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The Sidon Constant for Ordinary Dirichlet Series

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Abstract. We obtain the asymptotic formula of the Sidon constant for ordinary Dirichlet series using the Bohnenblust–Hille inequality and estimates on smooth numbers. We moreover give precise estimates for the error term.

Sammendrag. Ved å benytte Bohnenblust–Hille-ulikheten og estimater for glatte tall oppnår vi den asymptotiske formelen til Sidon-konstanten for ordinære Dirichlet-rekker. Videre angir vi presise estimater for feilleddet.

Preface

This master's thesis was written under the supervision of Professor Kristian Seip, and marks the conclusion of five years of mathematics studies at NTNU. I am deeply grateful to Professor Seip, not only for his excellent suggestion of topic for the thesis, but also for the enlightening discussions in our weekly meetings and his detailed feedback on my drafts.

My intention was to prove every result needed to "bridge the gap" between what (I think) every master student in analysis should know and cutting-edge results. With one notable exception, I feel I have accomplished this.

Even though there is nothing groundbreaking in this thesis, I feel I have made some small improvements and simplifications here and there. All in all I am reasonably satisfied with the result.

Ole Fredrik Brevig
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Introduction

We begin by giving a short introduction to the topic at hand, through General Dirichlet Series and Sidon Constants. After this we give a brief overview of the thesis and discuss some asymptotic notations employed.

General Dirichlet Series

Suppose that $\Lambda = \{\lambda_1,\, \lambda_2,\, \lambda_3,\, \ldots\}$ is an increasing sequence of real numbers such that $\lambda_n \to \infty$. We say that Λ defines a *General Dirichlet Series* by

$$\mathscr{F}(s) = \sum_{n=1}^{\infty} a_n e^{-s\lambda_n},$$

where the coefficients a_n and the variable $s = \sigma + it$ are complex numbers. For each defining set Λ we consider the sequence of *Sidon Constants*, given by

$$S_\Lambda(N) = \inf\left\{C \,:\, \sum_{n=1}^{N} |a_n| \leq C \sup_{t\in\mathbb{R}} \left|\sum_{n=1}^{N} a_n e^{-it\lambda_n}\right|,\ \forall a_n \in \mathbb{C}\right\}.$$

Combining the inequality

$$\left(\sum_{n=1}^{N} |a_n|^2\right)^{\frac{1}{2}} = \left(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left|\sum_{n=1}^{N} a_n e^{-it\lambda_n}\right|^2 \, dt\right)^{\frac{1}{2}} \leq \sup_{t\in\mathbb{R}} \left|\sum_{n=1}^{N} a_n e^{-it\lambda_n}\right|$$

with the Cauchy–Schwarz inequality yields the upper bound $S_\Lambda(N) \leq \sqrt{N}$, since

$$\sum_{n=1}^{N} |a_n| \leq \sqrt{N} \left(\sum_{n=1}^{N} |a_n|^2\right)^{\frac{1}{2}} \leq \sqrt{N} \sup_{t\in\mathbb{R}} \left|\sum_{n=1}^{N} a_n e^{-it\lambda_n}\right|.$$

If we take $\Lambda = \{0,\, 1,\, 2,\, \ldots\}$ we obtain *Power Series* in the variable $z = e^{-s}$ since

$$F(z) = \sum_{n=1}^{\infty} a_n z^n = \sum_{n=1}^{\infty} a_n \left(e^{-s}\right)^n.$$

The supremum is taken along the imaginary axis, which implies that we need to consider *Fourier Series*, since

$$F\left(e^{-it}\right) = \sum_{n=1}^{\infty} a_n e^{-int}.$$

This was exploited by Kahane in [**24**], where he used methods from probability theory to get the asymptotic equality

$$S(N) \sim \sqrt{N}$$

as $N \to \infty$, and the upper bound is obtained. The random trigonometric polynomials Kahane showed existed are called *ultra-flat*, since their supremum is as small as possible compared to the absolute sum of their coefficients, in view of the general bound $S_\Lambda(N) \leq \sqrt{N}$. Explicit ultra-flat polynomials were later found by Bombieri and Bourgain in [**10**], who also improved Kahane's estimates and obtained

$$S(N) = \sqrt{N}\left(1 + \mathcal{O}\left(N^{-1/9+\epsilon}\right)\right).$$

A natural question is whether ultra-flat polynomials can be found for any defining set. Let us take $\Lambda = \{\log(1), \log(2), \log(3), \ldots\}$. This yields the *Ordinary Dirichlet Series*, which we write as

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

The Sidon constant for the set $\Lambda = \{\log(1), \log(2), \log(3), \ldots\}$ was recently estimated as

$$S(N) = \sqrt{N} \exp\left(\left(-\frac{1}{\sqrt{2}} + o(1)\right)\sqrt{\log N \log\log N}\right),$$

combining contributions from several mathematicians, including Bohr, Littlewood, Bohnenblust–Hille, Kahane, Hildebrand–Tenenbaum, Konyagin–Queffélec, de la Breteche, and finally Defant–Frerick–Ortega-Cerdà–Ounaïes–Seip. The estimate implies that ultra-flat Dirichlet polynomials cannot exist, since

$$\lim_{N\to\infty} \frac{S(N)}{\sqrt{N}} = 0 \neq 1.$$

The main goal of this thesis is to retrace the steps of the mathematicians listed above and obtain the asymptotic formula for $S(N)$. In addition, some precise estimates for the $o(1)$-term will be provided.

Remark. In the remainder of this thesis, we will restrict our study to the set $\Lambda = \{\log(1), \log(2), \log(3), \ldots\}$. Furthermore, when we say *Dirichlet series* and the *Sidon constant*, we have this set in mind.

Overview of the Thesis

The thesis is divided into four chapters and two appendices. The first chapter may be considered an extended introduction, but we also obtain some necessary tools. The following two chapters provide crucial theorems, needed for the fourth and final chapter where the main result is proven. The first appendix supplies some general inequalities, while the second provides some properties of the Riemann Zeta Function.

Chapter 1. The first chapter is devoted to the study of convergence properties of Dirichlet series. We obtain Cahen–Bohr formulae for ordinary Dirichlet series, and show how these allow us to view the Sidon constant as a statement on the relationship between uniform and absolute convergence of the series. Perron's Formula is also obtained, which is an important inversion formula.

Chapter 2. The second chapter contains the main number theoretic part of the thesis, and concerns itself with the study of smooth numbers. Number theory is closely connected with Dirichlet series, and Perron's Formula is crucial here.

Chapter 3. In the third chapter, we study multilinear forms and homogenous polynomials. Starting with Littlewood's 4/3-Inequality, we combine Khinchine–Type Inequalities with Blei's Inequality to prove several hypercontractive versions of the Bohnenblust–Hille Inequality.

Chapter 4. In the fourth chapter, we combine Bohr's Correspondence with Rankin's Trick and the Salem–Zygmund Inequality to obtain the main result of the thesis. We give as precise estimates as we can obtain, and mention some related open problems.

Asymptotics

Let us give a brief overview of the different types of asymptotic notation that is used in the text. We will employ both Landau's notation $f(x) = \mathcal{O}(g(x))$ and Vinogradov's notation $f(x) \ll g(x)$ interchangeably when there is some constant $C > 0$ such that

$$|f(x)| \leq C|g(x)|,$$

for all x in some domain. Unless otherwise stated, we often assume this is globally, as $x \to \infty$. If we have both $f(x) \ll g(x)$ and $g(x) \ll f(x)$ we will write $f(x) \asymp g(x)$. We will also consider limit comparisons, say

$$L = \lim_{x\to\infty} \frac{f(x)}{g(x)}.$$

If $L = 1$ we have asymptotic equality, and write $f(x) \sim g(x)$. Furthermore, if $L = 0$ we say $f(x) = o(g(x))$. Clearly, if $L = \pm\infty$ we have the converse and $g(x) = o(f(x))$.

CHAPTER 1

Some Convergence Properties of Dirichlet Series

A familiar and important concept in mathematics are power series, which are studied at university courses in Complex Analysis. Power series are given by

$$F(z) = \sum_{n=0}^{\infty} c_n (z-c)^n. \tag{1.1}$$

The series converges trivially for $z = c$. In addition, the *radius of convergence* around this point can be computed by the Cauchy–Hadamard formula

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|c_n|}. \tag{1.2}$$

The power series (1.1) converges absolutely inside the radius of convergence, uniformly on any compact subset inside the radius of convergence, and diverges outside. We are mainly interested in Dirichlet series, which are defined by

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \tag{1.3}$$

for complex numbers $s = \sigma + it$. The main goal of this chapter is to investigate where the series converges, converges absolutely and converges uniformly, and obtain analogous formulae to (1.2) for Dirichlet series [**1, 20**].

1.1. Summation by Parts

To better understand the Dirichlet series (1.3), we will study the sequence of truncated *Dirichlet polynomials*, given by

$$f_N(s) = \sum_{n=1}^{N} \frac{a_n}{n^s}.$$

On these Dirichlet polynomials, we introduce the following quantities

$$\|f_N\|_0 = \left|\sum_{n=1}^{N} a_n\right| \tag{1.4}$$

$$\|f_N\|_\infty = \sup_{t\in\mathbb{R}} \left|\sum_{n=1}^{N} \frac{a_n}{n^{it}}\right| = \sup_{t\in\mathbb{R}} |f_N(it)| \tag{1.5}$$

$$\|\widehat{f_N}\|_1 = \sum_{n=1}^{N} |a_n| \tag{1.6}$$

which will be associated with the different types of convergence. Clearly

$$\|f_N\|_0 \leq \|f_N\|_\infty \leq \|\widehat{f_N}\|_1,$$

where the first inequality is obtained by taking $t = 0$ in the supremum, and the second by the triangle inequality.

Remark. It should be noted that only (1.5) and (1.6) defines *norms* on the set of Dirichlet polynomials. In general, (1.4) fails to separate points, and therefore only a *semi-norm*. This should not cause any confusion, we adopt this notation to obtain a sense of similarity in the Cauchy–Hadamard-type formulae.

We now consider any function $a : \mathbb{N} \to \mathbb{C}$ and define the following truncated sums

$$A(x) = \sum_{n\leq x} a(n)$$

$$A(x, y) = \sum_{y<n\leq x} a(n)$$

If $x < 1$ we take $A(x) = 0$ and similarly if $y \geq x$ we take $A(x, y) = 0$. Using these definitions, we are ready to state and prove two technical results, which will be applied to prove the central results in the following sections.

Lemma 1.1 (Abel Summation). *Suppose $\phi \in C^1([y, x])$ with $0 < y < x$. Then*

$$\sum_{y<n\leq x} a(n)\phi(n) = A(x)\phi(x) - A(y)\phi(y) - \int_y^x A(t)\phi'(t)\, dt. \tag{1.7}$$

Proof. Since A is a step function we can formulate the sum as a Riemann–Stieltjes integral. We apply integration by parts to obtain

$$\begin{aligned}\sum_{y<n\leq x} a(n)\phi(n) = \int_y^x \phi(t)\, dA(t) &= A(x)\phi(x) - A(y)\phi(y) - \int_y^x A(t)\, d\phi(t) \\ &= A(x)\phi(x) - A(y)\phi(y) - \int_y^x A(t)\phi'(t)\, dt,\end{aligned}$$

where the final equality is due by the fact that ϕ is continuously differentiable. □

LEMMA 1.2 (Partial Summation). *Let $a, b : \mathbb{N} \to \mathbb{C}$. Then*

$$\sum_{y<n\leq x} a(n)b(n) = \sum_{y<n\leq x-1} A(n,y)[b(n) - b(n+1)] + A(x,y)b([x]).$$

PROOF. We compute

$$\begin{aligned}\sum_{y<n\leq x} a(n)b(n) &= \sum_{y<n\leq x} \left[A(n,y) - A(n-1,y)\right] b(n) \\ &= \sum_{y<n\leq x} A(n,y)b(n) - \sum_{y-1<n\leq x-1} A(n,y)b(n+1) \\ &= A(x,y)b([x]) + \sum_{y<n\leq x-1} A(n,y)\left[b(n) - b(n+1)\right],\end{aligned}$$

and since $A([y],y) = 0$, the first term in the second sum disappears. □

1.2. The Abscissa of Convergence

We begin by investigating where the Dirichlet series converges. Our first result is an important lemma which shows how the real part of $s = \sigma + it$ is the dominating element in the question of convergence.

LEMMA 1.3. *Assume that the Dirichlet series*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

converges for $s_0 = \sigma_0 + it_0$. Then $f(s)$ converges for all $s = \sigma + it$ with $\sigma > \sigma_0$.

PROOF. To apply Lemma 1.1 we take $a(n) = a_n/n^{s_0}$ and $\phi(t) = t^{s-s_0}$. We plug this into (1.7) to obtain

$$\sum_{y<n\leq x} \frac{a_n}{n^s} = \frac{A(x)}{x^{s-s_0}} - \frac{A(y)}{y^{s-s_0}} - (s-s_0)\int_y^x \frac{A(t)}{t^{s-s_0+1}}\, dt.$$

Since $f(s_0)$ converges, the partial sums are bounded, say $|A(t)| \leq M$. We apply the triangle inequality and obtain

$$\begin{aligned}\left|\sum_{y<n\leq x} \frac{a_n}{n^s}\right| &\leq Mx^{\sigma_0-\sigma} + My^{\sigma_0-\sigma} + M|s-s_0| \int_y^x t^{\sigma_0-\sigma-1}\, dt \\ &= M\left(x^{\sigma_0-\sigma} + y^{\sigma_0-\sigma}\right) + M\frac{|s-s_0|}{\sigma-\sigma_0}\left(y^{\sigma_0-\sigma} - x^{\sigma_0-\sigma}\right) \\ &\leq M\left(1 + \frac{|s-s_0|}{\sigma-\sigma_0}\right)\left(y^{\sigma_0-\sigma} + x^{\sigma_0-\sigma}\right) \leq 2My^{\sigma_0-\sigma}\left(1 + \frac{|s-s_0|}{\sigma-\sigma_0}\right),\end{aligned}$$

since $\sigma_0 < \sigma$ and $y < x$ implies $x^{\sigma_0-\sigma} < y^{\sigma_0-\sigma}$. We let $y \to \infty$, and hence the partial sums of $f(s)$ forms a Cauchy sequence and $f(s)$ converges. □

THEOREM 1.4. *Assume that the Dirichlet series $f(s)$ does not converge everywhere or diverge everywhere. Then there exists a real number σ_c, such that the series converges in the half-plane $\{s : \sigma > \sigma_c\}$ and diverges in the half-plane $\{s : \sigma < \sigma_c\}$.*

PROOF. Consider the quantity

$$\sigma_c = \inf_{s \in \mathbb{C}} \left\{ \sigma : \sum_{n=1}^{\infty} \frac{a_n}{n^s} \text{ converges} \right\}. \tag{1.8}$$

Since the series does not diverge everywhere, the set is non-empty and the infimum exists. Since the series does not converge everywhere, the set is bounded below. Hence the infimum is a finite real number. By Theorem 1.3 the number σ_c gives the half-planes of convergence and divergence by its definition. □

DEFINITION. The real number σ_c as defined by (1.8) is called *the abscissa of convergence* for the Dirichlet series. If the series converges everywhere, we say that $\sigma_c = -\infty$ and if it diverges everywhere we say $\sigma_c = \infty$.

We are now ready to prove the first Dirichlet series analogy to (1.2), which will give the abscissa of convergence.

THEOREM 1.5. *Assume that the Dirichlet series*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

diverges at $s = 0$. Then $\sigma_c = \sigma_0$ where

$$\sigma_0 = \limsup_{N \to \infty} \frac{\log \|f_N\|_0}{\log N} = \limsup_{N \to \infty} \frac{\log |a_1 + a_2 + \cdots + a_N|}{\log N}. \tag{1.9}$$

PROOF. We take σ_0 as above, and since the series diverges at $s = 0$,

$$\lim_{x \to \infty} A(x)$$

diverges. By Lemma 1.3, we only consider real s, and take $s = \sigma > 0$.

Step 1. Let $\epsilon > 0$ be arbitrary. There is some $N_\epsilon \in \mathbb{N}$ such that

$$\frac{\log \|f_N\|_0}{\log N} \leq \sigma_0 + \epsilon$$

for all $N \geq N_\epsilon$ by the definition of σ_0. We take care of the finite number of $N < N_\epsilon$ by adding a suitable constant C_ϵ such that

$$\frac{\log \|f_N\|_0}{\log N} \leq \sigma_0 + \epsilon + \frac{\log C_\epsilon}{\log N}$$

holds for all $N \in \mathbb{N}$. This implies the estimate

$$\|f_N\|_0 \leq C_\epsilon N^{\sigma_0+\epsilon}. \tag{1.10}$$

Now, we take $\sigma = \sigma_0 + 2\epsilon$ and apply Abel summation with $a(n) = a_n$ and $\phi(t) = 1/t^\sigma$, which yields

$$\sum_{y<n\leq x} \frac{a_n}{n^\sigma} = \frac{A(x)}{x^\sigma} - \frac{A(y)}{y^\sigma} - \sigma \int_y^x \frac{A(t)}{t^{1+\sigma}}\, dt.$$

The estimate (1.10) implies $|A(t)| \leq C_\epsilon t^{\sigma_0+\epsilon}$. We take absolute values and apply the triangle inequality to obtain

$$\begin{aligned}\left|\sum_{y<n\leq x} \frac{a_n}{n^\sigma}\right| &\leq C_\epsilon \left(\frac{x^{\sigma_0+\epsilon}}{x^\sigma} + \frac{y^{\sigma_0+\epsilon}}{y^\sigma} + \sigma \int_y^x \frac{t^{\sigma_0+\epsilon}}{t^{1+\sigma}}\, dt\right) \\ &= C_\epsilon \left(\frac{1}{x^\epsilon} + \frac{1}{y^\epsilon} + \sigma \int_y^x \frac{dt}{t^{1+\epsilon}}\right) \leq \frac{C_\epsilon}{y^\epsilon}\left(2 + \frac{\sigma}{\epsilon}\right),\end{aligned}$$

which demonstrates that the Dirichlet series converges at $s = \sigma$. Thus we see that $\sigma_c \leq \sigma = \sigma_0 + 2\epsilon$ and hence $\sigma_c \leq \sigma_0$.

Step 2. Now, fix $\epsilon > 0$ and let $\sigma = \sigma_c + \epsilon$. Clearly $f(\sigma)$ converges, and hence the partial sums $B(x)$ are bounded, say by a constant M. Again by Abel summation we obtain

$$\sum_{n\leq x} a_n = \sum_{n\leq x} \frac{a_n}{n^\sigma} n^\sigma = B(x)x^\sigma - \sigma \int_1^x A(t)t^{\sigma-1}\, dt.$$

Absolute values and the triangle inequality yields

$$\left|\sum_{n\leq x} a_n\right| \leq M\left(x^\sigma + \sigma \int_1^x t^{\sigma-1}\, dt\right) = M(2x^\sigma - 1) \leq 2Mx^\sigma.$$

This estimate is valid for $x = N$ and by (1.9) we obtain

$$\sigma_0 = \limsup_{N\to\infty} \frac{\log \|f_N\|_0}{\log N} \leq \limsup_{N\to\infty} \frac{\log 2M + \sigma \log N}{\log N} = \sigma = \sigma_c + \epsilon.$$

Hence $\sigma_0 \leq \sigma_c$ and we are done. □

1.3. The Abscissa of Absolute Convergence

We immediately obtain the domain of absolute convergence, since we can appeal to the familiar convergence tests from calculus regarding positive series.

THEOREM 1.6. *Assume that the Dirichlet series $f(s)$ does not converge absolutely everywhere or diverge absolutely everywhere. Then there exists a real number σ_a, such that the series converges absolutely in the half-plane $\{s : \sigma > \sigma_a\}$ and diverges absolutely in the half-plane $\{s : \sigma < \sigma_a\}$.*

PROOF. We observe that

$$\sum_{n=1}^{\infty}\left|\frac{a_n}{n^s}\right| = \sum_{n=1}^{\infty}\frac{|a_n|}{n^\sigma}.$$

By the comparison test we immediately see that if the series converges absolutely for some $s_0 = \sigma_0 + it_0$, it converges absolutely for all $s = \sigma + it$ where $\sigma > \sigma_0$. As in the proof of Theorem 1.4 we consider the quantity

$$\sigma_a = \inf_{\sigma\in\mathbb{R}}\left\{\sigma : \sum_{n=1}^{\infty}\frac{|a_n|}{n^\sigma} < \infty\right\}. \tag{1.11}$$

Since the series does not diverge absolutely everywhere, the set is non-empty and the infimum exists. Since the series does not converge absolutely everywhere, the set is bounded below. Hence the infimum is a finite real number. By the observation above, the number σ_a gives the half-planes of absolute convergence and divergence by its definition. □

DEFINITION. The real number σ_a as defined by (1.11) is called *the abscissa of absolute convergence* for the Dirichlet series. If the series converges absolutely everywhere, we say that $\sigma_a = -\infty$ and if it diverges absolutely everywhere we say $\sigma_a = \infty$.

We can apply the formula for the abscissa of convergence to obtain the formula for the abscissa of absolute convergence easily.

COROLLARY 1.7. *Suppose that the Dirichlet series $f(s)$ diverges absolutely at $s = 0$. Then*

$$\sigma_a = \limsup_{N\to\infty}\frac{\log\|\widehat{f_N}\|_1}{\log N} = \limsup_{N\to\infty}\frac{\log\left(|a_1| + |a_2| + \cdots + |a_N|\right)}{\log N}. \tag{1.12}$$

PROOF. This follows from Theorem 1.5 by replacing the coefficients a_n with their absolute values $|a_n|$. □

THEOREM 1.8. *Assume that the Dirichlet series $f(s)$ does not converge everywhere or diverge everywhere. Then $0 \le \sigma_a - \sigma_c \le 1$.*

PROOF. The lower bound is trivial since $\sigma_a \geq \sigma_c$. It is sufficient to show that if the Dirichlet series converges for some $s_0 = \sigma_0 + it_0$ it converges absolutely for all $s = \sigma + it$ with $\sigma > \sigma_0 + 1$. Since the series

$$f(s_0) = \sum_{n=1}^{\infty} \frac{a_n}{n^{s_0}}$$

converges, its summands are bounded. Say $|a_n n^{-s_0}| \leq M$. Then clearly

$$\sum_{n=1}^{\infty} \left| \frac{a_n}{n^s} \right| = \sum_{n=1}^{\infty} \left| \frac{a_n}{n^{s_0}} \right| \left| \frac{1}{n^{s-s_0}} \right| \leq \sum_{n=1}^{\infty} \frac{M}{n^{\sigma-\sigma_0}},$$

converges since $p = \sigma - \sigma_0 > 1$ by p-test from calculus. □

REMARK. It should be noted that the above argument implies that if $\sigma_c = -\infty$ then $\sigma_a = -\infty$. Similarly, if $\sigma_a = \infty$ then clearly $\sigma_c = \infty$. Since we have $\sigma_c \leq \sigma_a$ the two remaining cases follow. If either σ_a or σ_c is infinite, then $\sigma_a = \sigma_c$.

THEOREM 1.9. *For any $0 \leq \alpha \leq 1$ there is a Dirichlet series with $\sigma_a - \sigma_c = \alpha$.*

PROOF. The case $\alpha = 0$ is given by the Riemann zeta function,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

By (1.9), the Dirichlet eta function

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s},$$

provides the case $\alpha = 1$. For $0 < \alpha < 1$ we consider a Dirichlet series $f(s)$ with $a_n \in \{-1, 1\}$. Hence we have $A = 1$, by (1.12). We want to show that we can choose a_n such that it converges for all $\sigma > \sigma_\alpha = 1 - \alpha$ and diverges elsewhere, to obtain $\sigma_c = \sigma_\alpha$ and $\sigma_a - \sigma_c = \alpha$. We call upon Lemma 1.1 and apply (1.7) to obtain

$$\sum_{y<n\leq x} \frac{a_n}{n^\sigma} = \frac{A(x)}{x^\sigma} - \frac{A(y)}{y^\sigma} + \sigma \int_y^x \frac{A(t)}{t^{\sigma+1}} \,\mathrm{d}t.$$

We now want $A(t) \sim t^{\sigma_\alpha}$. This is easily done, by recursively choosing

$$a_n = \begin{cases} 1 & \text{if } n^{\sigma_0} \geq A(n-1) \\ -1 & \text{if } n^{\sigma_0} < A(n-1) \end{cases},$$

which implies $|A(t) - t^\sigma| < 3$, by the fact that t^{σ_α} is strictly increasing contraction on $[1, \infty)$. Thus it is clear that $\sigma_c = \sigma_\alpha$ and hence we have $\sigma_a - \sigma_c = \alpha$. □

1.4. The Abscissa of Uniform Convergence

We are now interested in obtaining the abscissa of uniform convergence. It was introduced by H. Bohr in [**6, 9**]. Our first result is a general result on uniform convergence, and can be seen as an improvement of Theorem 1.4.

LEMMA 1.10. *For any $s \in \mathbb{C}$ and $n \in \mathbb{N}$ we have*

$$\left|\frac{1}{n^s} - \frac{1}{(n+1)^s}\right| \leq \frac{|s|}{\sigma}\left(\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma}\right)$$

PROOF. We convert to an integral and estimate using the triangle inequality

$$\left|\frac{1}{n^s} - \frac{1}{(n+1)^s}\right| = \left|s\int_n^{n+1}\frac{dt}{t^{s+1}}\right| \leq |s|\int_n^{n+1}\frac{dt}{t^{\sigma+1}} = \frac{|s|}{\sigma}\left(\frac{1}{n^\sigma} - \frac{1}{(n+1)^\sigma}\right). \quad \square$$

THEOREM 1.11. *Suppose that the Dirichlet series $f(s)$ converges for some $s = s_0$. Then it converges uniformly in the angular region*

$$\operatorname{Arg}(s - s_0) \leq \theta < \frac{\pi}{2}. \tag{1.13}$$

PROOF. Let $\epsilon > 0$ be arbitrary. Since $f(s_0)$ converges, we can find an $N_\epsilon \in \mathbb{N}$ such that

$$\left|\sum_{y<n\leq x}\frac{a_n}{n^{s_0}}\right| < \epsilon\cos\theta \leq \epsilon,$$

for any x and any $y \geq N_\epsilon$. Summation by parts with $a(n) = a_n/n^{s_0}$ and $b(n) = 1/n^{s-s_0}$ yields

$$\sum_{y<n\leq x}\frac{a_n}{n^s} = \sum_{y<n\leq x-1}A(n,y)\left(\frac{1}{n^{s-s_0}} - \frac{1}{(n+1)^{s-s_0}}\right) + \frac{A(x,y)}{[x]^{s-s_0}}.$$

It is clear that $|A(n,y)| \leq \epsilon\cos\theta$. We take absolute values and apply the triangle inequality to obtain

$$\left|\sum_{y<n\leq x}\frac{a_n}{n^s}\right| \leq \epsilon\left(\sum_{y<n\leq x-1}\cos\theta\left|\frac{1}{n^{s-s_0}} - \frac{1}{(n+1)^{s-s_0}}\right|\right) + \frac{\epsilon\cos\theta}{[x]^{\sigma-\sigma_0}}$$

by Lemma 1.10 and the fact that $|s - s_0|/(\sigma - \sigma_0) \leq 1/\cos\theta$ in (1.13) we obtain

$$\leq \epsilon\left(\sum_{y<n\leq x-1}\cos\theta\frac{|s-s_0|}{\sigma-\sigma_0}\left(\frac{1}{n^{\sigma-\sigma_0}} - \frac{1}{(n+1)^{\sigma-\sigma_0}}\right) + \frac{1}{[x]^{\sigma-\sigma_0}}\right) \leq \epsilon$$

by summing the telescoping series and the fact that $y \geq N_\epsilon$. Since s was arbitrary in the angular region, we have proven uniform convergence. $\square$

Theorem 1.11 implies that there in general is no largest domain of uniform convergence. Hence we are required to take the abscissa of uniform convergence as a definition.

DEFINITION. Given a Dirichlet series which neither converge everywhere nor diverge everywhere, the *the abscissa of uniform convergence* is the unique real number σ_b such that the Dirichlet series converges uniformly in each closed half-plane $\{s : \sigma \geq \sigma_0 > \sigma_b\}$. If the Dirichlet series converges everywhere, we say that $\sigma_b = -\infty$ and if it diverges everywhere we say that $\sigma_b = \infty$.

Observe that Theorem 1.8 implies that this definition makes sense; and obviously $\sigma_c \leq \sigma_b \leq \sigma_a$ since absolute convergence implies uniform convergence, which in turn implies convergence. Our first goal is to obtain the formula for the abscissa of uniform convergence, analogous to Theorem 1.5 and Corollary 1.7 [**7**].

THEOREM 1.12. *Suppose that the Dirichlet series $f(s)$ diverges at $s = 0$, but does not diverge everywhere. Then $\sigma_b = \sigma_0$, where*

$$\sigma_0 = \limsup_{N\to\infty} \frac{\log \|f_N\|_\infty}{\log N} = \limsup_{N\to\infty} \frac{\log\left(\sup_{t\in\mathbb{R}}\left|\sum_{n=1}^N a_n/n^{it}\right|\right)}{\log N}. \tag{1.14}$$

PROOF. We want to prove that $\sigma_0 = \sigma_b$, which we will do in two steps. Since $f(s)$ diverges at $s = 0$ we have $\sigma_c \geq 0$ and from (1.14) it is clear that $\sigma_0 \geq 0$.

Step 1. Fix some $\epsilon > 0$. There is some $N_\epsilon \in \mathbb{N}$ such that for $N \geq N_\epsilon$ we have

$$\frac{\log \|f_N\|_\infty}{\log N} \leq \sigma_0 + \epsilon.$$

Choose a suitable $C_\epsilon > 0$ to compensate for the finite $N < N_\epsilon$, such that

$$\frac{\log \|f_N\|_\infty}{\log N} \leq \sigma_0 + \epsilon + \frac{\log C_\epsilon}{\log N}$$

holds for all N. We can use this to estimate

$$|f_N(it)| \leq \|f_N\|_\infty \leq C_\epsilon N^{\sigma_0+\epsilon}. \tag{1.15}$$

Now, take $s = \sigma + it$ where $\sigma \geq \sigma_0 + 2\epsilon$. We split the summand into a_n/n^{it} and $1/n^\sigma$ and apply partial summation to obtain

$$f_N(s) = \sum_{n=1}^N \frac{a_n}{n^s} = \sum_{n=1}^{N-1} f_n(it)\left(n^{-\sigma} - (n+1)^{-\sigma}\right) + f_N(it)N^{-\sigma}.$$

The final term goes uniformly to 0 by (1.15), since it is bounded by $C_\epsilon N^{-\epsilon}$. Hence,

$$f(s) = \sum_{n=1}^\infty f_n(it)\left(n^{-\sigma} - (n+1)^{-\sigma}\right).$$

We estimate the telescoping part by

$$n^{-\sigma} - (n+1)^{-\sigma} = \sigma \int_n^{n+1} \frac{dt}{t^{1+\sigma}} \le \frac{\sigma}{n^{1+\sigma}}.$$

The function

$$\phi_n(t) = \frac{t}{n^{1+t}}$$

has a global maximum at $t = 1/\log n$ and decreases strictly to 0 as t increases. For those $n \ge M_\epsilon$ such that

$$\frac{1}{\log n} \le \sigma_0 + 2\epsilon < \sigma,$$

we obtain

$$n^{-\sigma} - (n+1)^{-\sigma} \le \frac{\sigma_0 + 2\epsilon}{n^{1+\sigma_0+2\epsilon}}.$$

Choose $K \ge M_\epsilon$ and apply (1.15) to obtain

$$|f(s) - f_K(s)| \le \sum_{n \ge K} \frac{C_\epsilon(\sigma_0 + 2\epsilon)}{n^{1+\epsilon}},$$

which can be made smaller than any $\delta > 0$ by choosing large enough K. This implies uniform convergence for $\sigma \ge \sigma_0 + 2\epsilon$. Hence $\sigma_b \le \sigma_0 + 2\epsilon$ and $\sigma_b \le \sigma_0$.

Step 2. Fix some $\epsilon > 0$. We know that $f(s)$ converges uniformly on the vertical line $s = \sigma_b + \epsilon + it$, and hence its partial sums are uniformly bounded, that is

$$\left| \sum_{n=1}^{N} \frac{a_n}{n^{\sigma_b+\epsilon+it}} \right| \le M,$$

where $M > 0$ depends neither on N nor t. By taking the supremum over all $t \in \mathbb{R}$, we obtain

$$\|f_N\|_\infty = \sup_{t\in\mathbb{R}} \left| \sum_{n=1}^{N} \frac{a_n}{n^{it}} \right| \le \sup_{t\in\mathbb{R}} \left| N^{\sigma_b+\epsilon} \sum_{n=1}^{N} \frac{a_n}{n^{\sigma_b+\epsilon+it}} \right| \le MN^{\sigma_b+\epsilon}.$$

This immediately yields

$$\sigma_0 = \limsup_{N\to\infty} \frac{\log \|f_N\|_\infty}{\log N} \le \limsup_{N\to\infty} \frac{\log M + (\sigma_b + \epsilon)\log N}{\log N} = \sigma_b + \epsilon,$$

which clearly implies $\sigma_0 \le \sigma_b$. In total, we have $\sigma_0 = \sigma_b$. □

We easily obtain the bound $0 \le \sigma_a - \sigma_b \le 1$, from Theorem 1.8, and the fact that $\sigma_c \le \sigma_b$. In the following result, we improve this.

