NTNU – Trondheim
Norwegian University of
Science and Technology

The strong no loop conjecture

Michael Kyei

Master of Science in Mathematics
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Supervisor: Steffen Oppermann, MATH

Norwegian University of Science and Technology
Department of Mathematical Sciences

Abstract

The strong no loop conjecture states that a simple module of finite projective dimension over an artin algebra has no non-zero self extension. In this work I looked at the proof of the following result due to Kiyoshi Igusa, Shiping Liu and charles Paquette which confirms the strong no loop conjecture for finite dimensional algebras over an algebraically closed field. The point is to understand the background leading to the proof.

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Contents

Introduction

It has been known for a long time; see [11] and [13] that the quiver of a finite dimensional algebra Λ of finite global dimension does not contain any loop, or equivalently $\operatorname{Ext}^1_\Lambda(S,S)=0$ for simple Λ-modules, S: that is the "no loop conjecture". This conjecture is known for finite dimensional elementary K-algebras, (where K is a field) see [13], this can be derived from an earlier result of Lenzing on Hochschild homology, see [11]. Furthermore, this result has been recently, strengthen due to Zacharia; see [13] and [14], to state that a vertex in the extension quiver admits no loop if it has finite projective dimension: that is "the strong no loop conjecture".
The strong no loop conjecture is known for

- algebras with atmost two simples and radical cube zero (by Jensen [3]),
- mild algabras,"hence representation finite algebras" (by Skorodumov [6]),
- bound quiver algebras KQ/I such that for each loop $\alpha \in Q$ there exist an $n \in \mathbb{N}$ with $\alpha^n \in I/(IJ+JI)$ where J denotes the ideal generated by the arrows (by Green, Solberg and Zacharia [8]),
- special biserial algebras (by Liu and Morin [16]),
- truncated extensions of semisimple rings (by Marmaridis and Papistas [15]).

Following earlier work done by Lenzing, who used K-theoretic methods to obtain information on nilpotent elements in rings of finite global dimension. Skorodumov generalised and localised Lenzing's filtration to indecomposable projective modules. Enabling him to prove this conjecture for finite dimensional elementary algebra of finite representation type; see [6].
K.Igusa, S.Liu and C.Paquette localised Lenzing's trace function to endomorphisms of modules in $\operatorname{mod}\Lambda$ with e-bounded projective resolution, where e is an idempotent in Λ. Enabling them to obtain a local version of Lenzing's result which consequently provided the needed tool to solve the strong no loop conjecture for a large class of artin algebras including finite dimensional elementary algebras over any field and specially for finite dimensional algebras over an algebraically closed field.
The contents of the work, chapter by chapter are as follows.

Chapter 1, cover our preliminaries. Here we introduce some needed and basic concepts, such as radicals, semisimple modules, path algebras, then we show that any basic connected algebra is the quotient of a path algebra by an admissible ideal. A very important observation is that representation of a quiver are the same as modules over the underlying path algebra. Finally we end every thing with an important result (in propositon 1.4.5) for this work, that if $\Lambda = KQ/I$, a bound quiver algebra and $x, y \in Q_0$. There exists an isomorphism of K-vector spaces

$$\operatorname{Ext}^1_\Lambda(S(x), S(y)) \cong e_x(\operatorname{rad}\Lambda/\operatorname{rad}^2\Lambda)e_y.$$

In chapter 2, we basically look at some interesting result that leads us to the conjectures relating to the structure of the quiver Q_Λ. Which conjectures are the no loop and strong no loop conjecture.

In chapter 3, we present some results due to Hattori, Stallings and Lenzing giving us a tool in hand, which, combine together with chapter 4 will enable us to handle the strong no loop conjecture for the finite dimensional algebras over an algebraically closed field in chapter 5. Also the nice thing about these tools, we will use, is that we don't need to calculate any projective resolutions in order to make claims about the projective dimensions of the simple modules in view.

In chapter 4, here we go little step further. We recall Lenzing's extension of the Hattori-Stallings trace of endomorphism of projective modules to endomorphism of modules of finite projective dimension as before in chapter 3. Then localise this Lenzing's trace function to endomorphism of modules in $\operatorname{mod}\Lambda$ with e-bound projective resolution. This helps us to obtain a local version of the Lenzing's result. Finally, we will prove that the zeroth Hochschild homology group of Λ, $\operatorname{HH}_0(\Lambda)$ is radical trivial.

In chapter 5, here as consequence of chapter 4 we have the main result of the whole work: a proof of the strong no loop conjecture for large class of artin algebras includind finite dimensional elementary algebras over any field, and in particular for finite dimensional algebras over an algebraically closed field.

Chapter 1

Preliminaries

1.1 Basic facts from algebra and module theory

We will assume K to be an algebraically closed field, except otherwise stated. We recall that a K-algebra is a ring Λ with identity element such that Λ has a K-vector space structure compatible with the multiplication of the ring, that is, such that

$$\lambda(ab) = (a\lambda)b = a(\lambda b) = (ab)\lambda$$

for all $\lambda \in K$ and all $a, b \in \Lambda$. The algebra is called commutative if it is a commutative ring. We say that Λ is finite dimensional K-algebra if the dimension $\dim_K \Lambda$ of the K-vector space is finite. A morphism of K-algebras is a ring homomorphism which is linear over K.
Unless otherwise stated, all algebras will be assumed to be finite dimensional.
A right ideal of a K-algebra Λ is a K-vector subspace I such that $xa \in I$ for all $x \in I$ and $a \in \Lambda$. A left ideal is defined dually and a two-sided ideal or simply an ideal, is a K-vector subspace which is both a left and right ideal. A (right or left) I ideal is maximal if it is not equal to Λ and if $I \subset I'$ for an ideal I', then $I = I'$. It is straight to see that the K-vector space Λ/I is a K-algebra if I is an ideal and the quotient map is a morphism of K-algebras. Given an ideal I and $n \geq 1$, the ideal I^n consist of finite sums of elements of the form $x_1, ..., x_n$ with $x_i \in I$ and I is called nilpotent if for some n we have $I^n = 0$. This also make sense for right (or left) ideals

Definition. The (Jacobson) radical $\operatorname{rad}\Lambda$ of a K-algebra Λ is the intersection of all the maximal right ideals in λ.

Next we describe elements in the radical in the following.

Lemma 1.1.1. *Let Λ be a K-algebra and let $a \in \Lambda$. The following are equivalent*

a) $a \in \operatorname{rad}\Lambda$*;*

b) $a \in$ *to the intersection of all maximal left ideals of* Λ*;*

c) for any $b \in \Lambda$*, the element* $1 - ab$ *has a two sided inverse;*

d) for any $b \in \Lambda$*, the element* $1 - ab$ *has a right inverse;*

e) *for any* $b \in \Lambda$*, the element* $1 - ba$ *has a two-sided inverse;*

f) *for any* $b \in \Lambda$*, the element* $1 - ba$ *has a left inverse.*

Proof. (a) implies (d). Let $b \in \Lambda$ and assume to the contrary that $1 - ab$ has no right inverse. Then there exists a maximal right ideal of Λ such that $1 - ab \in I$. Because $a \in \operatorname{rad}\Lambda \subseteq I$, $ab \in I$ and $1 \in I$; this is a contradiction. This shows that $1 - ab$ has a right inverse.
(d) implies (a). Suppose to the contrary that $a \notin \operatorname{rad}\Lambda$ and let I be a maximal right ideal of Λ such that $a \notin I$. Then $\Lambda = I + a\Lambda$ and therefore there exist $x \in I$ and $b \in \Lambda$ such that $1 = x + ab$. It follows that $x = 1 - ab \in I$ has no right inverse, contrary to our assumption. The equivalence of (b) and (f) can be proof in a similar way.
The equivalence of (c) and (e) is a consequence of the following two implications:

- if $(1 - cd)x = 1$, then $(1 - dc)(1 + dxc) = 1$.
- If $y(1 - cd) = 1$, then $(1 + dyc)(1 - dc) = 1$.

(d) implies (c). Fix an element $b \in \Lambda$. By (d), there exist an element $c \in \Lambda$ such that $(1 - ab)c = 1$. Hence $c = 1 - a(-bc)$ and, according to (d), there exist $d \in \Lambda$ such that $1 = cd = d + abcd = d + ab$. It follows that $d = 1 - ab$, c is the left inverse of $1 - ab$ and (c) follows. That (f) implies (e) follows in a similar way. Because (c) implies (d) and (e) implies (f) obviously the lemma is proved.

Corollary 1.1.2. *Let* $\operatorname{rad}\Lambda$ *be the radical of an algebra* Λ*.*

a) $\operatorname{rad}\Lambda$ *is the intersection of all the maximal left ideals of* Λ*.*

b) $\operatorname{rad}\Lambda$ *is a two-sided ideal and* $\operatorname{rad}(\Lambda/\operatorname{rad}\Lambda) = 0$*.*

c) *If* I *is a two-sided nilpotent ideal of* Λ*, then* $I \subseteq \operatorname{rad}\lambda$*. If, inaddition, the algebra* Λ/I *is isomorphic to a product* $K \times ... \times K$ *of copies of* K*, then* $I = \operatorname{rad}\Lambda$*.*

Proof. The statement (a) easily follow from (1.1.1). To see that (b) holds, assume $a' \in \operatorname{rad}(\Lambda/\operatorname{rad}\Lambda)$. From (1.1.1) we see that for a representative a of a' and any $b \in \Lambda$ there exist $c \in \Lambda$ such that $(1 - ab)c = 1 - x$ for some $x \in \operatorname{rad}\Lambda$. Applying (1.1.1) to $1 - x$, we get an element $d \in \Lambda$ such that $(1 - x)d = 1$, hence $a \in \operatorname{rad}\Lambda$ and so $a' = 0 \in \Lambda/\operatorname{rad}\Lambda$.
To see (c) hold, Suppose that $I^n = 0$ for some $n > 0$. Let $x \in I$ and let a be an element of Λ. Then $ax \in I$ and therefore $(as)^r = 0$ for some $r > 0$. It follows that the equality $(1 + ax + (ax)^2 + ... + (ax)^{r-1})(1 - ax) = 1$ holds for any element $a \in \Lambda$, and, according to (1.1.1), the element x belongs to $\operatorname{rad}\Lambda$. Consequently, $I \subseteq \operatorname{rad}\Lambda$. To prove the reverse inclusion, assume that the algebra Λ/I is isomorphic to a product of copies of K. It follows that $\operatorname{rad}(\Lambda/I) = 0$. Next the natural surjective algebra homomorphism $\pi : \Lambda \to \Lambda/I$ carries $\operatorname{rad}\Lambda$ to $\operatorname{rad}(\Lambda/I) = 0$. Indeed, if $a \in \operatorname{rad}\Lambda$ and $\pi(b) = b + I$, with $b \in \Lambda$, is any element of Λ/I then, by (1.1.1), $1 - ba$ is invertible in Λ and therefore the element $\pi(1 - ba) = 1 - \pi(b)\pi(a)$ is invertible in Λ/I; thus $\pi(a) \in \operatorname{rad}\Lambda/I = 0$, by (1.1.1). This yields $\operatorname{rad}\Lambda \subseteq \ker\pi = I$.

Definition. Let Λ be a K-algebra. A right module over Λ is a pair $(M, .)$, where M is a K-vector space and $M \times \Lambda \to M$, $(m, a) \mapsto ma$, is a binary operation satisfying the following :

- $(x + y)a = xa + ya$;
- $x(a + b) = xa + xb$;
- $x(ab) = (xa)b$;
- $x1 = x$;
- $(x\lambda)a = x(a\lambda) = (xa)\lambda$

for all $x, y \in M$, $a, b \in \Lambda$ and $\lambda \in K$.

A left module over Λ is defined dually. Except otherwise stated, we will usually consider right modules from here on.
A module M is said to be finite dimensional if the dimension $\dim_K M$ of the underlying K-vector space of M is finite. A right Λ-module M is said to be generated by the elements $m_1, ..., m_s$ of M if any element $m \in M$ has the form $m = m_1a_1 + ... + m_sa_s$ for some $a_1, ..., a_s \in \Lambda$. In this case we write $M = m_1\Lambda + ... + m_s\Lambda$. A module M is said to be finitely generated if it is generated by a finite subset of elements of M. Also all well known notions such as submodules, module homomorphisms, etc., are the same as for modules over commutative rings. In particular, the category $\operatorname{Mod}\Lambda$ of all right modules is an abelian category. Given an algebra Λ, the opposite algebra is defined by reversing the order of the multiplication. It follows that $\operatorname{Mod}\Lambda^{op}$ is equivalent to the category of left modules over Λ and vice versa. The subcategory $\operatorname{mod}\Lambda$ of $\operatorname{Mod}\Lambda$ has as objects the finite dimensional modules.

Lemma (Nakayama's lemma). *Let Λ be a K-algebra, M be finitely generated right Λ-module, and $I \subseteq \operatorname{rad}\Lambda$ be a two-sided ideal of Λ. If $MI = M$, then $M = 0$.*

Proof. Suppose that $M = MI$ and $M = m_1\Lambda + ... + m_s\Lambda$, that is, M is generated by the elements $m_1, ..., m_s$. We proceed by induction on s. If $s = 1$, then the equality $m_1\Lambda = m_1I$ implies that $m_1 = m_1x_1$ for some $x_1 \in I$. Hence $m_1(1 - x_1) = 0$ and therefore $m_1 = 0$, because $1 - x_1$ is invertible. Consequently $M = 0$, as required.
Assume that $s \geq 2$. The equality $M = MI$ implies that there are elements $x_1, ..., x_s \in I$ such that $m_1 = m_1x_1 + m_2x_2 + ... + m_sx_s$. Hence $m_1(1 - x_1) = m_2x_2 + ... + m_sx_s$ and therefore $m_1 \in m_2\Lambda + ... + m_s\Lambda$ because $1 - x_1$ is invertible. This shows that $M = m_2\Lambda + ... + m_s\Lambda$ and the inductive hypothesis yields $M = 0$.

Corollary 1.1.3. *If Λ is a finite dimensional K-algebra, then $\operatorname{rad}\Lambda$ is nilpotent.*

Proof. Because $\dim_K \Lambda < \infty$, the chain

$$\Lambda \supseteq \operatorname{rad}\Lambda \supseteq (\operatorname{rad}\Lambda)^2 \supseteq ...(\operatorname{rad}\Lambda)^n \supseteq (\operatorname{rad}\Lambda)^{n+1} \supseteq ...$$

becomes stationary. It follows that $(\operatorname{rad}\Lambda)^n = (\operatorname{rad}\Lambda)^n \operatorname{rad}\Lambda$ for some n, and Nakayama's lemma yields $(\operatorname{rad}\Lambda)^n = 0$.

If Λ is a finite dimensional K algebra and $M \in \operatorname{mod}\Lambda$, consider the dual space $M^* = \operatorname{Hom}_K(M, K)$ endowed with the left Λ-module structure given by the formula $(a\phi)(m) = \phi(ma)$ for $\phi \in M^*$, $a \in \Lambda$ and $m \in M$, and to each Λ module homomorphism $h : M \to N$ the dual K-homomorphism $D(h) = \operatorname{Hom}_K(h, K) : D(N) \to D(M)$, $\phi \mapsto \phi h$, of left Λ-modules. One shows that D is a duality of categories , called the standard K-duality. The quasi-inverse to the duality is denoted by

$$D : \operatorname{mod}\Lambda^{op} \to \operatorname{mod}\Lambda$$

and is defined by attaching to each left Λ-module Y the K-vector space $D(Y) = Y^* = \operatorname{Hom}_K(Y, K)$ endowed with the right Λ-module structure given by the formula $(\phi a)(y) = \phi(ay)$ for $\phi \in \operatorname{Hom}_K(Y, K)$, $a \in \Lambda$ and $y \in Y$. A straight forward calculation shows that the evaluation K-linear map $ev : M \to M^{**}$ given by the formula $ev(m)(f) = f(m)$, where $m \in M$ and $f \in D(M)$, defines natural equivalences of functors $1_{\operatorname{mod}\Lambda} \cong D \circ D$ and $1_{\operatorname{mod}\Lambda^{op}} \cong D \circ D$.

Definition. Let Λ and Γ be two K-algebras. An $\Lambda - \Gamma$ bimodule is a triple ${}_\Lambda M_\Gamma = (M, *, .)$ such that ${}_\Lambda M = (M, *)$ is a left Λ-module, $M_\Gamma = (M, .)$ is a right Γ-module, and $(a * m).b = a * (m.b)$ for all $m \in M$, $a \in \Lambda$ and $b \in \Gamma$. Throughout, we write simply am and mb instead of $a * m$ and $m.b$, respectively.

Example. Any right module M can be considered as an $(\operatorname{End} M)\Lambda$-bimodule by noting that the left $\operatorname{End} M$-module structure is defined by $\phi m := \phi(m)$.

Note that if ${}_\Lambda M_\Gamma$ is an $\Lambda - \Gamma$-bimodule and N_Γ is a right Γ-module, the vector space $\operatorname{Hom}_\Gamma({}_\Lambda M_\Gamma, N_\Gamma)$ is a right Λ-module by setting $fa(m) := f(am)$ for all $a \in \Lambda$, $m \in M$ and $f \in \operatorname{Hom}_\Gamma({}_\Lambda M_\Gamma, N_\Gamma)$. Using this observation, we have covariant functor

$$\operatorname{Hom}_\Gamma({}_\Lambda M_\Gamma, -) : \operatorname{Mod}\Gamma \to \operatorname{Mod}\Lambda.$$

Similarly, we have a contravariant functor

$$\operatorname{Hom}_\Gamma(-, {}_\Lambda M_\Gamma) : \operatorname{Mod}\Gamma \to \operatorname{Mod}\Lambda^{op}.$$

Furthermore, given ${}_\Lambda M_\Gamma$ as above there are the tensor product functors

$$- \otimes_\Lambda M_\Gamma : \operatorname{Mod}\Lambda \to \operatorname{Mod}\Gamma,$$
$${}_\Lambda M \otimes_\Gamma - : \operatorname{Mod}\Gamma^{op} \to \operatorname{Mod}\Lambda^{op}$$

and an adjunction isomorphism

$$\operatorname{Hom}_\Gamma(X \otimes_\Lambda M_\Gamma, Z_\Gamma) \cong \operatorname{Hom}_\Lambda(X_\Lambda, \operatorname{Hom}_\Gamma({}_\Lambda M_\Gamma, Z_\Gamma)) \tag{1.1}$$

defined for a ϕ in the left hand space by by sending it to the map ψ given by $\psi(x)(m) = \phi(x \otimes m)$. The inverse map sends ψ in the right hand space to the map $\phi : x \otimes m \to \psi(x)(m)$. Formula (1.1) shows that the functor $- \otimes_\Lambda M_\Gamma$ is left adjoint to $\operatorname{Hom}_\Gamma(-, {}_\Lambda M_\Gamma)$ and $\operatorname{Hom}_\Gamma(-, {}_\Lambda M_\Gamma)$ is right adjoint to $- \otimes_\Lambda M_\Gamma$.

Definition. A right Λ-module S is simple if any submodule of S is either S or 0 module. A module M is semisimple if it is a direct sum of simple modules. A module is called indecomposable if in a decomposition $M = M_1 \oplus M_2$ either $M_1 = 0$ or $M_2 = 0$.

Lemma (Schur's lemma). *Any nonzero homomorphism between simple modules is an isomorphism.*

Proof. Let $f : S \to S'$ be a homomorphism from a simple module S to a simple module S'. Since $\ker f$ and $\operatorname{Im} f$ are submodules of S and S', respectively, $f \neq 0$ implies $\ker f \neq S$ and $\operatorname{Im} f \neq 0$. Since S and S' are simple modules, $\ker f = 0$ and $\operatorname{Im} f = S'$, thus f is both a monomorphism and an epimorphism, hence f is an issomorphism.

Corollary 1.1.4. *If S is a simple Λ-module, then $\operatorname{End}(S) \cong K$.*

Proof. By Schur's lemma, $\operatorname{End}(S)$ is a skew field. Since Λ is simple, any map $\Lambda \to S$ is an epimorphism, hence $\dim_K(S) < \infty$. Thus, also $\dim_K \operatorname{End}(S) < \infty$. Hence, for any $0 \neq \phi \in \operatorname{End}(S)$ there exist an irreducible polynomial $f \in K[t]$ such that $f(\phi) = 0$. Since K is algebraically closed, f is of degree 1, hence ϕ corresponds to a scalar $\lambda_\phi \in K^*$, which give the desired isomorphism.

Proposition (A.S.M). *The endomorphism ring of an artinian semisimple module is semisimple.*

Proof. We see first that if $M = M_1 \oplus M_2 \oplus ... \oplus M_r$ then the $\operatorname{End} M$ is isomorphic to the $r \times r$ matrix ring where (i, j)-th component of the matix is an element from $\operatorname{Hom}(M_i, M_j)$. (To see this, note that $\operatorname{Hom}(M_i, M_j) \cong \pi_i \operatorname{End} M \pi_j$, where π_l is the corresponding projection of M onto M_l. That is the required isomorphism is given by the standard Peirce-decomposition of the ring Λ : if e is an idempotent in Λ and $f = 1 - e$ then $\Lambda \cong \begin{pmatrix} e\Lambda e & e\Lambda f \\ f\Lambda e & f\Lambda f \end{pmatrix}$.) In our case. if M is semisimple artinian then $M = S_1^{n_1} \oplus S^{n_2} \oplus ... \oplus S^{n_s}$ where the modules S_i are pairwise nonisomorphic simple modules. Here $\operatorname{Hom}(S_i, S_j) = 0$ for $i \neq j$, and $\operatorname{Hom}(S_i, S_j) = D_i$ is a division ring for each i by Schur's lemma. Thus $\operatorname{End} \Lambda \cong M_{n_1}(D_1) \oplus M_{n_2}(D_2) \oplus ... \oplus M_{n_s}(D_s)$. This ring is semisimple because each of the matrix rings $M_{n_i}(D_i)$ is generated by the columns (as left ideals) which are simple modules.

Theorem (Wedderburn-Artin). *A ring Λ is semisimple if and only if Λ is a direct sum of finitely many ideals, each of which is full matrix ring over a division ring.*

Proof. If Λ is semisimple then $\Lambda \cong \operatorname{End}({}_\Lambda\Lambda)$ and the argument of Prop[A.S.M] showed that Λ is a direct sum of full matrix rings over division rings. Conversely, $M_n(D)$ is clearly generated by its columns as left ideals and it is easy to see these are simple modules. So the direct sum of full matrix rings is also semisimple.

Lemma 1.1.5. *A finite dimensional module M is semisimple if and only if for any submodule N of M there exists a submodule L of M such that $L \oplus N \cong M$. In particular, a submodule of a semisimple module is semisimple*

Proof. Suppose that $M = S_1 \oplus ... \oplus S_n$ where the S_i are simple modules. Let $0 \neq N \subseteq M$ be a submodule and consider the maximal family $\{S_{i_1}, ..., S_{i_k}\}$ of the S_i such that $N \cap L = 0$, where $L = S_{i_1} \oplus ... \oplus S_{i_k}$.Then $N \cap (L + S_t) \neq 0$, for any $t \notin \{i_1, ..., i_k\}$. From this it follows that $(L + N) \cap S_t \neq 0$ for all $t \notin \{i_1, ..., i_k\}$. Therefore, $M = L + N$ and hence $M = L \oplus N$. The reverse implication follows by induction on $\dim_K(M)$.

Definition. Let M be a right Λ-module the Jacobson radical of M is the intersection of all the maximal submodule.

We recollect the following main properties.

