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Degenerations and other partial orders on the space of representations of algebras

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Master of Science in Mathematics (for international students)
Submission date: April 2013
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Abstract

Let K be a field and Λ be an artin K-algebra. Let $rep_d\Lambda$ represent the set of all Λ-modules with the length equal to a natural number d as a K-vector space. The set of modules $rep_d\Lambda$ is equipped with the action of the general linear group. The corresponding Zariski-topology for algebraically closed field K then induce a partial order on $rep_d\Lambda$, which is called degeneration order and it is denoted by $\leq_{deg}$. Here for M and N, Λ-modules, the notion $M \leq_{deg} N$ mean that the orbit of N under the action of general linear group is contained in the closure of the orbit of M under the same group action. Another partial order on $rep_d\Lambda$ first showed by Riedtmann, is the virtual degeneration order, which is denoted by $\leq_{vdeg}$, are given by $M \leq_{vdeg} N$, if there is a Λ-module X such that $M \oplus X \leq_{deg} N \oplus X$. There are known examples where these two partial orders do not coincide. If K is an algebraically closed field, there is a geometric interpretation of these notions. However, there is also a module theoratical interpretation, which can be generalized to the general settings with K a commutative artin ring. Let Γ be the Kronecker quiver $1 \rightrightarrows 2$ and $\Lambda = \mathbb{Z}_2\Gamma$ be the path algebra of Γ over the field $\mathbb{Z}_2$ with two elements. In this work all degenerations between isomrphism classes of modules over Λ of dimension vector $(1,1)$, $(2,2)$ and $(3,3)$ are determined and the Hasse diagrams of the corresponding partial orders are given.

Contents

Chapter 1

Introduction and preliminaries

1.1 Introduction

Let K be a field and Λ be an artin K-algebra. Let $rep_d\Lambda$ represent the set of all Λ-modules with K-length equal to a natural number d. the set of modules $rep_d\Lambda$ is equipped with the action of the general linear group. The corresponding Zariski-topology for algebraically closed field K then induce a partial order on $rep_d\Lambda$, which is called degeneration order and it is denoted by $\leq_{deg}$. Here for M and N, Λ-modules the notion $M \leq_{deg} N$ mean that the orbit of N under the action of the group is contained in the closure of the orbit of M under the same group action. Another partial order on $rep_d\Lambda$ first showed by Riedtmann [1], is the virtual degeneration order, which is denoted by $\leq_{vdeg}$ is given by $M \leq_{vdeg} N$, if there is a Λ-module X such that $M \oplus X \leq_{deg} N \oplus X$. There are known examples where these two partial orders do not coincide.

If K is an algebraically closed field then there is a geometric interpretation of these notions. The theorems of Christine Riedmann and Grzegorz Zwara give a complete algebraic description, which can be generalized to the general settings with K a commutative artin ring. See [2, 1] for detail. This work deals with the notion of degeneration for non algebraically closed field K and the generalized pure module theoretical interpretation is used here. Chapter 2 is dedicated to describe these notions, especially the degeneration order.

Chapter 3 is devoted to explain some examples of degenration order of the modules for the kronecker quiver over the field of two $\mathbb{Z}_2$. The degenerations between isomorphism classes of modules of dimension vector (1,1), (2,2) and (3,3) are determined and the

Hasse diagrams will be used to present a graphical representation of the partial order.

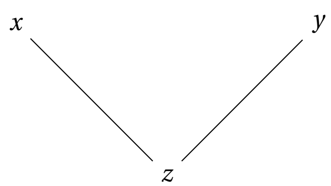
1.2 Preliminaries

Hasse diagram is a very intuitive tool to give a graphical representation of partial orders on finite sets consisting of vertices and line segments. A vertex represent each element of the partially ordered set and the line segements are drawn between these vertices according to the following rules:

- if $x < y$ in the poset, then the vertex corresponding to x appears above in the drawing than the vertex corresponding to y.
- The line segment between the vertices corresponding to any two elements x and y of the poset can only be included in the graphs if $x < y$ and $x < z < y$ implies that $z = x$ or $z = y$.

Example:

Let the set x, y, z be a partially ordered set where the relations between the elements are $x < z$ and $y < z$, then the Hasse diagram would look like this



Throughtout this dissertation, $modR$ denotes the the category of finitely generated R-modules, where R is a ring. The subcategory $indR \subseteq modR$ consists of exactly one representative of each isomorphism class of indecomposable modules in modR. The ring R is said to be of finite representation type if $indR$ is finite. An R-module is called artin if every descending chain of proper submodules is finite. We say R is artin if it is artin as an R-module. Let K be a commutative ring, then a K-module Λ is called a K-algebra if it is also a ring such that

$$a(xy) = (ax)y = x(ay)$$

for all $a \in K$ and $x, y \in \Lambda$.

Further if K is a commutative artin ring, then a K-algebra Λ is called an artin K-algebra if it is finitely generated as a K-module. Here we present some examples of algebras

Example:

In all these examples K is a commutative ring.

- Let $\Lambda = K$, then Λ not only is a K-algebra but also artin algebra if K is an artin ring.
- Let $\Lambda = K[X]$ the polynomial ring in one variable is a K-algebra. It is finitely generated as an algebra by $\{X\}$, however as a K-module the basis set $\left\{1_K, X, X^2, \ldots\right\}$ is not finite. Hence it is not an artin algebra over K.
- Let the ring of polynomials in n commuting variables $K[X_1, X_2, \ldots, X_n]$ is a K-algebra. It is finitely generated as an algebra by $\{X_1, X_2, \ldots, X_n\}$, but again for the similar reason is not finitely generated as a K-module and therefore is not artin algebra.

Path algebra

A path algebra is an important example of an algebra which is extensively used in this work. The starting point is a quiver Q which is a directed graph where loops and multiple arrows between vertices are allowed, i.e. a directed multidigraph. The quiver Q consists of a set of vertices Q_0 and a set Q_1 of oriented edges. The oriented edges are also often called arrows. We explain this with an example.

Example:

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

One can construct an algebra using all the oriented paths in this oriented graph including the paths of length zero at each vertex as a basis. By concatenating paths one makes a multiplication table for these base elements and in this way one obtains the path algebra. For the example of the quiver above, this will be a six dimensional algebra, with basis e_1, e_2, e_3, α, β and $\beta\alpha$. Here e_1, e_2 and e_3 represent the paths of length zero at the vertices 1, 2 and 3 respectively. One has to make a convention about how to represent a path and here one is using the convention that an oriented path is ordered from right to left. The multiplication table for this algebra is rather long, but for the convenience of the reader the complete table is included.

$e_1.e_1 = e_1$, $e_1.e_2 = 0$, $e_1.e_3 = 0$, $e_1.\alpha = 0$, $e_1.\beta = 0$, $e_1.\beta\alpha = 0$, $e_2e_1 = 0$, $e_2e_2 = e_2$, $e_2e_3 = 0$, $e_2\alpha = \alpha$, $e_2\beta = 0$, $e_2\beta\alpha = 0$, $e_3e_1 = 0$, $e_3e_2 = 0$, $e_3e_3 = e_3$, $e_3\alpha = 0$, $e_3\beta = \beta$, $e_3\beta\alpha = \beta\alpha$, $\alpha e_1 = \alpha$, $\alpha e_2 = 0$, $\alpha e_3 = 0$, $\alpha\alpha = 0$, $\alpha\beta = 0$, $\alpha\beta\alpha = 0$, $\beta e_1 = 0$, $\beta e_2 = \beta$, $\beta e_3 = 0$, $\beta\alpha = \beta\alpha$, $\beta\beta = 0$, $\beta\beta\alpha = 0$, $\beta\alpha e_1 = \beta\alpha$, $\beta\alpha e_2 = 0$, $\beta\alpha e_3 = 0$, $\beta\alpha\alpha = 0$, $\beta\alpha\beta = 0$, $\beta\alpha\beta\alpha = 0$.

