to find the constant input values $\boldsymbol{u}$ that minimize the measured steady-state plant performance $\bar{y}_{\boldsymbol{w}}$, yielding the optimization of the steady-state plant performance variable $\bar{\boldsymbol{e}}_{\boldsymbol{w}}$. In the context of extremum-seeking control, ideally the measured plant performance y and the measured steady-state plant performance $\bar{y}_{\boldsymbol{w}}$ are constant for constant inputs $\boldsymbol{u}$; this forms one of the basic assumptions in the extremum-seeking control literature [2], [7]. However, due to the time-varying nature of the disturbances $\boldsymbol{w}(t)$ in (1), in general, the measured plant performance y and steady-state plant performance $\bar{y}_{\boldsymbol{w}}$ are time-varying in nature.

To deal with time-varying plant outputs, consider the series connection of the plant Σ_p as in (1), the cost function Z as in (6), and additionally a filter, denoted by Σ_f, which reads

$$\Sigma_f : \begin{cases} \dot{\boldsymbol{z}}(t) = \alpha_{\boldsymbol{z}} \boldsymbol{h}(\boldsymbol{z}(t), y(t)) \\ l(t) = k(\boldsymbol{z}(t)), \end{cases} \tag{9}$$

where $\alpha_{\boldsymbol{z}} \in \mathbb{R}_{>0}$ is a tuning parameter, $\boldsymbol{z} \in \mathbb{R}^{n_{\boldsymbol{z}}}$ is the state of the filter, and $l \in \mathbb{R}$ is the output of the filter, see Fig. 1. We adopt the following assumption on the filter Σ_f, which can be satisfied by design.

Assumption 6: The functions $\boldsymbol{h} : \mathbb{R}^{n_{\boldsymbol{z}}} \times \mathbb{R} \to \mathbb{R}^{n_{\boldsymbol{z}}}$ and $k : \mathbb{R}^{n_{\boldsymbol{z}}} \to \mathbb{R}$ are twice continuously differentiable with respect to all arguments. Moreover, there exist constants $L_{\boldsymbol{hz}}$, $L_{\boldsymbol{h}y}$, $L_k \in \mathbb{R}_{>0}$ such that

$$\left\| \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{z}}(\boldsymbol{z},y) \right\| \leq L_{\boldsymbol{hz}}, \ \left\| \frac{\partial \boldsymbol{h}}{\partial y}(\boldsymbol{z},y) \right\| \leq L_{\boldsymbol{h}y}, \ \left\| \frac{\partial k}{\partial \boldsymbol{z}}(\boldsymbol{z}) \right\| \leq L_k, \tag{10}$$

for all $\boldsymbol{z} \in \mathbb{R}^{n_{\boldsymbol{z}}}$, and all $y \in \mathbb{R}$.

The series connection of the cost function Z in (6) and the filter Σ_f in (9), we call the *dynamic cost function*. We adopt the following assumption on the dynamic cost function.

Assumption 7: The dynamic cost function consisting of the cascade of Z and Σ_f, given by (6) and (9), respectively, is exponentially input-to-state convergent[2] for all constant inputs $\boldsymbol{u} \in \mathcal{U}$ and all $\alpha_{\boldsymbol{z}} \in \mathbb{R}_{>0}$.

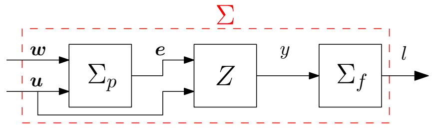

Fig. 1. Series connection of the nonlinear plant Σ_p, the user-defined cost function Z, and the to-be-designed filter Σ_f.

The series connection of the nonlinear plant Σ_p in (1), the user-defined cost function Z in (6), and the to-be-designed filter Σ_f in (9) is referred to as the extended plant Σ and is schematically depicted in Fig. 1. The dynamics of the extended plant is given by

$$\Sigma : \begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t),\boldsymbol{u}(t),\boldsymbol{w}(t)) \\ \dot{\boldsymbol{z}}(t) = \alpha_{\boldsymbol{z}} \boldsymbol{h}(\boldsymbol{z}(t), Z(\boldsymbol{g}(\boldsymbol{x}(t),\boldsymbol{u}(t),\boldsymbol{w}(t)),\boldsymbol{u}(t))) \\ l(t) = k(\boldsymbol{z}(t)). \end{cases} \tag{11}$$

By similar arguments as in the proof of Property 2.27 in [19], we can conclude from Assumptions 3 and 7 that the extended plant Σ in (11) is globally exponentially convergent for all constant inputs $\boldsymbol{u} \in \mathcal{U}$ and disturbances $\boldsymbol{w}(t) \in \mathcal{W}$. As such, there exists a unique steady-state solution of Σ_f, induced by the extended plant, which is defined and bounded on $t \in \mathbb{R}$ and GES. This steady-state solution is denoted by $\bar{\boldsymbol{z}}_{\boldsymbol{w}}(t,\boldsymbol{u},\alpha_{\boldsymbol{z}})$, emphasizing the dependency on time-varying disturbances $\boldsymbol{w}(t)$, constant inputs $\boldsymbol{u}$, and the tunable parameter $\alpha_{\boldsymbol{z}}$, and satisfies

$$\dot{\bar{\boldsymbol{z}}}_{\boldsymbol{w}}(t,\boldsymbol{u},\alpha_{\boldsymbol{z}}) = \alpha_{\boldsymbol{z}} \boldsymbol{h}(\bar{\boldsymbol{z}}_{\boldsymbol{w}}(t,\boldsymbol{u},\alpha_{\boldsymbol{z}}), \bar{y}_{\boldsymbol{w}}(t,\boldsymbol{u})). \tag{12}$$

For the steady-state solution of the extended plant Σ, we adopt the following assumption.

