

ROBUST NUMERICAL METHODS FOR NONLOCAL (AND LOCAL) EQUATIONS OF POROUS MEDIUM TYPE. PART II: SCHEMES AND EXPERIMENTS*

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Abstract. We develop a unified and easy to use framework to study robust fully discrete numerical methods for nonlinear degenerate diffusion equations $\partial_t u - \mathfrak{L}[\varphi(u)] = f(x, t)$ in $\mathbb{R}^N \times (0, T)$, where \mathfrak{L} is a general symmetric Lévy-type diffusion operator. Included are both local and nonlocal problems with, e.g., $\mathfrak{L} = \Delta$ or $\mathfrak{L} = -(-\Delta)^{\frac{\alpha}{2}}$, $\alpha \in (0, 2)$, and porous medium, fast diffusion, and Stefan-type nonlinearities φ . By robust methods we mean that they converge even for nonsmooth solutions and under very weak assumptions on the data. We show that they are L^p -stable for $p \in [1, \infty]$, compact, and convergent in $C([0, T]; L^p_{\text{loc}}(\mathbb{R}^N))$ for $p \in [1, \infty)$. The first part of this project is given in [F. del Teso, J. Endal, and E. R. Jakobsen, preprint, arXiv:1801.07148v1 [math.NA], 2018] and contains the unified and easy to use theoretical framework. This paper is devoted to schemes and testing. We study many different problems and many different concrete discretizations, proving that the results of Part I apply and testing the schemes numerically. Our examples include fractional diffusions of different orders and Stefan problems, porous medium, and fast diffusion nonlinearities. Most of the convergence results and many schemes are completely new for nonlocal versions of the equation, including results on high order methods, the powers of the discrete Laplacian method, and discretizations of fast diffusions. Some of the results and schemes are new even for linear and local problems.

Key words. fully discrete, numerical schemes, convergence, uniqueness, distributional solutions, nonlinear degenerate diffusion, porous medium equation, fast diffusion equation, Stefan problem, fractional Laplacian, Laplacian, nonlocal operators, existence, a priori estimates

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1. Introduction. We develop a unified and easy to use framework for fully discrete monotone numerical methods of finite difference type for a large class of possibly degenerate nonlinear diffusion equations of porous medium type:

$$(1.1) \quad \partial_t u - \mathfrak{L}^{\sigma, \mu}[\varphi(u)] = f(x, t) \quad \text{in} \quad Q_T := \mathbb{R}^N \times (0, T),$$

$$(1.2) \quad u(x, 0) = u_0(x) \quad \text{on} \quad \mathbb{R}^N,$$

where u is the solution, φ continuous and nondecreasing, and $T > 0$. The diffusion operator $\mathfrak{L}^{\sigma, \mu}$ is given as

$$(1.3) \quad \mathfrak{L}^{\sigma, \mu} := L^\sigma + \mathcal{L}^\mu$$

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with local and nonlocal (anomalous) parts,

$$(1.4) \quad L^\sigma[\psi](x) := \operatorname{tr}(\sigma\sigma^T D^2\psi(x)),$$

$$(1.5) \quad \mathcal{L}^\mu[\psi](x) := \int_{\mathbb{R}^N \setminus \{0\}} (\psi(x+z) - \psi(x) - z \cdot D\psi(x)\mathbf{1}_{|z|\leq 1}) \, d\mu(z),$$

where $\sigma = (\sigma_1, \dots, \sigma_P) \in \mathbb{R}^{N \times P}$ for $P \in \mathbb{N}$ and $\sigma_i \in \mathbb{R}^N$, D and D^2 are the gradient and Hessian, $\mathbf{1}_{|z|\leq 1}$ is a characteristic function, and μ is a nonnegative symmetric measure.

Remark 1.1. By the symmetry of μ , $\lim_{r \rightarrow 0^+} \int_{r < |z| \leq 1} z \, d\mu(z) = 0$, and we have an equivalent definition of \mathcal{L}^μ in (1.5) in terms of a principal value (P.V.) integral:

$$\mathcal{L}^\mu[\psi](x) = \lim_{r \rightarrow 0^+} \int_{|z| > r} (\psi(x+z) - \psi(x)) \, d\mu(z) = \text{P.V.} \int_{|z| > 0} (\psi(x+z) - \psi(x)) \, d\mu(z).$$

The assumptions we impose on $\mathfrak{L}^{\sigma, \mu}$ and φ are so mild that many different problems can be modeled by (1.1): Flow in porous media, nonlinear heat transfer, phase transitions, and population dynamics; see, e.g., [66] for local problems and [70, 57, 14, 67] for nonlocal problems. Important examples are strongly degenerate Stefan problems with $\varphi(u) = \max(0, au - b)$, $a \geq 0$, and the full range of porous media and fast diffusion equations with $\varphi(u) = u|u|^{m-1}$ for any $m \geq 0$. The class of diffusion operators $\mathfrak{L}^{\sigma, \mu}$ coincides with the generators of the *symmetric* Lévy processes [7, 64, 4] and includes, e.g., the Laplacian Δ , fractional Laplacians $-(-\Delta)^{\frac{\alpha}{2}}$, $\alpha \in (0, 2)$, relativistic Schrödinger operators $m^\alpha I - (m^2 I - \Delta)^{\frac{\alpha}{2}}$, and even discretizations of these. Since σ and μ may be degenerate or even identically zero, problem (1.1) can be purely nonlocal, purely local, or a combination. An additional challenge both analytically and numerically is the fact that solutions of (1.1) in general can be very irregular and even discontinuous.

The numerical schemes will be defined on a grid (x_β, t_j) as follows:

$$(1.6) \quad U_\beta^j = U_\beta^{j-1} + \Delta t_j (\mathcal{L}_1^h[\varphi_1^h(U^j)]_\beta + \mathcal{L}_2^h[\varphi_2^h(U^{j-1})]_\beta + F_\beta^j),$$

where $U_\beta^j \approx u(x_\beta, t_j)$, $\mathcal{L}_1^h + \mathcal{L}_2^h \approx \mathfrak{L}^{\sigma, \mu}$, $\varphi_i^h \approx \varphi$, $F_\beta^j \approx f(x_\beta, t_j)$, and h and Δt_j are the discretization parameters, and the discrete diffusion operators \mathcal{L}_i^h have a monotone difference representation

$$\mathcal{L}_i^h[\psi](x) = \sum_{\beta \neq 0} (\psi(x + z_\beta) - \psi(x)) \omega_{i, \beta} \quad \text{for } \omega_{i, \beta} \geq 0.$$

As we will see, different choices of $\varphi_1^h, \varphi_2^h, \mathcal{L}_1^h, \mathcal{L}_2^h$ lead to explicit, implicit, θ -methods, and various explicit-implicit methods. In a simple one dimensional case,

$$\partial_t u = \varphi(u)_{xx} - (-\partial_x^2)^{\alpha/2} \varphi(u),$$

an example of a discretization in our class is given by

$$\begin{aligned} U_m^j &= U_m^{j-1} + \frac{\Delta t}{h^2} \left(\varphi(U_{m+1}^j) - 2\varphi(U_m^j) + \varphi(U_{m-1}^j) \right) \\ &\quad + \Delta t \sum_{k \neq 0} \left(\varphi(U_{m+k}^{j-1}) - \varphi(U_m^{j-1}) \right) \int_{(k-\frac{1}{2})h}^{(k+\frac{1}{2})h} \frac{c_{N, \alpha} \, dz}{|z|^{N+\alpha}}. \end{aligned}$$

The main result of the first part of this project [36] was a unified, rigorous, and easy to use theoretical framework for these schemes. This novel analysis includes well-posedness, L^p -stability, equicontinuity, compactness, and L^p_{loc} -convergence results. These results are very general since they hold for local and nonlocal, linear and nonlinear, nondegenerate and degenerate, and smooth and nonsmooth problems. An important new idea is to work in a sufficiently general class of solutions of (1.1) that allows for atomic (nonabsolutely continuous) measures μ in the definition of $\mathfrak{L}^{\sigma,\mu}$. Since the discrete operator \mathcal{L}^h is a nonlocal operator \mathcal{L}^ν with $\nu := \sum_{\beta \neq 0} (\delta_{z_\beta} + \delta_{z_{-\beta}})\omega_\beta$, it is in the form of $\mathfrak{L}^{\sigma,\mu}$ and can be analyzed with the same powerful PDE techniques. This analysis requires recent uniqueness results for (1.1) obtained by the authors in [35, 34]—results for bounded distributional solutions or very weak solutions of (1.1) in the generality needed here. The fact that we can use such a weak notion of solution both simplifies the analysis and makes a global theory for all the different problems and schemes we consider here possible. At this point the reader should note that if (1.1) has more regular (bounded) solutions (weak, strong, mild, or classical), then our results still apply because these solutions will coincide with our (unique) distributional solution.

Schemes that converge in such general circumstances are often said to be *robust*. Consistent numerical schemes are not robust in general, i.e., they need not always converge, or can even converge to false solutions. Such issues are seen especially in nonlinear, degenerate, and/or low regularity problems. Our general results are therefore only possible because we have (i) identified a class of schemes with good properties (including monotonicity) and (ii) developed the new mathematical techniques needed to analyze these schemes in the current generality.

In this paper, which is the second part of this project, we have two main objectives: (1) to give many concrete discretizations that fall into the theoretical framework of the first part [36], and (2) to test and verify numerically a number of these schemes for a wide and representative number of examples of problems of the form (1.1).

The scheme (1.6) is essentially determined as soon as we specify \mathcal{L}_i^h and φ_i^h , the discretizations of $\mathfrak{L}^{\sigma,\mu}$ and φ . The whole of section 4 is devoted to such concrete discretizations. We start by splitting the diffusion operator $\mathfrak{L}^{\sigma,\mu}$ into local, singular nonlocal, and bounded nonlocal parts, and then explain how these parts can be discretized separately. For the local part, we consider classical finite difference methods and in this context new semi-Lagrangian methods. For the singular nonlocal part, we analyze the trivial discretization and the adapted vanishing viscosity approximation, and, finally, for the bounded nonlocal part, we consider quadrature methods obtained from interpolation in two different ways. For the first time we apply the so-called powers of the discrete Laplacian method (when $\mathfrak{L}^{\sigma,\mu} = -(-\Delta)^\alpha$) to diffusion problems, and we explain how non-Lipschitz (including fast diffusion) nonlinearities φ have to be approximated to get good explicit schemes.

In every case we check that the discretizations satisfy the conditions of the theoretical framework of [36], and hence we prove that schemes (1.6) involving these discretization are L^p -stable for $p \in [1, \infty]$ and L^p_{loc} -convergent for $p \in [1, \infty)$. We also compute the local truncation errors, and we explain how to combine the methods to get better than first order methods for problems involving fractional Laplace like operators and very high order methods for bounded nonlocal operators. The powers of the discrete Laplacian method is shown to be an order 2 method regardless of the value of $\alpha \in (0, 2)$. Many of these schemes and most of the convergence results are new in this context, sometimes even in the linear case. This is especially the case for nonlocal problems. Some important examples here are the following:

- (i) the first high order methods for nonlinear nonlocal diffusions of porous medium type (but see also [43]);
- (ii) the first time the powers of the discrete Laplacian method is applied to nonlinear problems; and
- (iii) the first numerical methods and simulations for nonlocal problems with non-Lipschitz (“fast diffusion”) nonlinearities.

We also mention that our results provide a rigorous justification for the numerical simulations of [10, section 7] for a nonlocal Stefan problem with discontinuous solutions; see Remark 5.1 for more details.

Numerical tests are presented in sections 5–6. We focus on nonlocal problems since there are many fewer results for such problems in the literature, especially for porous-medium-type equations. For simplicity, we take the diffusion operator $\mathfrak{L}^{\sigma,\mu}$ to be the fractional Laplacian $-(-\Delta)^{\frac{\alpha}{2}}$ in section 5. All the well-known one dimensional special cases of (1.1) are then considered: the linear fractional heat equation, the fractional porous medium equation [31], and fractional equations with fast diffusion and Stefan-type nonlinearities [31, 10, 3]. In each problem we test and compare four different numerical schemes for different powers α of the fractional Laplacian. Most test problems are set up to have smooth exact solutions, and the numerical tests confirm the theoretical results, in most cases also including the truncation error bounds and the expected convergence rates.

Note that in the Stefan case, we expect that $\varphi(u) \in C^\gamma$ for some $\gamma \in (0, 1]$, but this is not enough to ensure that $-(-\Delta)^{\frac{\alpha}{2}}[\varphi(u)]$ exists pointwise when $\alpha \geq \gamma$. If this is the case, the scheme does not converge in L^∞ , but it still converges in L^1 with the expected rates given by the local truncation error. See section 5.3 for the details.

We also produce a Stefan-type example where the numerical solution is nondifferentiable. Finally, in section 6 we test a much more complicated problem in two dimensions: A Stefan problem with degenerate local and nonlocal diffusion and non-smooth castle like initial data.

To perform the numerical computations mentioned above, we have restricted the scheme to a (large) bounded domain and set the numerical solution equal to zero outside. Convergence of the scheme then requires the size of the computational domain to increase as the grid is refined. We briefly discuss the error introduced by the restriction to a bounded domain in section 5.5.

Related work. In the local linear case, when $\varphi(u) = u$ and $\mu \equiv 0$ in (1.1), numerical methods and analysis can be found in undergraduate text books. In the nonlinear case there is a very large literature so we will focus only on some developments that are more relevant to this paper. For porous medium nonlinearities ($\varphi(u) = u|u|^{m-1}$ with $m > 1$), there are early results on finite element and finite difference interface tracking methods in [62] and [38] (see also [59]). There is extensive theory for finite volume schemes; see [48, section 4] and references therein for equations with locally Lipschitz φ . For finite element methods there is a number of results, including results for fast diffusions ($m \in (0, 1)$), Stefan problems, convergence for strong and weak solutions, and discontinuous Galerkin methods; see, e.g., [63, 45, 46, 44, 72, 61, 58]. Note that the latter paper considers the general form of (1.1) with $\mathfrak{L}^{\sigma,\mu} = \Delta$ and provides a convergence analysis in L^1 . A number of results on finite difference methods for degenerate convection-diffusion equations also yield results for (1.1) in special cases; see, e.g., [47, 13, 55, 54]. In particular the results of [47, 55] imply our convergence results for a particular scheme when φ is locally Lipschitz, $\mathfrak{L}^{\sigma,\mu} = \Delta$, and solutions have a certain additional BV-regularity. Finally, we mention very general results on so-called gradient schemes [40, 41] for doubly or triply degenerate parabolic

equations. This class of equations include local porous-medium-type equations as a special case.

In the nonlocal case, the literature is more recent and not so extensive. For the linear case we refer somewhat arbitrarily to [26, 50, 51, 60] and references therein. Here we also mention [28] and its novel finite element plus semigroup subordination approach to discretizing $\mathcal{L}^{\sigma,\mu} = -(-\Delta)^{\frac{\alpha}{2}}$. Some early results for nonlocal problems came from finite difference quadrature schemes for Bellman equations and fractional conservation laws; see [53, 17, 8] and [39]. For the latter case discontinuous Galerkin and spectral methods were later studied in [25, 23, 71]. The first results that include nonlinear nonlocal versions of (1.1) was probably given in [22]. There, convergence of finite difference quadrature schemes was proven for a convection-diffusion equation. This result is extended to more general equations and error estimates in [24] and a higher order discretization in [43]. In some cases our convergence results follow from these results (for two particular schemes, $\sigma = 0$, and φ locally Lipschitz). However, the analysis there is different and more complicated since it involves entropy solutions and Kruřkov doubling of variables arguments.

In the purely parabolic case (1.1), the behavior of the solutions and the underlying theory is different from the convection-diffusion case (especially so in the nonlocal case; see, e.g., [30, 31, 68, 29, 69] and [42, 18, 1, 22, 2, 52]). It is therefore important to develop numerical methods and analysis that are specific for this setting. The first (nonlocal) results in this direction seem to be [33, 37]. These papers are based on the extension method [15], and introduce and analyze finite difference methods for the fractional porous medium equation. The present work is possibly the first not to use the extension method or the regularity of the solution.

Outline. The next section is a short section where we collect the assumptions and well-posedness results for the porous-medium-type equation (1.1). In section 3 we formulate the numerical schemes and state the main theoretical results. This is a slightly simplified version of the theoretical framework of Part 1 of this project [36]. We also give a couple of new results that will greatly simplify the verification of the assumptions of this framework. The main contributions of this paper are then given in the two sections that follow. In section 4 we introduce the concrete discretizations and prove rigorously that they fall into our theoretical framework, while in sections 5–6 we present our numerical simulations for all the well-known special cases of (1.1).

2. Preliminaries. In this section we present the assumptions and well-posedness results for the initial value problem (1.1)–(1.2). In this paper we work in the setting of bounded distributional solutions. This is very convenient for numerical analysis since it leads to an easy to work with convergence theory that applies even to very bad problems. By uniqueness it also applies to situations where solutions are more regular, e.g., classical, strong, weak/energy, or mild solutions.