Here $e_1 + e_2 + e_3$ is the identity element.For this simple example, the path algebra is isomorphic to the K-algebra of lower three by three matrices over K. To see this let e_{ij} be the matrix with 1 in place ij and zero otherwise. Then an isomorphism can be given by sending e_1 in the path algebra to the matrix e_{11}, e_2 in the path algebra to the matrix e_{22}, e_3 in the path algebra to the matrix e_{33}, α in the path algebra to the matrix e_{21}, β in the path algebra to the matrix e_{32} and $\beta\alpha$ in the path algebra to the matrix e_{31}. An easy calculation now shows that this is a K-algebra isomorphism from the path algebra of this quiver to the algebra of lower three by three matrices over K.

Let R be a commutative artin algebra and Λ be R-algebra then we define the following:

Definition 1.1. A -module P in $Mod\Lambda$ is projective if for every module epimorphism $f : N \longrightarrow M$ and every module homomorphism $g : P \longrightarrow M$, there exists a homomorphism $h : P \longrightarrow N$ such that $fh = g$. An aribtrary module A is said to be preprojective if and only if $(DTr)^n A = 0$ for some nonnegative integer n.

Definition 1.2. A -module I in $Mod\Lambda$ is injective if for any module monomorphism $f : N \longrightarrow M$ and every module homomorphism $g : M \longrightarrow I$, there exists a homomorphism $h : N \longrightarrow I$ such that $hf = g$. An aribtrary module B is said to be preinjective if and only if $(TrD)^n B = 0$ for some nonnegative integer n.

Definition 1.3. A representation (V, f) of a quiver Q over a field K is a set of vector spaces $V(i) \mid i \in Q_0$ together with K-linear mapes $f_\alpha : V(i) \longrightarrow V(j)$ for each arrow $\alpha : i \longrightarrow j$. Further a representation (V', f') is called a subrepresentation of (V, f) , if $V'(i) \subset V(i)$ for all i_0 and $f'_\alpha = f_{\alpha|_{v(i)}}$ for each arrow $\alpha : i \longrightarrow j$.

Definition 1.4. In abstract algebra, a module is indecomposable if it is non-zero and cannot be written as a direct sum of two non-zero submodules. Equivalently, representation of an algebra is said to be indecomposable if it cannot be expressed as a direct sum of proper nonzero subrepresentations.

Example:

The Kronecker quiver is the quiver having two vertices 1, 2 and α_1, $\alpha_2 : 1 \to 2$. The representations of K consist of two vector spaces V and W together with linear maps f_1, $f_2 : V \to W$. The dimension vector is the pair $(dimV, dimW)$ of non-negative integers. The Kronecker quiver is represented as:

$$1 \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} 2$$

The isomorphism classes of representations with dimension vector (m, n) correspond bijectively to the r-tuples of $n \times m$ matrices, up to simultaneous multiplication by invertible $n \times n$ matrices on the left, and by invertible $m \times m$ matrices on the right. The pairwise non-isomorphic indecomposable representations up to isomorphism are given by the following representations for $m \in \mathbb{N}$ and $\lambda \in K$ (see [5]).

$$K^m \underset{J_{m,K}}{\overset{1_m}{\rightrightarrows}} K^m$$

where

$$1_m = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

is the $m \times m$ identity matrix and

$$J_{m,K} = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & & \\ & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}$$

is a Jordan block of size $m \times m$. The number of such indecomposables is infnite, but can be organized in a one-parameter family in every dimension.

Chapter 2

Some partial orders and their mutual relationship

This chapter is devoted to explain the notions of degeneration, virtual degeneration, Hom orders and their mutual relationship.

2.1 Degeneration

We start by discussing the geometric interpretation of degeneration order.

Definition 2.1. A homomorphism between two algebras, Λ_1 and Λ_2, over a field K, is a map such that for all $r \in K$ and $\lambda_1, \lambda_2 \in \Lambda_1$,

$F(r\lambda_1) = rF(\lambda_1)$

$F(\lambda_1 + \lambda_2) = F(\lambda_1) + F(\lambda_2)$

$F(\lambda_1\lambda_2) = F(\lambda_1)F(\lambda_2)$

Further, we define $rep_d\Lambda$ for a finitely generated K-algebra Λ and a fixed natural number d, by a set of all K-algebra-homomorphisms from Λ to $M_d(K)$.

Example:

- For $\Lambda = K$, $rep_d\Lambda$ consist of only one homomorphism sending 1_Λ to I_d where I_d is $d \times d$ identity matrix.

- For $\Lambda = K[X]$, $rep_d\Lambda \simeq M_d(K)$, by sending $f \in rep_d(\Lambda)$ to $f(X)$. This can easily be extended to $rep_d\Lambda \simeq M_d(K) \times M_d(K)$ for $\Lambda = K < X, Y >$ (see [6]).

For each $f \in rep_d\Lambda$ there is associated a d-dimensional Λ-module M_f . This module is K^d as a K-vector space and define Λ action on M by $\lambda.x = f(\lambda)x$ for all $\lambda \in \Lambda$ and $x \in M_f = K^d$.

Conversly, from a d-dimensional Λ-module M one can have a $f_M \in rep_d\Lambda$ by fixing a K-basis for M and identifying M with K^d through this basis. This setting then allow to construct f_M by letting $f_M(\lambda)$ be the matrix where the ith column is λ times the ith basis vector in M. It can easily be verified that f_M is K-algebra homorphism.

Any $f \in rep_d\Lambda$ is completely determined by its values on the generators of Λ. Since Λ is finitely generated, by letting $X_1, X_2, ..., X_n$ be a generating set, the n-tuple $(f(X_1), f(X_2), ..., f(X_n)) \in M_d(K)^n$ can be identified with f. Here, one realize that $rep_d\Lambda$ is an affine space.

We also define a group action of the set of invertible $d \times d$ matrices $Gl_d(K)$ on $rep_d\Lambda$ by

$$G * f = (Gf(X_1)G^{-1}, Gf(X_2)G^{-1}, ..., Gf(X_n)G^{-1})$$

where $G \in Gl_d(K)$.

We denote the orbit of f under the group action above by $O(f)$. If $\Gamma = KQ$ is a path algebra then there is a correspondence between the representations of dimension vector $\boldsymbol{d} = (d_i)_{i\in Q_0}$ and points in $rep_d\Lambda$ by choosing basis for each vectorspace in the representation where $\boldsymbol{d} = \sum d_i$. Two elements f and f' in $rep_d\Lambda$ represent isomorphic Λ-modules M_f and M'_f if and only if f and f' belong to the same orbit under the action of $Gl_d(K)$ on $rep_d\Lambda$. This follows from the following result:

Lemma 2.1. The orbits of $O(f)$ for $f \in rep_d\Lambda$ corresponds to the isomorphism classes of Λ-modules of dimension d or equivalently correspond to the isomorphism classes of representations with dimension vector $\boldsymbol{d}$, where $\boldsymbol{d} = (d_i)_{i_0}$ and $\boldsymbol{d} = \sum(d_i)$

Definition 2.2. Let $f \in rep_d\Lambda$ then Zariski closure of $O(f)$ is

$$\overline{O(f)} = \left\{g \in rep_d\Lambda / p(g) = 0 \text{ for all polynomials} p \text{ such that } p(O(f)) = 0\right\}.$$

Degeneration on $rep_d\Lambda$ is then defined as $f \leq_{deg} g \iff g \in \overline{O(f)}$. or equivalently let M and N be d-dimensional Λ-modules and let f_M and f_N be the corresponding elements in $rep_d\Lambda$ then $M \leq_{deg} N$, if $O(f_N) \subseteq \overline{O}(f_M)$.

Degeneration make a reflexive and transitive relation on the set of isomophism classes of d-dimensional Λ-modules. In the later part of this chapter we will see that it is also antisymmetric, and hence is a partial order on the set of isomorphism classes of d-dimensional Λ-modules and will be called the degeneration order throughout this work. This description of degeneration provide an algebraic interpretation. Thanks to results from C. Riedtmann [1] and G. Zwara [2] providing an algebraic description for the notion of degeneration. We start by the following result from Riedtmann [1]:

Proposition 2.1. Let Λ be a finitely generated K-algebra. If there exists a short exact sequence sequence

$$0 \to A \to A \oplus M \to N \to 0$$

of Λ-modules with A, M and N finite dimensional as Λ-modules, then $dim(M) = dim(N)$ and $M \leq_{deg} N$.