Proposition 1.1.6. *Let L, M and $N \in \Lambda$.*

a) *$m \in M$ belongs to $\operatorname{rad} M$ if and only if $f(m) = 0$ for any $f \in \operatorname{Hom}_\Lambda(M, S)$ and any simple right Λ-modules.*

b) $\operatorname{rad}(M \oplus N) = \operatorname{rad} M \oplus \operatorname{rad} N$.

c) *If $f \in \operatorname{Hom}_\Lambda(M, N)$, then $f(\operatorname{rad} M) \subseteq \operatorname{rad} N$.*

d) $M \operatorname{rad} \Lambda = \operatorname{rad} M$.

e) *If L and M are Λ-submodules of N. If $L \subseteq \operatorname{rad} N$ and $L + M = N$ then $M = N$.*

Proof. a) claim holds from the definition as $L \subseteq M$ is a maximal submodule if and only if M/L is semisimple.

b) This statement follows immediately from (a).

c) To prove (c), follows immediately from (a), by considering any map $g \in \operatorname{Hom}_\Lambda(N, S)$ and using fact that $gf(m) = 0$.

d) Let $m \in M$ and $f_m : \Lambda \to M$ be a homomorphism of right modules with $f_m(a) = ma$ for $a \in \Lambda$. From (c) we see that as $a \in \operatorname{rad} \Lambda$, we have $ma = f_m(a) \in f_m(\Lambda) \subseteq \operatorname{rad} M$. So then $M \operatorname{rad} \Lambda \subseteq \operatorname{rad} M$.To prove that $\operatorname{rad} M \subseteq M \operatorname{rad} \Lambda$ we know that $(M/M \operatorname{rad} \Lambda) \operatorname{rad} \Lambda = 0$ and so the $M/M \operatorname{rad} \Lambda$ is a module over the algebra $\Lambda/\operatorname{rad} \Lambda$ with respect to the action $(m + M \operatorname{rad} \Lambda)$. Thus $a + \operatorname{rad} \Lambda = ma + M \operatorname{rad} \Lambda$. The Wedderburn-Artin theorem tells us that an algebra $\Lambda/\operatorname{rad} \Lambda$ is semisimple and the finite dimensional $\Lambda/\operatorname{rad} \Lambda$-module $M/M \operatorname{rad} \Lambda$ is a direct sum of simple modules. Since the radical of any simple module is zero. (b) yields $\operatorname{rad}(M/M \operatorname{rad} \Lambda) = 0$.So by (c) the natural Λ -module epimorphism $\pi : M \to M/M \operatorname{rad} \Lambda$ annihilates $\operatorname{rad} M$ thus $\operatorname{rad} \subseteq \ker \pi = M \operatorname{rad} \Lambda$.

e) Let $L \subseteq \operatorname{rad} N$ and $L + M = N$ and suppose otherwise that $M \neq N$.Since N is finite dimensional, M is a submodule of a maximal submodule $X \neq N$ of N. So that $L \subseteq \operatorname{rad} N \subseteq X$ and yield $N = L + M \subseteq X + M = X$ contrary to our claim.

Corollary 1.1.7. *Let $M \in \operatorname{mod} \Lambda$*

a) The Λ*-module* $M/\operatorname{rad} M$ *is semisimple and it is a module over the* K*-algebra* $\Lambda/\operatorname{rad}\Lambda$*.*

b) If L *is a submodule of* M *such that* M/L *is semisimple, then* $\operatorname{rad} M \subseteq L$*.*

Proof. a) By (1.1.6d) $\operatorname{rad} M = M \operatorname{rad}\Lambda$. This yield $(M/\operatorname{rad} M)\operatorname{rad}\Lambda = 0$ and so the Λ-module $M/\operatorname{rad} M$ is a module over $\Lambda/\operatorname{rad}\Lambda$ with respect to the action $(m + M\operatorname{rad}\Lambda)(a + \operatorname{rad}\Lambda) = ma + M\operatorname{rad}\Lambda$. And by Wedderburn-Artin theorem,the algebra $\Lambda/\operatorname{rad}\Lambda$ is semisimple and the module $M/\operatorname{rad} M$ is semisimple.

b) Let L be a submodule of M such that M/L is semisimple. We have this natural epimorphism $\pi : M \to M/L$. Since (1.1.6c) gives $\pi(\operatorname{rad} M) \subseteq \operatorname{rad}(M/L) = 0$, $\operatorname{rad} M \subseteq \ker\pi = L$ and b) holds. Also by (1.1.6d) we have $(M/\operatorname{rad} M)\operatorname{rad}\Lambda = 0$ and so the module top $M = M/\operatorname{rad} M$ called top of M is a right $\Lambda/\operatorname{rad}\Lambda$ module with respect to the action of $\Lambda/\operatorname{rad}\Lambda$ defined by the formula $(m + \operatorname{rad} M)(a + \operatorname{rad} N) = ma + \operatorname{rad} M$.

Corollary 1.1.8. *a) A homomorphism* $f : M \to N$ *in* $\operatorname{mod}\Lambda$ *is surjective if and only if the homomorphism* $\operatorname{top} f : \operatorname{top} M \to \operatorname{top} N$ *is surjective.*

b) If S *is a simple* Λ *module, then* $S\operatorname{rad}\Lambda = 0$ *and* S *is a simple* $\Lambda/\operatorname{rad}\Lambda$*-module.*

c) A Λ *module* M *is semisimple if and only if* $\operatorname{rad} M = 0$*.*

Proof. a) Assume the top f is surjective. Then $\operatorname{Im} f + \operatorname{rad} N = N$ and therefore f is surjective, because (1.1.6e) yields $\operatorname{Im} f = N$. Since the converse implication is clear, (a) follows.

b) Statement (b) is clear, by Nakayama's lemma and since $S\operatorname{rad}\Lambda$ is a submodule of the simple module S.

c) If M is semisimple, then (b) yields $\operatorname{rad} M = 0$. The converse implication is a consequence of (1.1.6d) and (1.1.7a).

Let M be a module satisfying ascending and descending chain conditons (ACC and DCC). In other words every increasing sequence of submodules $M_1 \subset M_2 \subset \ldots$ and any decreasing sequence $M_1 \supset M_2 \supset \ldots$ are finite. Then it is easy to see that there exist a finite sequence

$$M = M_0 \supset M_1 \supset M_2 \supset \ldots \supset M_r = 0$$

such that M_i/M_{i+1} is a simple module. Such a sequence is called a composition series. We say the two composition series

$$M = M_0 \supset M_1 \supset M_2 \supset \ldots \supset M_r = 0,$$
$$M = N_0 \supset N_1 \supset N_2 \supset \ldots \supset N_s = 0$$

are equivalent if $r = s$ and for some permutation σ $M_i/M_{i+1} \cong N_{\sigma_i}/N_{\sigma(i+1)}$.

Theorem (Jordan-Hölder). *Any two composition series are equivalent.*

Proof. We will prove that if the statement is true for any submodule of M then it is true for M.(If M is simple, the statement is trivial.) If $M_1 = N_1$, then the statement is obvious. Otherwise, $M_1 + N_1 = M$, hence $M/M_1 \cong N_1/(M_1 \cap N_1)$ and $M/N_1 \cong M_1/(M_1 \cap N_1)$. Consider the series

$$M = M_0 \supset M_1 \supset M_1 \cap N_1 \supset K_1 \supset ... \supset K_\sigma = 0,$$
$$M = N_0 \supset N_1 \supset N_1 \cap M_1 \supset K_1 \supset ... \supset K_\sigma = 0$$

They are obviously equivalent, and by induction assumption the first series is equivalent to $M = M_0 \supset M_1 \supset M_2 \supset ... \supset M_r = 0$, and the second one is equivalent to $M = M_0 \supset M_1 \supset M_2 \supset ... \supset M_r = 0$. Hence they equivalent.

Thus we can define a length $l(M)$ of a module M satisfying ACC and DCC, and if M is a proper submodule of N, then $l(M) < l(N)$.

Definition (Idempotents and direct decompositions). An $e \in \Lambda$ is called an idempotent if $e^2 = e$. An idempotent e is said to be central if $\Lambda e = e\Lambda$ for all $\lambda \in \Lambda$. The idempotents $e_1, e_2 \in \Lambda$ are called orthogonal if $e_1 e_2 = e_2 e_1 = 0$. An idempotent e is said to be primitive if e cannot be written as a sum $e = e_1 + e_2$ where e_1 and e_2 are non zero orthogonal idempotents of Λ. Every algebra Λ has two trivial idempotents 0 and 1. If the idempotent e of Λ is non-trivial, then $1 - e$ is also a nontrivial idempotent, the idempotents e and $1 - e$ are orthogonal and there is a non trivial right Λ -module decomposition $\Lambda_\Lambda = e\Lambda \oplus (1-e)\Lambda$. Conversely, if $\Lambda_\Lambda = M_1 \oplus M_2$ is a non trivial Λ-module decomposition then $m_i \in M_i$ with $1 = m_1 + m_2$ are orthogonal idempotents and $M_i = e_i\Lambda$ is indecomposable module if and only if e_i is primitive.
If e is a central idempotent, then so is $1 - e$, and $e\Lambda$ and $(1 - e)\Lambda$ are two-sided ideals and they are easily shown to be $K - algebra$ with identity elements $e \in e\Lambda$ and $1 - e \in (1 - e)\Lambda$, respectively. In this case the decomposition $\Lambda_\Lambda = e\Lambda \oplus (1-e)\Lambda$ is a direct product decomposition of the algebra Λ. Since the algebra Λ is finite dimensional, the module Λ_Λ admits a direct sum decomposition $\Lambda_\Lambda = P_1 \oplus ... \oplus P_s$, where $P_1 ... P_s$ are indecomposable right ideals of Λ. So that $P_1 = e_1\Lambda, ..., P_s = e_s\Lambda$, with $e_1, ..., e_s$ as primitive pairwise orthogonal idempotents of Λ such that $1 = e_1 + ... + e_s$. Conversely every set of idempotents with the previous properties induces a decomposition $\Lambda_\Lambda = P_1 \oplus ... \oplus P_s$ with indecomposable right ideals $P_1 = e_1\Lambda, ..., P_s = e_s\Lambda$. Such a decomposition is called an indecomposable decomposition of Λ and such a set $\{e_1, ..., e_s\}$ is called a complete set of primitive orthogonal idempotens of Λ. So we say that an algebra Λ is connected(or indecomposable) if Λ is not a direct product of two algebras or equivalently if 0 and 1 are the only central idempotent.

Consider a right Λ-module M and an idempotent $e \in \Lambda$. Note that the K-vector subspace $e\Lambda e$ of Λ is a K-algebra with identity e. Also note it is subalgebra of Λ if and only if $e = 1$. We can define an $e\Lambda e$-module structure on the subspace Me of M by setting $me(eae) := meae$ for all $m \in M$ and $a \in \Lambda$. In particular, Λe is a right $e\Lambda e$-module and $e\Lambda$ is a left $e\Lambda e$-module. This implies that $\mathrm{Hom}_\Lambda(e\Lambda, M)$ is a right $e\Lambda e$-module with respect to the action $(\phi.eae)(x) = \phi(eaex)$ for $x \in e\Lambda$, $a \in \Lambda$ and $\phi \in \mathrm{Hom}_\Lambda(e\Lambda, M)$.
The following lemmas will be very useful.

Lemma 1.1.9. *Let Λ be a K-algebra, $e \in \Lambda$ be an idempotent, and M be a right Λ-module.*

a) *The K-linear map*

$$\omega_M : \mathrm{Hom}_\Lambda(e\Lambda, M) \to Me \tag{1.2}$$

defined by the formula $\psi \mapsto \psi(e) = \psi(e)e$ for $\psi \in \mathrm{Hom}_\Lambda(e\Lambda, M)$, is an isomorphism of right $e\Lambda e$-modules, and it is functorial in M.

b) *The isomorphism $\omega_{e\Lambda} : \mathrm{End}\, e\Lambda \xrightarrow{\sim} e\Lambda e$ of right $e\Lambda e$-modules induces an isomorphism of K-algebras.*

Proof. It is easy to see that the map ω_M is a homomorphism of right $e\Lambda e$-modules and it is functorial at at the variable M. We define a K-linear map $\omega'_M : Me \to \mathrm{Hom}_\Lambda(eM, M)$ by the formula $\omega'_M(me)(ea) = mea$ for $a \in \Lambda$ and $m \in M$. A straightforward calculation shows that, given $m \in M$, the map $\omega'_M(me) : e\Lambda \to M$ is well defined (does not depend on the choice of a in the presentation ea), it is a homomorphism of Λ-modules , moreover ω'_M is a homomorphism of $e\Lambda e$-modules and ω'_M is an inverse of ω_M. This proves (a). The statement (b) easily follows from (a)

Lemma 1.1.10. *For any K-algebra Λ the idempotents of the algebra $B = \Lambda/\operatorname{rad}\Lambda$ can be lifted modulo $\operatorname{rad}\Lambda$, that is, for any idempotent $f = g + \operatorname{rad}\Lambda \in B$, $g \in \Lambda$, there exist an idempotent e of Λ such that $g - e \in \operatorname{rad}\Lambda$.*

Proof. It follows from (1.1.3) that the $(\operatorname{rad}\Lambda)^n = 0$ for some $n > 1$. Because $f^2 = f$, $g - g^2 \in \operatorname{rad}\Lambda$ and therefore $(g - g^2)^n = 0$. Hence, by Newtons binomial formula, we get $0 = (g - g^2)^n = g^n - g^{n+1}t$, where $t = \sum_{i=1}^{n}(-1)^{i-1}\begin{pmatrix} n \\ i \end{pmatrix} g^{i-1}$. It follows that

a) $g^n = g^{n+1}t$;

b) $gt = tg$.
We claim that the element $e = (gt)^n$ is the idempotent lifting f. First, we note that $e = g^n t^n = g^{n+1}t^{n+1} = ... = g^{2n}t^{2n} = ((gt)^m)^2 = e^2$ and therefore e is an idempotent. Next, we note that

c) $g - g^n \in \operatorname{rad}\Lambda$,
because the relation $g - g^2 \in \operatorname{rad}\Lambda$ yields the inequalities $g - g^n = g(1 - g^{n-1}) = g(1 - g)(1 + g + ... + g^{n-2}) = (g - g^2)(1 + g + ... + g^{n-2}) \in \operatorname{rad}\Lambda$. Moreover, we have

d) $g - gt \in \operatorname{rad}\Lambda$,
because equalities (a)-(c) yield
$g + \operatorname{rad}\Lambda = g^n + \operatorname{rad}\Lambda = g^{n+1}t + \operatorname{rad}\Lambda = (g^{n+1} + \operatorname{rad}\Lambda)\ (t + \operatorname{rad}\Lambda) = (g^n + \operatorname{rad}\Lambda)(g + \operatorname{rad}\Lambda)(t + \operatorname{rad}\Lambda) = (g + \operatorname{rad}\Lambda)(g + \operatorname{rad}\Lambda)(t + \operatorname{rad}\Lambda) = (g^2 + \operatorname{rad}\Lambda)(t + \operatorname{rad}\Lambda) = (g + \operatorname{rad}\Lambda)(t + \operatorname{rad}\Lambda) = gt + \operatorname{rad}\Lambda$. Consequently, we get $e + \operatorname{rad}\Lambda = (gt)^n + \operatorname{rad}\Lambda = (gt + \operatorname{rad}\Lambda)^n = (g + \operatorname{rad}\Lambda)^n = g^n + \operatorname{rad}\Lambda = g + \operatorname{rad}\Lambda$ and our claim follows.

Proposition 1.1.11. *Let $B = \Lambda/\operatorname{rad}\Lambda$. The following statements hold.*

a) *Every right ideal ideal I of B is a direct sum of simple right ideals of the form eB, where e is a primitive idempotent of B. In particular, the right ideals B-module B_B is semisimple.*

b) *Any module N in $\operatorname{mod} B$ is isomorphic to a direct sum of simple right ideals of the form eB, where e is a primitive idempotent of B.*

c) *If $e \in \Lambda$ is a primitive idempotent of Λ, then the B-module $\operatorname{top} e\Lambda$ is simple and $\operatorname{rad} e\Lambda = e \operatorname{rad} \Lambda \subset e\Lambda$ is the unique maximal proper submodule of $e\Lambda$.*

Proof. a) Let S be a nonzero right ideal of B contained in I that is of minimal dimension. Then S is a simple B-module and $S^2 \neq 0$, because otherwise, in view of (1.1.2c), $0 \neq S \subseteq \operatorname{rad} B = 0$ and we get a contradiction. Hence $S^2 = S$ and there exists $x \in S$ such that $xS \neq 0$, $S = xS$ and $x = xe$ for nonzero $e \in S$. Then, according to Schur's lemma, the B-homomorphism $\psi : S \to S$ given by the formula $\psi(y) = xy$ is bijective. Because $\psi(e^2 - e) = x(e^2 - e) = xee - xe = xe - xe = 0, e^2 - e = 0$, the element $e \in S$ is a nonzero idempotent, and $S = eB$. It follows that $B = eB \oplus (1-e)B$ and $I = S \oplus (1-e)I$. Because $\dim_K(1-e)I < \dim_K I$, we can assume by induction that (a) is satisfied for $(1-e)I$ and therefore (a) follows.

b) Let N be a B-module generated by the elements $n_1, ..., n_s$ and consider the B-module epimorphism $h : B^s \to N$ defined by the formula $h(\delta_i) = n_i, where \delta_1, ..., \delta_s$ is the standard basis of the B-modules B^s. If N is simple, then $s = 1$ and (a) together with (1.1.5a) yields $N \cong eB$, where e is a primitive idempotent of B. Now suppose N is arbitrary. Then, by (a), B^s is a direct sum of simple right ideals of the form eB, where e is a primitive idempotent of B, and it follows from (1.1.5a) that $B^s = \ker h \oplus L$ for some B submodule L of B^s. Then h induces an isomorphism $L \cong N$ and (b) follows from (1.1.5b).

c) The element $\bar{e} = e + \operatorname{rad} \Lambda$ is an idempotent of b and $\operatorname{top} e\Lambda \cong \bar{e}B$. Assume to the contrary that $\bar{e}B$ is not simple. It follows from (a) that $\bar{e}B = \bar{e_1}B \oplus \bar{e_2}B$, where $\bar{e_1}, \bar{e_2}$ are nonzero idempotents of B such that $\bar{e} = \bar{e_1} + \bar{e_2}$ and $\bar{e_1}\bar{e_2} = \bar{e_2}\bar{e_1} = 0$. Because $\bar{e_1} = \bar{e_1}^2 = (\bar{e} - \bar{e_2}^2)\bar{e_1} = e\bar{e}_1$, $\bar{e_1} = g_1 + \operatorname{rad} \Lambda$ for some $g_1 \in e\Lambda$. By (1.1.10), there exist $t \in \Lambda$ and $n \in \mathbb{N}$ such that the element $e_1 = (g_1 t)^n$ is an idempotent of $\Lambda and \bar{e_1} = e_1 + \operatorname{rad} \Lambda$. It follows that $\operatorname{top} e\Lambda = \bar{e}B = \bar{e_1}B \oplus \bar{e_2}B$. Because $g_1 \in e\Lambda, e_1 \in e\Lambda$ and $e_1\Lambda \subseteq e\Lambda$. Then the decomposition $\Lambda_\Lambda = e_1\Lambda \oplus (1-e_1)\Lambda$ induces the decomposition $e\Lambda = e_1\Lambda \oplus \{(1-e_1)\Lambda \cap e\Lambda\}$. It follows that $e\Lambda = e_1\Lambda$, because the primitivity of e implies that $e\Lambda$ is indecomposable. Hence $\bar{e}B = \operatorname{top} e\Lambda = \operatorname{top} e_1\Lambda = \bar{e_1}B$ and therefore $\bar{e_2}B = 0$, contrary to our assumption. Consequently, the module $\operatorname{top} e\Lambda$ is simple and therefore $\operatorname{rad} e\Lambda = (e\Lambda) \operatorname{rad} \Lambda$ is a maximal proper Λ-submodule of $e\Lambda$, then $L + \operatorname{rad} e\Lambda = e\Lambda$ and (1.1.6e) yields $L = e\Lambda$, a contradicton. This shows that $\operatorname{rad} e\Lambda$ contains all proper submodules of $e\Lambda$. Hence proof.

Definition. An algebra is called local if it has a unique maximal right ideal.

Next we give characterisations of a local algebra.

Lemma 1.1.12. *Let Λ be a K-algebra. The following are equivalent:*

a) *Λ is local.*

b) *Λ has a unique maximal left ideal.*

c) *The set of all noninvertible elements of Λ is a two-sided ideal.*

d) *For any $a \in \Lambda$, either a or $1-a$ is invertible.*

e) *Λ has only two idempotents, namely 0 and 1.*

f) *The algebra $\Lambda/\operatorname{rad}\Lambda$ is isomorphic to K.*

Proof. (a) implies (b). If Λ is local, then $\operatorname{rad}\Lambda$ is the unique maximal right ideal of Λ. Hence, $x \in \operatorname{rad}\Lambda$ if and only if x has no right inverse. Now, if x is right invertible, so $xy = 1$ for some y, then $(1-yx)y = 0$. The element y has to have a right inverse, because otherwise $y \in \operatorname{rad}\Lambda$, so view of lemma (1.1.1) $1 - yx$ is invertible and we get $y = 0$, a contradiction. But if y has a right inverse, $1-yx = 0$, so x is invertible. Summarising, $x \in \operatorname{rad}\Lambda$ if and only if x has no right inverse or equivalently, if and only if x is not invertible. Similar arguments show that (b) implies (c).It is obvious that (c) implies (d). Next, if e is an idempotent, so is $1-e$ and $e(1-e) = 0$, so if (d) holds, then so does (e). If (e) holds, then the algebra $B = \Lambda/\operatorname{rad}\Lambda$ has only two idempotents. By (1.1.12), the module B_B is simple and by (1.1.4), $\operatorname{End}(B_B) = K$. Therefore, $B \cong \operatorname{End}(B_B) \cong K$, hence (e) implies (f). Finally, if (f) holds, then clearly so does (a).

Remark. Note that the algebra $K[t]$ has only two idempotents 0 and 1 but is not local. Hence the lemma does not hold for infinite dimensional algebras.

Corollary 1.1.13. *An idempotent $e \in \Lambda$ is primitive if and only if the algebra $e\Lambda e \cong \operatorname{End} e\Lambda$ has only two idempotents 0 and e, that is, the algebra is local.*

Corollary 1.1.14. *Let Λ be an arbitrary K-algebra and M a right module*

a) *If the algebra $\operatorname{End} M$ is local, then M is indecomposable.*

b) *If M is finite dimensional and indecomposable, then the algebra $\operatorname{End} M$ is local and any Λ-module endomorphism of M is nilpotent or is an isomorphism.*

Proof. a) If M decomposes as $M = X_1 \oplus X_2$ with both X_1 and X_2 nonzero, then there exist projections $p_i : M \to X_i$ and injections $u_i : X_i \to M$ (for $i = 1, 2$) such that $u_1p_1 + u_2p_2 = 1_M$. Because u_1p_1 and u_2p_2 are nonzero idempotents in $\operatorname{End} M$, the algebra $\operatorname{End} M$ is not local.

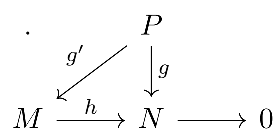
b) Suppose that M is finite dimensional and indecomposable. If $\operatorname{End} M$ is not local then, by (1.1.12), the algebra $\operatorname{End} M$ has a pair of nonzero idempotents $e_1, e_2 = 1 - e_1$ and therefore $M \cong \operatorname{Im} e_1 \oplus \operatorname{Im} e_2$ is a nontrivial direct sum decomposition. Consequently, the algebra $\operatorname{End} M$ is local. By (1.1.12) every noninvertible Λ-module endomorphism $f : M \to M$ belongs to the radical of $\operatorname{End} M$ and therefore f is nilpotent, because $\operatorname{End} M$ is finite dimensional, and it follows from (1.1.3).

Theorem 1.1.15. *Every finite dimensional module M over Λ has a decomposition $M \cong M_1 \oplus ... \oplus M_r$, where the M_i are indecomposable modules, and hence have local endomorphism algebras. Furthermore, if $M \cong M_1 \oplus ... \oplus M_r$ and $M \cong N_1 \oplus ... \oplus N_s$ with M_i and N_j indecomposable, then $m = n$ and there exist a permutation $\sigma of \{1, ..., r\}$ such that $M_i \cong N_{\sigma(i)}$ for all i.*

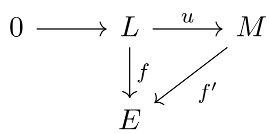
Proof. The first statment is clear, because $\dim_K M$ is finite. To see the second, we proceed by induction. If $n = 1$, then there is nothing to show. So suppose that $n > 1$ and consider $M' := \bigoplus_{i>1} M_i$. We have the decomposition $M = M_1 \oplus M'$ with the corresponding projections and injections p, p' and u, u' respectively. Denote the projections and injections corresponding to $M = \bigoplus N_j$ by p_j and u_j. We know that $1_{M_1} = pu = p(\sum_j u_j p_j)u = \sum_j p u_j p_j u$. Since $\operatorname{End} M_1$ is local, by (1.1.12d), there exist an index j for an invertible $v = p u_j p_j u$ say. Without lost of generality can be assumed to be 1, such that $v := p u_1 p_1 u$ is invertible. Now set $w := v^{-1} p u_1 : N_1 \to M_1$ and note that $w p_1 u = 1_{M_1}$. Hence, $p_1 u w$ is an idempotent in $\operatorname{End} N_1$. The latter is a local algebra, so $p_1 u w$ is 0 or 1. It cannot be equal to zero, because then $p_1 u = 0$, since w is an epimorphism, but $v := p u_1 p_1 u$ is invertible. Therefore, $p_1 u w = 1_{N_1}$ and hence $p_1 u$ gives $M_1 \cong N_1$. Writing $M \cong M_1 \oplus M' = N_1 \oplus N'$, where $N' := \bigoplus_{j>1} N_j$, we are done by induction if we can show that $M' \cong N'$. But this is clear, since N' is the kernel of $p_1 : M \to N_1$ and M' is the kernel of $p : M \to M_1$ and it is obvious that they coincide via the above isomorphism $p_1 u : M_1 \cong N_1$.