Following [34] (see also [11, 35]) we use the following assumptions:

- (A $_{\varphi}$) $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is nondecreasing and continuous;
- (A $_f$) f is measurable and $\int_0^T \|f(\cdot, t)\|_{L^1(\mathbb{R}^N)} + \|f(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} dt < \infty$;
- (A $_{u_0}$) $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$; and
- (A $_{\mu}$) μ is a nonnegative symmetric Radon measure on $\mathbb{R}^N \setminus \{0\}$ satisfying

$$\int_{|z| \leq 1} |z|^2 d\mu(z) + \int_{|z| > 1} 1 d\mu(z) < \infty.$$

Sometimes we will need stronger assumptions than (A_φ) and (A_μ) :

(Lip_φ) $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is nondecreasing and locally Lipschitz.

(A_{μ_α}) There are constants $\alpha \in (0, 2)$ and $C \geq 0$ such that for all $r \in (0, 1)$,

$$\int_{|z| < r} |z|^k d\mu(z) \leq Cr^{k-\alpha}, \quad k = 2, 3, 4, \quad \text{and} \quad \int_{r < |z| < 1} d\mu(z) \leq Cr^{-\alpha}.$$

(A_ν) ν is a nonnegative symmetric Radon measure satisfying $\nu(\mathbb{R}^N) < \infty$.

Remark 2.1.

(a) Without loss of generality, we can assume $\varphi(0) = 0$ (replace $\varphi(u)$ by $\varphi(u) - \varphi(0)$), and when (Lip_φ) holds, that φ is globally Lipschitz (since u is bounded). In the latter case we let L_φ denote the Lipschitz constant.

(b) Under assumption (A_μ) , for any $p \in [1, \infty]$ and any $\psi \in C_c^\infty(\mathbb{R}^N)$,

$$(2.1) \quad \|\mathcal{L}^{\sigma, \mu}[\psi]\|_{L^p} \leq c\|D^2\psi\|_{L^p} \left(|\sigma|^2 + \int_{|z| \leq 1} |z|^2 d\mu(z) \right) + 2\|\psi\|_{L^p} \int_{|z| > 1} d\mu(z).$$

(c) When μ is absolutely continuous w.r.t. the Lebesgue measure dz , assumption (A_{μ_α}) means that $d\mu(z) \leq \frac{C}{|z|^{N+\alpha}} dz$ for $|z| < 1$. The nonlocal operator \mathcal{L}^μ then typically would be a fractional differential operator of order $\alpha \in (0, 2)$ (a pseudodifferential operator), like, e.g., the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$.

(d) Assumption (A_f) is equivalent to requiring $f \in L^1(0, T; L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))$, an iterated L^p -space as in, e.g., [6]. Note that $L^1(0, T; L^1(\mathbb{R}^N)) = L^1(Q_T)$.

DEFINITION 2.1 (distributional solution). *Let $u_0 \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $f \in L^1_{\text{loc}}(Q_T)$. Then $u \in L^1_{\text{loc}}(Q_T)$ is a distributional (or very weak) solution of (1.1) if for all $\psi \in C_c^\infty(\mathbb{R}^N \times [0, T])$, $\varphi(u)\mathcal{L}^{\sigma, \mu}[\psi] \in L^1(Q_T)$ and*

$$(2.2) \quad \int_0^T \int_{\mathbb{R}^N} (u \partial_t \psi + \varphi(u)\mathcal{L}^{\sigma, \mu}[\psi] + f\psi) dx dt + \int_{\mathbb{R}^N} u_0(x)\psi(x, 0) dx = 0.$$

By Remark 2.1(b), $\varphi(u)\mathcal{L}^{\sigma, \mu}[\psi] \in L^1$ if, e.g., $u \in L^\infty$ and φ continuous. Distributional solutions exist and are unique in $L^1 \cap L^\infty$.

THEOREM 2.2 ([35, Theorem 2.8] and [34, Theorem 3.1]). *Assume (A_φ) , (A_f) , (A_{u_0}) , and (A_μ) . Then there exists a unique distributional solution u of (1.1)–(1.2) such that*

$$(2.3) \quad u \in L^1(Q_T) \cap L^\infty(Q_T) \cap C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N)).$$

Note that by (2.2) and (2.3), $u(x, t) \rightarrow u_0(x)$ in $L^1_{\text{loc}}(\mathbb{R}^N)$ as $t \rightarrow 0^+$.

3. Numerical schemes—general theory. We introduce and discuss the class of numerical methods that we consider and state the main results about well-posedness, stability, equicontinuity, compactness, and convergence. The proofs of most results in this section are given in [36].

3.1. The numerical method. Our schemes will be defined on time-space grids, nonuniform in time, but uniform in space for simplicity. Our discrete diffusion operators will then have weights and stencils not depending on the position x . Let $h > 0$, the cube $R_h = h(-\frac{1}{2}, \frac{1}{2}]^N$, and \mathcal{G}_h be the uniform spatial grid

$$\mathcal{G}_h := h\mathbb{Z}^N = \{x_\beta := h\beta : \beta \in \mathbb{Z}^N\}.$$

The nonuniform time grid is

$$\mathcal{T}_{\Delta t}^T = \{t_j\}_{j=0}^J \quad \text{for} \quad 0 = t_0 < t_1 < \dots < t_J = T.$$

Let $\mathbb{J} := \{1, \dots, J\}$, and denote the time steps by

$$\Delta t_j = t_j - t_{j-1}, \quad j \in \mathbb{J}, \quad \text{and} \quad \Delta t = \max_{j \in \mathbb{J}} \Delta t_j.$$

On the grid $\mathcal{G}_h \times \mathcal{T}_{\Delta t}^T$ we define a class of numerical approximations of (1.1) by discretizing in time and space using monotone finite difference (quadrature) approximations. Using a θ -method in time, the resulting scheme can be written as

$$(3.1) \quad U_\beta^j = U_\beta^{j-1} + \Delta t_j (\theta \mathcal{L}^h[\varphi(U^j)]_\beta + (1 - \theta) \mathcal{L}^h[\varphi^h(U^{j-1})]_\beta + F_\beta^j)$$

for $\beta \in \mathbb{Z}^N$ and $j \in \mathbb{J}$, where the discrete diffusion operator \mathcal{L}^h is given by

$$(FD) \quad \mathcal{L}^h[\psi](x) = \sum_{\beta \neq 0} (\psi(x + z_\beta) - \psi(x)) \omega_{\beta,h} \quad \text{with} \quad z_\beta \in \mathcal{G}_h.$$

We will always assume $z_\beta = -z_{-\beta}$, $\omega_{\beta,h} = \omega_{-\beta,h} \geq 0$, and $\sum_{\beta \neq 0} \omega_\beta < +\infty$; see Definition 3.1 and Lemma 3.1 below. Then \mathcal{L}^h is a monotone finite difference operator with *stencil* $\mathcal{S} = \{z_\beta\}_\beta$ and *weights* $\{\omega_{\beta,h}\}_\beta$. Note that the scheme is explicit when $\theta = 0$, implicit when $\theta = 1$, and Crank–Nicolson like when $\theta = \frac{1}{2}$.

Formally we want $U_\beta^j \approx u(x_\beta, t_j)$, $\mathcal{L}^h \approx \mathfrak{L}^{\sigma,\mu}$, $\varphi^h \approx \varphi$, and $F_\beta^j \approx f(x_\beta, t_j)$. For \mathcal{L}^h and φ^h this means that we have to impose *consistency* assumptions; see Definition 3.1(ii) and Definition 3.2(ii) below. But since u and f need not be continuous and point values are not always defined or useful, we will interpret U and F as piecewise polynomial approximations. In this paper we restrict ourselves to piecewise constant approximations defined from cell averages for simplicity. Hence as initial data for the scheme we take

$$U_\beta^0 := \frac{1}{h^N} \int_{x_\beta + R_h} u_0(x) \, dx, \quad F_\beta^j := \frac{1}{h^N \Delta t_j} \int_{t_j - \Delta t_j}^{t_j} \int_{x_\beta + R_h} f(x, \tau) \, dx \, d\tau.$$

Of course if f and u_0 are continuous, we could use $U_\beta^0 := u_0(x_\beta)$ and $F_\beta^j = f(x_\beta, t_j)$ instead and all the results below would remain valid.

3.2. The discretizations \mathcal{L}^h and φ^h . An admissible discretization \mathcal{L}^h of \mathfrak{L} should be (i) monotone, symmetric, (ii) consistent, and (iii) satisfy some uniform Levy integrability condition (which is trivial in the local case). In the next definition we will use that $\mathcal{L}^h = \mathcal{L}^{\nu_h}$, where \mathcal{L}^{ν_h} is a Levy operator like \mathfrak{L} defined as

$$(FD2) \quad \mathcal{L}^{\nu_h}[\psi] := \int_{|z|>0} (\psi(x + z) - \psi(x)) \, d\nu_h(z) \quad \text{with} \quad \nu_h(z) = \sum_{\beta \neq 0} \delta_{z_\beta}(z) \omega_{\beta,h}.$$

This surprising observation along with the sufficiently general well-posedness result in section 3, are key ingredients that make our theory work.

DEFINITION 3.1. A family $\{\mathcal{L}^h\}_{h>0}$ of discretizations of \mathfrak{L} is admissible if it is

- (i) in the class (A_ν) : $\mathcal{L}^h = \mathcal{L}^{\nu_h}$ for a measure ν_h satisfying (A_ν) for all $h > 0$;
- (ii) consistent: For every $\psi \in C_c^\infty(\mathbb{R}^N)$,

$$\|\mathfrak{L}[\psi] - \mathcal{L}^h[\psi]\|_{L^1(\mathbb{R}^N)} \rightarrow 0 \quad \text{as } h \rightarrow 0^+;$$

(iii) uniformly in (A_μ) :

$$(UL) \quad \sup_{h<1} \sum_{\beta \neq 0} (|z_\beta|^2 \wedge 1) \omega_{\beta,h} < +\infty.$$

Note that $\|\mathfrak{L}[\psi] - \mathcal{L}^h[\psi]\|_{L^1(\mathbb{R}^N)}$ is the local truncation error of \mathcal{L}^h (in L^1), and that in view of (FD2), condition (UL) can equivalently be written as

$$(UL2) \quad \sup_{h<1} \int_{|z|>0} |z|^2 \wedge 1 \, d\nu_h(z) < +\infty.$$

LEMMA 3.1. The operators $\{\mathcal{L}^h\}_{h>0}$ defined in (FD) are in the class (A_ν) if and only if $z_\beta = -z_{-\beta}$, $\omega_{\beta,h} = \omega_{-\beta,h} \geq 0$, and $\sum_{\beta \neq 0} \omega_\beta < +\infty$.

Proof. Since $\nu_h(z) = \sum_{\beta \neq 0} \delta_{z_\beta}(z) \omega_{\beta,h}$, equivalence for the symmetry and non-negativity part of (A_ν) follows immediately. Equivalence for the boundedness follows from $\nu_h(\mathbb{R}^N) = \sum_{\beta \neq 0} \delta_{z_\beta}(\mathbb{R}^N) \omega_{\beta,h} = \sum_{\beta \neq 0} \omega_{\beta,h}$. \square

Assumption (UL) may seem unusual, but it is in fact very natural in view of (A_μ) . It is trivial to verify for local problems, and we now provide a very easy to use sufficient condition for it to hold in the general case.

PROPOSITION 3.2. Assume (A_μ) , \mathfrak{L} is defined by (1.3)–(1.5), and $\{\mathcal{L}^{\nu_h}\}_{h>0}$ defined by (FD) is in the class (A_ν) . Then (UL2) holds if

$$(3.2) \quad \mathcal{L}^{\nu_h}[\psi](x) \xrightarrow{h \rightarrow 0^+} \mathfrak{L}[\psi](x) \quad \text{for all } x \in \mathbb{R}^N, \quad \psi \in C_c^\infty(\mathbb{R}^N).$$

Remark 3.3. (3.2) follows, e.g., from L^∞ -consistency, $\|\mathfrak{L}[\psi] - \mathcal{L}^{\nu_h}[\psi]\|_{L^\infty(\mathbb{R}^N)} \rightarrow 0$ as $h \rightarrow 0^+$ for all $\psi \in C_c^\infty(\mathbb{R}^N)$.

Proof of Proposition 3.2. By (FD2), the Taylor expansion

$$\psi(x+z) = \psi(x) + z \cdot D\psi(x) + \int_0^1 (1-t) z^T D^2\psi(x+tz) z \, dt,$$

and since $\int_{|z|<1} z \, d\nu_h(z) = 0$ by the symmetry of ν_h , we find that

$$\begin{aligned} \mathcal{L}^{\nu_h}[\psi](x) &= \int_{|z| \leq 1} \int_0^1 (1-t) z^T D^2\psi(x+tz) z \, dt \, d\nu_h(z) \\ &\quad + \int_{|z| > 1} (\psi(x+z) - \psi(x)) \, d\nu_h(z). \end{aligned}$$

Then we take $\psi \in C_c^\infty$ such that $\psi(x) = -1 + |x|^2$ for $|x| \leq 1$ and $\psi(x) \geq 0$ for $|x| > 1$.

Since $|tz| < 1$ in the first integral above,

$$\begin{aligned} \mathcal{L}^{\nu_h}[\psi](0) &= \int_{|z|\leq 1} |z|^2 d\nu_h(z) + \int_{|z|>1} \psi(z) d\nu_h(z) - \psi(0) \int_{|z|>1} d\nu_h(z) \\ &\geq \int_{|z|\leq 1} |z|^2 d\nu_h(z) + 0 + \int_{|z|>1} d\nu_h(z) \geq \int_{|z|>0} (|z|^2 \wedge 1) d\nu_h(z). \end{aligned}$$

By (A_μ) , the bound (2.1) holds, and then by (3.2) we conclude that

$$\begin{aligned} \sup_{h<1} \int_{|z|>0} (|z|^2 \wedge 1) d\nu_h(z) &\leq \sup_{h<1} \mathcal{L}^{\nu_h}[\psi](0) \\ &\leq |\mathfrak{L}[\psi](0)| + \sup_{h<1} |(\mathcal{L}^{\nu_h} - \mathfrak{L})[\psi](0)| < +\infty. \end{aligned}$$

The proof is complete. □

In most situations we can simply take the nonlinearity $\varphi_h = \varphi$, but sometimes it is useful to approximate φ also. We will see below that this is true especially for fast diffusions.

DEFINITION 3.2. A family $\{\varphi^h\}_{h>0}$ of approximation of φ is admissible if it is

- (i) in the class (Lip_φ) for every $h > 0$, φ^h satisfy (Lip_φ) ;
- (ii) consistent: $\varphi^h \rightarrow \varphi$ locally uniformly as $h \rightarrow 0^+$.

3.3. CFL condition for the explicit part. A crucial property in our convergence analysis is monotonicity. Our schemes are monotone under the CFL condition

$$(CFL) \quad \Delta t(1 - \theta)L_{\varphi^h}\nu^h(\mathbb{R}^N) \leq 1,$$

where L_{φ^h} denotes the Lipschitz constant of φ^h . Note that the condition always holds if $\theta = 1$ and the scheme is implicit. If the scheme has some explicit part, $\theta \in [0, 1)$, then this condition gives a relation between Δt and h . In the local case, we typically have $\nu_h(\mathbb{R}^N) = O(h^{-2})$, and the CFL condition becomes the classical

$$\Delta t \leq Ch^2.$$

In the (explicit and) nonlocal case, typically $\nu_h(\mathbb{R}^N) = O(h^{-\alpha})$ for some $\alpha \in (0, 2)$ (e.g., $\nu_h \sim |z|^{-N-\alpha}$), $\nu_h(\mathbb{R}^N) = O(|\log(h)|)$ (e.g., $\nu_h \sim |z|^{-N}e^{-|z|}$) or $\nu_h(\mathbb{R}^N) < C$ (e.g., $\nu_h \sim f \in L^1(\mathbb{R}^N)$), and the CFL condition becomes

$$\Delta t \leq Ch^\alpha, \quad \Delta t \leq C \frac{1}{|\log(h)|} \quad \text{or} \quad \Delta t \leq C.$$

We refer to [36] for more details and the origin of such conditions.

Remark 3.4. Note that φ^h has to be Lipschitz for the CFL condition to make sense. Hence if φ is not Lipschitz (the fast diffusion case), it must be replaced by a Lipschitz approximation to obtain a monotone explicit scheme. The Lipschitz constant L_{φ^h} will then blow up as $h \rightarrow 0$, and the overall CFL condition is worse than in the Lipschitz case. See section 5.4.2 for examples.

3.4. Comparison, stability, and convergence of the method. By [36] our numerical method has the following list of properties.

