NTNU – Trondheim
Norwegian University of
Science and Technology

Serre's Conjecture

Finitely generated projective modules over polynomial rings

Fredrik Sandbakk

Teacher Education with Master of Science
Submission date: December 2013
Supervisor: Petter Andreas Bergh, MATH

Norwegian University of Science and Technology
Department of Mathematical Sciences

Problem Description

In this thesis I present a proof of Serre's Conjecture, that is, all finitely generated projective modules over the polynomial ring $k[x_1, \ldots, x_n]$, where k is a field, are free.

Abstract

We start by proving that all finitely generated projective R-modules, where $R = k[x_1, \ldots, x_n]$ and k is a field, are stably free. Then we show that all stably free projective modules over a ring with the unimodular column property are free before showing that the polynomial ring R has the unimodular column property.

Samandrag

Vi byrjar med å vise at alle endeleggenererte projektive R-modular, der $R = k[x_1, \ldots, x_n]$ og k er ein kropp, er stabilt frie. Etterpå visar vi at alle stabilt frie projektive R-modular over ein ring med einingsmodulert kolonneeigenskap (unimodular column property) er frie før vi viser at R har einingsmodulert kolonneeigenskap.

Contents

Introduction **5**

1 Projective Modules **7**

2 Tensor Product and Flat Modules **21**
2.1 Tensor Product 21
2.2 Flat Modules 32

3 Stably Free Modules **37**

4 UCP and Serre's Conjecture **55**
4.1 UCP, Unimodular Column Property 55
4.2 Serre's Conjecture 62

Bibliography **65**

Acknowledgements

I would like to show my gratitude for my family, in particular my mum and dad for always supporting me, my supervisor Petter Bergh, who's help was invaluable and patience has to be unmatched, my friends and my fellow students.

Introduction

We will present a proof of Serre's Conjecture, that is, all finitely generated projective modules over $k[x_1, \ldots, x_n]$ are free. We start by reviewing some basic homological algebra. In Chapter 1 we take a look at projective modules, free modules, exact sequences and noetherian rings. In Chapter 2 we look into the tensor product and flat modules, which we will use to prove the first step in our goal, namely that finitely generated projective modules over $k[x_1, ..., x_n]$ are stably free, which we will do in Chapter 3. In Chapter 4 we will examine unimodular columns and the unimodular column property which we need to complete the proof.

Serre's Conjecture was proven by Suslin [4] and Quillen [2] independently of each other. In *An Introduction to Homological Algebra* [3] Rotman presents a proof based on Suslin's version and a sketch of Quillen's version. We will follow Rotman's account of Suslin's version.

All rings in this text are to be considered commutative.

Chapter 1

Projective Modules

Before we start working on the proof of Serre's Conjecture we will take a look at projective modules and some basic tools which we will need. We start by defining free modules.

Definition. Let R be a ring and F an R-module. We say F is a free R-module if F is isomorphic to a direct sum of copies of R. In other words there is an index set B, possibly infinitely large, where $R_b = (b) \cong R$ and $F \cong \bigoplus_{b \in B} R_b$. We call B a basis for F.

The basis of a free module has some similarities with the basis of a vector space. There is a theorem in linear algebra which states that linear transformations can be described by matrices. That theorem can also be stated as a mapping defined by the basis elements, that is, if T is a linear transformation $T : V \to W$ defined by a matrix, where $\{v_1, \ldots, v_n\}$ is a basis for the vector space V and $\{w_1, \ldots, w_n\}$ is a basis for the vector space W, then it can also be described by mapping elements of the basis of V to elements of the basis of W. The following proposition is a parallel to that, giving a mapping from a free module F to a module M.

Proposition 1.1. *Given a ring R and a free R-module F with a basis B. If $f : B \to M$ is a map to any R-module M then there exists a unique*

R-homomorphism $\tilde{f} : F \to M$ with $\tilde{f}u = f$, where $u : B \hookrightarrow F$ is the injection.

$$\begin{tikzcd}
F \arrow[dr, dashed, "\tilde{f}"] & \\
B \arrow[u, "u"] \arrow[r, "f"'] & M
\end{tikzcd}$$

Proof. Since B is a basis for F for every $v \in F$ we can uniquely express $v = \sum_{b\in B} r_b b$ where $r_b \in R$ and $b \in B$, and there is a well defined map $\tilde{f} : F \to M$ by $\tilde{f}(v) = \sum_{b\in B} r_b f(b)$. If $s \in R$ then

$$\tilde{f}(sv) = \sum_{b\in B} sr_b f(b) = s \sum_{b\in B} r_b f(b) = s\tilde{f}(v).$$

If $v' = \sum_{b\in B} r'_b f b \in F$ then $v + v' = \sum_{b\in B}(r_b + r'_b)b$ and

$$\tilde{f}(v + v') = \sum_{b\in B}(r_b + r'_b)f(b) = (\sum_{b\in B} r_b f(b)) + (\sum_{b\in B} r'_b f(b)) = \tilde{f}(v) + \tilde{f}(v').$$

This shows that $\tilde{f}$ is an R-map. If we assume that there exists another such map $\tilde{g}$, that is $\tilde{g}u = f$, then for all $v \in F$

$$\tilde{f}(v) = \tilde{f}(\sum_{b\in B} r_b u(b)) = \sum_{b\in B} r_b \tilde{f}u(b) = \sum_{b\in B} r_b f(b),$$

$$= \sum_{b\in B} r_b \tilde{g}u(b) = \tilde{g}(\sum_{b\in B} r_b u(b)) = \tilde{g}(v),$$

so

$$\tilde{g}(v) = \tilde{f}(v).$$

Hence the R-map is unique. $\square$

We will use this to show a property of free modules that we will later generalize into a basis free property. The notation $\to 0$, in the following proposition, will become clear after we have defined exact sequences.

Theorem 1.2. *Given a free R-module F and a surjective map $p : A \to A'$, then for every map $h : F \to A$ there exists a map $g : F \to A$ such that the following diagram commutes.*

$$\begin{tikzcd} & F \arrow[dl, dashed, "g"'] \arrow[d, "h"] & \\ A \arrow[r, "p"] & A' \arrow[r] & 0 \end{tikzcd}$$

Proof. Let B be a basis for F. For every $b \in B$ there exist an element $p(a_b) \in A'$ such that $h(b) = p(a_b)$ where $a_b \in A$. It follows that we have a map $u : B \to A$ where $u(b) = a_b \ \forall \ b \in B$. By Theorem 1.1 there exists a map $g : F \to A$ where $pg = h$ $\square$

The next theorem shows another of the similarities between vector spaces and free modules. A vector space over a field k is a finitely generated k-module if and only if it is finite dimensional.

Theorem 1.3. *Every R-module M is a quotient of a free R-module F. The module M is also finitely generated if and only if F can be chosen to be finitely generated.*

Proof. Let X be a generating set for M, and F be a free module where the set $\{b_x\}_{x\in X}$ forms a basis of F. By Theorem 1.1 there exists a map $g : F \to M$ such that $g(b_x) = x \ \forall \ x \in X$. Since $X \subseteq \operatorname{Im} g$ then g is surjective and $F/\ker g \cong M$.

If M is finitely generated by X then F is finitely generated by $\{b_x\}_{x\in X}$ since X is a finite set. If F is finitely generated then M is finitely generated, since the image of a finitely generated module is itself finitely generated. $\square$

Definition. A submodule of an R-module M is a retract of M if there exists an R-homomorphism $\rho : M \to S$, called a retraction, with $\rho(s) = s \ \forall \ s \in S$. It is equivalent to say that ρ is a retraction if and only if $\rho i = 1_S$, where $i : S \hookrightarrow M$ is the inclusion.

Proposition 1.4. *A submodule S of an R-module M is a direct summand of M if and only if there exists a retraction $\rho : M \to S$.*

Proof. First assume $\rho : M \to S$ is a retraction. We want to show that $M = S \oplus T$, where $T = \ker \rho$. If $m \in M$ then $\rho(m - \rho(m)) = \rho(m) - \rho(m) = 0$, which gives us $m - \rho(m) \in \ker \rho = T$. We see that $m = (m - \rho(m)) + \rho(m)$ where $\rho(m) \in S$ so $M = S + T$. If $m \in S$ then $\rho(m) = m$, and if $m \in T$ then $\rho(m) = 0$. Therefore $S \cap T = 0$ and $M = S \oplus T$.

Now assume $M = S \oplus T$. We can then write every $m \in M$ uniquely as $m = s + t$ for $s \in S$ and $t \in T$. Let $\rho : M \to S$ be a map where $s + t \mapsto s$.

$$\rho((s_1 + t_1) + (s_2 + t_2)) = s_1 + s_2$$

$$\rho(s_1 + t_1) + \rho(s_2 + t_2) = s_1 + s_2$$

$$\rho((s_1 + t_1) + (s_2 + t_2)) = \rho(s_1 + t_1) + \rho(s_2 + t_2)$$

Clearly ρ is a retraction. □

Definition. A lifting of a map $h : C \to A'$ is a map $g : C \to A$ such that the following diagram commutes.

$$\begin{tikzcd}
 & C \arrow[dl, "g"'] \arrow[d, "h"] \\
A \arrow[r, "p"] & A'
\end{tikzcd}$$

That is $pg = h$.

We will use the definition of a lifting to extend the notion of Theorem 1.2 into a basis free property. We define modules with this property to be projective modules.

Definition. Let R be a ring and P be an R-module. We say P is a projective module if given a map $h : P \to A'$ and a surjective map $p : A \to A'$ then there is a map $g : P \to A$ such that the following diagram commutes.

$$\begin{tikzcd} & P \arrow[dl, dashed, "g"'] \arrow[d, "h"] & \\ A \arrow[r, "p"] & A' \arrow[r] & 0 \end{tikzcd}$$

That is g is a lifting of h.

From Theorem 1.2 we see that every free R-module is projective. The converse is not true in general, but, as we will show, it is true for finitely generated projective $k[x_1, \ldots, x_n]$-modules, where k is a field.

A very useful concept is exact sequences. They will help us formulate several definitions, theorems and proofs.

Definition. A, possibly infinite, sequence of R-modules and R-maps,

$$\cdots \xrightarrow{f_{i-2}} M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} \cdots$$

where $\operatorname{Im} f_j = \ker f_{j+1} \ \forall\ j$, is called an exact sequence.

Definition. An exact sequence of the form

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

is called short exact.

We see directly from the definition of exact sequences that the maps i and p, in the above diagram, are respectively injective and surjective.

Definition. We say a short exact sequence,

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

where there exists a map $j : C \to B$ such that $pj = 1_C$, splits.

Proposition 1.5. *If the short exact sequence*

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

splits, then $B \cong A \oplus C$.

Proof. Assuming the exact sequence splits there exists a map $j : C \to B$ such that $pj = 1_C$. If $b \in B$ then $p(b) \in C$. Since

$$p(b - jp(b)) = p(b) - pjp(b) = p(b) - 1_C p(b) = 0$$

then $b - jp(b) \in \ker p$, and since it is an exact sequence then there exists an element $a \in A$ such that $i(a) = b - jp(b)$. It follows that $B = \operatorname{Im} i + \operatorname{Im} j$. Next assume that $b \in \operatorname{Im} i \cap \operatorname{Im} j$. Since $b \in \operatorname{Im} i$ we have that $p(b) = 0$ and $b = i(a)$ for some $a \in A$, and since the sequence splits we have $b = j(c)$ for some $c \in C$. So

$$j(c) = b,$$

and

$$pj(c) = p(b) = 0.$$

Since

$$pj(c) = 1_C c.$$

we get

$$c = pj(c) = 0,$$

and

$$b = j(c) = j(0) = 0.$$

Hence we have that $B = \operatorname{Im} i \oplus \operatorname{Im} j \cong A \oplus C$. □

Proposition 1.6. *A module P is projective if and only if every short exact sequence ending in P splits.*

Proof. Consider the short exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} P \longrightarrow 0$$

where P is projective. We can modify the diagram to the following.

$$\begin{tikzcd}
 & & & P \arrow[dl, dashed, "j"'] \arrow[d, "1_P"] & \\
0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r, "p"] & P \arrow[r] & 0
\end{tikzcd}$$

Since P is projective there exists a map $j : P \to B$ such that $pj = 1_P$ so the sequence splits by definition.