Definition. a) A right Λ-module F is free if is isomorphic to a direct sum of copies of the module Λ_Λ.

b) A right Λ-module P is projective if for any epimorphism $h : M \to N$ and a homomorphism $g : P \to N$ there is an homomorphism $g' : P \to M$ such that the following diagram commute.



c) A right Λ-module E is injective if for any monomorphism $u : L \to M$ and any homomorphism $f : L \to E$ there is a homomorphism $f' : M \to E$ such that the following diagram commute.



Lemma 1.1.16. *a) A right Λ module P is projective if and only if there exist a free Λ-module F and Λ-module P' such that $P \oplus P' \cong F$.*

b) Suppose that $\Lambda_\Lambda = e_1\Lambda \oplus ... \oplus e_s\Lambda$ is a decomposition of Λ_Λ into indecomposable submodules.If a right Λ-module P is projective, then $P = P_1 \oplus ... \oplus P_r$ where every summand P_i is indecomposable and isomorphic to some $e_z\Lambda$.

c) *Let M be an arbituary right Λ-module. Then there exist an exact sequence*

$$\dots \longrightarrow P_r \xrightarrow{h_r} P_{r-1} \xrightarrow{h_{r-1}} \dots \longrightarrow P_1 \xrightarrow{h_1} P_0 \xrightarrow{h_0} M \longrightarrow 0 \tag{1.3}$$

in $\operatorname{Mod}\Lambda$, *with P_i projective Λ-modules for $i \geq 0$.If also M in* $\operatorname{mod}\Lambda$, *then there exist an exact sequence 1.3) with P_i projective module in* $\operatorname{mod}\Lambda$ *for $i \geq 0$.*

Proof. a) It is easy to check that any free module is projective and a direct summand of a free module is projective. Conversely suppose that P is a projective module generated by elements $\{m_i | i \in I\}$. If $F = \bigoplus_{i \in I} x_i\Lambda$ is a free module with the set $\{x_i | i \in I\}$ of free generators and $f : F \to P$ is the epimorphism defined by $f(x_i) = m_i$ then since P is projective there exist a split epi $s : P \to F$ of f and hence $F \cong P \oplus \ker f$.

b) Let P be projective then from a) there exist a free Λ-module F and a right Λ- module P' such that $P \oplus P' \cong F$. From our assumption F is a direct sum of copies of the indecomposable modules $e_1\Lambda, \dots, e_s\Lambda$.

c) It was shown in a) for any $M \in \operatorname{mod}\Lambda$ there is an epimorphism $f : F \to M$, where F is a free module in $\operatorname{Mod}\Lambda$ (or in $\operatorname{mod}\Lambda$) respectively. We set $P_0 = F$ and $h_0 = f$.Let $f_1 : F_1 \to \ker h_0$ be an epimorphism with a free module F_1 in $\operatorname{Mod}\Lambda$.We set $P_1 = F_1$ and we take for h_1 the composition of f_1 with the embedding $\ker h_0 \subseteq P_0$. If $M \in \operatorname{mod}\Lambda$, then the free module F_1 can be chosen in $\operatorname{mod}\Lambda$, since Λ is finite dimensional, hence $\dim_K M$ and $\dim_K F_0$ are finite and so $\ker h_0$ is in $\operatorname{mod}\Lambda$. Continuing this procedure, we construct by induction the required exact sequence (1.3).

Definition (Projective resolution:). We define a projective resolution of a right Λ-module M to be the complex

$$P. : \dots \longrightarrow P_r \xrightarrow{h_r} P_{r-1} \xrightarrow{h_{r-1}} \dots \longrightarrow P_1 \xrightarrow{h_1} P_0 \longrightarrow 0$$

of projective Λ-modules together with an epimorphism $h_0 : P_0 \to M$ of right Λ-modules such that the sequence (1.3) is exact. For the sake of simplicity, we call the sequence (1.3) a projective resolution of the Λ-module M. By the lemma any module M in $\operatorname{mod}\Lambda$ admit a projective resolution in $\operatorname{mod}\Lambda$.

Definition (Injective resolution:). We define an injective resolution of M to be a complex

$$I. : 0 \longrightarrow I_0 \xrightarrow{d_1} I_1 \xrightarrow{d_2} \dots \longrightarrow I_r \xrightarrow{d_{r+1}} I_{r+1} \longrightarrow \dots$$

of injective Λ modules together with a monomorphism $d_0 : M \to I_0$ of right Λ-modules such that the sequence

$$0 \longrightarrow M \xrightarrow{d_0} I_0 \xrightarrow{d_1} I_1 \xrightarrow{d_2} \dots \longrightarrow I_r \xrightarrow{d_{r+1}} I_{r+1} \longrightarrow \dots$$

is exact. For the sake of simplicity, we call this sequence an injective resolution.

Next we show that if Λ is a finite dimensional algebra over K, then any module $M \in \operatorname{mod}\Lambda$ admit an exact sequence (1.3) in $\operatorname{mod}\Lambda$, where the epimorphisms $h_i : P_i \to \operatorname{Im} h_i$ are minimal for all $i \geq 0$ in the following sense.

Definition.

- A Λ-submodule L of M is small if for every submodule X of M the equality $L + M = M$ implies $X = M$.
- A Λ-epimorphism $h : M \to N$ in $\operatorname{mod}\Lambda$ is minimal if $\ker h$ is small in M. An epimorphism $h : P \to M$ in $\operatorname{mod}\Lambda$ is called a projective cover of M if P is a projective module and h is a minimal epimorphism.

Lemma (characterisation of projective covers). *An epimorphism $h : P \to M$ is a projective cover of a Λ-module M, if and only if P is projective, and for any Λ-homomorphism $g : N \to P$, the surjectivity of hg, implies the surjectivity of g.*

Proof. Suppose that $h : P \to M$ is a projective cover of M, and let $g : N \to P$ be a homomorphism such that hg is surjective. It follows that $\operatorname{Im} g + \ker h = P$, and thus g is surjective, since by assumption $\ker h$ is small in P. Conversely assume that $h : P \to M$ has the stated property. Let N be a submodule of P such that $N + \ker h = P$. If $g : N \hookrightarrow P$ is the natural inclusion, then $hg : N \to M$ is surjective. So by the claim g is surjective. Thus $\ker h$ is small and finishes the proof.

Definition.

a) An exact sequence

$$P_1 \xrightarrow{h'_1} P_0 \xrightarrow{h'_0} M \longrightarrow 0$$

in $\operatorname{mod}\Lambda$ is called a minimal projective presentation of a Λ-module M if the Λ-module homomorphisms $h'_0 : P_0 \to M$ and $h'_1 : P_1 \to \ker P_0$ are projective covers.

b) An exact sequence (1.3) in $\operatorname{mod}\Lambda$ is called a minimal projective resolution of M if $h_i : P_i \to \operatorname{Im} h_i$ is a projective cover for all $i \geq 1$ and $h_0 : P_0 \to M$ is a projective cover.

From (1.1.16) and a consequence of (1.1.15) we recall that a module P is projective if and only if it is a direct summand of a free module.

Corollary 1.1.17. *Assume $\Lambda_\Lambda = e_1\Lambda \oplus ... \oplus e_s\Lambda$ is a decomposition with respect to a complete set of primitive orthogonal idempotents. Then the indecomposable projective modules are precisely the modules $P(i) = e_i\Lambda$.*

Theorem 1.1.18. *Let Λ be a finite dimensional K-algebra and let $\Lambda_\Lambda = e_1\Lambda \oplus ... \oplus e_s\Lambda$, where $\{e_1, ..., e_s\}$ is a complete set of primitive orthogonal idempotents of Λ. For any Λ-module M in $\operatorname{mod}\Lambda$ there exist a projective cover,*

$$P(M) \xrightarrow{h} M \longrightarrow 0$$

where $P(M) \cong (e_1\Lambda)^{n_1} \oplus ... \oplus (e_s\Lambda)^{n_s}$ and $n_1 \geq 0, ..., n_s \geq 0$. The homomorphism h induces an isomorphism $P(M)/\operatorname{rad} P(M) \cong M/\operatorname{rad} M$.

Proof. We set $B = \Lambda/\operatorname{rad}\Lambda$, $\bar{e_i} = e_i + \operatorname{rad}\Lambda \in B$ and let $\pi : \Lambda \to B$ be the residual class K-algebra epimomorphism. Because $\{e_1, ..., e_s\}$ is a complete set of primitive orthogonal idempotents of Λ, $\{\bar{e_1}, ..., \bar{e_s}\}$ is a complete set of primitive orthogonal idempotents of B and $B_B = \bar{e_1}B \oplus ... \oplus \bar{e_s}B$ is an indecomposable decomposition. Then we have by (1.1.11c) that $\operatorname{rad} e_i\Lambda \subset e_i\Lambda$ is the unique maximal Λ-submodule of $e_i\Lambda$, and $\operatorname{top} e_i\Lambda \cong \bar{e_i}B$ is a simple B module and the epimorphism $\pi_i : e_i\Lambda \to \operatorname{top} e_i\Lambda$ induced by π is a projective cover of $\operatorname{top} e_i\Lambda$. Let M be a module in $\operatorname{mod}\Lambda$. Then $\operatorname{top} M = M/\operatorname{rad} M$ is a module in $\operatorname{mod} B$ and by (1.1.5) and (1.1.11) there exist B-module isomorphisms.

$$\operatorname{top} M \cong (\bar{e_1}B)^{n_1} \oplus ... \oplus (\bar{e_s}B)^{n_s} \cong (\operatorname{top} e_1\Lambda)^{n_1} \oplus ... \oplus (\operatorname{top} e_s\Lambda)^{n_s},$$

for some $n_1 \geq 0, ..., n_s \geq 0$. We set $P(M) = (e_1\Lambda)^{n_1} \oplus ... \oplus (e_s\Lambda)^{n_s}$. By the projectivity of the module $P(M)$, there exist a Λ-module homomorphism $h : P(M) \to M$ making the following diagram commute

$$\begin{array}{ccc} P(M) & \xrightarrow{\;h\;} & M \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t'} \\ \operatorname{top} P(M) & \xrightarrow{\operatorname{top} h} & \operatorname{top} M. \end{array}$$

Where t and t' are canonical epimorphisms. It follows that $\operatorname{top} h$ is an isomorphism and from (1.1.8c) we infer that h is an epimorphism. Furthermore, the commutativity of the diagram yields $\ker h \subseteq \ker t = (\operatorname{rad} e_1\Lambda)^{n_1} \oplus ... \oplus (\operatorname{rad} e_s\Lambda)^{n_s} = \operatorname{rad} P(M)$. Because according (1.1.6e), the module $\operatorname{rad} P(M)$ is small in $P(M)$, $\ker h$ is also small in $P(M)$. Therefore the epimorphism h is a projective cover of M. Summarising, for any module M in $\operatorname{mod}\Lambda$ there exists a projective cover $P(M)$ and $P(M)/\operatorname{rad} P(M) \cong M/\operatorname{rad} M$.
The next step is to show that the projective cover is unique, thus if $P' \xrightarrow{h'} M \longrightarrow 0$ is a projective cover, then $P' \cong P(M)$. The projectivity of P' gives us a morphism $g : P' \to P(M)$ such that $hg = h'$. Since h' is surjective, $\operatorname{Im} g + \ker h = P(M)$. Since $\ker h = \operatorname{rad} M$, this implies the surgectivity of g. Therefore, $l(P') \geq l(P(M))$. Reversing the situation, we get $l(P(M)) \geq l(P')$, hence an equality. Thus, $P' \cong P(M)$.

Summarising:

Proposition 1.1.19. *Any module M in $\operatorname{mod}\Lambda$ has a unique projective cover $P(M)$ satisfying $P(M)/\operatorname{rad} P(M) \cong M/\operatorname{rad} M$.*

Corollary 1.1.20. *If P is a projective module in $\operatorname{mod}\Lambda$, then $P \to \operatorname{top} P$ is a projective cover. In particular, $e_i\Lambda \to \operatorname{top} e_i\Lambda$ is a projective cover for any primitive idempotent e_i of Λ. By the uniqueness of projective covers, $e_i\Lambda \cong e_j\lambda$ if and only if $\operatorname{top} e_i\Lambda \cong \operatorname{top} e_j\Lambda$.*

Corollary 1.1.21. *The simple modules in $\operatorname{mod}\Lambda$ are precisely the modules $S(i) = \operatorname{top} e_i\Lambda = \operatorname{top}(P(i))$.*

Proof. Given a simple module S. It has a projective cover $P(S)$ which is a direct sum of copies of the $P(i)$. Since $P(S)/\operatorname{rad} P(S) \cong S$, the left hand side is a direct sum of the $S(i)$. But S is simple so the claim follows.

Definition. Let Λ be an algebra with a complete set of primitive idempotents $\{e_1, ..., e_s\}$. The algebra is called basic if $e_i\Lambda$ is not isomorphic to $e_j\Lambda$ for all $i \neq j$.

Clearly, a local algebra is basic. Basicness of an algebra Λ can be detected by the algebra $\Lambda/\operatorname{rad}\Lambda$:

Proposition 1.1.22. *A finite dimensional K algebra Λ is basic if and only if $B = \Lambda/\operatorname{rad}\Lambda \cong K \times ... \times K$.*

Proof. Let $\Lambda_\Lambda = \bigoplus_{i=1}^{s} e_i\Lambda$ for a complete set of primitive orthogonal idempotents and $B_B = \bigoplus_{i=1}^{s} \pi(e_i)B$ the correspondng decomposition. Since $e_i\Lambda \cong e_j\Lambda$ if and only if $\pi(e_i)B \cong \operatorname{top} e_i\Lambda \cong \operatorname{top} e_j\Lambda \cong \pi(e_j)B$, we conclude that B is basic if Λ is. Schur's lemma gives that $\operatorname{Hom}(\pi(e_i)B, \pi(e_j)B) = 0$ for $i \neq j$ and, since these modules are simple, $\operatorname{End}(\pi(e_i)B) \cong K$ for all i. Using this, we get

$$B \cong \operatorname{End}_B(B_B) \cong \bigoplus_{i=1}^{s} \operatorname{End}(\pi(e_i)B) \cong K \times ... \times K.$$

For the converse, assume that B is isomorphic to a product of s copies of K. Then B is a commutative algebra and admit s central primitive pairwise orthogonal idempotents $\bar{e_i}$. Hence, e_iB is $\ncong e_j\Lambda$ and therefore $P(e_iB) \cong e_i\Lambda \ncong P(e_jB) \cong e_j\Lambda$ for $i \neq j$.

Corollary 1.1.23. *Any simple module S over a basic algebra is one dimensional.*

Proof. First note that a simple module S' over any algebra Λ satisfies $S' \operatorname{rad}\Lambda = 0$ by (1.1.8), and consequently, S' is a simple module $\Lambda/\operatorname{rad}\Lambda$. Indeed, Nakayama's lemma gives that $S' \neq S' \operatorname{rad}\Lambda$, hence latter has to be zero, since S' is simple. Using this and (1.1.24), we see that S is a simple module over the algebra $\Lambda/\operatorname{rad}\Lambda \cong K \times ... \times K$ and the corollary follows.

Definition. Let Λ be an algebra with a complete set of primitive idempotents $\{e_1, ..., e_s\}$. A basic algebra associated to Λ is the algebra $\Lambda^b = e_\Lambda \Lambda e_\Lambda$, where $e_\Lambda = e_{j_1} + ... + e_{j_a}$ are chosen such that $e_{j_i} \ncong e_{j_t}$ for $i \neq t$ and each module $e_r\Lambda$ is isomorphic to one of the modules $e_{j_1}\Lambda ... e_{j_a}\Lambda$.

In other words, we consider all modules $e_k\Lambda$ and if $e_k \cong e_l\Lambda$, only e_k or e_l will be part of e_Λ. Hence prior, Λ^b is not unique, since it depends on which idempotents we keep.

Lemma 1.1.24. *Let Λ^b be a basic algebra associated to Λ. The element $e_\Lambda \in \Lambda^b$ is the identity of Λ^b and $\Lambda^b \cong \operatorname{End}(e_{j_1}\Lambda + ... + e_{j_a}\Lambda)$. Furthermore, the algebra Λ^b does not depend on the choice of the sets $(e_i)i$ and $e_{j_1}, ..., e_{j_a}$.*

Proof. To prove the first part, we apply (1.1.9) to the Λ-module $M = e_\Lambda\Lambda$, there is a K-algebra isomorphism $\operatorname{End} e_\Lambda\Lambda \cong e_\Lambda\lambda e_\lambda$. Because there exists a Λ module isomorphism $e_\Lambda\Lambda \cong e_{j_1}\Lambda + ... + e_{j_a}\Lambda$, we derive K-algebra isomorphisms,

$$\Lambda^b = e_\Lambda\Lambda e_\Lambda \cong \operatorname{Hom}_\Lambda(e_\Lambda\Lambda, e_\Lambda\Lambda) \cong \operatorname{End}(e_{j_1}\Lambda + ... + e_{j_a}\Lambda).$$

To see the second apply (1.1.9) to $e_\Lambda\Lambda$ and use that $e_\Lambda\Lambda \cong (e_{j_1}\Lambda + ... + e_{j_a}\Lambda$. then (1.1.15) tells as that $e_\Lambda\Lambda$ does not depend on the choice of the set $\{e_1, ..., e_s\}$ and $\{(e_{j_1}, ..., e_{j_a}\}$ up to isomorphism of Λ-modules. Then the second statement is a consequence of the K-algebra isomorphisms $\Lambda^b \cong e_\Lambda\Lambda e_\Lambda \cong \operatorname{End}(e_{j_1}\Lambda + ... + e_{j_a}\Lambda)$.

For an idempotent $e \in \Lambda$. Consider the algebra $B := e\Lambda e \cong \operatorname{End} e\Lambda$ with identity e. Given a Λ-module M, note that Me is a B-module. If $f : M \to M'$ is a homomorphism of Λ-modules, we get a homomorphism between the B-modules Me and $M'e$ by setting $me \mapsto f(m)e$. This defines a restriction functor

$$\operatorname{res}_e : \operatorname{mod}\Lambda \to \operatorname{mod} B.$$

We now define two functors from $\operatorname{mod} B$ to $\operatorname{mod}\Lambda$ as follows. We have seen before that $e\Lambda$ is a left $B = e\Lambda e$-module. It is, of course, also a right Λ-module. Therefore, we have the functor $T_e(-) := - \otimes_B e\Lambda$. On the other hand, Λe is a left Λ-module and a right $e\Lambda e$-module, hence we have the functor $L_e(-) := \operatorname{Hom}_B(\Lambda e, -)$.
Next we collect some properties of these functors.

Proposition 1.1.25. *Let Λ be an algebra, let e be an idempotent of Λ and $B = e\Lambda e$. Then the following holds:*

a) *T_e and L_e are fully faithful K-linear functors such that $\operatorname{res}_e T_e \cong id_{\operatorname{mod} B} \cong \operatorname{res}_e L_e$, the functor L_e is right adjoint to res_e and T_e is left adjoint to res_e.*

b) *T_e is right exact, L_e is left exact and res_e is exact.*

c) *T_e and L_e preserve indecomposability, T_e respect projectives and and L_e respect injectives.*

d) *A right Λ-module M is in the image of T_e if and only if there exist an exact sequence $P_1 \xrightarrow{h} P_0 \longrightarrow M \longrightarrow 0$, where P_1 and P_0 are direct sums of summands of $e\Lambda$.*

Proof. a) We recall from (1.1.9), that we have a functorial B-module isomorphism,

$$\operatorname{Hom}_\Lambda(e\Lambda, M) \cong Me$$

for any right Λ-module M. Using the adjointness properties of tensor and Hom functors we have, for a B-module N,

$$\begin{aligned}\mathrm{Hom}_\Lambda(T_e(N), M) &\cong \mathrm{Hom}_\Lambda(N \otimes_B e\Lambda, M)\\ &\cong \mathrm{Hom}_B(N, \mathrm{Hom}_\Lambda(e\Lambda, M)\\ &\cong \mathrm{Hom}_B(N, Me) \cong \mathrm{Hom}_B(N, \mathrm{res}_e(M)).\end{aligned}$$

Hence, T_e is left adjoint to res_e. We note also

$$\mathrm{res}_e\, T_e(N) = (N \otimes_B e\Lambda)e \cong N \otimes_B B \cong N,$$

and $\mathrm{res}_e\, L_e(N) \cong N$. consequently,

$$\begin{aligned}\mathrm{Hom}_B(N, N') &\cong \mathrm{Hom}_B(N, \mathrm{res}_e\, T_e(N'))\\ &\cong \mathrm{Hom}_\Lambda(T_e(N), T_e(N')).\end{aligned}$$

and $\mathrm{Hom}_B(N, N') \cong \mathrm{Hom}_\Lambda(L_e(N), L_e(N'))$

Hence T_e and L_e is fully faithful.

b) The exactness of the functor res_e is obvious. The functor T_e is right exact, because the tensor product functor is right exact. Since the functor $\mathrm{Hom}_\Lambda(M, -)$ is left exact, the functor L_e is left exact and (b) hold.

c) Since T_e and L_e are fully faithful, $\mathrm{End}(N) \cong \mathrm{End}(T_e(N)) \cong \mathrm{End}(L_e(N))$. So if N is indecomposable, then its endomorphism algebra is local, hence the same holds for $T_e(N)$ and $L_e(N)$ and these modules are indecomposable by (1.1.14).
Now consider a projective B-module P and an epimorphism $h : M \to M'$ in $\mathrm{mod}\,\Lambda$. We have the following commutative diagram

$$\begin{array}{ccc}\mathrm{Hom}_\Lambda(T_e(P), M) & \longrightarrow & \mathrm{Hom}_\Lambda(T_e(P), M')\\ \downarrow\cong & & \downarrow\cong\\ \mathrm{Hom}_B(P, \mathrm{res}_e(M)) & \longrightarrow & \mathrm{Hom}_B(P, \mathrm{res}_e(M')).\end{array}$$

Since P is projective, the lower map is an epimorphism, hence so is the upper map. Therefore, $T_e(P)$ is a projective Λ-module if P is a projective B-module. Dually, we can show the statement for L_e.

d) Assume that $e = (e_{j_1} + ... + e_{j_s}$ and $e_{j_1}, ..., e_{j_s}$ are primitive idempotents. This implies $B = e_{j_1}B \oplus ... \oplus e_{j_s}B$ and the modules $e_{j_1}B, ..., e_{j_s}B$ are indecomposable.
Consider the map

$$m_{j_i} : e_{j_i}B \otimes_B e\Lambda \to e_{j_i}\Lambda, e_{j_i}x \otimes ea \mapsto e_{j_i}xea.$$

Note that this map is the restriction of the Λ-module isomorphism $B \otimes_B e\Lambda \to e\Lambda$ to the direct summand $e_{j_i}B \otimes_B e\Lambda$, hence it is well defined homomorphism of Λ-modules and injective and $e_{j_i}\Lambda$ is the image of the restriction. Therefore m_{j_i} is an isomorphism. Now assume that $Q_1 \longrightarrow Q_0 \longrightarrow N \longrightarrow 0$ is an exact sequence in $\operatorname{mod} B$, Q_1, Q_0 are projective. Applying the right exact functor T_e to this sequence, we have:

$$T_e(Q_1) \longrightarrow T_e(Q_0) \longrightarrow T_e(N)$$

in $\operatorname{mod} B$ is exact and the modules $T_e(Q_i)$ are projectives satisfy the properties required in(d) because by (1.1.16), the modules Q_1 and Q_0 are direct sums of indecomposable modules isomorphic to some of the modules $e_{j_1}B, ..., e_{j_s}B$.
Conversely, assume a sequence as in (d) is given. Observe that P_1e and P_0e are projective B-modules, since res_e is exact. Applying T_e gives back P_1 and P_0. Denote by N the cokernel of the restriction $he : P_1e \to P_0e$ of h to $\operatorname{res}_e(P_1) = P_1e$, then we derive a commutative diagram

$$\begin{array}{ccccccc}
P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & & & \\
T_e(P_1e) & \longrightarrow & T_e(P_0e) & \longrightarrow & T_e(N) & \longrightarrow & 0
\end{array}$$

Hence, $M \cong T_e(N)$.