THEOREM 1.13. *Suppose that the Dirichlet series $f(s)$ does not converge everywhere or diverge everywhere. Then $\sigma_a - \sigma_b \le 1/2$.*

PROOF. A simple computation yields

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left(\frac{m}{n}\right)^{it}dt=\begin{cases}1 & \text{if } m=n\\ 0 & \text{if } m\neq n\end{cases}, \tag{1.16}$$

for $m, n = 1, 2, \ldots$. We begin by applying (1.16) to compute

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f_N(it)|^2\,dt &= \sum_{n,m=1}^{N}a_n\overline{a_m}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left(\frac{m}{n}\right)^{it}dt\\ &= \sum_{m,n=1}^{N}a_n\overline{a_m}\delta_{mn}=\sum_{n=1}^{N}|a_n|^2.\end{aligned} \tag{1.17}$$

Using the Cauchy–Schwarz inequality and (1.17), we obtain

$$\begin{aligned}\|\widehat{f_N}\|_1 &= \sum_{n=1}^{N}|a_n| \le \left(\sum_{n=1}^{N}1\right)^{\frac12}\left(\sum_{n=1}^{N}|a_n|^2\right)^{\frac12}=\sqrt{N}\left(\sum_{n=1}^{N}|a_n|^2\right)^{\frac12}\\ &= \sqrt{N}\left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f_N(it)|^2\,dt\right)^{\frac12}\le\sqrt{N}\|f_N\|_\infty.\end{aligned} \tag{1.18}$$

By Theorem 1.5 and Theorem 1.12, this implies

$$\sigma_a-\sigma_b\le\limsup_{N\to\infty}\left(\frac{\log\|\widehat{f_N}\|_1}{\log N}-\frac{\log\|f_N\|_\infty}{\log N}\right)\le\limsup_{N\to\infty}\frac{\log\sqrt{N}}{\log N}=\frac12,$$

which completes the proof. □

Theorem 1.13 suggests that one way to understand the quantity $\sigma_a - \sigma_b$ is to study the ratio $\|\widehat{f_N}\|_1/\|f_N\|_\infty$ for different N and different choices of coefficients $\{a_n\}$. In particular, we would like to maximize this to obtain a lower bound for $\sigma_a - \sigma_b$. This leads naturally to the definition of the *Sidon Constant*,

$$S(N)=\sup_{\{a_n\}\neq 0}\frac{\|\widehat{f_N}\|_1}{\|f_N\|_\infty}.$$

We are now in a position to state the main result of this thesis, namely

$$S(N)=\sqrt{N}\exp\left(\left(-\frac{1}{\sqrt{2}}+o(1)\right)\sqrt{\log N\log\log N}\right).$$

The $o(1)$ term is taken as $N\to\infty$. Since $\sqrt{\log N\log\log N}<\log N$ this implies that the bound $\sigma_a-\sigma_b\le 1/2$ indeed is optimal, by similar considerations as in the proof above.

1.5. The Mellin Transformation

The goal of this section is to provide an analytic correspondence between the Dirichlet series and the summatory function of the coefficients, which we recall are given by:

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$
$$A(x) = \sum_{n \leq x} a_n.$$

To do this, we apply integral transformations. We begin by proving a version of Kronecker's lemma for Dirichlet series.

LEMMA 1.14 (Kronecker's Lemma). *Consider the Dirichlet series $f(s)$ and let $s = \sigma + it$ with $\sigma > \max(\sigma_c, 0)$. Then*

$$\lim_{x\to\infty} \frac{A(x)}{x^s} = 0. \tag{1.19}$$

PROOF. Fix $s = \sigma + it$ with $\sigma > \max(\sigma_c, 0)$. Let

$$B(x) = \sum_{n \leq x} \frac{a_n}{n^s}.$$

Since $\sigma > \sigma_c$ the partial sums of $f(s)$ are bounded, say $|B(x)| \leq M/2$ for some $M > 0$. Abel summation yields

$$A(x) = \sum_{n \leq x} \frac{a_n}{n^s} \cdot n^s = B(x)x^s - \int_0^x B(t)st^{s-1}\,dt = \int_0^x (B(x) - B(t))\,st^{s-1}\,dt.$$

Now, let $\epsilon > 0$ be arbitrary. Since $\sigma > \sigma_c$ we know there exists some $B \in \mathbb{C}$ such that

$$|B(x) - B| \leq \epsilon/2,$$

for all $x > x_0(\epsilon)$. We split the integral at $x_0(\epsilon)$ and obtain

$$\begin{aligned} |A(x)| &\leq \int_0^{x_0(\epsilon)} |B(x) - B(t)|\sigma t^{\sigma-1}\,dt + \int_{x_0(\epsilon)}^{x} |B(x) - B(t)|\sigma t^{\sigma-1}\,dt \\ &\leq \int_0^{x_0(\epsilon)} M\sigma t^{\sigma-1}\,dt + \int_{x_0(\epsilon)}^{x} \epsilon\sigma t^{\sigma-1}\,dt = x_0(\epsilon)^\sigma(M-\epsilon) + \epsilon x^\sigma, \end{aligned}$$

since $\sigma > 0$. Clearly this implies

$$\lim_{x\to\infty} \left|\frac{A(x)}{x^s}\right| \leq \epsilon,$$

which gives (1.19) since $\epsilon > 0$ was arbitrary. □

We are now ready to obtain the first part of the correspondence, using Abel summation and Kronecker's lemma.

THEOREM 1.15. *Consider the Dirichlet series $f(s)$ and the summatory coefficient function $A(x)$. Let $s = \sigma + it$ with $\sigma > \max(\sigma_c, 0)$. Then*

$$f(s) = s \int_1^\infty \frac{A(x)}{x^{s+1}}\, dx. \tag{1.20}$$

PROOF. By Abel summation we obtain

$$\sum_{n \leq y} \frac{a_n}{n^s} = \frac{A(y)}{y^s} + s \int_1^y \frac{A(x)}{x^{s+1}}\, dx,$$

since $A(x) = 0$ for $x < 1$. We want to take $y \to \infty$, and apply Kronecker's lemma, which yields

$$f(s) = \lim_{y \to \infty} \left(\frac{A(y)}{y^s} + s \int_1^y \frac{A(x)}{x^{s+1}}\, dx \right) = s \int_1^\infty \frac{A(x)}{x^{s+1}}\, dx,$$

as required. □

EXAMPLE 1.16. Consider the Riemann zeta function $\zeta(s)$. Clearly,

$$A(x) = [x],$$

since $a_n = 1$. We apply (1.20) to obtain

$$\zeta(s) = s \int_1^\infty \frac{[x]}{x^{s+1}}\, dx = \frac{s}{s-1} - s \int_1^\infty \frac{\{x\}}{x^{s+1}}\, dx,$$

since $x = [x] + \{x\}$. This yields an analytical continuation of $\zeta(s)$, since the integral is absolutely convergent for $\sigma > 0$. Furthermore, the continuation has a simple pole at $s = 1$ with residue 1.

Inspired by (1.20), we define the following integral transformation, defined for more general functions than summatory functions of Dirichlet series.

DEFINITION. Suppose $f : [1, \infty) \to \mathbb{C}$ is locally Lebesgue integrable, and satisfies the growth condition $|f(x)| \leq Ax^B$. The *Mellin transformation* of f is defined as

$$\widehat{f}(s) = \mathcal{M}\{f\}(s) = s \int_1^\infty \frac{f(x)}{x^{s+1}}\, dx.$$

If we can invert the Mellin transformation, we will be able to to obtain the converse of (1.20). To do this, we try to connect the Mellin transformation to some familiar integral transformations.

DEFINITION. For any $f \in L^1(\mathbb{R})$ we define Fourier transformation

$$\widehat{f}(\xi) = \mathcal{F}\{f\}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-ix\xi}\,dx. \tag{1.21}$$

The Fourier transformation exists for any $\xi \in \mathbb{R}$ and is continuous. The proof of the following theorem is omitted, but can be found in any standard text on Fourier Analysis [**18**].

THEOREM (Inverse Fourier Transformation). *Suppose that $f, f' \in L^1(\mathbb{R})$, and that f is piecewise continuously differentiable. Then*

$$f^*(x) = \frac{f(x^+) + f(x^-)}{2} = \mathcal{F}^{-1}\{\widehat{f}\}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\xi)e^{ix\xi}\,d\xi, \tag{1.22}$$

at the least in the Cauchy principal value-sense.

DEFINITION. Suppose that $f : [0,\infty) \to \mathbb{R}$ is locally Lebesgue integrable and satisfies the growth condition $|f(x)| \leq Ae^{Bx}$. We define the Laplace transformation by

$$\widehat{f}(s) = \mathcal{L}\{f\}(s) = \int_0^{\infty} f(x)e^{-xs}\,dx.$$

We may use the Fourier Inversion Theorem, to obtain the Laplace Inversion Theorem. We can later apply this theorem to invert the Mellin transformation, and obtain the desired formula.

LEMMA 1.17. *Suppose that both $|f|$ and $|f'|$ satisfy the growth condition $\leq Ce^{Dx}$ and that f is piecewise continuously differentiable. Then*

$$f^*(x) = \mathcal{L}^{-1}\{\widehat{f}\}(x) = \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} \widehat{f}(s)e^{xs}\,ds, \tag{1.23}$$

for any $\kappa > D$ at the least in the Cauchy principal value-sense.

PROOF. By (1.21) and (1.22) we obtain

$$g^*(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(y)e^{-iy\xi}e^{ix\xi}\,dyd\xi,$$

for any piecewise continuously differentiable function g, with $g, g' \in L^1(\mathbb{R})$. The outer integral may be taken in the Cauchy principal value sense if necessary. We choose

$$g(x) = H(x)e^{-\kappa x}f(x),$$

where $H(x)$ is the Heaviside's function, with one small addition: We take the mean value at $x = 0$ to obtain the correct value in the principal value integrals,

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1/2 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}.$$

The function $g(x)$ satisfies the demands by the growth condition. Clearly, for $x > 0$ and by the substitution $s = \kappa + i\xi$ we obtain

$$\begin{aligned} f^*(x) &= \frac{e^{\kappa x}}{2\pi} \int_{-\infty}^{\infty} e^{ix\xi} \int_0^{\infty} f(y) e^{-y(\kappa+i\xi)}\, dy d\xi \\ &= \frac{e^{\kappa x}}{2\pi} \int_{\kappa-i\infty}^{\kappa+i\infty} e^{x(s-k)} \int_0^{\infty} f(y) e^{-ys}\, dy \, \frac{ds}{i} = \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} e^{xs} \widehat{f}(s)\, ds, \end{aligned}$$

where the integral is taken in the Cauchy principal value-sense if necessary. The case $x = 0$ follows similarly. □

THEOREM 1.18. *Suppose that both $|f|$ and $|f'|$ satisfy the growth condition $\leq Ax^B$ and that f is piecewise continuously differentiable. Then*

$$f^*(x) = \mathcal{M}^{-1}\{\widehat{f}\}(s) = \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \widehat{f}(s) \frac{x^s}{s}\, ds, \tag{1.24}$$

at the least in the Cauchy principal value-sense, for any $\kappa > B$.

PROOF. We define $g(x) = f(e^x)$. By the growth conditions, we can compute

$$\mathcal{L}\{g\}(s) = \int_0^{\infty} g(x) e^{-xs}\, ds = \int_0^{\infty} f(e^x) e^{-xs}\, ds = \int_1^{\infty} \frac{f(x)}{x^s} \frac{ds}{x} = \frac{1}{s} \mathcal{M}\{f\}(s).$$

Similarly, the demands for (1.23) are met, and we obtain

$$f^*(x) = g^*(\log x) = \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \widehat{g}(s) e^{s \log x}\, ds = \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} \widehat{f}(s) \frac{x^s}{s}\, ds,$$

where the integral is taken in the Cauchy principal value-sense if necessary. □

Observe that (1.24) implies that

$$A(x) = \frac{1}{2\pi i} \int_{\kappa-i\infty}^{\kappa+i\infty} f(s) \frac{x^s}{s}\, ds$$

is formally the converse to (1.20). However, we see that the mean value is taken at the discontinuities $x = n$. In the next section, we will obtain several versions of Theorem 1.18 tailored to Dirichlet series. We also estimate error terms when restricting the integration to a finite part of the imaginary axis, say $|t| \leq T$.

1.6. Perron's Formulae

To satisfy the required mean value at the discontinuities, we introduce a weighted version of the summatory coefficient function,

$$A^*(x) = \frac{1}{2} \left(\sum_{n \leq x} a_n + \sum_{n < x} a_n \right). \tag{1.25}$$

The multiplicative version of $H(x)$ is the auxiliary function $h : \mathbb{R}^+ \to \mathbb{R}$ by

$$h(x) = \begin{cases} 1, & \text{if } x > 1 \\ 1/2, & \text{if } x = 1 \\ 0, & \text{if } 0 < x < 1 \end{cases}. \tag{1.26}$$

The following lemma regarding (1.26) will yield Perron's formula.

LEMMA 1.19. *For any positive* κ, T *and* T' *we have*

$$\left| h(x) - \frac{1}{2\pi i}\int_{\kappa - iT'}^{\kappa + iT} x^s \,\frac{ds}{s} \right| \le \frac{x^\kappa}{2\pi|\log x|}\left(\frac{1}{T} + \frac{1}{T'}\right), \tag{1.27}$$

$$\left| h(1) - \frac{1}{2\pi i}\int_{\kappa - iT}^{\kappa + iT} \frac{ds}{s} \right| \le \frac{\kappa}{T+\kappa}. \tag{1.28}$$

where $0 < x \neq 1$ *in* (1.27) *and* h *is defined as in* (1.26).

PROOF. Assume first that $x > 1$ and take $k > \kappa$. Let $\mathcal{R}_k$ be the rectangle given by its corners $\kappa - iT'$, $\kappa + iT$, $\kappa - k + iT$ and $\kappa - k - iT'$ oriented counter-clockwise. By the residue theorem,

$$\frac{1}{2\pi i}\int_{\mathcal{R}_k} x^s \,\frac{ds}{s} = 1 = h(x).$$

One of the edges is the integral in (1.27). We will estimate the other three, starting with

$$\left| \int_{\kappa + iT}^{\kappa - k + iT} x^s \,\frac{ds}{s} \right| \le \frac{1}{T}\int_{\kappa - k}^{\kappa} x^s\, ds = \frac{x^\kappa - x^{\kappa - k}}{T\log x} \le \frac{x^\kappa}{T\log x} = \frac{x^\kappa}{T|\log x|}.$$

The same holds for the opposite edge,

$$\left| \int_{\kappa - k + - iT'}^{\kappa - iT'} x^s \,\frac{ds}{s} \right| \le \frac{1}{T'}\int_{\kappa - k}^{\kappa} x^s\, ds = \frac{x^\kappa - x^{\kappa - k}}{T'\log x} \le \frac{x^\kappa}{T'\log x} = \frac{x^\kappa}{T'|\log x|}.$$

Finally we apply the triangle inequality to the third edge and obtain

$$\left| \frac{1}{2\pi i}\int_{\kappa - k - iT}^{\kappa - iT'} x^s \,\frac{ds}{s} \right| \le (T' + T)\frac{x^{\kappa - k}}{\kappa - k},$$

which disappears as $k \to \infty$. Combining this, we obtain (1.27) for $x > 1$. The same argument applies for $x < 1$, by replacing κ by $-\kappa$, since $h(x) = 0$. This proves (1.27). Now for the case $x = 1$, we observe that

$$\frac{1}{2\pi i}\int_{\kappa - iT}^{\kappa + iT} \frac{ds}{s} = \frac{1}{2\pi}\left(\arg(\kappa + iT) - \arg(\kappa - iT)\right) = \frac{1}{\pi}\arctan(T/\kappa).$$

We note that

$$0 \leq \frac{\pi}{2} - \arctan(y) = \int_y^\infty \frac{dt}{1+t^2} \leq \frac{2}{(1+t)^2}\, dt = \frac{2}{1+y},$$

and hence (1.28) follows by taking $y = T/\kappa$, since $2/\pi < 1$. □

THEOREM 1.20. *Consider the Dirichlet series*

$$f(s) = \sum_{n=1}^\infty \frac{a_n}{n^s},$$

with finite abscissa of absolute convergence σ_a, and suppose that $\kappa > \max(0, \sigma_a)$. Then

$$A^*(x) = \frac{1}{2\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} f(s) x^s \,\frac{ds}{s}, \tag{1.29}$$

which converges conditionally for $x \in \mathbb{R}\backslash\mathbb{N}$ and in the Cauchy principal value sense for for $x \in \mathbb{N}$.

PROOF. Since $\kappa > \sigma_a$, we have absolute convergence. This means that we can interchange integration and summation to obtain

$$\frac{1}{2\pi i} \int_{\kappa - iT'}^{\kappa + iT} f(s) x^s \,\frac{ds}{s} = \frac{1}{2\pi i} \sum_{n=1}^\infty a_n \int_{\kappa - iT'}^{\kappa + iT} \left(\frac{x}{n}\right)^s \frac{ds}{s}.$$

By (1.25) and (1.27) we then have

$$\left| A^*(x) - \frac{1}{2\pi i} \int_{\kappa - iT'}^{\kappa + iT} f(s) x^s \,\frac{ds}{s} \right| \leq \frac{x^\kappa}{2\pi} \left(\frac{1}{T} + \frac{1}{T'}\right) \sum_{n=1}^\infty \frac{|a_n|}{n^\kappa |\log(x/n)|},$$

valid for $x \in \mathbb{R}\backslash\mathbb{N}$. The infinite series converges since $\kappa > \sigma_a$ and since x is not an integer, which implies $0 < C \leq |\log(x/n)|$. Thus by letting $T', T \to \infty$ independently, we obtain (1.29). For $x \in \mathbb{N}$ we apply (1.28) and similarly obtain convergence in the Cauchy principal value sense, since we have $T' = T$. □

We now prove effective companions of Theorem 1.20, where we keep T finite and obtain error bounds, which will be very useful later.

LEMMA 1.21. *Consider the Dirichlet series $f(s)$ with finite abscissa of absolute convergence σ_a, and suppose that $\kappa > \max(0, \sigma_a)$, and take $T \geq 1$ and $x \geq 1$. Then*

$$A(x) = \frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} f(s) x^s \,\frac{ds}{s} + \mathcal{O}\left(x^\kappa \sum_{n=1}^\infty \frac{|a_n|}{n^\kappa \left(1 + T|\log(x/n)|\right)} \right). \tag{1.30}$$

PROOF. By the proof of Theorem 1.20 we note that it is sufficient to prove that for any fixed $\kappa > 0$ we have uniformly for $y > 0$ and $T > 0$ that

$$\left| h(y) - \frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} y^s \, \frac{ds}{s} \right| = \mathcal{O}\left(\frac{y^\kappa}{1 + T|\log y|} \right), \tag{1.31}$$

where we will take $y = x/n$. First, suppose that $T|\log y| > 1$. Then (1.31) follows immediately from (1.27). Now, assume that $T|\log y| \leq 1$. We write

$$\int_{\kappa - iT}^{\kappa + iT} y^s \, \frac{ds}{s} = y^\kappa \int_{\kappa - iT}^{\kappa + iT} \frac{ds}{s} + y^\kappa \int_{\kappa - iT}^{\kappa + iT} \left(y^{i\tau} - 1 \right) \frac{ds}{s}.$$

The first integral is bounded by π, since we obtain an arctangent as in the proof of Lemma 1.19. For any $x \in \mathbb{R}$ we have

$$|e^{ix} - 1| \leq |x|,$$

hence $|y^{i\tau} - 1| \leq |\tau \log y|$, and thus the second integral can be bounded by

$$\left| \int_{\kappa - iT}^{\kappa + iT} \left(y^{i\tau} - 1 \right) \frac{ds}{s} \right| \leq \int_{\kappa - iT}^{\kappa + iT} \left| \frac{\tau \log y}{s} \right| ds \leq 2T|\log y| \leq 2.$$

The error term is $\mathcal{O}(y^\kappa)$ which is of the right order since $T|\log y|$ is bounded. □

Let us provide two methods to further estimate the sum in the error term of Lemma 1.21.

LEMMA 1.22. *Suppose that $a_n \geq 0$. For any $\epsilon > 0$ and any $\kappa > \max(\sigma_a, 0)$ we have*

$$\sum_{|\log x/n| \leq \epsilon} a_n \left(\frac{x}{n} \right)^\kappa \leq C \epsilon x^\kappa \int_{-T}^{T} |f(\kappa + i\tau)| \, d\tau,$$

where $T = 1/\epsilon$ and $C = \operatorname{sinc}^{-2}(1/2)$, where f is associated to a_n.

PROOF. Suppose $w, \mathcal{F}\{w\} \in L^1(\mathbb{R})$. We begin by proving the auxiliary formula

$$\sum_{n=1}^{\infty} a_n \left(\frac{x}{n} \right)^\kappa w\left(\log \frac{x}{n} \right) = \frac{1}{2\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} f(s) x^s \widehat{w}(\tau) ds. \tag{1.32}$$

We have absolute convergence and may exchange integration and summation. We then use a substitution to obtain

$$\frac{1}{2\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} f(s) x^s \widehat{w}(\tau) ds = \sum_{n=1}^{\infty} a_n \left(\frac{x}{n} \right)^k \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{x}{n} \right)^{i\tau} \widehat{w}(\tau) d\tau,$$

which proves (1.32), since the integral is the Fourier inverse of $\widehat{w}(\tau)$ evaluated at $\log x/n$. Consider the following function:

$$\widehat{w}(\tau) = \begin{cases} 1 - |\tau| & \text{if } |\tau| \leq 1 \\ 0 & \text{if } |\tau| > 1 \end{cases}$$

We may compute the inverse Fourier transformation easily, since $\widehat{w} \in L^1(\mathbb{R})$,

$$w(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{w}(\tau) e^{it\tau} \, \mathrm{d}\tau = \frac{1}{2\pi} \operatorname{sinc}^2(t/2).$$

In particular, we also have $w \in L^1(\mathbb{R})$. Now, let $\epsilon > 0$ be arbitrary and define $w_\epsilon(t) = w(t/\epsilon)$. This yields $\widehat{w_\epsilon}(\tau) = \epsilon \widehat{w}(\epsilon\tau)$, and (1.32) is applied to obtain

$$\sum_{n=1}^{\infty} a_n \left(\frac{x}{n}\right)^{\kappa} w_\epsilon \left(\log \frac{x}{n}\right) = \frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} f(s) x^s \epsilon \widehat{w}(\epsilon\tau) ds \leq \frac{\epsilon x^\kappa}{2\pi} \int_{-T}^{T} |F(\kappa + i\tau)| \, d\tau,$$

since $a_n \geq 0$. The inequality holds if we consider only a finite number of the summands,

$$\sum_{|\log x/n| \leq \epsilon} a_n \left(\frac{x}{n}\right)^{\kappa} w_\epsilon \left(\log \frac{x}{n}\right) \leq \frac{\epsilon x^\kappa}{2\pi} \int_{-T}^{T} |F(\kappa + i\tau)| \, d\tau.$$

The proof is complete, since

$$\inf_{|\log x/n| \leq \epsilon} w_\epsilon \left(\log \frac{x}{n}\right) = \frac{1}{2\pi} \operatorname{sinc}^2(1/2). \qquad \square$$

LEMMA 1.23. *Suppose that $T > 1$ and $x > 0$, and $I = \{n : x/2 \leq n \leq x\}$. Then*

$$\mathscr{S} = \sum_{n \in I} \frac{1}{1 + T|\log x/n|} \leq \frac{2x}{T}(1 + \gamma + 2\log T/x),$$

where γ denotes the Euler–Mascheroni constant.

PROOF. By the mean value theorem, for some ξ between x and n,

$$|\log(x/n)| = |\log x - \log n| = \frac{|x - n|}{\xi} \geq \frac{|x - n|}{2x}.$$

We furthermore observe that $0 \leq |x - n| \leq x$, since $x/2 \leq n \leq 2x$. We split the sum at x/T and obtain

$$\begin{aligned} \mathscr{S} &\leq \sum_{0 \leq |x-n| \leq x/T} 1 + \sum_{x/T < |x-n| \leq x} \frac{2x}{T|x-n|} \\ &\leq \frac{2x}{T} + \frac{2x}{T}\left(\gamma + \int_{x/2}^{x - x/T} \frac{dt}{x - t} + \int_{x + x/T}^{2x} \frac{dt}{t - x}\right) = \frac{2x}{T}(1 + \gamma + 2\log T/x), \end{aligned}$$

which completes the proof. $\square$

CHAPTER 2

Smooth Numbers

This chapter concerns itself with estimating the smooth numbers, which the following definition introduces.

DEFINITION. Given any positive real number y, we say that the integer n is *y-smooth* if all the prime factors of n are smaller than or equal to y. For $x \geq y \geq 2$ we consider the number of y-smooth numbers less than x,

$$\Psi(x, y) = \operatorname{card}\{n \leq x \,:\, n \text{ is } y\text{-smooth}\}.$$

The goal of this chapter is to estimate $\Psi(x, y)$ precisely [**38, 22**]. To obtain the required estimates we shall apply the Saddle Point Method. The following example will illustrate the method.

EXAMPLE 2.1. The formula

$$n! = \int_0^\infty e^{-t} t^n \, dt$$

is well-known, or may be obtained by induction. We will use the Saddle Point Method to obtain Stirling's approximation for $n!$. We substitute $t = sn$ to obtain

$$n! = n^{n+1} \int_0^\infty e^{-sn} s^n \, ds = n^{n+1} \int_0^\infty e^{-nf(s)} \, ds,$$

where $f(s) = s - \log(s)$. As n gets big, one can expect that the main contribution will be where $f(s)$ is small, and thus we are interested in $0 = f'(s) = 1 - 1/s$, and hence $s = 1$. Using a Taylor polynomial we can approximate

$$f(s) \approx f(s_0) + f'(s_0)(s - s_0) + \frac{f''(s_0)}{2}(s - s_0)^2 = 1 + \frac{(s-1)^2}{2}.$$

for s near 1. Since the main contribution is near $s = 1$, we use the Taylor polynomial and extend the integration to the entire real line to obtain

$$n! \approx n^{n+1} e^{-n} \int_0^\infty e^{-n\frac{(s-1)^2}{2}} \, ds \approx n^{n+1} e^{-n} \int_{-\infty}^\infty e^{-n\frac{(s-1)^2}{2}} \, ds = n^{n+1} e^{-n} \sqrt{\frac{2\pi}{n}},$$

which is Stirling's approximation $n! \approx \sqrt{2\pi n} \cdot n^n e^{-n}$. It should be noted that this argument can be made more rigorous.

Following Example 2.1 we introduce the Gamma function, which is the natural generalization of the factorial. It is defined by

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\, dt, \tag{2.1}$$

for $\Re(s) > 0$. We require some properties of this function in what follows. We easily obtain

$$\Gamma(n+1) = n!,$$

and the functional equation

$$\Gamma(s+1) = s\Gamma(s)$$

using integration by parts. The following standard result on the Gamma function will be needed [**17**].

THEOREM 2.2. *The analytical continuation of the Gamma function satisfies*

$$\frac{1}{\Gamma(s)} = se^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n}, \tag{2.2}$$

where γ denotes the Euler–Mascheroni constant.

PROOF. We begin by letting $s > 0$ and considering the sequence of functions

$$\Gamma_n(s) = \int_0^n t^{s-1} \left(1 - \frac{t}{n}\right)^n dt.$$

The dominated convergence theorem implies that $\Gamma_n(s) \to \Gamma(s)$ as $n \to \infty$. By the substitution $r = t/n$ and integration by parts, we may recursively compute

$$\begin{aligned}
\Gamma_n(s) = n^s \int_0^1 r^{s-1}(1-r)^n\, dr &= \frac{1}{s}\left(\frac{n}{n-1}\right)^{s+1} \Gamma_{n-1}(s+1) \\
= \Gamma_1(s+n-1) \prod_{k=0}^{n-2} \frac{1}{s+k} \left(\frac{n-k}{n-(k+1)}\right)^{s+k+1} &= \frac{n^s n!}{s(s+1)\cdots(s+n)},
\end{aligned}$$

where the integral $\Gamma_1(s+n-1)$ is easily computed. After taking the reciprocal, this product can be rewritten as

$$\frac{1}{\Gamma_n(s)} = \frac{s}{n^s} \prod_{k=1}^{n} \left(1 + \frac{s}{k}\right) = se^{\gamma_n s} \prod_{k=1}^{n} \left(1 + \frac{s}{k}\right) e^{-s/k},$$

where $\gamma_n \to \gamma$. This is done to ensure convergence for any $s \in \mathbb{C}$, which gives (2.2) as we let $n \to \infty$. This can be shown by taking logarithms and estimating the sum using Taylor's theorem. Hence (2.2) extends to $\mathbb{C}$ by analytical continuation. $\square$

2.1. Dickman's Function

In this section, we study *Dickman's function.* The function $\rho : \mathbb{R} \to \mathbb{R}$ is defined by the initial condition $\rho(u) = 1$ for $0 \leq u \leq 1$ and recursively

$$\rho(u) = \rho(k) + \int_k^u \rho(v-1)\,\frac{dv}{v}, \qquad k \in \mathbb{N}. \tag{2.3}$$

We take $\rho(u) = 0$ for $u < 0$. This function is continuous except for a discontinuity at $u = 0$. We obtain the following properties of Dickman's function.

LEMMA 2.3. *The function ρ has the following properties:*

$$u\rho'(u) + \rho(u-1) = 0 \qquad u > 1 \tag{2.4}$$

$$u\rho(u) = \int_{u-1}^u \rho(v)\,dv \qquad u \in \mathbb{R} \tag{2.5}$$

$$\rho(u) > 0 \qquad u \geq 0 \tag{2.6}$$

$$\rho'(u) < 0 \qquad u > 1 \tag{2.7}$$

$$\rho(u) \leq \frac{1}{\Gamma(u+1)} \qquad u \geq 0 \tag{2.8}$$

PROOF. We proceed as follows:

(1) By differentiating (2.3) we obtain (2.4).
(2) We see that (2.5) holds trivially for any $u \leq 1$. Furthermore, it holds for $u > 1$ by continuity since both sides have the same derivative by (2.4).
(3) Define
$$u_0 = \inf\{u > 0 \,:\, \rho(u) = 0\}.$$
Clearly $u_0 > 1$. If we suppose that $u_0 < \infty$, then we have
$$0 = u_0\rho(u_0) = \int_{u_0-1}^{u_0} \rho(v)\,dv > 0,$$
by (2.5) and continuity. This is impossible and $u_0 = \infty$.
(4) We obtain (2.7) from (2.4) and (2.6) since
$$\rho'(u) = -\frac{\rho(u-1)}{u},$$
which is valid for $u > 1$.
(5) Clearly (2.8) is true for $0 \leq u \leq 1$. Inductively and by (2.5) and (2.7) we obtain
$$\rho(u) = \frac{1}{u}\int_{u-1}^u \rho(v)\,dv \leq \frac{1}{u}\,\frac{u-(u-1)}{\Gamma(u)} = \frac{1}{\Gamma(u+1)},$$
by applying the functional equation for $\Gamma(s)$.

This completes the proof. □

Our next task is to compute the Laplace transformation of Dickman's function.

LEMMA 2.4. *We have*

$$\widehat{\rho}(s) = e^{\gamma + E(-s)}, \tag{2.9}$$

where γ is the Euler–Mascheroni constant and

$$E(s) = \int_0^s \frac{e^t - 1}{t}\, dt. \tag{2.10}$$

PROOF. By (2.8) the integral

$$\widehat{\rho}(s) = \int_0^\infty \rho(u) e^{-us}\, du$$

is absolutely convergent for any $s \in \mathbb{C}$. By (2.5) and Fubini's theorem,

$$\begin{aligned} -\frac{d}{ds}\widehat{\rho}(s) &= \int_0^\infty u\rho(u)e^{-us}\, ds = \int_0^\infty \left(\int_{u-1}^u \rho(v)\, dv \right) e^{-us}\, du \\ &= \int_0^\infty \rho(v) \left(\int_{v-1}^v e^{-us}\, du \right) dv = \frac{1 - e^{-s}}{s}\widehat{\rho}(s). \end{aligned}$$

This separable ordinary differential equation has the solution

$$\widehat{\rho}(s) = C \exp\left(- \int_0^s \frac{1 - e^{-t}}{t}\, dt \right) = Ce^{E(-s)}.$$

By logarithmic differentiation of (2.2) we obtain $-\gamma = \Gamma'(1)$. Now, by (2.1)

$$\Gamma'(1) = \int_0^1 \frac{e^{-t} - 1}{t}\, dt + \int_1^\infty \frac{e^{-t}}{t}\, dt = \int_0^s \frac{e^{-t} - 1}{t}\, dt + \int_1^s \frac{1}{t}\, dt + \int_s^\infty \frac{e^{-t}}{t}\, dt,$$

where $s > 1$. Combining this we obtain the identity

$$0 = \gamma + E(-s) + \log(s) + E_1(s), \tag{2.11}$$

where we have

$$E_1(s) = \int_s^\infty \frac{e^{-t}}{t}\, dt = \int_0^\infty \frac{e^{-(t+s)}}{t+s}\, dt. \tag{2.12}$$

Using integration by parts we obtain

$$s\widehat{\rho}(s) = \int_0^\infty \rho(u) s e^{-us}\, du = \rho(0^+) + \int_1^\infty \rho'(u) e^{-us}\, du,$$

and as $s \to \infty$ the final integral disappears. Thus, by (2.11) we compute

$$1 = \lim_{u \to 0^+} \rho(u) = \lim_{s \to \infty} s\widehat{\rho}(s) = \lim_{s \to \infty} Ce^{-\gamma - E_1(s)} = Ce^{-\gamma},$$

and hence $C = e^\gamma$ and we are done. $\square$

By Lemma 1.17 the inverse Laplace transformation is applicable for $\kappa \in \mathbb{R}$,

$$\rho(u) = \frac{1}{2\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} \widehat{\rho}(s) e^{us} \, ds,$$

and $u > 1$, since ρ is continuously differentiable for $u > 1$. We would expect the main contribution at $|\tau| < \delta$ by choosing κ to be a zero of the derivative of the integrand. Hence we require

$$0 = \frac{d}{ds} \left(\log(\widehat{\rho}(s)) + us \right) = \frac{e^{-s} - 1}{s} + u,$$

by (2.10). We want $\kappa = -\xi(u)$, where ξ is positive and satisfies

$$e^{\xi} = 1 + u\xi. \tag{2.13}$$

The function $\xi(u)$ is non-elementary, but we can obtain the following properties.