Assumption 8: There exists a twice continuously differentiable function $\boldsymbol{q}_{\boldsymbol{w}} : \mathbb{R}^{n_{\boldsymbol{u}}} \to \mathbb{R}^{n_{\boldsymbol{z}}}$, referred to as the constant performance cost, such that

$$\lim_{\alpha_{\boldsymbol{z}} \to 0} \bar{\boldsymbol{z}}_{\boldsymbol{w}}(t,\boldsymbol{u},\alpha_{\boldsymbol{z}}) = \boldsymbol{q}_{\boldsymbol{w}}(\boldsymbol{u}), \tag{13}$$

for all $t \in \mathbb{R}$ and all $\boldsymbol{u} \in \mathcal{U}$ and disturbances $\boldsymbol{w}(t) \in \mathcal{W}$. Moreover, there exists a positive constant $\delta_{\boldsymbol{w}} \in \mathbb{R}_{>0}$, related to the disturbances $\boldsymbol{w}(t) \in \mathcal{W}$ and the extended plant Σ, such that

$$\|\bar{\boldsymbol{z}}_{\boldsymbol{w}}(t,\boldsymbol{u},\alpha_{\boldsymbol{z}}) - \boldsymbol{q}_{\boldsymbol{w}}(\boldsymbol{u})\| \leq \alpha_{\boldsymbol{z}} \delta_{\boldsymbol{w}}, \tag{14}$$

for all $t \in \mathbb{R}$, all $\boldsymbol{u} \in \mathcal{U}$ and all $0 < \alpha_{\boldsymbol{z}} \leq \epsilon_{\boldsymbol{z}}$ for some $\epsilon_{\boldsymbol{z}} \in \mathbb{R}_{>0}$. In addition, there exists a constant $L_{\boldsymbol{z}} \in \mathbb{R}_{>0}$ such that

$$\left\| \frac{\partial \bar{\boldsymbol{z}}_{\boldsymbol{w}}}{\partial \boldsymbol{u}}(t,\boldsymbol{u},\alpha_{\boldsymbol{z}}) - \frac{d\boldsymbol{q}_{\boldsymbol{w}}}{d\boldsymbol{u}}(\boldsymbol{u}) \right\| \leq \alpha_{\boldsymbol{z}} L_{\boldsymbol{z}}, \tag{15}$$

for all $t \in \mathbb{R}$, all $\boldsymbol{u} \in \mathcal{U}$ and all $0 < \alpha_{\boldsymbol{z}} \leq \epsilon_{\boldsymbol{z}}$.

[2] For definitions of convergent systems the reader is referred to [19].

Remark 3: To illustrate Assumption 8, consider, e.g., $\bar{y}_{\boldsymbol{w}}(t,u) = ((u-u^*)\sin(t))^2 + \gamma$ with $\gamma \in \mathbb{R}$, $u \in \mathcal{U}$ with $\mathcal{U} = \{u \in \mathbb{R} : |u - u^*| \leq L_u\}$, and Σ_f being a linear low-pass filter $\dot{z}(t) = \alpha_z(y(t) - z(t))$, $l(t) = z(t)$. The steady-state solution $\bar{z}_{\boldsymbol{w}}(t,u,\alpha_z)$ then reads

$$\bar{z}_{\boldsymbol{w}}(t,u,\alpha_z) = \gamma + \frac{(u-u^*)^2}{2}\left(1 - \alpha_z \frac{\alpha_z\cos(2t) + 2\sin(2t)}{\alpha_z^2 + 4}\right).$$

The function $q_{\boldsymbol{w}}$ then yields

$$\lim_{\alpha_z \to 0} \bar{z}_{\boldsymbol{w}}(t,u,\alpha_z) = q_{\boldsymbol{w}}(u) = \gamma + \frac{(u-u^*)^2}{2},$$

which is indeed twice continuously differentiable. As a result, the difference between $\bar{z}_{\boldsymbol{w}}(t,u,\alpha_z)$ and $q_{\boldsymbol{w}}$ can be bounded as follows:

$$\|\bar{z}_{\boldsymbol{w}}(t,u,\alpha_{\boldsymbol{z}}) - q_{\boldsymbol{w}}(u)\| \leq \frac{1}{2}\alpha_z L_u^2,$$

such that (14) is satisfied with $\delta_{\boldsymbol{w}} = \frac{L_{\boldsymbol{u}}^2}{2}$. In Assumption 8, (14) can be understood as a bound on the difference between the time-varying steady-state solution of the extended plant Σ and the constant performance cost of the extended plant, which is bounded and tunable by $\alpha_{\boldsymbol{z}}$. For constant disturbances $\boldsymbol{w}$, this difference will be zero (i.e., $\delta_{\boldsymbol{w}} = 0$) for any value of the tuning parameter $\alpha_{\boldsymbol{z}}$, as the steady-state solution will be independent of time. In case $\delta_{\boldsymbol{w}} > 0$, the tuning parameter $\alpha_{\boldsymbol{z}}$ should be tuned small in order to have a sufficiently close approximation of the constant performance cost $q_w(u)$.

Hence, by Assumption 8, steady-state conditions of the plant Σ_p, the cost function Z, the filter Σ_f, the limit $\alpha_{\boldsymbol{z}} \to 0$, and for constant inputs $\boldsymbol{u} \in \mathcal{U}$, we have that the parameter-to-steady-state performance cost of the plant can be characterized by the static input-to-output map

$$F_{\boldsymbol{w}}(\boldsymbol{u}) := k(\boldsymbol{q}_{\boldsymbol{w}}(\boldsymbol{u})), \quad \forall\, \boldsymbol{u} \in \mathcal{U}. \tag{16}$$

We refer to the map $F_{\boldsymbol{w}}$ as the objective function. To minimize the steady-state plant performance y, we aim to find the plant parameter values for which the objective function in (16) is minimal. We further assume that the dynamic cost function is designed such that there exists a unique minimum of the objective function $F_{\boldsymbol{w}}$ on the compact set $\mathcal{U}$ for any (time-varying) disturbance $\boldsymbol{w}(t) \in \mathcal{W}$ satisfying Assumption 1, where the minimum of the map $F_{\boldsymbol{w}}$ corresponds to the optimal plant performance. This assumption is formulated as follows.