Theorem 1.1.26. *Let $\Lambda^b = e_\Lambda \Lambda e_\Lambda$ be a basic algebra associated with Λ. The algebra Λ^b is basic and the functor T_{e_Λ} gives an equivalence $\operatorname{mod}\Lambda^b \cong \operatorname{mod}\Lambda$, with quasi-inverse res_e.*

Proof. We know that $\Lambda^b = e_\Lambda\Lambda^b = e_{j_1}\Lambda^b \oplus ... \oplus e_{j_a}\Lambda^b$ and $e_{j_t}\Lambda^b e_{j_t} = e_{j_t}\Lambda e_{j_t}$ for all t. It follows from (1.1.13) that the algebra $\operatorname{End}(e_{j_t}\Lambda^b) \cong e_{j_t}\Lambda^b e_{j_t}$ is local because $e_{j_t}\Lambda$ is indecomposable in $\operatorname{mod}\Lambda$. Hence e_{j_t} is a primitive idempotent of Λ^b. Now assume that $e_{j_t}\Lambda^b \cong e_{j_r}\Lambda^b$. Using the isomorphisms m_{j_i} from (1.1.25), we have that

$$e_{j_t}\Lambda \cong e_{j_t}\Lambda^b \otimes_{\Lambda^b} e_\Lambda\Lambda \cong e_{j_r}\Lambda^b \otimes_{\Lambda^b} e_\Lambda\Lambda \cong e_{j_r}\Lambda,$$

therefore $t = r$ by the choice of $e_{j_1}, ..., e_{j_a}$. We know already that T_e is fully faithful. Now any module $M \in \operatorname{mod}\Lambda$ admit an exact sequence $P' \longrightarrow P \longrightarrow M \longrightarrow 0$, where P', P are projective. It remain to note that P' and P are direct sums of summands of $e_\Lambda\Lambda$. So it follows from (1.1.25d), T_e is essentially surjective and hence an equivalence.

Remark. The theorem tells us that if we are interested in finite dimensional modules, then we can restrict our attention to basic algebras.

1.2 QUIVERS, PATH ALGEBRAS AND THEIR QUOTIENT FORM

Definition. A quiver $Q = (Q_0, Q_1, s, t)$ is a quadruple consisting of two sets: Q_0 called the vertex set and Q_1 called the arrow set, and two maps $s, t : Q_1 \to Q_0$ which associate to each arrow $\alpha \in Q_1$ its source $s(\alpha) \in Q_0$ and its target $t(\alpha) \in Q_0$, respectively. A quiver is called finite if Q_0 and Q_1 are finite sets.
A subquiver of a quiver $Q = (Q_0, Q_1, s, t)$ is a quiver $Q' = (Q'_0, Q'_1, s', t')$ such that $Q'_0 \subseteq Q_0$, $Q'_1 \subseteq Q_1$ and s', t' are the restrictions of s, t to Q'_1. A subquiver is called full if if Q'_1 equals the set of all those arrows in Q_1 whose source and target both belong to Q'_0.
If x and y are elements in Q_0, a path from x to y of length l is a sequence of arrows $\alpha_1, ..., \alpha_l$ such that $s(\alpha_1) = x, t(\alpha_k) = s(\alpha_{k+1})$ for all $1 \leq k < l$ and $t(\alpha_l) = y$. We will write this as $\alpha_1...\alpha_l$.
A cycle is a path such that source and target coincide. A cycle is a loop if it is of length 1. A quiver is called acyclic if it contains no cycles.
For any vertex x we have the stationary path ϵ_x of length 0.

Definition. Let Q be a quiver. The path algebra KQ of Q is the K-algebra whose underlying K-vector space has its basis the set of all paths $(x|\alpha_1, ..., \alpha_l|y)$ of length $l \geq 0$ in Q and such that the product of two basis vectors $(x|\alpha_1, ..., \alpha_l|y)$ and $(u|\beta_1, ..., \beta_k|v)$ is defined by

$$(x|\alpha_1, ..., \alpha_l|y)(u|\beta_1, ..., \beta_k|v) = \delta_{yu}(x|\alpha_1, ..., \alpha_l, \beta_1, ..., \beta_k|v),$$

where δ_{yu} denotes the Kronecker delta. That is the product of two paths $\alpha_1...\alpha_l and \beta_1...\beta_k$ is zero if $t(\alpha_l) \neq s(\beta_1)$ and is equal to the composed path $\alpha_1...\alpha_l\beta_1...\beta_k$ if $t(\alpha_l) = s(\beta_1)$.
Note the product of basis elements can be extended to arbitrary elements of KQ by distributivity. Thus, there is a direct sum decomposition

$$KQ = KQ_0 \oplus KQ_1 \oplus KQ_2 \oplus ... \oplus Q_l \oplus ...$$

of the K-vector space KQ, where, for each $l \geq 0$, KQ_l is the subspace generated by all paths of length l. So then, we can see that $(KQ_n).(KQ_m) \subseteq KQ_{n+m}$ for all $n, m \geq 0$, since the product in KQ of path of length n by a path of length m is either zero or a path of length $n + m$. By this, is sometimes expressed by saying that KQ is an associative graded algebra.

Lemma 1.2.1. *Let Q be a quiver and KQ its path algebra.*

a) The algebra KQ has an identity element if and only if Q_0 is finite and

b) KQ is finite dimensional if and only if Q is finite and acyclic.

Proof. a) If Q_0 is finite, say $Q_0 = \{1, ..., n\}$, then it is easily checked that $\sum_{i=1}^{n} \epsilon_i$ is the identity of KQ. To see the converse, assume that Q_0 is infinite and let $1 = \sum_{i=1} \lambda_i w_i$, where $\lambda_i \in K$ and w_i are paths, be the identity element. The paths w_i have only finitely many sources, so let x be a vertex not in this set. Then $\epsilon_x 1 = 0$, a contradiction.

b) If Q is finite and acyclic, there are only finitely many paths, hence KQ is finite dimensional. To see the converse, if Q_0 is infinite, then so is KQ. If Q is not acyclic, then take a cycle w in Q. Considering all its powers gives that KQ is infinite dimensional.

Proposition 1.2.2. *Let Q ba a finite quiver. The set of all stationary paths $\epsilon_x, x \in Q_0$, is a complete set of primitive orthogonal idempotents of KQ.*

Proof. It is clear that the ϵ_x are orthogonal idempotents. To see that they are primitive, it is enough to show that the algebra $B = \epsilon_x KQ \epsilon_x$ is local, see (1.1.13). We note that this algebra is clearly K if Q has no cycles. In any case, an idempotent ϵ of B can be written as $\epsilon = \lambda\epsilon_x + w$, where $\lambda \in K$ and w is a linear combination of cycles through x of length at least 1. then we have:

$$0 = \epsilon^2 - \epsilon = (\lambda^2 - \lambda)\epsilon_x + (2\lambda - 1)w + w^2$$

which holds if $\lambda^2 = \lambda$ and $w = 0$, so $\lambda = 1$ or $\lambda = 0$. Hence $\epsilon = \epsilon_x$ or $\epsilon = 0$

Lemma 1.2.3. *Let Λ be an algebra and assume that that $\{e_1, ..., e_n\}$ is a complete set of primitive orthogonal idempotents. Then Λ is connected if and only if there does not exist a nontrivial partition $I \amalg J$ of the set $\{1, .., n\}$ such that for any $i \in I$ and $j \in J e_i \Lambda e_j = 0 = e_j \Lambda e_i$.*

Proof. Suppose that such a partition does not exist and let $z = \sum_{j=J} \epsilon_j$. By the assumption z is nontrivial. Moreover, it is an idempotent, $ze_i = e_i z = 0$ for each $i \in I$ and $ze_j = e_j z = e_j$ for $j \in J$. By our hypothesis, $e_i x e_j = 0 = e_j x e_i$ for any $a \in \Lambda$. Thus,

$$\begin{aligned} zx &= \sum_{j=J} \epsilon_j x = (\sum_{j=J} \epsilon_j x).1 \\ &= (\sum_{j=J} \epsilon_j x)(\sum_{i=I} \epsilon_i + \sum_{k=J} \epsilon_k) = \sum_{j,k} \epsilon_j x e_k \\ &= (\sum_{j} \epsilon_j + \sum_{i} \epsilon_i) x (\sum_{k=J} \epsilon_k) = xz. \end{aligned}$$

Hence, z is a nontrivial central idempotent and so Λ is not connected. To see the converse, if Λ is not connected, there exist a central nontrivial idempotent z. Because z is central, we have $z = \sum_{i=I}^{n} \epsilon_j z \epsilon_i$. Let $c_i = e_i z e_i$. Then $z_i^2 = z_i$, so $z_i \in e_i \Lambda e_i$ is an idempotent. Because e_i is primitive, $z_i = 0$ or $z_i = e_i$. We set $I = \{i | z_i = 0\}$ and $J = \{j | z_i = e_i\}$. This obviously is a partition of $\{1, ..., n\}$ and because $ze_j = e_j = e_j z$ and $ze_i = 0 = e_i z$, gives us $e_i \Lambda e_j = 0 = e_j \Lambda e_i$.

By this lemma we can now prove (1.2.4)

Lemma 1.2.4. *Let Q be a finite quiver. The path algebra KQ is connected if and only if Q is a connected quiver, which, by definition, means that the graph obtained by forgetting the orientation of the arrows is connected.*

Proof. If Q is not connected, let Q' be a connected component and let Q'' be the full subquiver of Q having as vertices $Q_0 \backslash Q_0'$. Let $x \in Q_0'$ and $y \in Q_0''$. Any path in Q is either contained in

Q_0' or Q_0''. Hence, either, $w_{\epsilon_y} = 0$ or $e_{\epsilon_x} w = 0$. In any case, $\epsilon_x w \epsilon_y = 0$. By (1.2.3) KQ is not connected. To see the converse, let Q be connected, but not in KQ. That is we have a partition $Q_0 = Q_0' \amalg Q_0''$ as in (1.2.3). Because Q is connected, there exist $x \in Q_0'$ and $y \in Q_0''$ with an arrow α from x to y. Then $\alpha = \epsilon_x \alpha \epsilon_y = 0$, a contradiction.

Next we record the following obvious properties.

Proposition 1.2.5. *Let Q be a finite connected quiver and Λ an associative algebra with identity. For any pair of maps $g_0 : Q_0 \to \Lambda$ and $g_1 : Q_1 \to \Lambda$ satifying (a)$\sum_{x \in Q_0} g_0(x) = 1$,(b) $g_0(x)^2 = g_0(x)$, (c) $g_0(x) \neq g_0(y)$ for $x \neq y$ and (d) if $\alpha : x \to y$, then $g_1(x) = g_0(x) g_1(\alpha) g_0(y)$, there exist a unique K-algebra homomorphism $g : KQ \to \Lambda$ such that $g(\epsilon_x) = g_0(x)$ for any $x \in Q_0$ and $g(\alpha) = g_1(\alpha)$ for any $\alpha \in Q_1$.*

Definition. Let Q be a finite and connected quiver. The two-sided ideal of KQ generated by the arrows of Q is called the arrow ideal and denote by R_Q.

Clearly, $R_Q = KQ_1 \oplus KQ_2 \oplus ...$ as a K-vector space. This implies that $R_Q^l = \otimes_{m \geq l} KQ_m$.

Proposition 1.2.6. *Let Q be a finite connected quiver, R_Q the arrow ideal of KQ and ϵ_x the stationary paths associated to the vertices of Q. Consider the canonical algebra homomorphism $\pi : KQ \to KQ/R_Q$ and the set of the images $\epsilon_x := \pi(\epsilon_x)$. Then this is a complete set of primitive orthogonal idempotents for KQ/R_Q and the latter algebra is isomorphic to $K \times ... \times K$. If Q is acyclic, then* $\operatorname{rad} KQ = R_Q$ *and KQ is a finite dimensional basic algebra.*

Proof. As a K-vector space we have

$$KQ/R_Q = \bigoplus_{x,y \in Q_0} e_x (KQ/R_Q) e_y = \bigoplus_{x \in Q_0} e_x (KQ/R_Q) e_x,$$

where the second equality stems from the fact that R_Q contains all paths of length at least 1. Hence, KQ/R_Q is a Q_0-dimensional vector space. The elements e_x give a complete set of primitive orthogonal dempotents of KQ/R_Q and every piece $e_x(KQ/R_Q)e_x$ is isomorphic to K. Thus, the first statement holds.
Assume Q is acyclic, then KQ is finite dimensional and the length of paths in Q is bounded by some integer l. Hence, $R_Q^{l+1} = 0$, so by (1.1.2) $R_Q \subseteq \operatorname{rad} KQ$. Because $KQ/R_Q \cong K \times ... \times K$, (1.1.2) gives that $R_Q = \operatorname{rad} KQ$ and it follows from (1.1.22) that KQ is basic.

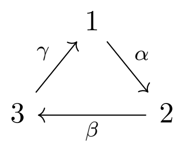
Remark. Assume Q is not acyclic, then $\operatorname{rad} KQ$ need not be equal to R_Q. For example consider the quiver with one vertex and one loop. Then the radical is trivial, but R_Q is not.

Definition 1.2.7. Let Q be a finite quiver and R_Q be the arrow ideal of the path algebra KQ. A two-sided ideal I of KQ is called admissible if there exist an $m \geq 2$ such that $R_Q^m \subseteq I \subseteq R_Q^2$. If I is an admissible ideal of KQ, we call the pair (KQ, I) a bound quiver. The quotient algebra KQ/I is said to be a bound quiver algebra.

It is clear from the definition that an I ideal $I \subseteq R_Q^2$ is admissible if and only if it contains all paths whose length is large enough. Infact. this is the case if and only if for each cycle σ there exists an $s \geq 1$ such that $\sigma^s \in I$. In particular, assuming Q is acyclic, any ideal $I \subseteq R_Q^2$ is admissible.

Definition. Let Q be a quiver. A relation ρ in Q with coefficients in K is a K-linear combination of paths w_i of length at least two having the same source and target. We write, $\rho \sum_{i=1}^{n} \lambda_i w_i$. If $(\rho_j)_{j\in J}$ is a set of relations such that the ideal they generate is admissible, then we say the quiver Q is bound by the relations $\rho_j = 0$ for all $j \in J$.

Example. Let Q be the quiver



and I an ideal of KQ generated by $\alpha\beta$. Then I is an admissible ideal of KQ containing R_Q^4, and the associated bound quiver algebra KQ/I is a 9-dimensional K-algebra. Thus KQ/I has K basis $\{e_1 + I, e_2 + I, e_3 + I, \alpha + I, \beta + I, \gamma + I, \beta\alpha + I, \gamma\alpha + I,\ \beta\gamma\alpha + I\}$.

Lemma 1.2.8. *Let Q be a finite quiver and I be an admissible ideal of KQ. The set $e_x = \pi(\epsilon_x)$, where $\pi : KQ \to KQ/I$, is a complete set of primitive orthogonal idempotents of KQ/I.*

Proof. It is clear that the given set is a complete set of orthogonal idempotents under the canonical map,π. It therefore remains to check that each e_x is primitive, or equivalently, that the algebra $e_x(KQ/I)e_x$ has only the trivial idempotents 0 and 1 for any $x \in Q_0$. Note any idempotent e in the algebra $e_x(KQ/I)e_x$ can be written in the form $e = \lambda\epsilon_x + w + I$, where w is a linear combination of cycles through x of length ≥ 1 and $\lambda \in K$. Because $e^2 = e$, we get

$$(\lambda^2 - \lambda)\epsilon_x + (2\lambda - 1)w + w^2 \in I.$$

Because $I \subseteq R_Q^2, \lambda^2 - \lambda = 0$, hence $\lambda = 1$ or $\lambda = 0$. Let consider, $\lambda = 0$, then $e = w + I$, so w is idempotent modulo I. Since $R_Q^m \subseteq I$ for some $m \geq 2$, $w^m \in I$, so $w \in I$ and hence $e = 0$. If $\lambda = 1$, then $e_x - e = -w + I$ is an idempotent in $e_x(KQ/I)e_x$, so w is idempotent modulo I, that is nilpotent as before, so is an element in I. Thus $e_x = e$.

Lemma 1.2.9. *Let Q be a finite quiver and I be an admissible ideal of KQ. The bound quiver algebra KQ/I is connected if and only if Q is connected quiver.*

Proof. Let assume Q is not connected, then neither is KQ, so there exists a central nontrivial idempotent γ which is a sum of paths of length 0. Then its image is central nontrivial idempotent in KQ/I, since if $\pi(\gamma) = 1$, then $1 - a \in I$, which is impossible, since $I \supseteq R_Q^2$.
The reverse implication is proved in (1.2.4)

Proposition 1.2.10. *Let Q be a finite quiver and I an admissible ideal. Then KQ/I is a finite dimensional algebra*

Proof. We have a surjective homomorphism $KQ/R_Q^m \to KQ/I$. The former algebra is finite dimensional, because the finitely many paths of length at most m form a basis of KQ/R_Q^m as a K-vector space

Lemma 1.2.11. *Let Q be a finite quiver. Every admissible ideal I of KQ is finitely generated.*

Proof. Let consider the short exact sequence of KQ-modules

$$0 \longrightarrow R_Q^m \longrightarrow I \longrightarrow I/R_Q^m \longrightarrow 0$$

Clearly, R_Q^m is the KQ-module generated by the paths of length m, and since there are only finitely many such paths, R_Q^m is finitely generated. On the other hand, I/R_Q^m is an ideal of the finite dimensional algebra KQ/R_Q^m (see(1.2.9)). Therefore I/R_Q^m is a finite dimensional K-vector space, hence a finitely generated KQ-module.

Corollary 1.2.12. *If I is an admissible ideal of a finite quiver Q, then it is generated by a finite set of relations.*

Proof. We know that I has a finite generating set $\{\sigma_1, ..., \sigma_n\}$ by (1.2.10), but the σ_i need not have the same source and target. Also for any i such that $1 \le i \le n$ $x, y \in Q_0$, the term $\epsilon_x \sigma_i \epsilon_y$ is either zero or a relation. Since $\sigma_i = \sum_{x,y\in Q_0} \epsilon_x \sigma_i \epsilon_y$, for $i \le n$, the nonzero elements among the set $\{\epsilon_x \sigma_i \epsilon_y | 1 \le i \le n; x, y \in Q_0\}$ gives a finite set of relations generating I.

Lemma 1.2.13. *Let Q be a finite quiver and I an admissible ideal of KQ. Then $R_Q/I = \operatorname{rad}(KQ/I)$. Moreover, the algebra KQ/I is basic.*

Proof. We know that $R_Q^m \subseteq I$ for some $m \ge 2$. Consequently, $(R_Q/I)^m = 0$ and $R_Q/I \subseteq \operatorname{rad}(KQ/I)$. Because $(KQ/I)/(R_Q/I) \cong KQ/R_Q \cong K \times ... \times K$, the claim follow by (1.1.2) and (1.1.22).

Remark 1.2.14. For each $l \ge 1$ we have $\operatorname{rad}^l(KQ/I) = (R_Q/I)^l$. So by this and (1.2.12) we have,

$$\operatorname{rad}(KQ/I)/\operatorname{rad}^2(KQ/I) = (R_Q/I)/(R_Q/I)^2 = R_Q/R_Q^2.$$

Next we show that any basic and connected finite dimensional algebra can be described as the bound quiver algebra of a finite connected quiver.

Definition. Let Λ be a basic and connected finite dimensional algebra and $\{e_1, ..., e_n\}$ be a complete set of primitive orthogonal idempotents. The (ordinary) quiver of Λ, denoted by Q_Λ, is defined as follows:

- The vertices of Q_Λ are the numbers $\{1, ..., n\}$.
- Given two points $x, y \in (Q_\Lambda)_0$ the arrows $\alpha : x \to y$ are in bijective correspondence with the vectors in a basis of $e_x(\operatorname{rad}\Lambda/\operatorname{rad}^2\Lambda)e_y$.

Note the Q_Λ is finite, because Λ is finite dimensional and therefore the vector spaces $e_x(\operatorname{rad}\Lambda/\operatorname{rad}^2\Lambda)e_y$ are also finite dimensional.

Lemma 1.2.15. *Let Λ be as in the definition. Then*

a) *The quiver Q_Λ does not depend on the choice of a complete set of primitive orthogonal idempotents of Λ.*

b) *For any pair e_x, e_y of primitive orthogonal idempotents of Λ the K-linear map*

$$\psi : e_x(\operatorname{rad}\Lambda)e_y/e_x(\operatorname{rad}^2\Lambda)e_y \to e_x(\operatorname{rad}\Lambda/\operatorname{rad}^2\Lambda)e_y$$

defined by

$$e_x a e_y + e_x \operatorname{rad}^2\Lambda e_y \mapsto e_x(a + \operatorname{rad}^2\Lambda)e_y$$

is an isomorphism for all $a \in$ rad.

Proof. a) By (1.1.15), the number of points of Q_Λ is uniquely determined, since it equals the number of indecomposable direct summands of Λ_Λ. The same (1.1.15) also gives that for distinct complete sets of primitive orthogonal idempotents, say e_x and e'_x, there is a bijection $e_x\Lambda \cong e'_x\Lambda$ for all x. Define a Λ-module homomorphism $\phi : e_x(\operatorname{rad}\Lambda/\operatorname{rad}^2\Lambda)$ by $e_x a \mapsto e_x(a + \operatorname{rad}^2\Lambda)$ admit $e_x(\operatorname{rad}^2\Lambda)$ as a kernel. Hence

$$e_x(\operatorname{rad}\Lambda/\operatorname{rad}^2\Lambda) \cong e_x(\operatorname{rad}\Lambda)/e_x(\operatorname{rad}^2\Lambda) \cong \operatorname{rad}(e_x\Lambda)/\operatorname{rad}^2(e_x\Lambda).$$

Thus we have sequence of K-vector isomorphisms

$$\begin{aligned} e_x(\operatorname{rad}\Lambda/\operatorname{rad}^2\Lambda)e_y &\cong (\operatorname{rad}(e_x\Lambda)/\operatorname{rad}^2(e_x\Lambda))e_y \\ &\cong \operatorname{Hom}_\Lambda(e_y\Lambda, \operatorname{rad}(e_x\Lambda)/\operatorname{rad}^2(e_x\Lambda)) \\ &\cong \operatorname{Hom}_\Lambda(e'_y\Lambda, \operatorname{rad}(e'_x\Lambda)/\operatorname{rad}^2(e'_x\Lambda)) \\ &\cong (\operatorname{rad}(e'_x\Lambda)/\operatorname{rad}^2(e'_x\Lambda))e'_y \\ &\cong e'_x(\operatorname{rad}\Lambda/\operatorname{rad}^2\Lambda)e'_y. \end{aligned}$$

b) It is clear that the K-linear map $e_x(\operatorname{rad}\Lambda)e_y \to e_x(\operatorname{rad}\Lambda/\operatorname{rad}^2\Lambda)e_y$ defined by $e_x a e_y \mapsto e_x(a+\operatorname{rad}^2\Lambda)e_y$ admits $e_x(\operatorname{rad}^2\Lambda)e_y$ as a kernel. Therefore we conclude that the ψ defined in the second statement is an isomorphism.

Lemma 1.2.16. *For each arrow $\alpha : i \to j \in (Q_\Lambda)_1$, let $a_\alpha \in (\operatorname{rad}\Lambda)e_j$ be such that the set $\{a_\alpha + \operatorname{rad}^2\Lambda | \alpha : i \to j\}$ is a basis of $e_i(\operatorname{rad}\Lambda/\operatorname{rad}^2\Lambda)e_j$. Then*

a) *for any two points $x, y \in (Q_\Lambda)_0$, every element $a \in e_x(\operatorname{rad}\Lambda)e_y$ can be written in the the form $a = \sum a_{\alpha_1}a_{\alpha_2}...a_{\alpha_l}\lambda_{\alpha_1...\alpha_l} \in K$ and the sum is taken over all the paths $\alpha_1\alpha_2...\alpha_l \in Q_\Lambda$ from x to y.*

b) *for each arrow* $\alpha : i \to j$, *the element* a_α *uniquely determines a nonzero nonisomorphism* $\tilde{a_\alpha} \in \operatorname{Hom}_\Lambda(e_j\Lambda, e_i\Lambda)$ *such that* $\tilde{a_\alpha}(e_j) = a_\alpha$, $\operatorname{Im}\tilde{a_\alpha} \subseteq e_i(\operatorname{rad}\Lambda)$ *and* $\operatorname{Im}\tilde{a_\alpha} \nsubseteq e_i(\operatorname{rad}^2\Lambda)$.

