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# The Infinity Laplace Equation

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I would like to thank my parents for their support during my years at NTNU. I would also like to thank my supervisor, Peter Lindqvist, for his patience and guidance.

# Abstract

In this thesis, we prove that the $\infty$-Laplace equation has a unique solution in the viscosity sense. We prove existence by approximating the equation by the $p$-Laplace equation, and uniqueness will be shown by use of the Theorem on Sums. We will also show that the viscosity solutions of the $\infty$-Laplace equation enjoys comparison with cones, and vice versa.

# Sammendrag

I denne oppgaven viser vi at $\infty$-Laplace ligningen har en unik viskositetsløsning. Vi viser eksistens via approksimering med $p$-Laplace ligningen, og unikhet blir bevist via det såkalte Theorem on Sums. Vi tar også for oss sammenigning med kjegler, og viser at de funksjonene som sammenligner med kjegler er nøyaktig viskositetsløsningene av $\infty$-Laplace ligningen.

# Notation

Throughout this thesis, $\Omega$ means an open, bounded and simply connected subset of $\mathbb{R}^n$, unless explicitly stated otherwise. $|a|$ is the usual Euclidean norm of the vector $a \in \mathbb{R}^n$, while

$$\langle a, b \rangle$$

denotes the Euclidian inner product of the vectors $a, b \in \mathbb{R}^n$. For a function $u : \Omega \to \mathbb{R}$,

$$Du = \left( \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n} \right)$$

is the gradient of $u$, while

$$D^2 u = \begin{pmatrix} u_{x_1 x_1} & u_{x_1 x_2} & \cdots & u_{x_1 x_n} \\ u_{x_2 x_1} & u_{x_2 x_2} & \cdots & u_{x_2 x_n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{x_n x_1} & u_{x_n x_2} & \cdots & u_{x_n x_n} \end{pmatrix}$$

Is the Hessian of $u$. $\mathbb{S}^n$ is the space of all symmetric $n \times n$ matrices. Balls will be denoted by $B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$.

For any measurable subset $A$ of $\mathbb{R}^n$, we denote the $n$-dimentional Lebesgue measure of $A$ by $\mu(A)$.

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# Introduction

In 1967, Gunnar Aronsson considered the following problem: Given an open set $\Omega$ and a Lipschitz-continuous $f$ defined on $\partial\Omega$, is it possible to extend $f$ into $\Omega$ in the following way: given any open subset $U$ of $\Omega$, we have that the Lipschitz constant for $u$ on $U$ is less than the Lipschitz constant for $u$ on $\Omega$. Such a function is called an *absolutely* Lipschitz minimizing extension.

Aaronson proved that if $\Omega$ is a convex, bounded domain of $\mathbb{R}^n$, then an absolutely Lipschitz minimizing function $u$ must satisfy the celebrated $\infty$-Laplace equation:

$$\Delta_\infty u = \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j} = 0. \tag{1}$$

in $\Omega$, with $u = f$ on $\partial\Omega$, see [Aro67].

However, it was not possible to prove uniqueness of solutions of (1) with the tools available at the time. Furthermore, it became evident that solutions of the $\infty$-Laplace equation are not necessarily $C^2$. For example in $\mathbb{R}^2$, Aronsson provided the example

$$u(x, y) = x^{4/3} - y^{4/3}$$

which is absolutely Lipschitz minimizing, while $u$ is not twice differentiable at the coordinate axes. Thus, a more liberal concept of what being a solution of (1) means is needed. This concept turns out to be the viscosity solutions introduced in 1983 by Michael G. Crandall and Pierre-Louis Lions in [CL83]. With this machinery available, R. Jensen was able to prove the uniqueness of absolutely Lipschitz minimizing functions. Further applications of the equation is discussed in Section 6.2.

In Chapter 1, we will follow Aronsson's reasoning in deriving the $p$-Laplace equation as the Euler-Lagrange equation for a family of variational integrals of the type $I[u] = \|u\|_p^p$, and introduce the concept of weak solutions of this equation.

Having established existence and uniqueness of solutions of the $p$-Laplace equation, we introduce the notion of viscosity solution for this PDE in Chapter 2. There we also prove an important comparison principle for $p$-harmonic functions, and deduce from this that weak solutions of the $p$-Laplace equation are in fact viscosity solutions. Using these results, we will prove that as $p \to \infty$, the solutions of the $p$-Laplace equation converge to the solutions of the $\infty$-Laplace equation.

Having established these vital facts, we turn in Chapter 3 to two auxiliary equations, which will be deduced in the same way as the $p$-Laplace equation. The results obtained in this chapter will be important in the next chapter

Chapter 4 is dedicated to proving the uniqueness of $\infty$-harmonic functions. This will be done by considering two obstacle problems introduced by R. Jensen, together with the Theorem on Sums.

In Chapter 5, we introduce the concept of *comparison with cones*, first investigated by Crandall et. al. in [CEG01]. We prove that every $\infty$-harmonic function enjoys comparison with cones, and vice versa. These tools, together with Perron's method, will allow us to prove the existence of $\infty$-harmonic functions on *unbounded* domains of $\mathbb{R}^n$, provided we restrict the growth of the boundary function $f$.

Some comments regarding the regularity of the solutions, and some practical applications of the equation are presented in Chapter 6.

# Chapter 1

# The p-Laplace Equation

Following the strategy outlined in the introduction, we will in this chapter consider the problem of minimizing a family of variational integrals, and show that a minimizer must satisfy the $p$-Laplace equation. We also prove that the Diriclet problem

$$\begin{cases} \Delta_p u &= 0 \quad \text{in}\Omega \\ u &= f \quad \text{on}\partial\Omega \end{cases} \tag{1.1}$$

is well-posed and has a unique solution.

## 1.1 The Extension problem

We will first prove the existence of the largest and smallest *global* Lipschitz extensions.

Define the functional $\mathcal{L}$ by

$$\mathcal{L}(u,\Omega) = \inf\{L \ : \ |u(x) - u(y) \leq L|x-y| \ \forall x,y \in \Omega\}$$

Where, by definition, $\inf\emptyset = +\infty$. We then consider the following problem: Find a $u$ such that

$$\begin{cases} u \in C(\Omega), \ u = f \text{ on } \partial\Omega \\ \mathcal{L}(u,\Omega) = \min\{\mathcal{L}(v,\Omega) : v \in C(\bar{\Omega}), v = f \text{ on } \partial\Omega\} \end{cases} \tag{1.2}$$

where $f$ is a given continuous function on $\partial\Omega$. Since $\mathcal{L}(u,\partial\Omega) \leq \mathcal{L}(u,\Omega)$, it is clear that if $\mathcal{L}(f,\partial\Omega) = +\infty$, then any continuous $u$

agreeing with $f$ on $\partial\Omega$ will solve (1.2). However, if $\mathcal{L}(f, \partial\Omega) = L < \infty$, we have the following:

**Proposition 1.1.** *The functions $f^*(x)$ and $f_*(x)$ defined by*

$$f^*(x) = \inf_{z \in \partial\Omega} \{f(z) + L|x - z|\} \tag{1.3}$$

$$f_*(x) = \sup_{y \in \partial\Omega} \{f(y) - L|x - y|\} \tag{1.4}$$

*provide the maximal and minimal solution to the extension problem* (1.2)*. Thus, $f_* \le u \le f^*$ for any minimizer $u$.*

*Proof.* It is clear from the definition that $f^*(z) = f_*(z) = f(z)$ for $z \in \partial\Omega$. Furthermore, since the infimum and supremum of functions with Lipschitz constant $L$ have the same constant, we see that $\mathcal{L}(f^*(x), \Omega) = \mathcal{L}(f_*(x), \Omega) = L$. As for the minimal and maximal part, any function $u$ with Lipschitz constant $L$ must satisfy

$$\begin{aligned} f(z) - L|x - z| = u(z) - L|x - z| \le u(x) \\ f(y) + L|x - y| = u(y) + L|x - y| \ge u(x) \end{aligned}$$

for $y, z \in \partial\Omega$ and $x \in \Omega$. From this we conclude that

$$\sup_{z \in \partial\Omega} \{f(z) - L|x - z|\} \le u(x) \le \inf_{y \in \partial\Omega} \{f(y) + L|x - y|\}$$

and the proof is complete. □

The extensions $f^*(x)$ and $f_*(x)$ are called the *McShane-Whitney extensions.*

However, this solution is not very satisfactory. There is no reason for the maximal and minimal solution to coincide, in particular, the functional $\mathcal{L}$ is not *local*, that is if $u$ is a continuous function it does not follow that $\mathcal{L}(u, \Omega) \le \mathcal{L}(v, U)$ for every function defined on $V$ such that $u = v$ on $\partial\Omega$ whenever $U \subset \Omega$. This leads us to consider the following concept:

**Definition 1.2.** *A continuous function $u : \Omega \to \mathbb{R}$ is said to be an absolute minimizer for the functional $\mathcal{L}$ provided*

$$\mathcal{L}(u, U) \le \mathcal{L}(v, U) \text{ whenever } U \subset \Omega \text{ and } u = v \text{ on } \partial U \tag{1.5}$$

Functions with the property (1.5) are referred to as *absolutely Lipschitz minimizing.*

Now, define another functional

$$I_\infty(u,\Omega) = \|Du\|_{L^\infty(\Omega)} \tag{1.6}$$

It is clear from the Mean Value Theorem that $I_\infty(u,\Omega) \geq \mathcal{L}(u,\Omega)$, however equality may not hold in general. This implies that $u$ is absolutely minimizing for $\mathcal{L}$ if $u$ is absolutely minimizing for $I_\infty$. This means that we can can consider the the problem of minimizing the functional (1.6) instead of (1.5). This will be discussed in the next section.

## 1.2 The Variational Integral

We start by considering the variational integral

$$I[u] = \int_\Omega |Du(x)|^p \mathrm{d}x = \|Du\|_p^p \tag{1.7}$$

Where $p$ is a natural number, $1 < p < \infty$. If $p = 2$, we have the well-known Dirichlet integral, and minimizing this will lead to the Laplace equation.

We want to (1.7) over the class of admissible functions in the Sobolev space $W^{1,p}(\Omega)$, and boundary values $f : \partial\Omega \to \mathbb{R}$ in Sobolev's sense, that is $\mathcal{A} = \{u \in W^{1,p}(\Omega),\ u - f \in W_0^{1,p}(\Omega)\}$.

The aim of this section is to derive the Euler-Lagrange equation for (1.7). In the following, we assume that the boundary function $f$ is Lipschitz continuous with Lipschitz constant $L$:

$$L = \|f\|_{\mathrm{Lip}} = \inf\Big\{L \ : \ |f(x) - f(y)| \leq L|x-y| \ \ \forall x, y \in \Omega\Big\}$$

First, we derive the Euler-Lagrange equation for (1.7).