Lemma 2.5. *For any $u > 1$ the equation* (2.13) *has an unique positive solution. For $u > 3$ we have*

$$\xi(u) = \log u + \log\log u + \mathcal{O}\left(\frac{\log\log u}{\log u} \right), \tag{2.14}$$

as $u \to \infty$ and furthermore for $u > 1$ we have

$$\xi'(u) = \frac{\xi}{1 + u\xi - u} \asymp \frac{1}{u}. \tag{2.15}$$

Proof. Let

$$f(x) = \frac{e^x - 1}{x} = \int_0^1 e^{tx} \, dt. \tag{2.16}$$

It is clear that $f(\xi) = u$, if ξ solves (2.13). Furthermore $f(0^+) = 1$, and clearly $f'(x) > 0$. Thus f is strictly increasing on $[0, \infty)$, and $f(0) = 1$. This implies that for any $u > 1$ there exists some unique positive ξ such that $f(\xi) = u$, and hence (2.13) has a unique positive solution. The simple estimate $\xi(u) = \mathcal{O}(\log u)$ is valid for $u > 3$, and can be used to prove (2.14). We apply (2.13) iteratively

$$\begin{aligned} \xi = \log(1 + u\xi) &= \log u + \log\left(\xi + \frac{1}{u} \right) \\ &= \log u + \log\left(\log u + \log\left(\xi + \frac{1}{u} \right) + \frac{1}{u} \right) \\ &= \log u + \log\log u + \mathcal{O}\left(\frac{\log\log u}{\log u} \right), \end{aligned}$$

by the simple estimate and the fact that $\log(1 + x) = \mathcal{O}(x)$. We need to take $u > 3$ to ensure that the iterated logarithms are defined. The first equality in

(2.15) follows by implicit differentiation with respect to u in (2.13). Applying the inverse function theorem to (2.16) yields $\xi'(u) = 1/f'(\xi)$. Now, we compute

$$f(\xi) \geq \int_0^1 te^{t\xi}\,dt = f'(\xi) \geq \frac{1}{2}\int_{1/2}^1 e^{t\xi}\,dt \geq \frac{1}{4}\int_0^1 e^{t\xi}\,dt = \frac{f(\xi)}{4}, \tag{2.17}$$

which implies $\xi'(u) \asymp 1/u$ as required. □

The following lemma provides estimates on $\widehat{\rho}(s)$, given the choice of $\kappa = -\xi(u)$.

Lemma 2.6. *Let $u > 1$, $\xi = \xi(u)$ and $s = -\xi + i\tau$. Then we have the estimates:*

$$\widehat{\rho}(s) \ll \exp\left(E(\xi) - \frac{\tau^2 u}{2\pi^2}\right) \qquad |\tau| \leq \pi, \tag{2.18}$$

$$\widehat{\rho}(s) \ll \exp\left(E(\xi) - \frac{u}{\xi^2 + \pi^2}\right) \qquad |\tau| > \pi, \tag{2.19}$$

$$\widehat{\rho}(s) = \frac{1}{s}\left(1 + \mathcal{O}\left(\frac{1+u\xi}{|\tau|}\right)\right) \qquad |\tau| > 1 + u\xi. \tag{2.20}$$

Proof. For (2.18) and (2.19) we consider the quantity

$$H(\xi;\tau) = E(\xi) - \Re\left(E(-s)\right) = \int_0^1 e^{t\xi}\frac{1-\cos t\tau}{t}\,dt.$$

If $|\tau| \leq \pi$ we have $1 - \cos(\tau t) \geq 2\tau^2 t^2/\pi^2$. Following (2.17) we get

$$H(\xi;\tau) \geq \frac{2\tau^2}{\pi^2}\int_0^1 te^{t\xi}\,dt \geq \frac{\tau^2}{\pi^2}\int_{1/2}^1 e^{t\xi}\,dt \geq \frac{\tau^2}{2\pi^2}\int_0^1 e^{t\xi}dt = \frac{\tau^2 u}{2\pi^2}.$$

Combining this with (2.9) yields (2.18). Suppose now that $|\tau| > \pi$. Then a computation of the integral and estimation yields

$$H(\xi;\tau) = \int_0^1 e^{t\xi}\frac{1-\cos t\tau}{t}\,dt \geq \int_0^1 e^{t\xi}\left(1-\cos t\tau\right)\,dt \geq \frac{u}{\xi^2+\pi^2} - \frac{2}{\pi}.$$

Combining this with (2.9) yields (2.19). To prove (2.20), it suffices to show

$$e^{-E_1(s)} = 1 + \mathcal{O}\left(\frac{1+u\xi}{|\tau|}\right),$$

by first applying (2.11). This follows from the estimate $e^z = 1 + \mathcal{O}(z)$, which is applicable if z is bounded. For $s = -\xi + i\tau$ we obtain

$$|E_1(s)| \leq e^{\xi}\int_0^\infty \frac{e^{-t}}{|s+t|}\,dt \leq \frac{e^{\xi}}{|\tau|}\int_0^\infty e^{-t}\,dt = \frac{1+u\xi}{|\tau|} \leq 1,$$

by the assumption $|\tau| > 1 + u\xi$, which completes the proof. □

We are now ready to obtain the asymptotic formula for Dickman's function. We apply the inverse Laplace transformation, using Lemma 2.6 and various estimates to obtain the required result.

THEOREM 2.7. *For $u > 1$ we have*

$$\rho(u) = \sqrt{\frac{\xi'(u)}{2\pi}} \exp\left(\gamma - u\xi + E(\xi)\right)\left(1 + \mathcal{O}\left(\frac{1}{u}\right)\right). \tag{2.21}$$

PROOF. We take $\kappa = -\xi$ and want to use the inverse Laplace transformation

$$\rho(u) = \frac{1}{2\pi i}\int_{\kappa - i\infty}^{\kappa + i\infty} \widehat{\rho}(s)e^{us}\, ds.$$

Choose $\delta = \pi\sqrt{2\log(u+1)/u}$ and divide the line into four parts:

$$I_1 = \{-\xi + i\tau \,:\, |\tau| \le \delta\}, \qquad I_2 = \{-\xi + i\tau \,:\, \delta < |\tau| \le \pi\},$$
$$I_3 = \{-\xi + i\tau \,:\, \pi < |\tau| \le 1 + u\xi\}, \qquad I_4 = \{-\xi + i\tau \,:\, 1 + u\xi < |\tau|\}.$$

The main contribution is expected to come from I_1, and the other three parts should be absorbed in the error term of (2.21). By Lemma 2.5, we have that $\xi'(u) \asymp 1/u$. This implies that the error term in (2.21) is of order

$$\frac{e^{-u\xi + E(\xi)}}{u^{3/2}}.$$

We begin by computing the main contribution at I_1, and follow up by estimating the other parts of the integral.

Part 1. By Lemma 2.4 we can rewrite the integrand

$$\widehat{\rho}(s)e^{us} = \exp\left(\gamma + E(\xi - i\tau) - u\xi + iu\tau\right) = e^{\gamma - u\xi + E(\xi)} \cdot e^{E(\xi - i\tau) - E(\xi) + iu\tau}.$$

The first exponential is taken outside the integral and appears in (2.21). For the second exponential, we apply Taylor's theorem to $E(\xi - i\tau)$ to obtain

$$E(\xi - i\tau) - E(\xi) + iu\tau = -\frac{\tau^2}{2}E''(\xi) + \frac{i\tau^3}{6}E^{(3)}(\xi) + \mathcal{O}\left(\tau^4 \max_{|t| \le |\tau|}\left|E^{(4)}(\xi + i\tau)\right|\right),$$

since a computation shows $E'(\xi) = u$, by (2.10) and (2.16). Now, since

$$\left|E^{(k)}(\xi + i\tau)\right| \le \int_0^1 t^k e^{\xi t}\, dt \le \int_0^1 e^{\xi t}\, dt = u,$$

and $|\tau| \le \delta$ the final two terms are $\mathcal{O}(1)$. This means we can apply the estimates

$$e^z = 1 + z + \mathcal{O}(z^2) = 1 + \mathcal{O}(z),$$

respectively. Thus the two final terms are

$$= \left(1 + \frac{i\tau^3}{6}E^{(3)}(\xi) + \mathcal{O}(u^2\tau^6)\right)\left(1 + \mathcal{O}(u\tau^4)\right) = 1 + \frac{i\tau^3}{6}E^{(3)}(\xi) + \mathcal{O}\left(u\tau^4 + u^2\tau^6\right).$$

Since I_1 is symmetrical in τ, the τ^3 term disappears when integrated. Combining what we have obtained so far, we have

$$\int_{I_1} \widehat{\rho}(s)e^{us}\,ds = e^{\gamma-u\xi+E(\xi)}\int_{-\delta}^{\delta} e^{-\frac{\tau^2}{2}E''(\xi)}\left(1+\mathcal{O}\left(u\tau^4+u^2\tau^6\right)\right)\,d\tau.$$

We begin by showing that the $\mathcal{O}$-term is absorbed in the error term of (2.21). The substitution $t = \tau^2/2E''(\xi)$ allows us to write

$$\int_{-\delta}^{\delta} e^{-\frac{\tau^2}{2}E''(\xi)}\mathcal{O}\left(u\tau^4+u^2\tau^6\right)\,d\tau = \int_{-\epsilon}^{\epsilon}\frac{e^{-|t|}}{\sqrt{|t|}}\mathcal{O}\left(\frac{ut^2}{(E''(\xi))^2}+\frac{u^2|t|^3}{(E''(\xi))^3}\right)\,dt,$$

where $\epsilon = \pi^2\log(u+1)E''(\xi)/u$. A computation yields

$$E''(\xi) = \frac{d}{d\xi}\frac{e^\xi-1}{\xi} = u\left(1-\frac{1}{\xi}\right) = \frac{1}{\xi'(u)} \asymp u,$$

by (2.17). Thus

$$\int_{-\delta}^{\delta} e^{-\frac{\tau^2}{2}E''(\xi)}\mathcal{O}\left(u\tau^4+u^2\tau^6\right)\,d\tau = \frac{1}{u^{3/2}}\int_{-\epsilon}^{\epsilon}\frac{e^{-|t|}}{\sqrt{|t|}}\mathcal{O}\left(t^2+|t|^3\right)\,dt = \mathcal{O}\left(\frac{1}{u^{3/2}}\right),$$

since the integral is clearly bounded by a constant independent of ϵ. As for the main term, we write

$$\int_{-\delta}^{\delta} e^{-\frac{\tau^2}{2}E''(\xi)}\,d\tau = \int_{-\infty}^{\infty} e^{-\frac{\tau^2}{2}E''(\xi)}\,d\tau - 2\int_{\delta}^{\infty} e^{-\frac{\tau^2}{2}E''(\xi)}\,d\tau.$$

By the same substitution as above, and the estimate $E''(\xi)\asymp u$ we obtain the introduced error

$$2\int_{\delta}^{\infty} e^{-\frac{\tau^2}{2}E''(\xi)}\,d\tau = \frac{\sqrt{2}}{\sqrt{E''(\xi)}}\int_{\epsilon}^{\infty}\frac{e^{-t}}{\sqrt{t}}\,dt \ll \frac{1}{\sqrt{E''(\xi)}}\int_{\log u}^{\infty} e^{-t}\,dt,$$

which clearly is $\mathcal{O}\left(u^{-3/2}\right)$, as required. We finally compute the modified main term, which now is a Gaussian integral

$$\frac{\sqrt{2}}{\sqrt{E''(\xi)}}\int_{-\infty}^{\infty} e^{-\frac{\tau^2}{2}E''(\xi)}\,d\tau = \sqrt{\frac{2\pi}{E''(\xi)}} = \sqrt{2\pi\xi'(u)}.$$

Part 2. For I_2 we may estimate using (2.18) to obtain

$$\begin{aligned}\left|\int_{I_2}\widehat{\rho}(s)e^{us}\,ds\right| &\ll e^{-u\xi+E(\xi)}\int_{\delta}^{\pi} e^{-\frac{\tau^2 u}{2\pi^2}}\,dt = e^{-u\xi+E(\xi)}\frac{\pi}{\sqrt{u}}\int_{\log(u+1)}^{u/2}\frac{e^{-t}}{\sqrt{t}}\,dt\\ &\ll \frac{e^{-u\xi+E(\xi)}}{\sqrt{u}}\int_{\log u}^{\infty} e^{-t}\,dt = \frac{e^{-u\xi+E(\xi)}}{u^{3/2}}.\end{aligned}$$

Part 3. Similarly we apply (2.19) to I_3, which yields

$$\left|\int_{I_3} \widehat{\rho}(s)e^{us}\,ds\right| \ll e^{-u\xi+E(\xi)} \int_{\pi}^{1+u\xi} \exp\left(-\frac{u}{\xi^2+\pi^2}\right) d\tau$$
$$\ll e^{-u\xi+E(\xi)} \exp\left(-\frac{u}{\xi^2+\pi^2}+\xi\right)$$

by (2.13). By Lemma 2.5 we have the estimate $\xi = \mathcal{O}(\log u)$. This shows that the final term of this estimate is of order $u^{-3/2}$, as required.

Part 4. Finally, for I_4 we apply (2.20) to obtain

$$\left|\int_{I_3} \widehat{\rho}(s)e^{us}\,ds\right| \ll e^{-u\xi}\left(\left|\int_{1+u\xi}^{\infty} \frac{e^{iu\xi}}{-\xi+i\tau}\,d\tau\right| + (1+u\xi)\int_{1+u\xi}^{\infty} \frac{1}{\tau^2}\,d\tau\right)$$
$$\ll e^{-u\xi}(1+u\xi) = e^{-u\xi+\xi} \ll \frac{e^{-u\xi+E(\xi)}}{u^{3/2}}$$

by the fact that $E(\xi) \geq \xi + \frac{3}{2}\log u$ for large enough u. $\square$

We provide the following corollary for later use.

Corollary 2.8. *For $u > 1$ and $0 \leq v \leq u$*

$$\rho(u-v) \ll \rho(u)e^{v\xi(u)}. \tag{2.22}$$

Proof. For $u > 1$ we may write (2.21) in the form

$$\rho(u) = \sqrt{\frac{\xi'(u)}{2\pi}} \exp\left(\gamma - \int_1^u \xi(t)\,dt\right)\left(1+\mathcal{O}\left(\frac{1}{u}\right)\right).$$

If $0 \leq v < u-1$ we may apply this to obtain

$$\rho(u-v) = \rho(u)\sqrt{\frac{\xi'(u-v)}{\xi'(u)}} \exp\left(\int_{u-v}^{u} \xi(t)\,dt\right)\left(1+\mathcal{O}\left(\frac{1}{u-v}\right)\right).$$

Using the fact that $\xi'(t) \asymp 1/t$ we obtain

$$\frac{v^2}{2u} \leq \int_{u-v}^{u} \frac{t-u+v}{t}\,dt \asymp \int_{u-v}^{u} \xi'(t)(t-u+v)\,dt = v\xi(u) - \int_{u-v}^{u} \xi(t)\,dt.$$

By applying the same estimate in the square root we may write

$$\rho(u-v) \ll \rho(u)\exp\left(v\xi(u) - c\frac{v^2}{u} + \frac{1}{2}\log\left(\frac{u}{u-v}\right)\right) \ll \rho(u)e^{v\xi(u)},$$

for some positive $c > 0$. The case $u-1 \leq v \leq u$ follows immediately, since the left side of (2.22) is 1 in this range, while the right side is increasing in v. $\square$

2.2. Dirichlet Convolution

In this section we give some brief remarks on Dirichlet convolutions and the von Mangoldt function, which is given by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some prime } p \text{ and integer } k \geq 1 \\ 0 & \text{otherwise} \end{cases}. \tag{2.23}$$

Definition. An arithmetic function $a : \mathbb{N} \to \mathbb{C}$ is *completely multiplicative* if

$$a(mn) = a(m)a(n).$$

Definition. The *Dirichlet convolution* of two arithmetic functions a and b is

$$(a * b)(n) = \sum_{d|n} a(n)b\left(\frac{n}{d}\right).$$

Dirichlet convolution is useful when pointwise multiplying Dirichlet series,

$$\left(\sum_{m=1}^{\infty} \frac{a(m)}{m^s}\right)\left(\sum_{n=1}^{\infty} \frac{b(n)}{n^s}\right) = \sum_{k=1}^{\infty} \frac{1}{k^s} \sum_{mn=k} a(m)b(n) = \sum_{k=1}^{\infty} \frac{(a*b)(k)}{k^s}, \tag{2.24}$$

where both sides converge.

Lemma 2.9. *If the Dirichlet series f has its coefficients given by the completely multiplicative arithmetic function a, say $a_n = a(n)$, then*

$$\frac{f'(s)}{f(s)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)a(n)}{n^s}, \tag{2.25}$$

where both sides converge.

Proof. We introduce the constant function $\mathbf{1}(n) = 1$, which is trivially completely multiplicative. If $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, we may compute

$$(\Lambda * \mathbf{1})(n) = \sum_{d|n} \Lambda(d) = \sum_{i=1}^{k} \alpha_i \log p_i = \log n, \tag{2.26}$$

which also holds if $n = 1$. Thus $(\Lambda * \mathbf{1})(n) = \log n$. Now, by differentiating the Dirichlet series term wise, we obtain

$$f'(s) = -\sum_{n=1}^{\infty} \frac{a(n)\log n}{n^s}.$$

By multiplying both sides of (2.25) by $f(s)$, and applying (2.24) we need to prove $(\Lambda a * a)(n) = a(n)\log n$. This follows from the fact that a is completely multiplicative and a computation similar to (2.26). ☐

2.3. Smooth Zeta Functions

The main goal of this chapter is to precisely estimate the function $\Psi(x,y)$. We shall see that the desired approximation of $\Psi(x,y)$ is de Bruijn's function

$$\Lambda(x,y) = \begin{cases} x\displaystyle\int_{-\infty}^{\infty} \rho(u-v)\, d\left(\frac{[y^v]}{y^v}\right) & x \notin \mathbb{N} \\ \Lambda\left(x^+,y\right) & x \in \mathbb{N} \end{cases}, \tag{2.27}$$

with the convention $u = \log x/\log y$ and the assumptions $2 \le y \le x$. To obtain the required estimates, we shall apply Perron's formula and the inverse Laplace transformation. The argument may be split into three main steps. Each of these

$$\begin{array}{ccc} \Psi(x,y) & \overset{\sim}{\dashrightarrow} & \Lambda(x,y) \\ \mathcal{M}\big\downarrow & & \big\uparrow \mathcal{L}^{-1} \\ \zeta(s,y) & \xrightarrow{\ \sim\ } & F(s,y) \end{array}$$

steps will be a lemma, and in the final proof we will combine them. We begin with the first step, the Mellin transformation. A natural way to consider the function $\Psi(x,y)$ is as the summatory coefficient function of some Dirichlet series. We let $\chi(n,y)$ denote the characteristic function of the y-smooth numbers, and define the *y-smooth zeta function* by

$$\zeta(s,y) = \prod_{p \le y} \left(1 - \frac{1}{p^s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{\chi(n,y)}{n^s}. \tag{2.28}$$

The Euler product representation is finite and hence $\sigma_a = 0$. Perron's formula in the form of Theorem 1.20 would imply that

$$\Psi(x,y) = \sum_{n \le x} \chi(n,y) = \frac{1}{2\pi i} \int_{\kappa - i\infty}^{\kappa + i\infty} \zeta(s,y) \frac{x^s}{s}\, ds,$$

for $\kappa > 0$ and $x \notin \mathbb{N}$. We are looking for an estimate, so we seek to apply the effective version given in Lemma 1.21. The following lemma provides this estimate.

LEMMA 2.10. *Let $x \ge y \ge 2$. For any $0 < \kappa \le 1$ and $T \ge 1$ we have*

$$\Psi(x,y) = \frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} \zeta(s,y) \frac{x^s}{s}\, ds + \mathcal{O}(R),$$

where

$$R = \frac{x^\kappa}{\sqrt{T}} \zeta(\kappa,y) + \min\left(\frac{x}{\sqrt{T}}, \frac{x^\kappa}{\sqrt{T}} \int_{-\sqrt{T}}^{\sqrt{T}} |\zeta(\kappa + it, y)|\, dt\right).$$

PROOF. Perron's formula (1.30) provides the required formula with the error term

$$R = \sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^{\kappa} \frac{\chi(n,y)}{1 + T|\log(x/n)|}.$$

To obtain the required estimate, we split the sum at $|\log(x/n)| = 1/\sqrt{T}$, with $R = R_1 + R_2$. When $|\log(x/n)| \geq 1/\sqrt{T}$ we obtain

$$R_1 \ll \sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^{\kappa} \frac{\chi(n,y)}{\sqrt{T}} = \frac{x^{\kappa}}{\sqrt{T}} \zeta(\kappa, y).$$

Now we turn to $\leq$, which is estimated as

$$R_2 \ll \sum_{|\log x/n| \leq 1/\sqrt{T}} \left(\frac{x}{n}\right)^{\kappa} \chi(n,y).$$

We estimate the quantity R_2 in two different ways.

(1) Since κ is bounded, the summands of the sum in R_2 are bounded. Thus we may estimate

$$R_2 \ll \operatorname{card}\left\{n \,:\, |\log(x/n)| \leq 1/\sqrt{T}\right\}.$$

Since $1/\sqrt{T}$ is $\mathcal{O}(1)$ we may take exponentials and estimate

$$1 - \frac{1}{\sqrt{T}} \leq \exp\left(-\frac{1}{\sqrt{T}}\right) \leq \frac{n}{x} \leq \exp\left(\frac{1}{\sqrt{T}}\right) \leq 1 + \mathcal{O}\left(\frac{1}{\sqrt{T}}\right), \tag{2.29}$$

and hence $R_2 \ll x/\sqrt{T}$.

(2) By Lemma 1.23 with $\epsilon = 1/\sqrt{T}$ we immediately obtain

$$R_2 \ll \frac{x^{\kappa}}{\sqrt{T}} \int_{-\sqrt{T}}^{\sqrt{T}} |\zeta(\kappa + it, y)|\, dt.$$

At any given point, we need only consider one of these estimates, and hence we may take

$$R_2 = \min\left(\frac{x}{\sqrt{T}}, \frac{x^{\kappa}}{\sqrt{T}} \int_{-\sqrt{T}}^{\sqrt{T}} |\zeta(\kappa + it, y)|\, dt\right),$$

as required. This completes the proof. □

Lemma 2.10 has few demands on x, y, T and κ. In each step that follows, we will be more demanding. Our next goal is to approximate $\zeta(s, y)$.

2.4. Approximate Functional Equations

The product representation given in (2.28) implies

$$\lim_{y\to\infty} \zeta(s,y) = \zeta(s).$$

The approximation of $\zeta(s,y)$ can be expected to be related to $\zeta(s)$. In this section, we study approximate functional equations for the Riemann zeta function.

THEOREM 2.11. *For $\sigma > 0$ and $t \neq 0$ we have*

$$\zeta(s) = \sum_{n\le x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + \mathcal{O}\left(\frac{s}{x^{1+s}}\right). \tag{2.30}$$

PROOF. We begin by using Abel summation in a similar manner to the proof of Theorem 1.15 and Example 1.16 to obtain

$$\zeta(s) - \sum_{n\le x} \frac{1}{n^s} = -x^{1-s} + s\int_x^\infty \frac{[y]}{y^{1+s}}\,dy = -\frac{x^{1-s}}{1-s} - s\int_x^\infty \frac{\{y\}}{y^{s+1}}\,dy,$$

valid if $\sigma > 1$. What remains is to estimate the final integral. We easily compute

$$\phi(x) = \int_0^x \{y\}\,dy = \int_0^{[x]} \{y\}\,dy + \int_{[x]}^x \{y\}\,dy = \frac{[x]}{2} + \mathcal{O}(1) = \frac{x}{2} + \mathcal{O}(1).$$

Thus for any fixed s with $\Re(s) > 0$ we compute

$$\begin{aligned}\int_x^\infty \frac{\{y\}}{y^{s+1}}\,dy &= -\frac{\phi(x)}{x^{1+s}} + (s+1)\int_x^\infty \frac{\phi(y)}{y^{2+s}}\,dy \\ &= -\frac{1}{2x^s} + \mathcal{O}\left(\frac{1}{x^{1+s}}\right) + \frac{1}{2x^s} + \mathcal{O}\left(\frac{1}{x^{1+s}}\right) = \mathcal{O}\left(\frac{1}{x^{1+s}}\right).\end{aligned}$$

Hence we have obtained (2.30). By analytic continuation it continues to be valid for $\sigma > 0$, if $t \neq 0$. □

Our main applications of Theorem 2.11 will be the following estimates.

COROLLARY 2.12. *For $0 < \sigma \le 2 \le |t|$ we have*

$$\zeta(s) = \sum_{n\le |t|} \frac{1}{n^s} + \mathcal{O}\left(|t|^{-\sigma}\right),$$

and for $\sigma > 1/2$ and $|t| \ge 1$ we have $\zeta(s) = \mathcal{O}\left(\sqrt{|t|}\right)$.

PROOF. The first claim follows from setting $x = |t|$ in Theorem 2.11. The second follows by comparing the sum in the first with the corresponding integral. □

THEOREM 2.13. *For $\epsilon > 0$, let*

$$L_\epsilon(y) = \exp\left((\log y)^{3/5-\epsilon}\right). \tag{2.31}$$

We have the approximate functional equation

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n\le y} \frac{\Lambda(n)}{n^s} - \frac{y^{1-s}}{1-s} + \mathcal{O}\left(\frac{1}{L_\epsilon^2(y)}\right), \tag{2.32}$$

under the conditions $y \ge y_0(\epsilon)$, $|t| \le L_\epsilon(y)$ and

$$1 - (\log y)^{-2/5-\epsilon} \le \sigma \le 1. \tag{2.33}$$

PROOF. Lemma 2.9 suggests that we should consider

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s},$$

and apply Perron's formula to the Dirichlet series with translated coefficients $a_n = \Lambda(n)/n^s$. We apply Lemma 1.21 and obtain

$$\sum_{n\le y} \frac{\Lambda(n)}{n^s} = \frac{-1}{2\pi i}\int_{\kappa-iT}^{\kappa+iT} \frac{\zeta'(s+w)}{\zeta(s+w)}\frac{y^w}{w}\,dw + \mathcal{O}\left(y^\kappa \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\kappa+\sigma}(1+T|\log y/n|)}\right),$$

under the conditions $T \ge 2$ and $y \ge 2$, and $\kappa = 1-\sigma+1/\log y$, where $s = \sigma + it$. To estimate the sum in the error term, we split it according to $|\log(y/n)| = \log 2$, say R_1 and R_2. For the $\le$, we estimate using Lemma 1.23

$$R_1 \le \frac{4^\kappa}{y^\sigma}\cdot \log(2y)\cdot \frac{2y}{T}\cdot \log y \ll \frac{y^{1-\sigma}(\log y)^2}{T}.$$

The case $\ge$ is estimated by comparison with an integral

$$R_2 \le y^\kappa \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{\kappa+\sigma}(1+T\log 2)} \le \frac{y^\kappa}{T\log 2}\int_1^\infty \frac{\log t}{t^{\kappa+\sigma}}\,dt = \frac{ey^{1-\sigma}}{T\log 2}(\log y)^2.$$

Thus, the error term is of order $y^{1-\sigma}(\log y)^2/T$. To obtain the required terms in (2.32), we want to extend the path of integration to obtain the residues of the poles at $w = 1-s$ and $w = 0$. We move the integration as far left as

$$-\eta = 1 - \sigma - \frac{\log T}{\log y}.$$

First, to obtain $w = 1-s$ we require $|t| \le L_\epsilon(y)$ and $T > L_\epsilon(y)$. Furthermore, $w = 0$ requires $\kappa > 0$ which implies $\sigma \le 1$ and $-\eta < 0$, which is satisfied since

$$1 - \frac{\log T}{\log y} \le 1 - \frac{\log L_\epsilon(y)}{\log y} = 1 - (\log y)^{-2/5-\epsilon} \le \sigma.$$

This gives the bounds of (2.33), as well as the required bounds on $|t|$. To avoid any zeroes of the Riemann zeta function, we want to stay within Vinogradov's Zero Free Region (consult APPENDIX B), and hence require

$$-\eta + \sigma = 1 - \frac{\log T}{\log y} \geq 1 - C\left(\log T\right)^{-2/3}\left(\log\log T\right)^{-1/3}.$$

Take $T = L_\epsilon^4(y)$ and $y \geq y_0(\epsilon)$ to satisfy this. In view of the residue theorem,

$$\sum_{n \leq y} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)} + \frac{y^{1-s}}{1-s} + \mathcal{O}\left(\frac{y^{1-\sigma}\left(\log y\right)^2}{T} + R\right),$$

where R denotes the integral along $\mathcal{W}$, the polygonal path connecting $\kappa - iT$, $-\eta - iT$, $-\eta + iT$ and $\kappa + iT$. See APPENDIX B for the estimate

$$\frac{\zeta'(s+w)}{\zeta(s+w)} \ll \log|\Im(s+w)| \ll \log T \ll \log y,$$

valid in the zero free region. The horizontal parts are estimated by

$$\frac{y^\kappa}{T}(\log y)(\kappa - \eta) = \mathcal{O}\left(\frac{y^{1-\sigma}\left(\log y\right)^2}{T}\right),$$

by the definition of κ. The vertical part may also be estimated by

$$y^{-\eta}\log y \int_{-T}^{T} \frac{dt}{|w|} \leq C\frac{y^{1-\sigma}}{T}\left(\log y\right)^2 = \mathcal{O}\left(\frac{y^{1-\sigma}\left(\log y\right)^2}{T}\right).$$

We are done, by choosing $T = L_\epsilon^4(y)$, since $y^{1-\sigma} \leq L_\epsilon(y)$ in (2.33). $\square$

LEMMA 2.14. *Let $\epsilon > 0$. We have*

$$\zeta(s,y) = F(s,y)\left(1 + \mathcal{O}\left(\frac{1}{L_\epsilon(y)}\right)\right), \tag{2.34}$$

$$F(s,y) = \zeta(s)(s-1)(\log y)\widehat{\rho}((s-1)\log y). \tag{2.35}$$

under the conditions $y \geq y_0(\epsilon)$, $|t| \leq L_\epsilon(y)$ and

$$1 - \left(\log y\right)^{-2/5-\epsilon} \leq \sigma \leq 1,$$

where $L_\epsilon(y)$ is given by (2.31).

PROOF. We begin by logarithmically differentiating (2.28) and using Lemma (2.9) to obtain

$$-\frac{\zeta'(s,y)}{\zeta(s,y)} = \sum_{n=1}^{\infty} \frac{\chi(n,y)\Lambda(n)}{n^s} = \sum_{n \leq y} \frac{\Lambda(n)}{n^s} + \sum_{n > y} \frac{\chi(n,y)\Lambda(n)}{n^s}.$$

The sum for $n \le y$ may be estimated using the approximate functional equation given in Theorem 2.13. We will show that final sum is absorbed in the error term of (2.32). Observe that that

$$\sum_{n>y} \frac{\chi(n,y)\Lambda(n)}{n^s} = \sum_{p\le\sqrt{y}} \log p \sum_{\nu\,:\,p^\nu>y} p^{-\nu\sigma} + \sum_{\sqrt{y}<p\le y} \log p \sum_{\nu\,:\,p^\nu>y} p^{-\nu\sigma}.$$

We recall that both σ and y are bounded below by a constant depending on ϵ, and hence

$$\sum_{p\le\sqrt{y}} \log p \sum_{\nu\,:\,p^\nu>y} p^{-\nu\sigma} = \sum_{p\le y} \frac{y^{-\sigma}\log p}{1-1/p^\sigma} \ll_\epsilon y^{-\sigma} \sum_{p\le y} \log p \ll_\epsilon y^{1/2-\sigma},$$

by the Prime Number Theorem. For the second sum, we observe that $y > p$ implies $\nu > 1$, and hence we similarly obtain

$$\sum_{\sqrt{y}<p\le y} \log p \sum_{\nu\,:\,p^\nu>y} p^{-\nu\sigma} \le \sum_{\sqrt{y}<p\le y} \log p \sum_{\nu=2}^{\infty} p^{-\nu\sigma} \ll_\epsilon \sum_{\sqrt{y}<p\le y} \frac{\log p}{p^{2\sigma}} \ll_\epsilon y^{1-2\sigma},$$

again by the Prime Number Theorem. Since $\sigma > 1/2$, the first term is the largest. Both are clearly bounded by the error term in the functional equation (2.32),

$$y^{1/2-\sigma} \le \frac{L_\epsilon(y)}{\sqrt{y}} \le \frac{1}{L_\epsilon(y)},$$

for $y \ge y_0(\epsilon)$, and hence we obtain

$$\frac{\zeta'(s,y)}{\zeta(s,y)} = \frac{\zeta'(s)}{\zeta(s)} - \frac{y^{1-s}}{1-s} + \mathcal{O}\left(\frac{1}{L_\epsilon^2(y)}\right) = \frac{F'(s,y)}{F(s,y)} + \mathcal{O}\left(\frac{1}{L_\epsilon^2(y)}\right),$$

where the final equality follows from logarithmic differentiation of $F(s,y)$, using

$$w\widehat{\rho}(w) = \exp\left(-\int_w^\infty \frac{e^{-t}}{t}\,dt\right),$$

by (2.12) for $w = (s-1)\log y$. Thus, by integrating from s to 1 we obtain

$$\frac{\zeta(s,y)}{\zeta(1,y)} = \frac{F(s,y)}{F(1,y)} \exp\left(\frac{1+|s|}{L_\epsilon^2(y)}\right) = \frac{F(s,y)}{F(1,y)}\left(1+\mathcal{O}\left(\frac{1}{L_\epsilon(y)}\right)\right).$$

The proof is completed by noting that

$$\zeta(1,y) = \prod_{p\le y}\left(1-\frac{1}{p}\right)^{-1} = e^\gamma \log y\left(1+\mathcal{O}\left(\frac{1}{L_\epsilon(y)}\right)\right),$$

by Theorem B.5, and furthermore computing

$$F(1,y) = (\log y)\widehat{\rho}(0) \lim_{s\to1} \zeta(s)(s-1) = e^\gamma \log y,$$

by Lemma 2.4 and Example 1.16, which proves (2.34). $\square$

2.5. de Bruijn's Function

Our next goal is to study the function $F(s,y)$ as defined by (2.35). In this section, it will become apparent that $F(s,y)$ can be considered the Laplace transformation of de Bruijn's function $\Lambda(x,y)$, as introduced in (2.27).