Proof. We recall that $\operatorname{rad}\Lambda$ is nilpotent and, as a K-vector space, $\operatorname{rad}\Lambda \cong (\operatorname{rad}\Lambda/\operatorname{rad}^2\Lambda) \oplus \operatorname{rad}^2\Lambda$, we have $e_x(\operatorname{rad}\Lambda)e_y \cong e_x(\operatorname{rad}\Lambda/\operatorname{rad}^2\Lambda)e_y \oplus e_x(\operatorname{rad}^2\Lambda)e_y$. Thus we can be write

$$a' = a - \sum_{\alpha : x \to y} a_\alpha \lambda_\alpha \in e_x(\operatorname{rad}^2\Lambda)e_y,$$

for $\lambda_\alpha \in K$. By the decomposition of $\operatorname{rad}\Lambda = \otimes_{i,j} e_i(\operatorname{rad}\Lambda)e_j$ we have

$$e_x(\operatorname{rad}^2\Lambda)e_y = \sum_{z \in (Q_\Lambda)_0} (e_x(\operatorname{rad}\Lambda)e_z)(e_z(\operatorname{rad}\Lambda)e_y),$$

so that $a' = \sum_{z \in (Q_\Lambda)_0} a'_z b'_z$, where $a'_z \in e_x(\operatorname{rad}\Lambda)e_z$ and $b'_z \in e_z(\operatorname{rad}\Lambda)e_y$. We now apply the previous consideration to a'_z and b'_z and get

$$a = \sum_{\alpha : x \to y} a_\alpha \lambda_\alpha + \sum_{\beta : x \to z} \sum_{\alpha : z \to y} a_\beta a_\alpha \lambda_\beta \lambda_\alpha$$

modulo $e_x(\operatorname{rad}^3\Lambda)e_y$. We complete the proof by an obvious induction using the nilpotency of $\operatorname{rad}\Lambda$.

By our assumption, the element $a_\alpha \in e_i(\operatorname{rad}\Lambda)e_j$ is nonzero and maps to a nonzero element $\tilde{a_\alpha}$ by the K-linear isomorphism $e_i(\operatorname{rad}\Lambda)e_j \cong \operatorname{Hom}_\Lambda(e_j\Lambda, e_i(\operatorname{rad}\Lambda))$(see equation 1.2). It follows that $\tilde{a_\alpha}(e_j) = a_\alpha, \operatorname{Im}\tilde{a_\alpha} \subseteq e_i(\operatorname{rad}\Lambda)$, and $\operatorname{Im}\tilde{a_\alpha} \nsubseteq e_i(\operatorname{rad}^2\Lambda)$. Hence proof!

Corollary 1.2.17. *If* Λ *is a basic connected algebra, then* Q_Λ *is connected*

Proof. Assume that this is not the case, then the set $(Q_\Lambda)_0$ of points of Q_Λ can be written as the disjoint union of two nonempty sets Q'_0 and Q''_0 such that the points of Q'_0 are not connected to those of Q''_0. We will show that for $i \in Q'_0$ and $j \in Q''_0$ we have $e_i\Lambda e_j = 0 = e_j\Lambda e_i$, which means that Λ is not connected, a contradiction. We have already that $M \operatorname{rad}\Lambda = \operatorname{rad} M$ for any right module, so the $\operatorname{rad}(e_i\Lambda) = e_i \operatorname{rad}\Lambda$. Moreover, $e_i\Lambda e_j \cong \operatorname{Hom}(e_j\Lambda, e_i\Lambda)$ and $\operatorname{Hom}(e_j\Lambda, \operatorname{rad} e_i\Lambda) \cong e_i(\operatorname{rad}\Lambda)e_j$. The latter space is zero by our assumption and by (1.2.14). Hence, we are done if we can show that $\operatorname{Hom}(e_j\Lambda, e_i\Lambda) \cong \operatorname{Hom}(e_j\Lambda, \operatorname{rad} e_i\Lambda)$.
We racall that, given an idempotent $e \in \Lambda, rad(e\Lambda)$ is the unique maximal submodule of $e\Lambda$ by (1.1.11). This implies that $e\Lambda/\operatorname{rad}(e\Lambda) \cong e\Lambda/e\operatorname{rad}\Lambda$ is simple. Let now take a map $\psi : e_j \to e_i\Lambda$. If it is not surjective, we are done, because the image has to be $\operatorname{rad} e_i\Lambda$. If ψ is surjective, then $e_j\Lambda/\ker\psi \cong e_i\Lambda$. Since $\ker\psi \subset \operatorname{rad}(e_j\Lambda)$, this gives a map $e_j\Lambda \to S(i) = e_i\Lambda/\operatorname{rad}(e_i\Lambda)$ which is surjective. factoring out its kernel, we get a nontrivial map $S(j) \to S(i)$, a contradiction by Schur's lemma, because $S(j)$ cannot be to $S(i)$ by the assumption that Λ is basic and (1.1.20).

Lemma 1.2.18. *Let* Q *be a finite connected quiver,* I *be an admissible ideal and* $\Lambda = KQ/I$. *Then* $Q_\Lambda = Q$.

Proof. By (1.2.7), the set $\{e_x = \epsilon_x + I\}$ is a complete set of primitive orthogonal idempotents of $\Lambda = KQ/I$. So the points of Q_Λ are in bijective correspondence with those of Q. On the other hand, by (1.2.13), the arrows from x to y in Q are in bijective correspondence with the vectors in a basis of $e_x(\operatorname{rad}\Lambda/\operatorname{rad}^2\Lambda)e_y$, that is, with the arrows from x to y in Q_Λ.

Theorem 1.2.19. *Let Λ be basic and connected finite dimensional K-algebra. There exist an admissible ideal I of KQ_Λ such that $\Lambda \cong KQ_\Lambda/I$.*

Proof. First we construct an algebra homomorphism $\phi : KQ_\Lambda \to \Lambda$, and show that ϕ is surjective and its kernel $I = \ker\phi$ is an admissible ideal of KQ_Λ
Let $\alpha : i \to j \in (Q_\Lambda)_1$, and choose $a_\alpha \in \operatorname{rad}$ such that $\{a_\alpha + \operatorname{rad}^2\Lambda | \alpha : i \to j\}$ forms basis in $e_i(\operatorname{rad}\Lambda/\operatorname{rad}^2\Lambda)e_j$. Consider the following maps:

$$\phi_0 : (Q_\Lambda)_0 \to A; x \mapsto e_x,$$
$$\phi_1 : (Q_\Lambda)_1 \to \Lambda; \alpha \mapsto a_\alpha.$$

By (1.2.5) we get an algebra homomorphism $\phi : KQ_\Lambda \to \Lambda$. It remains to check that ϕ is surjective ant it kernel is an admissible ideal of KQ_Λ. The Wedderburn-Malcev theorem tells us that $\Lambda \cong \Lambda/\operatorname{rad}\Lambda \oplus \operatorname{rad}\Lambda$. The former space is generated by the e_x, while any element of rad is in the image by (1.2.15). So, ϕ is surjective. By definition, $\phi(R_Q) \subseteq \operatorname{rad}\Lambda$, hence $\phi(R_Q^l) \subseteq \operatorname{rad}^l\Lambda$ for any $l \geq 1$. Because $\operatorname{rad}\Lambda$ is nilpotent, there exists an $n \geq 1$ such that $R_Q^n \subseteq \ker\phi = I$. It remains to check that $I \subseteq R_Q^2$. Any $a \in I$ can be written as

$$a = \sum_{x\in(Q_\Lambda)_0} \epsilon_x\lambda_x + \sum_{\alpha\in(Q_\Lambda)_1} \alpha\mu + b,$$

Where $\lambda_x, \mu_\alpha \in K$ and $b \in R_Q^2$. If $\phi(a) = 0$, we have

$$\sum_{x\in(Q_\Lambda)_0} \epsilon_x\lambda_x = -\sum_{\alpha\in(Q_\Lambda)_1} x_\alpha\mu_\alpha - \phi(b) \in \operatorname{rad}\Lambda.$$

Because $\operatorname{rad}\Lambda$ is nilpotent, and e_x are orthogonal idempotents, we infer that $\lambda_x = 0$, for any $x \in (Q_\Lambda)_0$. Similarly $\sum_{\alpha\in(Q_\Lambda)_1} a_\alpha\mu_\alpha = -\phi(b) \in \operatorname{rad}^2\Lambda$, so the equality $\sum_{\alpha\in(Q_\Lambda)_1}(a_\alpha + \operatorname{rad}^2\Lambda)\mu_\alpha = 0$ holds in $\operatorname{rad}\Lambda/\operatorname{rad}^2\Lambda$. But the set $\{a_\alpha + \operatorname{rad}^2\Lambda|\alpha \in (Q_\Lambda)_1\}$ is, by construction, a basis of $\operatorname{rad}\Lambda/\operatorname{rad}^2\Lambda$. So all the μ_α have to be zero, hence $a = b \in R_Q^2$.

Remark 1.2.20. We say two algebras Λ and Λ' are Morita equivalent if $\operatorname{mod}\Lambda \cong \operatorname{mod}\Lambda'$. Since any algebra Λ is Morita equivalent to a basic algebra by (1.1.26). And (1.2.18) implies, in particular, that any connected algebra is Morita equivalent to a bound quiver algebra.

1.3 REPRESENTATIONS OF BOUND QUIVERS

Definition. Let Q be a finite quiver. A K-linear representation M of Q comprises the following data. For each point $x \in Q_0$ a vector space M_x and for every arrow $\alpha : x \to y$ in Q_1 a K-linear map $\phi_\alpha : M_x \to M_y$. A representaion is called finite dimensional if every M_x is a finite dimensional vector space.
A morphism between representations M and M' comprises linear maps $f_x : M_x \to M'_x$ for every $x \in Q_0$ such that the following diagram commute

$$\begin{array}{ccc} M_x & \xrightarrow{\phi_\alpha} & M_y \\ \downarrow{\scriptstyle f_x} & & \downarrow{\scriptstyle f_y} \\ M'_x & \xrightarrow{\phi'_\alpha} & M'_y \end{array}$$

for all x, y and α .

It is clear that maps of representations can be composed and that there exist identity maps, thus there is a category $\text{Rep}(Q)$ of representations of Q. We can define direct sums, kernels and images componentwise and it is easily checked that this makes $\text{Rep}(Q)$ into an abelian category. The full abelian subcategory of finite dimensional representations will be denoted by $\text{rep}(Q)$.

Example. Let Q be the quiver $1 \longrightarrow 2 \longrightarrow 3$. A representation of Q is, for example N=[$K \xrightarrow{id} K \xrightarrow{0} K$.] and another one is $N' =$ [$0 \xrightarrow{0} K \xrightarrow{0} 0$.]. It is easily checked that $\text{Hom}(N, N') = 0$, and $\text{Hom}(N', N) \cong K$.

Definition. If $w = \alpha_1\alpha_2...\alpha_l$ is a nontrivial path from x to y in a finite quiver Q, the evaluation of w is the K-linear map from M_x to M_y defined by

$$\phi_w = \phi_{\alpha_l}\phi_{\alpha_{l-1}}...\phi_{\alpha_2}\phi_{\alpha_1}.$$

This extends to K-linear combinations of paths with the same source and target. If I is an admissible of KQ, a representation M of Q is said to satisfy the relations in I or to be bounded by I if $\phi_\rho = \sum_{i=1}^{n} \lambda_i \phi_{w_i} = 0$ for all relations $\rho = \sum_{i=1}^{n} \lambda_i w_i$ in I.

The full subcategory of $\text{Rep}(Q)$ comprises representations satisfying the relations in I will be denoted by $\text{Rep}(Q, I)$, and similarly for $\text{rep}(Q)$.

Theorem 1.3.1. *Let Q be a finite connected quiver, I be an admissible ideal of KQ and $\Lambda = KQ/I$. There exist a K-linear equivalence*

$$F : \text{Mod}\,\Lambda \cong \text{Rep}(Q, I)$$

that restricts to an equivalence of categories $F : \text{mod}\,\Lambda \cong \text{rep}(Q, I)$.

Proof. We start with the construction of a functor $F : \operatorname{Mod}\Lambda \to \operatorname{rep}(Q, I)$. Let $M \in \operatorname{Mod}\Lambda$ and $x \in Q_0$. Set M_x to be Me_x, where e_x is the image of the stationary path ϵ_x under the canonical projection $KQ \to KQ/I$. Next, if $\alpha : x \to y$ is an arrow and $a \in M_x = Me_x$, let $\phi_\alpha(a) = a\bar{\alpha}$, where $\bar{\alpha}$ is the residual class of α modulo I. $\rho = \sum_{i=1} \lambda_i w_i$ is a relation in I, then $\phi_\rho(a) = \sum_{i=1} \lambda_i \phi_{w_i}(a) = a\bar{\rho} = 0$. Hence, $F(M)$ is indeed a representation bound by I.
Let $f : M \to M'$ be a homomorphism of Λ-modules. For any $x \in Q_0$ and $a = ae_x \in M_x$. Then

$$f(ae_x) = f(ae_x^2) = f(ae_x)e_x \in M'e_x = M'_x.$$

That is, we get a K-linear map $f_x : M_x \to M'_x$ for any $x \in Q_0$ which just a restriction of f. Given an arrow $\alpha : x \to y$ and $a \in M_x$, we now calculate

$$f_y\phi_\alpha(a) = f(a\bar{\alpha}) = f(a)\bar{\alpha} = \phi'_\alpha f_x(a).$$

It is obvious that F is a K-linear functor. Moreover, it restricts to a functor $\operatorname{mod}\Lambda \to \operatorname{rep}(Q, I)$.
We now define a functor $G : \operatorname{Rep}(Q, I) \to \operatorname{Mod}\Lambda$. Let M be a representation bound by I. We set $G(M) = \bigoplus_{x \in Q_0} M_x$ and define a Λ-module structure on $G(M)$ in two steps, first by specyfying a KQ-module structure and then show that it is annihilated by I. To define a KQ-module structure on $G(M)$, it suffices to define the product of the form aw, where w is a path in Q. If $w = \epsilon_x$ is the stationary path in x, we set $aw = x\epsilon_x = a_x$. If w is a nontrivial path from x to y, we define aw to be the component of $\phi_w(a)$ in M_y. This endows $G(M)$ with a KQ-module structure. If $\rho \in I$, by definition $a\rho = 0$, hence $G(M)$ is a Λ-module.
Next, given a morphism $(f_x)_{x \in Q_0}$ from $M = (M_x, f_x)$ to $M' = (M'_x, f'_x)$, there exists a K-linear map

$$f = \bigoplus_{x \in Q_0} f_x : G(M) = \bigoplus_{x \in Q_0} M_x \to G(M') = \bigoplus_{x \in Q_0} M'_x.$$

We claim that this map is Λ-homomorphism. It suffices to show that the statement for $a = a_x \in M_x \subset G(M)$ and $\bar{w} \in KQ/I$, where w is a path from x to y in Q. Then

$$f(a_x\bar{w}) = f_y\phi_w(a_x) = \phi'_w f_x(a_x) = f(a)\bar{w}.$$

The functor G is obviously K-linear and restricts to a functor $\operatorname{rep}(Q, I) \to \operatorname{mod}\Lambda$. If is easy to check that $FG \cong 1_{\operatorname{Rep}(Q,I)}$ and $GF \cong 1_{\operatorname{Mod}\Lambda}$. Finally, we note that a representation M of a finite quiver is finite dimensional if and only if M_x is finite dimensional for all $x \in Q_0$, which proves that F and G restrict to equivalences of smaller categories.

Next we recall from (1.1.18) and (1.1.21) classify the indecomposable projective and simple modules in $\operatorname{mod}\Lambda$, where Λ is any finite dimensional algebra.
We now consider the following situation. Let Q be a finite connected quiver with n vertices, be an admissible ideal of KQ and let KQ/I be the associated path algebra, which we know is basic and connected, to have R_Q/I as radical and $\pi(\epsilon_x) = e_x$, for $x \in Q_0$ as a complete set of primitive orthogonal idempotents. Here we want to understand the indecomposable projective/injective

and simple modules in mod $\Lambda \to \text{rep}(Q, I)$.
We also deduce some interesting results of the following description.
Let $x \in Q_0$, we denote by $S(x)$ the representation $S(x)_y$, defined by $S(x)_y = \delta_{xy}K$, where δ_{xy} is the Kronecker delta and $y \in Q_0$. In other words, $S(x)$ only has the vector space K over the vertex x. Hence, all the linear maps in $S(x)$ are zero.

Let

Lemma 1.3.2. *Let $\Lambda = KQ/I$ be the bound qiver algebra of (Q, I). The Λ-modules $S(x)$ is isomorphic to top $e_x\Lambda$. In particular, the set $\{S(x)|x \in Q_0\}$ contains precisely the simple Λ-modules*

Proof. The K-vector space $S(x)$ is one dimensional for all $x \in Q_0$ and defines a simple Λ-module. We also have $\text{Hom}_\Lambda(e_x\Lambda, S(x)) \cong S(x)e_x \cong S(x)_x \neq 0$, then there exist a nonzero map $e_x\Lambda \to S(x)$. The map is surjective by Schurs's lemma and its kernel is a maximal submodule of $e_x\Lambda$, hence isomorphic to rad $e_x\Lambda$. This proves the first statement. To prove the second statement: we see clearly that $\text{Hom}(S(x), S(y)) = 0$ for $x \neq y$, so the $S(x)$ are pairwise nonisomorphic.

Remark. **Warning!**, in contrast to the description of the simple modules of (finite dimensional) bound quiver algebra KQ/I given in the secaond statement of (1.3.2), any path algebra $\Lambda = KQ$ of a finite quiver Q with an oriented cycle has infinitely many pairwise nonisomorphic simple modules of finite dimension, distinct from the modules $S(x)$, with $x \in Q_0$. For example, take Q to be

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$$

We have the simple modules $S(1) = [\ K \underset{0}{\overset{0}{\rightleftarrows}} 0\ ]$ and $S(2) = [\ 0 \underset{0}{\overset{0}{\rightleftarrows}} K\ ]$. But also $S_\lambda = [\ K \underset{\lambda}{\overset{id}{\rightleftarrows}} K\ ]$ for $\lambda \in K$ are simple nonisomorphic modules.

Before we state the next result, We define the socle of a module M, denoted by soc M, to be the submodule of M generated by all simple submodules of M. Moreover, we say that a vertex of a quiver is sink if no arrow starts in this vertex.

Lemma 1.3.3. *Let $M = (M_x, \phi_\alpha)$ be a bound representation of (Q, I). Then*

a) *M is semisimple if and only if $\phi_\alpha = 0$ for all $\alpha \in Q_1$.*

b) soc $M = N$ *where $N = (N_x, \psi_\alpha)$ is the representation where $N_x = M_x$ if x is a sink, whereas*

$$N_x = \bigcap_{\alpha : x \to y} \ker(\phi_\alpha : M_x \to M_y)$$

if x is not a sink, and $\psi_\alpha = 0$ for every arrow α.

c) $\operatorname{rad} M = J$, *where* $J = (J_x, \gamma_\alpha)$ *with*

$$J_x = \sum_{\alpha: y \to x} \operatorname{Im}(\phi_\alpha : M_y \to M_x)$$

and $\gamma_\alpha = \phi_\alpha|_{J_x}$ *for every arrow* α *of source* x.

d) $\operatorname{top} M = L$, *where* $L = (L_x, \psi_\alpha)$ *with* $L_x = M_x$ *if* x *is a source, while*

$$L_x = \sum_{\alpha: y \to x} \operatorname{coker}(\phi_\alpha : M_y \to M_x)$$

if x *is not a source, and* $\psi_\alpha = 0$ *for any arrow* α.

Proof. a) The first part follows easily from fact that $\phi_\alpha = 0$ for every $\alpha \in Q_1$ if and only if M is a direct sum of sopies of $S(x)$.

b) Obviously, N is a semisimple submodule of M. Let S be a simple submodule of M, which has to be isomorphic to some $S(x)$. We thus have, for each arrow $\alpha : x \to y$, the following commutative diagram

$$\begin{array}{ccc} K \cong S(x)_x & \longrightarrow & S(x)_y = 0 \\ \downarrow & & \downarrow \\ M_x & \xrightarrow{\phi_\alpha} & M_y. \end{array}$$

It follows that $S(x)_x \subseteq \ker \phi_\alpha$ for all arrows $\alpha : x \to y$, so $S(x)_x \subseteq N_x$. This shows that $S(x) \subseteq N$, hence $N = \operatorname{soc} M$.

c) Since $\operatorname{rad} \Lambda = R_Q/I$ is generated as a two-sided ideal by the residual class modulo I of the arrows $\alpha \in Q_1$, it follows from (1.3.1) that

$$J = \operatorname{rad} M = M. \operatorname{rad} \Lambda = M.(R_Q/I) = \sum_{\alpha \in Q_1} M\bar{\alpha},$$

where the sum is taken over all arrows of target x. For such an arrow $\alpha : y \to x$, we have $M\bar{\alpha} = Me_y\bar{\alpha} = M_y\bar{\alpha} = \phi_\alpha(M_y) = \operatorname{Im} \phi_\alpha$, since the action of ϕ_α corresponds to the right multiplication by $\bar{\alpha}$. Therefore J_x is as claimed and since J is a submodule of M, we have $\gamma_\alpha = \phi_\alpha|_{J_x}$.

d) Follows from (c), since $L = M/(M \operatorname{rad} \Lambda)$= top M

Lemma 1.3.4. *Let* (Q, I) *be a bound quiver,* $\Lambda = KQ/I$ *and* $P(x) = e_x\Lambda$, *where* $e_x = \epsilon_x + I$ *and* $x \in Q_0$. *We have the decomposition* $\Lambda_\Lambda = \bigoplus_{x \in Q_0} e_x\Lambda$ *corresponding to the complete set of primitive orthogonal idempotents* $\{e_x | x \in Q_0\}$.

a) If $P(x) = (P(x)_y, \phi_\beta)$, then $P(x)_y$ is the vector space with basis the set of all $\bar{w} = w + I$ with w a path from x to y, and for an arrow $\beta : y \to z$, the map $\phi_\beta : P(x)_y \to P(x)_z$ is given by the right multiplication with $\bar{\beta} = \beta + I$.

b) Let $\operatorname{rad} P(x) = (P'(x)_y, \phi'_\beta)$. Then $P'(x)_y = P(x)_y$ for $y \neq x$, $P'(x)_x$ is the vector space with basis set of all $\bar{w} = w + I$ with w a nontrivial path from x to x, $\phi'_\beta = \phi_\beta$ for any arrow of source $y \neq x$ and ϕ'_α is the restriction of ϕ_α to $P'(x)_x$ for any arrow α with source x.

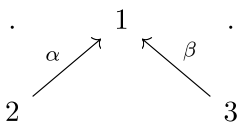
Proof. It suffices to prove (a), because (b) follows from it and (1.3.3c). We have

$$P(x)_y = P(x)e_y = e_x \Lambda e_y = e_x(KQ/I)e_y = (\epsilon_x KQ \epsilon_y)/(\epsilon_x I \epsilon_y).$$

This proves the (a). It follows immediately from the construction of the functor F, that for an arrow $\beta : y \to z$, the K-linear map ϕ_β is given by the right multiplication with $\bar{\beta}$, proving (b).

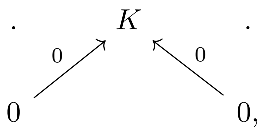
Remark. If $I = 0$ and Q is acyclic, the space $P(x)_y$ has basis the set of all paths from x to y.