**Proposition 1.3.** *A function $u \in \mathcal{A}$ minimizes* (1.7) *if and only if*

$$\int_\Omega |Du|^{p-2} Du \cdot D\phi \ \mathrm{d}x = 0 \tag{1.8}$$

*for all test functions $\phi \in C_0^\infty(\Omega)$.*

**Remark.** In fact, we can consider test functions $\phi$ in $W_0^{1,p}(\Omega)$ instead of $C_0^\infty(\Omega)$ in the formulation above. The proof of this is given in Lemma 1.10 at the end of this chapter.

*Proof.* First, assume that $u$ minimizes (1.7). Employing the trick by Lagrange, let $\phi$ be a test function, and $\epsilon$ a real number. We know that the function $u+\epsilon\phi$ is admissible, and that the function $\psi(\epsilon) = I[u+\epsilon\phi]$ attains its minimum at $\epsilon = 0$, since $u$ is assumed to be minimizing. This means that $\psi'(0) = 0$, and calculating:

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} \int_\Omega |Du + \epsilon D\phi|^p \,\mathrm{d}x = p \int_\Omega |Du + \epsilon D\phi|^{p-2} Du \cdot D\phi \,\mathrm{d}x,$$

which, for $\epsilon = 0$ gives

$$p \int_\Omega |Du|^{p-2} Du \cdot D\phi \,\mathrm{d}x = 0.$$

Division by $p$ gives (1.8)

Now, assume that $u$ satisfies (1.8). For all vectors $a, b \in \mathbb{R}^n$ and all $p \geq 1$, we have the inequality

$$|b|^p \geq |a|^p + p|a|^{p-2} a \cdot (b - a).$$

This follows from the fact that the function $x \mapsto |x|^p$ is convex for $p \geq 1$. This implies that, for any $v \in W^{1,p}(\Omega)$, we have

$$|Dv|^p \geq |Du|^p + p|Du|^{p-2} Du \cdot D(v - u). \tag{1.9}$$

Integrating this inequality over $\Omega$, we get

$$\int_\Omega |Dv|^p \,\mathrm{d}x \geq \int_\Omega |Du|^p \,\mathrm{d}x + p \int_\Omega |Du|^{p-2} Du \cdot D(v - u) \,\mathrm{d}x.$$

Since (1.8) holds for any $\phi \in W_0^{1,p}(\Omega)$, it also holds if we choose $v - u$ as our test function, and the last integral disappears. Thus

$$\int_\Omega |Dv|^p \,\mathrm{d}x \geq \int_\Omega |Du|^p \,\mathrm{d}x.$$

Since $v$ was arbitrary, it follows that $u$ minimizes the variational integral. $\square$

Under suitable restrictions on $u$, for example demanding that it is $C^2$, we can use integration by parts and get

$$\int_\Omega D\big(|Du|^{p-2}Du\big)\phi \,\mathrm{d}x = 0,$$

since $\phi$ is compactly supported.

By the Variational Lemma, this implies that

$$D \cdot (|Du|^{p-2}Du) = \operatorname{div}(|Du|^{p-2}Du) = \Delta_p u = 0.$$

for $p \geq 2$. Writing this out, we have

$$\begin{aligned} \Delta_p u &= |Du|^{p-4}\Big(|Du|^2\Delta u + (p-2)\sum_{i,j}^{n} u_{x_i}u_{x_j}u_{x_i u_{x_j}}\Big) && (1.10)\\ &= |Du|^{p-4}(|Du|^2\Delta u + (p-2)\Delta_\infty u) = 0 && (1.11) \end{aligned}$$

This is the $p$-Laplace equation. Notice that if $p = 2$, we are left with the Laplace equation, as mentioned earlier. Here we also see the $\infty$-Laplace operator, defined as

$$\Delta_\infty u = \sum_{i,j}^{n} u_{x_i}u_{x_j}u_{x_i u_{x_j}}.$$

Note that this can also be written as

$$\langle D^2uDu, Du\rangle.$$

This formulation will be useful to us later.

**Definition 1.4.** *We say that $u \in W^{1,p}(\Omega)$ is a weak subsolution of the p-Laplace equation if, for all test functions $\phi \in C_0^\infty(\Omega)$:*

$$\int_\Omega |Du|^{p-2}Du \cdot D\phi \,\mathrm{d}x \geq 0.$$

*Similarly, we say that $u \in W^{1,p}(\Omega)$ is a weak supersolution of the p-Laplace equation if, for all test functions $\phi \in C_0^\infty(\Omega)$:*

$$\int_\Omega |Du|^{p-2}Du \cdot D\phi \,\mathrm{d}x \leq 0.$$

*If $u$ is both a weak sub- and a supersolution, we simply say that $u$ is a weak solution.*

*If $u$ also is continuous, we say that $u$ is* p-harmonic in $\Omega$.

**Remark.** We have that if $\Omega \subset \mathbb{R}^n$ and $u \in W^{1,p}(\Omega)$ for $p > n$, then there exists a $u^* \in C(\Omega)$ such that

$$u = u^* \text{ a.e.}$$

This is a consequence of Morrey's Inequality, the proof of which can be found in [Jos02, p.267]. Since we will consider the $p$-Laplace equation for $p \to \infty$, we can assume that $p > n$, and thus we can always identify $u$ with its continuous representative, and we will do so without further comment.

We will now prove that there exists a minimizer to (1.7), and that it is unique. We start with uniqueness:

**Proposition 1.5.** *If a function $u$ that minimizes* (1.7) *exists, it is unique.*

*Proof.* We argue by contradiction. Assume that $u, v \in W_0^{1,p}(\Omega)$ are such that $u \neq v$, and that both minimize (1.7). Since both are admissible functions, the function

$$\frac{u+v}{2}$$

is admissible. The inequality

$$\left(\frac{|Du + Dv|}{2}\right)^p \leq \frac{|Du|^p + |Dv|^p}{2}$$

is strict if $Du \neq Dv$. If $Du \neq Dv$ in a set of positive measure, it follows that

$$\begin{aligned} M = \int_\Omega |Du|^p \, \mathrm{d}x &\leq \int_\Omega \frac{|Du + Dv|^p}{2} \, \mathrm{d}x \\ &< \frac{1}{2} \int_\Omega |Du|^p \, \mathrm{d}x + \frac{1}{2} \int_\Omega |Dv|^p \, \mathrm{d}x = M. \end{aligned}$$

This is impossible, and we must have $Du = Dv$, and thus $u = v + k$ for some constant $k$. But $u = v = f$ on $\partial\Omega$, and we are forced to conclude that $u = v$ in $\Omega$. $\square$

**Proposition 1.6.** *There exists a unique $u$ that minimizes* (1.7) *among all the admissible functions in $\mathcal{A}$.*

*Proof.* We will employ the so-called direct method due to Lebesgue, see for example [Jos02, p. 281]. Define

$$m = \inf \left\{ \int_\Omega |Du(x)|^p \mathrm{d}x \right\} = \inf \left\{ ||Du(x)||_p^p \right\},$$

where the infimum is taken over all admissible functions $u - f \in W_0^{1,p}(\Omega)$ Since $u$ is in $W^{1,p}(\Omega)$ , we have that $0 \leq m < \infty$ since $\|Du\|_{L^p} \leq \|u\|_{W^{1,p}}$. Now let $\{u_\nu\}_{\nu=1}^\infty$ be a sequence of functions in $W^{1,p}(\Omega)$ such that

$$\int_\Omega |Du_\nu|^p \, \mathrm{d}x \to m$$

as $\nu \to \infty$. Such a minimizing sequence exists by the definition of infimum. We can assume that

$$\int_\Omega |Du_\nu|^p \, \mathrm{d}x \leq m + 1$$

for every $\nu$. This implies that the sequence $\{\|Du_\nu\|_p\}_{\nu=1}^\infty$ is bounded. Furthermore, since $f \in W^{1,p}(\Omega)$ we get the estimate

$$\|u_\nu\|_p \leq \|f\|_p + \|f - u_\nu\|_p$$

by Minkowski's inequality. Using Sobolev's inequality [Jos02, p. 257] and Minkowski again, we get the bound

$$\begin{aligned} \|u_\nu\|_p &\leq \|f\|_p + C_\Omega \|Df - Du_\nu\|_p \\ &\leq \|f\|_p + C_\Omega \Big( (m+1)^{1/p} + \|Df\|_p \Big), \end{aligned}$$

where $C_\Omega$ is a constant depending upon $\Omega$. Since this bound is independent of $\nu$, Theorem B.2 gives the existence of functions $u$ and $w$ such that

$$\begin{aligned} \lim_{j\to\infty} u_{\nu_j} &= u \\ \lim_{j\to\infty} Du_{\nu_j} &= w \end{aligned}$$

weakly in $L^p(\Omega)$, for some subsequence of $\{u_\nu\}$. By the definition of the weak derivative, we must have $w = Du$. Since $u_{\nu_j} - f$ is in $W_0^{1,p}(\Omega)$, we get that $u - f \in W_0^{1,p}(\Omega)$ since this space is closed under weak convergence, that is, Theorem B.7.

Furthermore, we have that for all j

$$\int_\Omega |Du_{\nu_j}|^p \, \mathrm{d}x \geq \int_\Omega |Du|^p \, \mathrm{d}x + p \int_\Omega |Du|^{p-2} Du \cdot D(u_{\nu_j} - u) \, \mathrm{d}x$$

by (1.9). Notice that since $Du \in L^p(\Omega)$, we have that $|Du|^{p-2}Du \in L^q(\Omega)$ where $q = p/(p-1)$. Since $Du_{\nu_j} \rightharpoonup Du$ in $L^p(\Omega)$, we have that the last integral converges to 0 as $j \to \infty$. We have thus proved that the weak limit of the $u_{\nu_j}$'s is the minimizer sought for. □

**Remark.** In proving that $u$ is the minimizer, we could also have used the fact that the $L^p$-norm is weakly lower semicontinuous. In fact, we proved this in the calculation above.

## 1.3 Comparison Principle

Having derived the weak form of the $p$-Laplace equation, and shown existence and uniqueness of solution, we now prove the fundamental Comparison Principle. This is a a stronger result than simply uniqueness, and will be important to us later. First, however, we need a inequality which will also be useful to us later.