THEOREM 3.5 (existence and uniqueness). *Assume (A_f) , (A_{u_0}) , and (A_φ) , \mathcal{L}^h defined in (FD) satisfies (A_ν) , φ^h is in the class (Lip_φ) , and $h, \Delta t > 0$ are such that (CFL) holds. Then there exists a unique solution U_α^j of (3.1) such that*

$$\sum_{j \in \mathbb{J}} \sum_{\beta} |U_\alpha^j| < \infty.$$

THEOREM 3.6 (properties and convergence). *Assume (A_μ) , (A_φ) , u_0, v_0 satisfy (A_{u_0}) , f, g satisfy (A_f) , $\{\mathcal{L}^h\}_{h>0}$ and $\{\varphi^h\}_{h>0}$ are admissible approximations of \mathfrak{L} and φ , $\Delta t = o_h(1)$ such that (CFL) holds, and U_β^j, V_β^j are solutions of the scheme (3.1) with data u_0, v_0 and f, g . Then,*

(a) (Monotone) *If $U_\beta^0 \leq V_\beta^0$ and $F_\beta^j \leq G_\beta^j$ for all β , then $U_\beta^j \leq V_\beta^j$ for all β , $j \geq 0$.*

(b) (L^1 -stable) $\sum_{\beta} |U_\beta^j| \leq \sum_{\beta} |U_\beta^0| + \sum_{l=1}^j \sum_{\beta} |F_\beta^l| \Delta t_l.$

(c) (L^∞ -stable) $\sup_{\beta} |U_\beta^j| \leq \sup_{\beta} |U_\beta^0| + \sup_{\beta} \sum_{l=1}^j |F_\beta^l| \Delta t_l.$

(d) (Conservative) *If φ^h satisfy (Lip_φ) , $\sum_{\beta} U_\beta^j = \sum_{\beta} U_\beta^0 + \sum_{l=1}^j \sum_{\beta} F_\beta^l \Delta t_l.$*

(e) (L^1 -contractive) $\sum_{\beta} (U_\beta^j - V_\beta^j)^+ \leq \sum_{\beta} (U_\beta^0 - V_\beta^0)^+ + \sum_{l=1}^j \sum_{\beta} (F_\beta^l - G_\beta^l)^+ \Delta t_l.$

(f) (Equicontinuity in time) *For all compact sets $K \subset \mathbb{R}^N$ there exists a modulus of continuity Λ_K (independent of h and Δt) such that*

$$h^N \sum_{x_\beta \in \mathcal{G}_h \cap K} |U_\beta^j - U_\beta^{j-k}| \leq \Lambda_K(t_j - t_{j-k}) + |K| \int_{t_{j-k}}^{t_j} \|f(\cdot, \tau)\|_{L^\infty(\mathbb{R}^N)} d\tau.$$

(g) (Convergence) *There exists a unique distributional solution $u \in L^1(Q_T) \cap L^\infty(Q_T) \cap C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$ of (1.1)–(1.2) and for all compact sets $K \subset \mathbb{R}^N$,*

$$\|U - u\|_K := \max_{t_j \in \mathcal{T}_{\Delta t}^T} \left\{ \sum_{x_\beta \in \mathcal{G}_h \cap K} h^N |U_\beta^j - u(x_\beta, t_j)| \right\} \rightarrow 0 \quad \text{as } h \rightarrow 0^+.$$

Note that our schemes are stable in L^p for any $p \in [1, \infty]$ by interpolation. The discrete norm convergence results are equivalent to convergence in $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$ for interpolants of the numerical solution (piecewise constant in space and piecewise linear in time); see [36] for the details. Convergence was proved through a compactness argument in this space, where equicontinuity results like Theorem 3.6(f) and (g) were needed. By the L^p stability, convergence also holds in $C([0, T]; L^p_{\text{loc}}(\mathbb{R}^N))$ for all $p \in [1, \infty)$.

3.5. Some extensions. As shown in [36], the results of Theorems 3.5 and 3.6 also hold for a larger class of schemes,

$$U_\beta^j = U_\beta^{j-1} + \Delta t_j \left(\sum_{k=1}^n \mathcal{L}_k^h[\varphi_k^h(U^j)]_\beta + \sum_{k=n+1}^m \mathcal{L}_k^h[\varphi_k^h(U^{j-1})]_\beta + F_\beta^j \right),$$

where $n, m \in \mathbb{N}$ with $n \leq m$ and

$$\sum_{k=1}^n \mathcal{L}_k^h[\varphi_k^h(U_h^j)](x) + \sum_{k=n+1}^m \mathcal{L}_k^h[\varphi_k^h(U_h^{j-1})](x) \approx \mathfrak{L}[\varphi(u)](x, t_j).$$

Depending on the choices of \mathcal{L}_k^h and φ_k^h , we can then get many different schemes:

- (1) Discretizing separately the different parts of the operator

$$\mathfrak{L} = \mathfrak{L}^{\sigma, \mu} = L^\sigma + \mathcal{L}_{\text{sing}}^\mu + \mathcal{L}_{\text{bnd}}^\mu,$$

e.g., the local, singular nonlocal, and bounded nonlocal parts, corresponds to different choices for \mathcal{L}_k^h . See section 4 for a detailed discussion.

- (2) Explicit schemes ($\theta = 0$), implicit schemes ($\theta = 1$), or combinations like Crank–Nicolson ($\theta = \frac{1}{2}$), follow by the choices

$$\mathcal{L}_1^h = \theta \mathcal{L}^h \quad \text{and} \quad \mathcal{L}_2^h = (1 - \theta) \mathcal{L}^h.$$

- (3) Combinations of type (1) and (2) schemes, e.g., implicit discretization of the unbounded part of $\mathfrak{L}^{\sigma, \mu}$ and explicit discretization of the bounded part.

4. Numerical schemes—discretizations. In this section we explore known, and find new, approximations of \mathfrak{L} and φ and, in every case, we show that they are admissible in the sense of Definitions 3.1 and 3.2 (see also Lemma 3.1) and hence yield monotone, stable, and convergent numerical schemes for (1.1)–(1.2) by Theorem 3.6. Many of these schemes are completely new in this setting, and even for many of the known schemes, the convergence results are new. For the diffusion operator $\mathfrak{L} = \mathfrak{L}^{\sigma, \mu} = \mathcal{L}^\mu + L^\sigma$, we present a series of possible discretizations of both the nonlocal part \mathcal{L}^μ of the form (1.5) satisfying (A_μ) and the local second order elliptic operator L^σ given by (1.4). Most of these discretizations apply to all such operators and are not restricted to the fractional Laplacian/Laplacian. We end the exposition by discussing how to handle non-Lipschitz merely continuous nonlinearities φ and, hence, also fast diffusions.

The nonlocal operator \mathcal{L}^μ has a possibly singular part and a nonsingular part that can (and often should) be discretized separately: for $r > 0$,

$$\begin{aligned} \mathcal{L}^\mu[\psi](x) &= \int_{0 < |z| \leq r} (\psi(x+z) - \psi(x) - z \cdot D\psi(x)) \, d\mu(z) \\ &\quad + \int_{|z| > r} (\psi(x+z) - \psi(x)) \, d\mu(z) \\ &=: \mathcal{L}_r^\mu[\psi](x) + \mathcal{L}^{\mu, r}[\psi](x). \end{aligned}$$

When we discretize this operator, we have to take $h \leq r = o_h(1)$, where h is the discretization in space parameter. Often we can simply take $r = h$, but in some cases a different choice can produce higher order discretizations. We will present admissible discretizations for general measures μ and state their local truncation error. We also give the local truncation error when the nonlocal operator is a fractional derivative in the sense of (A_{μ_α}) .

4.1. Lagrange interpolation. Let $\{p_\beta^k\}_{\beta \in \mathbb{Z}^N}$ be the basis of piecewise Lagrange polynomials of order k on the grid \mathcal{G}_h , i.e., $\sum_\beta p_\beta^k(x) \equiv 1$ for all $x \in \mathbb{R}^N$ and $p_\beta^k(z_\gamma) = 1$ for $\beta = \gamma$ and zero otherwise. Since the grid is uniform, we may define these functions

in a tensorial way for $N > 1$ (direction by direction). On \mathcal{G}_h the Lagrange polynomial interpolant of order k of a function ψ is given by

$$(4.1) \quad I_h^k[\psi](z) := \sum_{\beta \neq 0} \psi(z_\beta) p_\beta^k(z),$$

and if $\psi \in C_c^\infty(\mathbb{R}^N)$, the corresponding interpolation error is

$$(4.2) \quad \|I_h^k[\psi] - \psi\|_{L^p(\mathbb{R}^N)} = C \|D^{k+1}\psi\|_{L^p(\mathbb{R}^N)} h^{k+1},$$

where $C = C(k, p)$ and $p = \{1, \infty\}$ (cf., e.g., [19]). Since the grid is uniform, the p_β^k -basis will have a lot of symmetries. E.g., when $k = 1$ (linear interpolation), $0 \leq p_\beta^1(x) = p_0^1(x - z_\beta)$ for $z_\beta = \beta h \in \mathcal{G}_h$ and $\beta \in \mathbb{Z}^N$, and $p_0^1(x_1, \dots, x_i, \dots, x_N) = p_0^1(x_1, \dots, -x_i, \dots, x_N)$ for $x_i \in \mathbb{R}$ and $i = 1, \dots, N$.

4.2. Discretizations of the local operator L^σ . The operator L^σ given by (1.4) is a *local*, self-adjoint, and possibly degenerate operator that can be written as

$$(4.3) \quad L^\sigma[\psi](x) := \text{tr}(\sigma \sigma^T D^2 \psi(x)) = \sum_{i=1}^P \sigma_i^T D^2 \psi(x) \sigma_i = \sum_{i=1}^P (\sigma_i^T D)^2 \psi(x),$$

where $\sigma = (\sigma_1, \dots, \sigma_P) \in \mathbb{R}^{N \times P}$ and $\sigma_i \in \mathbb{R}^N$. For $\eta > 0$, we approximate L^σ by

$$(4.4) \quad L_\eta^\sigma[\psi](x) := \sum_{i=1}^P \frac{\psi(x + \sigma_i \eta) + \psi(x - \sigma_i \eta) - 2\psi(x)}{\eta^2}.$$

In general $x + \sigma_i \eta \notin \mathcal{G}_h$, not even when $x \in \mathcal{G}_h$ and, hence, this discretization is in the form (FD) only for special choices of σ . We can overcome this problem by replacing ψ by its interpolant on \mathcal{G}_h ,

$$(4.5) \quad L_{\eta,h}^\sigma[\psi](x) = \sum_{i=1}^P \frac{I_h^1[\psi](x + \sigma_i \eta) + I_h^1[\psi](x - \sigma_i \eta) - 2\psi(x)}{\eta^2},$$

where I_h^1 denotes the first order Lagrange interpolation defined in (4.1) for $k = 1$. This type of discretization have been studied before, e.g., in [16, 35].

Remark 4.1.

(a) If $\eta = h$ and $\sigma_i = e_i$, the standard basis in \mathbb{R}^N , then

$$L^\sigma = \Delta \quad \text{and} \quad L_h^\sigma = L_{h,h}^\sigma = \Delta_h,$$

where Δ_h is the classical second order finite difference approximation

$$(4.6) \quad \Delta_h \psi(x) := \sum_{i=1}^N \frac{\psi(x + h e_i) - 2\psi(x) + \psi(x - h e_i)}{h^2}.$$

(b) By a coordinate transformation $x = Ay$ (diagonalization), L^σ can always be transformed into

$$L^{I_0}, \quad \text{where} \quad I_0 := \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{N \times N},$$

where I is an identity matrix. L^{I_0} corresponds to a Laplacian operator in \mathbb{R}^M for some $M \leq N$. In the transformed coordinates y , $L^{I_0} = \Delta_{\mathbb{R}^M}$, the \mathbb{R}^M -Laplacian, for some $M \leq N$, and again $L_h^{I_0} = L_{h,h}^{I_0} = \Delta_{\mathbb{R}^M,h}$ (see [35]).

We have the following general result.

LEMMA 4.2. *Let $h, \eta > 0$, $h = o(\eta)$, and L^σ be defined by (1.4). The family of operators $\{L_{\eta,h}^\sigma\}_{\eta,h>0}$ given by (4.5) is an admissible approximation of L^σ with local truncation error $O(\frac{h^2}{\eta^2} + \eta^2)$ (or $O(h)$ with the optimal $\eta = \sqrt{h}$).*

Lemma 4.2 is close to results, e.g., in [32], but below we give a proof for completeness. For the discretization Δ_h we have the following result.

LEMMA 4.3. *Let $h > 0$. The family of operators $\{\Delta_h\}_{h>0}$ given by (4.6) is an admissible approximation of Δ with local truncation error $O(h^2)$.*

Admissibility and the improved (and classical!) rate follows as in the proof of Lemma 4.2 since there is no table interpolation now.

Proof of Lemma 4.2. We show that $L_{\eta,h}^\sigma$ can be written in the finite difference form (FD). For simplicity, let us set $\sigma_{-i} = -\sigma_i$ and let $x_\alpha \in \mathcal{G}_h$. Then

$$\begin{aligned} L_{\eta,h}^\sigma[\psi](x_\alpha) &= \sum_{i=1}^P \frac{I_h^1[\psi](x_\alpha + \sigma_i\eta) + I_h^1[\psi](x_\alpha - \sigma_i\eta) - 2\psi(x_\alpha)}{\eta^2} \\ &= \sum_{0<|i|\leq P} \left(\sum_{\beta} \psi(z_\beta) p_0^1(x_\alpha + \sigma_i\eta - z_\beta) - \psi(x_\alpha) \right) \frac{1}{\eta^2} \\ &= \sum_{0<|i|\leq P} \left(\sum_{\gamma} \psi(x_\alpha + z_\gamma) p_0^1(\sigma_i\eta - z_\gamma) - \psi(x_\alpha) \right) \frac{1}{\eta^2} \\ &= \sum_{\gamma \neq 0} (\psi(x_\alpha + z_\gamma) - \psi(x_\alpha)) \left(\frac{1}{\eta^2} \sum_{0<|i|\leq P} p_0^1(\sigma_i\eta - z_\gamma) \right). \end{aligned}$$

The weights are $\omega_{\eta,h,\beta} = \frac{1}{\eta^2} \sum_{0<|i|\leq P} p_0^1(\sigma_i\eta - z_\beta)$, and we immediately find that $0 \leq \omega_{\eta,h,\beta} = \omega_{\eta,h,-\beta}$ since $0 \leq p_0^1$ is even, $\sigma_{-i} = -\sigma_i$, and $z_{-\beta} = -z_\beta$. Moreover,

$$\sum_{\beta \neq 0} \omega_{\eta,h,\beta} = \frac{1}{\eta^2} \sum_{0<|i|\leq P} \sum_{\beta \neq 0} p_0^1(\sigma_i\eta) = \frac{1}{\eta^2} \sum_{0<|i|\leq P} (1 - p_0^1(\sigma_i\eta)) \leq \frac{2P}{\eta^2} < +\infty.$$

To show consistency we split the error in the following way:

$$\|L^\sigma \psi - L_{\eta,h}^\sigma[\psi]\|_{L^1(\mathbb{R}^N)} \leq \|L^\sigma \psi - L_\eta^\sigma[\psi]\|_{L^1(\mathbb{R}^N)} + \|L_\eta^\sigma[\psi] - L_{\eta,h}^\sigma[\psi]\|_{L^1(\mathbb{R}^N)},$$

where $L_\eta^\sigma[\psi]$ is given by (4.4). The first term on the right-hand side is the classical error of a second order approximation of second derivatives,

$$(4.7) \quad \|L^\sigma \psi - L_\eta^\sigma[\psi]\|_{L^1(\mathbb{R}^N)} \leq C \|D^4 \psi\|_{L^1(\mathbb{R}^N)} \eta^2 |\sigma|^4.$$

For the second term, we have (cf. (4.2))

$$\begin{aligned} \|L_\eta^\sigma[\psi] - L_{\eta,h}^\sigma[\psi]\|_{L^1(\mathbb{R}^N)} &= \frac{1}{\eta^2} \left\| \sum_{0<|i|\leq P} (I_h^1[\psi](\cdot + \sigma_i\eta) - \psi(\cdot + \sigma_i\eta)) \right\|_{L^1(\mathbb{R}^N)} \\ &\leq \frac{1}{\eta^2} \sum_{0<|i|\leq P} \|I_h^1[\psi](\cdot + \sigma_i\eta) - \psi(\cdot + \sigma_i\eta)\|_{L^1(\mathbb{R}^N)} \\ &= \frac{1}{\eta^2} \sum_{0<|i|\leq P} C \|D^2 \psi\|_{L^1(\mathbb{R}^N)} h^2 = O\left(\frac{h^2}{\eta^2}\right). \end{aligned}$$

Thus, for any choice of $h = o(\eta)$ we get a consistent scheme. Moreover, one can observe that the last two estimates also hold in L^∞ by a trivial adaptation. By Proposition 3.2 it then follows that the uniform integrability condition (UL) holds. In view of Lemma 3.1, the scheme is admissible (cf. Definition 3.1) and the proof is complete. \square

4.3. Discretizations of the singular nonlocal operator \mathcal{L}_r^μ . We present discretizations of the singular/unbounded part of the nonlocal operator (recall Remark 1.1)

$$(4.8) \quad \mathcal{L}_r^\mu[\psi](x) = \int_{0 < |z| \leq r} (\psi(x+z) - \psi(x) - z \cdot D\psi(x)) \, d\mu(z), \quad r \in [0, 1].$$

We start with the *trivial discretization* where we discretize \mathcal{L}_r^μ by

$$(4.9) \quad \mathcal{L}^h[\psi](x) \equiv 0.$$

This crude discretization is computationally efficient, and depending on the order of the operator and the other discretizations involved, the error could be satisfactory.