Assumption 9: The objective function $F_{\boldsymbol{w}} : \mathbb{R}^{n_{\boldsymbol{u}}} \to \mathbb{R}$ in (16) is twice continuously differentiable and exhibits a unique minimum in the compact set $\mathcal{U}$. Let the corresponding optimal input $\boldsymbol{u}^*$ be defined as

$$\boldsymbol{u}^* = \underset{u \in \mathcal{U}}{\arg\min}\, F_{\boldsymbol{w}}(\boldsymbol{u}). \tag{17}$$

Furthermore, there exists a constant $L_{F1} \in \mathbb{R}_{>0}$ such that

$$\frac{dF_{\boldsymbol{w}}}{d\boldsymbol{u}}(\boldsymbol{u})(\boldsymbol{u} - \boldsymbol{u}^*) \geq L_{F1}\|\boldsymbol{u} - \boldsymbol{u}^*\|^2 \tag{18}$$

for all $\boldsymbol{u} \in \mathcal{U}$.

From Assumption 9, it follows that $F_{\boldsymbol{w}}(\boldsymbol{u}^*)$ is the unique minimum of the objective function. In addition, it follows that the vector of tunable plant parameters $\boldsymbol{u}$ will converge to optimal input $\boldsymbol{u}^*$ for any initial value $\boldsymbol{u}(0) \in \mathcal{U}$ if we are able to design a controller that drives the tunable plant parameters in opposite direction of the gradient of the objective function in (16). However, since the steady-state solutions of the plant in (1) and the filter in (9) and the objective function $F_{\boldsymbol{w}}$ are unknown, we typically cannot design a such a gradient-descent controller. Information of the objective function can only be obtained through measured outputs l of the extended plant in (11). The measured output of the extended plant differs from the objective function $F_{\boldsymbol{w}}$ in three ways; i) due to the dynamics of the plant in (1) and the filter in (9) not being in steady-state, ii) due to the presence of (time-varying) disturbance $\boldsymbol{w}(t)$, and iii) due to the design parameter $\alpha_{\boldsymbol{z}}$ which is typically designed to be small, but still non-zero and positive. Nevertheless, we aim to steer the inputs $\boldsymbol{u}$ to their performance optimizing values $\boldsymbol{u}^*$ by using the measured extended plant output $l(t)$ as feedback to an extremum-seeking controller that is introduced in the next section.

III. EXTREMUM-SEEKING CONTROLLER

The controller design proposed here is inspired by the one in [12, Ch. 2]. In Section III-A, a dither signal design is presented, in Section III-B, a model of the input-to-output behavior of the plant is presented to be used as a basis for gradient estimation, in Section III-C, a least-squares observer to estimate the state of that model (and therewith the gradient) and a normalized optimizer to steer the plant parameters $\boldsymbol{u}$ to the minimizer $\boldsymbol{u}^*$ are presented, and, in Section III-D, tuning guidelines are provided for the closed-loop system composed of the extended plant Σ in (11) and the extremum-seeking controller.

A. Dither signal

To estimate the gradient of the objective function and use this estimated gradient to drive $\boldsymbol{u}$ towards $\boldsymbol{u}^*$ by an optimizer, we define the following input signal:

$$\boldsymbol{u}(t) = \hat{\boldsymbol{u}}(t) + \alpha_{\boldsymbol{\omega}}\boldsymbol{\omega}(t), \tag{19}$$

where $\alpha_{\boldsymbol{\omega}}\boldsymbol{\omega}$ is a vector of perturbation signals with amplitude $\alpha_{\boldsymbol{\omega}} \in \mathbb{R}_{>0}$, and $\hat{\boldsymbol{u}}$ is referred to as the nominal input to be generated by the extremum-seeking controller. The vector $\boldsymbol{\omega}$ is defined by $\boldsymbol{\omega}(t) = [\omega_1(t), \omega_2(t), ..., \omega_{n_{\boldsymbol{u}}}(t)]^T$, with

$$\omega_i(t) = \begin{cases} \sin\left(\frac{i+1}{2}\eta_{\boldsymbol{\omega}} t\right), & \text{if } i \text{ is odd}, \\ \cos\left(\frac{i}{2}\eta_{\boldsymbol{\omega}} t\right), & \text{if } i \text{ is even}, \end{cases} \tag{20}$$

for $i = \{1, 2, ..., n_{\boldsymbol{u}}\}$, where $\eta_{\boldsymbol{\omega}} \in \mathbb{R}_{>0}$ is a tuning parameter. The purpose of the perturbation signal is to provide sufficient excitation to accurately estimate the gradient of the objective function. The nominal plant parameters $\hat{\boldsymbol{u}}$ can be regarded as an estimate of the minimizer $\boldsymbol{u}^*$.

B. Model of input-to-output behavior of the extended plant

To obtain an estimate of the gradient of the objective function, we model the input-to-output behavior of the extended plant in (11), that is, from the nominal input $\hat{\boldsymbol{u}}$ to the measured output of the extended plant l, in a general form. Let the state of the model be given by

$$\boldsymbol{m}(t)=\begin{bmatrix} F_{\boldsymbol{w}}(\hat{\boldsymbol{u}}(t)) \\ \alpha_{\boldsymbol{\omega}}\frac{dF_{\boldsymbol{w}}}{d\boldsymbol{u}^T}(\hat{\boldsymbol{u}}(t)) \end{bmatrix}. \tag{21}$$

The measured output of the extended plant l in (11) can be written as

$$l(t)=k(\boldsymbol{z}(t))-k(\bar{\boldsymbol{z}}_{\boldsymbol{w}}(t,\boldsymbol{u},\alpha_{\boldsymbol{z}}))+F_{\boldsymbol{w}}(\boldsymbol{u}(t))+d(t), \tag{22}$$

with the signal $d(t)$ defined as

$$d(t):=k(\bar{\boldsymbol{z}}_{\boldsymbol{w}}(t,\boldsymbol{u},\alpha_{\boldsymbol{z}}))-k(\boldsymbol{q}_{\boldsymbol{w}}(\boldsymbol{u}(t))). \tag{23}$$