**Lemma 1.7.** *For any vectors $a, b \in \mathbb{R}^n$, the following holds for $p \geq 2$:*

$$\left(|b|^{p-2}b - |a|^{p-2}a)\right) \cdot \ (b - a) \geq 2^{2-p}|b - a|^p \tag{1.12}$$

*Proof.* By direct calculation, the following holds

$$\begin{aligned}\left(|b|^{p-2}b - |a|^{p-2}a\right) \cdot \ (b - a) &= \frac{1}{2}\left(|b|^{p-2} - |a|^{p-2}\right)\left(|b|^2 - |a|^2\right) \\ &+ \frac{1}{2}\left(|b|^{p-2} + |a|^{p-2}\right)|b - a|^2.\end{aligned}$$

This gives directly

$$\left(|b|^{p-2}b - |a|^{p-2}a\right) \cdot \ (b - a) \geq \frac{1}{2}\left(|b|^{p-2} + |a|^{p-2}\right)|b - a|^2,$$

which in turn implies that

$$\big(|b|^{p-2}b - |a|^{p-2}a)\big) \cdot \ (b-a) \geq 2^{2-p}|b-a|^p,$$

and the inequality is proved. □

We are now in a position to prove the following:

**Proposition 1.8** (Comparison Principle)**.** *Assume that $u$ and $v$ are $p$-harmonic in $\Omega$, and that $u \leq v$ on $\partial\Omega$. Then $u \leq v$ in $\Omega$.*

*Proof.* We argue by contradiction. Assume that there exists some subset $A$ of $\Omega$ with positive measure where $u > v$: $A = \{x \in \Omega \; |u > v\}$ Notice that $u = v$ on $\partial A$. By definition of weak solution we have

$$\int\limits_{\Omega} |Du|^{p-2} Du \cdot D\phi \, \mathrm{d}x = 0$$

$$\int\limits_{\Omega} |Dv|^{p-2} Dv \cdot D\phi \, \mathrm{d}x = 0$$

for any test function $\phi \in W_0^{1,p}(\Omega)$. Subtracting these two equations, we get

$$\int\limits_{\Omega} \Big( Du|Du|^{p-2} - Dv|Du|^{p-2} \Big) \cdot D\phi \, \mathrm{d}x = 0.$$

Choose $\phi = \max\{u-v, 0\} = (u-v)^+$. This function is in $W_0^{1,p}(\Omega)$, and we have that

$$D\phi = \begin{cases} D(u-v) & \text{if } u > v \\ 0 & \text{if } u \leq v \end{cases}.$$

That is, $D(u-v)$ is zero outside of A. Inserting this into our equation, we arrive at

$$\int\limits_{A} \Big( Du|Du|^{p-2} - Dv|Du|^{p-2} \Big) \cdot D(u-v) \, \mathrm{d}x = 0.$$

Since $p \geq 2$, we can use (1.12) on the vectors $Du$ and $Dv$, and see that the integrand above is positive. This implies that

$$\Big( Du|Du|^{p-2} - Dv|Du|^{p-2} \Big) \cdot D(u-v) = 0$$

almost everywhere in $A$. This in turn implies that $u(x) = v(x) + K$ for a.e. $x \in A$. But since $u$ and $v$ are equal on $\partial A$, this implies that $K = 0$ and $u = v$ within $A$, contradicting our assumption that $u > v$ in $A$. $\square$

From this Proposition, we get the following corollary:

**Corollary 1.9.** *The Dirichlet problem* (1.1) *has a unique solution.*

**Remark.** The statement of the proposition is also true if $u$ is a weak supersolution and $v$ is a weak subsolution of the $p$-Laplace equation. In this case, we have

$$\int_{\Omega} |Du|^{p-2} Du \cdot D\phi \, \mathrm{d}x \leq 0$$
$$\int_{\Omega} |Dv|^{p-2} Dv \cdot D\phi \, \mathrm{d}x \geq 0$$

Subtracting these equations and choosing the same $\phi$ as above, the proof above carries over to this case with no modification.

In the proofs above, we have chosen test-functions that are not necessarily smooth. The following lemma assures us that this is possible for the $p$-Laplace equation.

**Lemma 1.10.** *Any $w \in W_0^{1,p}(\Omega)$ is a viable test function in the weak formulation for the $p$-Laplace equation, that is*

$$\int_{\Omega} |Du|^{p-2} Du \cdot Dw \quad \mathrm{d}x = 0 \ \forall \ w \in W_0^{1,p}(\Omega),$$

*if and only if*

$$\int_{\Omega} |Du|^{p-2} Du \cdot D\phi \quad \mathrm{d}x = 0 \ \forall \ \phi \in C_0^{\infty}(\Omega).$$

*Proof.* The first implication is clear, since $C_0^\infty(\Omega) \subset W_0^{1,p}(\Omega)$.

For the other, we have by the definition of a function $w$ being in $W_0^{1,p}(\Omega)$, given an $\epsilon > 0$ there exists a $\psi \in C_0^\infty(\Omega)$ such that

$$\|w - \psi\|_{W_0^{1,p}} < \epsilon.$$

Then we have

$$\begin{aligned}\left|\int_\Omega |Du|^{p-2} Du \cdot Dw \, \mathrm{d}x\right| &\leq \left|\int_\Omega |Du|^{p-2} Du \cdot D(w-\psi) \, \mathrm{d}x\right| \\ &+ \left|\int_\Omega |Du|^{p-2} Du \cdot D\psi \, \mathrm{d}x\right| = \left|\int_\Omega |Du|^{p-2} Du \cdot D(w-\psi) \, \mathrm{d}x\right|\end{aligned}$$

since the last integral is equal to zero by the definition of weak solution. Employing the triangle inequality and then Hölder's inequality, we get that

$$\begin{aligned}&\left|\int_\Omega |Du|^{p-2} Du \cdot Dw \, \mathrm{d}x\right| \leq \int_\Omega |Du|^{p-1} |D(w-\psi)| \, \mathrm{d}x \\ &\leq \||Du|\|_p^{p-1} \|D(w-\psi)\|_p \leq \||Du|\|_p^{p-1} \|w-\psi\|_{W_0^{1,p}(\Omega)} < \||Du|\|_p^{p-1} \epsilon\end{aligned}$$

Since $\epsilon$ was arbitrary, this implies that any $w \in W_0^{1,p}(\Omega)$ is a usable test function. $\square$

# Chapter 2

# Viscosity Solutions

In this chapter we begin by defining the concept of viscosity solutions for a partial differential equation, and prove that weak solutions of the $p$-Laplace equation are viscosity solutions. This makes it possible to prove that as $p \to \infty$, the viscosity solutions of the $p$-Laplace equations converge to a viscosity solution of the $\infty$-Laplace equation.

## 2.1 Definitions

We start by defining the concept of a viscosity solution for the $p$-Laplace equation, and prove that weak solutions are in fact viscosity solutions. For a brief introduction to the general theory of viscosity solutions, see Appendix A.

**Definition 2.1** (Viscosity Solution)**.** *We say that a lower semi-continuous function $u$ is a viscosity subsolution of $\Delta_p u = 0$ if, for any point $x \in \Omega$ and any $\psi \in C^2(\Omega)$ we have that*

$$\Delta_p \psi \geq 0$$

*provided that $u - \psi$ attains it maximum at $x$. Likewise, we say that a upper semi-continuous function $u$ a viscosity supersolution of the equation if*

$$\Delta_p \psi \leq 0$$

*provided $u - \psi$ attains its minimum at $x$.*

If $u$ is both a viscosity sub- and supersolution, it is continuous, and we simply say that $u$ is a *viscosity solution*.

**Remark.** Notice the disparity with definition A.2. This comes from the fact that $F(x, u, Du, D^2u) = \Delta_p u$ does not define a positive operator, while $F(x, u, Du, D^2u) = -\Delta_p u$ does. Thus all inequalities are reversed. One could have considered $-\Delta_p u = 0$ in the following, but we will stick to the convention above.

If $\psi \leq u$ at $x_0$, we can modify it by setting

$$\psi^*(x) = \psi(x) + |x - x_0|^4.$$

Then $\psi^*$ is also twice continuously differentiable, and $\psi^* < \psi \leq u$. Furthermore,

$$\begin{aligned}
&\psi^*(x_0) = \psi(x_0),\\
&D\psi^*(x_0) = D\psi(x_0),\\
&D^2\psi^*(x_0) = D^2\psi(x_0).
\end{aligned}$$

Thus we can always assume that $u - \psi$ attains its *strict* maximum or minimum in the definitions above.

**Proposition 2.2.** *If $u$ is a weak solution of the $p$-Laplace equation as defined in section 1.3, then $u$ is also a viscosity solution.*

*Proof.* We will prove that if $u$ is a weak supersolution, then $u$ is also a viscosity supersolution. The case for subsolutions is analogous. We will make the following *antithesis*:

Assume that $u - \psi$ has a maximum at $x_0 \in \Omega$, while $\Delta_p\psi(x_0) < 0$.

Since $\psi \in C^2(\Omega)$, there exists a $B_\delta(x_0)$ where $\psi$ is $p$-superharmonic. Define

$$m_\delta = \max_{x \in \partial B_\delta(x_0)} \{u - \psi\}.$$

We then have that $m_\delta > 0$, since $u < \psi$ in $\Omega \setminus \{x_0\}$. Consider the function $\overline{\psi} = \psi + m_\delta/2$. We then have that $\overline{\psi}$ is $p$-superharmomic, and $\overline{\psi} < u$ on $\partial B_\delta(x_0)$. Consider the set

$$A = \{x \mid u < \overline{\psi}\}.$$

Note that $x_0 \in A \subset B_\delta(x_0)$ by our antithesis. But the Comparison Principle implies that since $u$ is $p$-subharmonic in $\Omega$, and $\overline{\psi} < u$ on $\partial B_\delta(x_0)$, we must have $\overline{\psi} < u$ in $B_\delta(x_0)$, but this in turn implies that $A = \emptyset$, a contradiction. $\square$

**Remark.** In fact, it can be shown that viscosity solutions of the $p$-Laplace equation also are weak solutions. But since the proof is fairly difficult, and we will not need it, it is not included here. See [Juu01] for the proof.

## 2.2 Limit Procedure

We now want to prove that as $p \to \infty$, the viscosity solutions of the $p$-Laplace equation converge to the viscosity solution of the $\infty$-Laplace equation. Before we proceed, we need the following lemma, which is a consequence of Hölder's Inequality.