Proof. A proof of this proposition is available in [1] and therefore is not given here. □

Proposition 2.2. Let Λ be a finitely generated K-algebra and M and N in $rep_d\Lambda$ with $M \leq_{deg} N$. Then there exists an exact sequence $0 \to A \to A \oplus M \to N \to 0$ of Λ-modules where A is finite-dimensional as a K-module.

These two results combined give a complete algebraic despcription and therefore this notion can be extended to algebras over commutative rings. From now on K will be a commutative artin ring, Λ will be a finitely generated algebra over K.

Here we also present a result from [6] which states that

Lemma 2.2. Let M and N be non-isomorphic Λ-module. Then there exists a Λ-module X such that $\ell(Hom_\Lambda(X,M)) \neq \ell(Hom_\Lambda(X,N))$.

The above two results can be used to show that degeneration is antisymmetric. If M, $N \in mod(\Lambda)$ such that $M \leq_{deg} N$ and $N \leq_{deg} M$ implies

$$0 \to A \to A \oplus M \to N \to 0$$

$$0 \to B \to B \oplus N \to M \to 0$$

Where A, $B \in mod(\Lambda)$. For any $X \in mod(\Lambda)$ one get two new short exact sequences

$$0 \to Hom_\Lambda(N,X) \to Hom_\Lambda(A \oplus M, X) \to Hom_\Lambda(A,X)$$

$$0 \to Hom_\Lambda(M,X) \to Hom_\Lambda(B \oplus N, X) \to Hom_\Lambda(B,X)$$

This follows

$$dim_K(Hom_\Lambda(M,X)) + dim_K(Hom_\Lambda(A,X)) \leq dim_K(Hom_\Lambda(A,X)) + dim_K(Hom_\Lambda(N,X))$$

$$\implies dim_K(Hom_\Lambda(M,X)) \leq dim_K(Hom_\Lambda(N,X))$$

In a similar way we get

$$\implies dim_K(Hom_\Lambda(N,X)) = dim_K(Hom_\Lambda(M,X))$$

for any X. Using above lemma we conclude that $M \simeq N$ and therefore degeneration is a partial order on the set of isomorphism classes of $rep_d \Lambda$.

Later G. Zwara [2] proved that the converse of Riedmann's result holds.

2.2 Virtual Degeneration and Hom-order

This section is devoted to introduce two other important partial orders on $rep_d \Lambda$. Virtual degeneration is a generalization of degeneration. In general one cannot cancel common direct summands from M and N when $M \leq_{deg} N$ (courtesy: an axemple due to J. Carlson)[1]), and obtain a degeneration of the remaining complements. But virtual degeneration provides this generalization and thus it is formally defined as

Definition 2.3. Let M and N be Λ-modules, M virtually degenerates to N if $M \oplus X \leq_{deg} N \oplus X$ for some $X \in mod(\Lambda)$. It is denoted as $M \leq_{vdeg} N$.

One can choose X to be the zero module, so obviously $\leq_{deg} \implies \leq_{vdeg}$.

Definition 2.4. Let M and N are Λ-modules, We write $M \leq_{Hom} N$ if $\ell_K(Hom_\Lambda(X, M)) \leq \ell_K(Hom_\Lambda(X, N))$ for all $X \in mod(\Lambda)$.

Hom-order provide an opportunity to extend the notion of virtual degeneration to situations where finitely generated algebra Λ over a commutative artin ring K instead of over an algebraically closed field(Using the charaterization of virtual degeneration given by short exact sequences). The relationship between virtual degeneration and Hom-order is established in following proposition.

Proposition 2.3. Let Λ be a finitely generated K-algebra. If the Λ-module M virtually degenerates to the Λ-module N, i.e. there is an exact sequence of Λ-modules

$$0 \to A \to A \oplus B \oplus M \to B \oplus N \to 0$$

which are of finite length as K-module, then

$$\ell_K(Hom_\Lambda(X, M)) \leq \ell_K(Hom_\Lambda(X, N))$$

for each Λ-module X which has finite length as a K-module.

Proof. Since $M \leq_{vdeg} N$ then for any Λ-module X there exists an exact sequence

$$0 \to Hom_\Lambda(A, X) \to Hom_\Lambda(A \oplus B \oplus M, X) \to Hom_\Lambda(B \oplus N, X)$$

From this we get

$$\ell(Hom_\Lambda(A \oplus B \oplus M, X)) \leq \ell(Hom_\Lambda(A, X)) + \ell(Hom_\Lambda(B \oplus N, X))$$

By subtracting

$$(\ell(Hom_\Lambda(A, X)) + \ell(Hom_\Lambda(B, X)))$$

from each side of the above inequality we get

$$\ell(Hom_\Lambda(M, X)) \leq \ell(Hom_\Lambda(N, X))$$

for each Λ module X.

□

As mentioned earlier degeneration does not implies virtual degeneration in general due to the example of J. Carlson. So in general, there is no equivalence between these three relations, but again Zawara come up with an important example when these three notions are equivalent.

Theorem 2.1. If Λ is an artin K-algebra of finite representation type, and M and N are two Λ-modules of the same length as K-modules, then the following three statements are equivalent:

- $M \leq_{deg} N$.
- $M \leq_{vdeg} N$.
- $M \leq_{hom} N$.

Chapter 3

Examples of Degeneration Order Of The Modules for the Kronecker Quiver Over The Field Of Two Elements

Let Γ be the Kronecker quiver $1 \rightrightarrows 2$ and $\Lambda = \mathbb{Z}_2\Gamma$ be the path algebra of Γ over the field $\mathbb{Z}_2$ with two elements. This chapter is dedicated to derive all degenerations between isomorphism classes of modules over Λ of dimension vector $(1,1)$, $(2,2)$ and $(3,3)$ and the Hasse diagrams of the corresponding partial orders.

Definition 3.1. A degeneraton $M \leq_{deg} N$ is called minimal if there does not exist a Λ-module $M^{'}$ such that $M \leq_{deg} M^{'} \leq_{deg} N$ with $N \neq M^{'} \neq M$.

The degeneration order is only determined for the modules of dimension vector $(1,1)$, $(2,2)$ and $(3,3)$. Hasse diagram are very intuitive tools for dealing with partial orders on finite sets and therefore is chosen here to represent the degeneration orders determined in these examples. The transitive reduction here is the minimal degeneration as the covering relation on the finite set of modules. Given an exact sequence $0 \to Y \xrightarrow{f} M \xrightarrow{g} N \to 0$ of Λ-modules, it follows that there always exist an exact sequence $0 \to Y \xrightarrow{(f)} Y \oplus M \xrightarrow{(g,0)} Y \oplus N \to 0$ which implies $M \leq_{deg} Y \oplus N$. This fact will be extensively used in this article.

The map between two modules are represented here as $\begin{array}{ccc} \mathbb{Z}_2^{n_1} & & \mathbb{Z}_2^{m_1} \\ \downarrow\downarrow & \xrightarrow{A,B} & \downarrow\downarrow \\ \mathbb{Z}_2^{n_2} & & \mathbb{Z}_2^{m_2} \end{array}$ with A and B are matrices of order $m_1 \times n_1$ and $m_2 \times n_2$ respectively.

There are only four Λ-modules of dimension vector $(1,1)$ and all are non-isomorphic.