LEMMA 4.4. *Assume (A_μ) , $h \leq r = o_h(1)$, and \mathcal{L}_r^μ is given by (4.8). Then $\{\mathcal{L}^h\}_{h>0}$ given by (4.9) is an admissible approximation of \mathcal{L}_r^μ . Moreover, if also (A_{μ_α}) holds, then the local truncation error is $O(r^{2-\alpha})$.*

Proof. Since $\mathcal{L}^h = \mathcal{L}^{\nu_h}$ with $\nu_h \equiv 0$, it is a discretization of the form (FD) in the class (A_ν) . It is consistent by the dominated convergence theorem,

$$\begin{aligned} \|\mathcal{L}_r^\mu[\psi] - \mathcal{L}^h[\psi]\|_{L^1(\mathbb{R}^N)} &= \|\mathcal{L}_r^\mu[\psi]\|_{L^1(\mathbb{R}^N)} \\ &\leq \frac{1}{2} \|D^2\psi\|_{L^1(\mathbb{R}^N)} \int_{0 < |z| \leq r} |z|^2 \, d\mu(z) \rightarrow 0 \quad \text{as } h \rightarrow 0^+. \end{aligned}$$

If also (A_{μ_α}) holds, then the local truncation error is $O(r^{2-\alpha})$. Moreover (UL) also holds since $\sup_{h < 1} \sum_{\beta \neq 0} (|z_\beta|^2 \wedge 1) \omega_{\beta,h} = 0 < +\infty$. \square

We now show how to get a more accurate discretization of \mathcal{L}_r^μ through the *adapted vanishing viscosity discretization* [5, 26, 53]: Approximate \mathcal{L}_r^μ by a local second order operator and then discretize. To do this, note that by Taylor's theorem,

$$\psi(x+z) - \psi(x) - z \cdot D\psi(x) = \sum_{2 \leq |\beta| \leq 3} \frac{1}{\beta!} D^\beta \psi(x) z^\beta + \sum_{|\beta|=4} R_\beta(x, z) z^\beta,$$

where $R_\beta(x, z) = \frac{|\beta|}{\beta!} \int_0^1 (1-s)^{|\beta|-1} D^\beta \psi(x+sz) \, ds$. Since terms with $|\beta| = 3$ are odd and the measure μ is symmetric,

$$\mathcal{L}_r^\mu[\psi](x) = T_2^r(x) + 0 + R_4^r(x),$$

where $R_4^r(x) = \sum_{|\beta|=4} \int_{|z| < r} R_\beta(x, z) z^\beta \, d\mu(z)$,

$$T_2^r(x) = \sum_{|\beta|=2} \frac{D^\beta \psi(x)}{\beta!} \int_{|z| < r} z^\beta \, d\mu(z) = \frac{1}{2} \text{tr}[Z D^2 \psi(x)],$$

and $Z_{ij} = \int_{|z| < r} z_i z_j \, d\mu(z)$. Observe that $Z = Z(r)$ is a symmetric and positive semidefinite matrix, $\xi^T Z \xi = \int_{|z| < r} \xi^T (z z^T) \xi \, d\mu(z) = \int_{|z| < r} (\xi \cdot z)^2 \, d\mu(z) \geq 0$ for

$\xi \in \mathbb{R}^N$, and that $Z(r) \rightarrow 0$ as $r \rightarrow 0$ by the dominated convergence theorem. Hence it has a square root \sqrt{Z} with entries $(\sqrt{Z})_{ij}$ and columns $(\sqrt{Z})_i$ (our notation). This allows us to write T_2^r as a sum of directional derivatives

$$T_2^r(x) = \frac{1}{2} \sum_{i=1}^N ((\sqrt{Z})_i \cdot D)^2 \psi(x).$$

This is an operator of the form L^σ with $\sigma_i = (\sqrt{Z})_i$, $i = 1, \dots, N$ (cf. (4.3)), and we discretize it as in section 4.2:

$$(4.10) \quad \mathcal{L}^{\eta,h}[\psi](x) = \sum_{i=1}^P \frac{I_h^1[\psi](x + (\sqrt{Z})_i \eta) + I_h^1[\psi](x - (\sqrt{Z})_i \eta) - 2\psi(x)}{\eta^2}.$$

Note that (4.10) depends on r through the terms $(\sqrt{Z})_i$, and that (linear) interpolation is needed since the $x + (\sqrt{Z})_i \eta$ do not belong to the grid \mathcal{G}_h , in general.

LEMMA 4.5. *Assume (A_μ) , $h, \eta, r > 0$, $h = o(\eta)$, $h \leq r = o_h(1)$, and \mathcal{L}_r^μ is defined by (4.8). Then the family $\{\mathcal{L}^{\eta,h}\}_{h,\eta>0}$ given by (4.10) is an admissible approximation of \mathcal{L}_r^μ with local truncation error $O(\frac{h^2}{\eta^2} + \eta^2|Z(r)|^2 + r^2)$. Moreover, if also (A_{μ_α}) holds, then the local truncation error is $O(\frac{h^2}{\eta^2} + \eta^2 r^{4-2\alpha} + r^{4-\alpha})$.*

In general, the optimal choice will turn out to be $\eta = r = h^{\frac{1}{2}}$ which gives a linear rate $O(h)$. If also (A_{μ_α}) holds, the optimal choice will turn out to be $r = h^{\frac{1}{2}}$ and $\eta = h^{\frac{\alpha}{4}}$ which gives the superlinear rate $O(h^{2-\frac{\alpha}{2}})$. The proof will be given below. Now we show how this rate can be improved when the measure μ satisfies stronger symmetry assumptions.

DEFINITION 4.1. *A Borel measure μ is symmetric in coordinate directions if $\mu(A) = \mu(T_{x_i}(A))$ for all $i = 1, \dots, N$ and Borel sets A , where $T_{x_i}(A) = \{(x_1, \dots, -x_i, \dots, x_N) : (x_1, \dots, x_i, \dots, x_N) \in A\}$.*

Remark 4.6. Obviously, radial symmetry \Rightarrow symmetry in coordinate directions \Rightarrow symmetry ($\mu(A) = \mu(-A)$). All reverse implications are false. If, e.g., $d\mu(z_1, z_2) = \text{sign}(z_1 z_2)^+ dz_1 dz_2$, then μ is symmetric but not symmetric in coordinate directions.

If μ is symmetric in coordinate directions, then mixed derivatives vanish and $T_2^r = \frac{1}{2} \sum_{i=1}^N \frac{\partial^2 \psi}{\partial x_i^2} \int_{|z|<r} z_i^2 d\mu(z)$. A natural difference approximation of \mathcal{L}_r^μ is then

$$(4.11) \quad \mathcal{L}^h[\psi](x) := \frac{1}{2} \sum_{i=1}^N \frac{\psi(x + e_i h) + \psi(x - e_i h) - 2\psi(x)}{h^2} \int_{|z|<r} z_i^2 d\mu(z).$$

If μ is also radially symmetric we even have $\int_{|z|<r} z_i^2 d\mu(z) = \frac{1}{N} \int_{|z|<r} |z|^2 d\mu(z)$.

LEMMA 4.7. *Assume (A_μ) , μ is symmetric in coordinate directions, $h, r > 0$, $h \leq r = o_h(1)$, and \mathcal{L}_r^μ is defined by (4.8). Then the family $\{\mathcal{L}^h\}_{h>0}$ given by (4.11) is an admissible approximation of \mathcal{L}_r^μ with local truncation error $O(h^2 + r^2)$. Moreover, if also (A_{μ_α}) holds, then the local truncation error is $O(h^2 r^{2-\alpha} + r^{4-\alpha})$.*

The optimal choice will turn out to be $r = h$ which gives a quadratic rate $O(h^2)$ or superquadratic rate $O(h^{4-\alpha})$ if (A_{μ_α}) also holds. The proof is similar to the proof of Lemma 4.5.

Proof of Lemma 4.5. Since $\mathcal{L}^{\eta,h}$ is in the form of (4.5), it is already in finite difference form (FD). By Lemma 4.2 it is in the class (A_ν) . We now check consistency:

$$\|\mathcal{L}_r^\mu[\psi] - \mathcal{L}^{\eta,h}[\psi]\|_{L^1(\mathbb{R}^N)} \leq \|T_2^r - \mathcal{L}^{\eta,h}[\psi]\|_{L^1(\mathbb{R}^N)} + \|R_4^r\|_{L^1(\mathbb{R}^N)},$$

and by Lemma 4.2 and inequality (4.7) with $\sigma_i = (\sqrt{Z})_i$, the definition of R_4^r , and Fubini,

$$\begin{aligned} \|T_2^r - \mathcal{L}^{\eta,h}[\psi]\|_{L^1(\mathbb{R}^N)} &\leq C\|D^2\psi\|_{L^1(\mathbb{R}^N)} \frac{h^2}{\eta^2} + C\|D^4\psi\|_{L^1(\mathbb{R}^N)} \eta^2 |Z|^2 \\ &= O\left(\frac{h^2}{\eta^2} + \eta^2 |Z|^2\right), \\ \|R_4^r\|_{L^1(\mathbb{R}^N)} &\leq C\|D^4\psi\|_{L^1(\mathbb{R}^N)} \int_{|z|<r} |z|^4 d\mu(z) = O(r^2), \end{aligned}$$

leading to the desired local truncation error. Since $|Z_i| \leq C \int_{|z|<r} |z|^2 d\mu(z)$, the modifications when also (A_{μ_α}) holds are obvious. Once again, the estimates also hold in L^∞ with a trivial adaptation and thus Proposition 3.2 ensures that (UL) is satisfied. \square

4.4. Discretizations of the bounded nonlocal operator $\mathcal{L}^{\mu,r}$. We present discretization of the bounded/nonsingular part of the nonlocal operator, i.e., of

$$(4.12) \quad \mathcal{L}^{\mu,r}[\psi](x) := \int_{|z|>r} (\psi(x+z) - \psi(x)) d\mu(z), \quad r \in [0, 1].$$

Note that $\mathcal{L}^{\mu,r}$ is an operator in the form (A_μ) with measure $d\mu_r(z) = \mathbf{1}_{|z|>r} d\mu(z)$.

4.4.1. Midpoint quadrature rule. The idea is to approximate ψ on each cube $z_\beta + R_h$ by its midpoint value $\psi(z_\beta)$. This gives the following quadrature formula for the cube,

$$\int_{z_\beta + R_h} (\psi(x+z) - \psi(x)) d\mu(z) \simeq (\psi(x+z_\beta) - \psi(x)) \mu(z_\beta + R_h),$$

and a discretization of $\mathcal{L}^{\mu,r}$ given by

$$(4.13) \quad \mathcal{L}^h[\psi](x) = \sum_{\beta \neq 0} (\psi(x+z_\beta) - \psi(x)) \mu(z_\beta + R_h).$$

LEMMA 4.8. *Assume (A_μ) , $h \leq r = o_h(1)$, and $\mathcal{L}^{\mu,r}$ is defined by (4.12). The family $\{\mathcal{L}^h\}_{h>0}$ given by (4.13) is an admissible approximation of $\mathcal{L}^{\mu,r}$. Moreover, if also (A_{μ_α}) holds, then the local truncation error is $O(h + r^{2-\alpha})$.*

The result was proved in [35, Lemmas 5.3 and 5.4]. Note that since μ is not translation invariant as the Lebesgue measure, the midpoint rule is no longer a second order method in general. Moreover, the local truncation error is dominated by the contribution from the integral over $|z| > 1$ and can therefore not be improved by assuming (A_{μ_α}) .

Remark 4.9. Lemma 4.8 is consistent with the numerical experiments in section 5 for $\alpha > 1$, while for $\alpha < 1$ the numerical results are better. Note that if $N = 1$, $\alpha = 1$, and $r = h$, then the midpoint rule coincides with the second order discretization of section 4.5 below. This time the observed local truncation error is $O(h^2)$.

4.4.2. Quadrature from interpolation—general μ . The idea is to replace $\psi(x+z) - \psi(x)$ by a Lagrange polynomial interpolant on \mathcal{G}_h (defined in section 4.1),

$$I_h^k[\psi(x + \cdot) - \psi(x)](z) = \sum_{\beta \neq 0} (\psi(x + z_\beta) - \psi(x)) p_\beta^k(z),$$

and integrate with respect to the measure μ to obtain the discretization

$$\begin{aligned} \mathcal{L}^h[\psi](x) &= \int_{|z|>r} I_h^k[\psi(x + \cdot) - \psi(x)](z) \, d\mu(z) \\ (4.14) \qquad &= \sum_{\beta \neq 0} (\psi(x + z_\beta) - \psi(x)) \int_{|z|>r} p_\beta^k(z) \, d\mu(z). \end{aligned}$$

This is a classical idea that has been used for nonlocal operators before, e.g., [8, 50].

To have a more understandable presentation, we divide the proof of admissibility into several lemmas. We begin by noting that the operator (4.14) obviously is in the finite difference form (FD) with symmetric weights

$$\omega_{\beta,h} = \int_{|z|>r} p_\beta^k(z) \, d\mu(z), \quad \beta \in \mathbb{Z}^N.$$

However, the nonnegativity of the weights is only guaranteed when $k = 0$ and $k = 1$ since these are the only cases where the basis functions p_β^k are all nonnegative.

LEMMA 4.10. *Assume (A_μ) , $h, r > 0$, and $k = 0$ or $k = 1$. Then the family $\{\mathcal{L}^h\}_{h>0}$ given by (4.14) is in the class (A_ν) .*

Proof. In view of the above discussion and Lemma 3.1, it remains to check that

$$\sum_{\beta} \omega_{\beta,h} = \sum_{\beta} \int_{|z|>r} p_\beta^k(z) \, d\mu(z) = \int_{|z|>r} \sum_{\beta} p_\beta^k(z) \, d\mu(z) = \int_{|z|>r} d\mu(z) < +\infty.$$

The proof is complete. □

Remark 4.11. Note that $p_\beta^k \geq 0$ implies $\omega_{\beta,h} \geq 0$, but the other implication depends on μ and is not true in general. But if μ were the Lebesgue measure supported on a cube, then our quadrature would coincide with the Newton–Cotes quadratures which are known to have nonnegative weights for orders $k \leq 6$.

Moreover, if $d\mu(z) = \frac{dz}{|z|^{N+\alpha}}$ (fractional Laplace), then explicit nonnegative weights are found and presented in a nice way in [50] for $N = 1$, $k = 1$, and $k = 2$. In this case, quadratic interpolation combined with (4.11) yields an admissible discretization of $-(\Delta)^{\frac{\alpha}{2}}$ with a local truncation error of $O(h^{3-\alpha})$. Numerical evidence for this rate is given in section 5.

The following result on local truncation error is valid for any $k \in \mathbb{N}$.

LEMMA 4.12. *Assume (A_μ) , $h \leq r = o_h(1)$, $k \geq 0$, and $\mathcal{L}^{\mu,r}$ is defined by (4.12). Then the family $\{\mathcal{L}^h\}_{h>0}$ given by (4.14) has local truncation error*

$$(4.15) \quad \|\mathcal{L}^{\mu,r}[\psi] - \mathcal{L}^h[\psi]\|_{L^p} \leq Ch^{k+1} \|D^{k+1}\psi\|_{L^p\mu(\{|z| > r\})}, \quad p \in \{1, \infty\}.$$

Moreover, if also (A_{μ_α}) holds, then the local truncation error is $O(h^{k+1}r^{-\alpha})$.

Proof. We use the notation $P_\psi(x, z) = I_h^k[\psi(x + \cdot) - \psi(x)](z)$ and use (4.2) to get

$$\begin{aligned} \|\mathcal{L}^{\mu, r}[\psi] - \mathcal{L}^h[\psi]\|_{L^p(\mathbb{R}^N)} &\leq \int_{|z|>r} \|P_\psi(\cdot, z) - (\psi(\cdot + z) - \psi(\cdot))\|_{L^p(\mathbb{R}^N)} d\mu(z) \\ &\leq C \|D^{k+1}\psi\|_{L^p(\mathbb{R}^N)} h^{k+1} \int_{|z|>r} d\mu(z). \end{aligned}$$

The proof is complete. \square

To be consistent we need to impose that $h^{k+1}\mu(\{|z| > r\}) \rightarrow 0$ as $h \rightarrow 0$. When also (A_{μ_α}) holds, this is always satisfied for $k \geq 1$ and $r = o_h(1)$, while for $k = 0$ we need $hr^{-\alpha} = o_h(1)$. We are now in a position to state and prove admissibility for the cases $k = 0$ and $k = 1$.

LEMMA 4.13. *Assume (A_μ) , $h \leq r = o_h(1)$, and $\mathcal{L}^{\mu, r}$ is defined by (4.12). If either $k = 0$ and $h = o_r(\frac{1}{\mu(\{|z|>r\})})$ or $k = 1$, then the family $\{\mathcal{L}^h\}_{h>0}$ given by (4.13) is an admissible approximation of $\mathcal{L}^{\mu, r}$.*

Proof. By the above discussion and Lemma 4.10, $\{\mathcal{L}^h\}_{h>0}$ is an (FD)-type discretization in the class (A_ν) . For consistency we use the error estimate (4.15). When $k = 0$ we can conclude from the extra assumption $h = o_r(\frac{1}{\mu(\{|z|>r\})})$. When $k = 1$, we observe that

$$h^{k+1} \int_{|z|>r} d\mu(z) = \int_{|z|>r} h^2 d\mu(z) \leq \int_{|z|>0} |z|^2 \wedge r^2 d\mu(z)$$

and conclude by the dominated convergence theorem since $|z|^2 \wedge r^2 \rightarrow 0$ pointwise as $r \rightarrow 0^+$ and $|z|^2 \wedge r^2 \leq |z|^2 \wedge 1$ ($r \leq 1$) which is integrable by (A_μ) . Uniform integrability (UL) follows from Proposition 3.2 and the L^∞ version of the above estimate. \square

For admissibility and higher order interpolation, see Remark 4.11.