**Lemma 2.3.** *Assume that $f : \Omega \to \mathbb{R}$ is a measurable function, and that $\mu(\Omega) < \infty$. For $0 < q < p \leq \infty$, the following inequality holds :*

$$\left\{ \frac{1}{\mu(\Omega)} \int_\Omega |f|^q \, \mathrm{d}x \right\}^{\frac{1}{q}} \leq \left\{ \frac{1}{\mu(\Omega)} \int_\Omega |f|^p \, \mathrm{d}x \right\}^{\frac{1}{p}}. \tag{2.1}$$

*And consequently, $L^p(\Omega) \subset L^q(\Omega)$*

*Proof.* Since

$$\frac{q}{p} + \frac{p-q}{p} = 1$$

they are conjugate pairs. This is why we needed $p < q$, to ensure that $(p-q)/p$ is positive. From Hölder's Inequality, we have

$$\int_\Omega |f|^q \, \mathrm{d}x \leq \left( \int_\Omega |f|^p \, \mathrm{d}x \right)^{\frac{q}{p}} \left( \int_\Omega 1 \, \mathrm{d}x \right)^{\frac{p}{p-q}} = \left( \int_\Omega |f|^p \, \mathrm{d}x \right)^{\frac{q}{p}} \mu(\Omega)^{\frac{p}{p-q}}$$

Taking the $q$-th root on both sides, we arrive at

$$\left( \int_\Omega |f|^q \, \mathrm{d}x \right)^{\frac{1}{q}} \leq \left( \int_\Omega |f|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \mu(\Omega)^{\frac{p-q}{pq}}.$$

Division by $\mu(\Omega)^{\frac{1}{q}}$ Gives (2.1). $\square$

Now, let $u_p$ denote the solutions of the $p$-Laplace equation, and let $u$ denote the solutions of the $\infty$-Laplace equation. As stated earlier, we want to prove that the sequence of solutions $\{u_p\}_{p=2}^{\infty}$ converge to a solution of the $\infty$-Laplace equation. To reach this result, we want to use the weak compactness of $L^p$ to find a convergent subsequence of $\{u_p\}_{p=2}^{\infty}$, and show that it converges to a function in $W^{1,\infty}(\Omega)$.

First, we need a uniform bound on $\|Du\|_p$. To find this, we fix $q$ so that $n < q < p$. We know from Section 1.3 that weak solutions are variational solutions, and that weak solutions are viscosity solutions. Hence we know that $u_p$ minimises the variational integral (1.7), and also that the function $f$ is admissible. Thus we get

$$\begin{aligned}\left\{\frac{1}{|\Omega|}\int_{\Omega}|Du_p|^q\,\mathrm{d}x\right\}^{\frac{1}{q}} &\leq \left\{\frac{1}{|\Omega|}\int_{\Omega}|Du_p|^p\,\mathrm{d}x\right\}^{\frac{1}{p}}\\ &\leq \left\{\frac{1}{|\Omega|}\int_{\Omega}|Df|^p\,\mathrm{d}x\right\}^{\frac{1}{p}}\\ &\leq \|Df\|_{\infty} = L.\end{aligned}$$

Where $L$ is the Lipschitz constant for $f$. This, together with Lemma 2.3, implies that

$$\||Du_p|\|_q \leq L\mu(\Omega)^{1/q}. \tag{2.2}$$

This bound does not depend on $p$. Furthermore, by Minkowski's and then Sobolev's inequalities:

$$\begin{aligned}\|u_p\|_q &\leq \|u_p - f\|_q + \|f\|_q\\ &\leq C\||Du_p - Df|\|_q + \|f\|_q \leq C(\||Du_p|\|_q + \||Df|\|_q) + \|f\|_q.\end{aligned}$$

This estimate, together with (2.2) implies that

$$\|u_p\|_q \leq CL(\mu(\Omega)^{1/q} + 1) + \|f\|_q, \tag{2.3}$$

for $p > q > n$.

The weak compactness of $L^q(\Omega)$, that is Theorem B.2, now implies the existence a subsequence (also labeled $\{u_p\}$) of $\{u_p\}$ and a function

$u$ such that that

$$\begin{aligned} u_p &\rightharpoonup u \\ Du_p &\rightharpoonup w \end{aligned}$$

in $L^q(\Omega)$, where again we must have that $w = Du$ by the definition of the weak derivative.

Since $Du_p \rightharpoonup w$ in *each* $L^q(\Omega)$, we have that $Du \in L^\infty(\Omega)$, and thus $u \in W^{1,\infty}(\Omega)$. Furthermore, the estimates (2.2) and (2.3) implies that

$$\|u_p\|_{W^{1,q}(\Omega)} \leq C,$$

and so the Rellich-Kondrachov Theorem [Jos02, p. 265] proves that the sequence $\{u_p\}$ in fact converges pointwise a.e. to $u$.

We now want to show the existence of a subsequence of $\{u_p\}$ that converges uniformly on compact sets. We will do this by proving the following:

**Proposition 2.4.** *The sequence $\{u_p\}$ is equicontinous.*

*Proof.* Fix any $q$ so that $n < q < p$. For any cube $Q \subseteq \Omega$, and $q > n$, Theorem 7.17 in [GT01] gives the following estimate:

$$|u_p(x) - u_p(y)| \leq \frac{2qn}{q-n}|x-y|^{1-n/q}\|Du_p\|_q.$$

Then, because of (2.3), we have:

$$|u_p(x) - u_p(y)| \leq \frac{2qn}{q-n}|x-y|^{1-n/q}L\mu(\Omega)^{1/q}.$$

By (2.2). Also, since $2n > \frac{2qn}{q-n}$ for all $q > n$, we get

$$|u_p(x) - u_p(y)| \leq 2nL\mu(\Omega)^{1/q}|x-y|^{1-1/q} \leq 2n\mu(\Omega)^{1/q}\mathrm{diam}(\Omega)^{1-1/q}.$$

We also need to know that $u_p$ is bounded in $\Omega$ for every $p$. To see that this is true, Theorem 20.15 in [Jos02] guarantees the existence of a constant $c = c(n,q)$ such that

$$\sup_{x\in\Omega}|u_p(x)| \leq c(n,q)\mu(\Omega)^{\frac{1}{n}-\frac{1}{q}}\|Du_p\|_q$$

for every $q > n$. As in the above, fix $q$. By (2.2) we have

$$\sup_{x\in\Omega} |u_p(x)| \le c(n,q)L\mu(\Omega)^{\frac{1}{n}}.$$

This bound is free of $p$, and so the sequence $\{u_p\}$ is uniformly bounded in $\Omega$. Together, we have that $\{u_p\}$ is uniformly equicontinous in $Q$, $\square$

The Arzela-Ascoli Theorem [Eva98, p. 718] then gives the existsence of a subsequence of $\{u_p\}$, say $\{u_{p_j}\}$ that converges uniformly to a continous function $u$ on compact subsets of $\Omega$. This $u$ will be our candidate for a viscosity solution of the $\infty$-Laplace equation.

**Proposition 2.5.** *As $p_j \to \infty$, the viscosity solutions of $\Delta_{p_j} u = 0$ converge to viscosity solutions of*

$$\Delta_\infty u = \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j} = 0. \tag{2.4}$$

*Proof.* We will prove that supersolutions of $\Delta_{p_j} u = 0$ converge to supersolutions of (2.4). The proof for subsolutions is similar.

To this end, let $u_j$ denote $u_{p_j}$ for simplicity. Assume that $x_0 \in \Omega$ and $\psi \in C^2(\Omega)$ are so that

$$(u-\psi)(x_0) > (u-\psi)(x)\ ,\ x \in \Omega \setminus \{x_0\},$$

that is, $u-\psi$ has a maximum at $x_0$. We want to show that this implies

$$\Delta_\infty \psi(x_0) \le 0.$$

Since $u_j$ converges uniformly to $u$, there exist points $\{x_j\}$ in $\Omega$ such that $u_j - \psi$ has a maximum at $x_j$. We then must have that $x_j \to x_0$, and furthermore, by the definition viscosity solution, we have

$$\Delta_{p_j} \psi(x_j) \le 0$$

for every $j \ge N$. Expanding the expression above, we arrive at

$$|D\psi(x_j)|^{p_j-2}\Delta\psi(x_j) + (p_j-2)|D\psi(x_j)|^{p_j-4}\Delta_\infty\psi(x_j) \le 0.$$

If $|D\psi(x_j)|$ is equal to 0 for any $j$, there is nothing to prove, since then $\Delta_\infty\psi(x_j) = 0$ and the inequality holds trivially. So we can safely assume that $|D\psi(x_j)| > 0$ for $j \geq N$. Dividing through by $(p_j - 2)|D\psi(x_j)|^{p_j-2} > 0$, we get

$$|D\psi(x_j)|^2 \frac{\Delta\psi(x_j)}{p_j - 2} + \Delta_\infty\psi(x_j) \leq 0.$$

Letting $j \to \infty$, we have that $p_j \to \infty$, and

$$D\psi(x_j) \to D\psi(x_0),$$
$$D^2\psi(x_j) \to D^2\psi(x_0)$$

by the continuity of $\psi$. This implies that, since $\Delta\psi(x_j) \to \Delta\psi(x_0)$:

$$\Delta_\infty\psi(x_0) \leq 0.$$

This shows that $u$ is a viscosity supersolution of (2.4). $\square$

Chapter 3

# Auxiliary Equations

In this chapter, we study the equations

$$\max\{\epsilon - |Dv|, \Delta_\infty v\} = 0 \quad \text{Upper Equation} \tag{3.1}$$

$$\min\{|Du| - \epsilon, \Delta_\infty u\} = 0 \quad \text{Lower Equation} \tag{3.2}$$

where the parameter $\epsilon$ is so that $0 < \epsilon < 1$. These equations were first introduced by Robert Jensen in [Jen93], and will be used to “squeeze" the solutions of (2.4) between the Lower and Upper equation, for a positive $\epsilon$ given that they all are equal to a function $f$ on $\partial\Omega$.

Inspired by Chapter 1.2, we will derive the equations by studying a family of variational integrals, and prove that viscosity solutions of (3.1) and (3.2) are the limit of a sequence of minimizers of these variational integrals.

## 3.1 The Upper Equation

First, we need to prove the existence of weak solutions of (3.1) and (3.2). To do this, consider the problem of minimizing the following variational integral

$$I[u] = \int_\Omega \frac{|Du|^p}{p}\,\mathrm{d}x - \int_\Omega \epsilon^{p-1} u \mathrm{d}x \tag{3.3}$$

among all $u \in W^{1,p}(\Omega)$ and $u = f$ on $\partial\Omega$ in Sobolev's sense, that is $(u - f) \in W_0^{1,p}(\Omega)$. Again employing the device by Lagrange, we get the following proposition:

**Proposition 3.1.** *If a function $u \in W^{1,p}(\Omega)$ minimizes (3.3), then*

$$\int_\Omega |Du|^{p-2} Du \cdot D\phi \, \mathrm{d}x = \int_\Omega \epsilon^{p-1} \phi \, \mathrm{d}x \tag{3.4}$$

*for all $\phi \in C_0^\infty(\Omega)$*

*Proof.* We have that if $u$ minimizes (3.3), the function $\sigma(\tau) = I[u+\tau\phi]$ has a minimum at $\tau = 0$. Calculating the derivative, we get

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\int_\Omega \frac{|Du+\tau\phi|^p}{p} \, \mathrm{d}x - \int_\Omega \epsilon^{p-1}(u+\tau\phi) \, \mathrm{d}x\right) \\ &= \int_\Omega |Du+\tau\phi|^{p-2} Du \cdot D\phi \, \mathrm{d}x - \int_\Omega \epsilon^{p-1}\phi \, \mathrm{d}x\end{aligned}$$