The only decomposable module is $P_2 \oplus I_1$, where P_2 is the simple projective corresponding to vertex 2 and I_1 is a simple injective corresponding to vertex 1. The remaining three modules are all indecomposables with endomorphism rings isomporphic to $\mathbb{Z}_2$ which is local, and the indecomposables can be described with a specific nomenclature such as

$$R_{0,1} = \begin{array}{c}\mathbb{Z}_2\\ 1 \downarrow\downarrow 0\\ \mathbb{Z}_2\end{array}, \qquad \mathrm{R}_{\infty,1} = \begin{array}{c}\mathbb{Z}_2\\ 0 \downarrow\downarrow 1\\ \mathbb{Z}_2\end{array}, \qquad \mathrm{R}_{1,1} = \begin{array}{c}\mathbb{Z}_2\\ 1 \downarrow\downarrow 1\\ \mathbb{Z}_2\end{array}.$$

Trivially, there always exist a short exact sequence

$0 \to P_2 \to R_{i,1} \to I_1 \to 0$, i= 0,1,$\infty$. Therefore $R_{i,1} \leq_{deg} P_2 \oplus I_1$ are minimal degenrations and hence the Hasse diagram of the associated partial order can be drawn as:

$$\begin{tikzcd}
R_{0,1} \arrow[dr, no head] & R_{1,1} \arrow[d, no head] & R_{\infty,1} \arrow[dl, no head] \\
& P_2 \oplus I_1 &
\end{tikzcd}$$

Figure 3.1: Hasse diagram of the degeneration order for modules of dimension vector $(1,1)$

The rows are determined with ascending dimensions of the endomorphism rings of Λ-modules and to simplify it more each module represent the class of modules with isomorphic endomorphism rings which in this case is trivial but will be helpful in coming examples. Finding isomorphism classes of Λ-modules of higher dimension vector can be a little tricky. This may be simplified by using the following description.

Let $G = Gl_n(\mathbb{Z}_2^n) \times Gl_n(\mathbb{Z}_2^n)$, where $Gl_n(\mathbb{Z}_2^n)$ is the set of all $n \times n$ invertible matrices and X is the set of all representation of dimension vector (n, n). Define a group action $o : G \times X \to X$ by $(g, h) \times (A, B) = (hAg^{-1}, hBg^{-1})$ where $(g, h) \in G$.

As discussed in chapter 2 there is bijection between the set of isomorphism classes of Λ-modules in X and the set of G-orbits. With the above group action o and using Burnside's lemma, one gets that there are 16 Λ-modules of dimension vector $(2,2)$ up to isomorphism. The indecomposables are found by the fact that their endomorphism rings are local. Four modules are found to be indecomposable up to isomophism.

There is only one module up to isomorphism that has endomorphism ring isomorphic to the field of 4-elements $GF(4)$, which is local. Further, it corresponds to the only available irreducible polynomial of degree 2 in $\mathbb{Z}_2[X]$ and we represent it by $R_{x^2+x+1,2} = \begin{array}{c}\mathbb{Z}_2^2\\ I \downarrow\downarrow \alpha_1\\ \mathbb{Z}_2^2\end{array}$, where I is the identity matrix and $\alpha_1 = \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}$ is the matrix of linear operator obtained by $\mathbb{Z}_2[x]/f \xrightarrow{x} \mathbb{Z}_2[x]/f$ relative to the basis induced by 1 and x, where $f = x^2+x+1$ is the only monic polynomial of degree 2. The remaining indecomposables are $R_{0,2} = \begin{array}{c}\mathbb{Z}_2^2\\ I \downarrow\downarrow \alpha_2\\ \mathbb{Z}_2^2\end{array}$, $R_{1,2} = \begin{array}{c}\mathbb{Z}_2^2\\ I \downarrow\downarrow \alpha_3\\ \mathbb{Z}_2^2\end{array}$ and $R_{\infty,2} = \begin{array}{c}\mathbb{Z}_2^2\\ \alpha_4 \downarrow\downarrow I\\ \mathbb{Z}_2^2\end{array}$. Where $\alpha_2 = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$, $\alpha_3 = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ and $\alpha_4 = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$. Their endomorphism rings are isomorphic to the local ring $\mathbb{Z}_2[X]/ < x^2 >$.

The remaining 12 isomorphism classes of Λ-modules are decomposable and can completely be determined by the list of indecomposable, projective, injective and simple Λ-modules that is $P_1 = \begin{array}{c}\mathbb{Z}_2\\ \begin{pmatrix}1\\0\end{pmatrix} \downarrow\downarrow \begin{pmatrix}0\\1\end{pmatrix}\\ \mathbb{Z}_2^2\end{array}$, $P_2 = S_2$, $I_1 = S_1$ and $I_2 = \begin{array}{c}\mathbb{Z}_2^2\\ \begin{pmatrix}1 & 0\end{pmatrix} \downarrow\downarrow \begin{pmatrix}0 & 1\end{pmatrix}\\ \mathbb{Z}_2\end{array}$. Which then easily leads to the list of all isomorphism classes of decomposable modules. The list includes $R_{i,1} \oplus R_{j,1}$, $i < j$, $P_1 \oplus S_1$, $I_2 \oplus S_2$, $R_{i,1} \oplus R_{i,1}$, $R_{i,1} \oplus S_1 \oplus S_2$ and $S_1 \oplus S_1 \oplus S_2 \oplus S_2$. Note that i and j varies as $0, 1, \infty$ and $i < j$ whenever they come together in a single module notation through out this paper. The corresponding dimensions of their endomorphism rings as $\mathbb{Z}_2$-modules are 2, 3, 3, 4, 5 and 8 respectively and further are given as:

1. $End_\Lambda(R_{i,1} \oplus R_{j,1}) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$

2. $End_\Lambda(R_{i,2}) \simeq \mathbb{Z}_2[x]/ < x^2 >$

3. $End_\Lambda(R_{x^2+x+1,2}) \simeq GF(4)$

4. $End_\Lambda(P_1 \oplus S_1) = \begin{pmatrix} \begin{matrix} a & 0 \\ b & c \end{matrix} & 0 \\ 0 & \begin{matrix} a & 0 \\ 0 & a \end{matrix} \end{pmatrix} \in M_4(\mathbb{Z}_2)$

5. $End_\Lambda(I_2 \oplus S_2) = \begin{pmatrix} \begin{matrix} a & 0 \\ 0 & a \end{matrix} & 0 \\ 0 & \begin{matrix} a & b \\ 0 & c \end{matrix} \end{pmatrix} \in M_4(\mathbb{Z}_2)$

6. $End_\Lambda(R_{i,1} \oplus R_{i,1}) \simeq \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_2)$

7. $End_\Lambda(R_{i,1} \oplus S_1 \oplus S_2) = \begin{pmatrix} \begin{matrix} a & 0 \\ b & c \end{matrix} & 0 \\ 0 & \begin{matrix} a & d \\ 0 & e \end{matrix} \end{pmatrix} \in M_4(\mathbb{Z}_2)$

8. $End_\Lambda(S_1 \oplus S_1 \oplus S_2 \oplus S_2) = \begin{pmatrix} \begin{matrix} a & b \\ c & d \end{matrix} & 0 \\ 0 & \begin{matrix} e & f \\ g & h \end{matrix} \end{pmatrix} \in M_4(\mathbb{Z}_2)$

Hasse diagram in Figure 2 is then drawn by arranging the modules in rows corresponding to the ascending dimension of their endomorphism rings and further each modules is the representatives of the class of modules with same endomorphism rings. The joining

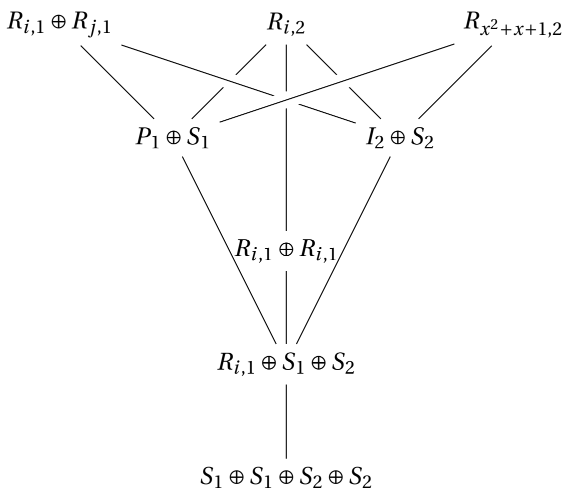


Figure 3.2: Hasse diagram for the degeneration order of modules of dimension vector $(2,2)$

lines represent all the minimal degenerations and they are determined by using the technique discussed earlier in this chapter. We have the following short exact sequences which correspond to the minimal degenerations for $i = 0$ and $j = 1$ in Figure 2:

1. $0 \to P_1 \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}} R_{0,1} \oplus R_{1,1} \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \end{pmatrix}} S_1 \to 0$

2. $0 \to S_2 \xrightarrow{\begin{pmatrix} 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}} R_{0,1} \oplus R_{1,1} \xrightarrow{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \end{pmatrix}} I_2 \to 0$

3. $0 \to R_{0,1} \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}} R_{0,2} \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \end{pmatrix}} R_{0,1} \to 0$

4. $0 \to P_1 \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} R_{0,2} \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \end{pmatrix}} S_1 \to 0$

5. $0 \to S_2 \xrightarrow{\begin{pmatrix}0\end{pmatrix},\begin{pmatrix}0\end{pmatrix}} R_{0,2} \xrightarrow{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\begin{pmatrix}1 & 0\end{pmatrix}} I_2 \to 0$

6. $0 \to P_1 \xrightarrow{\begin{pmatrix}1\\ 0\end{pmatrix},\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}} R_{x^2+x+1,2} \xrightarrow{\begin{pmatrix}0 & 1\end{pmatrix},\begin{pmatrix}0\end{pmatrix}} S_1 \to 0$

7. $0 \to S_2 \xrightarrow{\begin{pmatrix}0\end{pmatrix},\begin{pmatrix}0\\ 1\end{pmatrix}} R_{x^2+x+1,2} \xrightarrow{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\begin{pmatrix}1 & 0\end{pmatrix}} I_2 \to 0$

8. $0 \to S_2 \xrightarrow{\begin{pmatrix}0\\ 1\end{pmatrix},\begin{pmatrix}0\end{pmatrix}} P_1 \oplus S_1 \xrightarrow{\begin{pmatrix}1 & 0\end{pmatrix},\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}} R_{0,1} \oplus S_1 \to 0$

9. $0 \to R_{0,1} \xrightarrow{\begin{pmatrix}1\\ 0\end{pmatrix},\begin{pmatrix}1\\ 0\end{pmatrix}} I_2 \oplus S_2 \xrightarrow{\begin{pmatrix}0 & 1\end{pmatrix},\begin{pmatrix}0 & 1\end{pmatrix}} S_1 \oplus S_2 \to 0$

10. $0 \to S_2 \xrightarrow{\begin{pmatrix}0\end{pmatrix},\begin{pmatrix}1 & 0\end{pmatrix}} R_{0,1} \oplus R_{0,1} \xrightarrow{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\begin{pmatrix}0 & 1\end{pmatrix}} R_{0,1} \oplus S_1 \to 0$

11. $0 \to S_2 \xrightarrow{\begin{pmatrix}0\end{pmatrix},\begin{pmatrix}1 & 0\end{pmatrix}} R_{0,1} \oplus S_1 \oplus S_2 \xrightarrow{\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\begin{pmatrix}0 & 1\end{pmatrix}} S_1 \oplus S_1 \oplus S_2 \to 0$

Exact sequences can easily be found for other values of i and j in a similar fashion. No other minimal degeneration is possible. To prove this we recall a proposition from [6].

Proposition 3.1. Let M and N are Λ-modules then $M \nleq_{deg} N$ if $dim(Hom(X,M)) \nleq dim(Hom(X,N))$, for some X where M,N and X are Λ-modules.

By virtue of this proposition the following cases turn out to have no degenerations.

1. $R_{x^2+x+1,2} \not\leq_{deg} R_{i,1} \oplus R_{i,1}$,
   since $dim(Hom(X, R_{x^2+x+1,2})) \not\leq dim(Hom(X, R_{i,1} \oplus R_{i,1}))$ for $X = R_{x^2+x+1,2}$

2. $R_{i,1} \oplus R_{j,1} \not\leq_{deg} R_{i,1} \oplus R_{i,1}$ for $X = R_{j,1}$

3. $P_1 \oplus S_1 \not\leq_{deg} R_{i,1} \oplus R_{i,1}$ for $X = S_1$

Others only have zero homomorphisms between each other for instance say between Λ-modules M and N and therefore there cannot be an exact sequence of the form $0 \to Y \to Y \oplus M \to N \to 0$.

Now for the case of Λ-modules of dimension vector $(3,3)$, there are 52 Λ-modules up to isomorphism using the analogy as has been used in previous example. There are 5 indecomposable Λ-modules, and it turns out that their endomorphism rings are isomorphic either to $\mathbb{Z}_2[x]/<x^3>$ or $GF(8)$ (the field of 8-elements). Three of the 5 indecomposable Λ-modules of dimension vector $(3,3)$ have endomorphism rings isomorphic to $\mathbb{Z}_2[x]/<x^3>$ and they are represented and named as

$R_{0,3} = \begin{matrix}\mathbb{Z}_2^3\\ I\downarrow\downarrow\beta_1\\ \mathbb{Z}_2^3\end{matrix}$, $R_{1,3} = \begin{matrix}\mathbb{Z}_2^3\\ I\downarrow\downarrow\beta_2\\ \mathbb{Z}_2^3\end{matrix}$ and $R_{\infty,3} = \begin{matrix}\mathbb{Z}_2^3\\ \beta_1\downarrow\downarrow I\\ \mathbb{Z}_2^3\end{matrix}$, where $\beta_1 = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}$ and $\beta_2 = \begin{pmatrix} 1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}$.

The other two have endomorphism rings isomorphic to $GF(8)$ and they are $R_{p_1,3} = \begin{matrix}\mathbb{Z}_2^3\\ I\downarrow\downarrow\beta_3\\ \mathbb{Z}_2^3\end{matrix}$ and $R_{p_2,3} = \begin{matrix}\mathbb{Z}_2^3\\ I\downarrow\downarrow\beta_4\\ \mathbb{Z}_2^3\end{matrix}$, where $\beta_3 = \begin{pmatrix} 0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 1\end{pmatrix}$ and $\beta_4 = \begin{pmatrix} 0 & 0 & 1\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$ are linear operators obtained by $\mathbb{Z}_2[x]/f \xrightarrow{x} \mathbb{Z}_2[x]/f$ relative to the basis induced by $1, x$ and x^2. Where $f = p_1$ or p_2 and $p_1 = x^3 + x^2 + 1$ and $p_2 = x^3 + x + 1$ are the irreducible monic polynomials of degree 3.

Note that these are the only irreducible polynomials in $\mathbb{Z}_2[x]$ of degree 3. There is only one pre-projective Λ-module and one pre-injective Λ-module up to isomorphism that can be the summands of Λ-modules of dimension vector $(3,3)$ and they are $\tilde{P} = \begin{matrix} \mathbb{Z}_2^2 \\ \beta_5 \downarrow\downarrow \beta_6 \\ \mathbb{Z}_2^3 \end{matrix}$ and $\tilde{I} = \begin{matrix} \mathbb{Z}_2^3 \\ \beta_7 \downarrow\downarrow \beta_8 \\ \mathbb{Z}_2^2 \end{matrix}$ respectively, Where $\beta_5 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\beta_6 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\beta_7 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and $\beta_8 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

Figure 3 shows the Hasse diagram associated to the degeneration order of Λ-modules. Note that in Figure 3 p_r, $r = 1,2$ represents the two irreducible polynomials.