LEMMA 2.15. *Let $\epsilon > 0$, $x \geq x_0(\epsilon)$ and*

$$\exp\left((\log\log x)^{5/3+\epsilon}\right) \leq y \leq x. \tag{2.36}$$

We have

$$\Lambda(x,y) = \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} F(s,y)\frac{x^s}{s}\,dx + \mathcal{O}\left(\frac{x^\sigma}{\sqrt[4]{T}} + x^{3/4}\right), \tag{2.37}$$

for $\sigma = 1 - \xi(u)/\log y$ and $T \geq L_\epsilon^4(y)$.

PROOF. By the definition of $\Lambda(x,y)$, it suffices to prove (2.37) when x is not an integer. For $u = \log x/\log y$, we define

$$\lambda_y(u) = \Lambda(y^u,y)y^{-u} = \frac{\Lambda(x,y)}{x}.$$

By splitting the integration in (2.27) we obtain

$$\begin{aligned}|\lambda_y(u)| &\leq \left|\int_{0^-}^{u/2}\rho(u-v)\,d\left(\frac{[y^v]}{y^v}\right)\right| + \left|\int_{u/2}^{u}\rho(u-v)\,d\left(\frac{[y^v]}{y^v}\right)\right| \\ &\leq \rho(u/2)\frac{[y^{u/2}]}{y^{u/2}} + \left|\frac{[y^u]}{y^u} - \frac{[y^{u/2}]}{y^{u/2}}\right| \leq \rho(u/2) + \frac{3}{y^{u/2}},\end{aligned}$$

since $y \geq 2$ and $u \geq 0$. This implies that the Laplace transformation

$$\widehat{\lambda_y}(w) = \int_0^\infty e^{-wu}\lambda_y(u)\,du$$

exists when $\Re(w) > -\log(y)/2$. Assuming this is true, Fubini's theorem allows us to compute

$$\begin{aligned}\widehat{\lambda_y}(w) &= \int_0^\infty e^{-wu}\left(\int_{-\infty}^\infty \rho(u-v)\,d\left(\frac{[y^v]}{y^v}\right)\right)du \\ &= \int_{-\infty}^\infty\left(\int_0^\infty e^{-wu}\rho(u-v)\,du\right)d\left(\frac{[y^v]}{y^v}\right) = \widehat{\rho}(w)\int_{-\infty}^\infty e^{-wv}\,d\left(\frac{[y^v]}{y^v}\right).\end{aligned}$$

We apply the substitution $t = y^v$ and adopt the convention $s = 1 + w/\log y$ to rewrite

$$G_y(w) = \int_{-\infty}^\infty e^{-wv}\,d\left(\frac{[y^v]}{y^v}\right) = \int_{1^-}^\infty t^{1-s}\,d\left(\frac{[t]}{t}\right).$$

As distributions, we may apply the product rule to obtain

$$d\left(\frac{[t]}{t}\right) = \left(\frac{D(t)}{t} - \frac{[t]}{t^2}\right) dt,$$

where $D(t)$ denotes Dirac's comb with period 1 given by

$$D(t) = \sum_{n\in\mathbb{Z}} \delta(t-n).$$

Thus we may compute

$$G_y(w) = \int_{1^-}^{\infty} \frac{D(t)}{t^s}\, dt - \int_{1^-}^{\infty} \frac{[t]}{t^{1+s}} dt = \sum_{n=1}^{\infty} \frac{1}{n^s} - \frac{1}{s}\mathcal{M}([x])(s) = \left(1 - \frac{1}{s}\right)\zeta(s).$$

Combining this, we have computed

$$\widehat{\lambda_y}(w) = \widehat{\rho}(w) G_y(w) = \widehat{\rho}(w)\left(1 - \frac{1}{s}\right)\zeta(s) = \frac{F(s,y)}{s\log y}.$$

Using the inverse Laplace transformation, the appearance of $\widehat{\rho}(w)$ leads us to integrate along the line $\Re(w) = -\xi(u)$, which clearly is acceptable in the domain (2.36) by the bound

$$\xi(u) \le \log\log(x) + \mathcal{O}(1), \tag{2.38}$$

which we obtained from Lemma 2.5. Thus, we may conclude

$$\Lambda(x,y) = \frac{x}{2\pi i}\int_{-\xi(u)+i\infty}^{-\xi(u)+i\infty} \frac{F(s,y)}{s\log y} e^{uw}\, dw = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s,y)\frac{x^s}{s}\, ds,$$

by a substitution, where $\sigma = 1 - \xi(u)/\log y$. It remains to estimate the tails

$$\left|\int_{|t|>T} F(s,y)\frac{x^s}{s}\, ds\right|,$$

and show that they are contained in the error term of (2.37). We may apply (2.20), which yields

$$\log y(s-1)\widehat{\rho}(\log y(s-1)) = 1 + \mathcal{O}\left(\frac{1+u\xi}{|t|\log y}\right),$$

whenever $|t|\log y > 1 + u\xi$. This is clearly allowed since $1 + u\xi \le L_\epsilon(y)$, again by Lemma 2.5. We apply the estimate of Corollary 2.12 to estimate the contribution of the Riemann zeta function, and thus obtain

$$\left|\int_{|t|>T} F(s,y)\frac{x^s}{s}\, ds\right| \ll \left|\int_{|t|>T} \zeta(s)\frac{x^s}{s}\, ds\right| + x^\sigma L_\epsilon(y)\int_{|t|>T} \frac{dt}{|t|^{3/2}},$$

and this final term is of order $x^\sigma/\sqrt[4]{T}$, by the fact that $T^{1/4} \geq L_\epsilon(y)$. We may apply the first part of Corollary 2.12 to obtain

$$\int_{|t|>T} \zeta(s)\frac{x^s}{s}\,ds = \int_{|t|>T} \sum_{n\leq |t|} \left(\frac{x}{n}\right)^s \frac{ds}{s} + \mathcal{O}\left(\int_{|t|>T} \frac{x^s}{s}\frac{ds}{|t|^\sigma}\right).$$

The integral in the $\mathcal{O}$-term is easily estimated by

$$\int_{|t|>T} \frac{x^s}{s}\frac{ds}{|t|^\sigma} \ll x^\sigma \int_{|t|>T} \frac{dt}{|t|^{1+\sigma}} \ll \frac{x^\sigma}{T^\sigma},$$

which is clearly acceptable, since $\sigma > 1/4$. In the first integral, we may exchange integration and summation to obtain

$$\int_{|t|>T} \sum_{n\leq |t|} \left(\frac{x}{n}\right)^s \frac{ds}{s} = \sum_{n=1}^{\infty} \int_{|t|>T_n} \left(\frac{x}{n}\right)^s \frac{ds}{s},$$

where $T_n = \max(n, T)$. By an identical argument as that given in the proof of Lemma 1.21, it can be shown that

$$\sum_{n=1}^{\infty} \int_{|t|>T_n} \left(\frac{x}{n}\right)^s \frac{ds}{s} \ll \sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^\sigma \frac{1}{1+T_n|\log(x/n)|}.$$

We split the sum at $|\log x/n| = 1/\sqrt[4]{T_n}$. In the first part, the summands are bounded and we estimate by the number of summands.

$$\sum_{|\log(x/n)|\leq T_n^{-1/4}} \left(\frac{x}{n}\right)^\sigma \frac{1}{1+T_n|\log(x/n)|} \ll \operatorname{card}\left\{n \,:\, |\log(x/n)| \leq {T_n}^{-1/4}\right\}.$$

Using the same techniques as in (2.29) we obtain

$$\operatorname{card}\left\{n \,:\, |\log(x/n)| \leq {T_n}^{-1/4}\right\} \leq \operatorname{card}\left\{n \,:\, |\log(x/n)| \leq n^{-1/4}\right\} \ll x^{3/4}$$

Thus this is of order $x^{3/4}$, as required. When $|\log x/n| > 1/\sqrt[4]{T_n}$, we obtain

$$\sum_{|\log(x/n)|> T_n^{-1/4}} \left(\frac{x}{n}\right)^\sigma \frac{1}{1+T_n|\log(x/n)|} \ll \sum_{n\leq T} \left(\frac{x}{n}\right)^\sigma \frac{1}{T^{3/4}} + \sum_{n>T} \frac{x^\sigma}{n^{\sigma+3/4}},$$

both these sums are easily showed to be of the correct order, by comparing them with the corresponding integral. This completes the proof. □

It is clear that de Bruijn's function is closely related to Dickman's function. The following lemma shows an asymptotic equality.

LEMMA 2.16. *Let $\epsilon > 0$, $x \geq x_0(\epsilon)$ and suppose x and y satisfy (2.36), that is*

$$\exp\left((\log\log x)^{5/3+\epsilon}\right) \leq y \leq x.$$

Then

$$\Lambda(x,y) = x\rho(u)\left(1 + \mathcal{O}\left(\frac{\log u}{\log y}\right)\right).$$

PROOF. We recall that $\rho(u-v) = 0$ if $v > u$. We apply integration by parts and (2.8) to obtain

$$\begin{aligned}\Lambda(x,y) &= x\int_{-\infty}^{u} \rho(u-v)\, d\left(\frac{[y^v]}{y^v}\right) = [x] - x\int_0^u \frac{[y^v]}{y^v}\, d\rho(u-v),\\ &= [x] - x\int_0^u \rho'(u-v)\frac{[y^v]}{y^v}\, dv = [x] - x\int_0^{u-1} \rho'(u-v)\left(1 - \frac{\{y^v\}}{y^v}\right) dv,\end{aligned}$$

since we have $y^u = x$ and $[y^v] = 0$ for $v < 0$ and the fact that $\rho'(u-v) =$ when $0 < u - v < 1$. Thus we have

$$\Lambda(x,y) = x\rho(u) - \{x\} + x\int_0^{u-1} \rho'(u-v)\frac{\{y^v\}}{y^v}\, dv,$$

where we used $x = [x] + \{x\}$. We can ignore $\{x\}$, and have obtained the main term $x\rho(u)$. By (2.4) and Corollary 2.8 we obtain

$$\int_0^{u-1} \rho'(u-v)\frac{\{y^v\}}{y^v}\, dv = -\int_0^{u-1} \frac{\rho(u-v-1)}{u-v}\frac{\{y^v\}}{y^v}\, dv \ll \rho(u)\int_0^{u-1} \frac{e^{(v+1)\xi}}{u-v}\,\frac{dv}{y^v}.$$

What remain to show is that

$$\int_0^{u-1} \frac{e^{(v+1)\xi}}{u-v}\,\frac{dv}{y^v} \ll \frac{\log u}{\log y},$$

in (2.36). We obtain the estimate

$$\xi(u) \leq \log\log x + \mathcal{O}(1) \leq (\log y)^{3/5},$$

by (2.38) in the domain (2.36). This implies

$$0 \leq \int_0^{u-1} \frac{e^{(v+1)\xi}}{u-v}\,\frac{dv}{y^v} \leq (1+u\xi)\int_0^{u-1} \frac{y^{-3/5v}}{u-v}\, dv \leq \frac{3}{5}\frac{1+u\xi}{u\log y}.$$

The proof is completed by the estimate $\xi(u) \ll \log u$ of Lemma 2.5. $\square$

We are now ready to prove the main theorem. The reader is invited to review the following important results: Lemma 2.10, Lemma 2.14 and Lemma 2.15. We shall also rely on Lemma 2.16 and the results obtained concerning Dickman's function.

THEOREM 2.17. *Let $\epsilon > 0$ and $x \geq x_0(\epsilon)$ in the range* (2.36). *Then we have*

$$\Psi(x,y) = \Lambda(x,y)\left(1 + \mathcal{O}\left(\frac{1}{L_\epsilon(y)}\right)\right). \tag{2.39}$$

PROOF. We apply Lemma 2.10, Lemma 2.14 with $\epsilon/2$ and Lemma 2.15 to obtain

$$\begin{aligned}
\Psi(x,y) &= \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} \zeta(s,y)\frac{x^s}{s}\,ds + \mathcal{O}(R_1) \\
&= \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} F(s,y)\left(1+\mathcal{O}\left(\frac{1}{L_{\epsilon/2}(y)}\right)\right)\frac{x^s}{s}\,ds + \mathcal{O}(R_1) \\
&= \Lambda(x,y)\left(1+\mathcal{O}\left(\frac{1}{L_{\epsilon/2}(y)}\right)\right) + \mathcal{O}(R_1) + \mathcal{O}(R_2).
\end{aligned}$$

The error terms of Lemma 2.10 and Lemma 2.15 are

$$\begin{aligned}
R_1 &= \frac{x^\sigma}{\sqrt{T}}\zeta(\sigma,y) + \min\left(\frac{x}{\sqrt{T}}, \frac{x^\sigma}{\sqrt{T}}\int_{-\sqrt{T}}^{\sqrt{T}} |\zeta(\sigma+it,y)|\,dt\right), \\
R_2 &= \frac{x^\sigma}{\sqrt[4]{T}} + x^{3/4}.
\end{aligned}$$

The term of order $\Lambda(x,y)/L_{\epsilon/2}(y)$ is clearly absorbed in the error term of (2.39). When we applied Lemma 2.14 we implicitly demanded $T \leq L_{\epsilon/2}(y)$. This implies that we take

$$T = L_{\epsilon/2}(y),$$

which satisfies the demand $T \geq L_\epsilon^4(y)$ of Lemma 2.15, for $y \geq y_0(\epsilon)$ and hence $x \geq x_0(\epsilon)$. The choice of $\sigma = 1 - \xi(u)/\log y$ is also acceptable, since

$$1 > \sigma = 1 - \frac{\xi(u)}{\log y} \geq 1 - \frac{\log\log x + \mathcal{O}(1)}{\log y} \geq 1 - (\log y)^{-3/5-\epsilon/2}$$

in the domain (2.36), since we applied Lemma 2.14 with $\epsilon/2$. What remains is to show that R_1 and R_2 are contained in the error term of (2.39). To do this, we investigate the error term. By Lemma 2.16 and Theorem 2.7 we have

$$\Lambda(x,y) \asymp x\rho(u) \asymp \frac{x}{\sqrt{u}}\exp\left(-u\xi + E(\xi)\right) = \frac{x^\sigma e^{E(\xi)}}{\sqrt{u}},$$

since $\xi'(u) \asymp 1/u$ by Lemma 2.5. Hence the error term of (2.39) is of order

$$\frac{x^\sigma e^{E(\xi)}}{L_\epsilon(y)\sqrt{u}}.$$

We may estimate

$$E(\xi) = \int_0^\xi \frac{e^t-1}{t}\,dt \geq \frac{1}{\xi}\int_0^\xi e^t - 1\,dt = u - 1,$$

and thus $\exp(E(\xi))/\sqrt{u} \gg \exp(u/2)$. Both terms of R_2 are thus clearly acceptable, since $T^{1/4} = L_{\epsilon/2}^{1/4}(y) \gg L_\epsilon(y)$ and $x^{3/4} \ll x^\sigma/L_\epsilon(y)$. We now turn to R_1. We split the integral at $\sqrt[4]{T}$ and use $|\zeta(\sigma+it,y)| \le |\zeta(\sigma,y)|$ to obtain

$$R_1 \ll \frac{x^\sigma}{\sqrt[4]{T}}\zeta(\sigma,y) + \min\left(\frac{x}{\sqrt{T}},\, x^\sigma M\right),$$

where

$$M = \max_{\sqrt[4]{T}\le|t|\le\sqrt{T}} |\zeta(\sigma+it,y)|.$$

The first term is estimated using Lemma 2.14 and (2.18) which yields

$$\zeta(\sigma,y) \asymp F(\sigma,y) \asymp \zeta(\sigma)(1-\sigma)(\log y)\widehat{\rho}(-\xi) \asymp (\log y)e^{E(\xi)},$$

where we applied Example 1.16 to estimate the contribution of $\zeta(\sigma)$. This is clearly acceptable, since $u = \log x/\log y$ and

$$\exp\left(-\frac{1}{4}\,(\log y)^{3/5-\epsilon/2} + \log\log y + \frac{1}{2}\log u\right) \ll \exp\left(-\log y^{3/5-\epsilon}\right),$$

for in the domain (2.36) for $y \ge y_0(\epsilon)$, and hence $x \ge x_0(\epsilon)$. To estimate M,

$$\sqrt[4]{T} = L_{\epsilon/2}^{1/4}(y) \ge L_\epsilon(y) \ge 1 + u\xi$$

and may apply estimate (2.20) and compute using Lemma 2.14 to obtain

$$\zeta(\sigma+it,y) \asymp F(s+it,y) \asymp \zeta(\sigma+it).$$

Now, we observe that there is some small $\eta = \eta(\epsilon) > 0$ such that

$$1-\sigma = (\log y)^{-2/5-\epsilon/2} = (\log T)^{\frac{-2/5-\epsilon/2}{3/5-\epsilon/2}} \ge (\log T)^{-2/3-\eta},$$

since $T = L_{\epsilon/2}(y)$. Thus, by the estimate on Riemann's zeta function in Vinogradov's zero free region (consult Appendix B), we have $M \asymp \zeta(\sigma+it) \ll \log T$. What remains is to consider

$$\min\left(\frac{x}{\sqrt{T}},\, x^\sigma \log T\right).$$

Using $T \le y$ and Theorem 2.7 we obtain

$$x^\sigma \log T = xe^{-u\xi}\log T \ll x\rho(u)e^{-E(\xi)}\log y \ll x\rho(u)e^{-u/2}.$$

This is acceptable if $u \ge 2\log L_\epsilon(y)$. Now, if the converse it true, we estimate using $\xi \ll \log u$ and obtain

$$\frac{x}{\sqrt{T}} = x^\sigma\frac{e^{u\xi}}{\sqrt{L_{\epsilon/2}(y)}} = x^\sigma\exp\left(-\frac{1}{4}\,(\log y)^{3/5-\epsilon/2} + \mathcal{O}\left((\log y)^{3/5-\epsilon}\log\log y\right)\right)$$

$$\ll x^\sigma\exp\left(-\frac{1}{8}\,(\log y)^{3/5-\epsilon/2}\right) \ll \frac{x^\sigma}{L_\epsilon(y)},$$

which is acceptable. This completes the proof. □

Let us combine the results of CHAPTER 2 to obtain an effective estimate of $\Psi(x, y)$, which will be useful in our applications.

LEMMA 2.18. *Let $\epsilon > 0$ and $x \geq x_0(\epsilon)$ in the range* (2.36). *Then*

$$\Psi(x, y) = x \exp\left(-u\left(\log u + \log\log u + \mathcal{O}(1)\right)\right), \tag{2.40}$$

where

$$u = \frac{\log x}{\log y}.$$

PROOF. Combining Theorem 2.17, Lemma 2.16 and Theorem 2.7 yields the estimate

$$\Psi(x, y) \asymp \Lambda(x, y) \asymp x\rho(u) \asymp x\sqrt{\xi'(u)} \exp\left(E(\xi) - u\xi(u)\right).$$

By the substitution $s = \xi(t)$ we compute

$$\int_0^u t\xi'(t)\, ds = \int_1^{\xi(u)} t\, ds = \int_1^{\xi(u)} \frac{e^s - 1}{s}\, ds = E(\xi),$$

by the fact that

$$e^s = 1 + ts,$$

in view of the the definition of ξ. Now, by Lemma 2.5 we have the estimate $\xi'(u) \asymp 1/u$, which allows

$$E(\xi) = \int_0^u t\xi'(t)\, dt \asymp \int_0^u dt = u,$$

and hence $E(\xi) = \mathcal{O}(u)$. Combining this with the estimate

$$\xi(u) = \log u + \log\log u + \mathcal{O}(1),$$

again by Lemma 2.5 we obtain

$$E(\xi) - u\xi(u) = \int_0^u t\xi'(t)\, ds - u\xi(u) = \mathcal{O}(u) - u\left(\log u + \log\log u + \mathcal{O}(1)\right).$$

Furthermore, we have

$$\sqrt{\xi'(u)} \asymp \exp\left(-\frac{\log u}{2}\right),$$

which is absorbed in the error term. This completes the proof of (2.40). □

REMARK. When we computed Lemma 2.18, the sharp error term of Theorem 2.17 was ignored. Why did we need Vinogradov's zero free region, if we ignore the error term we obtained? We needed the estimate of Lemma 2.18 to be valid in the domain (2.36), that is for any $\epsilon > 0$ and $x \geq x_0(\epsilon)$,

$$\exp\left((\log\log x)^{5/3+\epsilon}\right) \leq y \leq x.$$

This allows us freedom in the choice of y, which is crucial.

CHAPTER 3

Multilinear Forms and Homogenous Polynomials

In this chapter we consider multilinear forms, which are multi-argument analogies of linear functionals on a vector space. Our main goal is to prove the hypercontractive Bohnenblust–Hille inequality for homogenous polynomials. We will follow the main ideas of their argument [**5**], but improve on their estimates.

DEFINITION. Let V be a vector space over a field $\mathbb{F}$. An *m-linear form* is a mapping $B : V \times V \times \cdots \times V \to \mathbb{F}$ which is linear in each of the m arguments.

We are particularly interested in the case $\mathbb{F} = \mathbb{C}$ and $V = \mathbb{C}^n$. Let us now see how these multilinear forms can be represented: Assume that m and n are positive integers strictly bigger than 1, and define

$$M(m,n) = \{i = (i_1,\, i_2,\, \ldots,\, i_m) \,:\, i_1,\, i_2,\, \ldots, i_m \in \{1,\, 2,\, \ldots,\, n\}\}\,.$$

Given any $i \in M(m,n)$ we let

$$i^k = (i_1,\, \ldots,\, i_{k-1},\, i_{k+1},\, \ldots i_m),$$

which then is in $M(m-1,n)$. For any permutation $\sigma \in S_m$ we say that

$$\sigma(i) = \big(i_{\sigma(1)},\, i_{\sigma(2)},\, \ldots,\, i_{\sigma(m)}\big)\,.$$

If we let $e^{(j)}$ denote the basis vector in $\mathbb{C}^n$ where $e^{(j)}_i = \delta_{ij}$, we obtain the following representation:

LEMMA 3.1. *Let $B : \mathbb{C}^n \times \mathbb{C}^n \times \cdots \times \mathbb{C}^n \to \mathbb{C}$ be a m-linear form. Then*

$$B\big(z^{(1)}, z^{(2)}, \ldots, z^{(m)}\big) = \sum_{i \in M(m,n)} a_i z^{(1)}_{i_1} z^{(2)}_{i_2} \cdots z^{(m)}_{i_m},$$

where $a_i = B\big(e^{(i_1)}, e^{(i_2)}, \ldots, e^{(i_m)}\big)$ for $i \in M(m,n)$.

PROOF. Write

$$z^{(k)} = \sum_{i_k=1}^{n} z^{(k)}_{i_k} e^{(i_k)}$$

and apply linearity in each of the m arguments. □

Let us now turn to polynomials in several variables, which we will represent using multi-index notation.

Definition. An m'th order *multi-index* on $\mathbb{C}^n$ is the vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ where $\alpha_i \in \{0, 1, \ldots, m\}$. For $z \in \mathbb{C}^n$ we take

$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n},$$

and furthermore $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n = m$ and $\alpha! = \alpha_1! \cdot \alpha_2! \cdots \alpha_n!$.

Any m'th degree polynomial on $\mathbb{C}^n$ can be represented as

$$P(z) = \sum_{|\alpha| \le m} a_\alpha z^\alpha,$$

under the assumption that there is some $a_\alpha \neq 0$ with $|\alpha| = m$.

3.1. Khinchine–Type Inequalities in the Polydisk

In this section, we introduce some properties of the polydisk and functions of several complex variables. For more on several complex variables and function theory in polydisks, consult Rudin [**35**]. We furthermore prove prove two Khinchine–Type inequalities in the polydisk [**26, 3**]. These inequalities replaces similar, but weaker, inequalities in the original proof of Bohnenblust–Hille.

Definition. Suppose $U \subset \mathbb{C}^n$ is an open set. A function $F : U \to \mathbb{C}$ is called *holomorphic* (in U) if it is continuous and holomorphic in each variable.

In one dimension, we study the unit disk and the unit torus:

$$\mathbb{D} = \{z \in \mathbb{C} \,:\, |z| < 1\},$$
$$\mathbb{T} = \{z \in \mathbb{C} \,:\, |z| = 1\}.$$

A simple fact is that the boundary $\partial\mathbb{D} = \mathbb{T}$, and by the Maximum Modulus Principle and continuity we obtain

$$\sup_{z \in \mathbb{D}} |F(z)| = \sup_{z \in \mathbb{T}} |F(z)|,$$

for any holomorphic function F on an open set which strictly contains $\mathbb{D}$. We would like to extend this to n dimensions. It is natural to consider:

$$\mathbb{D}^n = \{z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \,:\, z_i \in \mathbb{D}\},$$
$$\mathbb{T}^n = \{z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \,:\, z_i \in \mathbb{T}\}.$$

For $n > 1$ we observe that

$$\mathbb{T}^n \subsetneq \partial\mathbb{D}^n,$$

for example by $z = (1, 0, \ldots, 0)$, which is on $\partial\mathbb{D}^n$ but not on $\mathbb{T}^n$. In some sense the boundary $\mathbb{T}^n$ is the most important part of $\partial\mathbb{D}^n$, and we call it the *distinguished boundary.* The following result illustrates its importance.

LEMMA 3.2 (A Maximum Modulus Principle). *Suppose that F is holomorphic in some set U which strictly contains $\mathbb{D}^n$. Then*

$$\sup_{z\in\mathbb{D}^n} |F(z)| = \sup_{z\in\mathbb{T}^n} |F(z)|. \tag{3.1}$$

PROOF. The inequality $\geq$ in (3.1) is obvious by continuity. To show $\leq$, we fix $z = (z_1, z_2, \ldots, z_n) \in \overline{\mathbb{D}^n}$ and prove that there is some $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in \mathbb{T}^n$ such that $|F(z)| \leq |F(\zeta)|$. We begin by considering the one-variable function

$$F_1(w) = F(w, z_2, \ldots, z_n).$$

It is clearly holomorphic in some open set that contains $\overline{\mathbb{D}}$, and by the Maximum Modulus Principle, there exists some $\zeta_1 \in \mathbb{T}$ such that $|F_1(z_1)| \leq |F_1(\zeta_1)|$. Now, consider

$$F_2(w) = F(\zeta_1, w, z_3, \ldots, z_n).$$

By similarly considerations, we obtain $\zeta_2 \in \mathbb{T}$ such that $|F_2(z_2)| \leq |F_2(\zeta_2)|$. We continue in this way for $3, 4, \ldots, n$ and obtain

$$|F(z_1, z_2, \ldots, z_n)| \leq |F(\zeta_1, \zeta_2, \ldots, \zeta_n)|,$$

which shows $\leq$ in (3.1) and completes the proof. $\square$

This maximum modulus principle allows us to define the norms

$$\|P\|_\infty = \sup_{z\in\mathbb{D}^n} |P(z)| = \sup_{z\in\mathbb{T}^n} |P(z)|,$$
$$\|B\|_\infty = \sup_{z^{(k)}\in\mathbb{D}^n} \big|B\big(z^{(1)}, \ldots, z^{(m)}\big)\big| = \sup_{z^{(k)}\in\mathbb{T}^n} \big|B\big(z^{(1)}, \ldots, z^{(m)}\big)\big|.$$

by viewing the multilinear form as a polynomial in mn variables. Let us now turn to integration in the polydisk. We let μ^n and ν^n denote the normalized Lebesgue-measure on $\mathbb{T}^n$ and $\mathbb{D}^n$ respectively. We write $\mu^1 = \mu$ and $\nu^1 = \nu$.

LEMMA 3.3. *For any multi-indices α and β on $\mathbb{C}^n$ we have*

$$\int_{\mathbb{T}^n} z^\alpha \overline{z}^\beta \, d\mu^n(z) = \delta_{\alpha\beta}, \tag{3.2}$$

$$\int_{\mathbb{D}^n} z^\alpha \overline{z}^\beta \, d\nu^n(z) = \frac{\delta_{\alpha\beta}}{(1+\alpha_1)(1+\alpha_2)\cdots(1+\alpha_n)} \tag{3.3}$$

PROOF. For non-negative integers i and j we have

$$\int_{\mathbb{T}} z^i \overline{z}^j \, d\mu(z) = \delta_{ij},$$

by the orthogonality of the trigonometric system. This yields

$$\int_{\mathbb{D}} z^i \overline{z}^j \, d\nu(z) = 2\int_0^1 r^{i+j} \int_{\mathbb{T}} z^i \overline{z}^j \, d\mu(z)\, dr = \frac{2\delta_{ij}}{2+i+j} = \frac{\delta_{ij}}{1+i}.$$

Apply these for each of the n variables to obtain (3.2) and (3.3) respectively. $\square$

Let $H(\mathbb{D})$ denote the vector space of holomorphic functions on the unit disk. On this space we introduce the norms:

$$\|f\|_{A^p} = \left(\int_{\mathbb{D}} |f(z)|^p \, d\nu(z)\right)^{\frac{1}{p}}, \tag{3.4}$$

$$\|f\|_{H^p} = \left(\int_{\mathbb{T}} |f(z)|^p \, d\mu(z)\right)^{\frac{1}{p}}, \tag{3.5}$$

where the latter is taken as a radial limit if necessary. The following lemma provides a simple inequality, which we can build on.

LEMMA 3.4. *Suppose that $f \in H(\mathbb{D})$. Then we have $\|f\|_{A^4} \leq \|f\|_{H^2}$.*

PROOF. since f is holomorphic, we may write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

which converges in $\mathbb{D}$. Squaring this we obtain

$$f(z)^2 = \sum_{n=0}^{\infty} b_n z^n \qquad \text{where} \qquad b_n = \sum_{k=0}^{n} a_k a_{n-k}.$$

Applying (3.3) of Lemma 3.3 and the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}
\|f\|_{A^4}^4 = \|f^2\|_{A^2}^2 &= \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} b_n \overline{b_j} \int_{\mathbb{D}} z^n \overline{z}^j \, d\nu(z) = \sum_{n=0}^{\infty} \frac{|b_n|^2}{1+n} \\
&= \sum_{n=0}^{\infty} \frac{1}{n+1} \left|\sum_{k=0}^{n} a_k a_{n-k}\right|^2 \leq \sum_{n=0}^{\infty}\sum_{k=0}^{n} |a_k a_{n-k}|^2 \\
&= \left(\sum_{n=0}^{\infty} |a_n|^2\right)^2 = \left(\sum_{n=0}^{\infty}\sum_{j=0}^{\infty} a_n \overline{a_j} \int_{\mathbb{T}} z^n \overline{z}^j \, d\mu(z)\right)^2 = \|f\|_{H^2}^4,
\end{aligned}$$

as required. ☐

DEFINITION. A *Blaschke product* is a product of Möbius transforms of the form

$$B(z) = \prod_{k=1}^{\infty} \frac{\overline{a_k}}{|a_k|} \frac{z - a_k}{z\overline{a_k} - 1}, \qquad \text{for } a_k \in \mathbb{D}.$$

In what follows, we only consider finite Blaschke products. In this case, the only zeroes of B are a_k, and we observe that and $|B(z)| < 1$ in $\mathbb{D}$ and $|B(z)| = 1$ on $\mathbb{T}$, since each of the Möbius transforms satisfies these demands. Finally, $B \in H(\mathbb{D})$ since its poles lie outside $\mathbb{T}$.