Since we have that $\sigma'(0) = 0$, we get

$$\int_\Omega |Du|^{p-2} Du \cdot D\phi \, \mathrm{d}x = \int_\Omega \epsilon^{p-1} \phi \, \mathrm{d}x. \qquad \square$$

Assuming that $u$ is $C^2$, we can integrate by parts to arrive at the following Euler-Lagrange equation for the variational integral:

$$\Delta_p u = -\epsilon^{p-1}. \tag{3.5}$$

**Proposition 3.2.** *There exists a $u \in W^{1,p}(\Omega)$ that minimizes (3.3).*

*Proof.* We will again employ the method due to Lebesgue. Let

$$m = \inf\left\{\int_\Omega \frac{|Du|^p}{p} \, \mathrm{d}x - \int_\Omega \epsilon^{p-1} u \, \mathrm{d}x\right\}$$

where the infimum is taken over all $u - f \in W_0^{1,p}(\Omega)$. We have by the triangle inequality

$$\left|\int_\Omega \frac{|Du|^p}{p} \, \mathrm{d}x - \int_\Omega \epsilon^{p-1} u \, \mathrm{d}x\right| \le \frac{1}{p}\|Du\|_p^p + \epsilon^{p-1}\mu(\Omega)\sup(u) < \infty.$$

Likewise, we have that

$$\left| \int_\Omega \frac{|Du|^p}{p} \, \mathrm{d}x - \int_\Omega \epsilon^{p-1} u \, \mathrm{d}x \right| \geq \frac{1}{p} \|Du\|_p^p - \epsilon^{p-1} \mu(\Omega) \sup(u) > -\infty.$$

Showing that $-\infty < m < \infty$. By the definition of the infimum, there exists a sequence of functions $\{u_j\}$ in $W_0^{1,p}(\Omega)$ such that

$$\lim_{j \to \infty} I[u_j] = m$$

We can assume, for all $j$, that $I[u_j] \leq m + 1$. Employing first Minkowksi's-, then Sobolev's inequality, we get

$$\|u_j\|_p \leq \|u_j - f\|_p + \|f\|_p \leq K\|Du_j - Df\|_p + \|f\|_p.$$

For the term with $\|Du_j\|_p$, we have the following estimate:

$$\|Du_j\|_p = \left\{ \int_\Omega |Du_j|^p \, \mathrm{d}x \right\}^{1/p} \leq \left\{ \int_\Omega \frac{|Du_j|^p}{p} \, \mathrm{d}x \right\}^{1/p}$$

$$= \left\{ I[u_j] + \int_\Omega \epsilon^{p-1} u_j \, \mathrm{d}x \right\}^{1/p} \leq \left\{ m + 1 + \mu(\Omega)\epsilon^{p-1} \sup |u_j| \right\}^{1/p},$$

for all $j \in \mathbb{N}$. Since $u_j \in W^{1,p}(\Omega)$, we can in light of the remark following Definition 1.4 assume that $u_j$ is continuous, and hence $\sup |u_j| < \infty$. Together, this gives that

$$\|u_j\|_p \leq \|f\|_p + K(M + \|Df\|_p), \tag{3.6}$$

which is a bound independent of $j$. By the weak compactness of $L^p(\Omega)$, there exists a subsequence $\{u_{j_i}\}$ and a $u \in L^p(\Omega)$ such that $u_{j_i}$ converges weakly to $u$, and $Du_{j_i}$ converges to $Du$ weakly in $L^p(\Omega)$. Again, since $u_{j_i} - f$ is in $W_0^{1,p}(\Omega)$, so is the limit $u$.

Furthermore, in light of the bound (3.6), the Rellich-Kondrachev Theorem guarantees the existence of a subsequence that converges pointwise a.e to $u$. □

**Proposition 3.3.** *The limit $u$ found above is the unique minimizer of (3.3).*

*Proof.* First, we look at uniqueness. As in the proof of Proposition 1.6, we exploit the fact that if $u, v \in W_0^{1,p}(\Omega)$ both minimizes (3.3), we have that $(u-v)/2$ is an admissible function. We then get

$$\begin{aligned} m &= \int_\Omega \frac{|Du|^p}{p}\,\mathrm{d}x - \int_\Omega \epsilon^{p-1} u\,\mathrm{d}x \\ &\leq \int_\Omega \left|\frac{Du+Dv}{2}\right|^p \frac{\mathrm{d}x}{p} - \frac{1}{2}\int_\Omega \epsilon^{p-1}(u+v)\,\mathrm{d}x, \end{aligned}$$

since $u$ is assumed to be minimizing.

Using the inequality

$$\left|\frac{Du+Dv}{2}\right|^p \leq \frac{|Du|^p + |Dv|^p}{2},$$

again with strict inequality if $Du \neq Dv$. Accordingly, if $Du \neq Dv$ in a set of positive measure, we have

$$\begin{aligned} m &< \frac{1}{2}\int_\Omega |Du|^p\,\frac{\mathrm{d}x}{p} + \frac{1}{2}\int_\Omega |Dv|^p\,\frac{\mathrm{d}x}{p} - \frac{1}{2}\int_\Omega \epsilon^{p-1}u\,\mathrm{d}x - \frac{1}{2}\int_\Omega \epsilon^{p-1}v\,\mathrm{d}x \\ &= \frac{1}{2}m + \frac{1}{2}m = m \end{aligned}$$

after collecting the terms containing $u$ and $v$. This is an obvious contradiction, and we conclude that the minimizer is unique.

To prove that the $u$ found in 3.2 is the minimizer sought for, it suffices to show that

$$m = I[u] \leq \liminf_{j\to\infty} I[u_j]$$

where we write $u_j$ instead of $u_{j_i}$ for convenience. As in the proof of Proposition 1.6, we have

$$\int_\Omega \frac{|Du_j|^p}{p}\,\mathrm{d}x \geq \int_\Omega \frac{|Du|^p}{p}\mathrm{d}x + \int_\Omega |Du|^{p-2}Du\cdot\Big(Du_j - Du\Big)\,\mathrm{d}x$$

Since the last integral converges to 0 as $j\to\infty$, we have that

$$\liminf_{j\to\infty}\int_\Omega \frac{|Du_j|^p}{p}\,\mathrm{d}x \geq \int_\Omega \frac{|Du|^p}{p}\,\mathrm{d}x$$

Also, remembering that $u_j$ in fact converges pointwise a.e to $u$, Fatou's Lemma gives

$$\liminf_{j\to\infty} \int_\Omega \epsilon^{p-1} u_j \ \mathrm{d}x \geq \int_\Omega \epsilon^{p-1} u \ \mathrm{d}x$$

Collecting the terms, we have together that

$$I[u] \leq \liminf_{j\to\infty} I[u_j]$$

concluding the proof. □

## 3.2 Weak Solutions are Viscosity Solutions

We also need that weak supersolutions of (3.5) are viscosity supersolutions. To prove this, we start with the following comparison principle:

**Proposition 3.4.** *Assume that $u$ and $v$ are weak solutions of* (3.5)*, and that $u \leq v$ on $\partial\Omega$. Then $u \leq v$ in $\Omega$.*

Before we prove this proposition, we need that one can consider functions in $W_0^{1,p}(\Omega)$ as test-functions in (3.4).

**Lemma 3.5.** *Any $w \in W_0^{1,p}(\Omega)$ is a viable test-function in* (3.4)*, the same way as for the p-Laplace equation in Section 1.3.*

*Proof.* Again by the definition of $W_0^{1,p}(\Omega)$, we have that for any smooth $\phi$ that there exists a $w \in W_0^{1,p}(\Omega)$ such that

$$\|w - \phi\|_{W_0^{1,p}} < \epsilon$$

We then have by the triangle inequality

$$
\begin{aligned}
&\left| \int_\Omega |Du|^{p-2} Du \cdot Dw \ \mathrm{d}x - \int_\Omega \epsilon^{p-1} w \ \mathrm{d}x \right| \\
&\leq \left| \int_\Omega |Du|^{p-2} Du \cdot D(w - \phi) \ \mathrm{d}x - \int_\Omega \epsilon^{p-1} (\phi - w) \ \mathrm{d}x \right| \\
&+ \left| \int_\Omega |Du|^{p-2} Du \cdot D\phi \ \mathrm{d}x - \int_\Omega \epsilon^{p-1} \phi \ \mathrm{d}x \right| \\
&= \left| \int_\Omega |Du|^{p-2} Du \cdot D(w - \phi) \ \mathrm{d}x - \int_\Omega \epsilon^{p-1} (\phi - w) \ \mathrm{d}x \right|
\end{aligned}
$$

since $u$ is a weak solution of (3.4). This gives

$$
\begin{aligned}
&\left| \int_\Omega |Du|^{p-2} Du \cdot Dw \ \mathrm{d}x - \int_\Omega \epsilon^{p-1} \phi \ \mathrm{d}x \right| \\
&\leq \int_\Omega |Du|^{p-2} Du \cdot |D(w - \psi)| \ \mathrm{d}x + \int_\Omega \epsilon^{p-1} |\phi - w| \ \mathrm{d}x
\end{aligned}
$$

Let now $\epsilon^{p-1} = S$. Lemma 1.10 gives that

$$
\int_\Omega |Du|^{p-2} |Du \cdot D(w - \phi)| \ \mathrm{d}x < \epsilon
$$

and furthermore

$$
\int_\Omega S|\phi - w| \ \mathrm{d}x = S\|\phi - w\|_{L^1} \leq S\|\phi - w\|_{W^{1,p}} \leq S\epsilon
$$

implying that any $w \in W_0^{1,p}(\Omega)$ is viable in the formulation, and the lemma is proved. $\square$

With this fact in hand, we can prove Proposition 3.4.