4.4.3. Quadrature from interpolation—absolute continuous μ . If μ is absolutely continuous with respect to the Lebesgue measure dz with density also called $\mu(z)$, then (cf. [53]) we can approximate $\mathcal{L}^{\mu, r}$ by

$$\begin{aligned} \mathcal{L}^h[\psi](x) &:= \int_{|z|>r} I_h^k[(\psi(x + \cdot) - \psi(x))\mu(\cdot)](z) dz \\ (4.16) \quad &= \sum_{|z_\beta|>r} (\psi(x + z_\beta) - \psi(x))\mu(z_\beta) \int_{|z|>r} p_\beta^k(z) dz. \end{aligned}$$

Note that (4.16) is in the finite difference form (FD) with symmetric weights

$$\omega_{\beta, h} = \mu(z_\beta) \int_{|z|>r} p_\beta^k(z) dz \quad \text{for } |z_\beta| > r \quad \text{and} \quad \omega_{\beta, h} = 0 \quad \text{otherwise.}$$

The weights are nonnegative for $k \leq 6$ since $\mu \geq 0$ and $\tilde{\omega}_{\beta, h} = \int_{\mathbb{R}^N} p_\beta^k(z) dz$ coincides with Newton–Cotes quadrature weights that are known to be nonnegative. As we will describe later, this discretization can also be combined with (4.11) to further improve the order of accuracy.

LEMMA 4.14. *Assume (A_μ) , $0 \leq k \leq 6$, $h, r > 0$, and $d\mu(z) = \mu(z) dz$ for a density $\mu \in C_b(\mathbb{R}^N \setminus B_r)$. Then the family $\{\mathcal{L}^h\}_{h>0}$ given by (4.16) is in the class (A_ν) .*

Proof. By the previous discussion we know that \mathcal{L}^h is of the form (FD) with symmetric nonnegative weights, so by Lemma 3.1, we just need to check

$$\begin{aligned} \sum_{|z_\beta|>r} \mu(z_\beta) \int_{|z|>r} p_\beta^k(z) d\mu(z) &\leq \|\mu\|_{L^\infty(\mathbb{R}^N \setminus B_r)} \int_{|z|>r} \sum_\beta p_\beta^k(z) d\mu(z) \\ &= \|\mu\|_{L^\infty(\mathbb{R}^N \setminus B_r)} \int_{|z|>r} d\mu(z) < +\infty. \end{aligned}$$

The proof is complete. □

Since μ is also interpolated now, the local truncation error will depend on the regularity of μ . We state the result using standard Sobolov spaces $W^{k,1}$ and $W^{k,\infty}$.

LEMMA 4.15. *Assume (A_μ) , $0 \leq k \leq 6$, $h, r > 0$, and that $d\mu(z) = \mu(z) dz$ for a density $\mu \in C(\mathbb{R}^N \setminus B_r) \cap W^{k+1,1}(\mathbb{R}^N \setminus B_r)$. If $\mathcal{L}^{\mu,r}$ is defined by (4.12) and $\{\mathcal{L}^h\}_{h>0}$ by (4.16), then there is a constant $C > 0$ such that for $p \in \{1, \infty\}$,*

$$\|\mathcal{L}^{\mu,r}[\psi] - \mathcal{L}^h[\psi]\|_{L^p(\mathbb{R}^N)} \leq Ch^{k+1} \|\psi\|_{W^{k+1,p}(\mathbb{R}^N)} \|\mu\|_{W^{k+1,1}(\mathbb{R}^N \setminus B_r)}.$$

Proof. Let $Q_\alpha := z_\alpha + \frac{kh}{2}(-1, 1]^N$ for $\alpha \in \mathbb{Z}^N$, y_i^α denote the $(k+1)^N$ gridpoints in $Q_\alpha \cap \mathcal{G}_h$, and $F(x, z) = (\psi(x+z) - \psi(x))\mu(z)$. Note that then $I_h^k \psi(y_i^\alpha) = \psi(y_i^\alpha)$,

$$\bigcup_{\alpha \in k\mathbb{Z}^N} Q_\alpha = \mathbb{R}^N, \quad \text{and} \quad \mathcal{L}^h[\psi](x) := \int_{|z|>r} I_h^k[F(x, \cdot)](z) dz.$$

Using Taylor expansions with integral remainder terms at every point y_i^α , we get

$$\begin{aligned} &|\mathcal{L}^h[\psi](x) - \mathcal{L}^{\mu,r}[\psi](x)| \\ &\leq \sum_{\alpha \in k\mathbb{Z}^N, |z_\alpha|>r} \int_{Q_\alpha} |I_h^k[F(x, \cdot)](z) - F(x, z)| dz \\ &\leq h^{k+1} \sum_{i=1}^{(k+1)^N} \sum_{\alpha \in k\mathbb{Z}^N, |z_\alpha|>r} \int_{Q_\alpha} \int_0^1 |D_z^{k+1} F(x, z(1-s) + sy_i^\alpha)| ds dz. \end{aligned}$$

By Fubini, the definition of F , and the chain rule, for $p = \{1, \infty\}$ it follows that

$$\begin{aligned} &\|\mathcal{L}^h[\psi] - \mathcal{L}^{\mu,r}[\psi]\|_{L^p(\mathbb{R}^N)} \\ &\leq h^{k+1} \sum_{i=1}^{(k+1)^N} \sum_{\alpha \in k\mathbb{Z}^N, |z_\alpha|>r} \int_{Q_\alpha} \int_0^1 \|D_z^{k+1} F(\cdot, z(1-s) + sy_i^\alpha)\|_{L^p(\mathbb{R}^N)} ds dz \\ &\leq Ch^{k+1} \|\psi\|_{W^{k+1,p}(\mathbb{R}^N)} \sum_{i=1}^{(k+1)^N} \sum_{\alpha \in k\mathbb{Z}^N, |z_\alpha|>r} \sum_{l=0}^{k+1} \int_0^1 \int_{Q_\alpha} |D^l \mu(z(1-s) + sy_i^\alpha)| dz ds. \end{aligned}$$

Now we do the change of variables $y = z(1-s) + sy_i^\alpha$, which has the change in measure $dy = (1-s)^N dz$ and maps Q_α into $\tilde{Q}_\alpha = sy_i^\alpha + (1-s)Q_\alpha$. Then since $s \in [0, 1]$ and Q_α is convex, $\tilde{Q}_\alpha \subset Q_\alpha$ and

$$\int_{Q_\alpha} |D^l \mu(z(1-s) + sy_i^\alpha)| dz = \int_{\tilde{Q}_\alpha} (1-s)^N |D^l \mu(y)| dy \leq \int_{Q_\alpha} |D^l \mu(y)| dy.$$

For $0 \leq l \leq k + 1$,

$$\sum_{\alpha \in k\mathbb{Z}^N, |z_\alpha| > r} \int_{Q_\alpha} |D^l \mu(y)| \, dy \leq \|\mu\|_{W^{k+1,1}(\mathbb{R}^N \setminus B_r)},$$

so we conclude that for $p \in \{1, \infty\}$,

$$\|\mathcal{L}^h[\psi] - \mathcal{L}^{\mu,r}[\psi]\|_{L^p(\mathbb{R}^N)} \leq Ch^{k+1} \|\psi\|_{W^{k+1,p}(\mathbb{R}^N)} \|\mu\|_{W^{k+1,1}(\mathbb{R}^N \setminus B_r)}.$$

The proof is complete. \square

From Lemmas 4.14, 4.15, and 3.1, we have the following result on admissibility.

COROLLARY 4.16. *Assume (A_μ) , $0 \leq k \leq 6$, $0 < r = o_h(1)$, $d\mu(z) = \mu(z) \, dz$ for a density $\mu \in C(\mathbb{R}^N \setminus B_r) \cap W^{k+1,1}(\mathbb{R}^N \setminus B_r)$, and $\mathcal{L}^{\mu,r}$ is defined by (4.12). If $h^{k+1} \|\mu\|_{W^{k+1,1}(\mathbb{R}^N \setminus B_r)} \rightarrow 0$ as $h \rightarrow 0$, then the family $\{\mathcal{L}^h\}_{h>0}$ given by (4.16) is an admissible approximation of $\mathcal{L}^{\mu,r}$.*

We expect to have more precise results when μ comes from a fractional differential operator, i.e., a precise estimate on how $\|\mu\|_{W^{k+1,1}(\mathbb{R}^N \setminus B_r)}$ depends on r . However stronger assumptions than (A_{μ_α}) are needed here. It is easy to find such conditions in general, but for simplicity we only focus on the fractional Laplacian case.

COROLLARY 4.17 (fractional Laplace). *Let $\alpha \in (0, 2)$, $0 \leq h \leq r = o_h(1)$, $0 \leq k \leq 6$, $\mathcal{L}^{\mu,r}$ be defined by (4.12) with density $\mu(z) = \frac{1}{|z|^{N+\alpha}}$. If $h = o(r^{\frac{\alpha+k+1}{k+1}})$, then the family $\{\mathcal{L}^h\}_{h>0}$ given by (4.16) is an admissible approximation of $\mathcal{L}^{\mu,r}$. Moreover, for $p \in \{1, \infty\}$,*

$$\|\mathcal{L}^{\mu,r}[\psi] - \mathcal{L}^h[\psi]\|_{L^p(\mathbb{R}^N)} \leq Ch^{k+1} r^{-\alpha-k-1}.$$

Proof. Since $D^k \mu(z) = O(\frac{1}{|z|^{N+\alpha+k}})$ and $\int_{|z|>r} \frac{1}{|z|^{N+\alpha+k}} \, dz = c \int_r^\infty \frac{1}{r^{1+\alpha+k}} \, dr = C \frac{1}{r^{\alpha+k}}$, we find that $\|\mu\|_{W^{k+1,1}(\mathbb{R}^N \setminus B_r)} = O(\frac{1}{r^{\alpha+k+1}})$. \square

Remark 4.18 (fractional Laplace). Combining the discretizations of Corollary 4.17 (Newton–Cotes for the nonsingular part) and Lemma 4.5 (adapted vanishing viscosity for the singular part) we get a high order monotone discretization of the fractional Laplacian with (combined) local truncation error,

$$E = O(r^{4-\alpha} + h^{k+1} r^{-\alpha-k-1}).$$

The optimal choice of r is $r = h^{\frac{k+1}{k+5}}$, which leads to a local truncation error of $O(h^{\frac{k+1}{k+5}(4-\alpha)})$. Note that for all $\alpha \in (0, 2)$, this rate is increasing in k , superlinear for $k \geq 3$, and superquadratic in the limit $k \rightarrow \infty$. The best choice giving an admissible (monotone) scheme is $k = 6$ and $r = O(h^{\frac{7}{11}})$ with a local truncation error of $O(h^{\frac{7}{11}(4-\alpha)})$.

Remark 4.19 (random walk approximation). In [65, 12] the fractional heat equation is formally derived from a random walk approximation with arbitrarily long jumps,

$$U^j(x_\beta) = U^{j-1}(x_\beta) + \Delta t \sum_{\gamma \in \mathbb{Z}^N \setminus \{0\}} (U^{j-1}(x_\beta + z_\gamma) - U^{j-1}(x_\beta)) \omega_{\gamma,h},$$

where $\omega_{\gamma,h} := \mu^\alpha(z_\gamma) h^N$, $\mu^\alpha(z) := \frac{c_{N,\alpha}}{|z|^{N+\alpha}}$ for $z \neq 0$, $\mu^\alpha(0) = 0$, and $c_{N,\alpha}$ is such that $\sum_{\gamma \in \mathbb{Z}^N} \mu^\alpha(\gamma) = 1$. Note that here $\Delta t = h^\alpha$ and $\frac{\mu^\alpha(\gamma)}{h^\alpha} = h^N \mu^\alpha(z_\gamma)$. The scheme corresponds to the spatial discretization (4.16) with $k = 0$, and converges by Corollary 4.17 and Theorem 3.6 if we require $|z_\gamma| > r = ch^{\frac{1}{\alpha+1+\epsilon}}$ for some $c, \epsilon > 0$.

4.5. A second order discretization of the fractional Laplacian. The fractional Laplacian $-(-\Delta)^{\frac{\alpha}{2}}$ can be discretized by the corresponding power of the discrete Laplacian

$$\Delta_h \psi(x) = \frac{1}{h^2} \sum_{i=1}^N \psi(x + e_i h) + \psi(x - e_i h) - 2\psi(x)$$

defined via subordination as

$$(4.17) \quad (-\Delta_h)^{\frac{\alpha}{2}}[\psi](x) := \frac{1}{\Gamma(-\frac{\alpha}{2})} \int_0^\infty (e^{t\Delta_h} \psi(x) - \psi(x)) \frac{dt}{t^{1+\frac{\alpha}{2}}},$$

where $\Psi(x, t) := e^{t\Delta_h} \psi(x)$ is the solution of the semidiscrete heat equation

$$(4.18) \quad \begin{cases} \partial_t \Psi = \Delta_h \Psi & \text{for } (x, t) \in \mathbb{R}^N \times (0, \infty), \\ \Psi(x, 0) = \psi(x) & \text{for } x \in \mathbb{R}^N. \end{cases}$$

The solution of (4.18) has an explicit representation formula,

$$\Psi(x, t) := e^{t\Delta_h} \psi(x) = \sum_{\beta} \psi(x - z_{\beta}) G\left(\beta, \frac{t}{h^2}\right),$$

where $G(\beta, t) = e^{-2Nt} \prod_{i=1}^N I_{|\beta_i|}(2t)$ is the fundamental solution of (4.18) and I_m the modified Bessel function of first kind and order $m \in \mathbb{N}$. Moreover $G \geq 0$, $G(\beta, t) = G(-\beta, t)$, and $\sum_{\beta} G(\beta, t) = 1$; see, e.g., [20, 56].

The original idea of this discretization is due to Ciaurri et al. [20, 21]. Here the authors obtain a local truncation error of $O(h^{2-\alpha})$ for $N = 1$. These results were then extended to $N > 1$ in [28, 27] for various boundary value problems and the whole space case. Here the authors also improve the local truncation error to $O(h^2)$ independently of $\alpha \in (0, 2)$. This means that the approximation $(-\Delta_h)^{\frac{\alpha}{2}}$ preserves the $O(h^2)$ error bound of the discrete Laplacian Δ_h .

We now show that (4.17) is an admissible operator. First we express (4.17) in the form (FD). This result has essentially been proved in [20].

LEMMA 4.20. *Let $h > 0$, $\alpha \in (0, 2)$, and $N \geq 1$. Then $(-\Delta_h)^{\frac{\alpha}{2}}$ given by (4.17) is an operator of the form (FD),*

$$(4.19) \quad (-\Delta_h)^{\frac{\alpha}{2}}[\psi](x) = \sum_{\beta \neq 0} (\psi(x + z_{\beta}) - \psi(x)) K_{\beta, h},$$

where $K_{\beta, h} = \frac{1}{h^{\alpha}} \frac{1}{\Gamma(-\frac{\alpha}{2})} \int_0^\infty G(\beta, t) \frac{dt}{t^{1+\frac{\alpha}{2}}}$.

Proof. By (4.17), the representation formula for Ψ , and $\sum_{\beta} G(\beta, \frac{t}{h^2}) = 1$,

$$\begin{aligned} (-\Delta_h)^{\frac{\alpha}{2}}[\psi](x) &:= \frac{1}{\Gamma(-\frac{\alpha}{2})} \int_0^\infty \left(\sum_{\beta} \psi(x - z_{\beta}) G\left(\beta, \frac{t}{h^2}\right) - \psi(x) \right) \frac{dt}{t^{1+\frac{\alpha}{2}}} \\ &= \frac{1}{\Gamma(-\frac{\alpha}{2})} \int_0^\infty \sum_{\beta} (\psi(x - z_{\beta}) - \psi(x)) G\left(\beta, \frac{t}{h^2}\right) \frac{dt}{t^{1+\frac{\alpha}{2}}} \\ &= \sum_{\beta} (\psi(x - z_{\beta}) - \psi(x)) \frac{1}{\Gamma(-\frac{\alpha}{2})} \int_0^\infty G\left(\beta, \frac{t}{h^2}\right) \frac{dt}{t^{1+\frac{\alpha}{2}}}. \end{aligned}$$

The change of variables $\tau = t/h^2$ finishes the proof. □

Remark 4.21. In dimension $N = 1$ a more explicit expression for $K_{\beta,h}$ is given in [21]: $K_{j,h} := \frac{1}{h^\alpha} \frac{2^\alpha \Gamma(\frac{1+\alpha}{2}) \Gamma(|j| - \frac{\alpha}{2})}{\sqrt{\pi} \Gamma(-\frac{\alpha}{2}) |\Gamma(|j| + 1 + \frac{\alpha}{2})|}$ for $j \in \mathbb{Z}$, $j \neq 0$.