Lemma 3.5. *Suppose that $P \in \mathrm{Poly}(\mathbb{C})$. Then $\|P\|_{A^{2p}} \leq \|P\|_{H^p}$.*

Proof. Let P be a polynomial. If $P \equiv 0$ we are done. If $P \not\equiv 0$ then P has a finite number of zeroes. In particular, let $\omega_1, \omega_2, \ldots, \omega_n$ denote the zeroes of P in $\mathbb{D}$ numbered according to multiplicity. Form the Blaschke product of the zeroes,

$$B(z) = \prod_{k=1}^{n} \frac{\overline{\omega_k}}{|\omega_k|} \frac{z - \omega_k}{\overline{\omega_k} z - 1}.$$

We factor $P(z) = G(z)B(z)$. Now, since $|B(z)| = 1$ on $\mathbb{T}$ we have $\|P\|_{H^p} = \|G\|_{H^p}$. Furthermore, $G^{p/2}$ is in $H(\mathbb{D})$ since G does not vanish in $\mathbb{D}$. Then, since $|B(z)| < 1$ in $\mathbb{D}$, we apply Lemma 3.4

$$\|P\|_{A^{2p}} \leq \|G\|_{A^{2p}} = \left|\left|G^{p/2}\right|\right|_{A^4}^{2/p} \leq \left|\left|G^{p/2}\right|\right|_{H^2}^{2/p} = \|G\|_{H^p} = \|P\|_{H^p}. \qquad \square$$

For any function $f \in H(\mathbb{D}^n)$, we define n-dimensional versions of (3.4) and (3.5):

$$\|f\|_{A^p} = \left(\int_{\mathbb{D}^n} |f(z)|^p \, d\nu^n(z) \right)^{\frac{1}{p}},$$

$$\|f\|_{H^p} = \left(\int_{\mathbb{T}^n} |f(z)|^p \, d\mu^n(z) \right)^{\frac{1}{p}}.$$

Theorem 3.6. *Suppose that $P \in \mathrm{Poly}(\mathbb{C}^n)$. Then $\|P\|_{A^{2p}} \leq \|P\|_{H^p}$.*

Proof. We prove this using induction on n. The case $n = 1$ is already settled by Lemma 3.5. We write $z = (w, z_n)$ and compute

$$\|P\|_{A^{2p}} = \left[\int_{\mathbb{D}} \left(\int_{\mathbb{D}^{n-1}} |P(w, z_n)|^{2p} \, d\nu^{n-1}(w) \right) d\nu(z_n) \right]^{\frac{1}{2p}},$$

by the induction hypothesis

$$\leq \left[\int_{\mathbb{D}} \left(\int_{\mathbb{T}^{n-1}} |P(w, z_n)|^{p} \, d\mu^{n-1}(w) \right)^2 d\nu(z_n) \right]^{\frac{1}{2p}},$$

by Minkowski's inequality (see Theorem A.4)

$$\leq \left[\int_{\mathbb{T}^{n-1}} \left(\int_{\mathbb{D}} |P(w, z_n)|^{2p} \, d\nu(z_n) \right)^{\frac{1}{2}} d\mu^{n-1}(w) \right]^{\frac{1}{p}},$$

and finally and application Lemma 3.5

$$\leq \left[\int_{\mathbb{T}^{n-1}} \left(\int_{\mathbb{T}} |P(w, z_n)|^{p} \, d\mu(z_n) \right) d\mu^{n-1}(w) \right]^{\frac{1}{p}} = \|P\|_{H^2},$$

which completes the proof. $\square$

REMARK. It should be noted that both Lemma 3.5 and Theorem 3.6 can be generalized to hold for any $f \in H(\mathbb{D}^n)$, using the theory of inner and outer functions in Hardy Spaces. We avoid this, since we only require the polynomial version in our applications. Consult **[36]** for more on Hardy Spaces.

LEMMA 3.7 (Khinchine–Type Inequality for Polynomials). *Let P be any m'th degree polynomial on $\mathbb{C}^n$. Then*

$$\|P\|_{H^2} \leq \sqrt{2}^m \|P\|_{H^1}.$$

PROOF. We begin by computing:

$$\|P\|_{H^2} = \left(\sum_{|\alpha| \leq m} |a_\alpha|^2 \right)^{\frac{1}{2}}, \tag{3.6}$$

$$\|P\|_{A^2} = \left(\sum_{|\alpha| \leq m} \frac{|a_\alpha|^2}{(1+\alpha_1)(1+\alpha_2)\cdots(1+\alpha_n)} \right)^{\frac{1}{2}}. \tag{3.7}$$

We do this by writing

$$|P(z)|^2 = \sum_{|\alpha| \leq m} \sum_{|\beta| \leq m} a_\alpha \overline{a_\beta} z^\alpha \overline{z}^\beta,$$

then exchanging integration and summation. We then apply (3.2) and (3.3) to obtain (3.6) respectively (3.7). We combine these to obtain

$$\|P\|_{H^2}^2 = \sum_{|\alpha| \leq m} |a_\alpha|^2 \leq 2^m \|P\|_{A^2}^2,$$

since $1 + k \leq 2^k$. Furthermore, by taking $p = 1$ in Theorem 3.6 we conclude

$$\|P\|_{H^2} \leq \sqrt{2}^m \|P\|_{A^2} \leq \sqrt{2}^m \|P\|_{H^1},$$

which completes the proof. □

By again considering a multilinear form as a polynomial in mn variables, we extend the H^p-norm to multilinear forms

$$\|B\|_p = \left(\int_{\mathbb{T}^n \times \cdots \times \mathbb{T}^n} \left| B\left(z^{(1)}, \ldots, z^{(m)}\right) \right|^p d\mu^n\left(z^{(1)}\right) \cdots d\mu^n\left(z^{(m)}\right) \right)^{\frac{1}{p}}. \tag{3.8}$$

LEMMA 3.8 (Khinchine–Type Inequality for Forms). *Let B be a m-linear form on $\mathbb{C}^n$. Then we have the upper bound:*

$$\left(\sum_{i \in M(m,n)} |a_i|^2 \right)^{\frac{1}{2}} \leq \sqrt{2}^m \|B\|_1.$$

PROOF. We begin by computing

$$
\begin{aligned}
\|B\|_2^2 &= \int_{\mathbb{T}^n\times\cdots\times\mathbb{T}^n} \left|B\big(z^{(1)},\ldots,z^{(m)}\big)\right| d\mu^n\big(z^{(1)}\big)\cdots d\mu^n\big(z^{(m)}\big) \\
&= \sum_{\substack{i\in M(m,n)\\ j\in M(m,n)}} a_i\overline{a_j}\prod_{k=1}^{m}\int_{\mathbb{T}^n} z_{i_k}^{(k)}\overline{z_{j_k}^{(k)}}\,d\mu^n\big(z^{(k)}\big) = \sum_{i\in M(m,n)} |a_i|^2.
\end{aligned}
\tag{3.9}
$$

In light of this, it will suffice to show

$$\|B\|_2 \le \sqrt{2}^m \|B\|_1.$$

We will prove this using induction on m. The case $m=1$ follows from Lemma 3.7 since a 1-linear form actually is a polynomial. Assume hence that the inequality holds for $m-1$. Let $X=\mathbb{T}^n$ and $Y=\mathbb{T}^n\times\cdots\times\mathbb{T}^n$, where the product is taken $(m-1)$ times. Then

$$\|B\|_2 = \left[\int_X\left(\int_Y \left|B\big(z^{(1)},\ldots,z^{(m)}\big)\right|^2 d\mu^n\big(z^{(2)}\big)\cdots d\mu^n\big(z^{(m)}\big)\right) d\mu^n\big(z^{(1)}\big)\right]^{\frac12},$$

and by the induction hypothesis squared

$$\le \left[\int_X 2^{m-1}\left(\int_Y \left|B\big(z^{(1)},\ldots,z^{(m)}\big)\right| d\mu^n\big(z^{(2)}\big)\cdots d\mu^n\big(z^{(m)}\big)\right)^2 d\mu^n\big(z^{(1)}\big)\right]^{\frac12},$$

using Minkowski's inequality with $p=2$ we obtain

$$\le \sqrt{2}^{m-1}\int_Y\left(\int_X \left|B\big(z^{(1)},\ldots,z^{(m)}\big)\right|^2 d\mu^n\big(z^{(1)}\big)\right)^{\frac12} d\mu^n\big(z^{(2)}\big)\cdots d\mu^n\big(z^{(m)}\big),$$

finally applying the case $m=1$ we obtain the required

$$
\begin{aligned}
&\le \sqrt{2}^{m-1}\int_Y \sqrt{2}\left(\int_X \left|B\big(z^{(1)},\ldots,z^{(m)}\big)\right| d\mu^n\big(z^{(1)}\big)\right) d\mu^n\big(z^{(2)}\big)\cdots d\mu^n\big(z^{(m)}\big) \\
&= \sqrt{2}^m\int_{X\times Y} \left|B\big(z^{(1)},\ldots,z^{(m)}\big)\right| d\mu^n\big(z^{(1)}\big)\cdots d\mu^n\big(z^{(m)}\big) = \sqrt{2}^m\|B\|_1,
\end{aligned}
$$

which completes the proof. ∎

This completes our initial study of the polydisk, and we return to multilinear forms.

3.2. Symmetric Multilinear Forms and Polarization

Let us consider two special cases of polynomials and multilinear forms:

Definition. A *homogenous polynomial* on $\mathbb{C}^n$ is a polynomial where all the terms have the same degree, say m, and we write

$$P(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha.$$

Definition. A multilinear form B on $\mathbb{C}^n$ is called *symmetric* if the coefficients are symmetric, that is $a_{\sigma(i)} = a_i$ for any $i \in M(m,n)$ and any $\sigma \in S_m$.

To connect these concepts, we take any $i \in M(m,n)$, and associate i to the m'th order multi-index

$$\alpha(i) = \big(\#\{i_k = 1\},\, \#\{i_k = 2\},\, \dots,\, \#\{i_k = n\}\big) = \sum_{k=1}^{n} e^{(i_k)} \tag{3.10}$$

It is immediately clear that $\alpha(i) = \alpha(\sigma(i))$ for any permutation $\sigma \in S_m$, since this would only change the order of summation in (3.10). We take each variable of a symmetric multilinear form B equal to the same z to obtain

$$B(z,\dots,z) = \sum_{i\in M(m,n)} a_i z^{\alpha(i)} = \sum_{|\alpha|=m} a_\alpha z^\alpha = P(z). \tag{3.11}$$

The coefficient a_α is the sum of all coefficients a_i where $\alpha(i) = \alpha$. If i is one of these indices, the only others possibilities are $\sigma(i)$, by (3.10). However, some of these indices may be equal, if one of the factors z_i appears more than once. This implies that the total number of such indices are the multinomial

$$\Upsilon_i = \binom{m}{\alpha(i)} = \frac{m!}{\alpha(i)!}. \tag{3.12}$$

Since B is assumed to be symmetric, these a_i are identical. Combining this observation with (3.11) and (3.12) we obtain the coefficient relationship

$$a_{\alpha(i)} = \Upsilon_i a_i. \tag{3.13}$$

Thus, for any symmetric m-linear form B, we can construct an m-homogenous polynomial $P(z) = B(z,\dots,z)$. Furthermore, (3.13) implies that we have a one-to-one correspondence between symmetric m-linear forms on $\mathbb{C}^n$ and m-homogenous polynomials on $\mathbb{C}^n$. Finally, it is clear that

$$\|P\|_\infty = \sup_{z\in\mathbb{D}^n} |P(z)| \le \sup_{z^{(k)}\in\mathbb{D}^n} \big|B\big(z^{(1)},\dots,z^{(m)}\big)\big| = \|B\|_\infty. \tag{3.14}$$

Here $\|P\|_\infty$ is bounded by $\|B\|_\infty$. We would like the opposite implication, possibly with a constant depending on m. We can do this using polarization.

THEOREM 3.9. *Let P be the m-homogenous polynomial associated with the symmetric m-linear form B. Then we have the representation*

$$B\big(z^{(1)},\dots,z^{(m)}\big) = \frac{1}{m!}\int_{\mathbb{T}^m} P\big(\zeta_1 z^{(1)} + \cdots + \zeta_m z^{(m)}\big)\overline{\zeta_1\cdots\zeta_m}\,d\mu^m(\zeta).$$

PROOF. We first observe by the m-homogeneity of P that this indeed is a symmetric m-linear form. We also note that we have

$$(\zeta_1+\zeta_2+\cdots+\zeta_m)^m = \cdots + m!\zeta_1\zeta_2\cdots\zeta_m + \cdots\,,$$

and by (3.2) of Lemma 3.3, this is the only term that will be non-zero when integrated with $\overline{\zeta_1\cdots\zeta_m}$. It suffices then to show

$$\begin{aligned} B(z,\dots,z) &= \frac{1}{m!}\int_{\mathbb{T}^m} P\big(z(\zeta_1+\cdots+\zeta_m)\big)\overline{\zeta_1\cdots\zeta_m}\,d\mu^m(\zeta) \\ &= \frac{P(z)}{m!}\int_{\mathbb{T}^m} (\zeta_1+\cdots+\zeta_m)^m\,\overline{\zeta_1\cdots\zeta_m}\,d\mu^m(\zeta) = P(z), \end{aligned}$$

by the one-to-one correspondence. □

We can immediately apply Theorem 3.9 to give the following estimate, which will give the converse of (3.14), as we asked for above.

COROLLARY 3.10. *Let P be the m-homogenous polynomial associated with the symmetric m-linear form B. Then we have the following estimate:*

$$\|B\|_\infty = \sup_{z^{(k)}\in\mathbb{D}^n}\big|B\big(z^{(1)},\dots,z^{(m)}\big)\big| \le \frac{m^m}{m!}\sup_{z\in\mathbb{D}^n}|P(z)| = \frac{m^m}{m!}\|P\|_\infty.$$

PROOF. Follows from Theorem 3.9, since $z^{(1)}\zeta_1+\cdots+z^{(m)}\zeta_m$ is in $(m\mathbb{D})^n$. □

We also state the following addendum, which will be useful later.

LEMMA 3.11. *Let P be the m-homogenous polynomial associated with the symmetric m-linear form B. Then we have:*

$$B\,(z,w,\dots,w) = \frac{1}{m(m-1)^{m-1}}\int_{\mathbb{T}^2} P\,(\zeta_1 z + (m-1)\zeta_2 w)\,\overline{\zeta_1\zeta_2^{m-1}}\,d\mu^2(\zeta)$$

$$\sup_{z,w\in\mathbb{D}^n}|B\,(z,w,\dots,w)\,| \le \left(1+\frac{1}{m-1}\right)^{m-1}\sup_{z\in\mathbb{D}^n}|P(z)|$$

PROOF. Similar as the proof of Theorem 3.9 and Corollary 3.10. □

3.3. Littlewood's 4/3–Inequality

Let us for a moment consider a different problem. Let $\phi : \ell^\infty \to \mathbb{C}$ be a linear functional. By arguing like in Lemma 3.1, we can represent

$$\phi(z) = \sum_{i=1}^{\infty} a_i z_i,$$

where $a_i = \phi\big(e^{(i)}\big)$. We recall that ϕ is bounded if

$$\sup_{z \in \mathbb{D}^\infty} |\phi(z)| < \infty,$$

and thus if we let $z_i \to \overline{a_i}/|a_i|$ we obtain

$$\sum_{i=1}^{\infty} |a_i| < \infty.$$

Thus, if the linear functional is bounded, the series of coefficients converge absolutely. Can we say the same for multilinear forms?

Example 3.12. We consider the bilinear (or 2-linear) form B on ℓ^∞, which we can represent by the coefficients

$$b_{ij} = \begin{cases} \frac{x_i y_j}{i-j} & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases},$$

where we take

$$x_i = \frac{1}{\sqrt{i+1}\log(i+1)} \qquad \text{and} \qquad y_j = \frac{1}{\sqrt{j+1}\log(j+1)}.$$

Using the integration estimate we obtain

$$\sum_{i=1}^{\infty} |x_i|^2 = \sum_{j=1}^{\infty} |y_j|^2 \leq \frac{1}{2(\log 2)^2} + \int_1^\infty \frac{dt}{(t+1)(\log(t+1))^2} = \frac{1}{2(\log 2)^2} + \frac{1}{\log 2}.$$

If we apply Hilbert's inequality (see Theorem A.5) we obtain

$$\begin{aligned} \sup_{z,w \in \mathbb{D}^\infty} |B(z,w)| &= \sup_{z,w \in \mathbb{D}^\infty} \left| \sum_{i,j} b_{ij} z_i w_j \right| \\ &\leq \pi \left(\sum_i |x_i|^2 \right)^{\frac{1}{2}} \left(\sum_j |y_j|^2 \right)^{\frac{1}{2}} \leq \frac{\pi}{\log 2} \left(1 + \frac{1}{2(\log 2)^2} \right). \end{aligned}$$

Thus B is bounded in the unit disk. However, we have $x_i \geq x_{i+k}$ which implies

$$\begin{aligned}\sum_{i,j} |b_{ij}| &= \sum_{i=1}^{\infty}\sum_{j\neq i} \frac{x_i y_j}{|i-j|} \geq \sum_{k=1}^{\infty}\frac{1}{k}\sum_{i=1}^{\infty} x_i y_{i+k} \geq \sum_{k=1}^{\infty}\frac{1}{k}\sum_{i=1}^{\infty} x_{i+k} y_{i+k} \\ &= \sum_{k=1}^{\infty}\frac{1}{k}\sum_{i=k+1}^{\infty}\frac{1}{(i+1)(\log(i+1))^2} \geq \sum_{k=1}^{\infty}\frac{1}{k}\int_{k+1}^{\infty}\frac{dt}{(t+1)(\log(t+1))^2} \\ &= \sum_{k=1}^{\infty}\frac{1}{k\log(k+2)} \geq \sum_{k=1}^{\infty}\frac{1}{(k+2)\log(k+2)} \geq \int_3^{\infty}\frac{dt}{t\log t} = \infty.\end{aligned}$$

This example leads us to pose the following problem: Is there some exponent $\rho > 1$ such that

$$\sum_{i,j} |b_{ij}|^{\rho} < \infty$$

for every bilinear form B, bounded in the unit disk. Furthermore, what is the smallest exponent ρ we can take? Littlewood [**28**] solves this problem, in fact he does more:

THEOREM (Littlewood's 4/3-Inequality). *Let B be a bilinear form on $\mathbb{C}^n$. Then there is some absolute positive C such that*

$$\left(\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^{\frac{4}{3}}\right)^{\frac{3}{4}} \leq C \sup_{z,w\in\mathbb{D}^n} |B(z,w)|.$$

Furthermore, the exponent $4/3$ is optimal, in the sense that any smaller exponent will make C dependent on n.

We would like to extend this bilinear inequality to general multilinear forms. Hence we would like an inequality of the type

$$\left(\sum_{i\in M(m,n)} |a_i|^{\rho}\right)^{\frac{1}{\rho}} \leq C_m \sup_{z^{(k)}\in\mathbb{D}^n} \left|B\big(z^{(1)}, z^{(2)}, \dots, z^{(m)}\big)\right| = C_m \|B\|_{\infty} \tag{3.15}$$

for some exponent ρ and constant C_m, which both may depend on m, but not on n. Clearly, by Example 3.12 we need $\rho > 1$. We can also give an upper bound. In light of (3.9) we have

$$\left(\sum_{i\in M(m,n)} |a_i|^2\right)^{\frac{1}{2}} = \|B\|_2 \leq \|B\|_{\infty},$$

by the fact that the measures in (3.8) are normalized. In combination, we now have $1 < \rho \leq 2$. Bohnenblust–Hille was able to find ρ and prove its optimality.

3.4. Rudin-Shapiro Polynomials

The Bohnenblust–Hille inequality for homogenous polynomials can be stated as

$$\left(\sum_{|\alpha|=m} |a_\alpha|^\rho\right)^{\frac{1}{\rho}} \leq D_m \sup_{z\in\mathbb{D}^n} |P(z)|\,. \tag{3.16}$$

In this section we obtain a sharp lower bound for ρ which depends on m. We aim to construct some polynomials having some very special properties, which will give the lower bound [**30**]. The *Hadamard* matrices are given by the recursive relation

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad A_{k+1} = \begin{bmatrix} A_k & A_k \\ A_k & -A_k \end{bmatrix} \quad \text{for} \quad k \in \mathbb{N}.$$

The Hadamard matrix A_k is square of dimension $q = 2^k$ and its coordinates are either 1 or -1. Furthermore, we have

$$A_k A_k^T = qI_q.$$

We want to use the Hadamard matrices to recursively define homogenous polynomials, which will give the required lower bound. Fix some positive integer m bigger than 1 and $k \in \mathbb{N}$. Let $q = 2^k$. We shall recursively construct homogenous polynomials in $n = mq$ variables until they are of degree m. We begin by letting $z \in \mathbb{C}^n$ and for $0 \leq j \leq m$ considering the polynomial vectors

$$\mathcal{P}_j(z) = \begin{bmatrix} P_{j,1}(z) & P_{j,2}(z) & \cdots & P_{j,q}(z) \end{bmatrix}^T$$

where we take $\mathcal{P}_0(z) = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T$, and recursively introduce q new variables and apply A_k to obtain

$$\mathcal{P}_{j+1}(z) = A_k \begin{bmatrix} z_{jq+1}P_{j,1}(z) & z_{jq+2}P_{j,2}(z) & \cdots & z_{jq+q}P_{j,q}(z) \end{bmatrix}^T.$$

Observe that we can also write $\mathcal{P}_{j+1}(z) = A_k D_j(z)\mathcal{P}_j(z)$, where $D_j(z)$ is the appropriate diagonal matrix. We do this procedure m times, and then we have used all our $n = mq$ variables. We obtain the following result regarding the final polynomials. These polynomials $P_{m,l}$ are called *Rudin-Shapiro polynomials.*

LEMMA 3.13. *The polynomials $P_{m,j}(z)$ are m-homogenous and have q^m non-zero terms with coefficients ± 1. Furthermore, they are absolutely bounded by $q^{\frac{m+1}{2}}$ in the unit disk.*

PROOF. It is clear that $P_{m,j}$ is m-homogenous, since we at each iteration increase the degree of each term in each polynomial by 1. The number of non-zero terms in each polynomial are increased by q each iteration, and since there are only 1 to start with, we have q^m terms. They cannot take any value other than the values in A_k, since we at each iteration introduce q new variables, so we have no overlap. Now, for the bound, we take $z \in \mathbb{T}^n$ first prove the auxiliary formula

$$\mathscr{S}(z) = |P_{m,1}(z)|^2 + |P_{m,2}(z)|^2 + \cdots + |P_{m,q}(z)|^2 = q^{m+1}.$$

We apply the inner product and obtain

$$\begin{aligned}\mathscr{S}(z) &= \langle \mathcal{P}_m(z), \mathcal{P}_m(z)\rangle = \langle A_k D_m(z)\mathcal{P}_{m-1}(z), A_k D_m(z)\mathcal{P}_{m-1}(z)\rangle \\ &= \langle \mathcal{P}_{m-1}(z), D_m^*(z)A_k^* A_k D_m(z)\mathcal{P}_{m-1}(z)\rangle = q\,\langle \mathcal{P}_{m-1}(z), \mathcal{P}_{m-1}(z)\rangle \\ &= q^m \langle \mathcal{P}_0(z), \mathcal{P}_0(z)\rangle = q^{m+1}.\end{aligned}$$

Now, by the maximum modulus principle of Lemma 3.2 it is clear that

$$\sup_{z\in\mathbb{D}^n} |P_{m,j}(z)| \le \sup_{z\in\mathbb{T}^n} \sqrt{|P_{m,j}(z)|^2} \le \sup_{z\in\mathbb{T}^n} \sqrt{\mathscr{S}(z)} = q^{\frac{m+1}{2}},$$

which completes the proof. □

We can now obtain the required lower estimate for both Bohnenblust–Hille inequalities, in one theorem by polarization.

THEOREM 3.14. *Let m be a fixed positive integer. Suppose that ρ satisfies* (3.15) *or* (3.16). *Then $\rho \ge 2m/(m+1)$.*

PROOF. Recall that $q = 2^k$ and consider the Rudin-Shapiro polynomial $Q_k(z) = P_{m,1}(z)$ which is of $n = mq$ variables. By Lemma 3.13 we see that

$$\left(\sum_{|\alpha|=m} |a_\alpha|^\rho\right)^{\frac{1}{\rho}} = q^{\frac{m}{\rho}} \qquad \text{and} \qquad \sup_{z\in\mathbb{D}^n} |Q_k(z)| \le q^{\frac{m+1}{2}}.$$

Now, if ρ is supposed to satisfy (3.16) we obtain, for any $q = 2^k$,

$$q^{\frac{m}{\rho}} \le D_m q^{\frac{m+1}{2}} \implies \frac{2m}{m+1} \le \rho,$$

since D_m does not depend on $n = mq$. Furthermore, we let B_k be the symmetric polynomial associated to Q_k. By (3.13) and the fact that $\Upsilon_i \le m!$, we obtain

$$\left(\sum_{|\alpha|=m} |a_\alpha|^\rho\right)^{\frac{1}{\rho}} = \left(\sum_{i\in M(m,n)} \frac{1}{\Upsilon_i}|\Upsilon_i a_i|^\rho\right)^{\frac{1}{\rho}} \le (m!)^{1-\frac{1}{\rho}} \left(\sum_{i\in M(m,n)} |a_i|^\rho\right)^{\frac{1}{\rho}}.$$

Combining this with the polarization estimate given in Corollary 3.10, if ρ is to satisfy (3.15) we must have

$$(m!)^{\frac{1-\rho}{\rho}} q^{\frac{m}{\rho}} \le C_m \frac{m^m}{m!} q^{\frac{m+1}{2}} \implies \frac{2m}{m+1} \le \rho,$$

which completes the proof since C_m does not depend on $n = mq$. □

3.5. Blei's Inequality

Our final lemma is a powerful inequality due to Blei [**4**]. Recall the special versions of Hölder's inequality and Minkowski's inequality of (A.2) and (A.3).

LEMMA 3.15 (Blei's Inequality). *For families of complex numbers* $\{c_i\}_{i\in M(m,n)}$ *we have*

$$\left(\sum_{i\in M(m,n)} |c_i|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}} \leq \prod_{k=1}^{m}\left[\sum_{i_k=1}^{n}\left(\sum_{i^k\in M(m-1,n)} |c_i|^2\right)^{\frac{1}{2}}\right]^{\frac{1}{m}}.$$

PROOF. The proof works in m steps, one for each of the m variables we sum over. In step k we apply the Hölder once and the Minkowski $k-1$ times. Let

$$\mathscr{S} = \sum_{i\in M(m,n)} |c_i|^{\frac{2m}{m+1}}.$$

Step 1. We extract i_1 from the sum, and take $a_1(i_1) = \cdots = a_m(i_2) = |c_i|^{\frac{2}{m+1}}$, which yields

$$\mathscr{S} = \sum_{i_2,\dots,i_m}\sum_{i_1=1}^{n} a_1(i_1)\cdots a_m(i_1) \leq \sum_{i_2,\dots,i_m}\left(\sum_{i_1=1}^{n}|c_i|^2\right)^{\frac{m-1}{m+1}}\left(\sum_{i_1=1}^{n}|c_i|\right)^{\frac{2}{m+1}}.$$

Step 2. We now consider i_2 and take

$$a_1(i_2) = \left(\sum_{i_1=1}^{n}|c_i|\right)^{\frac{2}{m+1}} \quad\text{and}\quad a_2(i_2) = a_3(i_2) = \cdots = a_m(i_2) = \left(\sum_{i_1=1}^{n}|c_i|^2\right)^{\frac{1}{m+1}}.$$

We apply Hölder's inequality, and again obtain

$$\leq \sum_{i_3,\dots,i_m}\left(\sum_{i_2=1}^{n}\left(\sum_{i_1=1}^{n}|c_i|\right)^2\right)^{\frac{1}{m+1}}\left(\sum_{i_1,i_2=1}^{n}|c_i|^2\right)^{\frac{m-2}{m-1}}\left(\sum_{i_2=1}^{n}\left(\sum_{i_1=1}^{n}|c_i|^2\right)^{\frac{1}{2}}\right)^{\frac{2}{m+1}},$$

and if we apply Minkowski's inequality to the first factor, we obtain

$$\leq \sum_{i_3,\dots,i_m}\left(\sum_{i_1=1}^{n}\left(\sum_{i_2=1}^{n}|c_i|^2\right)^{\frac{1}{2}}\right)^{\frac{2}{m+1}}\left(\sum_{i_2=1}^{n}\left(\sum_{i_1=1}^{n}|c_i|^2\right)^{\frac{1}{2}}\right)^{\frac{2}{m+1}}\left(\sum_{i_1,i_2=1}^{n}|c_i|^2\right)^{\frac{m-2}{m-1}}.$$

Step 3. We turn to i_3 and choose

$$a_1(i_3) = \left(\sum_{i_1=1}^{n}\left(\sum_{i_2=1}^{n}|c_i|^2\right)^{\frac{1}{2}}\right)^{\frac{2}{m+1}},$$

$$a_2(i_3) = \left(\sum_{i_2=1}^{n}\left(\sum_{i_1=1}^{n}|c_i|^2\right)^{\frac{1}{2}}\right)^{\frac{2}{m+1}},$$

$$a_3(i_3) = a_4(i_3) = \cdots = a_m(i_3) = \left(\sum_{i_1,i_2=1}^{n}|c_i|^2\right)^{\frac{1}{m-1}}.$$

Another application of Hölder's inequality and two Minkowski's inequalities to the two first factors yield

$$\leq \sum_{i_4,\dots,i_m}\left(\sum_{i_1=1}^{n}\left(\sum_{i_2,i_3=1}^{n}|c_i|^2\right)^{\frac{1}{2}}\right)^{\frac{2}{m+1}}\left(\sum_{i_2=1}^{n}\left(\sum_{i_1,i_3=1}^{n}|c_i|^2\right)^{\frac{1}{2}}\right)^{\frac{2}{m+1}}\cdots$$
$$\cdots\left(\sum_{i_3=1}^{n}\left(\sum_{i_1,i_2=1}^{n}|c_i|^2\right)^{\frac{1}{2}}\right)^{\frac{2}{m+1}}\left(\sum_{i_1,i_2,i_3=1}^{n}|c_i|^2\right)^{\frac{m-3}{m+1}}.$$

Step k. We continue to consider the $2/(m+1)$-factors by themselves and apply Minkowski's inequality to these $k-1$ factors. The final term $2/(m+1)$-term always comes from the application of Hölder's inequality. We continue in this way until we have used up all the identical terms, that is until $k = m$. Then

$$\leq \left(\sum_{i_1=1}^{n}\left(\sum_{i^1\in M(m-1,n)}|c_i|^2\right)^{\frac{1}{2}}\right)^{\frac{2}{m+1}}\left(\sum_{i_2=1}^{n}\left(\sum_{i^2\in M(m-1,n)}|c_i|^2\right)^{\frac{1}{2}}\right)^{\frac{2}{m+1}}\cdots$$
$$\cdots\left(\sum_{i_m=1}^{n}\left(\sum_{i^m\in M(m-1,n)}|c_i|^2\right)^{\frac{1}{2}}\right)^{\frac{2}{m+1}} = \prod_{k=1}^{m}\left[\sum_{i_k=1}^{n}\left(\sum_{i^k\in M(m-1,n)}|c_i|^2\right)^{\frac{1}{2}}\right]^{\frac{2}{m+1}}.$$

The proof is completed by taking $2m/(m+1)$-roots on both sides. $\square$

The geometric upper bound provided by Blei's inequality is very useful, as we shall see in the following section.

3.6. Bohnenblust–Hille Inequalities

We are finally ready to prove the Bohnenblust-Hille inequality for homogenous polynomials. This section shows why multilinear forms are necessary to obtain the results we seek for polynomials. Linearity is essential in the proof. We begin by considering a homogenous polynomial, then the corresponding symmetric multilinear and finally back to the homogenous polynomial by polarization.

THEOREM 3.16. *Let P be a m-homogenous polynomial on $\mathbb{C}^n$. Then there exists some positive constant D_m such that*

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}} \leq D_m \sup_{z\in\mathbb{D}^n} |P(z)|,$$

and the exponent $2m/(m+1)$ is optimal in the sense that any smaller exponent ρ will make D_m depend on n, and

$$D_m \leq \sqrt{m}\left(1+\frac{1}{m-1}\right)^{m-1}\sqrt{2}^{m-1}.$$

PROOF. The optimality of $2m/(m+1)$ was already decided by Theorem 3.14. First we observe that for any $i \in M(m,n)$ and any k we obtain

$$\frac{\Upsilon_i}{\Upsilon_{i^k}} = \frac{m}{\alpha_k(i)!} \leq m. \tag{3.17}$$

Let $\mathscr{P}$ denote the left side of the inequality. By the fact that $1 \leq \Upsilon_i$ we get

$$\mathscr{P}^{\frac{2m}{m+1}} = \sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}} = \sum_{i\in M(m,n)} \frac{1}{\Upsilon_i}|\Upsilon_i a_i|^{\frac{2m}{m+1}} \leq \sum_{i\in M(m,n)} \left|\sqrt{\Upsilon_i} a_i\right|^{\frac{2m}{m+1}}.$$

By taking $(m+1)/2m$-roots and then applying Blei's inequality

$$\mathscr{P} \leq \left(\sum_{i\in M(m,n)} \left|\sqrt{\Upsilon_i} a_i\right|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}} \leq \prod_{k=1}^{m}\left[\sum_{i_k=1}^{n}\left(\sum_{i^k\in M(m-1,n)} \Upsilon_i |a_i|^2\right)^{\frac{1}{2}}\right]^{\frac{1}{m}}$$

and an application of (3.17), where we proceed to factor out the m,

$$\leq \sqrt{m}\prod_{k=1}^{m}\left[\sum_{i_k=1}^{n}\left(\sum_{i^k\in M(m-1,n)} \Upsilon_{i^k} |a_i|^2\right)^{\frac{1}{2}}\right]^{\frac{1}{m}}.$$

We let $a_{\beta_{i_k}}$ be the coefficients of $P_{i_k}(z) = B(z, \dots, z, e^{(i_k)}, z, \dots, z)$, where $e^{(i_k)}$ is inserted in the k'th argument. We apply Lemma 3.7 to obtain

$$= \sqrt{m} \prod_{k=1}^{m} \left[\sum_{i_k=1}^{n} \left(\sum_{|\beta_{i_k}|=m-1} |a_{\beta_{i_k}}|^2 \right)^{\frac{1}{2}} \right]^{\frac{1}{m}} \leq \sqrt{m} \prod_{k=1}^{m} \left[\sum_{i_k=1}^{n} \sqrt{2}^{m-1} \|P_{i_k}\|_1 \right]^{\frac{1}{m}}.$$

We see that we are done if, for any k, we can prove

$$\sum_{i_k=1}^{n} \|P_{i_k}\|_1 \leq \left(1 + \frac{1}{m-1}\right)^{m-1} \|P\|_\infty.$$

By choosing a unimodular function $\lambda_{i_k}(z)$ we can write $\|P_{i_k}\|_1$ as

$$\int_{\mathbb{T}^n} \left|B(z, \dots, e^{(i_k)}, \dots, z)\right| d\mu^n(z) = \int_{\mathbb{T}^n} B(z, \dots, \lambda_{i_k}(z) e^{(i_k)}, \dots, z)\, d\mu^n(z)$$

We add these up to obtain

$$\tau_k(z) = \sum_{i_k=1}^{n} \lambda_{i_k}(z) e^{(i_k)},$$

which assumes values on $\mathbb{T}^n$. Now, by Lemma 3.11 and the maximum modulus principle we obtain

$$\begin{aligned} \sum_{i_k=1}^{n} \|P_{i_k}\|_1 &= \int_{\mathbb{T}^n} B(z, \dots, \tau_k(z), \dots, z)\, d\mu^n(z) \leq \sup_{z \in \mathbb{T}^n} B(z, \dots, \tau_k(z), \dots, z) \\ &\leq \sup_{z, w \in \mathbb{D}^n} |B(z, \dots, z, w, z, \dots, z)| \leq \left(1 + \frac{1}{m-1}\right)^{m-1} \|P\|_\infty, \end{aligned}$$

which completes the proof. □

Remark. As we have stated earlier, this inequality is *hypercontractive*. This means that the coefficient D_m does not grow faster than exponential in m. In fact, we may obtain

$$D_m \leq \sqrt{m} \left(1 + \frac{1}{m-1}\right)^{m-1} \sqrt{2}^{m-1} \leq e^m,$$

for $m \geq 2$.