*Proof.* Since $u$ and $v$ are weak solutions, we have that

$$\int_\Omega |Du|^{p-2} Du \cdot D\phi \, \mathrm{d}x = \int_\Omega \epsilon^{p-1} \phi \, \mathrm{d}x, \tag{3.7}$$

$$\int_\Omega |Dv|^{p-2} Dv \cdot D\phi \, \mathrm{d}x = \int_\Omega \epsilon^{p-1} \phi \, \mathrm{d}x, \tag{3.8}$$

for all $\phi \in C_0^\infty(\Omega)$. As in the proof for Proposition 1.8, define

$$A = \{x \in \Omega \mid u > v \}.$$

Subtracting (3.7) from (3.8), and choosing $\phi = (u - v)^+$, we get

$$\int_A \Big( Du|Du|^{p-2} - Dv|Du|^{p-2} \Big) \cdot D(u - v) \, \mathrm{d}x = 0.$$

Again, since the integrand must be non-negative, we have that $D(u - v) = 0$, which implies that $u = v + K$, where K is a constant. But $K = 0$, since $u = v$ on $\partial\Omega$. Thus $u = v$ in $A$, a contradiction. □

We are now ready to prove the following fact:

**Proposition 3.6.** *Weak super-solutions to* (3.5) *are viscosity supersolutions.*

*Proof.* We will employ the same strategy as in the proof of Proposition 2.2. To this end, assume that $u$ is a weak super-solution to (3.5), that is

$$\int_\Omega |Du|^{p-2} Du \cdot D\phi \, \mathrm{d}x - \int_\Omega \epsilon^{p-1} \phi \, \mathrm{d}x \le 0$$

We then make the *antithesis*: At some point $x_0 \in \Omega$, we have that $u - \psi$ has a maximum, and

$$\Delta_p \psi(x_0) - \epsilon^{p-1} > 0$$

That is, $\psi$ is a weak subsolution at $x_0$. Since $\psi$ is twice continously differentiable, we also have that $\psi$ is a weak subsolution in $B_\delta(x_0)$ for some $\delta > 0$, i.e:

$$\int_{B_\delta(x_0)} |D\psi|^{p-2} D\psi \cdot D\phi \, \mathrm{d}x - \int_{B_\delta(x_0)} \epsilon^{p-1} \phi \, \mathrm{d}x \ge 0$$

Define the positive number $m_\delta$ by

$$m_\delta = \min_{\partial B_\delta(x_0)} \{u - \psi\}.$$

The open set

$$A = \{x \mid \psi(x) + \frac{m_\delta}{2} > u(x)\}$$

contains $x_0$, and as in the proof of Proposition 2.2, we get that $\psi(x) + m_\delta/2$ is a supersolution to (3.5), and $\psi(x) + m_\delta/2 < u$ on $\partial B_\delta(x_0)$. The comparason principle 3.4 then implies that $\psi(x) + m_\delta/2 < u$ in $B_\delta(x_0)$, implying that $A = \emptyset$, a contradiction. $\square$

## 3.3 Limit Procedure Again

We now turn to the problem of showing that as $p$ goes to $\infty$, the viscosity solutions of (3.5) will converge to viscosity solutions of (3.1). To this end, let $u_p$ denote viscosity solutions of (3.1). We need to show that the sequence $\{u_p\}_{p=2}^{\infty}$ is uniformly bounded.

**Proposition 3.7.** *For any viscosity solution $u_p$ of (3.5) and $p > q$, we have that*

$$\|u_p\|_{W^{1,q}(\Omega)} < \infty, \tag{3.9}$$

*and the weak compactness of $L^q$ gives the existence of functions $u$ and $Du$ such that $u_p \rightharpoonup u$ and $Du_p \rightharpoonup Du$ in every $L^q$.*

*Proof.* Since $u_p$ minimizes (3.3), we have that $u_p$ satisfies

$$\int_\Omega \frac{|Du_p|^p}{p} \,\mathrm{d}x - \int_\Omega \epsilon^{p-1} u_p \,\mathrm{d}x \le \int_\Omega \frac{|Df|^p}{p} \,\mathrm{d}x - \int_\Omega \epsilon^{p-1} f \,\mathrm{d}x,$$

since $f$ is admissible. This gives that

$$\int_\Omega |Du_p|^p \,\mathrm{d}x \le \int_\Omega |Df|^p \,\mathrm{d}x + \int_\Omega p\epsilon^{p-1}|u_p - f| \,\mathrm{d}x.$$

Furthermore we have that $u_p - f \in W_0^{1,p}(\Omega)$, and we can assume it is continuous. This implies that

$$\int_\Omega p\epsilon^{p-1}|u_p - f| \,\mathrm{d}x \le \mu(\Omega)p\epsilon^{p-1}\|u_p - f\|_\infty = \mu(\Omega)p\epsilon^{p-1}M \tag{3.10}$$

for every $p$. From Lemma 2.1 and 3.10, we get the following estimate for every $q < p$,

$$\begin{aligned}\left\{\frac{1}{\mu(\Omega)}\int_{\Omega}|Du_p|^q \,\mathrm{d}x\right\}^{\frac{1}{q}} &\leq \left\{\frac{1}{\mu(\Omega)}\int_{\Omega}|Du_p|^p \,\mathrm{d}x\right\}^{\frac{1}{p}}\\ &\leq \left\{\frac{1}{\mu(\Omega)}\int_{\Omega}|Df|^p \,\mathrm{d}x + p\epsilon^{p-1}M\right\}^{\frac{1}{p}}\\ &\leq \left\{L^p + p\epsilon^{p-1}M\right\}^{\frac{1}{p}},\end{aligned}$$

where $L$ is the Lipschitz constant for $f$. Let $A = \max(L, \epsilon)$. We then have the estimate

$$\left\{L^p + p\epsilon^{p-1}M\right\}^{\frac{1}{p}} \leq A\left\{1 + p\frac{\epsilon^p M}{\epsilon}\right\}^{\frac{1}{p}}.$$

Rewriting this, we have

$$\left\{1 + p\frac{pM\epsilon^p}{\epsilon}\right\}^{\frac{1}{p}} = \exp\left\{\frac{1}{p}\ln\left(1 + p\frac{\epsilon^p M}{\epsilon}\right)\right\}$$

The function inside the exponential is strictly decreasing in $p$, and it attains its maximum in $p = 1$. Therefore, we can assume that

$$\left\{L^p + p\epsilon^{p-1}M\right\}^{\frac{1}{p}} \leq A$$

for all $p$, and hence we have

$$\|Du_p\|_q \leq \mu(\Omega)^{1/q}\max(L, \epsilon)(1 + M) \tag{3.11}$$

We now have

$$\begin{aligned}\|u_p\|_q &\leq \|u_p - f\|_q + \|f\|_q \leq K_1\||Du_p - Df|\|_q + K_2\\ &\leq K_1(M + L) + K_2\end{aligned}$$

by the computations above and Sobolev's inequality. This, together with 3.11, gives that $\|u\|_{W^{1,q}(\Omega)} < \infty$. □

Using the same argument as in Chapter 2.2, we have the existence of a subsequence of $u_p$, labeled $\{u_{p_j}\}$, for convenience, that converges in *every* $L^q$, and it follows that $u \in W^{1,\infty}(\Omega)$. Furthermore, the Rellich-Kondrachov Theorem we have the existence of a subsequence of $\{u_{p_j}\}$ that converges pointwise a.e in $L^q(\Omega)$.

Again we need the convergence to be uniform on compact subsets, and to this end we have the following:

**Proposition 3.8.** *The sequence $u_{p_j}$ is equicontinous on compact cubes $K \subset \Omega$, and hence the convergence is uniform here.*

*Proof.* As in the proof of Proposition 2.4, we fix any $s$ so that $n < s < p_j$. Then we have

$$|u_{p_j}(x) - u_{p_j}(y)| \leq \frac{2sn}{s-n}|x-y|^{1-n/s}\|Du_{p_j}\|_s$$

Employing 3.9, we get

$$|u_{p_j}(x) - u_{p_{p_j}}(y)| \leq 2s|x-y|^{1-n/s}M.$$

Hence we have that the sequence $u_{p_j}$ is equicontinous in cubes. The Arzela-Ascoli Theorem then gives that $\{u_{p_j}\}$ converges uniformly. □

Let $u = \lim_{j\to\infty} u_{p_j}$. This is our candidate for a viscosity solution of the Upper Equation, as will be proven in the following proposition. But before we continue, we need to define the concept of viscosity solutions of (3.1).

**Definition 3.9.** *A continous function $u$ is a viscosity supersolution of* (3.1)*, if whenever $\psi \in C^2(\Omega)$ and $x_0 \in \Omega$, we have*

$$\begin{aligned}&\psi(x_0) = u(x_0),\\ &\psi(x) < u(x) \text{ for } x \in \Omega \setminus \{x_0\}\end{aligned}$$

*then*

$$\epsilon - |D\psi(x_0)| \leq 0 \text{ and } \Delta_\infty\psi(x_0) \leq 0.$$

We are now ready to prove the main proposition of this section.

**Proposition 3.10.** *As $p_j \to \infty$, viscosity solutions of* (3.5) *converge to viscosity solutions of* (3.1)*.*

*Proof.* Assume that $x_0 \in \Omega$ and $\psi \in C^2(\Omega)$ are so that

$$(u-\psi)(x_0) > (u-\psi)(x)\ ,\ x \in \Omega \setminus \{x_0\}$$

that is, $u-\psi$ has a maximum at $x_0$. We want to show that this implies

$$\Delta_\infty \psi(x_0) + \epsilon^{p-1} \leq 0$$

As in the proof of Proposition 2.5, we have that if $u_{p_j} - \psi$ attains its maximum at $x_j$, then $x_j \to x_0$ as $p_j \to \infty$, since $u_{p_j}$ converges uniformly to $u$.

By the definition of viscosity solution, we have then that

$$|D\psi_j(x_j)|^{p_j-2}\Delta\psi_j(x_j) + (p_j-2)|D\psi_j(x_j)|^{p_j-4}\Delta_\infty\psi_j(x_j) + \epsilon^{p_j-1} \leq 0.$$

Dividing through by $|D\psi_j(x_j)|(p_j-2)$ and re-arranging, we get

$$\frac{\Delta\psi_j(x_j)}{p_j-2} + \Delta_\infty\psi_j(x_j) + \frac{\epsilon^3}{p_j-2}\left(\frac{\epsilon}{|D\psi_j(x_j)|}\right)^{p_j-4} \leq 0.$$

Letting $p_j \to \infty$, we see that the first two terms in the equation converge to $\Delta_\infty\psi(x_0)$, while for the part with $\epsilon$ we have

$$\frac{\epsilon^3}{p_j-2}\left(\frac{\epsilon}{|D\psi_j(x_j)|}\right)^{p_j-4} \to \begin{cases} 0 & \text{if } \frac{\epsilon}{|D\psi_j(x_j)|} \leq 1 \\ \infty & \text{otherwise} \end{cases}.$$

Since $\Delta_\infty\psi_j(x_j)$ remains bounded, we must demand that

$$\frac{\epsilon}{|D\psi_j(x_j)|} \leq 1,$$

or equivalently, $\epsilon - |D\psi| \leq 0$. This proves that the limit $u$ is a viscosity supersolution of (3.1). □

## 3.4 The Lower Equation

So far, we have only derived the desired properties for the Upper Equation. The procedure for the Lower Equation is almost analogous, but for completeness we will comment on it here.

For the Lower Equation, (3.2), the existence is reached by considering the following variational integral:

$$I[u] = \int_\Omega \frac{|Du|^p}{p}\ \mathrm{d}x + \int_\Omega \epsilon^{p-1} u\ \mathrm{d}x. \tag{3.12}$$

With Euler-Lagrange equation

$$\Delta_p u = \epsilon^{p-1} \tag{3.13}$$

or, in its weak form

$$\int_\Omega |Du|^{p-2} Du \cdot D\phi\ \mathrm{d}x = -\int_\Omega \epsilon^{p-1}\phi \mathrm{d}x.$$

for all $\phi \in C_0^\infty(\Omega)$. The same estimates as in Section 3 gives boundedness of the variational integral, and the direct method works in the same way. the proof for uniqueness and existence of a minimizer is also analogous to what we did in the proof for Proposition 3.3. Thus we have the following Proposition.