The rows have Λ-modules with endomorphim rings of increasing dimensions: 3, 4, 5, 6, 8, 9, 10, 13 and 18 as one goes down the rows. Following is the list of endomorphism rings of all modules in the Figure 3 except the indecomposables which are already mentioned. This information is vital in order to draw the above Hasse diagram:

1. $End_\Lambda(R_{0,1} \oplus R_{1,1} \oplus R_{\infty,1}) = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

2. $End_\Lambda(R_{i,1} \oplus R_{j,2}) = \mathbb{Z}_2[x]/ < x^2 > \times \mathbb{Z}_2$

3. $End_\Lambda(R_{x^2+x+1,2} \oplus R_{i,1}) = GF(4) \times \mathbb{Z}_2$

4. $End_\Lambda(\tilde{I} \oplus S_2) = \begin{pmatrix} a & 0 & 0 & & & \\ 0 & a & 0 & & 0 & \\ 0 & 0 & a & & & \\ & & & a & 0 & b \\ & 0 & & 0 & a & c \\ & & & 0 & 0 & d \end{pmatrix} \in M_6(\mathbb{Z}_2)$

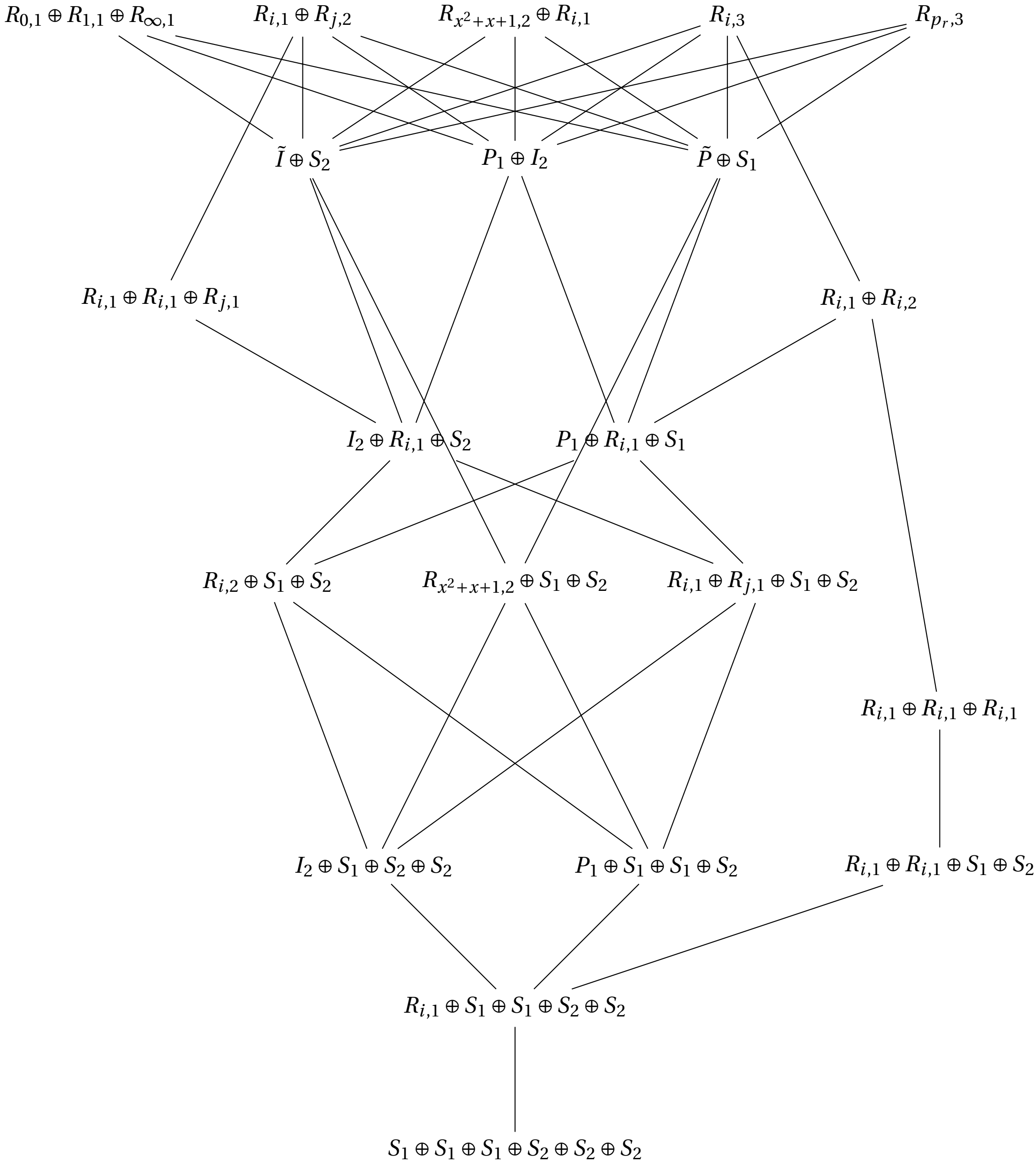


Figure 3.3: Hasse diagram for modules of dimension vector $(3,3)$

5. $$End_\Lambda(P_1 \oplus I_2) = \begin{pmatrix} a & 0 & b & & & \\ 0 & a & c & & 0 & \\ 0 & 0 & d & & & \\ & & & a & b & c \\ & 0 & & 0 & d & 0 \\ & & & 0 & 0 & d \end{pmatrix} \in M_6(\mathbb{Z}_2)$$

6. $$End_\Lambda(\tilde{P} \oplus S_1) = \begin{pmatrix} a & 0 & 0 & & & \\ b & d & 0 & & 0 & \\ c & 0 & d & & & \\ & & & a & 0 & 0 \\ & 0 & & 0 & a & 0 \\ & & & b & c & d \end{pmatrix} \in M_6(\mathbb{Z}_2)$$

7. $$End_\Lambda(R_{i,1} \oplus R_{i,1} \oplus R_{j,1}) \simeq \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{pmatrix} \in M_3(\mathbb{Z}_2)$$

8. $$End_\Lambda(R_{i,1} \oplus R_{i,2}) \simeq \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & d & e \end{pmatrix} \in M_3(\mathbb{Z}_2)$$

9. $$End_\Lambda(I_2 \oplus R_{i,1} \oplus S_2) = \begin{pmatrix} a & 0 & 0 & & & \\ 0 & a & b & & 0 & \\ 0 & 0 & d & & & \\ & & & a & b & c \\ & 0 & & 0 & d & e \\ & & & 0 & 0 & f \end{pmatrix} \in M_6(\mathbb{Z}_2)$$

10. $$End_\Lambda(P_1 \oplus R_{i,1} \oplus S_1) = \begin{pmatrix} a & 0 & 0 & & & \\ b & c & 0 & & 0 & \\ d & e & f & & & \\ & & & a & 0 & 0 \\ & 0 & & 0 & a & 0 \\ & & & 0 & b & c \end{pmatrix} \in M_6(\mathbb{Z}_2)$$

11. $$End_\Lambda(R_{i,2} \oplus S_1 \oplus S_2) = \begin{pmatrix} a & b & 0 & & & \\ 0 & a & 0 & & 0 & \\ f & d & c & & & \\ & & & a & b & 0 \\ & 0 & & 0 & a & g \\ & & & 0 & 0 & h \end{pmatrix} \in M_6(\mathbb{Z}_2)$$

12. $$End_\Lambda(R_{x^2+x+1,2} \oplus S_1 \oplus S_2) = \begin{pmatrix} a & b & 0 & & & \\ b & a & 0 & & 0 & \\ f & e & c & & & \\ & & & a+b & b & d \\ & 0 & & b & a & g \\ & & & 0 & 0 & h \end{pmatrix} \in M_6(\mathbb{Z}_2)$$

13. $$End_\Lambda(R_{i,1} \oplus R_{j,1} \oplus S_1 \oplus S_2) = \begin{pmatrix} a & 0 & 0 & & & \\ 0 & b & 0 & & 0 & \\ d & f & c & & & \\ & & & a & 0 & e \\ & 0 & & 0 & b & g \\ & & & 0 & 0 & h \end{pmatrix} \in M_6(\mathbb{Z}_2)$$

14. $$End_\Lambda(R_{i,1} \oplus R_{i,1} \oplus R_{i,1}) \simeq \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \in M_3(\mathbb{Z}_2)$$

15. $$End_\Lambda(I_2 \oplus S_1 \oplus S_2 \oplus S_2) = \left(\begin{array}{cccccc} a & 0 & 0 & & & \\ 0 & a & 0 & & 0 & \\ d & b & c & & & \\ & & & a & e & f \\ & 0 & & 0 & g & h \\ & & & 0 & k & l \end{array}\right) \in M_6(\mathbb{Z}_2)$$