LEMMA 4.22. *Assume $h > 0$, $\alpha \in (0, 2)$, and $N \geq 1$. The family $\{-(-\Delta_h)^{\frac{\alpha}{2}}\}_{h>0}$ given by (4.17) (or (4.19)) is an admissible approximation of $-(-\Delta)^{\frac{\alpha}{2}}$ with local truncation error*

$$\|(-\Delta_h)^{\frac{\alpha}{2}}[\psi] - (-\Delta)^{\frac{\alpha}{2}}[\psi]\|_{L^p(\mathbb{R})} = O(h^2)$$

for $p = \{1, \infty\}$ and $\psi \in C_c^\infty(\mathbb{R}^N)$.

Proof. By Lemma 4.20, $-(-\Delta_h)^{\frac{\alpha}{2}}$ is an operator in the form (FD) explicitly given by (4.19). We show that it is in the class (A_ν) . It is clear that $K_{\beta,h}$ is nonnegative and symmetric in β since these properties are shared by G . Next,

$$C_2 := \sum_{\beta \neq 0} \int_1^\infty G(\beta, t) \frac{dt}{t^{1+\frac{\alpha}{2}}} = \int_1^\infty \sum_{\beta \neq 0} G(\beta, t) \frac{dt}{t^{1+\frac{\alpha}{2}}} \leq \int_1^\infty \frac{dt}{t^{1+\frac{\alpha}{2}}} = \frac{2}{\alpha}$$

and

$$C_1 := \sum_{\beta \neq 0} \int_0^1 G(\beta, t) \frac{dt}{t^{1+\frac{\alpha}{2}}} = \int_0^1 \sum_{\beta \neq 0} G(\beta, t) \frac{dt}{t^{1+\frac{\alpha}{2}}} = \int_0^1 (1 - G(0, t)) \frac{dt}{t^{1+\frac{\alpha}{2}}}.$$

By regularity and the properties of G , $|1 - G(0, t)| = |G(0, 0) - G(0, t)| \leq Ct$ for $C = \max_{\xi \in [0, t]} \{\partial_t G(0, t)\}$, and then $C_1 \leq C \int_0^1 t \frac{dt}{t^{1+\frac{\alpha}{2}}} = C \frac{2}{2-\alpha}$. We conclude that

$$\sum_{\beta \neq 0} K_{\beta,h} = \frac{1}{h^\alpha} \frac{1}{\Gamma(-\frac{\alpha}{2})} (C_1 + C_2) < +\infty,$$

and that $-(-\Delta_h)^{\frac{\alpha}{2}}$ is in the class (A_ν) by Lemma 3.1.

Now we need to show that the discretization is consistent. We proceed as in [28, 27]. Using directly the semigroup formulation (4.17), we get that

$$(-\Delta_h)^{\frac{\alpha}{2}}[\psi](x) - (-\Delta)^{\frac{\alpha}{2}}[\psi](x) = \frac{1}{\Gamma(-\frac{\alpha}{2})} \int_0^\infty (e^{t\Delta_h} \psi(x) - e^{t\Delta} \psi(x)) \frac{dt}{t^{1+\frac{\alpha}{2}}},$$

where $e^{t\Delta} \psi(x)$ is the solution of the heat equation with initial condition ψ . Assume for the moment that the following estimate holds for $p = \{1, \infty\}$:

$$(4.20) \quad \|e^{t\Delta_h} \psi(\cdot) - e^{t\Delta} \psi(\cdot)\|_{L^p(\mathbb{R}^N)} \leq \begin{cases} Cth^2 & \text{for } 0 \leq t \leq 1, \\ C\frac{h^2}{t} & \text{for } t \geq 1 \end{cases}$$

for some $C > 0$ depending on ψ . Then,

$$\begin{aligned} \|(-\Delta_h)^{\frac{\alpha}{2}}[\psi] - (-\Delta)^{\frac{\alpha}{2}}[\psi]\|_{L^p(\mathbb{R}^N)} &\leq \frac{1}{\Gamma(-\frac{\alpha}{2})} \int_0^\infty \|e^{t\Delta_h} \psi(\cdot) - e^{t\Delta} \psi(\cdot)\|_{L^p(\mathbb{R}^N)} \frac{dt}{t^{1+\frac{\alpha}{2}}} \\ &\leq Ch^2 \int_0^1 \frac{dt}{t^{\frac{\alpha}{2}}} + Ch^2 \int_1^\infty \frac{dt}{t^{2+\frac{\alpha}{2}}} \leq \tilde{C}h^2. \end{aligned}$$

From the L^∞ -estimate and Proposition 3.2, we also have that (UL) holds.

It only remains to prove estimate (4.20). Recall that

$$e^{t\Delta}\psi(x) = \int_{\mathbb{R}^N} \psi(x-y)F(y,t) dy \quad \text{with} \quad F(x,t) = \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|x|^2}{4t}},$$

and set $\tau(x,t) := \partial_t e^{t\Delta}\psi(x) - \Delta_h e^{t\Delta}\psi(x)$. Since $e^{t\Delta}\psi$ is smooth, a Taylor expansion argument and the properties of F, D^4F show that

$$(4.21) \quad \|\tau(\cdot, t)\|_{L^p} \leq h^2 \|D^4 e^{t\Delta}\psi\|_{L^p} \leq \begin{cases} h^2 \|D^4\psi\|_{L^p} \|F\|_{L^1} \leq Ch^2 & \text{for } 0 \leq t \leq 1, \\ h^2 \|\psi\|_{L^p} \|D^4F\|_{L^1} \leq C\frac{h^2}{t^2} & \text{for } t \geq 1. \end{cases}$$

Now let $E(x,t) := e^{t\Delta_h}\psi(x) - e^{t\Delta}\psi(x)$ and note that $\partial_t E(x,t) = \Delta_h E(x,t) + \tau(x,t)$ and $E(x,0) = 0$. The weak maximum principle (for (4.18)) and a standard L^1 bound then immediately yield

$$\|E(\cdot, t)\|_{L^p} \leq \int_0^t \|\tau(\cdot, s)\|_{L^p} ds,$$

and (4.20) follows from (4.21). □

4.6. Approximation of the nonlinearity. As we saw in section 3.3, we need to impose the condition (CFL) when the schemes have some explicit part (i.e., when $0 \leq \theta < 1$). This condition requires the nonlinearity φ to be (locally) Lipschitz. But our results can handle merely continuous φ . If φ is not locally Lipschitz as, e.g., in the fast diffusion case, we must replace it by a Lipschitz approximation to get explicit monotone schemes. To be precise, we approximate φ by a sequence of nondecreasing Lipschitz functions φ^ϵ converging locally uniformly as $\epsilon = o_h(1) \rightarrow 0^+$. The (CFL) condition is then

$$(CFL) \quad \Delta t(1-\theta)L_{\varphi^\epsilon\nu_h}(\mathbb{R}^N) \leq 1.$$

Note that $L_{\varphi^\epsilon} \rightarrow \infty$ as $\epsilon \rightarrow 0^+$ making (CFL) a more and more restrictive condition as h approaches zero.

There are several ways of choosing the nonlinearity φ^ϵ in an admissible way. Two simple and general choices (cf., e.g., [36, 66]) are

$$\varphi^\epsilon(\xi) := (\varphi * \omega_\epsilon)(\xi) - (\varphi * \omega_\epsilon)(0) \quad \text{and} \quad \varphi^\epsilon(\xi) + \epsilon\xi,$$

where ω_ϵ is a standard mollifier in \mathbb{R} . However, in many applications φ is non-Lipschitz only at the origin. A well-known example is the fast diffusion equation, where $\varphi(\xi) = \xi^m$ for $0 < m < 1$. In this case an easier and more efficient choice is

$$\varphi^\epsilon(\xi) = \begin{cases} \varphi(\xi + \epsilon) - \varphi(\epsilon) & \text{if } \xi \geq 0, \\ \varphi(\xi - \epsilon) - \varphi(-\epsilon) & \text{if } \xi < 0. \end{cases}$$

Clearly $\varphi^\epsilon \rightarrow \varphi$ locally uniformly and φ^ϵ is Lipschitz with Lipschitz constant

$$L_{\varphi^\epsilon} = |(\varphi^\epsilon)'(0)| = \frac{m}{\epsilon^{1-m}}.$$

Moreover, this approximation enjoys the very interesting property of preserving the zero level sets of the solution since $\varphi^\epsilon(0) = 0 = \varphi(0)$.

5. Numerical experiments in 1 dimension. In this section, we test our numerical schemes on interesting special cases of (1.1)–(1.2) in one space dimension that involve the fractional Laplacian,

$$(5.1) \quad \partial_t u(x, t) + (-\Delta)^{\frac{\alpha}{2}} [\varphi(u(\cdot, t))](x) = g(x, t) \quad \text{in} \quad Q_T := \mathbb{R} \times (0, T)$$

for $\alpha \in (0, 2)$. All the schemes are of the form (3.1), and since our initial data and right-hand sides will be smooth, we simply take $U_\beta^0 = u_0(x_\beta)$ and $F_\beta^j = f(x_\beta, t_j)$. We consider explicit ($\theta = 0$) and implicit ($\theta = 1$) schemes and the following spatial discretizations of the nonlocal operators:

- 1) MpR = midpoint rule + trivial discretization for singular part;
- 2) FOI = first order interpolation + trivial discretization for singular part;
- 3) SOI = second order interpolation + adapted vanishing viscosity;
- 4) PDL = powers of the discrete Laplacian.

All experiments have been run on equidistant grids in space and time, with Δt such that (CFL) holds and the overall order of convergence is determined by the spatial discretization. To compute the solutions we restrict the grids to a (sufficiently) large bounded spatial domain I and set the numerical solution equal to zero outside. See section 5.5 for some numerical tests on how the size of the domain affects the error.

The error is calculated either in $L^1(I)$ or $L^\infty(I)$ at a certain time $T > 0$. To compute the error, we consider examples with (i) known exact solutions, or (ii) we force a nice function v to be a solution by taking

$$(5.2) \quad g(x, t) = \partial_t v(x, t) + (-\Delta)^{\frac{\alpha}{2}} [\varphi(v(\cdot, t))](x),$$

or (iii) we compute the errors numerically. In the latter case we assume the error E satisfies $E = Ch^\gamma$, take $h_j = c2^{-j}$, and compute an estimate of the rate γ as

$$\gamma = \log_2 \left(\frac{E_{j-1}}{E_j} \right).$$

5.1. Fractional heat equation (explicit scheme). We consider (5.1) with $\varphi(\xi) = \xi$, $\alpha = 1$, and $g \equiv 0$. This is a fractional heat equation with explicit fundamental solution $K(x, t) = \frac{t}{t^2 + |x|^2}$. We take $u_0(x) = K(x, 1)$ so that the exact solution is

$$(5.3) \quad u(x, t) = K(x, t + 1).$$

We also take $I = [-5000, 5000]$. By (5.3), $u \sim 10^{-8}$ outside of I , and hence is negligible in the error analysis. In Figure 1 and Table 1 we show the error and rates at time $T = 1$.

Conclusion: The MpR and the PDL coincide when $\alpha = 1$ and $N = 1$ (see section 4). The theoretical convergence rates are confirmed for all tested methods. Note that when $h = 1.56\text{e-}2$, the rate for MpR and PDL is $\gamma = 1.61$ instead of $\gamma \sim 2$. This is a consequence of how the rates are calculated since the previous value $\gamma = 2.40$ is much better than predicted. If we calculate the rate for $h = 1.56\text{e-}2$ with respect to $h = 6.25\text{e-}2$, we get $\gamma = 2.01$.

5.2. Fractional porous medium equation (explicit scheme). We consider (5.1) with $\varphi(\xi) = \xi^2$. An explicit solution of that problem is known when $\alpha = \frac{1}{3}$ (cf. [49]), but its slow decay at infinity makes it difficult to find a reasonably small computational domain I . To overcome this issue we impose $v = (t + 1)e^{-|x|^2}$ as a

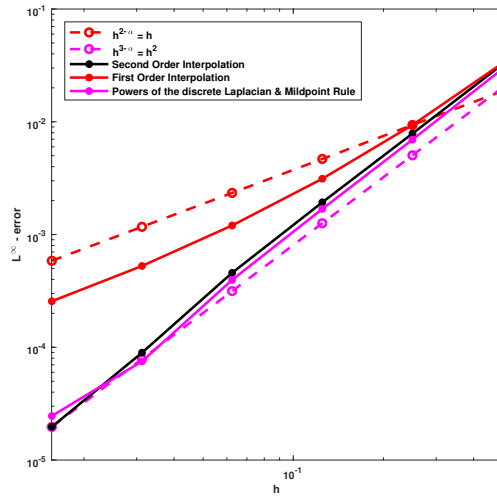


FIG. 1. L^∞ -error at $T = 1$ with $I = [-5000, 5000]$ for the exact solution (5.3) of (5.1) with $\alpha = 1$, $\varphi(\xi) = \xi$, and $g \equiv 0$.

TABLE 1

L^∞ -error at $T = 1$ with $I = [-5000, 5000]$ for the exact solution (5.3) of (5.1) with $\alpha = 1$, $\varphi(\xi) = \xi$, and $g \equiv 0$.

h	MpR		FOI		SOI		PDL	
	Error	γ	Error	γ	Error	γ	Error	γ
5.00e-1	2.95e-2		3.31e-2		3.24e-3		2.95e-2	
2.50e-1	6.94e-3	2.08	9.40e-3	1.82	7.89e-3	2.04	6.94e-3	2.08
1.25e-1	1.68e-3	2.04	3.12e-3	1.58	1.93e-3	2.03	1.68e-3	2.04
6.25e-2	3.95e-4	2.09	1.20e-4	1.37	4.57e-4	2.08	3.95e-4	2.09
3.13e-2	7.50e-5	2.40	5.26e-4	1.19	8.96e-5	2.35	7.50e-5	2.40
1.56e-2	2.45e-5	1.61	2.56e-4	1.03	1.97e-5	2.18	2.45e-5	1.61

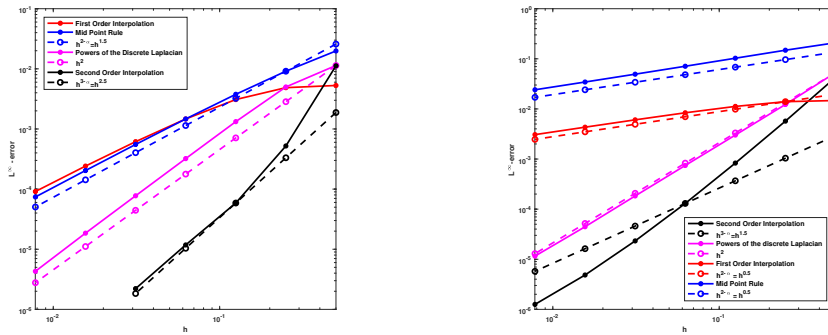


FIG. 2. L^∞ -error at $T = 1$ with $I = [-100, 100]$ for (5.1) with $\varphi(\xi) = \xi^2$ and g by (5.2) with $v(x, t) = (t + 1)e^{-|x|^2}$. Left: $\alpha = 0.5$. Right: $\alpha = 1.5$.

solution by taking g according to (5.2). Note that now the solution has exponential decay. The errors and rates of convergence are shown for $I = [-100, 100]$ at time $T = 1$ for $\alpha = 0.5$ (resp., $\alpha = 1.5$) in Figure 2 and Table 2 (resp., Table 3).

TABLE 2

L^∞ -error at $T = 1$ with $I = [-100, 100]$ for (5.1) with $\alpha = 0.5$, $\varphi(\xi) = \xi^2$, and g by (5.2) with $v(x, t) = (t + 1)e^{-|x|^2}$.

h	MPR		FOI		SOI		PDL	
	Error	γ	Error	γ	Error	γ	Error	γ
5.00e-1	1.98e-2		5.29e-3		1.12e-2		1.14e-2	
2.50e-1	9.13e-3	1.22	4.86e-3	0.12	5.21e-4	4.42	5.00e-3	1.19
1.25e-1	3.77e-3	1.28	3.09e-3	0.65	5.87e-5	3.15	1.32e-3	1.91
6.25e-2	1.47e-3	1.36	1.47e-3	1.07	1.18e-5	2.31	3.21e-4	2.04
3.13e-2	5.53e-4	1.40	6.15e-4	1.25	2.21e-6	2.41	7.74e-5	2.05
1.56e-2	2.04e-4	1.44	2.41e-4	1.35	—	—	1.85e-5	2.06
7.81e-3	7.41e-5	1.46	9.13e-5	1.40	—	—	4.30e-6	2.11

TABLE 3

L^∞ -error at $T = 1$ with $I = [-100, 100]$ for (5.1) with $\alpha = 1.5$, $\varphi(\xi) = \xi^2$, and g by (5.2) with $v(x, t) = (t + 1)e^{-|x|^2}$.

h	MPR		FOI		SOI		PDL	
	Error	γ	Error	γ	Error	γ	Error	γ
5.00e-1	2.10e-1		1.47e-2		4.26e-2		5.30e-2	
2.50e-1	1.49e-1	0.49	1.40e-2	0.08	5.71e-3	2.90	1.23e-2	2.11
1.25e-1	1.03e-1	0.53	1.12e-2	0.31	8.30e-4	2.78	3.01e-3	2.03
6.25e-2	7.11e-2	0.53	8.37e-3	0.42	1.30e-4	2.67	7.44e-4	2.16
3.13e-2	4.93e-2	0.53	6.05e-3	0.47	2.32e-5	2.48	1.83e-4	2.02
1.56e-2	3.44e-2	0.52	4.32e-3	0.49	4.85e-6	2.25	4.46e-5	2.04
7.81e-3	2.41e-2	0.51	3.07e-3	0.49	1.25e-6	1.94	1.16e-5	1.95

Conclusion: When $\alpha = 0.5$ all the expected rates are recovered. For the SOI, we excluded the last two rows because the error was much smaller compared to the other methods; see Table 2. It is also worth noting that when $\alpha = 1.5$ (see Table 3), the expected rate of convergence for SOI is $\gamma = 1.5$ while the rate obtained by the experiment is ~ 2.0 .