**Proposition 3.11.** *There exists a unique function $v \in W_0^{1,p}(\Omega)$ that minimizes* (3.12) *among all the admissible functions.*

For the Lower Equation only the subsolutions count. Mimicking the proof of Proposition 3.4, only considering subsolutions instead of supersolutions, we get this result.

**Proposition 3.12.** *Assume that $u$ and $v$ are weak solutions of* (3.13), *and that $u \geq v$ on $\partial\Omega$. Then $u \geq v$ in $\Omega$. Furthermore, weak subsolutions are viscosity subsolutions.*

We are now in position to examine the behavior of (3.13) as $p \to \infty$. Here there are some differences from the Upper Equation, so we will do the proof in more detail. We start of with a definition:

**Definition 3.13.** *A continuous function $v$ is a viscosity subsolution of* (3.2) *if, for every $\psi \in C^2(\Omega)$ that touches $v$ from above at $x_0$, we have:*

$$|D\psi(x_0)| - \epsilon \geq 0 \text{ and } \Delta_\infty \psi(x_0) \geq 0.$$

**Proposition 3.14.** *As $p \to \infty$, we have that viscosity solutions of* (3.13) *converges to viscosity solutions of the Lower Equation.*

*Proof.* As before, we expand (3.13) and get

$$|D\psi(x_j)|^{p_j-2}\Delta\psi(x_j) + (p_j - 2)|\psi(x_j)|^{p_j-4}\Delta_\infty\psi(x_j) - \epsilon^{p_j-1} \geq 0.$$

We can again assume that $|D\psi(x_j)| > 0$ for $j > N$, otherwise there would be nothing to prove. Dividing by $(p_j - 2)|D\psi(x_j)|^{p_j-2}$, and re-arranging, we get

$$\frac{\Delta\psi(x_j)}{p_j - 2} + \Delta_\infty\psi(x_j) - \frac{\epsilon^3}{p_j - 2}\left(\frac{\epsilon}{|D\psi(x_j)|}\right)^{p_j-4} \geq 0.$$

We have that the first term in the inequality approaches $\Delta_\infty\psi(x_0)$ because of continuity. For the term containing $\epsilon$, we have

$$\frac{-\epsilon^3}{p_j - 2}\left(\frac{\epsilon}{|D\psi(x_j)|}\right)^{p_j-4} \to \begin{cases} 0 & \text{if } \frac{\epsilon}{|D\psi_j(x_j)|} \leq 1 \\ -\infty & \text{otherwise} \end{cases}.$$

Since $\Delta_\infty\psi(x_j)$ remains bounded, we must have that $|D\psi| - \epsilon \geq 0$. It follows that the limit $v$ is a viscosity subsolution. $\square$

Chapter 4

# Comparison Principle and Uniqueness

In this chapter, we will prove that the intial value problem

$$\begin{cases} \Delta_\infty u & = 0 \text{ in } \Omega \\ u & = f \text{ on } \partial\Omega \end{cases}$$

has a unique solution. To do this, we aim at proving the following Comparison Principle:

**Proposition 4.1.** *If $u$ is a viscosity subsolution of* (2.4) *such that $u \leq f = u^+$ on $\partial\Omega$, then $u \leq u^+$ in $\Omega$.*

We we will do this by proving that the solutions to the $\infty$-Laplace equation lies between the solutions of (3.1) and (3.2), and then use the Theorem on Sums to show Proposition 4.1.

## 4.1 Preliminary Estimates

We start by proving some estimates regarding the solutions to the Upper and Lower equation.

From the proof of Proposition 3.9, we have that

$$\int_\Omega |Du_p|^p \, \mathrm{d}x \leq \int_\Omega |Df|^p \, \mathrm{d}x + \int_\Omega p\epsilon^{p-1}(u_p - f) \, \mathrm{d}x.$$

Taking the $p$-root on both sides and letting $p \to \infty$, we get that

$$\|Du\|_\infty \leq \|Df\|_\infty + \epsilon. \tag{4.1}$$

Also, from (3.2) and (3.1), we see that

$$\begin{aligned} \epsilon - |Du^+| &\leq 0, \\ |Du^-| - \epsilon &\geq 0. \end{aligned}$$

From this and the limit derived above, we have proved the following:

**Lemma 4.2.** *A variational solution of the Upper or Lower Equation satisfies*

$$\epsilon \leq \|Du^\pm\|_\infty \leq L,$$

*where $L$ is the Lipschitz constant of $f$.*

Let $u_p^+$ denote weak solutions of the Upper Equation (3.5), and $u_p^-$ solutions of the Lower Equation (3.13). Since they all have the same value $f$ on $\partial\Omega$, and the fact that weak solutions of (3.5) are $p$-superharmonic while weak solutions of (3.13) are $p$-subharmonic, we get by the Comparison Principle 1.8 that

$$u_p^- \leq u_p \leq u_p^+$$

Hence the name Lower and Upper Equation.

We have

$$\begin{aligned} \int_\Omega |Du_p^-|^{p-2} Du_p^- \cdot D\phi \,\mathrm{d}x &= -\int_\Omega \epsilon^{p-1}\phi \mathrm{d}x \\ \int_\Omega |Du_p^+|^{p-2} Du_p^+ \cdot D\phi \,\mathrm{d}x &= \int_\Omega \epsilon^{p-1}\phi \mathrm{d}x \end{aligned}$$

Subtracting these to equations, and selecting $\phi = u_p^+ - u_p^-$, we get

$$\begin{aligned} \int_\Omega \left(|Du_p^+|^{p-2} Du_p^+ - |Du_p^-|^{p-2} Du_p^-\right) \cdot D\left(Du_p^+ - Du_p^-\right) \,\mathrm{d}x \\ = 2\epsilon^{p-1} \int_\Omega (u_p^+ - u_p^-) \,\mathrm{d}x. \end{aligned}$$

Using the fact that the integrand on the left side is greater than or equal to $2^{p-2}|Du_p^+ - Du_p^-|^p$, we get that

$$2^{p-2}\int_\Omega |Du_p^+ - Du_p^-|^p \, \mathrm{d}x \le 2\epsilon^{p-1}\int_\Omega (u_p^+ - u_p^-)\, \mathrm{d}x.$$

Taking the $p$'th root on both sides, and letting $p \to \infty$, we arrive at

$$\sup\left|\frac{Du^+ - Du^-}{2}\right| \le \epsilon,$$

or simply

$$\sup\left|Du^+ - Du^-\right| \le \epsilon'.$$

Sobolev's Inequality then gives that

$$\|u^+ - u^-\|_\infty \le K\epsilon \tag{4.2}$$

For some constant $K$ only depending upon $\Omega$. We have that

$$u^- \le u \le u^+ \le u^- + \mathcal{O}(\epsilon).$$

We have only showed that for a selected subsequence of $p$'s, we can arrange (4.2), but we are not guaranteed the same solutions if we select *different* subsequence. The answer to this question requires more refined methods.

## 4.2 Proof of the Comparison Principle

To prove Proposition 4.1, we will use the *antithesis*:

$$\max_\Omega\{u - u^+\} > \max_{\partial\Omega}\{u - u^+\}, \tag{4.3}$$

where $u^+$ is a solution to the Upper Equation (3.1), and $u$ is a solution of the $\infty$-Laplace equation. We can assume that $u^+ > 0$ by adding a constant.

To get the desired contradiction, we will construct a strict supersolution, $w = g(u^+)$ of the upper equation such that

$$\max_\Omega\{u - w\} > \max_{\partial\Omega}\{u - w\}, \tag{4.4}$$

and

$$\Delta_\infty \psi \leq -\mu \quad \text{in } \Omega$$

for some constructed, positive number $\mu$, whenever $\psi \in C^2(\Omega)$ touches $w$ from below.

To achieve this, we consider the following function:

$$g(t) = \ln(1 + A(\mathrm{e}^t - 1)),\ t \in [0, \infty),$$

Where $A > 1$. This is an approximation of the identity function, and we have the following properties:

$$0 < g(t) - t < A - 1 \tag{4.5}$$

$$1 < g'(t) < A - 1, \tag{4.6}$$

for $t \geq 0$.

Calculating formally, we get that:

$$\begin{aligned} w_{x_i} &= g'(u^+)u^+_{x_i}, \\ w_{x_i x_j} &= g''(u^+)u^+_{x_i}u^+_{x_j} + g'(u^+)u^+_{x_i x_j}. \end{aligned}$$

Inserting this into the $\infty$-Laplace equation, we get that

$$\Delta_\infty w = g'(u^+)^3 \Delta_\infty u^+ + g'(u^+)^2 g''(u^+)|Du^+|^4.$$

So, by multiplying the upper equation for supersolutions by $g'(u^+)^3$, and move the parts containing $u^+$ to the other side, we arrive at

$$\Delta_\infty w \leq g'(u^+)^2 g''(u^+)|Du^+|^4.$$

We see that all terms on the right are positive, apart from $g''(u^+)$. For this we have

$$g''(u^+) = \frac{A(1-A)\mathrm{e}^{u^+}}{[1 + A(\mathrm{e}^{u^+} - 1)]^2}.$$

In order to force the right hand side of (4.2) to be negative, we have to choose $A$ strictly greater than 1. Also, we want the functions $w$ and $u^+$ to be close enough not to contradict (4.4). But we have

$$0 < w - u^+ = g(u^+) - u^+ < A - 1$$

from (4.5). So we not only have to choose the parameter $A > 1$, we have to choose it so that $A - 1 < \delta$, where $\delta$ is so small that (4.4) still holds. We also have that

$$\|Du^+\|_\infty \leq \|Df\|_\infty + \epsilon.$$

by equation 4.1. This implies that

$$\|u^+\|_\infty \leq K\|Du^+\|_\infty \leq K(\|Df\|_\infty + \epsilon) = K(L + \epsilon),$$

and hence $u^+$ is bounded.

With this in mind, we get that

$$\Delta_\infty w \leq \epsilon^4 A^3(1 - A)\mathrm{e}^{-\|u^+\|_\infty} \equiv -\mu.$$

Now, if $\psi \in C^2(\Omega)$ is a test-function touching $u^+$ from below at $x_0$, we know that $\phi = g(\psi)$ touches $w = g(u^+)$ from below at $x_0$, and hence we can replace $u^+$ with $\phi$ in the calculations above. This implies that

$$\Delta_\infty \phi(x_0) \leq -\mu$$

whenever $\phi$ touches $w$ from below at $x_0$.

Having constructed the desired supersolution $w$, we aim at using the so-called *Theorem on Sums* stated in terms of the sub-and super-jets, see [M.B95, p. 31]. This will give us the desired contradiction.