In our application of the Bohnenblust–Hille inequality, we are mainly interested in the version for homogenous polynomials. However, the inequality for multilinear forms are within reach, and we will provide the proof for completeness.

THEOREM 3.17. *Let B be a m-linear form on $\mathbb{C}^n$. Then there exists some positive constant C_m such that*

$$\left(\sum_{i \in M(m,n)} |a_i|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}} \leq C_m \sup_{z^{(k)} \in \mathbb{D}^n} \left|B\big(z^{(1)}, z^{(2)}, \ldots, z^{(m)}\big)\right|,$$

and the exponent $2m/(m+1)$ is optimal in the sense that any smaller exponent ρ will make C_m depend on n, and $C_m \leq \sqrt{2}^{m-1}$.

PROOF. The optimality of $2m/(m+1)$ was already decided by Theorem 3.14. We can proceed almost as in Theorem 3.16: We can immediately apply Blei's inequality, to obtain

$$\left(\sum_{i \in M(m,n)} |a_i|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}} \leq \prod_{k=1}^{m} \left[\sum_{i_k=1}^{n} \left(\sum_{i^k \in M(m-1,n)} |a_i|^2\right)^{\frac{1}{2}}\right]^{\frac{1}{m}}.$$

We use Lemma 3.8 identically to how we used Lemma 3.7 in the proof of the previous theorem; now to the $(m-1)$-linear form $B_{i_k} = B\big(z^{(1)}, \ldots, e^{(i_k)}, \ldots, z^{(m)}\big)$ to obtain

$$\leq \sqrt{2}^{m-1} \prod_{k=1}^{m} \left[\sum_{i_k=1}^{n} \|B_{i_k}\|_1\right]^{\frac{1}{m}}.$$

We choose a unimodular λ_{i_k}, and since we need not apply polarization, hence the factor

$$\left(1 + \frac{1}{m-1}\right)^{m-1}$$

does not appear. The rest of the proof is identical to that of Theorem 3.16. □

REMARK. The proofs we presented are actually very similar to the one presented by Bohnenblust–Hille. They did not have Blei's inequality and used weaker "power mean value"-inequality essentially due to Littlewood. This introduced a factor $m^{1/\rho}$. Their version of the Khinchine–Type inequality yielded

$$\sqrt{3}^{m-1}.$$

The main improvement of their argument comes from Lemma 3.11, and the specialized argument for Theorem 3.16 [**14**], which provides a hypercontractive inequality for homogenous polynomials. In the original proof, a factor of

$$\frac{m^m}{\sqrt{m!}} \leq m^{m/2},$$

appeared in D_m, which is not sufficient for our applications.

Let us for a moment turn our attention to general polynomials in n variables of degree m, say

$$P(z) = P(z_1, z_2, \ldots, z_n) = \sum_{|\alpha| \leq m} a_\alpha z^\alpha.$$

Does the Bohnenblust–Hille inequality hold for these polynomials? This is easily obtained as a corollary of the Bohnenblust–Hille inequality for homogenous polynomials using a simple homogenization process and the maximum modulus principle.

LEMMA 3.18. *Given any m'th degree polynomial $P(z)$ in n variables, there exists some m-homogenous polynomial $Q(z)$ in $n+1$ variables with the same coefficients and the same supremum in $\mathbb{D}^{n+1}$.*

PROOF. Let us define

$$Q(z, w) = w^m P\left(\frac{z_1}{w}, \frac{z_2}{w}, \ldots, \frac{z_n}{w}\right).$$

The computation

$$Q(z, w) = w^m \sum_{|\alpha| \leq m} a_\alpha \frac{z^\alpha}{w^{|\alpha|}} = \sum_{|\alpha| \leq m} a_\alpha w^{m-|\alpha|} z^\alpha$$

shows that $Q(z, w)$ is m-homogenous in $n+1$ variables with the same coefficients as $P(z)$. To obtain the supremum, we only need to consider $\mathbb{T}^n$ and $\mathbb{T}^{n+1}$ in view of the maximum modulus principle. Clearly,

$$Q(z, 1) = P(z)$$

and hence $\|P\|_\infty \leq \|Q\|_\infty$. Suppose that $Q(z, w)$ attains is maximal modulus at (z, w). Clearly, since $|w| = 1$, we have

$$|Q(z, w)| = \left|P\left(\frac{z_1}{w}, \frac{z_2}{w}, \ldots, \frac{z_n}{w}\right)\right|,$$

and thus $\|Q\|_\infty \leq \|P\|_\infty$. □

COROLLARY 3.19. *The Bohnenblust–Hille inequality of Theorem 3.16 holds for any m'th degree polynomial $P(z)$, that is*

$$\left(\sum_{|\alpha| \leq m} |a_\alpha|^\rho\right)^{\frac{1}{\rho}} \leq D_m \sup_{z \in \mathbb{D}^n} |P(z)|.$$

PROOF. This follows directly from Theorem 3.16 and Lemma 3.18. □

3.7. A Real Bohnenblust–Hille Inequality

Let us furthermore consider real homogenous polynomials,

$$P(x) = P(x_1, x_2, \ldots, x_n) = \sum_{|\alpha|=m} a_\alpha x^\alpha, \qquad a_\alpha \in \mathbb{R}.$$

We want to prove a Bohnenblust–Hille inequality for these polynomials, but we restrict the supremum to $[-1,1]^n$. The following Chebyshev type inequality will be useful **[39]**.

Lemma 3.20. *Let $P(x)$ be an m-homogenous polynomial $P(x)$ in n variables. Then*

$$\sup_{x\in[-1,1]^n} |P(x)| \geq \frac{1}{2^{m-1}} \sup_{z\in\mathbb{D}^n} |P(z)|.$$

Proof. Let $t \in [0,2\pi]^n$ and introduce

$$\mathscr{P}(t) = P(\cos(t_1), \ldots, \cos(t_n)) = \frac{P\left(e^{it_1}, \ldots, e^{it_n}\right) + P\left(e^{-it_1}, \ldots, e^{-it_n}\right)}{2^m}.$$

There is some $t_0 \in [0,2\pi]^n$ such that $P\left(e^{it_1}, \ldots, e^{it_n}\right)$ attains its maximal modulus and is positive at t_0: Take any maximal point and rotate each coordinate with θ/m for some suitable θ to obtain positivity. Clearly, by the fact that

$$P\left(e^{-it_1}, \ldots, e^{-it_n}\right) = \overline{P\left(e^{it_1}, \ldots, e^{it_n}\right)},$$

this is also of maximal modulus and positive at t_0. Thus,

$$\sup_{x\in[-1,1]^n} |P(x)| = \sup_{t\in[0,2\pi]^n} |\mathscr{P}(t)| \geq \frac{1}{2^{m-1}} \sup_{z\in\mathbb{T}^n} |P(z)| = \frac{1}{2^{m-1}} \sup_{z\in\mathbb{D}^n} |P(z)|,$$

by the maximum modulus principle. □

Combining this with Theorem 3.16 yields a hypercontractive Bohnenblust–Hille inequality for real polynomials. However, more can be done **[12]**.

Corollary 3.21. *Suppose that $P(x)$ is a real m-homogenous polynomial in n variables. Then*

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}} \leq E_m \sup_{x\in[-1,1]^n} |P(x)|,$$

the exponent $2m/(m+1)$ is optimal. Furthermore, hypercontractivity is both necessary and sufficient.

PROOF. Optimality of the exponent $2m/(m+1)$ is provided by the Rudin–Shapiro polynomials, and $E_m \leq (2e)^m/2$ follows from Theorem 3.16 and Lemma 3.20. The simple polynomial $P(x_1, x_2) = x_1^2 + x_1x_2 - x_2^2$ is absolutely bounded by $5/4$ in $[-1,1]^2$. Consider therefore

$$Q(x) = Q(x_1, x_2, \ldots, x_m) = P(x_1, x_2)P(x_3, x_4)\cdots P(x_{m-1}, x_m),$$

for any even number m. This is clearly m-homogenous, and furthermore

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}} = \left(3^{\frac{m}{2}}\right)^{\frac{m+1}{2m}} \qquad \text{and} \qquad \sup_{x\in[-1,1]^m} |Q(x)| = \left(\frac{5}{4}\right)^{\frac{m}{2}}.$$

This immediately implies that hypercontractivity is necessary since it demands

$$\sqrt[4]{3}\left(\frac{2\sqrt[4]{3}}{\sqrt{5}}\right)^m \leq E_m,$$

and $2\sqrt[4]{3} > \sqrt{5}$. □

Can we do something similar with the complex Bohnenblust–Hille inequality to prove that hypercontractivity is optimal? Using the maximum modulus principle, we may easily compute

$$\sup_{z\in\mathbb{D}^2} \left|z_1^2 + z_1z_2 - z_2^2\right| = \sup_{z\in\mathbb{T}^2} |z_1\overline{z_2} + 1 - \overline{z_1}z_2| = \sup_{z\in\mathbb{T}^2} |1 + 2i\Im\left(z_1\overline{z_2}\right)| = \sqrt{5},$$

and hence an identical computation to that of Corollary 3.21 may not be used in the complex case, since $\sqrt{5} > \sqrt{3}$. In fact, any similar construction to that of Corollary 3.21 will fail for complex polynomials: Suppose that $P(z)$ is any polynomial of degree d with r non-zero coefficients. Let

$$Q\big(z^{(1)}, z^{(2)}, \ldots, z^{(k)}\big) = P\big(z^{(1)}\big)P\big(z^{(2)}\big)\cdots P\big(z^{(k)}\big).$$

Here $m = kd$ and Q has r^k non-zero coefficients. We write $a_\alpha = a_\alpha \cdot 1$ and use Hölder's inequality to obtain

$$\left(\sum_{|\alpha|\leq m} |a_\alpha|^\rho\right)^{\frac{1}{\rho}} \leq \left(\sum_{|\alpha|\leq m} |a_\alpha|^2\right)^{\frac{1}{2}} \left(\sum_{a_\alpha\neq 0} 1\right)^{\frac{1}{\rho}\left(1-\frac{\rho}{2}\right)} \leq \left(\sum_{|\alpha|\leq m} |a_\alpha|^2\right)^{\frac{1}{2}} r^{\frac{1}{2d}}.$$

Furthermore, using (3.6) that the measures are normalized yields

$$\frac{1}{\|Q\|_\infty}\left(\sum_{|\alpha|\leq m} |a_\alpha|^\rho\right)^{\frac{1}{\rho}} \leq \frac{1}{\|Q\|_2}\left(\sum_{|\alpha|\leq m} |a_\alpha|^2\right)^{\frac{1}{2}} r^{\frac{1}{2d}} = r^{\frac{1}{2d}}.$$

and hence the construction would yield at best $r^{\frac{1}{2d}} \leq E_m$. Hypercontractivity of the complex inequality must be decided by some other means.

CHAPTER 4

Estimating the Sidon Constant

In Chapter 1 we defined the Sidon constant

$$S(N) = \sup_{\{a_n\}\neq 0} \frac{\|\widehat{f}\|_1}{\|f\|_\infty},$$

where we considered Dirichlet polynomials of the type

$$f(s) = \sum_{n=1}^{N} \frac{a_n}{n^s}.$$

We also stated the asymptotic formula for $S(N)$, which is our main theorem.

Theorem 4.1. *We have*

$$S(N) = \sqrt{N} \exp\left(\left(-\frac{1}{\sqrt{2}} + s(N)\right)\sqrt{\log N \log\log N}\right),$$

as $N \to \infty$. Furthermore, the $s(N)$-term satisfies

$$-\frac{1}{2}\frac{\log\log\log N}{\log\log N} + \mathcal{O}\left(\frac{1}{\log\log N}\right) \leq s(N) \leq \frac{3}{\sqrt{2}}\frac{\log\log\log N}{\log\log N} + \mathcal{O}\left(\frac{1}{\log\log N}\right).$$

In Chapter 2 we estimated the number of y-smooth numbers less than x and in Lemma 2.18 we obtained the effective estimate

$$\Psi(x,y) = x\exp\left(-u\left(\log u + \log\log u + \mathcal{O}(1)\right)\right), \tag{4.1}$$

which is valid for $x \geq x_0(\epsilon)$ and $\exp\left((\log\log x)^{5/3+\epsilon}\right) \leq y \leq x$. We shall also need the main result of Chapter 3; namely the hypercontractive Bohnenblust–Hille inequality for homogenous polynomials of Theorem 3.16,

$$\left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}} \leq e^m \sup_{z\in\mathbb{D}^n} |P(z)|, \tag{4.2}$$

To prove Theorem 4.1 we will combine these results with three new results. Rankin's Trick, Bohr's Correspondence and the Salem–Zygmund inequality.

4.1. Euler Products and Rankin's Trick

In this section, we study absolute convergent series of completely multiplicative functions. We obtain the Euler product representation of Dirichlet series and Rankin's trick, which allows us to estimate the cardinality of a certain set of rough numbers.

LEMMA 4.2. *Suppose that $b : \mathbb{N} \to \mathbb{C}$ is a completely multiplicative function such that the series $\sum b(n)$ is absolutely convergent. Then the series can be represented as a product over the prime numbers,*

$$\sum_{n=1}^{\infty} b(n) = \prod_{p} \frac{1}{1-b(p)}.$$

PROOF. Since $\sum b(n)$ is absolutely convergent, clearly

$$\sum_{k=1}^{\infty} \left|b\left(p^k\right)\right| = \sum_{k=1}^{\infty} |b(p)|^k = \frac{1}{1-|b(p)|},$$

by the fact that b is completely multiplicative. Hence $|b(p)| \leq \delta < 1$ for all primes p. Exploiting this fact, we may compute the finite product

$$P(y) = \prod_{p\leq y} \frac{1}{1-b(p)} = \prod_{p\leq y} \sum_{k=1}^{\infty} b(p)^k = \prod_{p\leq y} \left(1 + b(p) + b\left(p^2\right) + b\left(p^3\right) + \cdots\right).$$

Since the product is finite and the geometric series are absolutely convergent, we may rearrange the terms as we see fit. A general term in the expansion of $P(y)$ is of the form

$$b\left(p_1^{\alpha_1}\right) b\left(p_2^{\alpha_2}\right) \cdots b\left(p_m^{\alpha_m}\right) = b\left(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}\right) = b(n),$$

where n is y-smooth. Hence, for $B = \{n \in \mathbb{N} : n \text{ is } y\text{-smooth}\}$, we have

$$P(y) = \sum_{n\in B} b(n).$$

Now, let $\epsilon > 0$ be arbitrary. We may estimate

$$\left|\sum_{n=1}^{\infty} b(n) - P(y)\right| = \left|\sum_{n=1}^{\infty} b(n) - \sum_{n\in B} b(n)\right| \leq \sum_{n\in\mathbb{N}\setminus B} |b(n)| \leq \sum_{n>y} |b(n)| < \epsilon,$$

for $y \geq y_0(\epsilon)$ by the fact that $\sum b(n)$ is absolutely convergent. This completes the proof. ☐

We may immediately apply Lemma 4.2 to obtain the familiar product representation of Dirichlet series defined by completely multiplicative arithmetic functions.

THEOREM 4.3 (Euler Product Representation). *Suppose that $a : \mathbb{N} \to \mathbb{C}$ is completely multiplicative and defines the Dirichlet series*

$$f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}.$$

If $\sigma_a < \infty$, $f(s)$ can be represented as a product over the prime numbers:

$$f(s) = \prod_p \left(1 - \frac{a(p)}{p^s}\right)^{-1},$$

valid in the half plane $\Re(s) > \sigma_a$.

PROOF. Apply Lemma 4.2 with $b(n) = a(n)/n^s$ for $\Re(s) > \sigma_a$. $\square$

The crucial observation in the proof of Lemma 4.2 is the fact that

$$P(y) = \prod_{p \leq y} \frac{1}{1 - b(p)} = \sum_{n \in B} b(n),$$

where B is the set of all y-smooth numbers. Now, suppose that $b(n) \geq 0$ and $A \subseteq B$. Clearly we have the inequality

$$\sum_{n \in A} b(n) \leq \sum_{n \in B} b(n) = P(y), \tag{4.3}$$

which will serve as an inspiration for the next result. For $N \subseteq \mathbb{N}$ the *prime divisor set* of N is defined as

$$P(N) = \{p \text{ prime} : \text{there is some } n \in N \text{ such that } p|n\}.$$

We are now ready to prove a generalization of (4.3).

LEMMA 4.4 (Rankin's Trick). *Let $N \subseteq \mathbb{N}$ be a set and suppose that $b(n)$ is a completely multiplicative non-negative function such that $b(p) \leq \delta < 1$ for all $p \in P(N)$. Then*

$$\sum_{n \in N} b(n) \leq \prod_{p \in P(N)} \frac{1}{1 - b(p)}.$$

PROOF. By similar considerations as in the proof of Lemma 4.2 we obtain

$$\prod_{p \in P(N)} \frac{1}{1 - b(p)} = \sum_{n \in B} b(n),$$

where B is the set of all numbers of the form

$$n = \prod_{p \in P(N)} p^{\alpha_p}.$$

By the definition of $P(N)$, we have that $N \subseteq B$, since any $n \in N$ is of the required form. The proof is completed by the fact that $b(n) \geq 0$ like in (4.3). $\square$

In CHAPTER 2 we studied y-smooth numbers. Let us define a related property.

DEFINITION. Given any positive real number y, we say that the integer n is *y-rough* if all the prime factors of n are strictly larger than y. For $x \geq y \geq 0$ we consider the quantity

$$T(x,y) = \{n \leq x \,:\, n \text{ is } y\text{-rough}\}.$$

We shall require some estimates on y-rough numbers, but we are also interested in y-rough numbers with a specified number of divisors. Let us furthermore introduce the quantities

$$T(x,y,m) = \{n \in T(x,y) \,:\, \Omega(n) = m\}, \tag{4.4}$$

$$N(x,y,M) = \sum_{m \geq M} |T(x,y,m)|, \tag{4.5}$$

where $\Omega(n)$ denotes the number of prime divisor of n, counting multiplicity. We want to apply Rankin's trick to estimate $N(x,y,M)$. The following result is a slightly weaker version of an inequality in [**2**], which will not improve our estimates.

LEMMA 4.5. *Let $x \geq y \geq 2$ and $M \geq 1$. Then*

$$N(x,y,M) \leq \frac{x}{(y/2)^M} (\log x)^y \exp\left(\mathcal{O}(y)\right). \tag{4.6}$$

PROOF. We begin by proving an auxiliary inequality: For $0 \leq t \leq 1/2$ we have

$$e^{-2t} \leq 1 - 2t + \frac{(-2t)^2}{2!} = 1 - t + (2t^2 - t) \leq 1 - t, \tag{4.7}$$

by Taylor's theorem and the fact that $2t^2 - t \leq 0$ in the range $0 \leq t \leq 1/2$. Now, to prove (4.6) we need to force a completely multiplicative function to appear. Let $c = y/2 \geq 1$. Using (4.5) and the fact that $n \leq x$,

$$N(x,y,M) = \sum_{m \geq M} \sum_{n \in T(x,y,m)} 1 \leq \frac{x}{c^M} \sum_{m \geq M} \sum_{n \in T(x,y,m)} \frac{c^{\Omega(n)}}{n} \leq \frac{x}{c^M} \sum_{n \in T(x,y)} \frac{c^{\Omega(n)}}{n}.$$

We may take $b(n) = c^{\Omega(n)}/n$, which satisfies $b(p) = y/(2p) < 1/2$ for any $p \in P(T(x,y))$, since $\Omega(p) = 1$. Hence, by Rankin's trick and the inverse of (4.7),

$$N(x,y,M) \leq \frac{x}{(y/2)^M} \prod_{y < p \leq x} \left(1 - \frac{y}{2p}\right)^{-1} \leq \frac{x}{(y/2)^M} \exp\left(y \sum_{y < p \leq x} \frac{1}{p}\right).$$

By Mertens's estimate of Lemma B.3 we obtain

$$\sum_{y < p \leq x} \frac{1}{p} \leq \sum_{p \leq x} \frac{1}{p} = \log\log x + \mathcal{O}(1),$$

which completes the proof. $\square$

4.2. Bohr's Correspondence

In this section, we introduce a new way to view Dirichlet polynomials, which is due to Bohr [**8**]. Fix N and consider the Dirichlet polynomial

$$f(s) = \sum_{n=1}^{N} \frac{a_n}{n^s}. \tag{4.8}$$

We want to introduce a new way to study (4.8), to simplify the computation of

$$\|f\|_\infty = \sup_{t\in\mathbb{R}} |f(it)| = \sup_{t\in\mathbb{R}} \left|\sum_{n=1}^{N} \frac{a_n}{n^{it}}\right|.$$

Our first attempt may be obtained by noticing that

$$\frac{1}{n^{it}} = e^{-it\log n},$$

which is of unit modulus. Hence, we could study the related polynomial

$$G(z) = \sum_{n=1}^{N} a_n z_n,$$

where we apply N independent variables, $z = (z_1, z_2, \ldots, z_N)$, and study the supremum on the torus $\mathbb{T}^N$. Each natural number is translated into a variable. However, if we take $z_n = \overline{a_n}/|a_n|$, it is clear that

$$\sup_{z\in\mathbb{T}^N} |G(z)| = \sum_{n=1}^{N} |a_n| = \|\widehat{f}\|_1,$$

which will not work. In some sense, we have introduced too many variables. However, if we can find a way to reduce the number of variables, we may succeed. The fundamental theorem of arithmetic allows us to uniquely factor any integer into prime factors

$$n = \prod_{k=1}^{\pi(n)} p_k^{\alpha_k}. \tag{4.9}$$

This allows us to write

$$n^{-it} = \prod_{k=1}^{\pi(n)} p_k^{-i\alpha_k t} = \prod_{k=1}^{\pi(n)} e^{-it\alpha_k \log p}.$$

If we now translate each prime number into a variable, we will have at most $\pi(N)$ variables in the corresponding polynomial. The factorization (4.9) allows us to bijectively associate each integer to a finite multi-index

$$n \longleftrightarrow \alpha(n) = \big(\alpha_1, \alpha_2, \ldots, \alpha_{\pi(n)}\big)\,, \tag{4.10}$$

and hence obtain Bohr's correspondence

$$f(s) = \sum_{n=1}^{N} \frac{a_n}{n^s} \longleftrightarrow F(z) = \sum_{n=1}^{N} a_n z^{\alpha(n)}, \tag{4.11}$$

which yields a polynomial of at most $\pi(N)$ variables. The main result of this section is to prove that the correspondence (4.11) indeed provides the correct supremum on the torus. To obtain this, we employ the essential supremum:

DEFINITION. Suppose $f : X \to \mathbb{R}$ is a measurable function. We define the *essential supremum* as

$$\text{ess sup} f = \inf \{a \in \mathbb{R} : \mu\{x \in X : f(x) > a\} = 0\}.$$

If f is continuous, the essential supremum is equal to the "ordinary" supremum.

LEMMA 4.6. *Let (X, Σ, μ) be a finite measure space, and suppose $f : X \to \mathbb{C}$ is essentially bounded and integrable. Then*

$$\lim_{p\to\infty} \|f\|_p = \|f\|_\infty = \text{ess sup} |f|.$$

PROOF. Let $\epsilon > 0$. Since f is essentially bounded and integrable, $\|f\|_p$ exists. Furthermore, the set

$$A_\epsilon = \{x \in X : |f(x)| \geq (1-\epsilon)\|f\|_\infty\}$$

has measure $\mu(A_\epsilon) > 0$. Clearly,

$$\|f\|_\infty (1-\epsilon)\mu(A_\epsilon)^{\frac{1}{p}} \leq \left(\int_X |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}} \leq \|f\|_\infty.$$

If we let $p \to \infty$ and then $\epsilon \to 0$, we are done. □

Lemma 4.6 is applicable for polynomials on the torus $\mathbb{T}^k$, since they are continuous and the measures are normalized.

LEMMA 4.7. *Suppose the Dirichlet polynomial $f(s)$ corresponds to the k-variable polynomial $F(z)$ according to (4.11). Then, for any $q \in \mathbb{N}$, we have*

$$\int_{\mathbb{T}^k} |F(z)|^{2q} \, d\mu^k(z) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(it)|^{2q} \, dt. \tag{4.12}$$

PROOF. We apply the fact that $|w|^2 = w\overline{w}$ and then (4.12) follows from (3.2) and (1.17), that is

$$\int_{\mathbb{T}^k} z^{\alpha(n)} \overline{z}^{\alpha(m)} \, d\mu^k(z) = \delta_{mn} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left(\frac{m}{n}\right)^{it} dt,$$

since $z^{\alpha(r)} \cdot z^{\alpha(s)} = z^{\alpha(rs)}$ by (4.10). □

THEOREM 4.8. *Suppose $F(z)$ is the k-variable polynomial corresponding to the Dirichlet polynomial $f(s)$, according to* (4.11). *Then $\|F\|_\infty = \|f\|_\infty$.*

PROOF. By Lemma 4.7 we have

$$\|F\|_{2q} = \left(\int_{\mathbb{T}^k} |F(z)|^{2q} \, d\mu^k(z) \right)^{\frac{1}{2q}} = \left(\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(it)|^{2q} \, dt \right)^{\frac{1}{2q}} \leq \|f\|_\infty.$$

If we take $p = 2q$ and let $q \to \infty$, and apply Lemma 4.6 we obtain

$$\|F\|_\infty = \lim_{q \to \infty} \|F\|_{2q} \leq \|f\|_\infty.$$

The other direction is by comparison trivial, since the supremum clearly is taken over a bigger set

$$\|f\|_\infty = \sup_{t \in \mathbb{R}} \left| \sum_{n=1}^{N} a_n e^{-it \log n} \right| \leq \sup_{z \in \mathbb{T}^k} \left| \sum_{n=1}^{N} a_n z^{\alpha(n)} \right| = \|F\|_\infty,$$

which completes the proof. □

In our preliminary efforts, we observed that if we used "too many" variables in our polynomial representation, $\|f\|_\infty$ would become too big. To provide a lower bound for $S(N)$, we want to maximize the ratio $\|\widehat{f}\|_1 / \|f\|_\infty$. Hence it is natural to control the number of variables, say $k = \pi(y)$ for some $y \geq 2$. Let us obtain our first application of Bohr's correspondence.

LEMMA 4.9. *Fix some $x \geq y \geq 2$ and write $n = \kappa\lambda$ where κ is y-smooth and λ is y-rough. When splitting the Dirichlet polynomial as*

$$f(s) = \sum_{n \leq x} \frac{a_n}{n^s} = \sum_{\kappa} \left(\sum_{\lambda} \frac{a_n}{\lambda^s} \right) \frac{1}{\kappa^s} = \sum_{\kappa} \frac{f_\kappa(s)}{\kappa^s},$$

we have $\|f_\kappa\|_\infty \leq \|f\|_\infty$.

PROOF. We move to the polydisk using Bohr's correspondence, so let F_κ correspond to f_κ. The restriction to y-smooth numbers allows us to control the number of variables in F_κ. Let us write

$$z = (z_1, z_2) = \left((z_1, \ldots, z_{\pi(y)}), \, (z_{\pi(y)+1}, \ldots, z_{\pi(x)}) \right).$$

Indeed, this allows the decomposition

$$F(z) = F(z_1, z_2) = \sum_{\kappa} F_\kappa(z_2) z_1^{\alpha(\kappa)},$$

and using orthogonality we obtain

$$F_\kappa(z_2) = \int_{\mathbb{T}^{\pi(y)}} F(z_1, z_2) z_1^{-\alpha} \, d\mu^{\pi(y)}(z_1).$$

By taking absolute values and taking supremum over z_1, we obtain

$$|F_\kappa(z_2)| \leq \sup_{z_1} |F(z_1, z_2)| .$$

Hence we obtain $\|F_\kappa\|_\infty \leq \|F\|_\infty$, by taking the supremum over z_2. This implies $\|f_\kappa\|_\infty \leq \|f\|_\infty$ by Theorem 4.8. □

In view (4.9), we may write the prime divisor counting function

$$\Omega(n) = \sum_{k=1}^{\pi(n)} \alpha_k(n) = |\alpha(n)|.$$

The Bohnenblust–Hille inequality is only valid for homogenous polynomials. Let us introduce the m-homogenous Dirichlet polynomials, which can be written as

$$f(s) = \sum_{\Omega(n)=m} \frac{a_n}{n^s}.$$

The following lemma splits a Dirichlet polynomial into parts of the same number of prime divisors.

Lemma 4.10. *Consider the Dirichlet polynomial*

$$f(s) = \sum_{n \leq x} \frac{a_n}{n^s} = \sum_m \sum_{\Omega(n)=m} \frac{a_n}{n^s} = \sum_m f_m(s).$$

Then $\|f_m\|_\infty \leq \|f\|_\infty$.

Proof. Due to Bohr's correspondence, we consider the Taylor polynomial

$$F(z) = \sum_{n \leq x} a_n z^{\alpha(n)} = \sum_m \sum_{|\alpha|=m} a_\alpha z^\alpha = \sum_m F_m(z).$$

It is enough to prove $\|F_m\|_\infty \leq \|F\|_\infty$. Fix $z \in \mathbb{T}^k$, where $k = \pi(x)$, and introduce

$$\phi(\theta) = F\left(e^{i\theta} z_1, e^{i\theta} z_2, \ldots, e^{i\theta} z_k\right) = \sum_m \left(e^{i\theta}\right)^m \sum_{|\alpha|=m} a_\alpha z^\alpha = \sum_m e^{im\theta} F_m(z).$$

The homogenous polynomials $F_m(z)$ appear as the Fourier coefficients of the function $\phi(\theta)$. Thus, for any $z \in \mathbb{T}^k$,

$$|F_m(z)| = \left| \frac{1}{2\pi} \int_0^{2\pi} \phi(\theta) e^{-im\theta}\, d\theta \right| \leq \sup_\theta |\phi(\theta)| \leq \|F\|_\infty,$$

by the definition of $\phi(\theta)$. □

4.3. The Salem–Zygmund Inequality

We introduce some elements of probability theory based on Kolmogorov's axiomatic development. We follow [**31**], but our exposition is limited to those concepts needed to prove the Salem–Zygmund inequality [**25**].

Definition. A *probability space* is a measure space $(\Omega, \mathcal{A}, \mathbb{P})$, with $\mathbb{P}(\Omega) = 1$. A measurable set $A \in \mathcal{A}$ is called an *event*, and $\mathbb{P}$ is called a *probability measure*. The probability of A occurring is $\mathbb{P}(A)$.

Definition. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. A function $X : \Omega \to \mathbb{R}$ is called a *random variable* if it is Borel measurable, that is

$$\{X \in \mathcal{B}\} = \{\omega \in \Omega \,:\, X(\omega) \in \mathcal{B}\} \in \mathcal{A},$$

where $\mathcal{B}$ denotes the set of Borel measurable sets on $\mathbb{R}$.

Using this definition, we can compute the probability that the random variable X is contained in some Borel set by $\mathbb{P}(X \in \mathcal{B})$, for example $\mathbb{P}(X \geq 0)$.