Example 1.3.5. Let Q be the quiver

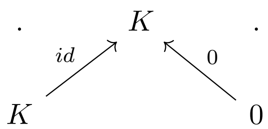


The indecomposable projective KQ-modules are given by:

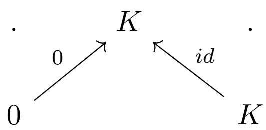
$P(1)=$



$P(2)=$



and $P(3)=$



Proposition 1.3.6. *Every indecomposable injective module in* mod Λ *is isomorphic to $I(j) = D(\Lambda e_j)$ for some j. Dually to the case of projective modules, the modules $I(j)$ is the injective envelope of the simple module $S(j)$ for all j.*

Again looking at our quiver. Note that, since $\operatorname{Hom}(e\Lambda, M) \cong Me$ for any idempotent e in an algebra Λ, we have $\operatorname{Hom}(\Lambda e_x, \Lambda) = D(\Lambda e_x) = I(x)$, since the Λe_x are the projective modules in Λ^{op}. Hence,

Proposition 1.3.7. *If $\Lambda = KQ$ is a bound quiver algebra, the indecomposable injective modules are precisely $I(x) = D(\Lambda e_x)$ for $x \in Q_0$.*

Lemma 1.3.8. a) *Given $x \in Q_0$, the simple module $S(x)$ is isomorphic to the simple socle of $I(x)$.*

b) *If $I(x) = (I(x)_y, \phi_\beta)$, then $I(x)_y$ is the dual of the K-vector space with basis the set of all $\bar{w} = w+I$ with w a path from y to x, and for an arrow $\beta : y \to z$ the map $\phi_\beta : I(x)_y \to I(x)_z$ is given by the dual of the left multiplication by $\bar{\beta}$.*

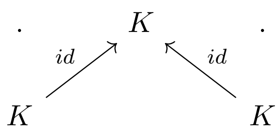
c) *Let $I(x)/S(x) = (L_y, \psi_\beta)$. Then L_y is the quotient space of $I(x)_y$ spanned by the residual classes of paths from y to x of length at least one, and ψ_β is the induced map.*

Proof. a) Since $S(x) \cong \operatorname{top} e_x\Lambda$, and this is isomorphic to the socle of $I(x)$ by duality. Alternatively we can apply (1.3.3b).

b) We have $I(x)_y = I(x)e_y = D(\Lambda e_x)e_y \cong D(e_y\Lambda e_x) \cong D(\epsilon_y(KQ)\epsilon_x/\epsilon_y I\epsilon_x)$, the first statement follows from (1.3.4). similarly, if $\beta : y \to z$ is an arrow, the K-linear map $\phi_\beta : D(\epsilon_y(KQ)\epsilon_x/\epsilon_y I\epsilon_x) \to D(\epsilon_z(KQ)\epsilon_x/\epsilon_z I\epsilon_x)$ is defined as follows: Let $\mu_\beta : D(\epsilon_z(KQ)\epsilon_x/\epsilon_z I\epsilon_x)$ be the left multiplication $\bar{w} \mapsto \bar{\beta}\bar{w}$, then $\phi_\beta = D(\mu_\beta)$ is given by $\phi_\beta(f) = f\mu_\beta$ for $f \in D(\epsilon_y(KQ)\epsilon_x/\epsilon_y I\epsilon_x)$,that is $\phi_\beta(f)(\bar{w}) = f(\bar{\beta}\bar{w})$.

c) Statement (c) is a consequence of (b).

Example. Let Q be the quiver as in Example (1.3.5). Then $I(2) = S(2)$, $I(3) = S(3)$ and the indecomposable injective $I(1)$ is



Thus $I(2)/S(2) = 0$, $I(3)/S(3) = 0$, while $I(1)/S(1) \cong S(2) \oplus S(3)$.

1.4 Existence of an expression of the quiver of Λ in terms of the extensions between simple modules

The previous results show that for each $x \in Q_0$ correspond to an idecomposable projective Λ-module $P(x)$ and an indecomposable injective module $I(x)$. The connection between them can be expressed by means of an endofunctor of the module category.

Definition 1.4.1. Let Λ be an algebra. The Nakayama functor of mod Λ is defined to be the endofunctor $\nu = D\operatorname{Hom}_\Lambda(-,\Lambda) : \operatorname{mod} \to \operatorname{mod}\Lambda$

Lemma 1.4.2. *The Nakayama functor is right exact and isomorphic to the functor $- \otimes_\Lambda D\Lambda$.*

Proof. First we note that ν is the composition of two contravariant left exact functors, so it is right exact. We define a functorial morphism $\varphi : - \otimes_\Lambda D\Lambda \to \nu = D\operatorname{Hom}_\Lambda(-, \Lambda)$ for Λ-module M by

$$\varphi_M : M \otimes_\Lambda D\Lambda \to D\operatorname{Hom}_\Lambda(M, \Lambda), a \otimes f \mapsto (\phi \mapsto f(\phi(a)),$$

for $a \in M$, $f \in D\Lambda$, and $\phi \in \operatorname{Hom}_\Lambda(M, \Lambda)$. If $M_\Lambda = \Lambda_\Lambda$, then φ_M is an isomorphism. φ_M an isomorphism if M_Λ is a projective Λ-module since both functors are K-linear. Now let M be arbitrary, and

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \longrightarrow 0$$

be a projective presentation for M. Since $- \otimes_\Lambda D\Lambda$ and ν are both right exact we apply both functors to a projective presentation of M to get the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} P_1 \otimes_\Lambda D\Lambda & \longrightarrow & P_0 \otimes_\Lambda D\Lambda & \longrightarrow & M \otimes_\Lambda D\Lambda & \longrightarrow & 0 \\ \downarrow{\scriptstyle \varphi_{P_1}} & & \downarrow{\scriptstyle \varphi_{P_0}} & & \downarrow{\scriptstyle \varphi_M} & & \\ \nu P_1 & \longrightarrow & \nu P_0 & \longrightarrow & \nu M & \longrightarrow & 0. \end{array}$$

Since φ_{P_1} and φ_{P_0} are isomorphism, so is φ_M

Proposition 1.4.3. *The Nakayama functor establishes an equivelence between the full subcategory of projective modules and the full subcategory of injective modules. The quasi-inverse is given by* $\operatorname{Hom}_\Lambda(D({}_\Lambda\Lambda), -)$.

Proof. For $x \in Q_0$, we have $\nu P(x) = D\operatorname{Hom}(e_x\Lambda, \Lambda) \cong D(\Lambda e_x) = I(x)$. On the other hand,

$$\begin{aligned} \operatorname{Hom}_\Lambda(D({}_\Lambda\Lambda), I(x)) &= \operatorname{Hom}_\Lambda(D({}_\Lambda\Lambda), D(\Lambda e_x)) \\ &\cong \operatorname{Hom}_{\Lambda^{op}}(\Lambda e_x, \Lambda) \cong e_x\Lambda = P(x). \end{aligned}$$

Lemma 1.4.4. *Let $\Lambda = KQ/I$ be a bound quiver algebra. For every Λ-module and $x \in Q_0$. There are functorial isomorphisms of K-vector spaces* $\operatorname{Hom}_\Lambda(P(x), M) \cong Me_x \cong D\operatorname{Hom}_\Lambda(M, I(x))$.

Proof. Since $P(x) = e_x\Lambda$, the first isomorphism is obvious. The second isomorphism is the composition

$$\begin{aligned} D\operatorname{Hom}_\Lambda(M, I(x)) \cong& D\operatorname{Hom}_\Lambda(M, D(\Lambda e_x)) \cong \operatorname{Hom}_{\Lambda^{op}}(\Lambda e_x, DM) \\ \cong& D(e_x DM) \cong D(DM)e_x \cong Me_x \end{aligned}$$

As a consequence, we obtain an expression of the quiver of Λ interms of the extensions between simple modules as a main result in this section.

Proposition 1.4.5. *Let $\Lambda = KQ/I$ and $x, y \in Q_0$. There exists an isomorphism of K-vector spaces*

$$\operatorname{Ext}^1_\Lambda(S(x), S(y)) \cong e_x(\operatorname{rad}\Lambda/\operatorname{rad}^2\Lambda)e_y.$$

Proof. Let

$$\dots \longrightarrow P_2 \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} S \longrightarrow 0$$

be a minimal projective resolution of the simple module S. Let take S' to be another simple module. By definiton to compute Ext^1, we have to apply the functor $\operatorname{Hom}(-, S')$ to the complex

$$\dots \longrightarrow P_2 \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0 \longrightarrow 0$$

and compute the homology of the resulting complex, thus we have

$$0 \to \operatorname{Hom}_\Lambda(P_0, S') \xrightarrow{\operatorname{Hom}_\Lambda(p_1,S')} \operatorname{Hom}_\Lambda(P_1, S') \xrightarrow{\operatorname{Hom}_\Lambda(p_2,S')} \operatorname{Hom}_\Lambda(P_2, S') \to \dots$$

We Show that $\operatorname{Hom}_\Lambda(p_{j+1}, S') = 0$ for every $j \geq 0$. Let $f \in \operatorname{Hom}_\Lambda(P_j, S')$ be a nonzero homomorphism. Since S' is simple, f is surjective so there exist an indecomposable summand P' of P_j such that f is the composition $P_j \to P' \to P'/\operatorname{rad} P' \cong S'$. Since we assumed the resolution to be minimal, we have $P_j/\operatorname{rad} P_j \cong \operatorname{Im} p_j/\operatorname{rad}(\operatorname{Im} p_j)$, so there exist a surjection from $\operatorname{Im} p_j = P_j/\ker p_j$ to $P_j/\operatorname{rad} P_j$, hence $\operatorname{Im} p_{j+1} = \ker p_j \subseteq \operatorname{rad} P_j$. Since the map $\operatorname{Hom}_\Lambda(p_{j+1}, S')$: $\operatorname{Hom}_\Lambda(p_j, S') \to \operatorname{Hom}_\Lambda(p_{j+1}, S')$ is given by precomposing with p_{j+1}, we obtain for any $j \geq 0$ and any $a \in P_j$,

$$\operatorname{Hom}_\Lambda(p_{j+1}, S')(f)(a) = fp_{j+1}(a) \in f(\operatorname{Im} p_{j+1}) \subseteq f(\operatorname{rad} P_j) = 0.$$

Hence $\operatorname{Hom}_\Lambda(p_{j+1}, S')(f)(a) = 0$ as desired. In particular, we have $\operatorname{Ext}^1_\Lambda(S, S') \cong \ker \operatorname{Hom}_\Lambda(p_2, S')/\operatorname{Im}\operatorname{Hom}_\Lambda(p_1, S') \cong \operatorname{Hom}_\Lambda(P_1, S')$. If $S = S(x)$. The semisimple module $\operatorname{rad} P(x)/\operatorname{rad}^2 P(x)$ is a direct sum of simple modules, thus

$$\operatorname{rad} P(x)/\operatorname{rad}^2 P(x) = \bigoplus_{z\in Q_0} S(z)^{n_z},$$

for some integers n_z. Now in the construction of a minimal projective resolution of $S(x)$. First, we take the projective cover of $S(x)$= top $P(x)$ which is just $P(x)$ and the map $P(x) \to S(x)$ is the natural projection. Next, we consider the ker of this map, namely $\operatorname{rad} P(x) = M_1$ and take it projective cover. The approach was to consider the semisimple $\Lambda/\operatorname{rad}\Lambda = B$-module $M_1/\operatorname{rad} M_1$, take its decomposition and then "lift" to Λ. Hence, in our case this gives that the next term in the resolution is precisely $\bigoplus_{z\in Q_0} S(z)^{n_z}$. Hence, $\operatorname{Ext}^1_\Lambda(S(x), S(y)) = \operatorname{Hom}_\Lambda(\bigoplus_{z\in Q_0} S(z)^{n_z}, S(y))$. Note that for a simple module S and any module M we have $\operatorname{Hom}_\Lambda(M, S) \cong \operatorname{Hom}_\Lambda(M/\operatorname{rad} M, S)$, because any nontrivial map from M to S sends $\operatorname{rad} M$ to 0. Applying this to $M = \bigoplus_{z\in Q_0} S(z)^{n_z}$, we obtain $\operatorname{Ext}^1_\Lambda(S(x), S(y)) = \operatorname{Hom}_\Lambda(\operatorname{rad} P(x)/\operatorname{rad}^2 P(x), S(y))$. Because $\operatorname{rad} P(x)/\operatorname{rad}^2 P(x)$ is semisimple it is equal to its socle. On the other hand, $S(y)$ is the socle of $I(y)$. Since any map between modules maps the socle into socle, we have that, $\operatorname{Hom}_\Lambda(\operatorname{rad} P(x)/\operatorname{rad}^2 P(x), S(y)) \cong \operatorname{Hom}_\Lambda(\operatorname{rad} P(x)/\operatorname{rad}^2 P(x), I(y))$. So that

$$\begin{aligned}
\operatorname{Ext}^1_\Lambda(S(x),S(y)) &\cong \operatorname{Hom}_\Lambda(\bigoplus_{z\in Q_0} S(z)^{n_z}, S(y)) \cong \operatorname{Hom}_\Lambda(\operatorname{rad} P(x)/\operatorname{rad}^2 P(x), S(y)) \\
&\cong \operatorname{Hom}_\Lambda(\operatorname{rad} P(x)/\operatorname{rad}^2 P(x), I(y)) \cong D\operatorname{Hom}_\Lambda(P(y), \operatorname{rad} P(x)/\operatorname{rad}^2 P(x)) \\
&\cong D\operatorname{Hom}_\Lambda(e_y\Lambda, e_x(\operatorname{rad}\Lambda/\operatorname{rad}^2\Lambda)) \cong D(e_x(\operatorname{rad}\Lambda/\operatorname{rad}^2\Lambda)e_y) \\
&\cong e_x(\operatorname{rad}\Lambda/\operatorname{rad}^2\Lambda)e_y.
\end{aligned}$$

Where the fourth isomorphism is from (1.4.3), the fifth isomorphism applies the equality $M\operatorname{rad}\Lambda = \operatorname{rad} M$ to $Me_x\Lambda$ and the sixth is from (1.1.9).

Chapter 2

Algebras of finite global dimension:acyclic quivers

The motivating thing in this chapter is to lead us to the following conjectures relating to the structure of the quiver Q_Λ.

- **No loop conjecture** :If gldim$\Lambda < \infty$, then Λ has no loop in the quiver or equivalently $\mathrm{Ext}_\Lambda(S, S) = 0$ for all simple Λ-modules.
- **Strong no loop conjecture** :If S is a simple Λ-module of finite projective dimension then Q_Λ does not have a loop at the vertex corresponding to S or equivalently $\mathrm{Ext}^1_\Lambda(S, S) = 0$.

2.1 Definitions and comments

To start with, we recall some needed terminology. Let Λ be an artinian ring and mod- Λ to be the category of finitely generated left Λ modules. Let $M \in \mathrm{mod}\,\Lambda$ then M is both artinian and noetherian and hence has finite length $l(M)$. Let $S(\Lambda)$ be the set of isomorphism classes of simple Λ-modules. By definition, the Gabriel quiver Q_Λ has $S(\Lambda)$ as it set of vertices. There is an arrow $S \to S'$ if $\mathrm{Ext}^1_\Lambda(S, S') \neq 0$. An acyclic quiver Q_Λ is one without oriented cycles. Let for $n \geq 0$ P_n be the minimal projective resolution of $M \in \mathrm{mod}\,\Lambda$. We set $P_{-1} = M$ and for $n \geq 1$, we write

$$\Omega^n(M) = \ker(P_{n-1} \to P_{n-2})$$

which is unique up to an isomorphism in $\mathrm{mod}\,\Lambda$.

Definition 2.1.1. If M is a non-zero Λ-module. We say the projective dimension of M, denoted by $\mathrm{pd}(M)$ is as follows:

$$\mathrm{pd}(M) = \sup\{n \geq 0 | \Omega^n(M) \neq 0\} \in \mathbb{N}_0 \cup \{\infty\}$$

Remark. We observe that projective dimension of 0 module is 0. Also we note that modules of projective dimension zero are the projectives. That is the projective dimension of M measures the degree of departure from projectivity. We also recall, for every simple Λ-module S,

$$\operatorname{Ext}^n_\Lambda(M,S) \cong \operatorname{Hom}_\Lambda(\Omega^n(M),S). \tag{2.1}$$

As a result we have the following,

$$\operatorname{pd}(M) = \sup\{n \geq 0 | \operatorname{Ext}^n_\Lambda(M,-) \neq 0\}.$$

Let $M \in \operatorname{mod}\Lambda$ and S a simple Λ-module, we set $[M:S]$ to be the multiplicity of S in a composition series of M. Then the long exact cohomology sequence now shows that

$$\operatorname{pd}(M) \leq \max\{\operatorname{pd}(S) | [M:S] \neq 0\}.$$

Definition 2.1.2. The global dimension of Λ, denoted by $\operatorname{gldim}\Lambda$, is define as follows,

$$\operatorname{gldim}\Lambda = \max\{\operatorname{pd}(S) | S, \textit{simple}\} \in \mathbb{N}_0 \cup \{\infty\}.$$

2.2 Some useful results

Let J be the Jacobson radical of Λ and $P(S)$ the projective cover of a simple Λ-module S. Then for $n = 1$ and every simple Λ module S', formula 2.1 specialises to

$$\operatorname{Ext}^1_\Lambda(S,S') \cong \operatorname{Hom}_\Lambda(JP(S)/J^2P(S),S'). \tag{2.2}$$

Lemma 2.2.1. *Given S and S' to be simple Λ- modules. If $[JP(S):S'] \neq 0$, then there exists a path in Q_Λ of length ≥ 1 that starts at S and ends in S'.*

Proof. Let B be a factor module of $P(S)$ of maximal length, subject to all composition factors of JB being endpoints of paths of lengths ≥ 1 that has originate in S. Let assume $P(S)$ is not simple, otherwise there is nothing to be shown. Alternatively formula 2.2.2, implies that $l(B) \geq 2$. Which yields a short exact sequence,

$$0 \longrightarrow A \longrightarrow P(S) \longrightarrow B \longrightarrow 0.$$

Suppose $B \neq P(S)$, we choose a maximal submodule $C \subset A$ and consider the induced exact sequence

$$0 \longrightarrow A/C \longrightarrow P(S)/C \longrightarrow B \longrightarrow 0.$$

As $P(S)/JP(S) \cong S$ is simple, the middle term is indecomposable, so the sequence does not split. Set $A'' = A/C$, then we have $\mathrm{Ext}^1_\Lambda(B, A'') \neq 0$, and standard homological algebra provide a composition factor A' of B with $\mathrm{Ext}^1_\Lambda(A', A'') =\neq 0$. If $A' \cong S$, then there is a path from S to A'' of length 1. Alternatively, $[JB : A'] \neq 0$, then there is a path from S to A', and also from S to A''. As a result, all composition factors of $J(P(S)/C)$ are endpoints of paths originating in S. Since, $l(P/C) = l(B) + 1$, this contradict the maximality of $l(B)$. Hence $B = P(S)$

Lemma 2.2.2. *Let S be a simple Λ-module and for $n \geq 0$ P_n be a minimal projective resolution of S. If $P(S')$ is a direct summand of P_n, then there exist a path of length $\geq n$ originating in S and terminating in S'.*

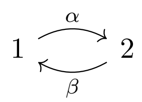
Proof. We apply induction on n, the case $n = 0$ is trivial. Let $n \geq 1$ and note that P_n is the projective cover of $\Omega^n(S) = ker(P_{n-1} \to P_{n-2}) \subseteq JP_{n-1}$ (Here we set $P_{-1} = S$). As a result, $P_n/JP_n \cong \Omega^n(S)/J\Omega^n(S)$, thus $P(S')$ being a summand of P_n implies $[JP_{n-1} : S'] \neq 0$. So there exists a summand $P(S'') of P_{n-1}$ with $[JP(S'') : S'] \neq 0$. Lemma 2.2.1 provides a path $S'' \to S'$ of length ≥ 1. By inductive hypothesis, there is a path $S \to S''$ of length $\geq n - 1$ and concatenation yields the desired path from S to S'.

Theorem 2.2.3. *Given Q_Λ is acyclic, then* $\mathrm{gldim}\,\Lambda \leq |S(\Lambda)| - 1$.

Proof. Let S be a simple Λ-module with minimal projective resolution P_n for $n \geq 0$. As Q_Λ is acyclic, a path $\in Q_\Lambda$ has length $\leq |S(\Lambda)| - 1 = n$. From lemma 2.2.2 we get, $P_{n+1} = 0$, whiles $\Omega^{n+1}(S) \cong \mathrm{Im}(P_{n+1} \to P_n) = 0$. Hence $\mathrm{pd}(S) \leq n$, so that $\mathrm{gldim}\,\Lambda \leq n$.

Remark. The proof shows that the projective dimension $pd(S)$ of the simple Λ module S is bounded by the maximum length of all paths originating in S. The following examples shows that algebras of finite global dimension also occur for quivers that admit oriented cycles.

Example. Let k be a field and consider $\Lambda = kQ/I$, where Q is the quiver



and I is the ideal in kQ generated by $\alpha\beta$. There are two simple modules S_1 and S_2. And we have $\Omega(S_1) = P(S_1)$ and $\Omega^2(S_2) = P(S_1)$, thus $\mathrm{pd}(S_1) = 1$ and $\mathrm{pd}(S_2) = 2$ whiles the $\mathrm{gldim}\,\Lambda = 2$.

Remark 2.2.4. Our formula 2.2 readily yields $Q_\Lambda = Q_\Lambda/J^2$. Thus we can hope to get more information for algebras satisfying $J^2 = 0$. Next we record the following observation:

Corollary 2.2.5. *If $J^2 = 0$. Then the following statements hold:*

a) *If* $\mathrm{gldim}\,\Lambda < \infty$, *then Q_Λ has no oriented cycles.*

b) *If Λ has only one simple module, then Λ is simple.*

Proof. a) Let S be a simple Λ-module. As $J^2 = 0$, the module $\Omega(S) = JP(S) = \bigoplus_{nS''} S''$ is semisimple and formula 2.2 implies

$$_{nS'} \operatorname{Hom}_\Lambda(S', S') \cong \operatorname{Hom}_\Lambda(JP(S), S') \cong \operatorname{Ext}^1_\Lambda(S, S').$$

Thus $ns' \neq 0$ whenever there is an arrow $S \to S'$, and in that case our Ext-criterion yields

$$\operatorname{pd}(S') \leq \max\{\operatorname{pd}(S'')|_{nS''} \neq 0\} = \operatorname{pd}(JP(S)) < \operatorname{pd}(S).$$

As a result Q_Λ has no oriented cycles.

b) Part (a) implies that Q_Λ has no arrows. So Λ is semi-simple and has only one simple module. By Artin-Wedderburn Theorem, Λ is simple.

Remark. We recall that an arrow starting and terminating at the same vertex is called a loop. There are two conjectures relating the structure of the quiver Q_Λ to the various dimensions introduced before.

- **No loop conjecture**:If $\operatorname{gldim}\Lambda < \infty$, then Q_Λ has no loop.
- **Strong no loop conjecture**:If S is a simple Λ-module of finite projective dimension then Q_Λ does not have a loop at the vertex corresponding to S or equivalently $\operatorname{Ext}^1_\Lambda(S, S) = 0$ for all simple Λ modules.

The subsequent chapters, provide the tools to prove the latter conjecture which is the reason for this work.

Chapter 3

Hattori-Stallings trace and Lenzing's results

Let Λ stands for a (basic) finite-dimensional algebra over an algebraically closed field. All modules are finitely generated right Λ-modules. We denote $\mathrm{HH}_0(\Lambda)$ to be the zeroth Hochschild homology group of Λ. It is well known that $\mathrm{HH}_0(\Lambda) = \Lambda/[\Lambda,\Lambda]$ is the quotient of Λ by the additive subgroup $[\Lambda,\Lambda]$ generated by all elements of the form $\lambda_1\lambda_2$-$\lambda_2\lambda_1$ where $\lambda_1, \lambda_2 \in \Lambda$.
Firstly, as a main result of this section, we recall the notion of the trace of an endomorphism f of a projective module P in $\operatorname{mod}\Lambda$, as defined by Hattori and Stallings; see [1],[12],[14] and [15]. Thus if $f \in \mathrm{End}(P)$. Write, $P \cong e_1\Lambda \oplus ... \oplus e_n\Lambda$, where the e_i are primitive idempotents in Λ. Then $f = (a_{ij})_{n\times n}$, where $a_{ij} \in e_i\Lambda e_j$. We define the trace of f as :

$$\mathrm{tr} : \mathrm{End}(P) \to \Lambda/[\Lambda,\Lambda];$$
$$f \mapsto \mathrm{tr}(f) = \sum_{i=1}^{n} a_{ii} + [\Lambda,\Lambda].$$

We will later see in section 3.2 that this definition does not depend on the decomposition of P. This function enjoys the usual properties of a trace which we recall in the following:

Proposition (HATTORI-STALLINGS). *Let P, P' be projective modules in* $\operatorname{mod}\Lambda$

(HS1) *if $f, g \in \mathrm{End}_\Lambda(P)$ then* $\mathrm{tr}(f+g) = \mathrm{tr}(f) + \mathrm{tr}(g)$

(HS2) *if $f : P \to P'$ and $g : P' \to P$ are Λ - linear then* $\mathrm{tr}(fg) = \mathrm{tr}(gf)$

(HS3) *if* $f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \in \mathrm{End}_\Lambda(P \oplus P')$ *then* $\mathrm{tr}(f) = \mathrm{tr}(f_{11}) + \mathrm{tr}(f_{22})$

(HS4) *if $g \in \mathrm{Hom}_\Lambda(P, P')$ is an isomorphism and $f \in End_\Lambda(P)$ then* $\mathrm{tr}(gfg^{-1}) = tr(f)$

(HS5) *if $f \in End_\Lambda(\Lambda)$ is the left multiplication by $a \in \Lambda$ then* $\mathrm{tr}(f) = a + [\Lambda,\Lambda]$.

Secondly, since this will be the main result, we aim to prove these properties in section 3.1. Here, we will start by developing some background statements of traces of endomorphisms of projective modules Λ-modules.