For the converse statement assume every short exact sequence ending in P splits. Consider the diagram

$$\begin{tikzcd}
 & P \arrow[d, "f"] & \\
B \arrow[r, "p"] & C \arrow[r] & 0
\end{tikzcd}$$

where p is surjective. By Theorem 1.3 there exists a free R-module F and a surjective map $h : F \to P$, so we can construct a short exact sequence and get the following modified diagram.

$$\begin{tikzcd}
0 \arrow[r] & \ker h \arrow[r] & F \arrow[r, shift left, "h"] \arrow[d, "g_0"'] & P \arrow[l, shift left, "j"] \arrow[r] \arrow[d, "f"] & 0 \\
 & & B \arrow[r, "p"] & C \arrow[r] & 0
\end{tikzcd}$$

The map $j : P \to F$ exists by our assumption that every short exact sequence ending in P splits. Since F is free F is also projective and there is a map $g_0 : F \to B$ with $pg_0 = fh$. We can define a map $g : P \to B$ by $g = g_0 j$, hence P is projective. $\square$

With the next theorem we will try to characterize projective modules.

Theorem 1.7.

1. *A R-module is projective if and only if it is a direct summand of a free R-module.*

2. *A finitely generated R-module is projective if and only if it is a direct summand of R^n for some n.*

Proof. First we will prove 1. Assume P is a projective R-module. Since every module is a quotient of a free module there is a free module F and a surjective map $g : F \to P$. Therefore there exists an exact sequence

$$0 \longrightarrow \ker g \xrightarrow{i} F \xrightarrow{g} P \longrightarrow 0$$

where i is the inclusion. By Proposition 1.6 we have that the exact sequence splits hence P is a summand in F.

Next assume an R module P is a direct summand of a free module F. Then, by Proposition 1.4, there exists maps $q : F \to P$ and $j : P \to F$ such that $qj = 1_P$. Let $f : P \to C$ and $p : B \to C$ be maps where p is surjective, and consider the following diagram.

$$\begin{tikzcd} F \arrow[r, shift left, "q"] \arrow[d, dashed, "h"] & P \arrow[l, shift left, "j"] \arrow[d, "f"] & \\ B \arrow[r, "p"] & C \arrow[r] & 0 \end{tikzcd}$$

The module F is free and therefore projective. Since the composition $qf : F \to C$ is a map from F to C and p is surjective there exists a map $h : F \to B$ where $fq = ph$ so $phj = fqj = f1_P = f$. Hence there exists a map $g : P \to B$ where $g = hj$ and $f = pg$ so P is projective.

Next we prove 2. If P is a summand of R^n then it follows immediately from part 1 that P is a finitely generated projective module. We can prove the other implication by assuming P is projective and letting $P = (a_1, \ldots, a_n)$. Let the set $x_1, \ldots, x_n$ denote the basis for R^n. We define the map $f : R^n \to P$ by $x_i \mapsto a_i$. This gives us the following short exact sequence where i is the inclusion.

$$0 \longrightarrow \ker f \xrightarrow{i} R^n \xrightarrow{f} P \longrightarrow 0$$

Since P is projective the sequence splits, by Proposition 1.6, and by Proposition 1.5 we get $R^n = P \oplus \ker f$. $\square$

Another important concept for finitely generated modules is noetherian rings. The next preposition will determine some equivalent properties that we will use to define noetherian rings.

Proposition 1.8. *Given an R-module M the following are equivalent:*

1. *Every ascending chain of submodules of M stabilizes, that is, there exists an n such that*

$$S_1 \subseteq S_2 \subseteq S_3 \subseteq \cdots \subseteq S_n = S_{n+1} = \dots$$

   where S_i is a submodule of M for every i.

2. *Every non-empty collection C of submodules has a maximal element. In other words there is a $S_0 \in C$ such that there is no $S \in C$ with $S_0 \subsetneq S$.*

3. *Every submodule of M is finitely generated.*

Proof. We begin by showing that point 1 implies point 2. Let C be a non-empty collection of submodules and $S_0 \in C$. Assuming point 2 is not true there exist $S_i \in C$ such that $S_{i-1} \subsetneq S_i$ for $i \geq 1$. This gives us the chain

$$S_1 \subseteq S_2 \subseteq S_3 \subseteq \cdots \subseteq S_n \subseteq \cdots$$

which does not stop contradicting point 1. Hence point 1 implies point 2.

Next we will show that point 2 implies point 3. Let S be a submodule of M, and C the collection of all finitely generated submodules contained in S, which is non-empty since $\{0\}$ is finitely generated. By assumption we have that there exists a maximal element $S^* \in C$ with $S^* \subseteq M$. If S is not finitely generated there is a $s \in S$ such that $s \notin S^*$. We construct the finitely generated submodule $(S^*, s) \in C$, but clearly $S^* \subsetneq (S^*, s)$ which contradicts point 2, since S^* is supposed to be the maximal element, therefore S is finitely generated and point 2 implies point 3.

Lastly we will see that point 3 implies point 1. Let

$$S_1 \subseteq S_2 \subseteq S_3 \subseteq \dots$$

be a ascending chain of submodules, that is $S_i \subseteq M \ \forall \ i \geq 1$. We define $S^* = \cup_{n \geq 1} S_n$. Clearly $S^* \subseteq M$ and therefore finitely generated, by point 3, which means that $S^* = (s_1, \dots, s_q)$ where every $s_i \in S_{n_i}$ for some n_i. Let

$$m = \max\{n_1, \dots, n_q\}.$$

Then $s_{n_i} \in S_m$ for all i, hence $S^* \subseteq S_m$. Then $S_m = S_{m+1} = \cdots$, hence the chain stabilizes. Thus point 3 implies point 1. □

Definition. Let R be a ring. If Proposition 1.8 is true for $M = R$, we say that the ring R is noetherian.

Corollary 1.9. *If R is a noetherian ring and S is a submodule of an R-module M which is finitely generated, then S is finitely generated.*

Proof. We will prove this by induction on the number of generating elements of $M = (x_1, \dots, x_n)$. Let $n = 1$. We begin by defining the R-map $f : R \to M$ by $r \mapsto rx_1$ which is clearly surjective since x_1 generates M. We denote the kernel of the map $\ker f = I$, which is an ideal in R, and note that $M \cong R/I$. Since R is noetherian then R/I is finitely generated. There is a bijection between the submodules of R/I and submodules of M since they are isomorphic. A submodule of R/I is on the form J/I, where J is an ideal in R and $I \subseteq J \subseteq R$, and are therefore finitely generated. If $S \subseteq M$ is a submodule then the bijection between submodules ensures that $S \cong J/I$ is finitely generated.

Next assume that the hypothesis holds for $n > 1$. Let $M = (x_1, \dots, x_{n+1})$ and $M' = (x_1, \dots, x_n)$. We have that submodules of M' are finitely generated, by our assumption, and that submodules of $M/M' \cong (x_{n+1})$ are finitely generated, by our base step $n = 1$. Let $S \subseteq M$ be a submodule. Consider the exact sequence

$$0 \longrightarrow S \cap M' \xrightarrow{\ i\ } S \xrightarrow{\ p\ } S/(S \cap M') \longrightarrow 0$$

where $S \cap M' \subseteq M'$ and $S/(S \cap M') \cong (S + M')/M' \subseteq M/M'$ are finitely generated submodules. Since p is surjective then there is a $z \in S$ for every $a \in S/(S \cap M')$ such that $p(z) = a$. We claim S is generated by the elements in the set $\{i(x_1), \ldots, i(x_m), z_1, \ldots, z_{m'}\}$, where $S \cap M' = (x_1, \ldots, x_m)$ and $p(z_i) = a_i$ where $(S \cap M') = (a_1, \ldots, a_{m'})$. Let $s \in S$ then $p(s) = a = \sum_{i=1}^{m'} r_i a_i$ where $r_i \in R$. Clearly

$$p(s - \sum_{i=1}^{m'} r_i z_i) = 0$$

so

$$s - \sum_{i=1}^{m'} r_i z_i \in \ker p = \operatorname{Im} i.$$

Hence

$$s - \sum_{i=1}^{m'} r_i z_i = i(\sum_{i=1}^{m} r_i x_i),$$

and

$$s = \sum_{i=1}^{m'} r_i z_i + \sum_{i=1}^{m} r_i i(x_i).$$

Therefore $S = (i(x_1), \ldots, i(x_m), z_1, \ldots, z_{m'})$ and so finitely generated. $\square$

Definition. For a ring R we let $R[x]$ denote the polynomial ring where we adjoin the indeterminate x to R where x commutes with every $r \in R$.

In Theorem 1.11 we will show that if R is a noetherian ring then the polynomial ring $R[x]$ is also noetherian. This will be very useful since Serre's Conjecture concerns $k[x_1, \ldots, x_n]$ where k is a field and therefore noetherian.

Lemma 1.10. *A ring R is noetherian if and only if for every sequence of elements $a_1, \dots, a_n, \dots, \in R$ there are elements $r_1, \dots, r_m \in R$ with $m \geq 1$ such that*

$$a_{m+1} = \sum_{i=1}^{m} r_i a_i$$

Proof. Let R be a noetherian ring. Let $a_1, \dots, a_n, \dots \in R$ be a sequence of elements and $I_r = (a_1, \dots, a_r)$, which is an ideal in R. Then there is an ascending chain of ideals

$$I_1 \subseteq I_2 \subseteq \cdots$$

Since R is noetherian there is an $m \geq 1$ such that $I_m = I_{m+1}$. Therefore if $a_{m+1} \in I_{m+1} = I_m$ then $a_{m+1} = \sum_{i=1}^{m} r_i a_i$.

Next assume that the statement on sequence of elements in R holds. If R is not noetherian there is a sequence of ideals that does not end.

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

If we disregard repetitions we get a new sequence.

$$I'_1 \subsetneq I'_2 \subsetneq I'_3 \subsetneq \cdots$$

We choose $a_i \in I'_i$ such that $a_i \notin I_{i-1}$. This will give us a sequence of elements that contradicts our assumption, and we conclude that R is noetherian. □

Theorem 1.11. *If R is a noetherian then $R[x]$ is noetherian.*

Proof. Let R be a noetherian ring. Assume I is a ideal in $R[x]$ which is not finitely generated. Let $f_0(x) \in R[x]$ be the polynomial of minimal degree in I and $f_{n+1} \in R[x]$ be the polynomial of minimal degree in $I - (f_0, \dots, f_n)$. Since I is not finitely generated f_i exists for all $i \geq 0$. We note that

$$\deg(f_0) \geq \deg(f_1) \geq \cdots$$

Let $a_i \in R$ denote the leading coefficient in f_i. From Lemma 1.10 there is an $m \geq 1$ such that $a_{m+1} = \sum_{i=1}^{m} r_i a_i$. We define

$$f^*(x) = f_{m+1}(x) - \sum_{i=0}^{m} x^{d_{m+1}-d_1} r_i f_i(x)$$

where $d_i = \deg(f_i)$. We note that $\deg(f^*) < \deg(f_{m+1})$. This is because the leading term of

$$\sum_{i=0}^{m} x^{d_{m+1}-d_1} r_i f_i(x)$$

is

$$\sum_{i=0}^{m} x^{d_{m+1}-d_1} r_i a_i x^{d_i} = \sum_{i=0}^{m} x^{d_{m+1}} r_i a_i = a_{m+1} x^{d_{m+1}}$$

which is also the leading term of f_{m+1}. Since $f_{m+1} \notin (f_1, \ldots, f_m)$ then $f^* \in I - (f_1, ..., f_m)$, but this contradicts the assumption that f_{m+1} is of minimal degree. Hence $R[x]$ is noetherian. $\square$

Chapter 2

Tensor Product and Flat Modules

Some very important functors in homological algebra are the Hom functor, the tensor product, and the functors which are derived from these. Flat modules and tensor products are essential for some of the following proofs and results. We will define flat modules by the tensor product therefore we will start with the tensor product before we examine flat modules.