16. $$End_\Lambda(R_{i,1} \oplus R_{i,1} \oplus S_1 \oplus S_2) = \left(\begin{array}{cccccc} a & b & 0 & & & \\ c & h & 0 & & 0 & \\ d & e & g & & & \\ & & & a & b & f \\ & 0 & & c & h & k \\ & & & 0 & 0 & l \end{array}\right) \in M_6(\mathbb{Z}_2)$$

17. $$End_\Lambda(P_1 \oplus S_1 \oplus S_1 \oplus S_2) = \left(\begin{array}{cccccc} h & 0 & 0 & & & \\ a & b & c & & 0 & \\ d & e & f & & & \\ & & & h & 0 & g \\ & 0 & & 0 & h & k \\ & & & 0 & 0 & l \end{array}\right) \in M_6(\mathbb{Z}_2)$$

18. $End_\Lambda(R_{i,1} \oplus S_1 \oplus S_1 \oplus S_2 \oplus S_2) = \begin{pmatrix} g & 0 & 0 & & & \\ a & b & c & & 0 & \\ d & e & f & & & \\ & & & g & h & k \\ & 0 & & 0 & l & m \\ & & & 0 & n & p \end{pmatrix} \in M_6(\mathbb{Z}_2)$

19. $End_\Lambda(S_1 \oplus S_1 \oplus S_1 \oplus S_2 \oplus S_2 \oplus S_2) = \begin{pmatrix} a & b & c & & & \\ d & e & f & & 0 & \\ g & h & i & & & \\ & & & a' & b' & c' \\ & 0 & & d' & e' & f' \\ & & & g' & h' & i' \end{pmatrix} \in M_6(\mathbb{Z}_2).$

Many of the degenerations in Figure 3 follow directly using the degenerations for Λ-modules of direction vector $(2,2)$ as if $M \leq_{deg} N$ implies $M \oplus Y \leq_{deg} N \oplus Y$ for M, N and Y are the Λ-modules. The following exact sequences describe these degenerations;

1. $0 \to R_{i,1} \to R_{i,1} \oplus R_{j,2} \to R_{i,1} \oplus R_{i,1} \to 0$
2. $0 \to S_2 \to P_1 \oplus I_2 \to I_2 \oplus R_{i,1} \to 0$
3. $0 \to P_1 \oplus R_{i,1} \to P_1 \oplus I_2 \to S_1 \to 0$
4. $0 \to S_2 \to R_{i,1} \oplus R_{i,1} \oplus R_{j,1} \to I_2 \oplus R_{i,1} \to 0$
5. $0 \to P_1 \to R_{i,1} \oplus R_{i,2} \to R_{i,1} \oplus S_1 \to 0$
6. $0 \to R_{j,1} \to I_2 \oplus R_{i,1} \oplus S_2 \to R_{i,1} \oplus S_1 \oplus S_2 \to 0$
7. $0 \to S_2 \oplus S_2 \to R_{i,2} \oplus S_1 \oplus S_2 \to I_2 \oplus S_1 \to 0$
8. $0 \to S_1 \oplus S_2 \to R_{x^2+x+1,2} \oplus S_1 \oplus S_2 \to I_2 \oplus S_1 \to 0$
9. $0 \to P_1 \to R_{x^2+x+1,2} \oplus S_1 \oplus S_2 \to S_1 \oplus S_1 \oplus S_2 \to 0$

10. $0 \to S_2 \to R_{i,1} \oplus R_{j,1} \oplus S_1 \oplus S_2 \to I_2 \oplus S_1 \oplus S_2 \to 0$

11. $0 \to P_1 \to R_{i,1} \oplus R_{j,1} \oplus S_1 \oplus S_2 \to S_1 \oplus S_1 \oplus S_2 \to 0$

12. $0 \to S_2 \to R_{i,1} \oplus R_{i,1} \oplus R_{i,1} \to R_{i,1} \oplus R_{i,1} \oplus S_1 \to 0$

13. $0 \to R_{i,1} \to I_2 \oplus S_1 \oplus S_2 \oplus S_2 \to S_1 \oplus S_1 \oplus S_2 \oplus S_2 \to 0$

14. $0 \to S_2 \to R_{i,1} \oplus R_{i,1} \oplus S_1 \oplus S_2 \to R_{i,1} \oplus S_1 \oplus S_1 \oplus S_2 \to 0$

15. $0 \to S_2 \to P_1 \oplus S_1 \oplus S_1 \oplus S_2 \to R_{i,1} \oplus S_1 \oplus S_1 \oplus S_2 \to 0$

16. $0 \to S_2 \to R_{i,1} \oplus S_1 \oplus S_1 \oplus S_2 \oplus S_2 \to S_1 \oplus S_1 \oplus S_1 \oplus S_2 \oplus S_2 \oplus S_2 \to 0$

Other exact sequences corresponding to the minimal degenerations in Hasse diagram in Figure 3 for $i = 0$ and $j = 1$ are given below:

1. $0 \to S_2 \xrightarrow{\begin{pmatrix}0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}} R_{0,1} \oplus R_{1,1} \oplus R_{\infty,1} \xrightarrow{\begin{pmatrix}0&1&1\\0&1&0\\1&1&0\end{pmatrix},\begin{pmatrix}1&1&0\\0&1&0\end{pmatrix}} \tilde{I} \to 0$

2. $0 \to P_1 \xrightarrow{\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1&0\\1&1\\0&1\end{pmatrix}} R_{0,1} \oplus R_{1,1} \oplus R_{\infty,1} \xrightarrow{\begin{pmatrix}1&1&0\\0&1&1\end{pmatrix},\begin{pmatrix}1&1&1\end{pmatrix}} I_2 \to 0$

3. $0 \to \tilde{P} \xrightarrow{\begin{pmatrix}1&0\\1&1\\0&1\end{pmatrix},\begin{pmatrix}0&0&1\\1&1&1\\1&0&0\end{pmatrix}} R_{0,1} \oplus R_{1,1} \oplus R_{\infty,1} \xrightarrow{\begin{pmatrix}1&1&1\end{pmatrix},\begin{pmatrix}0\end{pmatrix}} S_1 \to 0$

4. $0 \to S_2 \xrightarrow{\begin{pmatrix}0\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}} R_{0,1} \oplus R_{1,2} \xrightarrow{\begin{pmatrix}0&0&1\\0&1&1\\1&1&0\end{pmatrix},\begin{pmatrix}1&1&0\\0&1&1\end{pmatrix}} \tilde{I} \to 0$

5. $0 \to P_1 \xrightarrow{\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1&0\\1&0\\1&1\end{pmatrix}} R_{0,1} \oplus R_{1,2} \xrightarrow{\begin{pmatrix}1&1&0\\0&1&1\end{pmatrix},\begin{pmatrix}1&1&0\end{pmatrix}} I_2 \to 0$

6. $0 \to \tilde{P} \xrightarrow{\begin{pmatrix}1&0\\1&1\\0&1\end{pmatrix},\begin{pmatrix}0&0&1\\0&1&1\\1&1&0\end{pmatrix}} R_{0,1} \oplus R_{1,2} \xrightarrow{\begin{pmatrix}1&1&1\end{pmatrix},\begin{pmatrix}0\end{pmatrix}} S_1 \to 0$

7. $0 \to S_2 \xrightarrow{\begin{pmatrix}0\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}} R_{x^2+x+1,2} \oplus R_{0,1} \xrightarrow{\begin{pmatrix}1&0&0\\1&1&0\\0&1&1\end{pmatrix},\begin{pmatrix}0&1&1\\1&1&0\end{pmatrix}} \tilde{I} \to 0$

8. $0 \to P_1 \xrightarrow{\begin{pmatrix}0\\1\\1\end{pmatrix},\begin{pmatrix}0&1\\1&1\\1&0\end{pmatrix}} R_{x^2+x+1,2} \oplus R_{0,1} \xrightarrow{\begin{pmatrix}1&1&1\\1&0&0\end{pmatrix},\begin{pmatrix}1&1&1\end{pmatrix}} I_2 \to 0$