Thirdly, in section 3.2, we aim to achieve the following results, about an alternating sum definition for our trace, and consequently, the following composition:

$$K_1' \xleftarrow{\sim} K_1 \xrightarrow{\sim} \Lambda/[\Lambda, \Lambda],$$

of trace maps that are isomorphic (see 3.1.3 and 3.2.1). Where K_1 and K_1' will denote the additive group given by generators and relations given in section 3.1.

Then finally, by combining results from 3.1 and 3.2 to a particular kind of filtration for the Λ-module M to obtain information on nilpotent elements (see, Theorem 3.2.4).

3.1 THE RELATIVE K-THEORY GROUP $K_1(\Lambda)$

Let Λ be a ring with 1_Λ and $\operatorname{mod}\Lambda$ the category of all finitely generated right Λ-modules. Let $P(\Lambda)$ be the full subcategory of projective Λ-modules and denote by $P_0(\Lambda)$ the full subcategory in $\operatorname{mod}\Lambda$ consisting of modules of finite projective dimension.

Definition 3.1.1. Let $K_1(\Lambda)$ respectively $K_1'(\Lambda)$ be additive group generated by pairs (M, f) with $M \in P(\Lambda)$ respectively $M \in P_0(\Lambda)$ and $f \in \operatorname{End}_\Lambda(M)$ that satisfy the following relations

a) $(M, f + g) = (M, f) + (M, g)$

b) $(M, f) + (N, g) = (L, h)$ for every commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & L & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & M & \longrightarrow & L & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

with exact rows.

c) $(M, fg) = (N, gf)$ if $f : M \to N$ and $g : N \to M$.

It will later turn out that this additive group $K_1(\Lambda)$ respectively $K_1'(\Lambda)$ are both isomorphic to the zeroth Hochshild group.

Lemma 3.1.2. *a)* $(M \oplus N, \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix}) = 0 \in K_1(\Lambda)$.

b) $(P, f) + (Q, g) = (P \oplus Q, \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}) \in K_1(\Lambda)$ *resp.* $K_1'(\Lambda)$

c) if $(P, f) \in K_1(\Lambda)$ *there exist* $a_f \in \Lambda$ *such that* $(P, f) = (\Lambda, \lambda_{a_f})$. *Here* $\lambda_a : \Lambda \to \Lambda$ *is the left multiplication with* a.

d) $(\Lambda, \lambda_a \lambda_{a'}) = (\Lambda, \lambda_{a'} \lambda_a)$ *for all* $a, a' \in \Lambda$. *Furthermore* $(\Lambda, \lambda_{(aa' - a'a)}) = 0$ *in* $K_1(\Lambda)$

Proof. a) Because

$$\begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I_N \end{pmatrix} = \begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix}$$

with I_N being the identity on N and

$$\begin{pmatrix} 0 & 0 \\ 0 & I_N \end{pmatrix} \begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

It then follows from 3.1.1 c that

$$(M \oplus N, \begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix}) = 0$$

therefore

$$(M \oplus N, \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix}) = (M \oplus N, \begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix}) + (M \oplus N, \begin{pmatrix} 0 & 0 \\ g & 0 \end{pmatrix}) = 0$$

b) The claim holds by definition 3.1.1 b).

c) As P is a projective Λ-module, there is a complement we will call Q and an isomorphism. $w : P \oplus Q \to \Lambda^n$. Now we set

$$f \oplus 0_Q := \begin{pmatrix} f & \\ & 0_Q \end{pmatrix} : P \oplus Q \to P \oplus Q.$$

Hence $w^{-1}(f \oplus 0_Q)w : \Lambda^n \to \Lambda^n$ is represented by a left multiplication by an $n \times n$ matrix $(a_{ij})_{i,j}$ with entries in Λ. Applying 3.1.1a,b for the first equality, 3.1.1c for the second equality, 3.1.1a,b for the third equaliity and 3.1.2a for last equality, we have

$$\begin{aligned} (P, f) &= (P \oplus Q, f \oplus 0_Q) \\ &= (\Lambda^n, w^{-1}(f \oplus 0_Q)w) \\ &= (\Lambda^n, ((a_{ij})_{i.j}) = (\Lambda, \sum_{i=1}^{n} \lambda a_{ii}) \end{aligned}$$

Hence, $a_f = \sum_{i=1}^n a_{ii}$.

d) The claim is trivial.

Definition 3.1.3. Let the trace map $\mathrm{Tr} : K_1(\Lambda) \to \Lambda/[\Lambda, \Lambda]$ be defined as follows :

a) For $f \in \mathrm{End}_\Lambda(\Lambda^n)$, with $f = (f_{ij})$, we define, $\mathrm{tr}(f) = \sum_{i=1}^n f_{ii}(1_\Lambda)$. We denote $\mathrm{Tr}(\Lambda^n, f) = \overline{\mathrm{tr}(f)}$ as the residual class of $\mathrm{tr}(f)$ in $\Lambda/[\Lambda, \Lambda]$.

b) For $f \in \mathrm{End}(F)$ with $w : F \xrightarrow{\sim} \Lambda^n$. Define $\mathrm{Tr}(F, f) = \mathrm{Tr}(\Lambda^n, w^{-1} f w)$

c) For $f \in \mathrm{End}_\Lambda(P)$, $P \oplus Q \simeq \Lambda^n$. Define $\mathrm{Tr}(P, f) = \mathrm{Tr}(P \oplus Q, f \oplus 0_Q)$.

Next we show that, Tr is a well-defined homomorphism:
Let $\mathrm{Tr} : K_1(\Lambda) \to \Lambda/[\Lambda,\Lambda]$ as in (3.1.3)

Proof. Let $\phi : P \oplus Q \to \Lambda^n$ and $\psi : P \oplus Q' \to \Lambda^m$ be isomorphisms. Without loss of generality let us assume that $m = n$ and $Q' = Q$, since

$$\begin{aligned}\mathrm{Tr}(\Lambda^n, \phi^{-1}(f \oplus 0_Q)\phi) = \mathrm{tr}(\phi^{-1}(f \oplus 0_Q)\phi) + [\Lambda,\Lambda] &= \mathrm{tr}(\phi^{-1}(f \oplus 0_Q)\phi \oplus 0_{\Lambda^k}) + [\Lambda,\Lambda] \\ &= \mathrm{tr}((\phi^{-1} \oplus I_{\Lambda^k})(f \oplus 0_Q \oplus 0_{\Lambda^k})(\phi \oplus I_{\Lambda^k})) + [\Lambda,\Lambda].\end{aligned}$$

It is well known that for matrices $A, B \in \Lambda^{n\times n}$ one has $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ modulo $[\Lambda,\Lambda]$. Hence

$$\begin{aligned}\mathrm{Tr}(\Lambda^m, \psi^{-1}(f \oplus 0_Q)\psi) = \mathrm{tr}(\psi^{-1}(f \oplus 0_Q)\psi) + [\Lambda,\Lambda] &= \mathrm{tr}((\phi^{-1}\psi)(\psi^{-1}(f \oplus 0_Q)\psi)(\psi^{-1}\phi)) + [\Lambda,\Lambda] \\ &= \mathrm{tr}(\phi^{-1}(f \oplus 0_Q)\phi) + [\Lambda,\Lambda] \\ &= \mathrm{Tr}(\Lambda^n, \phi^{-1}(f \oplus 0_Q)\phi).\end{aligned}$$

Lemma 3.1.4. *Given the trace map* $\mathrm{Tr} : K_1(\Lambda) \to \Lambda/[\Lambda,\Lambda]$.

a) $\mathrm{Tr}(P, f) + \mathrm{Tr}(P, g) = \mathrm{Tr}(P, f + g)$,

b) $\mathrm{Tr}(P, f) + \mathrm{Tr}(Q, g) = \mathrm{Tr}(T, h)$, *for every commutative diagram with exact rows:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P & \longrightarrow & T & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & P & \longrightarrow & T & \longrightarrow & Q & \longrightarrow & 0 \end{array}$$

c) $\mathrm{Tr}(P, fg) = \mathrm{Tr}(Q, gf)$ *for every sequence* $P \xrightarrow{f} Q \xrightarrow{g} P \xrightarrow{f} Q$.

Proof. a) $\mathrm{Tr}(P, f) + \mathrm{Tr}(P, g) = \mathrm{Tr}(P, f + g)$ holds since tr is additive.

b) If we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P & \longrightarrow & T & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & P & \longrightarrow & T & \longrightarrow & Q & \longrightarrow & 0 \end{array}$$

then $T = P \oplus Q$ and for some $\gamma : Q \to P$ choose $h = \begin{pmatrix} f & 0 \\ \gamma & g \end{pmatrix}$. Furthermore there are complements P' and Q' with $\phi : P \oplus P' \oplus Q \oplus Q' \xrightarrow{\sim} \Lambda^n$ .For

$$\psi = \begin{pmatrix} I_P & & & \\ & 0 & I_{P'} & \\ & I_Q & 0 & \\ & & & I_{Q'} \end{pmatrix} : P \oplus Q \oplus P' \oplus Q' \xrightarrow{\sim} P \oplus P' \oplus Q \oplus Q'$$

then $\psi^{-1}(f \oplus g \oplus 0_{P'} \oplus 0_{Q'})\psi = (f \oplus 0_{P'} \oplus g \oplus 0_{Q'})$.Now we derive

$$\begin{aligned}
\phi^{-1}(f \oplus 0_{P'} \oplus 0_Q \oplus 0_{Q'})\phi + \phi^{-1}(0_P \oplus 0_{P'} \oplus g \oplus 0_{Q'})\phi \\
&= \phi(f \oplus 0_{P'} \oplus g \oplus 0_{Q'})\phi \\
&= \phi^{-1}\psi^{-1}(f \oplus g \oplus 0_{P'} \oplus 0_{Q'})\psi\phi \\
&= (\psi\phi)^{-1}(h \oplus 0_{P' \oplus Q'})(\psi\phi).
\end{aligned}$$

Thus $\text{Tr}(P, f) + \text{Tr}(Q, g) = \text{Tr}(T, h)$.

c) let $f : P \to Q, g : Q \to P$, $\phi : P \oplus P' \xrightarrow{\sim} \Lambda^n$ and $\psi : Q \oplus Q' \xrightarrow{\sim} \Lambda^m$ be homomorphisms. Then using as before well-defineness we have:

$$\begin{aligned}
\text{tr}(\phi^{-1}(fg \oplus 0_{P'})\phi) &= tr(\phi^{-1}(f \oplus 0)\psi\psi^{-1}(g \oplus 0)\phi) \\
&= \text{tr}(\psi^{-1}(g \oplus 0)\phi\phi^{-1}(f \oplus 0)\psi) \\
&= \text{tr}(\psi^{-1}(gf \oplus 0_{Q'})\psi)
\end{aligned}$$

Thus $\text{Tr}(P, fg) = \text{Tr}(Q, gf)$.

Observation. By 3.1.3 and 3.1.4 it follows that the trace properties HS1,HS2,...,HS5 recalled in the beginning of chapter 3 hold.

Theorem 3.1.5. *For any ring Λ the map $\text{Tr} : K_1(\Lambda) \to \Lambda/[\Lambda, \Lambda]$; $[(P, f)] \mapsto \text{tr}(f)$ is an isomorphism.*

Proof. First Tr is surjective since $\text{Tr}(\Lambda, \lambda_a) = \bar{a} \in \text{HH}_0(\Lambda)$ for all $a \in \Lambda$. And to prove Tr is injective,it suffices to show that the $\ker \text{Tr}$ is 0. So let (P, f) satisfy $\text{Tr}(P, f) = 0$. Then it follows from lemma 3.1.2c) that $(P, f) = (\Lambda, \lambda_{a_f})$, hence $\text{tr}(\lambda_{a_f}) = a_f \in [\Lambda, \Lambda]$. Thus $(P, f) = 0$ in $K_1(\Lambda)$ by 3.1.2d)

3.2 LENZING'S THEOREM

Theorem 3.2.1. *For any ring Λ the inclusion functor $P(\Lambda) \to P_0(\Lambda)$ induces an isomorphism $K_1(\Lambda) \to K_1'(\Lambda)$.*

Proof. Let $\alpha : K_1(\Lambda) \to K_1'(\Lambda)$ be the homomorphism induced by the inclusion $P(\Lambda) \to P_0(\Lambda)$. We construct the inverse map $\beta : K_1'(\Lambda) \to K_1(\Lambda)$. First of all, we define β as a map from the free additive group given by the generators (M, f) such that the projective dimension of M is finite to $K_1(\Lambda)$. Let

$$P_\bullet : \; 0 \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \longrightarrow \ldots \longrightarrow P_1 \xrightarrow{d_1} P_0 \longrightarrow 0$$

(with $M = cokd_1$) be a projective resolution of M. Given a map$f : M \to M$ we choose a chain map $f. : P_\bullet \to P_\bullet$ a lifting of f. Let define $\beta(M,f) := \sum_{i=0}^{n}(-1)^i(P_i, f_i)$. First we show that β is well defined.

- If $g_\bullet : P_\bullet \to P_\bullet$ is another lifting of f, then there are some maps $h_i : P_i \to P_{i+1}$ such that,

$$g_i - f_i = h_i d_{i+1} + d_i h_{i-1}.$$

  That is,

$$\begin{aligned}
\sum_{i=0}^{n}(-1)^i(P_i, g_i) &= \sum_{i=0}^{n}(-1)^i(P_i, f_i + h_i d_{i+1} + d_i h_{i-1}) \\
&= \sum_{i=0}^{n}(-1)^i((P_i, f_i) + (P_i, h_i d_{i+1}) + (P_i, d_i h_{i-1})) \\
&= \sum_{i=0}^{n}(-1)^i(P_i, f_i) + \sum_{i=0}^{n}(-1)^i(P_i, h_i d_{i+1}) - \sum_{i=-1}^{n-1}(-1)^i(P_{i+1}, d_{i+1} h_i) \\
&= \sum_{i=0}^{n}(-1)^i(P_i, f_i) + \sum_{i=0}^{n-1}(-1)^i((P_i, h_i d_{i+1}) - (P_{i+1}, d_{i+1} h_i)) \\
&+ (-1)^n(P_n, h_n d_{n+1}) + (P_0, d_0 h_{-1}) \\
&= \sum_{i=0}^{n}(-1)^i(P_i, f_i)
\end{aligned}$$

- If $Q_\bullet$ is another projective resolution of M let $\phi_\bullet : P_\bullet \to Q_\bullet$ be a lifting of I_M and $f_\bullet : Q_\bullet \to P_\bullet$ a lifting of $f_\bullet$. Then $\phi_\bullet f_\bullet$ and $f_\bullet \phi_\bullet$ are liftings of f. So we have this set up $P_i \xrightarrow{\phi} Q_i \xrightarrow{f_i} P_i \xrightarrow{\phi} Q_i$ thus $(P_i, \phi_i f_i) = (Q_i, f_i \phi_i)$ for $i \geq 0$ and

$$\sum_{i=0}^{n}(-1)^i(P_i, \phi_i f_i) = \sum_{i=0}^{n}(-1)^i(P_i, f_i \phi_i).$$

Also we see this alternating sum definition does not depend the choice of the liftings $\{f_i\}$ or $P_\bullet$. Moroeover, β defines a surjective homomorphism. Next we check that β is injective, but prior to that we make the following observations relating to the relations defined in (3.1.1):

a) Obviously $f_\bullet + g_\bullet$ is a lifting of $f + g$ if $f_\bullet, g_\bullet : P_\bullet \to P_\bullet$ are liftings of f, g.

b) For a commutative diagram with exact rows :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & L & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow f & & \downarrow h & & \downarrow g & & \\
0 & \longrightarrow & M & \longrightarrow & L & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

Let $f_\bullet : P_\bullet \to P_\bullet$ and $g_\bullet : Q_\bullet \to Q_\bullet$ be lifting of f and g respectively. It is a well known see [5,p. 46] that there exist $\eta_i : Q_i \to P_i$ such that $h_\bullet = \begin{pmatrix} f_\bullet & 0 \\ \eta_\bullet & g_\bullet \end{pmatrix} : P_\bullet \oplus Q_\bullet \to P_\bullet \oplus Q_\bullet$ is a lifting of h. Thus $\beta(L,h) = \beta(M,f) + \beta(N,g)$ by lemma 3.1.2.

c) Since a lifting of a composition $fg : M \to M$ is the composition $f_\bullet g_\bullet$ of lifting $f_\bullet : P_\bullet \to P_\bullet$, $g_\bullet : Q_\bullet \to Q_\bullet$ the equality

$$\beta(M,fg) = \sum_{i=0}^{n}(-1)^i(P_i, f_i g_i) = \sum_{i=0}^{n}(-1)^i(Q_i, g_i f_i) = \beta(N,gf)$$

holds.

Therefore β induces a map $\beta : K_1'(\Lambda) \to K_1(\Lambda)$. Now we proof that β is injective. Thus, it suffices to verify that $\alpha \circ \beta = 1_{K_1'(\Lambda)}$. So let,

$$0 \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \longrightarrow \ldots \longrightarrow P_1 \xrightarrow{d_1} P_0 \longrightarrow 0$$

(with $M = \operatorname{coker} d_1$) be a projective resolution of M. Then

$$\alpha \circ \beta(M,f) = \alpha(\sum_{i=0}^{n}(-1)^i(P_i, f_i)) = \sum_{i=0}^{n}(-1)^i(P_i, f_i).$$

To see this holds, we apply induction on n. For $n = 0$ the claim is trivial. Let π be the projective cover $P_0 \to M$ then

$$0 \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \longrightarrow \ldots \longrightarrow P_1 \xrightarrow{d_1} \ker \pi \longrightarrow 0$$

is projective resolution of $\ker \pi$ and there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker \pi & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow f' & & \downarrow f_0 & & \downarrow f & & \\ 0 & \longrightarrow & \ker \pi & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

By the induction hypothesis we have $(\ker \pi, f') = \sum_{i=1}^{n}(-1)^i(P_i, f_i)$ and $(P_0, f_0) = (\ker \pi, f') + (M, f)$ by def 3.1.1 b. Thus

$$(M,f) = \sum_{i=0}^{n}(-1)^i(P_i, f_i) = (\alpha \circ \beta)(M,f).$$

So α is an isomorphism.

Definition 3.2.2. Let M be in $\operatorname{mod} \Lambda$, $f : M \to M$. An f- filtration of M is a finite filtration

$$M = M_0 \supset M_1 \supset \ldots \supset M_n = 0$$

by submodules with

$$f(M_i) \subset M_{i+1} \quad \forall i = 0, ..., n-1.$$

The f-filtration has finite projective dimension if $\operatorname{pdim}_\Lambda M_i < \infty$ holds for all $i = 0, ..., n-1$.

Proposition 3.2.3. *Suppose that $M \in P_0(\Lambda)$ has a filtration of finite projective dimension. Then $(M, f) = 0$ in $K_1'(\Lambda)$.*

Proof. We proceed by induction on n. If $n = 0$ the claim is trivial. Let $n \geq 1$ and consider the map $f_1 : M_1 \to M_1$ induced by the restriction of f, then $(M_1, f_1) = 0$ in $K_1'(\Lambda)$ by induction hypothesis. Since $f(M) \subset M_1$ then we have the following commutative diagram with exact rows :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_1 & \longrightarrow & M & \longrightarrow & M/M_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & M_1 & \longrightarrow & M & \longrightarrow & M/M_1 & \longrightarrow & 0
\end{array}$$

So $(M, f) = (M_1, f_1) + (M/M_1, 0) = 0$.

Finally we state a very important result-theorem 3.2.4, which will later turn to be a useful guide for us in sec 4.4 of chapter 4.

Theorem 3.2.4. *Let Λ be a ring with 1_Λ and $e \in \Lambda$ a primitive idempotent. Let $a \in e\Lambda e$ be a nilpotent element and denote by $\lambda_a : e\Lambda \to e\Lambda$ the left multiplication with a. If $e\Lambda$ has a λ_a-filtration of finite projective dimension, then $a \in [\Lambda, \Lambda]$.*

Proof. By Prop 3.2.3 $(e\Lambda, \lambda_a) = 0$ in $K_1'(\Lambda)$; hence by Theorem 3.2.1 $(e\Lambda, \lambda_a) = 0$ in $K_1(\Lambda)$ and $0 = \operatorname{Tr}(e\Lambda, \lambda_a) = a + [\Lambda, \Lambda] \in \operatorname{HH}_0(\Lambda)$. That means $a \in [\Lambda, \Lambda]$.

Chapter 4

Localised trace function

4.1 Lenzing's trace function

Throughout Λ stands for a (basic) finite-dimensional algebra over an algebraically closed field. Let J stands for the Jacobson radical of Λ. All modules are finitely generated right Λ-modules. We denote the zeroth Hochschild homology group of Λ, $\mathrm{HH}_0(\Lambda) = \Lambda/[\Lambda,\Lambda]$, thus, the quotient of Λ by the additive subgroup $[\Lambda,\Lambda]$ generated by all elements of the form $\lambda_1\lambda_2$-$\lambda_2\lambda_1$ where $\lambda_1, \lambda_2 \in \Lambda$.
As a main result of this chapter we will prove that $\mathrm{HH}_0(\Lambda)$ is radical trivial, thus $J \subseteq [\Lambda,\Lambda]$.
Firstly, we recall the trace of f defined to be

$$\mathrm{tr}(f) = \sum_{i=1}^{n} a_{ii} + [\Lambda,\Lambda] \in \Lambda/[\Lambda,\Lambda]$$

by Hattori and Stallings in beginning of chapter 3.
Secondly, from chapter 3, we have an alternating sum definition of our trace (see, 3.2), and consequently, the following composition:

$$K_1' \overset{\sim}{\leftarrow} K_1 \overset{\sim}{\rightarrow} \Lambda/[\Lambda,\Lambda],$$

of trace maps that are isomorphic (see 3.1.3 and 3.2.1). Where K_1 and K_1' as before denote the additive group given by generators and relations (see definition 3.1.1).
By these, we recall Lenzing's extension of this function to endomorphism of modules of finite projective dimension. Let M be a Λ module of finite projective dimension and $f : M \to M$. Then we have a finite projective resolution :

$$\dots \longrightarrow P_i \xrightarrow{d_i} P_{i-1} \longrightarrow \dots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0$$

For each $f : M \to M$, we have the following commutative diagram

$$\begin{array}{ccccccccccccc}
\dots \longrightarrow & P_i & \xrightarrow{d_i} & P_{i-1} & \longrightarrow \dots \longrightarrow & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & M & \longrightarrow 0 \\
& \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i-1}} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & \\
\dots \longrightarrow & P_i & \xrightarrow{d_i} & P_{i-1} & \longrightarrow \dots \longrightarrow & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & M & \longrightarrow 0
\end{array}$$

Where f_i for $i \geq 0$ is a lifting of f to P_M (where P_M denote the projective resolution of $M \in \operatorname{mod}\Lambda$) Let M be of finite projective dimension and assuming that P_M is bounded We define the trace of f, as

$$\operatorname{tr}(f) = \lim_{n\to\infty} \sum_{i=1}^{n} (-1)^i \operatorname{tr}(f_i) \in \Lambda/[\Lambda,\Lambda]$$

Which is independent of the choice of P_M and $\{f_i\}$ see section 3.2. Thirdly, the plan is to localise Lenzing's trace function to endomorphisms of Λ modules with an e- bounded projective resolution, where e is an idempotent in Λ. In our next section we localise this construction.

4.2 THE e-TRACE FUNCTION

Here we maintain as before the same settings. Also, let e denote an idempotent in Λ. We set $\Lambda_e = \Lambda/\Lambda(1-e)\Lambda$. The canonical algebra projection $\Lambda \to \Lambda_e$ induces a group homomorphism

$$h : \Lambda/[\Lambda,\Lambda] \to \Lambda_e/[\Lambda_e,\Lambda_e].$$

If $f : P \to P$ is an endomorphism of a projective Λ - module, then we define the e-trace by

$$\operatorname{tr}(f) \mapsto h(\operatorname{tr}(f) \in \Lambda_e/[\Lambda_e,\Lambda_e]$$

We denote $\operatorname{tr}_e(f) = h(\operatorname{tr}(f))$.
Clearly the e-trace function has the properties HS1,HS2,HS3 and HS4 stated in the beginning of chapter 3. That is, we have the following:

- $\operatorname{tr}_e(f+g) = \operatorname{tr}_e(f) + \operatorname{tr}_e(g)$
- $\operatorname{tr}_e(fg) = \operatorname{tr}_e(gf)$
- $\operatorname{tr}_e(f) = \operatorname{tr}_e(f_{11}) + \operatorname{tr}_e(f_{22})$
- $\operatorname{tr}_e(gfg^{-1}) = \operatorname{tr}_e(f)$
- $\operatorname{tr}_e(f) = \bar{a} + [\Lambda_e,\Lambda_e]$, where $\bar{a} = a + \Lambda\ (1-e)\Lambda$

Lemma 4.2.1. *Let e be idempotent $\in \Lambda$, and let P be a projective Λ module whose* top *is annihilated by e. If $f : P \to P$, then $\operatorname{tr}_e(f) = 0$.*

Proof. Let Suppose $P \neq 0$. We have $1 - e = e_1 + ... + e_n$. Where the e_i are pairwise orthogaonal primitive idempotents in Λ. Let $f : P \to P$. Let suppose, P is indecomposable by HS3, then, for some m such that $1 \leq m \leq n$, we have, $P \cong e_m\Lambda$. By HS4, we may assume that $P = e_m\Lambda$. Then, f is a left multiplication by some $a \in e_m\Lambda e_m$. It follows from HS5 that,

$$\mathrm{tr}_e(f) = h(a + [\Lambda, \Lambda] = \bar{a} + [\Lambda_e, \Lambda_e].$$

Where $\bar{a} = a + \Lambda\ (1-e)\Lambda$. Write, $a = e_m a e_m = (1-e)a(1-e)$, then a is in $\Lambda(1-e)\Lambda$. Hence $\mathrm{tr}(f) = 0$.