From Taylor's Theorem and (19), $F_{\boldsymbol{w}}(\boldsymbol{u}(t))$ can be written as

$$\begin{aligned} F_{\boldsymbol{w}}(\boldsymbol{u}(t)) &= F_{\boldsymbol{w}}(\hat{\boldsymbol{u}}(t))+\boldsymbol{\alpha}_{\boldsymbol{\omega}}\frac{dF_{\boldsymbol{w}}}{d\boldsymbol{u}}(\hat{\boldsymbol{u}}(t))\boldsymbol{\omega}(t) \\ &+\boldsymbol{\alpha}_{\boldsymbol{\omega}}^{2}\boldsymbol{\omega}^T(t)\int_0^1(1-\sigma)\frac{d^2F_{\boldsymbol{w}}}{d\boldsymbol{u}d\boldsymbol{u}^T}(\hat{\boldsymbol{u}}(t)+\sigma\boldsymbol{\alpha}_{\boldsymbol{\omega}}\boldsymbol{\omega}(t))d\sigma\boldsymbol{\omega}(t). \end{aligned} \tag{24}$$

The dynamics of the state in (21) is governed by

$$\begin{aligned} \dot{\boldsymbol{m}}(t)&=\boldsymbol{A}(t)\boldsymbol{m}(t)+\alpha_{\boldsymbol{\omega}}^2\boldsymbol{B}\boldsymbol{s}(t) \\ l(t)&=\boldsymbol{C}(t)\boldsymbol{m}(t)+\alpha_{\boldsymbol{\omega}}^2 v(t)+r(t)+d(t), \end{aligned} \tag{25}$$

with the matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ defined as

$$\boldsymbol{A}(t)=\begin{bmatrix} 0 & \frac{\dot{\hat{\boldsymbol{u}}}^T(t)}{\alpha_{\boldsymbol{\omega}}} \\ 0 & 0 \end{bmatrix},\ \boldsymbol{B}=\begin{bmatrix} 0 \\ \boldsymbol{I} \end{bmatrix},\ \boldsymbol{C}(t)=\begin{bmatrix} 1 & \boldsymbol{\omega}^T(t) \end{bmatrix}, \tag{26}$$

and the signals $\boldsymbol{s}$, v, r defined as

$$\begin{aligned} \boldsymbol{s}(t)&:=\frac{d^2F_{\boldsymbol{w}}}{d\boldsymbol{u}d\boldsymbol{u}^T}(\hat{\boldsymbol{u}}(t))\frac{\dot{\hat{\boldsymbol{u}}}(t)}{\alpha_{\boldsymbol{\omega}}}, \\ v(t)&:=\boldsymbol{\omega}^T(t)\int_0^1(1-\sigma)\frac{d^2F_{\boldsymbol{w}}}{d\boldsymbol{u}d\boldsymbol{u}^T}(\hat{\boldsymbol{u}}(t)+\sigma\alpha_{\boldsymbol{\omega}}\boldsymbol{\omega}(t))d\sigma\boldsymbol{\omega}(t), \\ r(t)&:=k(\boldsymbol{z}(t))-k(\bar{\boldsymbol{z}}_{\boldsymbol{w}}(t,\boldsymbol{u},\alpha_{\boldsymbol{z}})). \end{aligned} \tag{27}$$

The signals $\boldsymbol{s}$, v, r and d can be interpreted as unknown disturbances to the model. The influences of $\boldsymbol{s}$, v, r and d on the state and output of the model in (25) are small if i) $\hat{\boldsymbol{u}}$ is slowly time varying, if ii) $\alpha_{\boldsymbol{\omega}}$ is small, if iii) the states $\boldsymbol{x}$ of the plant in (1) and the states $\boldsymbol{z}$ of the filter in (9) are close to their steady-state values, and if iv) $\alpha_{\boldsymbol{z}}$ is small.

The state $\boldsymbol{m}$ in (21) contains an estimate of the gradient of the objective function, scaled by the perturbation amplitude $\alpha_{\boldsymbol{\omega}}$. Hence, an estimate of the gradient of the objective function can be obtained from an estimate of the state $\boldsymbol{m}$. Based on this gradient estimate, an optimizer can steer the plant parameters $\boldsymbol{u}$ to the minimizer $\boldsymbol{u}^*$. In the next section, a least-squares observer and an optimizer for this purpose are proposed.

C. Controller design

We introduce an extremum-seeking controller that is composed of a dither signal as in (19), an observer to estimate the state $\boldsymbol{m}$ of the model in (25), and an optimizer that uses the estimate of the state $\boldsymbol{m}$ of the observer, denoted by $\hat{\boldsymbol{m}}$, to steer the nominal plant inputs $\hat{\boldsymbol{u}}$ to their performance optimal values $\boldsymbol{u}^*$.

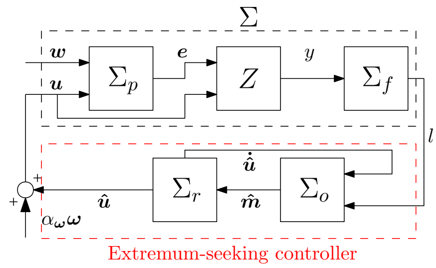

Fig. 2. The closed-loop system, composed of the extended plant Σ, the observer Σ_o, the optimizer Σ_r, and the dither signal $\alpha_{\boldsymbol{\omega}}\boldsymbol{\omega}$.

The observer, denoted by Σ_o, is given by

$$\Sigma_o:\begin{cases} \dot{\hat{\boldsymbol{m}}}(t) = \left(\boldsymbol{A}(t)-\eta_{\boldsymbol{m}}\sigma_r\boldsymbol{Q}(t)\boldsymbol{D}^T\boldsymbol{D}\right)\hat{\boldsymbol{m}}(t)+\alpha_{\boldsymbol{\omega}}^2\boldsymbol{B}\hat{\boldsymbol{s}}(t) \\ \qquad\quad +\eta_{\boldsymbol{m}}\boldsymbol{Q}(t)\boldsymbol{C}^T(t)(l(t)-\boldsymbol{C}(t)\hat{\boldsymbol{m}}(t)-\alpha_{\boldsymbol{\omega}}^2\hat{v}(t)) \\ \dot{\boldsymbol{Q}}(t) = \eta_{\boldsymbol{m}}\boldsymbol{Q}(t)+\boldsymbol{A}(t)\boldsymbol{Q}(t)+\boldsymbol{Q}(t)\boldsymbol{A}^T(t) \\ \qquad\quad -\eta_{\boldsymbol{m}}\boldsymbol{Q}(t)(\boldsymbol{C}^T(t)\boldsymbol{C}(t)+\sigma_r\boldsymbol{D}^T\boldsymbol{D})\boldsymbol{Q}(t), \end{cases} \tag{28}$$