2.1 Tensor Product

We will define the tensor product with the help of R-biadditive functions. Since the rings we consider are commutative then for any ring R a left R-module is also a right R-module, and the other way around, but since the tensor product and R-biadditive functions are defined for non-commutative rings as well we will define them by left and right R-modules specifically.

Definition. Given a (not necessarily commutative) ring R, a right R-module A_R, a left R-module ${}_RB$ (A_R and ${}_RB$ denotes that A and B are right and left R-modules respectively), and an abelian group G. A function $f : A \times B \to G$ is called R-biadditive, if for all $a, a' \in A$, $b, b' \in B$, and

$r \in R$, we have

$$f(a+a',b) = f(a,b) + f(a',b),$$
$$f(a,b+b') = f(a,b) + f(a,b'),$$

and

$$f(ar,b) = f(a,rb).$$

If R is commutative, and G is also an R-module, and

$$f(ar,b) = f(a,rb) = rf(a,b)$$

then f is R-bilinear.

Definition. Given a ring R, and R-modules A_R and ${}_RB$, their tensor product is an abelian group $A \otimes_R B$ and a R-biadditive function

$$h : A \times B \to A \otimes_R B$$

such that for every abelian group G and R-biadditive function $f : A \times B \to G$ there exists a unique $\mathbb{Z}$-homomorphism $\tilde{f} : A \otimes_R B \to G$ making the following diagram commute.

$$\begin{tikzcd}
A \times B \arrow[rr, "h"] \arrow[dr, "f"'] & & A \otimes_R B \arrow[dl, dashed, "\tilde{f}"] \\
& G &
\end{tikzcd}$$

The tensor product is defined such that it is an abelian group that admits a unique mapping that makes many diagrams commute. It is thus a solution to a universal mapping problem, and solutions, if they exists, are unique up to isomorphism.

Proposition 2.1. *If R is a ring, and A and B are R-modules then their tensor product exists.*

Proof. Let F be a free abelian group with basis $A \times B$ and let S be a subgroup of F generated by all elements of the following types:

$$(a, b + b') - (a, b) - (a, b'),$$

$$(a + a', b) - (a, b) - (a', b),$$

$$(ar, b) - (a, rb).$$

Where $a, a' \in A$, $b, b' \in B$ and $r \in R$. We define $A \otimes_R B = F/S$, $a \otimes b = (a, b) + S$, and the function

$$h : A \times B \to A \otimes_R B$$

by

$$h : (a, b) \mapsto a \otimes b.$$

Then we have the following identities in $A \otimes_R B$:

$$a \otimes (b + b') = a \otimes b + a \otimes b',$$

$$(a + a') \otimes b = a \otimes b + a' \otimes b,$$

$$ar \otimes b = a \otimes rb.$$

From these identities we get directly that h is R-biadditive.

Consider the following diagram where G is an abelian group, f is R-biadditive and i is the inclusion.

$$\begin{tikzcd}
A \times B \arrow[rr, "h"] \arrow[dr, "i"] \arrow[ddr, "f"'] & & A \otimes_R B \arrow[ddl, dashed, "\tilde{f}"] \\
& F \arrow[ur, "nat"'] \arrow[d, dashed, "\phi"] & \\
& G &
\end{tikzcd}$$

Since F is free abelian, with basis $A \times B$, there exists a homomorphism $\phi : F \to G$ with $\phi(a, b) = f(a, b)\ \forall\ (a, b)$, by Proposition 1.1. Since f is R-biadditive we have that

$$f((a, b + b') - (a, b) - (a, b')) = 0,$$

$$f((a + a', b) - (a, b) - (a', b)) = 0,$$

and

$$f((ar, b) - (a, rb)) = 0,$$

so $S \subseteq \ker \phi$. We define $\tilde{f} : A \otimes_R B \to G$ by $\tilde{f} : (a, b) + S \mapsto f(a, b)$. We need to check that $\tilde{f}$ is well defined. Since f is well defined we only need to check that $\tilde{f}(v) = \tilde{f}(v + s) = f(v)$ where $v \in F$ and $s \in S$. We have that

$$\tilde{f}(v + s) = f(v + s) = f(v) + f(s) = f(v) + 0 = \tilde{f}(v)$$

therefore the map is well defined. We also see that $\tilde{f}h = f$ so the diagram commutes.

All that is left to prove is that $\tilde{f}$ is unique. The group $A \otimes_R B$ is generated by the set of all $a \otimes b$. Let us denote that generating set $X = \{m_i\}_{i=1}^{n}$. If we assume there is another R-map $\tilde{g}$ with the same properties as $\tilde{f}$ then $\tilde{f}(m_i) = \tilde{g}(m_i)\ \forall\ m_i \in X$. For any $m \in A \otimes_R B$ we have

$$\tilde{g}(m) = \tilde{g}(\sum_{i=1}^{n} r_i m_i) = \sum_{i=1}^{n} r_i \tilde{g}(m_i) = \sum_{i=1}^{n} r_i \tilde{f}(m_i) = \tilde{f}(\sum_{i=1}^{n} r_i m_i) = \tilde{f}(m)$$

where $r_i \in R\ \forall\ i$, which shows that $\tilde{f}$ is unique. $\square$

Now we have shown that the tensor product actually exists, but we did so by defining the elements of the tensor product from the generators of the free group F. Therefore we had to take care and be sure $\tilde{f}$ was well defined, if $\tilde{f}(S) \neq \{0\}$ we would have had a problem. We could also have checked this by the following proposition.

Proposition 2.2. *Let $f : A \to A'$ and $g : B \to B'$ be maps of R-modules. Then there exists a unique $\mathbb{Z}$-homomorphism $f \otimes g : A \otimes_R B \to A' \otimes_R B'$ where*

$$a \otimes b \mapsto f(a) \otimes g(b)$$

Proof. We start by defining the map $\phi : A \times B \to A' \otimes_R B'$ by

$$(a, b) \mapsto f(a) \otimes g(b)$$

which is an R-biadditive function by the identities given in the proof of Proposition 2.1. It yields a unique homomorphism $f \otimes g : A \otimes_R B \to A' \otimes_R B'$ where

$$a \otimes b \mapsto f(a) \otimes g(b)$$

by the following diagram

$$\begin{tikzcd} A \times B \arrow[rr] \arrow[dr, "\phi"'] & & A \otimes_R B \arrow[dl, "f \otimes g"] \\ & A' \otimes_R B' & \end{tikzcd}$$

which is commutative. □

Now that we know that the tensor product actually exists and is well defined we will show that it is unique up to isomorphism.

Proposition 2.3. *If U and $A \otimes B$ are tensor products of A and B over a ring R then $A \otimes_R B \cong U$.*

Proof. Assume that $A \otimes_R B$ and U are tensor products of A and B corresponding to the following diagrams.

$$\begin{tikzcd} A \times B \arrow[rr, "h"] \arrow[dr, "f"'] & & A \otimes_R B \arrow[dl, "\tilde{f}"] \\ & G & \end{tikzcd} \qquad \begin{tikzcd} A \times B \arrow[rr, "n"] \arrow[dr, "g"'] & & U \arrow[dl, "g'"] \\ & G' & \end{tikzcd}$$

If we exchange G for U and f for n, and G' for $A \otimes_R B$ and g for h in the above diagrams we get

$$\begin{tikzcd}
A \times B \arrow[rr, "h"] \arrow[dr, "n"'] & & A \otimes_R B \arrow[dl, "\tilde{n}"] \\
& U &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \times B \arrow[rr, "n"] \arrow[dr, "h"'] & & U \arrow[dl, "h'"] \\
& A \otimes_R B &
\end{tikzcd}$$

which both commutes. Next consider the following diagrams.

$$\begin{tikzcd}
& A \otimes_R B \arrow[d, "\tilde{n}"] \arrow[dd, bend left, "1_{A \otimes_R B}"] \\
A \times B \arrow[ur, "h"] \arrow[r, "n"] \arrow[dr, "h"'] & U \arrow[d, "h'"] \\
& A \otimes_R B
\end{tikzcd}
\qquad
\begin{tikzcd}
& U \arrow[d, "h'"] \arrow[dd, bend left, "1_U"] \\
A \times B \arrow[ur, "n"] \arrow[r, "h"] \arrow[dr, "n"'] & A \otimes_R B \arrow[d, "\tilde{n}"] \\
& U
\end{tikzcd}$$

In the first diagram the two small triangles with vertices $A \times B$, $A \otimes_R B$ and U commutes making the larger triangle with vertices $A \times B$, $A \otimes_R B$ and $A \otimes_R B$ commute. The uniqueness of $\tilde{n}$ and h', from the definition of tensor product, leaves us with $1_{A \otimes_R B} = h'\tilde{n}$. A similar argument on the second diagram yields that $1_U = \tilde{n}h'$, therefore we have that $\tilde{n} : A \otimes_R B \to U$ is an isomorphism. $\square$

Corollary 2.4. *Given maps of modules*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[rr, bend right, "f'f"'] & A' \arrow[r, "f'"] & A''
\end{tikzcd}$$

and

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[rr, bend right, "g'g"'] & B' \arrow[r, "g'"] & B'' \end{tikzcd}$$

then we have that

$$(f \otimes g)(f' \otimes g') = f'f \otimes g'g$$

The arrows showing the compositions $f'f$ and $g'g$ are strictly not necessary, but are left there to better illustrate where the homomorphism $f'f \otimes g'g$, in the following proof, is derived from, and why it as well is unique.

Proof. If we consider the diagram

$$\begin{tikzcd} A \otimes_R B \arrow[r, "f \otimes g"] \arrow[rr, bend right, "f'f \otimes g'g"'] & A' \otimes_R B' \arrow[r, "f' \otimes g'"] & A'' \otimes_R B'' \end{tikzcd}$$

we can easily see that both $(f \otimes g)(f' \otimes g')$ and $f'f \otimes g'g$ takes $a \otimes b \mapsto f'f(a) \otimes g'g(b)$. From Proposition 2.2 the uniqueness of these homomorphism gives the equality. $\square$

We have already noted that the notion of exact sequences are relevant for some of the proofs and results in this text. The next theorem shows the effect of the tensor product on exact sequences.

Theorem 2.5. *Given an R-module A and an exact sequence of R-modules*

$$B' \xrightarrow{i} B \xrightarrow{p} B'' \longrightarrow 0$$

then the sequence

$$A \otimes_R B' \xrightarrow{1 \otimes i} A \otimes_R B \xrightarrow{1 \otimes p} A \otimes_R B'' \longrightarrow 0$$

is exact, and we say that the functor, in this case the tensor product, is right exact.

Proof. To prove this we need to show that $\operatorname{Im} 1 \otimes i = \ker 1 \otimes p$ and that $1 \otimes p$ is surjective.