Definition. Let X be a random variable on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The *expectation* of X, is defined by

$$\mathbb{E}(X) = \int_\Omega X(\omega)\, d\mathbb{P}(\omega),$$

provided it exists. If $X \in L^1(\Omega, \mathcal{A}, \mathbb{P})$ we say that X has *finite expectation.*

Lemma 4.11 (Markov's Inequality). *Let X be a nonnegative random variable on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with $0 < \mathbb{E}(X) < \infty$. For any $\kappa > 1$ we have*

$$\mathbb{P}\left(X \geq \kappa \mathbb{E}(X)\right) \leq \frac{1}{\kappa}. \tag{4.13}$$

Proof. Let $\gamma > 0$ be arbitrary and define $\Gamma = \{\omega \in \Omega \,:\, X(\omega) \geq \gamma\}$. Then

$$\mathbb{P}(X \geq \gamma) = \int_\Gamma d\mathbb{P}(\omega) \leq \frac{1}{\gamma} \int_\Gamma X(\omega)\, d\mathbb{P}(\omega) \leq \frac{1}{\gamma} \int_\Omega X(\omega)\, d\mathbb{P}(\omega) = \frac{\mathbb{E}(X)}{\gamma},$$

which proves (4.13) by setting $\gamma = \kappa \mathbb{E}(X)$ and $\kappa > 1$. $\square$

We want to consider series of functions with random coefficients. Let us consider the *Rademacher probability space*: Let $\Omega = \{-1, 1\}^n$, $\mathcal{A} = \mathcal{P}(\Omega)$ and $\mathbb{P}$ be the counting measure scaled by $1/2^n$ to make it a probability measure. This is called the *uniform probability measure* on Ω. Let $E \neq \emptyset$ be a set, and consider a sequence of functions $f_k : E \to \mathbb{C}$. We will consider the following random variables:

$$f(t) = f_\omega(t) = \sum_{k=1}^{n} \omega_k f_k(t), \tag{4.14}$$

$$\|f\|_\infty = \|f_\omega\|_\infty = \sup_{t\in E} \left| \sum_{k=1}^{n} \omega_k f_k(t) \right|. \tag{4.15}$$

We suppress the dependence on ω in the notation, and remark that (4.14) is a random variable for each $t \in E$.

Lemma 4.12. *Let $f(t)$ be given by (4.14) with the additional demand that the f_k's are real valued. For each $\lambda \in \mathbb{R}$ and any $t \in E$, we have*

$$\mathbb{E}\left(e^{\lambda f(t)}\right) \le \exp\left(\frac{\lambda^2}{2}\sum_{k=1}^{n} \|f_k\|_\infty^2\right).$$

Proof. The following Taylor series are valid for any $x \in \mathbb{R}$:

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \qquad \text{and} \qquad \exp(x^2/2) = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n(n!)}.$$

Clearly $2^n(n!) \le (2n)!$, which proves the inequality $\cosh(x) \le \exp(x^2/2)$. The uniform probability implies that the expectation is a uniform mean value:

$$\begin{aligned}\mathbb{E}\left(e^{\lambda f(t)}\right) &= \frac{1}{2^n}\sum_{\omega\in\Omega} e^{\lambda f(t)} = \frac{1}{2^n}\prod_{k=1}^{n}\left(e^{\lambda f_k(t)} + e^{-\lambda f_k(t)}\right) = \prod_{k=1}^{n}\cosh(\lambda f_k(t)) \\ &\le \prod_{k=1}^{n}\exp\left(\frac{\lambda^2}{2}f_k^2(t)\right) \le \prod_{k=1}^{n}\exp\left(\frac{\lambda^2}{2}\|f_k\|_\infty^2\right) = \exp\left(\frac{\lambda^2}{2}\sum_{k=1}^{n}\|f_k\|_\infty^2\right)\end{aligned}$$

This is valid for any fixed $t \in E$ and hence we are done. ☐

Let us now turn to the functions f_k. We now demand that (E, Σ, μ) is a finite measure space, and consider functions of the following type.

Definition. Let (E, Σ, μ) be a finite measure space. A set of functions, $\mathscr{B}$, from E into $\mathbb{C}$ is called a *Salem–Zygmund space* over E if the following hold:

(1) The functions in $\mathscr{B}$ are bounded and measurable.
(2) The space $\mathscr{B}$ is linear over $\mathbb{C}$ and closed under complex conjugation.
(3) There is some constant $\rho > 0$ with the following property: For any real valued $f \in \mathscr{B}$ there is some measurable set $I = I(f) \subseteq E$ such that

$$|f(t)| \ge \frac{\|f\|_\infty}{2},$$

for any $t \in I$ and furthermore $\mu(I) \ge \mu(E)/\rho$.

THEOREM 4.13. *Let $\mathscr{B}$ be a Salem–Zygmund space over E with constant ρ. Assume that $f_k \in \mathscr{B}$, and consider the random variable $\|f\|_\infty$ as defined by (4.15). For any $\kappa > 2$, we have*

$$\mathbb{P}\left(\|f\|_\infty \geq 3\sqrt{\log(2\rho\kappa) \sum_{k=1}^{n} \|f_k\|_\infty^2} \right) \leq \frac{2}{\kappa}. \tag{4.16}$$

PROOF. Let λ be a real number and write $\mathcal{M} = \exp\left(\lambda \|f\|_\infty / 2\right)$ and

$$\mathcal{F} = \sum_{k=1}^{n} \|f_k\|_\infty^2.$$

Suppose first that the functions f_k are real valued. Clearly, f is also real valued and we may apply property (3) in the definition of the Salem–Zygmund space. In particular, for $t \in I$ we have $\mathcal{M} \leq e^{\lambda f(t)} + e^{-\lambda f(t)}$, since at least one of the terms on the right side is bigger than $\mathcal{M}$ for each $t \in I$. This allows the estimate

$$\begin{aligned}
\mathcal{M} &= \frac{1}{\mu(I)} \int_I \mathcal{M}\, d\mu(t) \leq \frac{1}{\mu(I)} \int_I \left(e^{\lambda f(t)} + e^{-\lambda f(t)} \right) d\mu(t) \\
&\leq \frac{1}{\mu(I)} \int_E \left(e^{\lambda f(t)} + e^{-\lambda f(t)} \right) d\mu(t) \leq \frac{\rho}{\mu(E)} \int_E \left(e^{\lambda f(t)} + e^{-\lambda f(t)} \right) d\mu(t).
\end{aligned}$$

Taking the expectation of this estimate and using Fubini's theorem to interchange expectation and integration yields

$$\begin{aligned}
\mathbb{E}\left(\mathcal{M}\right) &\leq \frac{\rho}{\mu(E)} \int_E \mathbb{E}\left(e^{\lambda f(t)} + e^{-\lambda f(t)} \right) d\mu(t) \\
&\leq \frac{\rho}{\mu(E)} \int_E 2e^{\lambda^2 \mathcal{F}/2}\, d\mu(t) = 2\rho e^{\lambda^2 \mathcal{F}/2},
\end{aligned}$$

where Lemma 4.12 estimated the expectation. Using this estimate, we compute

$$\begin{aligned}
\mathbb{P}\left(\|f\|_\infty \geq \lambda \mathcal{F} + \frac{2}{\lambda} \log(2\kappa\rho) \right) &= \mathbb{P}\left(e^{\lambda \|f\|_\infty/2} \geq 2\kappa\rho e^{\lambda^2 \mathcal{F}/2} \right) \\
&\leq \mathbb{P}\left(e^{\lambda \|f\|_\infty/2} \geq \kappa \mathbb{E}\left(\mathcal{M}\right) \right) \leq \frac{1}{\kappa},
\end{aligned}$$

by Lemma 4.11. We take $\lambda = \sqrt{\mathcal{F} \log(2\rho\kappa)}$ which yields

$$\mathbb{P}\left(\|f\|_\infty \geq 3\sqrt{\log(2\rho\kappa) \sum_{k=1}^{n} \|f_k\|_\infty^2} \right) \leq \frac{1}{\kappa}. \tag{4.17}$$

Now, suppose f_k assumes complex values. Since $\mathscr{B}$ is closed under complex conjugation, the real and imaginary parts of f are in $\mathscr{B}$. We may apply (4.17) to the real and imaginary parts to obtain (4.16) with $2/\kappa$. $\square$

Let us consider Dirichlet series of the form

$$f(s) = \sum_{n \le x} \omega_n \frac{a_n}{n^s},$$

where $\omega_n \in \{-1, 1\}$. We want to apply the Salem–Zygmund inequality to investigate $\|f\|_\infty$ and $\|\widehat{f}\|_1$. In view of Theorem 4.8 we have

$$\|f\|_\infty = \|F\|_\infty = \sup_{z \in \mathbb{T}^k} \left| \sum_{n \le x} a_\alpha z^{\alpha(n)} \right|.$$

The supremum is taken over the torus $\mathbb{T}^k$, where k is the number of variables. We would like a Salem–Zygmund space that contains trigonometric polynomials of several variables. This is obtained in the following lemma.

Lemma 4.14. *The set $\mathscr{Q}$ of trigonometric polynomials of degree $\le m$ in k variables with complex coefficients is a Salem–Zygmund space over*

$$E = [0, 2\pi]^k,$$

equipped with the Lebesgue measure. Furthermore, we have $\rho \ge \left(\pi^2 m\right)^k$.

Proof. Demands (1) and (2) in the definition of the Salem–Zygmund space are clearly fulfilled. Let $Q \in \mathscr{Q}$ be arbitrary. Let $I = I(Q)$ be the set such that

$$|Q(t)| \ge \frac{\|Q\|_\infty}{2}.$$

There is some $t_0 \in E$ such that

$$\|Q\|_\infty = |Q(t_0)|.$$

This allows us to define $J = J(Q)$ as the periodic cube around t_0 of diameter $1/(\pi m)$, in the sense that the 2π-periodic distance from t_0 in each coordinate is at most $1/(\pi m)$. Clearly

$$\mu(J) = \left(\frac{2}{\pi m}\right)^k.$$

We employ the triangle inequality and Bernstein's inequality in the form of Corollary A.10 to compute

$$\|Q\|_\infty - |Q(t)| \le |Q(t_0) - Q(t)| \le \frac{\pi}{2} m \|t_0 - t\|_\infty \cdot \|Q\|_\infty \le \frac{\|Q\|_\infty}{2},$$

for any $t \in J$. Hence $J \subseteq I$ and $\mu(I) \ge \mu(J)$. In particular, the demand $\mu(I) \ge \mu(E)/\rho$ implies that

$$\rho \ge \frac{\mu(E)}{\mu(J)} = \left(m\pi^2\right)^k \ge \frac{\mu(E)}{\mu(I)},$$

and thus $\mathscr{Q}$ is a Salem–Zygmund space. $\square$

4.4. Proof of Theorem 4.1

We are now ready to prove Theorem 4.1. The proof is split into two parts. First we provide a lower estimate for $S(N)$ and then we prove an upper bound. In both parts, we shall need Bohr's correspondence and the estimate

$$\Omega(n) \leq \frac{\log n}{\log 2}, \tag{4.18}$$

which is obtained by noting that $\Omega(n)$ is largest if n is a power of 2. The first part is done using the Salem–Zygmund inequality as well as the estimates for $\Psi(x,y)$ we obtained in Chapter 2 [**13**].

Proof of Theorem 4.1 – Part I. For each x, we need to choose a suitable Dirichlet polynomial

$$f(s) = \sum_{n \leq x} \frac{a_n}{n^s},$$

to prove a lower bound for $S(x)$. The application of the Cauchy–Schwarz inequality in (1.18), would imply that $|a_n| = 1$ is optimal. To control the number of variables $x \geq y \geq 2$, we introduce the characteristic function of the y-smooth numbers, $\chi(n,y)$. To apply the Salem–Zygmund inequality, we choose

$$a_n = \chi(n,y)\omega_n.$$

Let us introduce the index set $S(x,y) = \{n \leq x \,:\, \chi(n,y) > 0\}$, and write

$$f(s) = \sum_{n \in S(x,y)} \frac{\omega_n}{n^s}.$$

Clearly $\|\widehat{f}\|_1 = |S(x,y)| = \Psi(x,y)$. In view of Bohr's correspondence we write

$$F(z) = \sum_{n \in S(x,y)} \omega_n z^{\alpha(n)} = \sum_{n \in S(x,y)} \omega_n F_n(z),$$

where $F_n(z)$ is a monomial. In particular

$$\mathcal{F} = \sum_{n \in S(x,y)} \|F_n\|_\infty^2 = |S(x,y)| = \Psi(x,y).$$

Since each variable corresponds to a prime number, $F(z)$ is a polynomial in $k = \pi(y)$ variables. Its degree is equal to the largest value $\Omega(n)$ assumes for $n \in S(x,y)$. Clearly, since $n \leq x$, this is bounded by $m = \log x/\log 2$ by (4.18). Lemma 4.14 allows us to use these functions in the Salem–Zygmund inequality, and we may take

$$\rho = \left(\pi^2 m\right)^k = \left(\pi^2 \frac{\log x}{\log 2}\right)^{\pi(y)}.$$

We now apply Theorem 4.13 with $\kappa = 4$ to obtain

$$\mathbb{P}\left(\|f\|_\infty \geq 3\sqrt{\log(8\rho)\Psi(x,y)}\right) \leq \frac{1}{2}.$$

Clearly there is some choice of ω such that the corresponding Dirichlet polynomial $f(s)$ does not satisfy the inequality in the probability, and hence

$$\|f\|_\infty \ll \sqrt{\Psi(x,y)\pi(y)\log\log x}.$$

Since $S(x)$ is defined as the supremum of the ratio $\|\widehat{f}\|_1/\|f\|_\infty$, we obtain

$$\Psi(x,y) = \|\widehat{f}\|_1 \leq S(x)\|f\|_\infty \ll S(x)\sqrt{\Psi(x,y)\pi(y)\log\log x}.$$

Applying the effective estimate (4.1) allows us to compute

$$\begin{aligned} S(x) &\gg \sqrt{\frac{\Psi(x,y)}{\pi(y)\log\log x}} = \sqrt{\frac{x\log y}{\log\log x}}\exp\left(\frac{1}{2}\log\rho(u) - \frac{\log y}{2}\right) \\ &\geq \sqrt{x}\exp\left(-\frac{u}{2}\left(\log u + \log\log u + \mathcal{O}(1)\right) - \frac{\log y}{2}\right), \end{aligned}$$

under the assumption $\log y \geq \log\log x$. Let us choose $y = \exp\left(\alpha\sqrt{\log x\log\log x}\right)$ for some $\alpha > 0$. We compute

$$\begin{aligned} \frac{u}{2} &= \frac{1}{2\alpha}\sqrt{\frac{\log x}{\log\log x}}, \\ \log u &= \frac{1}{2}\log\log x - \log\alpha - \frac{1}{2}\log\log\log x, \\ \log\log u &= \log\log\log x - \log 2 - \log\left(1 - \frac{2\log\alpha}{\log\log x} - \frac{\log\log\log x}{\log\log x}\right). \end{aligned}$$

The largest terms are of order $\sqrt{\log x\log\log x}$, which are

$$-\frac{1}{4\alpha}\sqrt{\log x\log\log x} - \frac{\log y}{2} = -\left(\frac{1}{4\alpha} + \frac{\alpha}{2}\right)\sqrt{\log x\log\log x}.$$

This is maximized when $\alpha = 1/\sqrt{2}$. Furthermore, $\log\log u = \log\log\log x + \mathcal{O}(1)$, which allows us to estimate the other terms

$$-\log\alpha - \frac{1}{2}\log\log\log x + \log\log u = \frac{1}{2}\log\log\log x + \mathcal{O}(1).$$

Combining everything, we have

$$S(x) \geq \sqrt{x}\exp\left(\left(-\frac{1}{\sqrt{2}} - \frac{1}{2}\frac{\log\log\log x}{\log\log x} + \mathcal{O}\left(\frac{1}{\log\log x}\right)\right)\sqrt{\log x\log\log x}\right),$$

which provides the required lower bound. $\square$

The second part of the proof uses Rankin's trick, in addition to the Bohnenblust–Hille inequality for homogenous polynomials of CHAPTER 3 [**27**].

Proof of Theorem 4.1 – Part II. Let $f(s)$ be any Dirichlet polynomial of the form

$$f(s) = \sum_{n \le x} \frac{a_n}{n^s}.$$

We want to prove

$$\|\widehat{f}\|_1 \le \|f\|_\infty \exp\left(\left(-\frac{1}{\sqrt{2}} + S(N)\right)\sqrt{\log x \log\log x}\right), \tag{4.19}$$

with

$$S(N) \le \frac{2}{\sqrt{3}} \frac{\log\log\log x}{\log\log x} + \mathcal{O}\left(\frac{1}{\log\log x}\right). \tag{4.20}$$

We split the proof into four steps.

Step 0. Suppose $x \ge y \ge 2$. In view of Lemma 4.9 we decompose

$$f(s) = \sum_{n \le x} \frac{a_n}{n^s} = \sum_{\kappa} \frac{f_\kappa(s)}{\kappa^s},$$

where $f_\kappa(s)$ contains only y-rough n and satisfies $\|f_\kappa\|_\infty \le \|f\|_\infty$. Suppose that we can prove that each $f_\kappa(s)$ satisfies (4.19) . Then, clearly

$$\|\widehat{f}\|_1 \le \|f\|_\infty \exp\left(\left(-\frac{1}{\sqrt{2}} + S(N)\right)\sqrt{\log x \log\log x}\right)\sum_{\kappa} 1,$$

where the sum is taken over all possible choices of y-smooth κ such that there is some $n \le x$ with $n = \kappa\lambda$. How many such choices are there? By (4.18), n has at most $\log x/\log 2$ prime factors, since $n \le x$. Since κ is y-smooth, each of these prime factors has to be $\le y$, so we have at most y choices for each factor. Hence

$$\sum_{\kappa} 1 \le \left(\frac{\log x}{\log 2}\right)^y \le \exp\left(y \log\log x\right) = \exp\left(\sqrt{\frac{\log x}{\log\log x}}\right), \tag{4.21}$$

for $y = \sqrt{\log x/(\log\log x)^3}$, which is the value we choose. This is absorbed in the $\mathcal{O}$-term of (4.20). Hence we are able to restrict our investigations to the case where f only contains y-rough n. Let us introduce

$$M_1 = \frac{1}{\sqrt{2}}\sqrt{\frac{\log x}{\log\log x}} \quad \text{and} \quad M_2 = 2\sqrt{2}\sqrt{\frac{\log x}{\log\log x}}.$$

With this in mind, let us further split the sum into three parts,

$$\|\widehat{f}\|_1 = \sum_{\Omega(n) \le M_1} |a_n| + \sum_{M_1 < \Omega(n) \le M_2} |a_n| + \sum_{M_2 < \Omega(n)} |a_n| = \Sigma_1 + \Sigma_2 + \Sigma_3.$$

Since x and y now are fixed, recall (4.4) and put $T_m = T(x,y,m)$. Let ut consider the polynomial

$$f_m(s) = \sum_{n\in T_m} \frac{a_n}{n^s}.$$

Under Bohr's correspondence, f_m is associated to a m-homogenous polynomial, with $\pi(y)$ variables and hence satisfies $\|f_m\|_\infty \le \|f\|_\infty$ by Lemma 4.10. We will estimate each part by itself.

Step 1. We consider Σ_1. We begin by using Hölder's inequality,

$$\Sigma_1 = \sum_{m\le M_1}\sum_{n\in T_m} |a_n| \le \sum_{m\le M_1}\left(\sum_{n\in T_m} |a_n|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}}\left(\sum_{n\in T_m} 1\right)^{\frac{m-1}{2m}},$$

the Bohnenblust–Hille inequality of (4.2) and the trivial estimate $|T_m| \le x$,

$$\begin{aligned}
&\le \sum_{m\le M_1} e^m \|f_m\|_\infty x^{\frac{m-1}{2m}} \le M_1 \|f\|_\infty e^{M_1}\sqrt{x}\exp\left(-\frac{\log x}{2M_1}\right)\\
&= \|f\|_\infty \sqrt{x}\exp\left(-\frac{1}{\sqrt{2}}\sqrt{\log x\log\log x} + \mathcal{O}\left(\sqrt{\frac{\log x}{\log\log x}}\right)\right),
\end{aligned}$$

by the definition of M_1. The error term is of the order of the $\mathcal{O}$-term of (4.20).

Step 2. This step is similar to the previous step. We again use Bohnenblust–Hille with Hölder's inequality to obtain

$$\Sigma_2 = \sum_{M_1<m\le M_2}\sum_{n\in T_m} |a_n| \le \sum_{M_1<m\le M_2} e^m \|f\|_\infty |T_m|^{\frac{m-1}{2m}}.$$

As above, the number of summands and e^m is absorbed in the error term, and we consider $|T_m|$. Since $M_1 < m \le M_2$ there is some $1/\sqrt{2} < \alpha \le 2\sqrt{2}$ such that

$$m = \alpha\sqrt{\frac{\log x}{\log\log x}}.$$

We will need a better estimate than $|T_m| \le x$. We may use Lemma 4.5 to obtain

$$|T_m| \le N(x,y,m) \le \frac{x}{(y/2)^m}(\log x)^y \exp\left(\mathcal{O}(y)\right) \le \frac{x}{y^m}\exp\left(\mathcal{O}\left(\sqrt{\frac{\log x}{\log\log x}}\right)\right),$$

by the definition of y and the fact that $m \le M_2$. The error term is fine, and hence

$$|T_m|^{\frac{m-1}{2m}} = \sqrt{x}\exp\left(-\frac{\log x}{2m} - \frac{m-1}{2}\log y + \mathcal{O}\left(\sqrt{\frac{\log x}{\log\log x}}\right)\right).$$

We recall that $y = \sqrt{\log x / (\log\log x)^3}$ and compute

$$-\frac{m-1}{2}\log y = -\frac{m}{4}\log\log x + \frac{3}{4}m\log\log\log x + \frac{\log y}{2}.$$

The first term is of order $\sqrt{\log x \log\log x}$ and is in the main term. Using $m \le M_2$ we bound the final two terms by

$$\frac{3}{4}m\log\log\log x + \frac{\log y}{2} \le \frac{3}{\sqrt{2}}\sqrt{\frac{\log x}{\log\log x}}\log\log\log x + \mathcal{O}\left(\sqrt{\frac{\log x}{\log\log x}}\right),$$

which is acceptable. What remains after ignoring the error terms is

$$\exp\left(-\frac{\log x}{2m} - \frac{m}{4}\log\log x\right) = \exp\left(-\sqrt{\log x \log\log x}\left(\frac{1}{2\alpha} + \frac{\alpha}{4}\right)\right).$$

This is largest at $\alpha = \sqrt{2}$, which yields the required $-1/\sqrt{2}$-term.

Step 3. Recalling (1.17) we estimate

$$\left(\sum_{n\in T_m} |a_n|^2\right)^{\frac{1}{2}} = \left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} |f_m(it)|^2\,dt\right)^{\frac{1}{2}} \le \|f_m\|_\infty.$$

Combining this with the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}\Sigma_3 = \sum_{M_2<m}\sum_{n\in T_m}|a_n| &\le \sum_{M_2<m}\left(\sum_{n\in T_m}|a_n|^2\right)^{\frac{1}{2}}\left(\sum_{n\in T_m}1\right)^{\frac{1}{2}} \\ &\le \sum_{M_2<m}\|f_m\|_\infty\sqrt{|T_m|} \le \|f\|_\infty\sqrt{N(x,y,M_2)}.\end{aligned}$$

By similar considerations as in Step 2, we easily estimate

$$N(x,y,M_2) \le \frac{x}{y^{M_2}}\exp\left(\mathcal{O}\left(\sqrt{\frac{\log x}{\log\log x}}\right)\right),$$

and furthermore we obtain

$$\sqrt{\frac{1}{y^{M_2}}} = \exp\left(-\frac{M_2}{2}\log y\right) = \exp\left(-\frac{1}{\sqrt{2}}\sqrt{\log x\log\log x} + \frac{3}{4}M_2\log\log\log x\right).$$

In total this yields

$$\frac{\Sigma_3}{\|f\|_\infty} \le \sqrt{x}\exp\left(-\frac{1}{\sqrt{2}}\sqrt{\log x\log\log x} + \frac{3}{\sqrt{2}}\sqrt{\frac{\log x}{\log\log x}}\left(\log\log\log x + \mathcal{O}(1)\right)\right),$$

as required. $\square$

Remark. It is possible to replace y by $\pi(y)$ in (4.21), but the term $y\log\log x$ also appears when Lemma 4.5 is applied, so this would not improve the theorem.

4.5. Some Related Open Problems

We end this chapter with some comments on a few open problems related to the topics discussed in this thesis.

The Bohnenblust–Hille Inequality. *Is hypercontractivity optimal for complex polynomial Bohnenblust–Hille inequality?* In the final section of CHAPTER 3 we studied a real version of the Bohnenblust–Hille inequality, and obtained that hypercontractivity was both necessary and sufficient. Furthermore, we showed that a similar approach would not prove hypercontractivity of the complex inequality.

Explicit flat Dirichlet polynomials. *Construct explicit Dirichlet polynomials to provide the lower estimate of Theorem 4.1.* The Salem–Zygmund inequality implies the existence of "flat enough" Dirichlet polynomials of the form

$$f(s) = \sum_{n \le x} \omega_n \frac{\chi(n,y)}{n^s}, \qquad \omega_n = \pm 1,$$

to provide the required lower bound for $S(N)$. It would be interesting to provide either explicit values $\omega_n(x,y)$ or some other sequence of Dirichlet polynomials providing the lower estimate in Theorem 4.1. This would mirror the work done by Bourgain and Bombieri on Kahane's ultra flat trigonometric polynomials.

Improving the estimate. *Improve the upper or lower bounds for $s(N)$ and decide the sign of this quantity, if any.* We were able to provide the bounds

$$-C_1 \frac{\log\log\log N}{\log\log N} \le s(N) \le C_2 \frac{\log\log\log N}{\log\log N},$$

for absolute positive constants C_1 and C_2, but it is not clear whether this is optimal or not. Furthermore, the ultimate sign of $s(N)$ is not decided, if there even is one.

p-Sidon Constants. *Study and estimate the p-Sidon constants for $0 < p < 2$.* We can define p-Sidon constants, $S_p(N)$, by replacing $\|f\|_\infty$ by the L^p-type norm

$$\|f\|_p = \left(\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(it)|^p \, dt \right)^{\frac{1}{p}}, \qquad p > 0,$$

in the ratio $\|\widehat{f}\|_1 / \|f\|_\infty$. In view of (1.17) is is clear that $S_2(N) = \sqrt{N}$. By Khinchine's inequality one can obtain

$$S_p(N) \ge \sqrt{N} \exp\left(-\mathcal{O}(1)\right),$$

for $p > 0$. Combining these observations yields $S_p(N) = \sqrt{N} \exp\left(-\mathcal{O}(1)\right)$ for $p > 2$. Thus the most interesting case is $p < 2$. In particular $p = 1$, where it is known that $\sqrt{N} \le S_1(N) \le (1 + o(1))\sqrt{N}\sqrt{\log N}$, and thus there is room for improvement **[37]**.

APPENDIX A

Inequalities

In this appendix, we state and prove some of the inequalities needed in the thesis, which are omitted from the main part due to their general or elementary nature. Our main reference is [**19**] and we refer to [**15**] for the measure theory.

A.1. Hölder's Inequality

Throughout this section, we assume that $p > 1$ and

$$\frac{1}{p} + \frac{1}{q} = 1,$$

unless otherwise is stated. We say that p and q are *Hölder conjugates.*

LEMMA A.1 (Young's Inequality). *Assume that a and b are non-negative real numbers, and $p \geq 1$. Then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

with equality if and only if $a^p = b^q$.

PROOF. If $a = 0$ or $b = 0$ we are done. Hence, assume $a, b > 0$. Now, it is known that $1 + x \leq e^x$ and thus $x \leq e^{x-1}$. In particular, for any $c > 0$, we obtain

$$\frac{ab}{c} = \left(\frac{a^p}{c}\right)^{\frac{1}{p}} \left(\frac{b^q}{c}\right)^{\frac{1}{q}} \leq \left(e^{\frac{a^p}{c}-1}\right)^{\frac{1}{p}} \left(e^{\frac{b^q}{c}-1}\right)^{\frac{1}{q}} = \exp\left(\frac{1}{c}\left(\frac{a^p}{p} + \frac{b^q}{q}\right) - 1\right) = 1.$$

if we take $c = a^p/p + b^q/q$ and thus the inequality follow by multiplying of c on both sides. Now, $x = e^{x-1}$ if and only if $x = 1$. Thus we require $a^p/c = 1$ and $b^q/c = 1$ and in particular $a^p = b^q$. $\square$

THEOREM A.2 (Hölder's Inequality). *Let (X, Σ, μ) be a measure space and suppose that f and g are Σ-measurable real or complex valued functions on X. Then*

$$\int_X |f(x)g(x)|\, d\mu(x) \leq \left(\int_X |f(x)|^p\, d\mu(x)\right)^{\frac{1}{p}} \left(\int_X |g(x)|^q\, d\mu(x)\right)^{\frac{1}{q}},$$

where $p \geq 1$ and $1/p + 1/q = 1$, with equality when $|f(x)|^p = |g(x)|^q$ almost everywhere.

Proof. Suppose that $\|f\|_p = 1$ and $\|f\|_q = 1$. Now, we apply Young's inequality to compute

$$\int_X |f(x)g(x)|\, d\mu(x) \leq \int_X \frac{|f(x)|^p}{p}\, d\mu(x) + \int_X \frac{|f(x)|^q}{q}\, d\mu(x) = \frac{1}{p} + \frac{1}{q} = 1.$$

Let now $\|f\|_p$ and $\|g\|_q$ be positive and finite. By normalizing we obtain

$$\int_X |f(x)g(x)|\, d\mu(x) = \|f\|_p \|g\|_q \int_X |\hat{f}(x)\hat{g}(x)|\, d\mu(x) \leq \|f\|_p \|g\|_q,$$

where equality follows from the equality condition in Young's inequality. The cases where one of $\|f\|_p$ and $\|g\|_q$ are zero or infinite are trivial. □

Corollary A.3. *Let m be a positive integer and f_j be Σ-measurable real or complex valued functions on X for $j = 1, 2, \ldots, m$. Assume $p_1 > 1$ and*

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} = 1.$$

Then we have

$$\int_X |f_1(x)f_2(x)\cdots f_m(x)|\ d\mu(x) \leq \prod_{j=1}^{m} \left(\int_X |f_j(x)|^{p_j}\, d\mu(x) \right)^{\frac{1}{p_j}}.$$

Proof. The case $m = 2$ follows from Hölder's inequality. We take q_k to be the Hölder conjugate of p_k. In particular we observe that

$$\frac{1}{q_k} = \sum_{j \neq k} \frac{1}{p_j}.$$

We use induction on m and apply Hölder's inequality to obtain

$$\int_x |f_1(x)\cdots f_{m+1}(x)|\, d\mu(x) \leq \|f_1\|_{p_1} \left(\int_x |f_2(x)\cdots f_{m+1}(x)|^{q_1}\, d\mu(x) \right)^{\frac{1}{q_1}}.$$

We take $r_i = p_i/q_1$ for $i = 2, 3, \ldots, m+1$ to complete the proof. □

Theorem A.4 (Minkowski's Inequality). *Let (X, Σ, μ) and $(Y, \mathcal{T}, \nu)$ be σ-finite measure spaces, and assume that $f : X \times Y \to \mathbb{C}$ is some $(\Sigma \otimes \mathcal{T})$-measurable function. For $1 \leq p < \infty$ we have*

$$\left[\int_X \left(\int_Y |f(x,y)|\, d\nu(y) \right)^p d\mu(x) \right]^{\frac{1}{p}} \leq \int_Y \left(\int_X |f(x,y)|^p\, d\mu(x) \right)^{\frac{1}{p}} d\nu(y). \tag{A.1}$$

Proof. Since our measure space are σ-finite and $|f(x,y)| \geq 0$ we can apply Tonelli's theorem where needed. In particular, the case $p = 1$ shows equality in (A.1) by Tonelli's theorem. Assume therefore that $p > 1$. Now, if

$$\int_X \left(\int_Y |f(x,y)|\, d\nu(y)\right)^p d\mu(x) = 0$$

we are done. Assume therefore that it is strictly positive and define

$$g(x) = \left(\int_Y |f(x,y)|\, d\nu(y)\right)^{p-1}.$$

We now observe that $(p-1)q = p$ and compute

$$\|g\|_q = \left[\int_X g(x)^q\, d\mu(x)\right]^{\frac{1}{q}} = \left[\int_X \left(\int_Y |f(x,y)|\, d\nu(y)\right)^p d\mu(x)\right]^{\frac{1}{q}} = I^{\frac{1}{q}},$$

where we again apply Tonelli's theorem and Hölder's inequality to

$$\begin{aligned} I &= \int_X \int_Y f(x,y) g(x)\, d\nu(y) d\mu(x) = \int_Y \int_X f(x,y) g(x)\, d\mu(x) d\nu(y) \\ &\leq \int_Y \left(\int_X |f(x,y)|^p\, d\mu(x)\right)^{\frac{1}{p}} \left(\int_X g(x)^q\, d\mu(x)\right)^{\frac{1}{q}} d\nu(y) \\ &= \int_Y \left(\int_X |f(x,y)|^p\, d\mu(x)\right)^{\frac{1}{p}} \|g\|_q\, d\nu(y) = I^{\frac{1}{q}} \left(\int_X |f(x,y)|^p\, d\mu(x)\right)^{\frac{1}{p}} d\nu(y). \end{aligned}$$

By dividing by $I^{\frac{1}{q}}$ we obtain (A.1), and complete the proof. □

Remark. The case of equality in Minkowski's inequality follows (if $p > 1$) from the application of Hölder's inequality, which requires

$$f(x,y)^p = g(x)^q = \left(\int_Y f(x,z)\, d\nu(z)\right)^{(p-1)q} \implies f(x,y) = \int_Y f(x,z)\, d\nu(z),$$

μ-almost everywhere for ν-almost every $y \in Y$.