Next we extend e-trace function and define a projective resolution P_M of M Λ-modules to be e-bounded if all but finitely many tops of the terms in P_M are annihilated by e

4.3 THE (e,n)- TRACE FUNCTION

Maintaining the same settings as before.

Definition 4.3.1.
- Let M be a module and

$$P_M = \ \dots \longrightarrow P_i \xrightarrow{d_i} P_{i-1} \longrightarrow \dots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0$$

  a projective resolution of M. We say that P_M is (e,n)-bounded if $\mathrm{top}(P_n).e = 0$.

- Let $f : M \to M$ with a lifting $(f_i)_{i\geq 0}$ to P_M, (with $\mathrm{tr}_e(f_i) = 0$ for all but finitely many i-lemma 4.2.1) We define the (e,n) trace of f by

$$\mathrm{tr}_e(f) = \sum_{i=1}^{\infty} (-1)^i \,\mathrm{tr}_e(f_i) \in \Lambda_e/[\Lambda_e, \Lambda_e]$$

Lemma 4.3.2. *Let e be an idempotent $\in \Lambda$. Then the e-trace is well defined for endomorphisms of modules $\in \mathrm{mod}\,\Lambda$ with an e-bounded projective resolution.*

Proof. Let M be a module $\in \mathrm{mod}\,\Lambda$ with following projective resolution that is e-bounded.

$$P_M = \ \dots \longrightarrow P_i \xrightarrow{d_i} P_{i-1} \longrightarrow \dots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0$$

Let $f \in \mathrm{End}_\Lambda(M)$.

- First we show that $\mathrm{tr}_e(f)$ does not depend on the choice of (f_i) the lifting of f. It will be enough to show that $\sum_{i=1}^{\infty} (-1)^i \,\mathrm{tr}_e(f_i) = 0$ (by HS1) for any lifting (f_i) of the zero endomorphism of M. So let $h_i : P_i \to P_{i+1}$ be some maps, such that $f_0 = d_1 h_0$ and $f_i = d_{i+1}h_i + h_{i-1}d_i$. By HS1 and HS2 $\mathrm{tr}_e(f_i) = \mathrm{tr}_e(d_{i+1}h_i) + \mathrm{tr}_e(h_{i-1}d_i)$

  $= \mathrm{tr}_e(d_{i+1}h_i) + \mathrm{tr}_e(d_i h_{i-1}) \forall i \geq 1$. Also on the other hand by assumption, $\exists$ some $s \geq 0$ such that $top(P_i).e = 0$, for $i \geq s$. Then by lemma 4.2.1 $\mathrm{tr}_e(d_{s+1}h_s) = 0$ and

$\mathrm{tr}_e(f_i) = 0 \quad \forall i \geq s$. And we have

$$\begin{aligned}\sum_{i=1}^{\infty}(-1)^i \mathrm{tr}_e(f_i) &= \mathrm{tr}(f_0) + \sum_{i=1}^{s}(-1)^i \mathrm{tr}_e(f_i) \\ &= \mathrm{tr}_e(d_1h_0) + \sum_{i=1}^{s}(-1)^i(\mathrm{tr}_e(d_{i+1}h_i) + \mathrm{tr}_e(d_ih_{i-1})) \\ &= (-1)^s \mathrm{tr}(d_{s+1}h_s) \\ &= 0\end{aligned}$$

- Next if M has another e- bounded projective resolution,

$$P'_M = \ \ldots \longrightarrow P'_i \xrightarrow{d'_i} P'_{i-1} \longrightarrow \ldots \longrightarrow P'_1 \xrightarrow{d'_1} P'_0 \xrightarrow{d'_0} M \longrightarrow 0.$$

Let $\alpha_i : P_i \to P'_i$ (for $i \geq 0$) be a lifting of I_M and $\beta_i : P'_i \to P_i$ (for $i \geq 0$) be a lifting of f. Then $(\alpha_i\beta_i)$ and $(\beta_i\alpha_i)$ are liftings of f. Actually, we have,

$$P_i \xrightarrow{\alpha_i} P'_i \xrightarrow{\beta_i} P_i \xrightarrow{\alpha_i} P'_i$$

and by HS2

$$\sum_{i=1}^{\infty}(-1)^i \mathrm{tr}_e(\alpha_i\beta_i) = \sum_{i=1}^{\infty}(-1)^i \mathrm{tr}_e(\beta_i\alpha_i)$$

done!

Next we look at a results which says that e-trace has this property that it is additive.

Proposition 4.3.3. *Let e be an idempotent $\in \Lambda$. If we have the following commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & L & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow h & & \downarrow g & & \\ 0 & \longrightarrow & M & \longrightarrow & L & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

with exact rows. If M, N have an e-bounded projective resolution then so does L and $\mathrm{tr}_e(h) = \mathrm{tr}_e(f) + \mathrm{tr}_e(g)$.

Proof. Let P_M and P_N be e-bounded projective resolutions of M and N respectively as follows:

$$P_M = \ \ldots \longrightarrow P_i \xrightarrow{d_i} P_{i-1} \longrightarrow \ldots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \longrightarrow 0$$

and

$$P_N = \ \ldots \longrightarrow P'_i \xrightarrow{d'_i} P'_{i-1} \longrightarrow \ldots \longrightarrow P'_1 \xrightarrow{d'_1} P'_0 \xrightarrow{d'_0} N \longrightarrow 0$$

Then by the horseshoe lemma there exist following set up with exact row where the squares commute

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & P_i & \xrightarrow{d_i} & P_{i-1} & \longrightarrow \ldots \longrightarrow & P_0 & \xrightarrow{d_0} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u_i} & & \downarrow{\scriptstyle u_{i-1}} & & \downarrow{\scriptstyle u_0} & & \downarrow{\scriptstyle u} & & \\
\ldots & \longrightarrow & P_i \oplus P'_i & \xrightarrow{d''_i} & P_{i-1} \oplus P'_{i-1} & \longrightarrow \ldots \longrightarrow & P_0 \oplus P'_0 & \xrightarrow{d''_0} & L & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle v_i} & & \downarrow{\scriptstyle v_{i-1}} & & \downarrow{\scriptstyle v_0} & & \downarrow{\scriptstyle v} & & \\
\ldots & \longrightarrow & P'_i & \xrightarrow{d'_i} & P'_{i-1} & \longrightarrow \ldots \longrightarrow & P'_0 & \xrightarrow{d'_0} & N & \longrightarrow & 0
\end{array}$$

where $u_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_i = (0, 1) \quad \forall i \geq 0$. We have that the middle sequence is an e-bounded projective resolution of L denoted by P_L. Therefore choosing (f_i) and (g_i) liftings of f and g respectively.it is a well known see [5,p. 46] that there exist a lifting (h_i) of h

such that following diagram commute

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_i & \xrightarrow{u_i} & P_i \oplus P'_i & \xrightarrow{v_i} & P'_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle h_i} & & \downarrow{\scriptstyle g_i} & & \\
0 & \longrightarrow & P_i & \xrightarrow{u_i} & P_i \oplus P'_i & \xrightarrow{v_i} & P'_i & \longrightarrow & 0
\end{array}$$

for every $i \geq 0$. As $h_i u_i = u_i f_i$ and $g_i v_i = v_i h_i$, we may choose to write h_i as a (2×2) matrix whose diagonal entries are f_i and g_i. So by HS3 $\mathrm{tr}_e(h_i) = \mathrm{tr}_e(f_i) + \mathrm{tr}_e(g_i)$. Hence $\mathrm{tr}_e(h) = \mathrm{tr}_e(f) + \mathrm{tr}_e(g)$.

4.4 Main result

Through out we let $S_e = e\Lambda/eJ$

Lemma 4.4.1. *Let e be an idempotent $\in \Lambda$, with S_e of finite injective dimension. Then the e-trace is defined for every endomorphism in* $\mathrm{mod}\,\Lambda$.

Proof. Let m be the injective dimension of S_e. Let M be a Λ-module. Where the following sequence

$$P_M = \ \ldots \longrightarrow P_i \longrightarrow P_{i-1} \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is the minimal projective resolution of M. Then it follows from our result in [chapter 1,2] that $\mathrm{Hom}_\Lambda(P_i, S_e)$= $\mathrm{Ext}^i_\Lambda(M, S_e)$ and for each $i \geq m$, $\mathrm{Ext}^i_\Lambda(M, S_e) = 0$. So, $\mathrm{top}(P_i).e = 0$. Thus P_M is e-bounded. Hence $\mathrm{tr}_e(f)$ is defined for every $f \in \mathrm{End}(M)$.

Next we state as main result of this chapter-theorem 4.4.2 which is needful in our main work in chapter 5.

Theorem 4.4.2. *Let Λ be an artin algebra, and let e be an idempotent $\in \Lambda$. If S_e is of finite injective dimension, then $\mathrm{HH}_0(\Lambda_e)$ is radical-trivial.*

Proof. Let S_e be of finite injective dimension. Then it follows from lemma 4.4.1, that the e-trace is defined for every endomorphism $\in \operatorname{mod} \Lambda$. Consider $z \in \Lambda$ such that $\bar{z}$ lies in the radical of Λ_e, which is $(eJe + \Lambda(1-e)\Lambda)/\Lambda(1-e)\Lambda$. Hence, $\bar{z} = \bar{a}$ for some $a \in eJe$. Let $n \geq 0$ be such that $a^n = 0$ in Λ. Consider the following chain of submodules of Λ:

$$0 = a^n\Lambda \subseteq a^{n-1}\Lambda \subseteq ... \subseteq a^1\Lambda \subseteq a^0\Lambda = \Lambda$$

Let $f_0 : \Lambda \to \Lambda$ be the left multitplication by a. Because $f_0(a^i\Lambda) \subseteq a^{i+1}\Lambda$, we have that, f_0 induces morphisms $f_i : a^i\Lambda \to a^i\Lambda$, for $i = 0, ..., n$. Thus, we have the following commutative diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & a^{i+1}\Lambda & \longrightarrow & a^i\Lambda & \longrightarrow & a^i\Lambda/a^{i+1}\Lambda & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_{i+1}} & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & a^{i+1}\Lambda & \longrightarrow & a^i\Lambda & \longrightarrow & a^i\Lambda/a^{i+1}\Lambda & \longrightarrow & 0. \end{array}$$

Then by proposition 4.3.3, $\mathrm{tr}_e(f_i) = \mathrm{tr}_e(f_{i+1})$ for $i = 0, 1, ..., n-1$. Applying HS5, we have,

$$0 = \mathrm{tr}_e(f_n) = ... = \mathrm{tr}_e(f_0) = \bar{a} + [\Lambda_e, \Lambda_e].$$

Hence $\bar{z} = \bar{a} \in [\Lambda_e, \Lambda_e]$.

Chapter 5

Proof of the strong no loop conjecture

5.1 Establishing the result for artin algebras

The main task is to apply the previously gathered results to solve the strong no loop conjecture for finite dimesional algebras over an algebraically closed field. We start with some needed terminology. Let Λ be an artin algebra and e a primitive idempotent in Λ. We say Λ is locally commutative at e if $e\Lambda e$ is commutative. Also Λ is locally commutative if it is locally commutative at every primitive idempotent. We also say e is basic if $e\Lambda$ is not isomorphic to any direct summand of $(1-e)\Lambda$.

Proposition 5.1.1. *Let Λ be an artin algebra, and let e be a basic primitive idempotent in Λ such that Λ/J^2 is locally commutative at $e+J^2$. Suppose S_e is of finite projective or injective dimension, then $\operatorname{Ext}^1_\Lambda(S_e, S_e) = 0$.*

Proof.

- Let first suppose S_e is of finite injective dimension. To prove $\operatorname{Ext}^1_\Lambda(S_e, S_e) = 0$, it suffice to show that $eJe/eJ^2e = 0$. Let z be in eJe. Then it follows from Theorem 4.4.2 that $z + \Lambda(1-e) \in [\Lambda_e, \Lambda_e]$. By assumption e is basic so $e\Lambda(1-e)\Lambda e \subsetneq eJ^2e$. Then we have the following algebra homomorphism

$$\phi : \Lambda_e \to e\Lambda e/eJ^2e;$$
$$y + \Lambda(1-e)\Lambda \mapsto eye + eJ^2e.$$

  So that $\phi(z+\Lambda(1-e)\Lambda) = eze+\Lambda(1-e)\Lambda$. And we have that $eze+\Lambda(1-e)\Lambda = z+\Lambda(1-e)\Lambda$ lies in the commutator group of $e\Lambda e/eJ^2$. As by assumption $(e+J^2)(\Lambda/J^2)(e+J^2)$ is commutative. It follows that $e\Lambda e/eJ^2e \cong (e+J^2)(\Lambda/J^2)(e+J^2)$ is commutative. So $z + eJ^2e = 0$, and we have $z \in eJ^2e$. Hence the result follows.

- Next, suppose S_e is of finite projective dimension. Let D be the standard duality between $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Lambda^{op}$. Then we have that $D(S_e)$ is a simple module in Λ^{op} of finite injective dimension with support from e^o, the primitive idempotent that correspond to e. Since e^o is basic, such that the quotient of Λ^{op} modulo the square of its radical is locally commutative at the class of e^o, we have $\operatorname{Ext}^1_{\Lambda^{op}}(D(S_e), D(S_e)) = 0$. Hence $\operatorname{Ext}^1_\Lambda(S_e, S_e) = 0$.

In our next theorem 5.1.2 we specialise the result to finite dimensional algebras over a field.

Theorem 5.1.2. *Let Λ be a finite dimensional algebra over a field K, and let S be a simple Λ-module of K-dimension one. If S is of finite projective or injective dimension, then* $\operatorname{Ext}^1_\Lambda(S,S) = 0$.

Proof. Let e be a primitive idempotent in Λ such that $S.e \neq 0$. Then Λ admits a complete set $\{e_1, ..., e_n\}$ of orthogonal primitive idempotents where $e = e_1$. If $e_1\Lambda, ..., e_s\Lambda$ for $1 \leq s \leq n$ are the non- isomorphic indecomposable projective modules $\in \operatorname{mod}\Lambda$. Then we have,

$$\Lambda/J \cong M_{n_1}(D_1) \times ... \times M_{n_s}(D_s).$$

Where $D_i = \operatorname{End}_\Lambda(e_i\Lambda/e_iJ)$ and n_i is the number of indices j for $1 \leq j \leq n$ such that $e_i\Lambda \cong e_j\Lambda$, with $i = 1, ..., s$. We observe that S is a simple $M_{n_1}(D_1)$-module, and thus $S \cong D_1^{n_1}$. As S is one dimensional over K, it is one dimensional over D_1. In particular, $n_1 = 1$. Thus e is a basic primitive dempotent. Furthermore, $e\Lambda e/eJe \cong S_e \cong K$. That is, for $y_1, y_2 \in e\Lambda e$ we write $y_i = \lambda_i e + z_i$, with $\lambda_i \in K$ and $z_i \in eJe$, for $i = 1, 2$. Consequently, $y_1y_2 - y_2y_1 = z_1z_2 - z_2z_1 \in eJ^2e$. Hence $e\Lambda e/eJ^2e$ is commutative and so is $(e+J^2)(\Lambda/J^2)(e+J^2)$. Therefore by proposition 5.1.1 it follows that $\operatorname{Ext}^1_\Lambda(S,S) = 0$.

Remark. We say a finite dimensional algebra over a field is elementary if its simple modules are all one dimensional over the base field, or equivalently, it is isomorphic to an algebra given by a quiver with relations; see[16]. It is a well known result that finite dimensional algebras over an algebraically closed field is Morita equivalent to an elementary algebra. In this regard theorem 5.2.1 in our next section confirms the stong no loop conjecture for finite dimensional elementary algebra over any field and in particular for finite dimensional algebras over an algebraically closed field.

5.2 SOME GENERALISATIONS

Before we extend our results further. We start with some terminology required.

- From now on, we let Λ stands for a finite dimensional elementary algebra over a field K which is isomorphic to an algebra given by a quiver with relations.
- Let Q be a finite quiver and I an admissible ideal in kQ.
- An element $\rho = \lambda_1 p_1 + ... + \lambda_s p_s$ in I. where the p_i are distinct paths in Q from one fixed vertex to another and λ_i are nonzero scalars in K. We say ρ is a minimal relation for Λ if no proper summand of ρ is in I or equivalently if $\sum_{i\in\omega} \lambda_i p_i \notin I$ for any $\omega \subset \{1, ..., s\}$.
- Given an oriented cycle $\sigma = \alpha_1\alpha_2...\alpha_s$ in Q where the α_i are arrows. We denote supp(σ) the set of vertices occuring as starting points of $\alpha_1\alpha_2...\alpha_s$ and define the idempotent supporting

σ to be the sum of all primitive idempotents $\in \Lambda$ associated to the vertices in supp(σ). Also we set $\sigma_1 = \sigma$, and for $2 \leq i \leq s$, we have $\sigma_i = \alpha_i \alpha_{i+1} ... \alpha_{i-1}$ as the cyclic permutations of σ.

- The idempotent supporting σ is the "smallest" idempotent e such that $e\sigma_i = \sigma_i$ for all i.

- We say that σ is cyclically non-zero (respectively, cyclically free) in Λ if none of the σ_i for $1 \leq i \leq s$ is a summand of a minimal relation. For example a loop in Q is always cyclically free in λ .

Theorem 5.2.1. *Let $\Lambda = kQ/I$ with Q a finite quiver and I an admissible ideal in kQ, and let σ be an oriented cycle in Q with supporting idempotent $e \in \Lambda$. If σ is cyclically free in Λ, then S_e is of infinite projective and injective dimensions.*

Proof. Let assume that σ be cyclically free in Λ. If σ is a power of a shorter oriented cycle δ, then we see that δ is also cyclically free in Λ and supp(σ)=supp(δ). Hence let suppose that σ is not a power of any shorter oriented cycle. It is a well known that the cyclic permutations $\sigma_1, ..., \sigma_s$ of σ, where $\sigma_1 = \sigma$, are pairwise distinct.
For any $y \in KQ$, we denote by $\bar{y}$ its class in Λ and by $\tilde{y}$ the class of $\bar{y}$ in Λ_e. Let V be the subspace of Λ_e spanned by the classes $\tilde{p}$, where p ranges over the paths in Q different from $\sigma_1, ...\sigma_s$. Then, $\exists$ paths $p_1, ..., p_t$ in Q different from $\sigma_1, ...\sigma_s$ such that $\{\tilde{p_1}, ..., \tilde{p_t}\}$ is a K-basis of V. The claim is that $\{\tilde{\sigma_1}, ..., \tilde{\sigma_s}, \tilde{p_1}, ..., \tilde{p_t}\}$ is a K-basis of Λ_e. To see that this spans Λ_e. Suppose that

$$\sum_{i=1}^{s} \lambda_i \tilde{\sigma_i} + \sum_{j=1}^{t} \mu_i \tilde{p_j} = \tilde{0}, \lambda_i, \quad \mu_j \in K$$

Thus $\sum_{i=1}^{s} \lambda_i \bar{\sigma_i} + \sum_{j=1}^{t} \mu_i \bar{p_j} \in \Lambda(1-e)\Lambda$. Then we can write

$$\sum_{i=1}^{s} \lambda_i \bar{\sigma_i} + \sum_{j=1}^{t} \mu_i \bar{p_j} = \sum_{l=1}^{r} \beta_l \bar{q_l}, \quad \beta_l \in K,$$

where $q_1, ..., q_r$ are some distinct paths in Q passing through a vertex $\in supp(\sigma)$. Put some m where $1 \leq m \leq s$ and let ϵ_m be the stationary path in Q associated to the starting point x_m of σ_m, then we have

$$\rho = \sum_{i=1}^{s} \lambda_i (\epsilon_m \sigma_i \epsilon_m) + \sum_{j=1}^{t} \mu_i (\epsilon_m p_j \epsilon_m) - \sum_{l=1}^{r} \beta_l (\epsilon_m q_l \epsilon_m) \in I$$

Also the non-zero elements of the $\epsilon_m \sigma_i \epsilon_m$, $\epsilon_m p_j \epsilon_m$, $\epsilon_m q_l \epsilon_m \in KQ$ are distinct oriented cycles from x_m to x_m. If $\lambda_m \neq 0$, then $\lambda(\epsilon_m \sigma_m \epsilon_m)$ would be a summand of a minimal non-zero summand ρ' of ρ where ρ' is in I. By definition ρ' is a minimal relation for Λ, but that will contradict the assumption that σ is cyclically free in Λ. So then λ_m must be zero. So we get $\lambda_1, ..., \lambda_s$ and consequently $\mu_1, ..., \mu_t$ to be all zero confirming our claim. Now let suppose that $\tilde{\sigma} \in [\Lambda_e, \Lambda_e]$. Then,

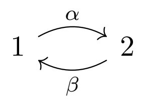
$$\tilde{\sigma} = \sum_{i=1}^{n} \nu_i (\tilde{a_i}\tilde{b_i} - \tilde{b_i}\tilde{a_i}) \tag{5.1}$$

with $\nu_i \in k$ and $a_i, b_i \in \{\sigma_1, ..., \sigma_s, p_1, ..., p_t\}$. For each $1 \leq i \leq n$, $a_i b_i \notin \{\sigma_1, ..., \sigma_s\}$ if and only if $b_i a_i \notin \{p_1, ..., p_t\}$, and in this case, $\tilde{a_i}\tilde{b_i} - \tilde{b_i}\tilde{a_i} \in V$. Then Equation 5.1 becomes

$$\tilde{\sigma} = \sum_{i=1}^{n} \nu_{ij} (\tilde{\sigma_i} - \tilde{\sigma_j}) + v, \tag{5.2}$$

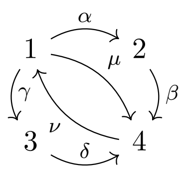
with $\nu_{ij} \in K$ and $v \in V$. Set T to be the linear form on Λ_e, which map each of $\tilde{\sigma_1}, ..., \tilde{\sigma_s}$ to 1 and vanishes on V. As $\sigma = \sigma_1$, and applying T to Equation 5.2 on the LHS $T(\tilde{\sigma}) = 1$ whiles on the RHS $T(\sum_{i=1}^{n} \nu_{ij}(\tilde{\sigma_i} - \tilde{\sigma_j}) + v)$ yields 0, but $1 = 0$, is a contradiction. Thus the class of $\tilde{\sigma}$ in the commutator group of Λ_e is non zero. As $\tilde{\sigma}$ lies in the radical of Λ_e, it then follows from Theorem 4.4.2 that S_e is of infinite projective and injective dimensions. Done!

Example 5.2.2. Consider $\Lambda = kQ/I$, where Q is the quiver



and I is the ideal in kQ generated by $\alpha\beta\alpha$. The oriented cycle $\beta\alpha$ is cyclically free in Λ. By theorem 5.2.1 one of the simple modules S_1,S_2 has infinite projective dimension and one has infinite injective dimension. Precisely $\text{pdim}(S_1) = \infty$ and $\text{Injdim}(S_2) = \infty$.

Example 5.2.3. Let $\Lambda = kQ/I$, where Q is the following quiver



and I is an ideal in kQ generated by $\alpha\beta - \gamma\delta$, $\beta\nu, \nu\mu\nu$. The oriented cycle $\mu\nu$ is cyclically free in Λ. By theorem 5.2.1 one of the simple modules S_1,S_4 has infinite projective dimension and one has infinite injective dimension. Precisely $\text{pdim}(S_1) = \infty$ and $\text{Injdim}(S_4) = \infty$.

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