First we observe

$$(1 \otimes p)(1 \otimes i) = (1 \otimes pi) = (1 \otimes 0)$$

which means $\operatorname{Im} 1 \otimes i \subseteq \ker 1 \otimes p$. This gives us an induced map

$$\tilde{p} : (A \otimes_R B)/K \to A \otimes_R B''$$

by

$$a \otimes b + K \mapsto a \otimes p(b),$$

where $a \in A$, $b \in B$ and $K = \operatorname{Im} 1 \otimes i$. Let

$$\pi : A \otimes_R B \to (A \otimes_R B)/K$$

be the natural map, then both $\tilde{p}\pi$ and $1 \otimes p$ sends $a \otimes b \mapsto a \otimes p(b)$ so we can form the following commutative diagram.

$$\begin{tikzcd}
A \otimes_R B \arrow[rr, "\pi"] \arrow[dr, "1 \otimes p"'] & & (A \otimes_R B)/K \arrow[dl, "\tilde{p}"] \\
& A \otimes_R B'' &
\end{tikzcd}$$

We have that $\ker 1 \otimes p = \ker \tilde{p}\pi$. If $\ker \tilde{p}\pi = \ker \pi$ then $\ker 1 \otimes p = \ker \pi = K = \operatorname{Im} 1 \otimes i$ and the proof is finished. The statement is true if $\tilde{p}$ is injective. To prove this we will construct an inverse map

$$\tilde{f} : A \otimes_R B'' \to (A \otimes_R B)/K$$

such that $\tilde{p}\tilde{f} = 1_{(A \otimes_R B)/K}$. First consider the map

$$f : A \times B'' \to (A \otimes_R B)/K$$

where

$$(a, b'') \mapsto a \otimes b + K$$

for $a \in A$, $b \in B$ and $b'' \in B''$ where $p(b) = b''$. This is a well defined map since if $p(b_1) = p(b_2) = b''$ then $p(b_1) - p(b_2) = 0$ hence $b_1 - b_2 \in \operatorname{Im} i$ and $a \otimes (b_1 - b_2) \in K$. Clearly f is R-biadditive therefore there exists a $\mathbb{Z}$-homomorphism $\tilde{f}$ by the definition of tensor product, making the following diagram commute.

$$\begin{tikzcd}
A \times B'' \arrow[rr, "h"] \arrow[dr, "f"'] & & A \otimes_R B'' \arrow[dl, "\tilde{f}"] \\
& (A \otimes_R B)/K &
\end{tikzcd}$$

By the equation

$$\tilde{f}\tilde{p}(a \otimes b + K) = \tilde{f}(a \otimes p(b)) = a \otimes b + K.$$

we see that $\tilde{f}\tilde{p} = 1$, hence $\ker 1 \otimes p = \operatorname{Im} 1 \otimes i$.

The last step of this proof is to show that $1 \otimes p$ is surjective. We have that if $b'' \in B''$ then there exists a $b \in B$ such that $p(b) = b''$. Since $1 \otimes p : a \otimes b \mapsto a \otimes p(b)$ we see that if $a \otimes b'' \in A \otimes_R B''$ then there exists $a \otimes b \in A \otimes_R B$ where $1 \otimes p(a \otimes b) = a \otimes b''$ hence the map is surjective. $\square$

So far we have only considered the tensor product as an abelian group, but is it under any circumstances a module? The rings we consider are commutative and in those cases, which we will show, it is also a module, but for non-commutative rings this is not always the case.

Proposition 2.6. *The tensor product of two R-modules (when R is a commutative ring) is a module.*

Proof. Let R be a ring, let A and B be two R-modules, and let $r \in R$, $a \in A$ and $b \in B$. If we let multiplication on the tensor product by R be the natural choice $r(a \otimes b) \mapsto ra \otimes b$ then the result follows from the module properties of A and B. $\square$

When we consider the tensor product as a module with scalar multiplication defined like in Proposition 2.6 we see that the R-biadditive functions from the definition of the tensor product are R-bilinear.

Proposition 2.7. *For every R-module M there exists an isomorphism*

$$\phi : R \otimes_R M \to M$$

such that $r \otimes m \mapsto rm$ for $r \in R$ and $m \in M$

Proof. The function $f : R \times M \to M$, defined by $r \times m \mapsto rm$, is a bilinear function. Consider the following diagram.

$$\begin{tikzcd}
R \times M \arrow[rr, "h"] \arrow[dr, "f"'] & & R \otimes_R M \arrow[dl, dashed, "\tilde{f}"] \\
& M &
\end{tikzcd}$$

Since all modules are abelian groups, by the definition of the tensor product there exists an R-homomorphism $\tilde{f}$ such that the diagram commutes. Hence $\tilde{f} : r \otimes m \mapsto rm$. To prove it is an isomorphism all we need is for $\tilde{f}$ to be injective, since it is clearly surjective. If we define $\tilde{f}^{-1} : M \to R \otimes_R M$ by $m \mapsto 1 \otimes m$ we find that it gives $\tilde{f}\tilde{f}^{-1} = 1_M$ and $\tilde{f}^{-1}\tilde{f} = 1_{R \otimes_R M}$ from the equations

$$\tilde{f}\tilde{f}^{-1}(m) = \tilde{f}(1 \otimes m) = m$$

and

$$\tilde{f}^{-1}\tilde{f}(r \otimes m) = \tilde{f}^{-1}(rm) = 1 \otimes rm = r \otimes m.$$

□

Another useful property of the tensor product that we will use is that it preserves direct sums.

Theorem 2.8. *Given a module A and a collection of modules $(B_i)_{i \in I}$ over a ring R we have that:*

1. *There is an R-isomorphism*

$$\tau : A \otimes_R (\oplus_{i\in I} B_i) \to \oplus_{i\in I}(A \otimes_R B_i)$$

where $\tau : a \otimes (b_i) \mapsto (a \otimes b_i)$.

2. *The map τ is a natural isomorphism, in other words, if $(C_j)_{j\in J}$ is a collection of R-modules, and for each $i \in I$ there exists a $j \in J$ and an R-map $\sigma_{ij} : B_i \to C_j$, then the diagram,*

$$\begin{tikzcd}
A \otimes_R (\oplus_{i\in I} B_i) \arrow[r, "1 \otimes \sigma"] \arrow[d, "\tau_B"] & A \otimes_R (\oplus_{j\in J} C_j) \arrow[d, "\tau_C"] \\
\oplus_{i\in I}(A \otimes_R B_i) \arrow[r, "\tilde{\sigma}"] & \oplus_{j\in J}(A \otimes_R C_j)
\end{tikzcd}$$

where $\sigma : (b_i) \mapsto (\sigma_{ij}(b_i))$ and $\tilde{\sigma} : (a \otimes b_i) \mapsto (a \otimes \sigma_{ij}(b_i))$, commutes.

Proof. First we prove 1. Let $f : A \times (\oplus_i B_i) \to \oplus_i(A \otimes_R B_i)$ be a bilinear map defined by $f : (a, (b_i)) \mapsto (a \otimes b_i)$. Then there exists a $\mathbb{Z}$-map $\tau : A \otimes_R (\oplus_i B_i) \to \oplus_i(A \otimes_R B_i)$ where $\tau : a \otimes (b_i) \mapsto (a \otimes b_i)$ such that the following diagram commutes.

$$\begin{tikzcd}
A \times (\oplus_i B_i) \arrow[rr, "h"] \arrow[dr, "f"'] & & A \otimes_R (\oplus_i B_i) \arrow[dl, "\tau"] \\
& \oplus_i(A \otimes_R B_i) &
\end{tikzcd}$$

.

For τ to be an isomorphism we need to check if it is an R-map and if it is injective. Let $r \in R$. We have that

$$\tau(r(a \otimes (b_i))) = f(r(a \otimes (b_i))) = rf((a \otimes (b_i))) = r\tau((a \otimes (b_i))).$$

Hence τ is a R-map. Next we define a set of injective maps $\lambda_k : B_k \to \oplus_i B_i$ by $\lambda_k : b_k \mapsto (..., 0, 0, b_k, 0, 0, ...)$ where b_k is the value of the kth coordinate. We then define a new map $\theta : \oplus_i(A \otimes_R B_i) \to A \otimes_R (\oplus_i B_i)$ by combining the set of maps and letting $\theta : (a \otimes b_i) \mapsto a \otimes \Sigma_i \lambda_i(b_i)$. We can see that

$$\theta\tau(a \otimes (b_i)) = \theta((a \otimes b_i)) = a \otimes (b_i)$$

which means that $\theta\tau = 1_{A \otimes_R (\oplus_i B_i)}$. Hence τ is injective and an R-isomorphism.

Next we prove 2. We can easily check that the diagram commutes directly. Going clockwise we get

$$a \otimes (b_i) \mapsto a \otimes (\sigma_{ij}(b_i) \mapsto (a \otimes \sigma_{ij}(b_i)),$$

and counter clockwise we get

$$a \otimes (b_i) \mapsto (a \otimes b_i) \mapsto (a \otimes \sigma_{ij}(b_i)),$$

hence the diagram commutes. □

2.2 Flat Modules

With what we now know about the tensor product we can start examining flat modules.

Definition. Let R be a ring and A an R-module. We say A is a flat R-module if for every exact sequence of R-modules

$$0 \longrightarrow B' \xrightarrow{i} B \xrightarrow{p} B'' \longrightarrow 0$$

the tensored sequence

$$0 \longrightarrow A \otimes_R B' \xrightarrow{A \otimes_R i} A \otimes_R B \xrightarrow{A \otimes_R p} A \otimes_R B'' \longrightarrow 0$$

is an exact sequence. That is, tensoring with A yields an exact functor $A \otimes_R \square$.

We have already shown, in Theorem 2.5, that the tensor product is a right exact functor. Thus we see that a module A is flat if and only if $1 \otimes i$, as in the definition, is an injection.

Proposition 2.9. *Given a ring R we have the following properties:*

1. *The ring itself is flat as an R-module.*
2. *A direct sum $\oplus_j M_j$ of R-modules is flat if and only if each M_j is flat.*
3. *Every projective module is flat.*

Proof. Consider the following diagram where B', B and B'' are R-modules.

$$\begin{tikzcd} 0 \arrow[r] & B' \arrow[r, "i"] \arrow[d, "\phi"] & B \arrow[r, "p"] \arrow[d, "\tau"] & B'' \arrow[r] \arrow[d, "\omega"] & 0 \\ & R \otimes_R B' \arrow[r, "R \otimes_R i"] & A \otimes_R B \arrow[r, "R \otimes_R p"] & A \otimes_R B'' & \end{tikzcd}$$

Where ϕ, τ and ω are natural isomorphisms from Proposition 2.7. If the diagram diagram commutes then $R \otimes_R i$ is injective and $R \otimes_R p$ is surjective. This makes the sequence of tensor products exact and R will be flat as an R-module. We start by examining the square with vertices B', B, $R \otimes_R B$ and $R \otimes_R B'$. Let $b' \in B'$ if we follow the diagram clockwise we get

$$b' \mapsto i(b') \mapsto 1 \otimes i(b'),$$

and counter clockwise

$$b' \mapsto 1 \otimes b' \mapsto 1 \otimes i(b'),$$

hence that part of the diagram commutes. A similar argument will show that the square with vertices B, B'', $R \otimes_R B''$ and $R \otimes_R B$ also commutes, but we already know this to be true since the tensor product is a right exact functor, making the whole diagram commute.

Next we will show that a direct sum of R-modules is flat if and only if every summand is flat. Let A and B be R-modules, let $\oplus_j M_j$ be a direct sum of R-modules, let $i : A \to B$ be an injective map and let $\phi : \oplus_j(M_j \otimes_R A) \to \oplus_j(M_j \otimes_R B)$ be the R-map composed of the collection of R-maps $M_j \otimes_R A \to M_j \otimes_R B$ defined by $m_j \otimes a \mapsto m_j \otimes i(a)$. By Theorem 2.8 there exists R-isomorphisms τ_A and τ_B such that the following diagram commutes.

$$\begin{tikzcd}
(\oplus_j M_j) \otimes_R A \arrow[r, "1 \otimes i"] \arrow[d, "\tau_A"] & (\oplus_j M_j) \otimes_R B \arrow[d, "\tau_B"] \\
\oplus_j(M_j \otimes_R A) \arrow[r, "\phi"] & \oplus_j(M_j \otimes_R B)
\end{tikzcd}$$

Since the diagram commutes we see that $1 \otimes i$ is injective if and only if ϕ is injective, which is injective if and only if each R-map it is composed of is injective, hence $\oplus_j M_j$ is flat if and only if each M_j is flat.