We will need the following special versions of Hölder's inequality (in the form of Corollary A.3) and Minkowski's inequality: For any family of positive positive numbers a and b we have:

$$\sum_{j=1}^n a_1(j) \cdots a_m(j) \leq \prod_{k=1}^m \left(\sum_{j=1}^n a_k(j)^{m+1}\right)^{\frac{1}{m+1}} \cdot \left(\sum_{j=1}^n a_m(j)^{\frac{m+1}{2}}\right)^{\frac{2}{m+1}}, \tag{A.2}$$

$$\sum_{j_1=1}^n \left(\sum_{j_2=1}^n b_{j_1 j_2}\right)^2 \leq \left(\sum_{j_2=1}^n \left(\sum_{j_1=1}^n b_{j_1 j_2}^2\right)^{\frac{1}{2}}\right)^2. \tag{A.3}$$

A.2. Hilbert's Inequality

THEOREM A.5. *Let $\{x_m\}_{m\in\mathbb{Z}}$ and $\{y_n\}_{n\in\mathbb{Z}}$ be families of complex numbers. Then we have*

$$\left|\sum_{m\in\mathbb{Z}}\sum_{n\neq m}\frac{x_m y_n}{m-n}\right| \leq \pi\left(\sum_{m\in\mathbb{Z}}|x_m|^2\right)^{\frac{1}{2}}\left(\sum_{n\in\mathbb{Z}}|y_n|^2\right)^{\frac{1}{2}}. \tag{A.4}$$

PROOF. Let K be any positive integer. If we can prove the inequality for $x_m = 0$ and $y_n = 0$ for $|m| > K$ and $|n| > K$, (A.4) will follow. Consider therefore

$$\mathscr{H}_K(x,y) = \sum_{n=-K}^{K}\sum_{\substack{n=-K\\ m\neq n}}^{K}\frac{x_m y_n}{m-n}.$$

We begin by computing the auxiliary integral

$$I(k) = \int_0^1\left(t-\frac{1}{2}\right)e^{2\pi ikt}\,dt = \begin{cases}\frac{1}{2\pi ik} & \text{if } k\neq 0\\ 0 & \text{if } k=0\end{cases}. \tag{A.5}$$

Now, we define

$$f(t) = \sum_{m=-K}^{K} x_m e^{2\pi imt} \quad\text{and}\quad g(t) = \sum_{n=-K}^{K} y_n e^{-2\pi int}.$$

Our first observation is that

$$\|x\|_2^2 = \sum_{m=-K}^{K}|x_m|^2 = \int_0^1 |f(t)|^2\,dt = \|f\|_2^2,$$

$$\|y\|_2^2 = \sum_{n=-K}^{K}|y_n|^2 = \int_0^1 |g(t)|^2\,dt = \|g\|_2^2.$$

by the orthogonality of the trigonometric system. We apply (A.5) to compute

$$J = \int_0^1\left(t-\frac{1}{2}\right)f(t)g(t)\,dt = \sum_{m=-K}^{K}\sum_{n=-K}^{K}x_m y_n I(m-n) = \frac{\mathscr{H}_K(x,y)}{2\pi i}.$$

But since

$$|t-1/2|\leq 1/2$$

for $t\in[0,1]$ we can apply the triangle inequality and the Cauchy–Schwarz inequality to obtain the required

$$|\mathscr{H}_K(x,y)| = 2\pi|J| \leq \pi\int_0^1|f(t)g(t)|\,dt \leq \pi\|f\|_2\|g\|_2 = \pi\|x\|_2\|y\|_2. \qquad \square$$

A.3. Bernstein's Inequality

In this section, we try to bound derivatives of trigonometric polynomials. We are in particular interested in trigonometric polynomials in several variables, and will obtain a corollary via the mean value theorem which will be required in our arguments [**33, 29**].

DEFINITION. An m'th degree *trigonometric polynomial* is a function on the form

$$T_m(x) = \sum_{k=-m}^{m} a_k e^{ikx},$$

for $a_k \in \mathbb{C}$ and $x \in [0, 2\pi)$ with $(a_{-m,}, a_m) \neq (0, 0)$.

We begin by making two important observations:

(1) Exploiting the fact that we can write any trigonometric polynomial as a rotation times a polynomial evaluated on $\mathbb{T}$,

$$T_m(x) = e^{-imx} P_{2m}(e^{ix}),$$

allows us to conclude that T_m has at most $2m$ distinct roots on $[0, 2\pi)$, since P_{2m} has at most $2m$ distinct roots in $\mathbb{C}$.

(2) The complex conjugate of a trigonometric polynomial is still a trigonometric polynomial, and hence

$$R_m(x) = \Re\left(T_m(x)\right) = \frac{1}{2} \sum_{k=-m}^{m} \frac{a_k + \overline{a_{-k}}}{2} \cdot e^{ikx},$$

is also a trigonometric polynomial.

We begin by proving a result on real-valued trigonometric polynomials.

LEMMA A.6. *Suppose that $T_m(x)$ is a real-valued n'th degree trigonometric polynomial with $2m$ distinct zeroes. Then $T_m'(x)$ has $2m$ distinct zeroes, all different from the zeroes of $T_m(x)$.*

PROOF. Since $T_m'(x)$ has degree $\leq m$ it is clear that it has at most $2m$ zeroes. If $f \in C^1[a, b]$ with $a < b$ such that $f(a) = f(b)$ there is some $a < c < b$ such that

$$0 = \frac{f(b) - f(a)}{b - a} = f'(c),$$

by the mean value theorem. This implies that there is at least one zero of $T_m'(x)$ between two consecutive zeroes of $T_m(x)$. There is also one zero of $T_m'(x)$ "between" the last and first zeroes of $T_m(x)$ by periodicity. Thus $T_m'(x)$ has exactly $2m$ distinct zeroes, which all are different from the zeroes of $T_m(x)$ since we have strict inequality in the mean value theorem, $a < c < b$. $\square$

Lemma A.7. *Let $T_m(x)$ be an m'th degree trigonometric polynomial over $\mathbb{C}$ satisfying $|T_m(x)| \leq M$. Then $|T_m'(x)| \leq mM$.*

Proof. We give a proof by contradiction. Suppose there is some $T_m(x)$ with

$$\sup_{x\in[0,2\pi)} |T_m'(x)| = nL$$

and $L > M$. There is some $0 \leq x_0 < 2\pi$ such that $|T_m'(x_0)| = mL$. By rotating the trigonometric polynomial and taking real values, we may assume that

$$R_m(x_0) = \Re\left(T_m(x_0)\right) = mL.$$

Furthermore $R_n''(x_0) = T_m''(x_0) = 0$. We consider the following m'th degree real trigonometric polynomial:

$$\begin{aligned} S_m(x) &= L\sin(m(x-x_0)) - R_m(x), \\ S_m'(x) &= mL\cos(m(x-x_0)) - R_m'(x), \\ S_m''(x) &= -m^2L\sin(m(x-x_0)) - R_m''(x). \end{aligned}$$

We begin by noting that $S_m(x)$ assumes alternating signs at the $2m$ points

$$x = x_0 + \frac{2k+1}{2m}\pi.$$

Hence, by the intermediate value theorem, $S_m(x)$ has $2m$ zeroes. By Lemma A.6, $S_m'(x)$ also has $2m$ zeroes. One of these zeroes has to be x_0 since a computation yields $S_m'(x_0) = 0$, by the definition of x_0. However, we also note that $S_m''(x_0) = 0$, which is impossible by Lemma A.6, since $S_m'(x)$ and $S_m''(x)$ have no zeroes in common. □

To extend Lemma A.7 to trigonometric polynomials of several variables, we will need the following result:

Lemma A.8. *For complex numbers z_k we have*

$$\sup_{\xi\in\{-1,1\}^n} \left|\sum_{k=1}^n \xi_k z_k\right| \geq \frac{2}{\pi}\sum_{k=1}^n |z_k|.$$

Proof. Let $z_k = r_k e^{i\theta_k}$. For any $\theta \in \mathbb{R}$ we obtain

$$\left|\sum_{k=1}^n \xi_k z_k\right| = \Re\left|\sum_{k=1}^n \xi_k z_k\right| \geq \Re\left(e^{i\theta}\sum_{k=1}^n \xi_k z_k\right) = \sum_{k=1}^n r_k\xi_k\cos(\theta+\theta_k).$$

We take the supremum over all $\xi \in \{-1,1\}^n$, which yields

$$\sup_{\xi\in\{-1,1\}^n} \left|\sum_{k=1}^n \xi_k z_k\right| \geq \sup_{\xi\in\{-1,1\}^n} \sum_{k=1}^n r_k\xi_k\cos(\theta+\theta_k) = \sum_{k=1}^n r_k|\cos(\theta+\theta_k)| = f(\theta).$$

Now, taking the supremum over all $\theta \in \mathbb{R}$ yields

$$\sup_{\xi \in \{-1,1\}^n} \left| \sum_{k=1}^{n} \xi_k z_k \right| \geq \sup_{\theta} f(\theta) \geq \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta = \frac{2}{\pi} \sum_{k=1}^{n} r_k = \frac{2}{\pi} \sum_{k=1}^{n} |z_k|. \quad \square$$

We will require some notation before we can state and prove Bernstein's inequality for trigonometric polynomials of several variables.

Definition. An m'th order *signed multi-index* on $\mathbb{C}^n$ is $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, where $\alpha_i \in \{0, \pm 1, \ldots, \pm m\}$ and $\|\alpha\| = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_n| = m$.

This allows us to write a m'th degree trigonometric polynomial of n variables as

$$Q(t) = Q(t_1, t_2, \ldots, t_n) = \sum_{\|\alpha\| \leq m} a_\alpha \exp\left(i\left(\alpha_1 t_1 + \alpha_2 t_2 + \cdots + \alpha_n t_n\right)\right). \tag{A.6}$$

Theorem A.9. *Let Q be an m'th degree trigonometric polynomial of n variables as defined by* (A.6). *Then*

$$\sum_{k=1}^{n} \left| \frac{\partial Q}{\partial t_k}(t) \right| \leq \frac{\pi}{2} m \|Q\|_\infty.$$

Proof. Fix $t = (t_1, t_2, \ldots, t_n)$. For $\xi \in \{-1,1\}^n$ and $u \in \mathbb{R}$ define

$$R(u) = Q\left(t_1 + \xi_1 u,\ t_2 + \xi_2 u,\ \ldots,\ t_n + \xi_n u\right),$$

which is a one variable trigonometric polynomial of degree $\leq m$. By Lemma A.7 we thus obtain

$$\left| \sum_{k=1}^{n} \xi_k \frac{\partial Q}{\partial t_k}(t) \right| = |R'(0)| \leq m\|R\|_\infty \leq m\|Q\|_\infty.$$

We may take the supremum over all $\xi \in \{-1,1\}^n$ and apply Lemma A.8 to complete the proof. $\square$

The following corollary will be our main application of Bernstein's inequality.

Corollary A.10. *Let Q be an m'th degree trigonometric polynomial of n variables as defined by* (A.6). *Then*

$$|Q(t) - Q(\tau)| \leq m \frac{\pi}{2} \sup_k |t_k - \tau_k|\, \|Q\|_\infty.$$

Proof. For $\alpha \in [0,1]$ consider $f(\alpha) = Q(t\alpha + (1-\alpha)\tau)$. By the mean value theorem there is some $\beta \in [0,1]$ with $T = \beta t + (1-\beta)\tau$ such that

$$|Q(t) - Q(\tau)| = |f'(\beta)| = \left| \sum_{k=1}^{n} \frac{\partial Q}{\partial t_k}(T)(t_k - \tau_k) \right| \leq \sup_k |t_k - \tau_k| \sum_{k=1}^{n} \left| \frac{\partial Q}{\partial t_k}(T) \right|.$$

The proof is completed by appealing to Theorem A.9. $\square$

APPENDIX B

The Riemann Zeta Function

In this appendix, we briefly outline some properties of the Riemann zeta function, which is defined as the analytical continuation of the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The Riemann zeta function has only one pole, at $s = 1$. This pole is simple with residue 1, see Example 1.16. Furthermore, we have the *trivial zeroes* at

$$s = -2, -4, -6, \ldots.$$

The remaining zeroes are all found in the *critical strip* $0 < \Re(s) < 1$. Using these properties, it is possible to prove the celebrated Prime Number Theorem:

THEOREM (The Prime Number Theorem). *Let $\pi(x)$ denote the number of primes less than or equal to the real number x. Then, as $x \to \infty$,*

$$\pi(x) \sim \frac{x}{\log x}.$$

These claims were stated and proved in the authors bachelor's thesis [**11**].

B.1. Vinogradov's Zero Free Region

In this section, all proofs are omitted. We invite the reader to consult [**23**] for a treatment. The best known zero free region in the critical strip is due to Korobov and Vinogradov. They established the upper bound

$$\zeta(s) = \mathcal{O}\left(\left(1 + |t|^{A(1-\sigma)^{3/2}}\right) \log |t|^{2/3}\right),$$

valid for $\sigma \geq 0$ and $|t| \geq 2$. They established that $\zeta(s) \neq 0$ in the region

$$\sigma \geq 1 - C\left(\log |t|\right)^{-2/3} \left(\log \log |t|\right)^{-1/3},$$

valid for $|t| \geq 3$. They also obtained the bound

$$\frac{\zeta'(s)}{\zeta(s)} = \mathcal{O}\left(\left(\log |t|\right)^{2/3} \left(\log \log |t|\right)^{1/3}\right),$$

valid in the zero free region. The quantities A and C are positive constants and we refer to [**16**] for some explicit estimates of these constants.

B.2. Estimating Chebyshev's Functions

In this section, we introduce Chebyshev's functions and show how to estimate them using Vinogradov's zero free region:

$$\psi(x) = \sum_{n \le x} \Lambda(n),$$
$$\vartheta(x) = \sum_{p \le x} \log p.$$

It should be noted that $\Lambda(n)$ is the von Mangoldt-function, as defined in (2.23). We now seek to estimate Chebyshev's functions precisely. For $\epsilon > 0$, let

$$L_\epsilon(x) = \exp\left((\log x)^{3/5-\epsilon}\right).$$

Following our study of the approximate functional equation of the logarithmic derivative of the Riemann zeta function, we obtain the following estimate.

THEOREM B.1. *Let $\epsilon > 0$ and suppose that $x \ge x_0(\epsilon)$. Then*

$$\psi(x) = x\left(1 + \mathcal{O}\left(\frac{1}{L_\epsilon(x)}\right)\right). \tag{B.1}$$

PROOF. This proof is very similar to the proof of the approximate functional equation of Theorem 2.13. Most of the computations and estimates are near identical and will be omitted. We begin by appealing to Lemma 1.21 and computing

$$\psi(x) = \sum_{n \le x} \Lambda(n) = \frac{-1}{2\pi i}\int_{\kappa - iT}^{\kappa + iT} \frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\, ds + \mathcal{O}\left(x^\kappa \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^\kappa\left(1 + T|\log x/n|\right)}\right),$$

where we take $\kappa = 1 + 1/\log x$ and $T \ge 2$ yet to be decided. This is different form Theorem 2.13, where we used the weighted coefficients $\Lambda(n)/n^s$. Now, by similar considerations as in the proof of Theorem 2.13 we observe that the error term is of order

$$\frac{x(\log x)^2}{T}.$$

We extend the integration as far to the left as $\eta = 1 - \log T/\log x$. We require this to be within Vinogradov's zero free region, which demands

$$\sigma = 1 - \frac{\log T}{\log x} \ge 1 - C\left(\log T\right)^{-2/3}\left(\log\log T\right)^{-1/3}.$$

To satisfy this demand, we choose $T = L_\epsilon^2(x)$ and $x \ge x_0(\epsilon)$. The residue at $s = 0$ is avoided, which is different from Theorem 2.13. The only residue inside the path of integration is the simple pole at $s = 1$. This implies that

$$\psi(x) = x + \mathcal{O}\left(\frac{x(\log x)^2}{T}\right) + R,$$

where R denotes the integral along the polygonal path connecting $\kappa - iT$, $\eta - iT$, $\eta + iT$ and $\kappa + iT$. These contributions may be estimated identically to Theorem 2.13, which in this case yields

$$R \ll \frac{x(\log x)^2}{T}.$$

The choice of $T = L_\epsilon^2(x)$ implies that we are done, since

$$\frac{(\log x)^2}{L_\epsilon^2(x)} = \exp\left(2\log\log x - 2(\log x)^{3/5-\epsilon}\right) \ll \exp\left(-(\log x)^{3/5-\epsilon}\right) = \frac{1}{L_\epsilon(x)}.$$

We obtain (B.1) by factoring out the x of the error term. $\square$

The following corollary shows that the error made replacing $\psi(x)$ with the $\vartheta(x)$ is insignificant in view of the estimate (B.1).

Corollary B.2. *Let $\epsilon > 0$ and suppose that $x \geq x_0(\epsilon)$. Then*

$$\vartheta(x) = x\left(1 + \mathcal{O}\left(\frac{1}{L_\epsilon(x)}\right)\right).$$

Proof. We observe that

$$0 \leq \psi(x) - \vartheta(x) = \sum_{p\leq\sqrt{x}} \log p \sum_{m=2}^{\log_p x} 1 \leq \sum_{p\leq\sqrt{x}} \log p \cdot \log_p x = \pi\left(\sqrt{x}\right)\log x \ll \sqrt{x},$$

by the Prime Number Theorem. It is clear that $\sqrt{x} \ll x/L_\epsilon(x)$, and hence the error made in replacing $\psi(x)$ by $\vartheta(x)$ is absorbed in the error term of (B.1). $\square$

B.3. Mertens's Formula

By rearranging the summands of the Riemann zeta function in the half-plane of absolute convergence as prime powers and summing the geometric series, we obtain the Euler product:

$$\zeta(s) = \left(1 + \frac{1}{2^s} + \frac{1}{4^s} + \cdots\right)\left(1 + \frac{1}{3^s} + \frac{1}{9^s} + \cdots\right)\cdots = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

The classical Mertens's formula **[32]**, which predates the Prime Number Theorem, can be stated as

$$\prod_{p\leq x}\left(1 - \frac{1}{p}\right)^{-1} = e^\gamma \log x\left(1 + \mathcal{O}\left(\frac{1}{\log x}\right)\right),$$

for $x \geq 2$. It may be considered as a statement regarding the pole at $s = 1$ of the Riemann zeta function in view of the Euler product. We shall require a stronger version. To obtain this, we apply the precise estimates on Chebyshev's function obtained in the previous section.

Lemma B.3. *Let $\epsilon > 0$ and suppose $x \geq x_0(\epsilon)$. Then*

$$\sum_{p \leq x} \frac{1}{p} = \log\log x + C_1 + \mathcal{O}\left(\frac{1}{L_\epsilon(x)}\right), \tag{B.2}$$

where C_1 is a constant.

Proof. We begin by letting $1/2 \leq \alpha \leq 1$ and taking $x \geq 2$, to estimate

$$\int_x^\infty \frac{\exp\left(-(\log y)^\alpha\right)}{y \log y}\, dy = \left(\frac{1}{\alpha} - 1\right) \int_{(\log x)^\alpha} \frac{e^{-t}}{t}\, dt \leq \exp\left(-(\log x)^\alpha\right).$$

We let $R(x) = \vartheta(x) - x$. Combining the estimated integral with Corollary B.2 we obtain

$$\int_x^\infty \frac{R(y)}{y^2 \log y}\, dy \ll \frac{1}{L_\epsilon(x)},$$

by setting $\alpha = 3/5 - \epsilon$. Furthermore, it implies that $R(y)/(y^2 \log y) \in L^1([2, \infty))$. We now apply Abel summation and $\vartheta(x) = x - R(x)$ to compute

$$\begin{aligned}\sum_{p \leq x} \frac{1}{p} = \sum_{p \leq x} \frac{\log p}{p \log p} &= \frac{\vartheta(x)}{x \log x} + \int_2^x \frac{\vartheta(y)(1 + \log y)}{(y \log y)^2}\, dy \\ &= \log\log x - \log\log 2 + \frac{1}{\log 2} + \frac{R(x)}{x \log x} + \int_2^x \frac{R(y)(1 + \log y)}{(y \log y)^2}\, dy.\end{aligned}$$

By Corollary B.2 it is clear that $R(x)/(x \log x)$ is absorbed in the error term of (B.2). What remains is to consider the integral, which we rewrite as

$$\int_2^\infty \frac{R(y)(1 + \log y)}{(y \log y)^2}\, dy - \int_x^\infty \frac{R(y)(1 + \log y)}{(y \log y)^2}\, dy = C' - \mathcal{O}\left(\frac{1}{L_\epsilon(x)}\right).$$

This implies (B.2) with $C_1 = 1/\log 2 - \log\log 2 + C'$. ☐

We may apply Lemma B.3 to compute a version of Mertens's formula with an unknown constant.

Lemma B.4. *Let $\epsilon > 0$ and suppose $x \geq x_0(\epsilon)$. Then*

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} = e^{C_2} \log x \left(1 + \mathcal{O}\left(\frac{1}{L_\epsilon(x)}\right)\right), \tag{B.3}$$

where C_2 is a constant.

PROOF. Using the Maclaurin series of $-\log(1-x)$ for $x = 1/p$ we obtain

$$-\sum_{p\le x}\log\left(1-\frac{1}{p}\right) = \sum_{p\le x}\sum_{n=1}^{\infty}\frac{1}{np^n} = \sum_{p\le x}\frac{1}{p} + \sum_{p}\sum_{n=2}^{\infty}\frac{1}{np^n} - \sum_{p>x}\sum_{n=2}^{\infty}\frac{1}{np^n}. \tag{B.4}$$

Now, we may easily estimate

$$\sum_{n=2}^{\infty}\frac{1}{np^n} \le \frac{1}{p^2}\sum_{n=0}^{\infty}\frac{1}{2p^n} \le \frac{1}{p^2}\sum_{n=1}^{\infty}\frac{1}{2^n} = \frac{1}{p^2},$$

and hence it is clear that (B.4) is of the form

$$-\sum_{p\le x}\log\left(1-\frac{1}{p}\right) = \sum_{p\le x}\frac{1}{p} + C'' - \mathcal{O}\left(\frac{1}{x}\right).$$

We obtain (B.3) using (B.2) and exponentiating, and setting $C_2 = C_1 + C''$. □

THEOREM B.5. *The constant C_2 of* (B.3) *is equal to γ, the Euler–Mascheroni constant.*

PROOF. Suppose $f(s)$ is the Dirichlet series associated with the coefficients a_n and abscissa of convergence σ_c. We introduce the scaled summatory function

$$B(x) = \sum_{n\le x}\frac{a_n}{n}.$$

The Dirichlet series with coefficients $b_n = a_n/n$ is $g(s) = f(s+1)$. We evaluate the Mellin transform of $g(s)$ at $s-1$, to obtain

$$f(s) = g(s-1) = (s-1)\int_1^{\infty}\frac{B(x)}{x^s}\,dx, \tag{B.5}$$

valid for $\Re(s) > \sigma_c$. We take $f(s) = \log\zeta(s)$. By integration of (2.25) we obtain the coefficients $a_n = \Lambda(n)/\log n$, and hence

$$B(x) = \sum_{n\le x}\frac{\Lambda(n)}{n\log n} = \sum_{p^m\le x}\frac{1}{mp^m}.$$

Let us turn to the product

$$P(x) = \prod_{p\le x}\left(1-\frac{1}{p}\right)^{-1}.$$

Using a geometric sum and the Prime Number Theorem, we may obtain

$$\log P(x) = -\sum_{p\le x}\log\left(1-\frac{1}{p}\right) = \sum_{p\le x}\sum_{n=1}^{\infty}\frac{1}{np^n} = B(x) + \sum_{p\le x}\sum_{m>\log_p x}\frac{1}{mp^m}$$
$$= B(x) + \mathcal{O}\left(\sum_{p\le x}\frac{1}{x}\right) = B(x) + \mathcal{O}\left(\frac{\pi(x)}{x}\right) = B(x) + \mathcal{O}\left(\frac{1}{\log x}\right).$$

Alternatively, by Lemma B.4, we may estimate

$$\log P(x) = \log\log x + C_2 + \mathcal{O}\left(\frac{1}{L_\epsilon(x)}\right),$$

and since $\log x \ll L_\epsilon(x)$ conclude that

$$B(x) = \log\log x + C_2 + \mathcal{O}\left(\frac{1}{\log x}\right),$$

for $x \ge 2$. Inserting this into (B.5), we obtain

$$\log\zeta(s) = f(s) = (s-1)\int_2^\infty \left(\log\log x + C_2 + \mathcal{O}\left(\frac{1}{\log x}\right)\right)\frac{dx}{x^s},$$

since $B(x) = 0$ for $1 \le x < 2$. We want to estimate this integral as $s \to 1^+$. Thus, let $1 < s \le 3/2$. We employ the substitution $y = (s-1)\log x$ to obtain

$$f(s) = \int_{(s-1)\log 2}^\infty \left(\log\frac{1}{s-1} + \log y + C_2 + \mathcal{O}\left(\frac{s-1}{y}\right)\right)e^{-y}\,dy.$$

The contribution of the $\mathcal{O}$-term may be estimated by considering

$$\int_{(s-1)\log 2}^\infty \frac{e^{-y}}{y}\,dy \le \int_{(s-1)\log 2}^1 \frac{dy}{y} + \int_1^\infty e^{-y}\,dy = \log\frac{1}{s-1} + \frac{1}{e} - \log\log 2,$$

which implies that it is of order $(s-1)\log 1/(s-1)$. For the remaining terms of the integral, we extend the integration to 0, which introduces an error of at most

$$\int_0^{(s-1)\log 2}\left(\log\frac{1}{s-1} + |\log y| + |C_2|\right)e^{-y}\,dy \ll (s-1)\log\frac{1}{s-1}.$$

This is the same order as the integrated $\mathcal{O}$-term, and is absorbed. By computing the extended integral we obtain the estimate

$$f(s) = \log\frac{1}{s-1} + C_2 + \int_0^\infty (\log y)\,e^{-y}\,dy + \mathcal{O}\left((s-1)\log\frac{1}{s-1}\right), \tag{B.6}$$

for $1 < s \le 3/2$. Now, we recall that $f(s) = \log\zeta(s)$. We now turn to Example 1.16 which allows us to estimate

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}\,dx = \frac{1}{s-1} + 1 + \mathcal{O}(1) = \frac{1}{s-1} + \mathcal{O}(1)$$

by the fact that $1 < s$. Taking logarithms and using $\log(1+x) = \mathcal{O}(x)$ for $0 < x \leq 1/2$ we may obtain

$$\text{(B.7)} \qquad \log\zeta(s) = \log\left(\frac{1}{s-1}(1+\mathcal{O}(s-1))\right) = \log\frac{1}{s-1} + \mathcal{O}(s-1),$$

when $1 < s \leq 3/2$. Equating (B.6) and (B.7), we may cancel $\log 1/(s-1)$ and take $s \to 1^+$ to obtain

$$C_2 = -\int_0^\infty (\log y)\, e^{-y}\, dy = -\Gamma'(1) = \gamma.$$

The second equality follows from the definition of the Gamma function and the third by logarithmic differentiation of its product representation in (2.2). □

Remark. The computation $C_2 = \gamma$ of Theorem B.5 also implies that

$$C_1 = \gamma - \sum_p \sum_{m=2}^{\infty} \frac{1}{mp^m} = \gamma + \sum_p \left(\frac{1}{p} + \log\left(1 - \frac{1}{p}\right)\right),$$

which is called the *Meissel–Mertens constant* and is ≈ 0.2614972.

B.4. The Riemann Hypothesis

In his celebrated paper [**34**], Riemann stated the following claim: $\zeta(s) \neq 0$ if $\Re(s) > 1/2$. This is one of the most famous open problems in mathematics. Assuming the Riemann Hypothesis to be true, we are able to replace the quantity

$$L_\epsilon(x) = \exp\left((\log x)^{3/5-\epsilon}\right)$$

by the much shaper estimate $\mathscr{L}_\epsilon(x) = x^{1/2-\epsilon}$ in the results of this appendix. By assuming the Riemann Hypothesis, we are able to move the integration as far left as $\eta = 1/2 + 1/\log x$ without encountering any zeroes. This is how the sharper estimate is obtained. The same improvement will also be true in Theorem 2.13 and Lemma 2.14. We are further able to extend these results to hold in the larger domain

$$\frac{1}{2} + \frac{1}{\log y} \leq \sigma \leq 1$$

and $|t| \leq \mathscr{L}_\epsilon(y)$. In [**21**] Hildebrand followed this argument to extend the domain (2.36) to

$$(\log x)^{2+\epsilon} \leq y \leq x$$

in Theorem 2.17. We do not supply proof of any of these claims.

Remark. The improved domain is sufficient for Lemmas 2.15 and 2.16, irregardless of the Riemann Hypothesis, but (2.36) is used throughout Chapter 3 for continuity.

Bibliography

1. Tom M. Apostol, *Introduction to analytic number theory*, Springer-Verlag, 1976.
2. Michel Balazard, *Remarques sur un théorème de G. Halász et A. Sárközy*, Bulletin de la Société Mathématique de France **117** (1989), no. 4.
3. Frédéric Bayart, *Hardy spaces of Dirichlet series and their composition operators*, Monatshefte für Mathematik **136** (2002), no. 3.
4. Ron C. Blei, *Fractional cartesian products of sets*, Annales de l'Institut Fourier **29** (1979), no. 2.
5. H.F. Bohnenblust and Einar Hille, *On the absolute convergence of Dirichlet series*, The Annals of Mathematics **32** (1931), no. 3.
6. Harald Bohr, *Sur la convergence des séries Dirichlet*, Comptes rendus Hebdomandaires des Séances de l'Académie des Sciences, Paris **151** (1910).
7. ______, *Darstellung der gleichmäßigen konvergenzabszisse einer Dirichletschen reihe* $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ *als funktion der koeffizienten der reihe*, Archiv der Mathematik und Physik **21** (1913), no. 3.
8. ______, *Über die bedeutnung der potenzreihen unendlich vielere variabeln in der theori der Dirichletscen reihen* $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$, Nachrichten von der Königlichen Gesellschaft der Wissenschafen zu Göttingen, Mathematisch–Physikalske Klasse (1913).
9. ______, *Über die gleichmaßige konvergenz Dirichletscher reihen*, Journal für die Reine und Angewandte Mathematik **143** (1913).
10. Enrico Bombieri and Jean Bourgain, *On Kahane's ultraflat polynomials*, Journal of the European Mathematical Society **11** (2009), no. 3.
11. Ole Fredrik Brevig, *The prime number theorem*, Norwegian University of Science and Technology (2010).
12. J. R. Campos, P. Jiménez-Rodríguez, G. A. Muõz-Fernàndez, D. Pellegrino, and J. B. Seoane-Sepùlveda, *The real polynomial Bohnenblust–Hille inequality is hypercontractive and its optimal asymptotic constant is 2*, ArXiv e-prints.
13. Régis de la Bretèche, *Sur l'ordre de grandeur des polynômes de Dirichlet*, Acta Arithmetica **134** (2008), no. 2.
14. Andreas Defant, Leonhard Frerick, Joaquim Ortega-Cerdà, Myriam Ounaïes, and Kristian Seip, *The Bohnenblust–Hille inequality for homogeneous polynomials is hypercontractive*, Annals of Mathematics **174** (2011), no. 1.
15. Gerald B. Folland, *Real analysis: Modern techniques and their applications*, Wiley, 1999.
16. Kevin Ford, *Vinogradov's integral and bounds for the Riemann zeta function*, Proceedings of the London Mathematical Society **85** (2002), no. 3.

17. Theodore W. Gamelin, *Complex analysis*, Springer-Verlag, 2001.
18. C. Gasquet and P. Witomski, *Fourier analysis and applications*, Springer-Verlag, 1998.
19. G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge University Press, 1952.
20. G. H. Hardy and Marcel Riesz, *The general theory of dirichlet series*, Cambridge University Press, 1915.
21. Adolf Hildebrand, *Integers free of large prime factors and the riemann hypothesis*, Mathematika **31** (1984), no. 2.
22. Adolf Hilebrand and Gérald Tenenbaum, *Integers without large prime factors*, Journal de Théori des Nombres Bordeux **5** (1993), no. 2.
23. Aleksander Ivic, *The riemann zeta function*, Wiley, 1985.
24. Jean-Pierre Kahane, *Sur les polynômes a coefficients unimodulaires*, Bulletin London Mathematical Society **12** (1980), no. 5.
25. ______, *Some random series of functions*, Cambridge University Press, 1985.
26. Sten Kaijser, *Some results in the metric theory of tensor products*, Studia Mathematica **63** (1978), no. 2.
27. S. V. Konyagin and H. Queffélec, *The translation $\frac{1}{2}$ in the theory of Dirichlet series*, Real Analysis Exchange **27** (2001), no. 2.
28. J. E. Littlewood, *On bounded bilinear forms in an infinite number of variables*, The Quarterly Journal of Mathematics **1** (1930), no. 1.
29. G. G. Lorentz, *Approximation of functions*, Holt, Rinehart and Wilson, 1966.
30. Brian Maurizi and Hervé Queffélec, *Some remarks on the algebra of bounded Dirichlet series*, Journal of Fourier Analysis and Applications **16** (2010), no. 5.
31. John H. McDonald and Neil A. Weiss, *A course in real analysis*, Academic Press, 1999.
32. Franz Mertens, *Ein beitrag zur analytischen zahlentheorie*, Journal für die reine und angewandte Mathematik **78** (1874).
33. Hervé Queffélec, *H. Bohr's vision of ordinary Dirichlet series; old a new results*, Journal of Analysis **3** (1995).
34. Bernhard Riemann, *Ueber die anzahl der primzahlen unter einer gegebenen grösse*, Monatsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin (1859).
35. Walter Rudin, *Function theory in polydiscs*, W. A. Benjamin, 1969.
36. ______, *Real and complex analysis*, McGraw-Hill, 1970.
37. Kristian Seip, *Estimates for Dirichlet polynomials*, EMS Lecturer (2012).
38. Gérald Tenenbaum, *Introduction to analytic and probabilistic number theory*, Cambridge University Press, 1990.
39. C. Visser, *A generalization of tchebychef's inequality to polynomials in more than one variable*, Proc. Nederlandse Academische Wetenschap **49** (1945), no. 8.