5.3. Stefan-type problem (explicit scheme). First we consider (5.1) with $\varphi(\xi) = \max\{0, \xi - 0.5\}$ (a globally Lipschitz function) and $g \equiv 0$. Solutions of such problems can lose regularity as we see in Figure 3 (below). Here we have used $C_c^\infty(\mathbb{R})$ initial data $u_0(x) = e^{-\frac{1}{4-x^2}} \mathbf{1}_{[-2,2]}(x)$.

Next we consider (5.1) with

$$(5.4) \quad \varphi(\xi) = \begin{cases} \xi & \text{if } \xi < 0.2, \\ 0.2 & \text{if } 0.2 \leq \xi < 0.4, \\ \xi - 0.2 & \text{if } \xi \geq 0.4, \end{cases}$$

and impose $v(x, t) = (t + 1)e^{-|x|^2}$ as the solution by taking g as in (5.2). We run experiments for $\alpha = 0.5$, $\alpha = 1$, and $\alpha = 1.5$ up to $T = 1$ with $I = [-100, 100]$ using the MPR. The results are given in Figure 4 (below) and Table 4 (below).

Conclusion: It is interesting to note that even when the solution $u \in C^\infty(\mathbb{R}^N)$, $\varphi(u)$ is just a Lipschitz function. Therefore $(-\Delta)^{\frac{1}{2}}[\varphi(u)]$ and $(-\Delta)^{\frac{3}{4}}[\varphi(u)]$ do not exist in every point and u is not a classical solution. This fact has a strong effect on the convergence rates in L^∞ , but not in L^1 ; see Figure 4 and Table 4.

Remark 5.1. In [10] the authors study a nonlocal Stefan problem of the form (1.1) with $\sigma \equiv 0$ and μ having a nonnegative continuous compactly supported density ((A_μ) holds). This case is easier than the above example since $\mathfrak{L}^{\sigma, \mu}$ is bounded. Nonetheless

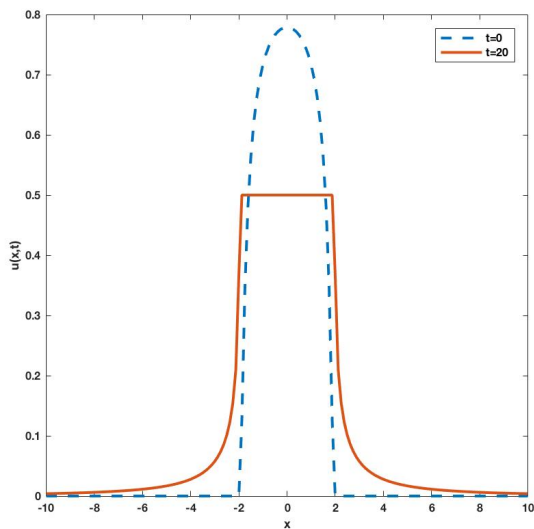


FIG. 3. Solution of (5.1) for $\alpha = 1$, $\varphi(\xi) = \max\{0, \xi - 0.5\}$, and $g \equiv 0$.

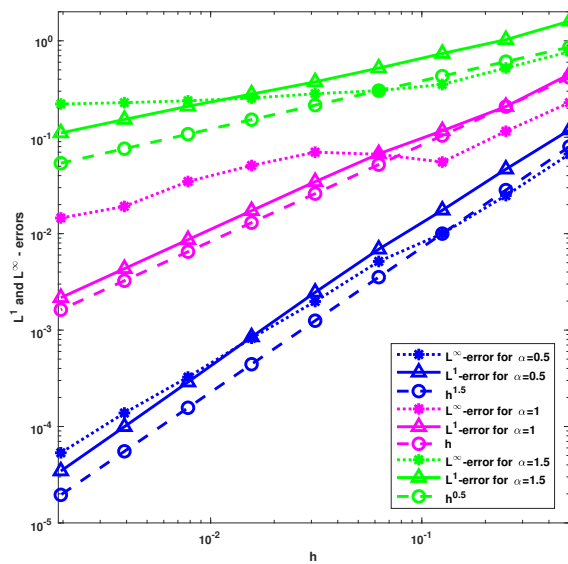


FIG. 4. L^∞ - and L^1 -errors with MpR at $T = 1$ with $I = [-100, 100]$ for (5.1) with φ given by (5.4) and g by (5.2) with $v(x, t) = (t + 1)e^{-|x|^2}$.

it has discontinuous solutions that are computed numerically in [10, Chapter 7]. Our results provide a rigorous justification for these computations.

5.4. Fractional fast diffusion equation. In the fast diffusion case a new difficulty appears: The nonlinearity, $\varphi(\xi) = \xi^m$ for $m \in (0, 1)$, is no longer locally Lipschitz, and the (CFL) condition can only hold for implicit schemes or under approximation of φ . Even in the local case there are few results for this case, and the results in this paper are, as far as we know, the first in the fractional case.

TABLE 4

L^∞ - and L^1 -errors with MpR at $T = 1$ with $I = [-100, 100]$ for (5.1) with φ given by (5.4) and g by (5.2) with $v(x, t) = (t + 1)e^{-|x|^2}$.

h	$\alpha = \frac{1}{2}$				$\alpha = 1$				$\alpha = \frac{3}{2}$			
	L^∞	γ	L^1	γ	L^∞	γ	L^1	γ	L^∞	γ	L^1	γ
5.0e-1	6.7e-2		1.3e-1		2.3e-1		4.5e-1		7.7e-1		1.6e-0	
2.5e-1	2.5e-2	1.4	4.7e-2	1.4	1.2e-1	1	2.1e-1	1.1	5.2e-1	0.6	1.0e-0	0.6
1.3e-1	1.0e-2	1.3	1.8e-2	1.4	5.5e-2	1.1	1.2e-1	0.8	3.5e-1	0.6	7.4e-1	0.5
6.3e-2	5.2e-3	1.0	6.9e-3	1.3	6.7e-2	<0	6.7e-2	0.8	3.0e-1	0.2	5.2e-1	0.5
3.1e-2	2.0e-3	1.4	2.5e-3	1.5	7.0e-2	<0	3.5e-2	1.0	2.8e-1	0.1	3.8e-1	0.5
1.6e-2	8.2e-4	1.3	8.5e-4	1.5	5.1e-2	0.5	1.7e-2	1.0	2.6e-1	0.2	2.8e-1	0.4
7.8e-3	3.3e-4	1.3	2.9e-4	1.6	3.5e-2	0.6	8.5e-3	1.0	2.4e-1	0.1	2.1e-1	0.4
3.9e-3	1.4e-4	1.2	9.9e-5	1.5	1.9e-2	0.9	4.3e-3	1.0	2.3e-1	0.1	1.5e-1	0.5
2.0e-3	5.3e-5	1.4	3.4e-5	1.5	1.5e-2	0.4	2.2e-3	1.0	2.2e-1	0.1	1.1e-1	0.5

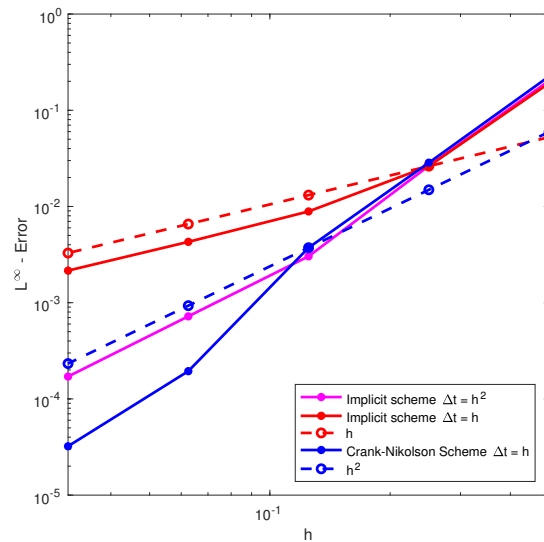


FIG. 5. L^∞ -error with MpR at $T = 1$ with $I = [-4, 4]$ for (5.1) with $\varphi(\xi) = \xi^{\frac{1}{2}}$, $\alpha = 1$, and g given by (5.2) with $v(x, t) = \sqrt{(t + 1)}e^{-|x|^8}$.

5.4.1. Implicit scheme. Implicit schemes automatically satisfy (CFL) at the price of having to solve a nonlinear system of equations at every time step. This is computationally very expensive. For simplicity we use the standard nonlinear solver “`fsolve`” in MATLAB. A more adapted solver could probably significantly reduce the computational time. Due to the computational cost, we take a very small domain and a very regular solution. We consider (5.1) with $\varphi(\xi) = \xi^{\frac{1}{2}}$, $\alpha = 1$, and g given by (5.2) with $v(x, t) = \sqrt{(t + 1)}e^{-|x|^8}$ (the solution). We run the experiments with the MpR up to time $T = 1$ in the domain $I = [-4, 4]$. For the time discretization we choose $\Delta t \sim h$ and $\Delta t \sim h^2$. We also consider the Crank–Nicolson method ($\theta = \frac{1}{2}$) with $\Delta t \sim h$ which means that (CFL) is satisfied. The results are shown in Figure 5 (above) and Table 5 (below).

Conclusion: For the implicit method with the MpR (a second order method when $\alpha = 1$), the expected error is $O(h^2 + \Delta t)$. When $\Delta t \sim h$ the error is clearly governed by Δt and so the rate of convergence is linear. For $\Delta t \sim h^2$, the error intro-

TABLE 5
 L^∞ -error with MpR at $T = 1$ with $I = [-4, 4]$ for (5.1) with $\varphi(\xi) = \xi^{\frac{1}{2}}$, $\alpha = 1$, and g given by (5.2) with $v(x, t) = \sqrt{(t+1)}e^{-|x|^8}$. Cr-Ni is Crank-Nicolson.

h	Implicit	$\Delta t \sim h$	Implicit	$\Delta t \sim h^2$	Cr-Ni	$\Delta t \sim h$
	error	γ	error	γ	error	γ
5.00e-1	1.91e-1		2.03e-1		2.25e-1	
2.50e-1	2.63e-2	2.86	2.64e-2	2.95	2.85e-2	2.98
1.25e-1	8.88e-3	1.56	3.03e-3	3.12	3.73e-3	2.93
6.25e-2	4.29e-3	1.05	7.21e-4	2.07	1.94e-4	4.26
3.13e-2	2.16e-3	1.00	1.71e-4	2.08	3.22e-5	2.59

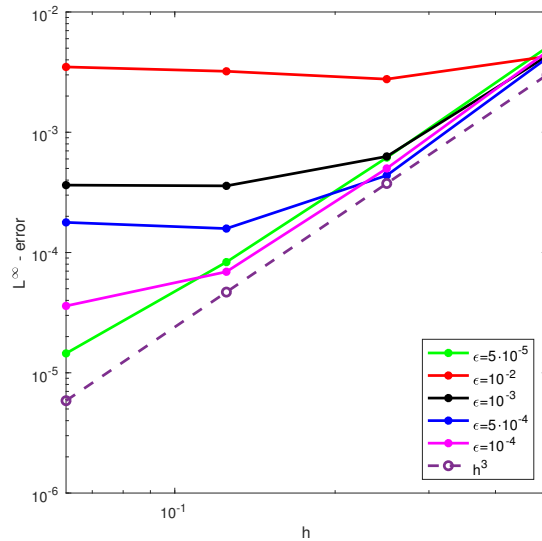


FIG. 6. L^∞ -errors with SOI at $T = 1$ with $I = [-1000, 1000]$ for the exact solution (5.5) of (5.1) with $g \equiv 0$, $\varphi(\xi) = \xi^{0.6}$, and $\alpha = 1.5$, approximating the nonlinearity by $\varphi^\epsilon(\xi) = (\xi + \epsilon)^{0.6} - \epsilon^{0.6}$.

duced by the time discretization is proportional to h^2 and so the rate of convergence is quadratic. For the Crank-Nicolson method, the expected error is $O(h^2 + \Delta t^2)$, so with $\Delta t \sim h$ we should see second order convergence. However the observed rates are better; see Table 5. The Crank-Nicolson method seems to be better than both implicit methods in terms of computational time and accuracy.

5.4.2. Explicit scheme approximating the nonlinearity. We consider (5.1) with $g \equiv 0$, $\varphi(\xi) = \xi^m$, $m = 0.6$, and $\alpha = 1.5$, and note that by [49] the solution is given by

$$(5.5) \quad v(x, t) = \lambda(t+1)^{-\beta} \left(1 + (|x|(t+1)^{-\beta})^2 \right)^{-\frac{\alpha+1}{2}}$$

for $\beta = \frac{1}{m-1+\alpha}$ and $\lambda = \left(\frac{2^{\alpha-1} \Gamma((1+\alpha)/2)}{\beta \Gamma((3-\alpha)/2)} \right)^{1/(1-m)}$. We use the approach of section 4.6 and implement an explicit method with the SOI + adapted vanishing viscosity (SOI) + approximate nonlinearity $\varphi^\epsilon(\xi) = (\xi + \epsilon)^{0.6} - \epsilon^{0.6}$. We run experiments for up to time $T = 1$ with $I = [-1000, 1000]$ for Δt 's satisfying (CFL) for φ^ϵ . The results are shown in Figure 6 (above) and Table 6 (below).

TABLE 6

L^∞ -errors with SOI at $T = 1$ with $I = [-1000, 1000]$ for the exact solution (5.5) of (5.1) with $f \equiv 0$, $\varphi(\xi) = \xi^{0.6}$, and $\alpha = 1.5$, approximating the nonlinearity by $\varphi^\epsilon(\xi) = (\xi + \epsilon)^{0.6} - \epsilon^{0.6}$. Note that we have only included the most accurate approximations here.

h	$\epsilon = 5e-4$			$\epsilon = 1e-4$			$\epsilon = 5e-5$		
	Δt	error	γ	Δt	error	γ	Δt	error	γ
5.00e-1	9.39e-3	4.14e-3		4.93e-3	4.67e-3		1.97e-3	5.09e-3	
2.50e-1	3.32e-3	4.38e-4	3.24	1.74e-3	5.00e-4	3.22	6.16e-4	6.16e-4	3.06
1.25e-1	1.17e-3	1.58e-4	1.47	6.17e-4	6.92e-5	2.85	8.32e-5	8.32e-5	2.89
6.25e-2	4.15e-4	1.78e-4	-0.17	2.18e-4	3.60e-5	0.94	1.44e-5	1.45e-5	2.52

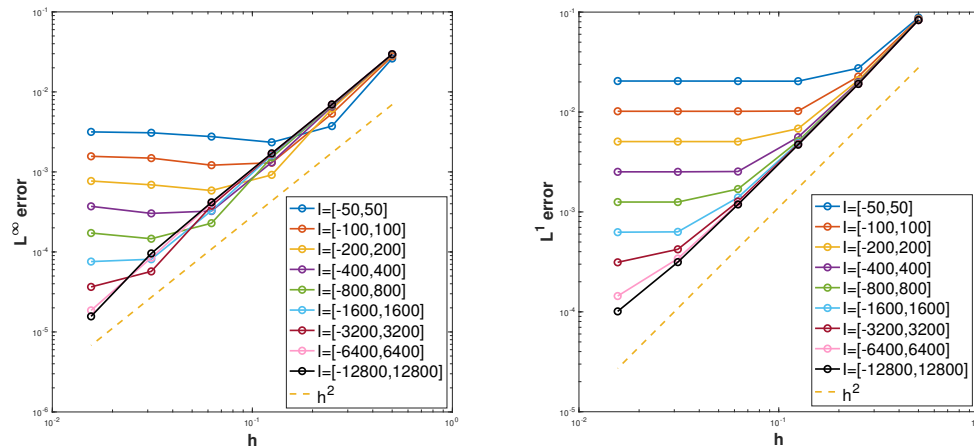


FIG. 7. L^∞ - and L^1 -errors with PDL at $T = 1$ with different interval sizes I for the exact solution (5.3) of (5.1) with $\alpha = 1$, $\varphi(\xi) = \xi$, and $g \equiv 0$.