From Theorem 1.7 we have that every projective module is a summand of a free module. From part 1 and 2 of this proposition we have that every free module is flat and every summand of a flat module is flat. Hence every projective module is flat. $\square$

Proposition 2.10. *Given a flat R-module A and an ideal I then the $\mathbb{Z}$-map $A \otimes_R I \to AI$, given by $a \otimes i \mapsto ai$, is an isomorphism.*

Proof. Let $\kappa : I \to R$ be the inclusion and $\phi_A : A \otimes_R R \to A$ be the isomorphism from Proposition 2.7. If we tensor

$$I \xrightarrow{\kappa} R$$

with A we get

$$A \otimes_R I \xrightarrow{A \otimes_R \kappa} A \otimes_R R$$

which we can extend with the map ϕ_A to

$$A \otimes_R I \xrightarrow{A \otimes_R \kappa} A \otimes_R R \xrightarrow{\phi_A} A$$

We can simply trace the diagram and see that

$$a \otimes i \mapsto a \otimes i \mapsto ai$$

and since A is flat we have that $A \otimes_R \kappa$ is injective, because it is injective if and only if i is injective, so the resulting composition of $A \otimes_R \kappa$ and ϕ_A will give us the isomorphism we are looking for. $\square$

Chapter 3

Stably Free Modules

We have now acquired enough tools to take the first big step in proving Serre's Conjecture Corollary 3.9. It states that every $k[x_1, ..., x_n]$-module, where k is a field, is stably free. The concept of families will help a lot with this. First we will define stably free modules and introduce the concept of a finite free resolution, which will be very useful for formulating some proofs and results in this chapter.

Definition. A finitely generated R-module P is stably free if there exists a finitely generated free R-module F such that $F \oplus P$ is free.

Clearly stably free modules are projective, by Theorem 1.7, since they are a summand of a free module.

Definition. A module M has FFR, finite free resolution, of length $\leq n$ if M is finitely generated and there exists an exact sequence:

$$0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

Where each F_i is a finitely generated free module.

Most of the work we are going to do in this chapter, and maybe of what we have done so far, is going into proving that every finitely generated projective $k[x_1, \ldots, x_n]$-module has FFR Corollary 3.11, a result of Theorem

3.10. The fact that they then also are stably free will follow immediately from the following proposition.

Proposition 3.1. *A finitely generated projective R-module P has FFR if and only if P is stably free.*

Proof. First assume P is stably free. then P is finitely generated and there exists a finitely generated free module F with $F \oplus P$ free. We get that P has FFR, of length ≤ 1, by the following exact sequence.

$$0 \longrightarrow F \longrightarrow F \oplus P \longrightarrow P \longrightarrow 0$$

Next assume that P has FFR. We will prove that P is stably free by induction on the length. Let P have FFR of length $n = 0$. Then there is an exact sequence

$$0 \longrightarrow F_0 \longrightarrow P \longrightarrow 0$$

where F_0 is finitely generated free and $F_0 \cong P$ by exactness. The module P is then free and thus stably free.

Assume the theorem holds for length $\leq n$. The free resolution

$$0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \xrightarrow{g} P \longrightarrow 0$$

has length $n+1$ and can be split into the two exact sequences

$$0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow K \longrightarrow 0$$

and

$$0 \longrightarrow K \longrightarrow F_0 \longrightarrow P \longrightarrow 0$$

where $K = \ker g$. The first sequence shows K has FFR of length $\leq n$. Since P is projective the second sequence splits. Therefore $F_0 \cong P \oplus K$ and K is finitely generated projective. By assumption K is stably free, that is, there is a finitely generated module Q with $K \oplus Q$ finitely generated free. Then P is stably free by

$$P \oplus (K \oplus Q) \cong (P \oplus K) \oplus Q \cong F_0 \oplus Q.$$

since $F_0 \oplus Q$ and $K \oplus Q$ are finitely generated free. $\square$

Similar to free resolutions, which are based on free modules, we can define projective resolutions based on projective modules.

Definition. A module M has a projective resolution if there exists an exact sequence

$$\cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where each P_i is a projective module.

Before we prove Theorem 3.10 we need some results considering noetherian rings and FFR.

Lemma 3.2. *Given a noetherian ring R and a finitely generated R-module A then there exists a projective resolution of A in which each module is finitely generated.*

Proof. We know there exists a finitely generated free R-module, P_0, and a surjective map $\epsilon : P_0 \to A$, from Theorem 1.3, and since A is finitely generated we can choose P_0 to be finitely generated. Since R is noetherian then $\ker \epsilon$ is finitely generated which means that there is another finitely generated free R-module P_1 giving us the exact sequence

$$0 \longrightarrow \ker d_1 \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \longrightarrow 0$$

where $d_1 : P_1 \to \ker \epsilon$ is a surjective map. Again $\ker d_1$ is also finitely generated so we can keep constructing in this manner and we get the sequence

$$
\begin{tikzcd}
... \arrow[r, "d_n"] & P_{n-1} \arrow[r] & ... \arrow[r] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "\epsilon"] & A \arrow[r] & 0
\end{tikzcd}
$$

which is a free resolution of A. Since free modules are projecive we are done. $\square$

That the projective resolution we just constructed in the proof Lemma 3.2 is also a free resolution will help us prove Proposition 3.5.

Lemma 3.3. *Given a diagram*

$$
\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & \\
 & P_1' \arrow[d] & & P_1'' \arrow[d] & \\
 & P_0' \arrow[d, "\epsilon'"] & & P_0' \arrow[d, "\epsilon''"] & \\
0 \arrow[r] & A' \arrow[r, "i"] & A \arrow[r, "q"] & A'' \arrow[r] & 0
\end{tikzcd}
$$

where the columns are projective resolutions of A' and A'', and the bottom row is exact, then there exists a projective resolution of A such that the three columns forms an exact sequence of complexes.

Proof. We will prove this by induction on the length of the projective resolutions. We start by checking for length $n = 0$. We want to show that we can form a 3×3 diagram with exact rows and columns, and with the exact sequence formed by A', A and A'' as the base row. Consider the following diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_0' \arrow[r] \arrow[d] & V_0 \arrow[r] \arrow[d] & K_0'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0' \arrow[r, "i_0"] \arrow[d, "\epsilon'"] & Q_0 \arrow[r, "q_0"] \arrow[d, "\epsilon"] & P_0'' \arrow[r] \arrow[d, "\epsilon''"] & 0 \\
0 \arrow[r] & A' \arrow[r, "i"] \arrow[d] & A \arrow[r, "q"] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

We define $Q_0 = P_0' \oplus P_0''$, and let $i_0 : P_0' \to Q_0$ and $q_0 : Q_0 \to P_0''$ by $x' \mapsto (x', 0)$ and $(x', x'') \mapsto x''$ respectively. Clearly

$$0 \longrightarrow P_0' \xrightarrow{i_0} Q_0 \xrightarrow{q_0} P_0'' \longrightarrow 0$$

is an exact sequence. Since P_0'' is projective we know there exists a map $\sigma : P_0'' \to A$ where $\epsilon'' = q\sigma$. We can use this to define $\epsilon : Q_0 \to A$ by $(x', x'') \mapsto i\epsilon(x') + \sigma(x'')$. By chasing the diagram we see that the square with vertices A, A'', P_0'' and Q_0 commutes, that is $\epsilon'' q_0 = q\epsilon$. We already know that q_0, q and ϵ'' are surjective hence ϵ is surjective. We define $V_0 = \ker \epsilon$, $K_0' = \ker \epsilon'$ and $K_0'' = \ker \epsilon''$, which immediately results in maps which will complete the diagram.

Next assume the proposition holds for n and let us denote the resulting diagram 'the diagram given by length n'. If we let $V_n = \ker(Q_n \to Q_{n-1})$, $K_n' = \ker \epsilon_n'$ and $K_n'' = \ker \epsilon_n''$ then we can construct the following diagram by the same reasoning as we used when $n = 0$.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K_0' \arrow[r] \arrow[d] & V_0 \arrow[r] \arrow[d] & K_0'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0' \arrow[r, "i_{n+1}"] \arrow[d, "\delta_{n+1}'"] & Q_0 \arrow[r, "q_{n+1}"] \arrow[d, "\delta_{n+1}"] & P_0'' \arrow[r] \arrow[d, "\delta_{n+1}''"] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

By composing the maps

$$\begin{tikzcd}
P_{n+1}' \arrow[d] & Q_{n+1} \arrow[d] & P_{n+1}'' \arrow[d] \\
K_n' \arrow[d] & V_n \arrow[d] & K_n'' \arrow[d] \\
P_n' & Q_n & P_n''
\end{tikzcd}$$

we can splice the diagram together with 'the diagram given by length n', and the proposition is true by induction. □

Proposition 3.4. *Let M be a module. If M has a projective resolution*

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \longrightarrow 0$$

where each P_i is a stably free, then M has FFR of length $\leq n+1$.

Proof. We will show this by induction on n. If we let $n = 0$ then we have an exact sequence,

$$0 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

so there exists an isomorphism $\epsilon : P_0 \to M$. Since P_0 is stably free we get that M is stably free and there exists finitely generated free modules F_0 and F_1 such that $F_0 \cong M \oplus F_1$. We can use this to construct an exact sequence

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

which shows that M has FFR of length 1.

Next we assume that the statement holds for $n < 0$. Let

$$0 \longrightarrow P_{n+1} \longrightarrow \cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \longrightarrow 0$$

be a projective resolution for M with each P_i stably free. Since P_0 is stably free there is a finitely generated free module $P_0 \oplus F$, where F is free. Therefore we can construct the exact sequence

$$0 \longrightarrow P_{n+1} \longrightarrow \cdots \longrightarrow P_2 \xrightarrow{d_2'} P_1 \oplus F \xrightarrow{d_1 \oplus 1_F} P_0 \oplus F \xrightarrow{\epsilon'} M \longrightarrow 0$$

where $d_2' : p_2 \mapsto (d_2(p_2), 0)$ and $\epsilon' : (p_0, f) \mapsto \epsilon(p_0)$. From this sequence we get a sequence ending in $\ker \epsilon'$ with n terms therefore $\ker \epsilon$ has FFR of length $\leq n+1$ by assumption. We can splice this sequence together with the exact sequence

$$0 \longrightarrow \ker \epsilon' \longrightarrow P_0 \oplus F \longrightarrow M \longrightarrow 0$$

giving us a finite free resolution for M with length $\leq (n+1)+1$ so the proposition is true by induction. $\square$

Proposition 3.5. *Given a noetherian ring R and a short exact sequence of R-modules*

$$0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0$$

if two of the modules have FFR then so does the third.

Before we begin the proof we note that for any free resolution of an R-module M^*,

$$\cdots \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M^* \longrightarrow 0$$

if we denote the kernel of $F_n \to F_{n-1}$ by K_n and K_n is stably free, then there exists an F such that $K_n \oplus F$ is finitely generated free and

$$0 \longrightarrow F \oplus K_n \longrightarrow F \oplus F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M^* \longrightarrow 0$$

is a finite free resolution for M^*.

Proof. If a module has FFR it is finitely generated therefore two of the modules in the above sequence are finitely generated. Since the ring R is noetherian the third module is finitely generated as well. Consider the case if M and M'' are finitely generated, then $M' \cong \operatorname{Im} i$ which is a submodule of M and thus finitely generated. If M' and M are finitely generated then M'' is the image of the finitely generated module M and thus finitely generated. If M' and M'' are finitely generated then we have shown, in Corollary 1.9, that in such an exact sequence M is finitely generated as well.

By Lemma 3.2 we can construct the following diagram where the columns are projective resolutions, which also are free resolutions by how they were constructed, of M' and M''.

$$\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & \\
 & F_1' \arrow[d] & & F_1'' \arrow[d] & \\
 & F_0' \arrow[d] & & F_0' \arrow[d] & \\
0 \arrow[r] & M' \arrow[r, "i"] & M \arrow[r, "p"] & M'' \arrow[r] & 0
\end{tikzcd}$$

By Lemma 3.3 we can create a projective resolution for M,

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

that gives us an exact sequence of complexes, where $F_i \cong F_i' \oplus F_i''$, by the proof of Lemma 3.3. Clearly it is also a free resolution. Let K_n', K_n and K_n'' denote the kernels of the following maps, respectively,

$$\begin{tikzcd}
F_n' \arrow[d] & F_n \arrow[d] & F_n'' \arrow[d] \\
F_{n-1}' & F_{n-1} & F_{n-1}''
\end{tikzcd}$$

We will show that for each of the three possibilities, where two of $\{M, M', M''\}$ has FFR, then the free resolution we just created for the remaining module will yield a finite free resolution.