9. $0 \to \tilde{P} \xrightarrow{\begin{pmatrix}0&1\\1&1\\1&0\end{pmatrix},\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}} R_{x^2+x+1,2} \oplus R_{0,1} \xrightarrow{\begin{pmatrix}1&1&1\end{pmatrix},\begin{pmatrix}0\end{pmatrix}} S_1 \to 0$

10. $0 \to S_2 \xrightarrow{\begin{pmatrix}0\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}} R_{0,3} \xrightarrow{\begin{pmatrix}0&0&1\\0&1&1\\1&1&0\end{pmatrix},\begin{pmatrix}1&1&0\\0&1&1\end{pmatrix}} \tilde{I} \to 0$

11. $0 \to P_1 \xrightarrow{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1&1\\0&1\\1&0\end{pmatrix}} R_{0,3} \xrightarrow{\begin{pmatrix}1&1&1\\0&1&0\end{pmatrix},\begin{pmatrix}1&1&1\end{pmatrix}} I_2 \to 0$

12. $0 \to \tilde{P} \xrightarrow{\begin{pmatrix}1&1\\0&1\\1&0\end{pmatrix},\begin{pmatrix}1&1&1\\0&1&0\\0&0&1\end{pmatrix}} R_{0,3} \xrightarrow{\begin{pmatrix}1&1&1\end{pmatrix},\begin{pmatrix}0\end{pmatrix}} S_1 \to 0$

13. $0 \to S_2 \xrightarrow{\begin{pmatrix}0\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}} R_{x^3+x+1,3} \xrightarrow{\begin{pmatrix}1&1&0\\1&0&1\\1&0&0\end{pmatrix},\begin{pmatrix}0&1&1\\1&0&1\end{pmatrix}} \tilde{I} \to 0$

14. $0 \to P_1 \xrightarrow{\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0&1\\1&0\\1&0\end{pmatrix}} R_{x^3+x+1,3} \xrightarrow{\begin{pmatrix}0&1&1\\0&0&1\end{pmatrix},\begin{pmatrix}0&1&1\end{pmatrix}} I_2 \to 0$

15. $0 \to \tilde{P} \xrightarrow{\begin{pmatrix}0&1\\1&1\\1&0\end{pmatrix},\begin{pmatrix}1&1&0\\1&1&1\\1&0&1\end{pmatrix}} R_{x^3+x+1,3} \xrightarrow{\begin{pmatrix}1&1&1\end{pmatrix},\begin{pmatrix}0\end{pmatrix}} S_1 \to 0$

16. $0 \to R_{0,1} \xrightarrow{\begin{pmatrix}1&0&0\end{pmatrix},\begin{pmatrix}1&0&0\end{pmatrix}} R_{0,3} \xrightarrow{\begin{pmatrix}0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}0&1&0\\0&0&1\end{pmatrix}} R_{0,2} \to 0$

17. $0 \to S_2 \oplus R_{0,1} \xrightarrow{\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}1&1\\0&0\\0&1\end{pmatrix}} \tilde{I} \oplus S_2 \xrightarrow{\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix},\begin{pmatrix}0&1&0\end{pmatrix}} I_2 \to 0$

18. $0 \to P_1 \xrightarrow{\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0&1\\1&0\\0&0\end{pmatrix}} \tilde{P} \oplus S_1 \xrightarrow{\begin{pmatrix}1&0&0\\0&0&1\end{pmatrix},\begin{pmatrix}0&0&1\end{pmatrix}} R_{0,1} \oplus S_1 \to 0$

19. $0 \to S_2 \xrightarrow{\begin{pmatrix}0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}} I_2 \oplus R_{0,1} \oplus S_2 \xrightarrow{\begin{pmatrix}0&1&0\\1&0&0\\1&1&1\end{pmatrix},\begin{pmatrix}1&0&1\\0&0&1\end{pmatrix}} R_{0,2} \oplus S_1 \to 0$

20. $0 \to S_2 \xrightarrow{\begin{pmatrix}0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix}} P_1 \oplus R_{0,1} \oplus S_1 \xrightarrow{\begin{pmatrix}0&1&0\\1&0&0\\1&1&1\end{pmatrix},\begin{pmatrix}0&1&1\\0&0&1\end{pmatrix}} R_{0,2} \oplus S_1 \to 0$

21. $0 \to R_{0,1} \oplus S_2 \xrightarrow{\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0&0\\1&1\\0&1\end{pmatrix}} P_1 \oplus R_{0,1} \oplus S_1 \xrightarrow{\begin{pmatrix}1&0&0\\1&0&1\end{pmatrix},\begin{pmatrix}1&0&0\end{pmatrix}} \oplus S_1 \oplus R_{0,1} \to 0$

Degenerations for other values of i and j can easily be determined.

Again by proposition **??** the following cases turn out to have no degenerations.

1. $R_{0,1} \oplus R_{1,1} \oplus R_{\infty,1} \not\leq_{deg} R_{i,1} \oplus R_{i,1} \oplus R_{j,1}$ for $X = R_{k,1}$, with $i < j$ and $i, j \neq k$, $k = 0, 1, \infty$
2. $R_{x^2+x+1,2} \oplus R_{i,1} \not\leq_{deg} R_{i,1} \oplus R_{i,1} \oplus R_{j,1}$ for $X = R_{x^2+x+1,2}$
3. $R_{x^2+x+1,2} \oplus R_{i,1} \not\leq_{deg} R_{i,1} \oplus R_{i,2}$ for $X = R_{x^2+x+1,2}$
4. $R_{p_s,3} \not\leq_{deg} R_{i,1} \oplus R_{i,2}$ for $X = p_s$
5. $\tilde{I} \oplus S_2 \not\leq_{deg} R_{i,1} \oplus R_{i,1} \oplus R_{j,1}$ for $X = \tilde{I}$
6. $\tilde{I} \oplus S_2 \not\leq_{deg} R_{i,1} \oplus R_{i,2}$ for $X = \tilde{I}$
7. $P_1 \oplus I_2 \not\leq_{deg} R_{i,1} \oplus R_{i,1} \oplus R_{j,1}$ for $X = I_2$
8. $P_1 \oplus I_2 \not\leq_{deg} R_{i,1} \oplus R_{i,2}$ for $X = I_2$
9. $\tilde{P} \oplus S_1 \not\leq_{deg} R_{i,1} \oplus R_{i,2}$ for $X = S_1$

10. $R_{i,2} \oplus S_1 \oplus S_2 \nleq_{deg} R_{i,1} \oplus R_{i,1} \oplus R_{i,1}$ for $X = S_1$

11. $\tilde{P} \oplus S_1 \nleq_{deg} R_{i,1} \oplus R_{i,1} \oplus R_{j,1}$ for $X = S_1$

12. $I_2 \oplus R_{i,1} \oplus S_2 \nleq_{deg} R_{x^2+x+1,2} \oplus S_1 \oplus S_2$ for $X = R_{i,1}$

13. $P_1 \oplus R_{i,1} \oplus S_1 \nleq_{deg} R_{x^2+x+1,2} \oplus S_1 \oplus S_2$ for $X = R_{i,1}$

14. $R_{x^2+x+1,2} \oplus S_2 \oplus S_1 \nleq_{deg} R_{i,1} \oplus R_{i,1} \oplus R_{i,1}$ for $X = R_{x^2+x+1,2}$

15. $R_{i,1} \oplus R_{j,1} \oplus S_1 \oplus S_2 \nleq_{deg} R_{i,1} \oplus R_{i,1} \oplus R_{i,1}$ for $X = R_{j,1}$

Here is the table of some data calculated:

dimension vector	(1,1)	(2,2)	(3,3)
no. of modules	4	256	262144
no. of indecomposables	3	66	29232
no. of isomorphic classes of modules	4	16	52
no. of isomorphic classes of indecomposable modules	3	4	5
$\lvert Gl_n(\mathbb{Z}_2)\rvert \oplus \lvert Gl_n(\mathbb{Z}_2)\rvert$	1	36	28224

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