with initial conditions $\hat{\boldsymbol{m}}(0)=\hat{\boldsymbol{m}}_0$ and $\boldsymbol{Q}(0)=\boldsymbol{Q}_0$ where $\boldsymbol{Q}_0$ is symmetric and positive definite, and where $\boldsymbol{D}=[\boldsymbol{0}\ \ \boldsymbol{I}]$, $\eta_{\boldsymbol{m}}$, $\sigma_r\in\mathbb{R}_{>0}$ tuning parameters related to the observer, referred to as a forgetting factor and a regularization constant, respectively, and signals $\hat{\boldsymbol{s}}$ and $\hat{v}$ being approximations of the signals $\boldsymbol{s}$ and v in (27), defined as

$$\hat{\boldsymbol{s}}:=\boldsymbol{H}(\hat{\boldsymbol{u}}(t))\frac{\dot{\hat{\boldsymbol{u}}}(t)}{\alpha_{\boldsymbol{\omega}}},\quad \hat{v}:=\frac{1}{2}\boldsymbol{\omega}^T(t)\boldsymbol{H}(\hat{\boldsymbol{u}}(t))\boldsymbol{\omega}(t), \tag{29}$$

with a user-defined function $\boldsymbol{H}:\mathbb{R}^{n_{\boldsymbol{u}}}\to\mathbb{R}^{n_{\boldsymbol{u}}\times n_{\boldsymbol{u}}}$ satisfying

$$\|\boldsymbol{H}(\hat{\boldsymbol{u}})\|\leq L_{\boldsymbol{H}}\quad\text{for all } \hat{\boldsymbol{u}} \text{ s.t. } \boldsymbol{u}\in\mathcal{U}, \tag{30}$$

with $L_{\boldsymbol{H}}\in\mathbb{R}_{>0}$.

The optimizer, denoted by Σ_r, is given by

$$\Sigma_r:\quad \dot{\hat{\boldsymbol{u}}}(t)=-\lambda_{\boldsymbol{u}}\frac{\eta_{\boldsymbol{u}}\boldsymbol{D}\hat{\boldsymbol{m}}(t)}{\eta_{\boldsymbol{u}}+\lambda_{\boldsymbol{u}}\|\boldsymbol{D}\hat{\boldsymbol{m}}(t)\|}, \tag{31}$$

with $\lambda_{\boldsymbol{u}}$, $\eta_{\boldsymbol{u}}\in\mathbb{R}_{>0}$ being tuning parameters related to the optimizer, and an initial estimate of the optimal nominal inputs denoted by $\hat{\boldsymbol{u}}(0)=\hat{\boldsymbol{u}}_0$. Normalization of the adaptation gain in (31) is done to prevent solutions of the closed-loop system of the extended plant and the extremum-seeking controller from having a finite escape time if the state estimate $\hat{\boldsymbol{m}}$ is inaccurate [12, Ch. 2]. The closed-loop system, composed of the extended plant Σ in (11), the observer Σ_o in (28), and the optimizer Σ_r in (31), is depicted in Fig. 2.

D. Tuning guidelines

For the closed-loop system to operate properly, we adopt the following design guidelines that guarantee time-scale separation:

1) The convergence of the solutions of the plant dynamics in (1) to its steady-state operation is assumed to be *fast*,
2) The tuning parameter $\alpha_{\boldsymbol{z}}$ of the filter in (9) is chosen small such that the difference between the time-varying steady-state solution of the extended plant Σ and the performance cost is small (see Assumption 8), however sufficiently large such that convergence of solutions of the filter dynamics is on a *medium-to-fast* time scale,

3) The dither frequencies parameterized by $\eta_{\boldsymbol{\omega}}$ are chosen slower than the filter dynamics to provide sufficient excitation, admitting a *medium* time-scale,
4) The observer should use a sufficiently long time history of the perturbation signals and measurement signal to be able to accurately extract the state of the model [12, Ch. 2]; the observer dynamics and its design parameter $\eta_{\boldsymbol{m}}$ should be associated with a *medium-to-slow* time scale compared to the dither signal,
5) The nominal plant parameters $\hat{\boldsymbol{u}}$, induced by the optimizer, should be slowly time varying with respect to the observer by proper design of the design parameters $\lambda_{\boldsymbol{u}}$ and $\eta_{\boldsymbol{u}}$, admitting a *slow* (optimizer) time-scale.

IV. STABILITY ANALYSIS

In this section, we will provide a stability result for the closed-loop system dynamics described in the previous sections. Due to the perturbation of the tunable parameter $\boldsymbol{u}$, the optimizer state $\hat{\boldsymbol{u}}$ will in general converge to a region of the performance-optimal value $\boldsymbol{u}^*$. The next result states conditions on the tuning parameters under which the extremum-seeking scheme guarantees that $\hat{\boldsymbol{u}}$ converge to an arbitrarily small set around the optimum $\boldsymbol{u}^*$.