Let M' and M'' have FFR of length n. If one of the lengths is 'shorter' we can always keep adding the zero module to the sequence until it is of the same length as the other. A consequence of Schanuel's Lemma [3, Proposition 3.12] is that if a module M^* has FFR of length n then for any free resolution

$$\cdots \longrightarrow F_n^* \longrightarrow \cdots \longrightarrow F_0^* \longrightarrow M^* \longrightarrow 0$$

we have that K_n^* is stably free where K_n^* is the kernel of $F_n^* \to F_{n-1}^*$. Therefore we have that both K_n' and K_n'' are stably free. By the exact sequence of complexes the following short exact sequence exists.

$$0 \longrightarrow K_n' \longrightarrow K_n \longrightarrow K_n'' \longrightarrow 0$$

Since K_n'' is stably free it is also projective, hence the sequence splits and

$$K_n \cong K_n' \oplus K_n''.$$

We have that K_n' and K_n'' are stably free therefore there exists free modules F' and F'' such that $K_n' \oplus F'$ and $F'' \oplus K_n''$ are free. By

$$(K_n' \oplus F') \oplus (F'' \oplus K_n'') \cong (K_n' \oplus K_n'') \oplus (F' \oplus F'') \cong K_n \oplus (F' \oplus F'')$$

we see that K_n is stably free and thus M has FFR.

Next let M and M'' have FFR of length n. Then K_n and K_n'' are stably free. Again the short exact sequence of kernels splits so

$$K_n \cong K_n' \oplus K_n''$$

and

$$K_n \oplus F \oplus F'' \cong K_n' \oplus ((K_n'' \oplus F'') \oplus F)$$

therefore K_n' is stably free thus M' has FFR.

Lastly let M' and M have FFR of length n. Then K_n' and K_n are stably free so there exists a free R-module F such that $K_n' \oplus F$ and $K_n \oplus F$ are finitely generated free. From the short exact sequence of kernels

$$0 \longrightarrow K_n' \longrightarrow K_n \longrightarrow K_n'' \longrightarrow 0$$

we get another short exact sequence

$$0 \longrightarrow K_n' \oplus F \longrightarrow K_n \oplus F \longrightarrow K_n'' \longrightarrow 0$$

which we can splice together with the exact sequence

$$0 \longrightarrow K_n'' \longrightarrow F_n'' \longrightarrow \cdots \longrightarrow F_0'' \longrightarrow M'' \longrightarrow 0$$

to get

$$0 \longrightarrow K_n' \oplus F \longrightarrow K_n \oplus F \longrightarrow F_n'' \longrightarrow \cdots \longrightarrow M'' \longrightarrow 0$$

a finite free resolution of M'', so M'' has FFR. □

What Proposition 3.5 states is that modules with FFR is what we call a family.

Definition. We define a family $\mathcal{F}$ to be a subclass of all R-modules such that for every exact sequence of R-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

if two of the terms lie in $\mathcal{F}$ then all three lie in $\mathcal{F}$.

Clearly $\{0\}$ is contained in all non-empty families since if $M \in \mathcal{F}$ then by the exact sequence

$$0 \longrightarrow M \longrightarrow M \longrightarrow 0 \longrightarrow 0$$

we have $\{0\} \in \mathcal{F}$.

Lemma 3.6. *Every intersection of families of R-modules is a family.*

Proof. Let $\mathcal{F}^* = \cap_\alpha \mathcal{F}_\alpha$, with each $\mathcal{F}_\alpha$ a family, and let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence with two terms in $\mathcal{F}^*$. This means that those two terms also are in each $\mathcal{F}_\alpha$ which means that the remaining term is in each $\mathcal{F}_\alpha$ and therefore in $\mathcal{F}^*$, making it a family. □

Definition. We define $\mathcal{F}(X)$, the family generated by X where X is a subclass of all R-modules, to be the intersection of all families containing X.

We define a child of X, or X-child, to be a module which occur in a short exact sequence where the two other terms are in X, we define $\mathcal{C}(X)$ to be the class of all X-children, we define $\mathcal{C}^0(X) = X$, and we define $\mathcal{C}^n(X) = \mathcal{C}(\mathcal{C}^{n-1}(X))$.

Lemma 3.7. *If X is a subclass of all R-modules then $\cup_{n=0}^{\infty}\mathcal{C}^n(X) = \mathcal{F}(X)$.*

Proof. The set X is contained in every family $\mathcal{F}$ making up $\mathcal{F}(X)$. From the definition of families and X-children we clearly get that $\mathcal{C}(X) \subseteq \mathcal{F}\ \forall\ \mathcal{F}$. Furthermore we have that $\mathcal{C}^n(X) \subseteq \mathcal{F}$ for all $\mathcal{F}$ and for any n. We see from the definition of $\mathcal{F}(X)$ that $\cup_{n=0}^{\infty}\mathcal{C}^n(X) \subseteq \mathcal{F}(X)$. To complete the proof we need to show that $\cup_{n=0}^{\infty}\mathcal{C}^n(X)$ is a family containing X. Consider the exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

where two terms lie in $\cup_{n=0}^{\infty}\mathcal{C}^n(X)$. This means that there is a n such that these two terms are contained in $\mathcal{C}^n(X)$ which means that there is a $\mathcal{C}^{n+1}$ which contains the third term. From this we clearly see that $\cup_{n=0}^{\infty}\mathcal{C}^n(X)$ is a family. Lastly we see that $X = \mathcal{C}^0(x) \subseteq \cup_{n=0}^{\infty}\mathcal{C}^n(X)$ and the lemma is true. □

Corollary 3.8. *Let R be a noetherian ring and X be the class of R-modules where each module has FFR. If $M \in \mathcal{F}(X)$ then M has FFR.*

Proof. Let $M \in \mathcal{F}(X)$ then, from Lemma 3.7, there is a $n \geq 0$ such that $M \in \mathcal{C}^n(X)$. We will prove the corollary by induction on n. Let $n = 0$. Then $M \in X$ and M has FFR.

Next assume the statements holds for n. Let $M \in \mathcal{C}^{n+1}(X)$. Then there exists a short exact sequence with M where the two other terms lie in $\mathcal{C}^n(X)$. By our assumption the two other terms have FFR, and from Proposition 3.5 we have that M has FFR making the corollary true by induction. $\square$

Definition. Let R be a ring and M an R-module. The subset $x \subseteq M$ is scalar closed if for every $x \in X$ and $r \in R$ we have that $rx \in X$

Definition. Given a scalar closed subset X we define the annihilator of x, $\operatorname{ann}(x)$, to be

$$\operatorname{ann}(x) = \{r \in R : rx = 0\}$$

We define the annihilator of the set X to be

$$\operatorname{ann}(X) = \{r \in R : rx = 0 \ \forall \ x \in X\}$$

and

$$\mathcal{A}(X) = \{\operatorname{ann}(x) : x \in X \text{ and } x \neq 0\}$$

Lemma 3.9. *Given a noetherian ring R, a non-zero finitely generated R-module M, and a non-empty set $X \subseteq M$ which is scalar closed. Then we have the following:*

1. *If I is a maximal ideal among $\mathcal{A}(X)$ then it is a prime ideal.*
2. *There exists a descending chain*

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n = 0$$

whose factor modules are of the form

$$M_i/M_{i+1} \cong R/\mathcal{P}_i$$

where $\mathcal{P}_i$ are prime ideals.

Proof. First we prove 1. Since R is a noetherian ring we have, by Corollary 1.8, that $\mathcal{A}(X)$ contains a maximal element which we will denote $I = \operatorname{ann}(x)$. let $a, b \in R$ where $ab \in I$ and $b \notin I$. We observe that $I \subseteq \operatorname{ann}(bx)$ and $Ra \subseteq \operatorname{ann}(bx)$, so $I \subseteq I + Ra \subseteq \operatorname{ann}(bx)$. If $a \notin I$ then we get $I \subsetneq I + Ra \subseteq \operatorname{ann}(bx)$, but $\operatorname{ann}(bx) \in \mathcal{A}(X)$, since X is scalar closed. This contradicts that I is the maximal element, hence $a \in I$ and I is prime.

Next we prove 2. Since R is noetherian then $\mathcal{A}(M)$ has a maximal element $\mathcal{P}_1 = \operatorname{ann}(x_1)$, which is a prime ideal by part 1. We define $M_1 = (x_1)$ and observe that $M_1 = R/\operatorname{ann}(x_1) = R/\mathcal{P}_1$. We repeat this construction for the maximal element of $\mathcal{A}(M/M_1)$ which we denote $\mathcal{P}_2 = \operatorname{ann}(x_2 + M_1)$. We let $M_2 = (x_1, x_2)$ and see that $M_2/M_1 \cong R/\operatorname{ann}(x_2 + M_1) = R/\mathcal{P}_2$, and $\{0\} \subseteq M_1 \subseteq M_2$. We can continue this process to create an ascending chain which must stop with some $M' \subseteq M$ where $M' = M$, since R is noetherian and by Corollary 1.9. □

Finally we are ready to show the main results of this chapter.

Theorem 3.10. *Let R be a noetherian ring. If every finitely generated R-module has FFR then every finitely generated $R[x]$-module has FFR.*

Proof. Let R be a noetherian ring and let every finitely generated R-module have FFR. Let X denote the class of all $R[x]$-modules M of the form $M \cong R[x] \otimes_R B$ where B is a finitely generated R-module. By assumption B has FFR so there exists an exact sequence

$$0 \longrightarrow F_m \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow B \longrightarrow 0$$

with each F_i free. We know $R[x]$ is flat as a R-module so if we tensor with $R[x]$ we get the following exact sequence.

$$0 \longrightarrow R[x] \otimes_R F_m \longrightarrow \cdots \longrightarrow R[x] \otimes_R F_0 \longrightarrow R[x] \otimes_R B \longrightarrow 0$$

For each $R[x] \otimes_R F_i$ we have, by Proposition 2.7 and Theorem 2.8, that

$$R[x] \otimes_R F_i \cong R[x] \otimes_R (\oplus_{i' \in I'} R) \cong \oplus_{i' \in I'} R[x] \otimes_R R \cong \oplus_{i' \in I} R[x],$$

that is to say, $R[x] \otimes_R F_i$ is a free $R[x]$-module. Thus $M \cong R[x] \otimes_R B$, and so all modules in X, has FFR. By Corollary 3.8 every module in $\mathcal{F}(X)$ have FFR. So what we need to prove is that every finitely generated $R[x]$-module lies in $\mathcal{F}(X)$.

Now let M be a finitely generated $R[x]$-module where $\mathrm{ann}(M) \cap R \neq 0$. Let $m \in M$ and $m \neq 0$. We observe that

$$0 \neq \mathrm{ann}(M) \cap R \subseteq \mathrm{ann}(m) \cap R.$$

We denote $\mathrm{ann}(m) \cap R = J$ and get $R/J \cong (m)_R$, giving us the exact sequence

$$0 \longrightarrow J \longrightarrow R \longrightarrow R/J \longrightarrow 0$$

which we can tensor with $R[x]$ to get the following exact sequence.

$$0 \longrightarrow R[x] \otimes_R J \longrightarrow R[x] \otimes_R R \longrightarrow R[x] \otimes_R R/J \longrightarrow 0$$

We have that $R/J \cong (m)_R$ and $R[x] \otimes_R R \cong R[x]$, by Proposition 2.7, so we have the following exact sequence.

$$0 \longrightarrow R[x] \otimes_R J \longrightarrow R[x] \longrightarrow R[x] \otimes_R (m)_R \longrightarrow 0$$

From Corollary 2.10 we have that $R[x] \otimes_R J \cong R[x]J$ and this is non-zero since $J \neq \{0\}$. By the exactness of the last sequence we have

$$R[x]/R[x]J \cong R[x] \otimes_R (m)_R.$$

Now $R[x]/R[x]J = (1 + R[x]J)$ is cyclic and $R[x] \otimes_R (m)_R$ is isomorphic to a submodule of M which then is cyclic and which we will denote (m_1) where $m_1 \in M$. We note that

$$(m_1) \cong R[x] \otimes_R (m)_R$$

and therefore $(m_1) \in (X) \subseteq \mathcal{F}(X)$. From the exactness of the last exact sequence, and since $(m_1) \cong R[x] \otimes_R (m)_R \cong R[x]J$, we get that $R[x] \otimes_R J \cong$

$\operatorname{ann}(m_1)$ and $\operatorname{ann}(m_1) \cap R \neq 0$. Now we can repeat this argument on $M/(m_1)$ and find a $m_2 + (m_1) \in M/(m_1)$ with $\operatorname{ann}(m_2 + (m_1)) \cap R \neq 0$ and $(m_1, m_2)/(m_1) \in X \subseteq \mathcal{F}(X)$. There is an exact sequence

$$0 \longrightarrow (m_1) \longrightarrow (m_1, m_2) \longrightarrow (m_1, m_2)/(m_1) \longrightarrow 0$$

making $(m_1, m_2) \in \mathcal{F}(X)$. Since R is noetherian and this is an ascending chain it has to stop by Corollary 1.9, hence we get that $M \in \mathcal{F}(X)$ if $\operatorname{ann}(M) \cap R \neq 0$.