Conclusion: Here the expected error is $O(h^{3-\alpha} + \Delta t)$ or $O(h^{1.5})$ when $\alpha = 1.5$ and $\Delta t \sim h^\alpha$. Since we are approximating the nonlinearity, the (CFL) condition becomes more and more restrictive when ϵ is decreased. When we fix ϵ and let $h, \Delta t \rightarrow 0^+$, the error stops decreasing at some point as can be seen in Figure 6 and Table 6. However, very good results are obtained for small ϵ . Compared with the implicit method of the previous section, the present method is better both in terms of computational times (we are able to deal with much bigger domains) and accuracy (we reach almost 10^{-6} with $h = 6.25e-2$ instead of $h = 3.13e-2$).

5.5. On the truncation of the domain. To test numerically the effect of the restriction to a large bounded domain, we consider the fractional heat equation and the explicit solution of section 5.1. An explicit PDL scheme is run up to $T = 1$ on a sequence of increasing domains. We take $\Delta t \sim h^2$ which satisfies (CFL) and ensures that the space discretization errors dominate. See Figure 7 for the results. We also test the minimal error that can be reached for a fixed domain for different values of α . Here we consider the fractional porous medium equation with $m = (3 - \alpha)/(1 + \alpha)$ and explicit solutions from [49]. See Figure 8.

Conclusion: The expected error for our schemes is $O(h^2 + \Delta t) = O(h^2)$ since $\Delta t \sim h^2$. For each fixed domain I , we see from Figure 7 that the errors decrease as $h \rightarrow 0^+$ down to some threshold below which there is no improvement. At these thresholds, the dominant error comes from the truncation of the domain. As expected,

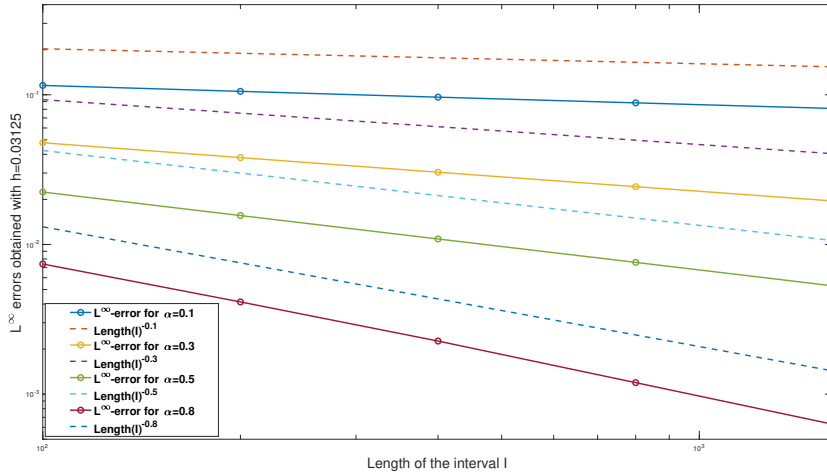


FIG. 8. Minimal L^∞ - and L^1 -errors with PDL at $T = 1$ with different interval sizes I for the exact solution of (5.1) with $\varphi(\xi) = \xi^{\frac{3-\alpha}{1+\alpha}}$, and $g \equiv 0$.

these thresholds decrease as the size of the domain increases. Figure 8 shows that when you vary α , the minimal reachable errors for a fixed domain I is of the order of $\text{length}(I)^{-\alpha}$. This could be an indication that the error due to the truncation of the domain is determined by the tail behavior of the Lévy measure—here $d\mu(z) \sim |z|^{-1-\alpha} dz$. Such behavior would be consistent with the analytical results of [9] for tempered Lévy models (which do not include the fractional Laplacian). Another analytical approach using Barenblatt solutions (cf. [68]) can be found in [33].

6. Numerical experiments in 2 dimensions. In this section we test our numerical schemes on more complicated problems in two space dimensions which has a much richer solution structure and a more interesting evolution. We consider Stefan problems of the type

$$(6.1) \quad \partial_t u(x, y, t) + \mathfrak{L}_i^{\sigma, \mu}[\varphi(u(\cdot, \cdot, t))](x, y) = 0 \quad \text{in} \quad Q_T := \mathbb{R}^2 \times (0, T)$$

for $i = 1, 2$, $\varphi(\xi) = \max\{0, \xi - 1\}$, and both x -directed nonlocal diffusion and “diagonal” $\vec{\sigma}^T = (\frac{1}{2}, \frac{47}{100})$ -directed local diffusion,

$$(6.2) \quad \mathfrak{L}_1^{\sigma, \mu}[\psi](x, y) := (\vec{\sigma} \cdot D)^2[\psi](x, y) + (-\partial_{xx})^{\frac{1}{4}}[\psi](x, y),$$

$$(6.3) \quad \mathfrak{L}_2^{\sigma, \mu}[\psi](x, y) := (\vec{\sigma} \cdot D)^2[\psi](x, y) + \frac{1}{10}(-\partial_{xx})^{\frac{1}{4}}[\psi](x, y).$$

Note that nonlocal diffusion is stronger in $\mathfrak{L}_1^{\sigma, \mu}$ than in $\mathfrak{L}_2^{\sigma, \mu}$.

As in section 5, we use an equidistant grid in space and time and restrict it to a (large) bounded spatial domain $I_1 \times I_2 \subset \mathbb{R}^2$ (setting the numerical solution equal to zero outside). The errors are computed numerically under the same assumptions as before. We run an explicit scheme ($\theta = 1$) with $\Delta t \sim h^2$ such that (CFL) holds and the overall order of convergence is determined by the spatial discretization. Since $x + \vec{\sigma}\eta$ is not aligned with the spatial grid \mathcal{G}_h , we discretize the local term by (4.5) and $\eta = \sqrt{h}$ which leads to $O(h)$ errors (cf. Lemma 4.2). For the nonlocal diffusion, we use the one dimensional version of PDL of section 4.5 which has $O(h^2)$ errors. We

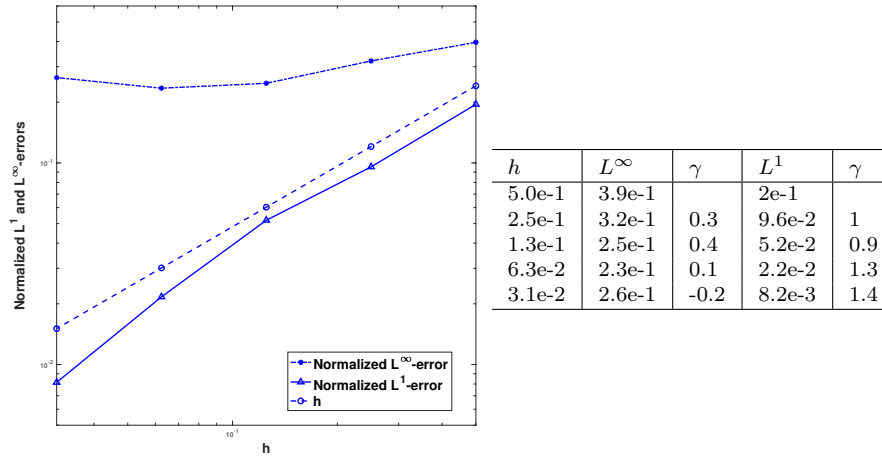


FIG. 9. Relative L^∞ - and L^1 -errors with (4.5) and PDL discretizations at $T = 1$ with $I_1 \times I_2 = [-100, 100] \times [-10, 10]$ for (6.1) with $\varphi(\xi) = \max\{0, \xi - 1\}$ and the diffusion operator given by (6.2).

also choose a rough initial datum (see Figures 10 and 11) given by

$$u_0(x, y) = 3(\mathbf{1}_{S_1}(x, y) - \mathbf{1}_{S_2}(x, y)) + 4\mathbf{1}_{S_3}(x, y),$$

where $S_1 = \{(x, y) \in \mathbb{R}^2 : |x| < 5 \text{ \& } |y| < 5\}$, $S_2 = \{(x, y) \in \mathbb{R}^2 : |x| < 2 \text{ \& } |y| < 2\}$, and $S_3 = \{(x, y) \in \mathbb{R}^2 : 3 < |x| < 4 \text{ \& } 3 < |y| < 4\}$. We run the experiments on the domain $I_1 \times I_2 = [-100, 100] \times [-10, 10]$. The different sizes of the domain in the x - and y -directions are adapted to the combination of compactly supported data and degenerate operators (6.2) and (6.3). These operators are nonlocal in the x -direction which requires a wide domain there. The size of the domain in the y -direction can be smaller because the Stefan problem with local diffusion has finite speed of propagation (see, e.g., the introduction of [10]).

In Figure 9 we list the relative L^∞ - and L^1 -errors (errors divided by the L^∞ - and L^1 -norms, respectively, of the solution) for the numerical solution of (6.1) with $i = 1$. In Figure 10 (resp., Figure 11) we plot, for different times, the numerical solution of (6.1) with $i = 1$ (resp., $i = 2$).

Conclusion: Figure 9 confirms the $O(h)$ convergence in L^1 predicted by Lemma 4.2. As in section 5.3, there is no convergence in L^∞ . In Figure 10 there are pronounced long “bands” due to the nonlocal diffusion, and the effect of the almost diagonal local diffusion is also visible. In Figure 11 the nonlocal diffusion is weaker and the local diffusion dominates. The “long bands” are not very pronounced. Both figures exhibit regions of discontinuity of the solution in the y -direction (where there is only local diffusion). Such behavior is well known for local Stefan problems.

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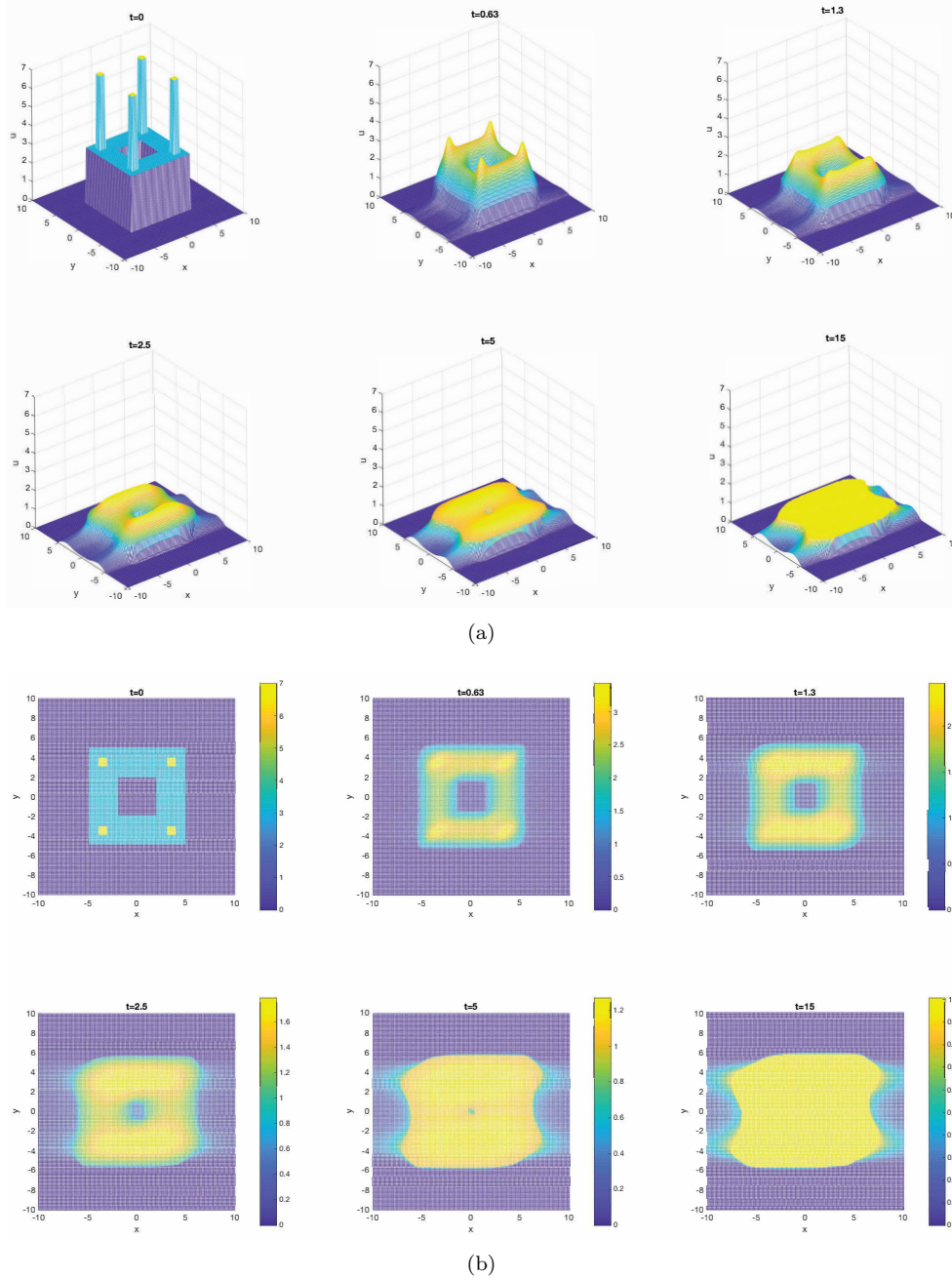


FIG. 10. Stronger nonlocal diffusion. Solution of (6.1) with $\varphi(\xi) = \max\{0, \xi - 1\}$ and the diffusion operator given by (6.2).

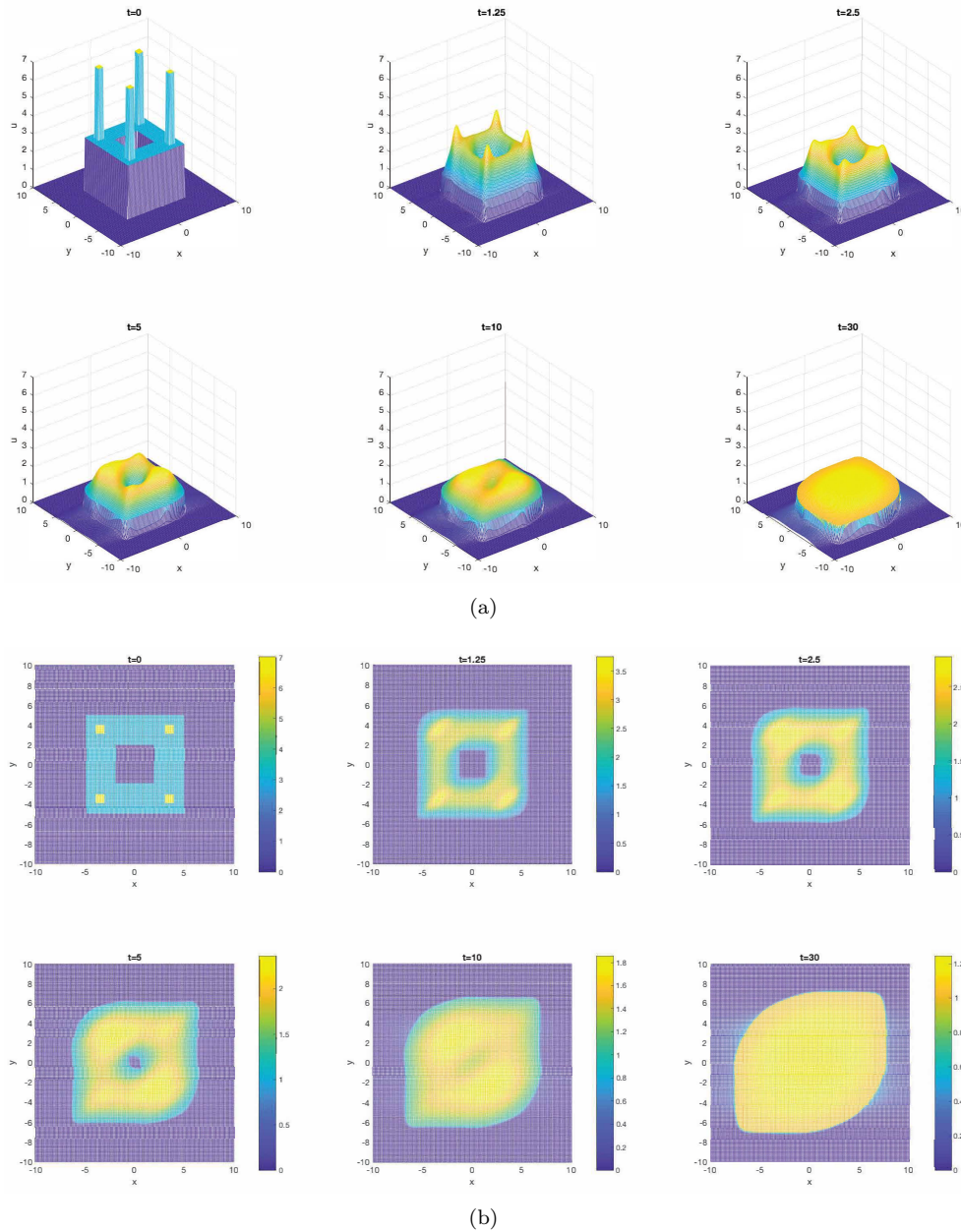


FIG. 11. Weak nonlocal diffusion. Solution of (6.1) with $\varphi(\xi) = \max\{0, \xi - 1\}$ and the diffusion operator given by (6.3).

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