Theorem 1: Under Assumptions 1-9, there exist (sufficiently small) constants $\epsilon_1, ..., \epsilon_6 \in \mathbb{R}_{>0}$, and $\epsilon_{\boldsymbol{z}} \in \mathbb{R}_{>0}$ such that the solutions of the closed-loop system of the extended plant in (11) (i.e., the series connection of the nonlinear plant Σ_p in (1), the cost function Z in (6), and the filter Σ_f in (9)) and the extremum-seeking controller (consisting of the dither signal in (19), the observer Σ_o in (28), and the optimizer Σ_r in (31)) are uniformly bounded for all $\boldsymbol{x}(0) \in \mathcal{X}_0$, all symmetric and positive-definite $\boldsymbol{Q}(0) \in \mathcal{Q}_0$, all $\hat{\boldsymbol{u}}(0) \in \mathcal{U}_0$, all $\boldsymbol{z}(0) \in \mathcal{Z}_0$, and all $\hat{\boldsymbol{m}}(0) \in \mathcal{M}_0$, where $\mathcal{X}_0 \subset \mathbb{R}^{n_{\boldsymbol{x}}}$, $\mathcal{U}_0 \subset \mathbb{R}^{n_{\boldsymbol{u}}}$, $\mathcal{Q}_0 \subset \mathbb{R}^{n_{\boldsymbol{u}}+1 \times n_{\boldsymbol{u}}+1}$, $\mathcal{Z}_0 \subset \mathbb{R}^{n_{\boldsymbol{z}}}$, $\mathcal{M}_0 \subset \mathbb{R}^{n_{\boldsymbol{u}}+1}$ are arbitrarily large compact sets, all $\alpha_{\boldsymbol{z}}, \alpha_{\boldsymbol{\omega}}, \eta_{\boldsymbol{u}}, \lambda_{\boldsymbol{u}}, \eta_{\boldsymbol{m}}, \eta_{\boldsymbol{\omega}} \in \mathbb{R}_{>0}$ and all $\sigma_r \in \mathbb{R}_{\geq 0}$ that satisfy $\alpha_{\boldsymbol{z}} \leq \epsilon_1$, $\eta_{\boldsymbol{\omega}} \leq \alpha_{\boldsymbol{z}}\epsilon_2$, $\eta_{\boldsymbol{m}} \leq \eta_{\boldsymbol{\omega}}\epsilon_3$, $\alpha_{\boldsymbol{\omega}}\lambda_{\boldsymbol{u}} \leq \eta_{\boldsymbol{m}}\epsilon_4$, $\eta_{\boldsymbol{u}} \leq \alpha_{\boldsymbol{\omega}}\eta_{\boldsymbol{m}}\epsilon_5$, $\sigma_r \leq \epsilon_6$, and $\alpha_{\boldsymbol{z}} \leq \epsilon_{\boldsymbol{z}}$. Moreover, the solutions $\hat{\boldsymbol{u}}(t)$ satisfy

$$\limsup_{t\to\infty} \|\tilde{\boldsymbol{u}}(t)\| \leq \max\left\{\alpha_{\boldsymbol{\omega}}c_1, \alpha_{\boldsymbol{\omega}}\eta_{\boldsymbol{w}}\alpha_{\boldsymbol{z}}c_2, \frac{\eta_{\boldsymbol{w}}}{\alpha_{\boldsymbol{z}}}c_3, \frac{\alpha_{\boldsymbol{z}}\delta_{\boldsymbol{w}}}{\alpha_{\boldsymbol{\omega}}}c_4\right\}, \tag{32}$$

for some constants $c_1, ..., c_4 \in \mathbb{R}_{>0}$, with $\tilde{\boldsymbol{u}}(t) = \hat{\boldsymbol{u}}(t) - \boldsymbol{u}^*$.

Sketch of proof: The proof of Theorem 1 is inspired by the one in [12, Ch. 2], and the full proof can be found in [23]. To prove Theorem 1, we introduce the following coordinate transformation:

$$\begin{aligned} \tilde{\boldsymbol{x}}(t) &= \boldsymbol{x}(t) - \bar{\boldsymbol{x}}_{\boldsymbol{w}}(t, \boldsymbol{u}(t)), \\ \tilde{\boldsymbol{Q}}(t) &= \boldsymbol{Q}^{-1}(t) - \Xi^{-1} - \tfrac{\eta_{\boldsymbol{m}}}{\eta_{\boldsymbol{\omega}}}\boldsymbol{n}(t), \\ \tilde{\boldsymbol{z}}(t) &= \boldsymbol{z}(t) - \bar{\boldsymbol{z}}_{\boldsymbol{w}}(t, \boldsymbol{u}(t), \alpha_{\boldsymbol{z}}), \\ \tilde{\boldsymbol{m}}(t) &= \hat{\boldsymbol{m}}(t) - \boldsymbol{m}(t), \\ \tilde{\boldsymbol{u}}(t) &= \hat{\boldsymbol{u}}(t) - \boldsymbol{u}^*, \end{aligned} \tag{33}$$

where $\boldsymbol{n}(t)$ and Ξ are defined in [12, Ch. 2]. The analysis of the stability properties of the closed-loop system can be divided into three temporal stages, where we defined some finite time instances t_1 and t_2:

- for $0 \leq t < t_1$ the solutions $\tilde{\boldsymbol{x}}$ and $\tilde{\boldsymbol{Q}}$ converge to a neighborhood of the origin and remain there, the solution $\tilde{\boldsymbol{z}}$ converges but may still be away from a neighborhood of the origin, while the solutions $\tilde{\boldsymbol{m}}$ and $\tilde{\boldsymbol{u}}$ may drift, but remain bounded.
- for $t_1 \leq t \leq t_2$, the solutions $\tilde{\boldsymbol{x}}$ and $\tilde{\boldsymbol{Q}}$ have already converged to a neighborhood of the origin, the solution $\tilde{\boldsymbol{z}}$ converges to a neighborhood of the origin, while the solutions $\tilde{\boldsymbol{m}}$ and $\tilde{\boldsymbol{u}}$ may drift, but remain bounded.
- for $t \geq t_2$, the solutions $\tilde{\boldsymbol{m}}$ and $\tilde{\boldsymbol{u}}$ also converge to a neighborhood of the origin.

We first derive bounds on each of the variables in (33) corresponding to these three temporal stages of convergence. It turns out that the dynamics of $\tilde{\boldsymbol{u}}$ and $\tilde{\boldsymbol{m}}$ can be seen as feedback-interconnected subsystems. To verify that the interconnected system exhibits uniformly bounded solutions, the cyclic-small-gain criterion in [22] is employed, which completes the proof.