Next we assume $\operatorname{ann}(M) \cap R = 0$. We know from Lemma 3.9 there is a chain of submodules

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n = \{0\}$$

where $M_i/M_{i+1} \cong R[x]/\mathcal{P}_i$ with $\mathcal{P}_i$ a prime ideal. For $i = n-1$ we get $M_{n-1}/M_n \cong R[x]/\mathcal{P}_{n-1}$, but $M_n = \{0\}$ therefore $M_{n-1} \cong R[x]/\mathcal{P}_{n-1}$. We construct the exact sequence

$$0 \longrightarrow M_{n-1} \longrightarrow M_{n-2} \longrightarrow M_{n-2}/M_{n-1} \longrightarrow 0$$

where both M_{n-1} and M_{n-2}/M_{n-1} are on the form $M' = R[x]/\mathcal{P}$ where $\mathcal{P}$ is a prime ideal. If we assume modules of the type M' have FFR then by Proposition 3.5 so does M_{n-2}. There is also an exact sequence

$$0 \longrightarrow M_{n-2} \longrightarrow M_{n-3} \longrightarrow M_{n-3}/M_{n-2} \longrightarrow 0$$

which yields that M_{n-3} has FFR. This process has to stop since R is noetherian hence M has FFR. Therefore the rest of the proof is reduced to showing that $M' = R[x]/\mathcal{P}$ has FFR. By assumption we have that

$$\operatorname{ann}(M') \cap R = \operatorname{ann}(R[x]/\mathcal{P}) \cap R \cong \mathcal{P} \cap R = \{0\}.$$

The ideal $\mathcal{P} \cap R$ is a prime ideal in R. A ring is a domain if and only if $\{0\}$ is a prime ideal in the ring, hence R, and so $R[x]$, are domains. Let $f(x)$ be a non-zero polynomial in $\mathcal{P} \subseteq R[x]$ and consider the following short exact sequence.

$$0 \longrightarrow (f) \longrightarrow \mathcal{P} \longrightarrow \mathcal{P}/(f) \longrightarrow 0$$

Since $R[x]$ is a domain then $(f) \cong R[x]$, and since $f(x) \in \mathcal{P}$ then $\operatorname{ann}(\mathcal{P}/(f)) \neq \{0\}$ hence $(f), \mathcal{P}/(f) \in \mathcal{F}$. Since $R[x], \mathcal{P} \in \mathcal{F}$ then $R[x]/\mathcal{P} \in \mathcal{F}$ □

Theorem 3.11. *Let k be a field. Then every finitely generated $k[x_1, \ldots, x_n]$-module has FFR.*

Proof. We will prove this by induction n. First let $n = 1$. Since k is a field by Theorem 3.10 every finitely generated $k[x]$-module has FFR.

Next assume the statement holds for $n > 1$. By Theorem 3.10 every finitely generated $k[x_1, \ldots, x_n, x_{n+1}]$-module has FFR and the theorem is true by induction. □

Corollary 3.12. *Let k be a field. If M is a finitely generated projective $k[x_1, \ldots, x_n]$-module then M is stably free.*

Proof. By Theorem 3.11 every finitely generated $k[x_1, \ dots, x_n]$ has FFR. From Theorem 3.1 we have that every finitely generated projective module with FFR is stably free. □

Chapter 4

UCP and Serre's Conjecture

Before we can prove Serre's Conjecture we will take a look at some results concerning linear transformations, polynomial rings and free modules. The key part of what we need to show is Theorem 4.2 which states that every stably free module over a ring with UCP is free.

4.1 UCP, Unimodular Column Property

Definition. Let R be a ring. We say $\alpha = (a_1, \dots, a_n) \in R^n$ is a unimodular column if there is $b_i \in R$ such that $a_1b_1 + \cdots + a_nb_n = 1$.

We say R has unimodular column property, which we will denote UCP, if, for every n, every unimodular column is the first column of some $n \times n$ invertible matrix over R.

Theorem 4.1. *Let R be a ring. If every finitely generated projective R-module is free then R has UCP.*

Proof. Let $\alpha = (a_1,\ dots, a_n) \in R^n$ be a unimodular column, then there are elements $b_i \in R$ such that $\sum_{i=1}^n r_i b_i = 1$. Define $\phi : R^n \to R$ by $(r_1, \dots, r_n) \mapsto \sum_{i=1}^n r_i b_i$. We have that $\phi(\alpha) = 1$ so there exists an exact sequence

$$0 \longrightarrow K \longrightarrow R^n \xrightarrow{\phi} R \longrightarrow 0$$

where $K = \ker \phi$. We know R is a projective module since it is a free R-module, so by Proposition 1.6, the sequence splits and by Proposition 1.5 we have that $R^n \cong K \oplus R$. Since α is a unimodular column the elements of α generates R hence $R^n \cong K \oplus (\alpha)$. The kernel $K \subseteq R^n$ is free by our hypothesis and of rank $n-1$. Let the set $\{\alpha_2, \ldots, \alpha_n\}$ be a basis for K. if we adjoin α then we get a basis for R^n. Let the set $\{\epsilon_1, \ldots, \epsilon_n\}$ denote the standard basis for R^n, in other words ϵ_i is the element of R^n with 1 in the i-th coordinate and 0 in the others. Then the R-map $T : R^n \to R^n$ defined by $T(\epsilon_1) = \alpha$ and $T(\epsilon_i) = \alpha_i$ for $i \geq 2$, can be represented by an invertible matrix T with α as it's first column. □

Theorem 4.1 is an example of the linear similarities we noted at the beginning of Chapter 1. The converse is not necessarily true, but we will show that it is so if $R = k[x_1, \ldots, x_n]$, where k is a field.

Proposition 4.2. *Let R be a ring with UCP. If P is a stably free R-module then P is free.*

Proof. If P is a stably free R-module then there is a direct sum $P \oplus F \cong R^n$ where F is free. Clearly $P \oplus R \oplus F' \cong R^n$ where F' is a free R-module of one less rank then F so it is enough to show that if $P \oplus R$ is free then P is free.

Let $P \oplus R \cong R^n$ and let π denote the R-map $\pi : R^n \to R$ where $\ker \pi = P$. Since π is surjective there is an element $\alpha = (a_1, \ldots, a_n) \in R^n$ such that $\pi(\alpha) = 1$. If we let the set $\{\epsilon_i\}_{i=1}^n$ denote the standard basis for R^n then $\pi(\alpha) = \sum_{i=1}^n \pi(a_i \epsilon_i) = \sum_{i=1}^n a_i \pi(\epsilon_i)$ which means α is a unimodular column. By our hypothesis there exists an invertible $n \times n$ matrix M with α as the first column. We denote the remaining columns $\alpha_2, \alpha_3, \ldots, \alpha_n$. We define the R-map $T : R^n \to R^n$ by $T(\epsilon_i) = M\epsilon_i$. We let $\alpha_j \to \alpha'_j$ denote the elementary column operation defined by $\alpha'_j = \alpha_j - \lambda_j \alpha$ where $\pi(\alpha_j) = \lambda_j \in R$ for $j \geq 2$. By applying $\alpha_j \to \alpha'_j$ to the matrix M we get the invertible matrix M' with columns $\alpha, \alpha'_2, \alpha'_3, \ldots, \alpha'_n$. We note that for $j \geq 2$ we have $\pi(\alpha'_j) = \pi(\alpha_j) - \lambda J \pi(\alpha) = \lambda_j - \lambda_j = 0$. The matrix M' induces an R-isomorphism by $T'(\epsilon_1) = \alpha$ and $T'(\epsilon_j) = \alpha'_j$ when $j \geq 2$ which satisfies $\alpha'_j = T'(\epsilon j) \in \ker \pi \ \forall\ j \geq 2$. Now let us take a look at the restriction $T^* = T|(\epsilon_2, \ldots, \epsilon_n) : (\epsilon_2, \ldots, \epsilon_n) \to P$. We have just observed that $\operatorname{Im} T^* \subseteq P$ and since T' is injective so is T^*. If we have that T^* is surjective then P is free and we are done. Let $\beta \in P$, then $\beta = T'(r_1\epsilon_1 + \delta)$ with $\delta = \sum_{i=2}^n r_i\epsilon_i$. Now $\beta - T'(\delta) \in P$, but $\beta - T'(\delta) = T'(\epsilon_1 r_1) = r_1 T'(\epsilon_1) = r_1\alpha \in (\alpha)$ so $\beta - T'(\delta) \in P \cap (\alpha) = \{0\}$. Hence $\beta = T'(\delta) \in \operatorname{Im}(T^*)$ and T^* is surjective. □

Corollary 4.3. *Let k be a field. If $k[x_1, \ldots, x_n]$ has UCP then every finitely generated projective $k[x_1, \ldots, x_n]$-module is free.*

Proof. By Corollary 3.12 every $k[x_1, \ldots, x_n]$-module is stably free and by Proposition 4.2 every stably free R-module where R has UCP is free. □

Clearly it only remains for us to show that $k[x_1, \ldots, x_n]$ has UCP, but we still need some more results before we can do that.

Definition. We define the total degree of a polynomial in $R = k[x_1, \ldots, x_n]$ as the maxiamal sum of powers of the variables of the terms of the polynomial.

To clarify, the total degree of $r_1x_1^1x_2^3x_3^4 + r_2x^4 \in R[x_1, \ldots, x_n]$ is $1+3+4 = 8$ since $8 > 4$.

Lemma 4.4. *Let k be a field, let $R = k[x_1, \ldots, x_n]$, let $a \in R$ have total degree δ and $b = \delta + 1$. Define*

$$y = x_m,$$

and for $1 \leq i \leq m-1$

$$y_i = x_i - x_m^{b^{m-i}} = x_i - y^{b^{m-i}}. \tag{4.1}$$

Then $a = ra'$ where $r \in k$ and $a' \in (k[y_1, \ldots, y_{m-1}])[y]$ is monic.

Proof. We get the defining equations by an automorphism in R with inverse $x_m \mapsto x_m$ and $x_i \mapsto x_i + x_m^{b^{m-i}}$ for $1 \leq n \leq m-1$, which restricts to an isomorphism $k[x_1, \dots, x_{m-1}] \to k[y_1, \dots, y_{m-1}]$ making $k[y_1, \dots, y_m]$ a polynomial ring.

We will denote $(j_1, \dots, j_m) \in \mathbb{N}^m$ by (j), and define $(j) \cdot (j') = \sum_{i=1}^m j_i j_i'$. Let us denote

$$(b^{m-1}, b^{m-2}, \dots, b, 1) = v$$

and a polynomial $a \in R$ with

$$a = \sum_{(j)} r_{(j)} x_1^{j_1} \cdots x_m^{j_m}$$

where $r_{(j)} \in k$ and $r_{(j)} \neq 0$. If we substitute the equations in Equation (4.1) into a then we get for the (j)-th monomial.

$$r_{(j)}(y_1 + y^{b^{m-1}})^{j_1}(y_2 + y^{b^{m-2}})^{j_2} \cdots (y_{m-1} + y^b)^{j_{m-1}} y^{j_m}.$$

We can expand and separate the pure power of y, that is,

$$r_{(j)}(y^{(j) \cdot v} + f_{(j)}(y_1, \dots, y_{m-1}, y))$$

where the polynomial $f_{(j)}$ has at least one positive power of some y_i which, for any y_i, can not be higher then the total degree, so it is strictly less then b. Therefore for any (j) we have $1 \leq j_i < b$ for each $j_i \in (j)$. If $(j) \neq (j')$ then $(j) \cdot v \neq (j') \cdot v$. Therefore there are no cancellations of terms in $\sum_{(j)} r_{(j)} y^{(j) \cdot v}$. Let D be the largest $(j) \cdot v$ then

$$a = r_D y^D + g(y_1, \dots, y_{m-1}, y)$$

for some polynomial g where the largest exponent of y in g is less then D, and r_D is a non-zero element in k. Since k is a field the inverse of r_D exists so $a = r_D a'$ where

$$a' = y^D + r_D^{-1} g(y_1, \dots, y_{m-1}, y)$$

is a monic polynomial in y. $\square$

Lemma 4.5. *Let B be a ring, let $s \geq 1$, and let $f, g \in B[y]$ be the polynomials:*

$$f(y) = y^s + a_1 y^{s-1} + \cdots + a_s$$

$$g(y) = b_1 y^{s-1} + \cdots + b_s$$

Then for each $1 \leq j \leq s-1$ the ideal $(f, g) \subseteq B[y]$ contains a polynomial of degree $\leq s-1$ with b_j as the leading coefficient.

Proof. Let I be the set containing $\{0\}$ and all the leading coefficients of $h(y) \in (f, g)$ where $\deg(h) \leq s-1$. Clearly $I \subseteq B$ is an ideal containing b_1. We will prove that I contains $b_1, \ldots, b_j$ $\forall\ j \leq s$. We define

$$g'(y) = yg(y) - b_1 f(y) = \sum_{i=1}^{s} (b_{i+1} - b_1 a_1) y^{s-1}$$

where $g'(y) \in (f, g)$ and we observe $(b_2 - b_1 a_1)$ is the leading term, therefore $(b_2 - b_1 a_1) \in I$. Since $b_1 \in I$ and $(b_2 - b_1 a_1) \in I$ then $b_2 \in I$. Next we define

$$g''(y) = yg'(y) - (b_2 - b_1 a_i) f(y)$$

whit the leading coefficient $(b_3 - b_1 a_3) - (b_2 - b_1 a_1) a_1$. Since $b_1, (b_2 - b_1 a_1) \in I$ then $b_3 \in I$. We can continue this process through all the coefficients in g, hence $b_j \in I$ $\forall\ j \leq s$ and we are done. $\square$

Definition. We define $\mathrm{GL(n, R)}$ to be the group of all invertible $n \times n$ matrices over R.

Definition. We define R to be a local ring if R has a unique maximal ideal.

Proposition 4.6. *Let B be a local ring and $R = B[y]$. If a_i is a monic polynomial where a_i is a coordinate in a unimodular column $\alpha = (a_i, ..., a_n) \in R^n$, then α is the first column of some matrix in* $\mathrm{GL(n, R)}$.

Proof. If we let $n = 1, 2$ it is easy to see that the statement is true. If $n = 1$ it will be an invertible element. If $n = 2$ then $a_1 b_1 + a_2 b_2 = 1$ and we observe that

$$\begin{bmatrix} a_1 & -b_2 \\ a_2 & b_1 \end{bmatrix}$$

is in $\mathrm{GL(n,R)}$. Therefore we can assume $n \geq 3$.

Let $\deg(a_1) = s$. We will prove the statement by induction on s. We can, without loss of generality, assume a_1 monic. If we let $s = 0$ then $a_1 = 1$. Let α be the first column in a $n \times q$ matrix L. If we perform elementary row operations on L it will yield a matrix NL, where N is invertible. Since $a_1 = 1$ we can perform elementary row operations on L such that the other entries in that column are equal to 0. Therefore $(NL)\epsilon_1 = \epsilon_1$ where N is invertible. The column vector α is the first column of L, hence $\alpha = L\epsilon_1 = N^{-1}\epsilon_1$ is the first column in an invertible matrix and the statement holds for $s = 0$.

Next assume $s > 0$. Since a_1 is monic we can perform elementary row operations on the column α such that $\deg(a_2) \leq s - 1 \ \forall\ i \geq 2$. Let m be the unique maximal ideal in B. The ideal mR consists of all polynomials with coefficients in m. The column $\bar{\alpha} = (a_1 + mR, ..., a_n + mR) \in R^n/mR^n$ is unimodular over $(B/m)[y]$ since α is a unimodular column. If $a_i \in mR \ \forall\ i \geq 2$ then $a_1 + m$ would be a unit in $(B/m)[y]$, but a_1 is not a constant, because $s > 0$, so it can not be a unit in the PID $(B/m)[y]$. Therefore we can assume at least one a_i where $i \geq 2$ is not in mR, and we can, without loss of generality assume $a_2 \notin mR$, that is

$$a_2 = b_1 y^{s-1} + ... + b_{s-1} y + b_s$$

where some of the $b_j \notin m$. From Lemma 4.5 the ideal $(a_1, a_2) \subseteq R$ contains a monic polynomial of degree $\leq s - 1$ so we can obtain a monic polynomial by adding a linear combinations of a_1 and a_2 into a_3, so the hypothesis applies and the statement is true by induction. □

Proposition 4.7. *Let $R = B[y]$ where B is a domain. If $\alpha = (a_1, ..., a_n) \in R^n$ is a unimodular column with at least one a_i monic then*

$$\alpha = M\beta$$

where $M \in \mathrm{GL(n,R)}$ and β is a unimodular column over B.

We will adopt a new notation for the proof of Proposition 4.7 to make the proof a bit more clear. If $R = B[y]$ we will denote a matrix M over R by $M(y)$, since the elements are polynomials in y, and if α is a unimodular column over R we denote it $\alpha(y)$.

Proof. We start by defining the ideal $I \subseteq B$ as

$$I = \{b \in B : \mathrm{GL(n,R)}\alpha(\mathrm{u+bv}) = \mathrm{GL(n,R)}\alpha(\mathrm{u})\ \forall\ \mathrm{u,v} \in \mathrm{R}\}$$

First we see that if $I = B$ then $1 \in I$. If we let $u = y$, $b = 1$ and $v = -y$ then

$$\mathrm{GL(n,R)}\alpha(\mathrm{y}) = \mathrm{GL(n,R)}\alpha(0)$$

that is

$$N\alpha(y) = N'\alpha(0)$$

for some $N, N' \in \mathrm{GL(n,R)}$, so $\alpha(y) = (N^{-1}N'\alpha(0)$. Since $\alpha(0)$ is a unimodular colum over B the statement is true in this case.

Next we assume that I is a proper ideal in B. Then $I \subseteq J$ for some maximal ideal J in B. By our hypothesis B is a domain and therefore a subring of the localization B_J. The column vector $\alpha(y)$ is also a unimodualar column over $B_J[y]$ and since B_J is local then $\alpha(y) = M(y)\epsilon_1$, by Proposition 4.6, where $M(y) \in \mathrm{GL(n, B_J[y])}$. We let z be a new indeterminate and adjoin it to $B_J[y]$. We define $N(y,z) \in \mathrm{GL(n, B_J[y,z])}$ to be

$$N(y,z) = M(y)M(y+z)^{-1}$$

We observe that $N(y,0) = I_n$, the identity matrix, and $\alpha(y+z) = M(y+z)\epsilon_1$, since $\alpha(y) = M(y)\epsilon_1$. It follows that

$$N(y,z)\alpha(y+z) = N(y,z)M(y+z)\epsilon_1 = M(y)\epsilon_1 = \alpha(y) \tag{4.2}$$

We also note that each entry of $N(y,z)$ is of the form $f_{ij}(y) + g_{ij}(y,z)$ where every term of g_{ij} involves a positive power of z. In other words $g_{ij}(y,0) = 0$. Since $N(y,0) = I_n$ then each $f_{ij}(y)$ is either equal to 1 or 0. Therefore there are no non-zero terms of the form λy^i, for $i > 0$ and $\lambda \in B_J$, of the entries in $N(y,z)$. So if we denote the entries of $N(y,z)$ by $h_{ij}(y,z)$

then $h_{ij}(y,z) = r + f_{ij}(y,z)$ where $r = 0$ or $r = 1$. Consider the $n \times n$ matrix $N(y,z)$. The entries of $N(y,z)$ are polynomials in the ring $B_J[y,z]$. Each such polynomial has coefficients in B_J, and these coefficients are by definition formal quotients of the form x/s where $x \in B$ and $s \in B \setminus J$. Let b be the product of all the denominators s of all the coefficients of all the n^2 entries in $N(y,z)$. Since J is a prime ideal we have that $b \notin J$ and hence $b \notin I$. If we let bz play the role of z in Equation 4.2 then

$$\mathrm{GL(n, B[y, z])\alpha(y + bz) = GL(n, B[y, z],)\alpha(y).}$$

Let $u, v \in R = B[y]$ and define a B-algebra map $\phi : B[y,z] \to B[y]$ by $\phi(y) = u$ and $\phi(z) = v$. By applying ϕ to the last equation we get

$$\mathrm{GL(n, R)\alpha(u + bv) = GL(n, R,)\alpha(u)}$$

thus $b \in I$ which is a contradiction. Hence I can not be a proper ideal in B. $\square$

4.2 Serre's Conjecture

Finally we can prove Serre's Conjecture.

Theorem 4.8. *Let k be a field. Every finitely generated projective $k[x_1, ..., x_m]$-module is free.*

Proof. By Corollary 4.3 it is enough for us to prove that $R = k[x_1, ..., x_m]$ has UCP, which we will do by induction on m. The polynomial ring $k[x]$ is a PID. The Structure Theorem for Finitely Generated Modules [1, Chapter 21, Theorem 1.1] over such rings implies, in particular, that finitely generated projective modules are free. By Proposition 4.1 every finitely generated projective $k[x]$-module then has UCP, which proves the case for the base step $m = 1$.

Next assume the statement holds for $m > 1$. Let $\alpha = (a_1, ..., a_n)$ be a unimodular column over $R = k[x_1, ..., x_{m+1}]$. Since α contains some non-zero a_i we can assume without loss of generality that $a_1 \neq 0$. By

Proposition 4.4 we have that there is a non-zero element $r \in k$ such that $a_1 = ra_1'$ where $a_1' \in k[y_1, ..., y_m](y)$ is a monic polynomial and the y_is and the y are as in Proposition 4.4. The element r is a unit, since k is a field, so we can assume $a_1 = a_1'$ without loss of generality. This allows us to apply Proposition 4.7 so

$$\alpha = M\beta$$

where $M \in \mathrm{GL(n, R)}$ and β is a unimodular over $B = k[y_1, ..., y_m]$. By our assumption B has UCP. Therefore $\beta = N\epsilon_1$ for some $N \in \mathrm{GL(n, B)}$, but $NM \in \mathrm{GL(n, R)}$ since $B \subseteq R$ so

$$\alpha = MN\epsilon_1$$

meaning α has the unimodular column property. Hence R has UCP by induction. $\square$

Bibliography

[1] Bhattacharya, P. B., Jain, S. K., and Nagpaul, S. R., *Basic Abstract Algebra*, 2d ed., Cambridge University Press, Cambridge, 1994.

[2] Quillen, D. , *Projective modules over polynomial rings*, Invent. Math. **36**, 167-171, 1976.

[3] Rotman, J. J., *An Introduction to Homological Algebra*, 2d ed., Springer, New York, 2009.

[4] Suslin, A. A., *Projective modules over polynomial rings are free*, (in Russian) Dokl. Akad. Nauk SSR **229**, 1063-1066, 1974.