Remark 4: Tuning guidelines. Under the conditions of Theorem 1, it follows that, if we are dealing with constant (or no) disturbances $\boldsymbol{w}(t)$, we have that $\delta_{\boldsymbol{w}} = 0$ (see Assumption 8), and the optimizer state $\hat{\boldsymbol{u}}$ converges to an arbitrarily small region of the performance-optimal value $\boldsymbol{u}^*$ if the tuning parameters $\alpha_{\boldsymbol{\omega}}$ and $\eta_{\boldsymbol{\omega}}$ related to the dither signal are chosen sufficiently small for an arbitrary bounded $\alpha_{\boldsymbol{z}}$. Choosing $\alpha_{\boldsymbol{z}}$ large in general allows faster convergence towards the performance-optimal value $\boldsymbol{u}^*$. If we are dealing with time-varying disturbances $\boldsymbol{w}(t)$, we have that $\delta_{\boldsymbol{w}} \neq 0$. To make the region to which $\hat{\boldsymbol{u}}$ converges arbitrarily small, see (32), we subsequently tune $\alpha_{\boldsymbol{\omega}}$ small to make the first term in the right-hand side of (32) arbitrarily small, tune $\alpha_{\boldsymbol{z}}$ small to make the fourth term in the right-hand side of (32) arbitrarily small, and finally tune $\eta_{\boldsymbol{\omega}}$ sufficiently small to make the second and third term in the right-hand side of (32) arbitrarily small.

V. ILLUSTRATIVE EXAMPLE

To illustrate the extremum-seeking control approach proposed in Section II, we consider an industrial case study of steady-state performance optimization of a closed-loop variable-gain controlled (VGC) motion stage as also studied in [20]. Herein, a VGC strategy is employed to overcome inherent performance limitations such as the waterbed effect [21]. In Section V-A, the VGC motion stage subject to time-varying disturbances is introduced, and in Section V-B the proposed extremum-seeking controller is employed to optimize the steady-state performance of the VGC motion stage, illustrating the effectiveness of the proposed extremum-seeking control approach.

A. Variable-gain controlled motion stage

The variable-gain controller structure is shown in Fig. 3. The scheme consists of a plant P, representing the dynamics of a short-stroke wafer stage of a wafer scanner in z-direction, and a nominal linear controller C, having transfer functions $P(s)$ and $C(s)$, respectively, with $s \in \mathbb{C}$ being the Laplacian variable, a (time-varying) force disturbance $f(t)$, a nonlinear control element $\varphi(\cdot)$, and a shaping filter $F(s)$. Furthermore, e denotes the tracking error, y_p denotes the output of the plant, and $-u$ denotes the output of the

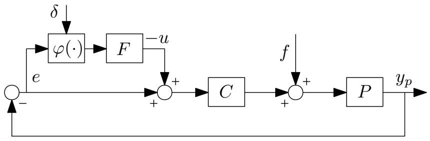

Fig. 3. The closed-loop variable-gain control scheme.

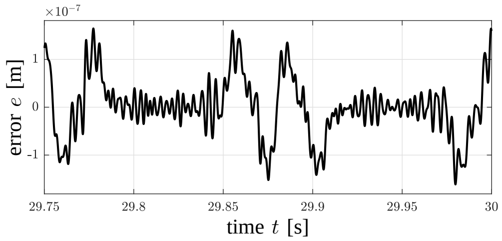

Fig. 4. A typical steady-state tracking error for the nominal low-gain linear control loop.

shaping filter F. The nonlinearity $\varphi(e)$, representing the variable-gain element with e as input, is given by a dead-zone characteristic

$$\varphi(e) = \begin{cases} \alpha(e+\delta) & \text{if} \quad e < -\delta, \\ 0 & \text{if} \quad |e| \leq \delta, \\ \alpha(e-\delta) & \text{if} \quad e > \delta, \end{cases} \tag{34}$$

where α and δ denotes the additional gain and the dead-zone length, respectively. The short-stroke wafer stage in z-direction is modelled as a 4^{th}-order mass-spring-damper-mass system, admitting the following transfer function representation

$$P(s) = \frac{m_1 s^2 + bs + k}{s^2\left(m_1 m_2 s^2 + b(m_1+m_2)s + k(m_1+m_2)\right)}, \tag{35}$$

where the following numerical values are used: $m_1 = 5$ kg, $m_2 = 17.5$ kg, $k = 7.5 \cdot 10^7$ N/m, $b = 90$ Ns/m. The nominal, and stabilizing linear controller consists of a PID-controller C_{pid}, a second-order low-pass filter C_{lp} and a notch filter C_n, i.e. $C(s) = C_{pid}(s)C_{lp}(s)C_n(s)$. The filters are given by $C_{pid}(s) = (k_p(s^2 + (\omega_i + \omega_d)s + \omega_i\omega_d))/(\omega_d s)$, where $k_p = 6.9 \cdot 10^6$ N/m, $\omega_d = 3.8 \cdot 10^2$ rad/s, and $\omega_i = 3.14 \cdot 10^2$ rad/s; $C_{lp}(s) = \omega_{lp}^2/(s^2 + 2\beta_{lp}\omega_{lp}s + \omega_{lp}^2)$, where $\omega_{lp} = 3.04 \cdot 10^3$ rad/s, and $\beta_p = 0.08$; $C_n(s) = (s^2 + 2\beta_z\omega_z s + \omega_z^2)/(s^2 + 2\beta_p\omega_p s + \omega_p^2)$, where $\omega_z = 4.39 \cdot 10^3$ rad/s, $\omega_p = 5.03 \cdot 10^3$ rad/s, $\beta_z = 2.7 \cdot 10^{-3}$, and $\beta_p = 0.88$. The shaping filter $F(s)$ is given by $F(s) = (s^2 + 2\beta_{z,F}\omega_{z,F}s + \omega_{z,F}^2)/(s^2 + 2\beta_{p,F}\omega_{p,F}s + \omega_{p,F}^2)$, with $\omega_{z,F} = \omega_{p,F} = 2.0 \cdot 10^3$ rad/s, $\beta_{z,F} = 0.6$, and $\beta_{p,F} = 4.8$.

The disturbance $f(t)$ consists of a low-frequency force disturbance contribution induced by setpoint accelerations in the x- and y-direction of the short-stroke wafer stage, and a high-frequency force disturbance, which is modelled as a signal containing multiple sinusoidal components with random frequencies between 200-500 Hz and random phases. Fig. 4 shows a certain time interval of the time-varying steady-state tracking error of the nominal linear low-gain control loop, induced by this disturbance $f(t)$.

From Theorem 1 in [20], which is based on circle criterion type arguments, it can be concluded that if the additional gain is chosen as $\alpha < 4.34$, then the variable-gain controlled motion system discussed above is exponentially convergent. The dead-zone length δ turns out to be a stability invariant tunable system parameter, however, the choice for δ does affect significantly the achievable tracking performance and is typically chosen in a heuristic manner. Moreover, as the disturbances are time-varying, an initially good choice for the value of δ may become sub-optimal after some time when the disturbance signature drifts. As such, we propose to tune the dead-zone length δ in real-time by the extremum-seeking control scheme presented in Sections II and III to optimize tracking performance.

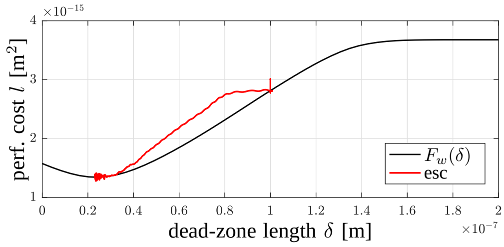

Fig. 5. Objective function F_w and minimization of the performance cost by tuning of δ through extremum-seeking control, with $\hat{\delta}(0) = 1 \cdot 10^{-7}$ m.

B. Performance optimization using extremum-seeking control

For the extremum-seeking control scheme as presented in Sections II and Section III, we choose the cost function $Z(e(t)) = \|e(t)\|^2$. The filter Σ_f is designed as a second-order low-pass filter admitting the following state-space formulation

$$\Sigma_f : \begin{cases} \dot{z}_1(t) &= \alpha_{\boldsymbol{z}} z_2(t) \\ \dot{z}_2(t) &= \alpha_{\boldsymbol{z}}\left(y(t) - 2\beta_{\boldsymbol{z}} z_2(t) - z_1(t)\right) \\ l(t) &= z_1(t), \end{cases} \tag{36}$$

which is of the form in (9). Furthermore, for $\alpha_z, \beta_z \in \mathbb{R}_{>0}$, Assumption 7 is satisfied. The objective function is depicted in Fig. 5. The parameters of the extremum-seeking controller are chosen as $\beta_{\boldsymbol{z}} = \frac{1}{2}\sqrt{2}$, $\alpha_{\boldsymbol{z}} = 3$, $\eta_\omega = 2$, $\alpha_\omega = 0.1 \cdot 10^{-8}$, $\eta_m = 0.05$, $\sigma_r = 1 \cdot 10^{-8}$, $\lambda_u = 1 \cdot 10^{10}$, $\eta_u = 1$, and $H = 0.55$. The initial conditions are chosen as $\boldsymbol{z}_0^T = [3 \cdot 10^{-15} \quad 0]$, $\hat{\boldsymbol{m}}_0^T = [3 \cdot 10^{-15} \quad 0.5 \cdot 10^{-17}]$, $\boldsymbol{Q}_0 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & \frac{2}{1+2\sigma_r} \end{smallmatrix}\right]$ and $\hat{\delta}_0 = 1 \cdot 10^{-7}$. The extremum-seeking controller is enabled at $t = 10$ seconds. Fig. 6 shows the dead-zone length δ and the measured performance cost $l(t)$ as a function of time, respectively. In here, results are shown for three cases; cases 1 and 2 in which two constant values for δ are used, namely $\delta = 2 \cdot 10^{-7}$ and $\delta = 0$, associated with a low-gain and high-gain linear controller, respectively, and case 3 in which δ is tuned by an extremum-seeking controller. It can be seen that the plant parameter δ and the corresponding performance cost l converges to the performance optimal region, which is also illustrated in Fig. 5. Fig. 7 shows the measured tracking error for the low-gain, high-gain, and optimally tuned variable-gain controller.

Remark 5: In Assumption 2, it is assumed that the dynamics of the plant (1) are twice continuously differentiable with respect to the vector of tunable plant parameters. The use of a dead-zone nonlinearity as presented in (34) actually

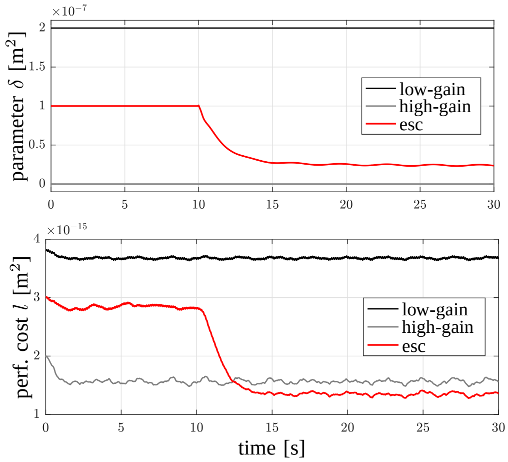

Fig. 6. Convergence of the tunable parameter δ towards δ^* and the associated performance cost l, and the performance cost in case of two constant values of δ, associated with a low-gain and a high-gain controller.

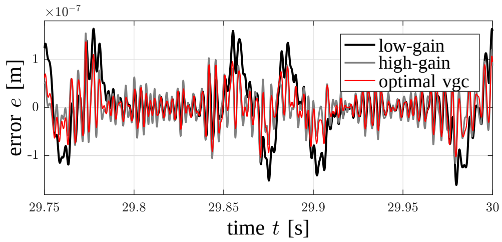

Fig. 7. Measured tracking error for the low-gain, high-gain, and optimized variable-gain controller.

violates this assumption. Although it is possible to define a sufficiently smooth nonlinearity $\varphi(\cdot)$ arbitrarily close to dead-zone nonlinearity, for ease of implementation and the fact that the conclusions with respect to convergence are similar, we use the non-smooth nonlinearity as in (34).

VI. CONCLUSIONS

In this work, we have introduced an extremum-seeking control method for steady-state performance optimization of general nonlinear plants with time-varying steady-state outputs. The proposed extremum-seeking controller includes a so-called dynamic cost function which allows for the characterization of a static input-to-output performance map, despite the presence of time-varying disturbances which induces time-varying steady-state plant outputs. We have shown that, under appropriate conditions, the extremum-seeking control scheme are uniformly ultimately bounded, and the region of convergence towards the optimal tunable plant parameters can be made arbitrarily small. An illustrative example is provided that shows the steady-state performance optimization of a closed-loop variable-gain controlled motion system subject to a time-varying force disturbance by means of the proposed extremum-seeking control method.

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