**Theorem 4.3** (Theorem on Sums)**.** *Let $\mathcal{O}$ be a open, bounded subset of $\mathbb{R}^n$, and let $u, v : \mathcal{O} \to \mathbb{R}$. Also, let $\phi \in C^2(\mathcal{O} \times \mathcal{O})$. Let*

$$w(x, y) = u(x) + v(y)$$

*and assume $(\hat{x}, \hat{y}) \in \mathcal{O} \times \mathcal{O}$ is a local maximum for*

$$w(x, y) - \phi(x, y)$$

*Then, for each $\tau > 0$, with $\tau D^2\phi(\hat{x}, \hat{y}) < I$, then there exists $X, Y \in \mathbb{S}^n$ such that*

$$(D_x(\phi(\hat{x}, \hat{y}), X) \in \overline{J}^{2,+}u(\hat{x})$$

*and*

$$(D_y(\phi(\hat{x}, \hat{y}), Y) \in \overline{J}^{2,-}v(\hat{y})$$

*and the matrices $X, Y$ satisfy*

$$-\frac{1}{\tau}I \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq (I - \tau D^2\phi(\hat{x}, \hat{y}))^{-1}D^2\phi(\hat{x}, \hat{y}).$$

We will consider the function

$$\phi_\epsilon(x,y) = \frac{1}{2\epsilon}|x-y|^2.$$

Let $(x_\epsilon, y_\epsilon) \in \overline{\Omega} \times \overline{\Omega}$ be a point such that

$$\Phi_\epsilon(x,y) = u(x_\epsilon) - w(y_\epsilon) - \frac{1}{2\epsilon}|x_\epsilon - y_\epsilon|^2$$

attains its maximum.

Calculating the derivatives, we get

$$A = \frac{1}{2\epsilon}D^2|x-y|^2 = \frac{1}{\epsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

at $x = x_\epsilon, y = y_\epsilon$. We also get

$$(I - \tau A)^{-1}A = \frac{1}{\epsilon - 2\tau}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Then the Theorem on Sums gives the existence of matrices $X_\epsilon, Y_\epsilon \in \mathbb{S}^n$ such that

$$\left(\frac{x_\epsilon - y_\epsilon}{\epsilon}, X_\epsilon\right) \in \overline{J}^{2,+}u(x_\epsilon),$$
$$\left(\frac{x_\epsilon - y_\epsilon}{\epsilon}, Y_\epsilon\right) \in \overline{J}^{2,-}w(y_\epsilon).$$

Furthermore, we see that since the matrix

$$\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$$

kills all vectors of the type $\begin{pmatrix} \xi \\ \xi \end{pmatrix}$ in $\mathbb{R}^{2n}$, we have that $X_\epsilon \leq Y_\epsilon$.

Using the equivalent definitions of viscosity supersolutions in Proposition A.6, we have that

$$\frac{1}{4\epsilon^2}\langle Y_\epsilon(x_\epsilon - y_\epsilon), x_\epsilon - y_\epsilon\rangle \leq -\mu,$$
$$\frac{1}{4\epsilon^2}\langle X_\epsilon(x_\epsilon - y_\epsilon), x_\epsilon - y_\epsilon\rangle \geq 0.$$

Subtracting these equations, and using the linearity of the inner product, we arrive at

$$\frac{1}{4\epsilon^2}\langle (Y_\epsilon - X_\epsilon)(x_\epsilon - y_\epsilon), x_\epsilon - y_\epsilon\rangle \leq -\mu.$$

But this implies, since $X_\epsilon \leq Y_\epsilon$, that $0 \leq -\mu$, a contradiction.

## 4.3 A Harnack Type Inequality

Having proved uniqueness of solutions for the $\infty$-Laplace equation, we are now ready to prove an important property of the solution. This is the Harnack Principle:

**Proposition 4.4.** *Assume $u$ is $\infty$-harmonic in $\Omega$. Then there exists a constant $C_\Omega$ only depending upon $\Omega$ such that*

$$\sup_{x\in\Omega} u \leq C_\Omega \inf_{x\in\Omega} u. \tag{4.7}$$

*Proof.* Assume that $u$ is a strictly positive minimizer of the integral (1.7). For $\epsilon > 0$, consider the function

$$u + \epsilon \frac{\phi^p}{u^{p-1}},$$

where $\phi \in C_0^\infty(\Omega)$ is a positive test function. We have, for all $i \in \{1, 2, \cdots n\}$:

$$\frac{\partial}{\partial x_i}\left(u + \epsilon \frac{\phi^p}{u^p - 1}\right) = \frac{\partial u}{\partial x_i} + \epsilon \frac{p\phi^{p-1}u^{p-1}\frac{\partial \phi}{\partial x_i} - (p-1)\phi^p u^{p-2}\frac{\partial u}{\partial x_i}}{u^{2(p-1)}}.$$

This implies that

$$D\left(u + \epsilon \frac{\phi^p}{u^{p-1}}\right) = Du + \epsilon p \left(\frac{\phi}{u}\right)^p D\phi - \epsilon(p-1)\left(\frac{\phi}{u}\right)^{p-1} Du. \tag{4.8}$$

If we set

$$\tau = \epsilon(p-1)\left(\frac{\phi}{u}\right)^p,$$

then we have that (4.8) is a convex combination of $Du$ and

$$\frac{u}{\phi}\frac{p}{p-1}D\phi,$$

provided we fix $\epsilon$ so that $0 \leq \tau \leq 1$ for all $x$. Since $x \mapsto |x|^p$ is convex, we have that

$$
\begin{aligned}
&\left| D\left( u + \epsilon \frac{\phi^p}{u^{p-1}} \right) \right|^p \\
&\leq \left( 1 - \epsilon(1-p) \left( \frac{\phi}{u} \right)^p \right) |Du|^p + \epsilon(p-1) \left( \frac{\phi}{u} \right)^p \left| \frac{u}{\phi} \frac{p}{p-1} \right|^p |D\phi|^p \\
&= \left( 1 - \epsilon(1-p) \left( \frac{\phi}{u} \right)^p \right) |Du|^p + \epsilon \frac{p^p}{(p-1)^p} |D\phi|^p.
\end{aligned}
$$

Integrating this inequality over $\Omega$, and remembering that $u$ is minimizing, we get that

$$
\begin{aligned}
\int\limits_\Omega |Du|^p \,\mathrm{d}x \leq \int\limits_\Omega |Du|^p \,\mathrm{d}x - \epsilon(p-1) \int\limits_\Omega \left( \frac{\phi}{u} \right)^p |Du|^p \,\mathrm{d}x \\
+ \epsilon \frac{p^p}{(p-1)^p} \int\limits_\Omega |D\phi|^p \,\mathrm{d}x.
\end{aligned}
$$

Re-arranging this and dividing by $\epsilon > 0$, we get

$$
(p-1) \int\limits_\Omega \left( \frac{\phi}{u} |Du| \right)^p \,\mathrm{d}x \leq \frac{p^p}{(p-1)^p} \int\limits_\Omega |D\phi|^p \,\mathrm{d}x.
$$

Here we see that we could have ignored $\epsilon$ from the start, since it will not affect the outcome of our calculation. Since $D(\ln u) = \frac{1}{u} Du$, we get at last

$$
(p-1) \int\limits_\Omega (\phi |D \ln u|)^p \,\mathrm{d}x \leq \frac{p^p}{(p-1)^p} \int\limits_\Omega |D\phi|^p \,\mathrm{d}x.
$$

Taking the $p$-th root of both sides of the above, and considering $u = u_p$ a minimizer of the variational integral (and thus also a viscosity solution of the $p$-Laplace equation), we get, as $p \to \infty$:

$$
\sup_\Omega \{\phi |D \ln u|\} \leq \sup_\Omega |D\phi|, \tag{4.9}
$$

where our $u$ is now a $\infty$-harmonic function. We now choose a clever test function. Consider the two concentric balls $B_{x_0}(r)$ and $B_{x_0}(R)$, where $r < R$. Choose $\phi$ so that $0 \leq \phi(x) \leq 1$, and $\phi(x) = 1$ for $x \in B_{x_0}0(r)$. Also, let $\phi$ go to zero in a smooth way at $R$ such that $|D\phi| \leq \frac{1}{R-r}$. With this choice in mind, we have from the Fundamental Theorem of Calculus:

$$|\ln u(x) - \ln u(y)| \leq |D \ln u||x-y|$$

for all $x, y \in B_{x_0}(r)$. Since $\phi = 1$ here, we get by (4.9):

$$|\ln u(x) - \ln u(y)| \leq \|D\phi\|_\infty |x-y| \leq \frac{1}{R-r} 2r.$$

Taking the exponential on both sides, this reduces to

$$u(x) \leq \exp\left(\frac{2r}{R-r}\right) u(y)$$

for all $x, y \in B_{x_0}(r)$. This implies that

$$\sup_{x \in B_{x_0}(r)} u \leq K \inf_{x \in B_{x_0}(r)} u, \tag{4.10}$$

where K only depends upon the domain considered. This is the desired Harnack inequality. □

**Remark.** The result is also true for $u \geq 0$. To see this, one can replace $u$ by $u + \delta$ for some positive $\delta$ in the calculations, and letting $\delta \to 0$ at the end.

Chapter 5

# A Question of Cones

In this chapter, we will explore the concept of *comparison with cones* for a continuous function $u : \Omega \mapsto \mathbb{R}$. This geometrical property of the $\infty$-Laplace equation was first described by Crandall et.al in [CEG01]. We prove the fact that $u$ is a viscosity solution of the $\infty$-Laplace equation if and only if it enjoys comparison with cones. This important property allows us to prove that the $\infty$-harmonic function are indeed absolutely Lipschitz minimizing. We also use the comparison property to show the existence of $\infty$-harmonic function on *unbounded* domains, via Perron's Method.

## 5.1 A New Perspective

We start with some definitions:

**Definition 5.1.** *The function $C(x) = a|x-z|$ is called a cone function with vertex $z$ and slope $a$.*

*A continuous function $u : \Omega \mapsto \mathbb{R}$ is said to enjoy comparison with cones from above if for every $a \in \mathbb{R}$, $U \subseteq \Omega$ and $z \notin U$, we have*

$$u(x) - a|x-z| \leq \max_{y \in \partial U}(u(y) - a|x-z|) \text{ for } x \in U. \tag{5.1}$$

*Similarly, we say that a function $v$ enjoys comparison with cones from below if $-u$ enjoy comparison with cones from above. Or more explicitly:*

$$v(x) - a|x-z| \geq \min_{y \in \partial U}(u(y) - a|x-z|) \text{ for } x \in U. \tag{5.2}$$

*If $u$ enjoys comparison with cones both from below and above, we simply say that $u$ enjoys comparison with cones.*

**Remark.** Notice that we can restate the definitions above as $u$ enjoys comparison with cones from above if, for $a, c \in \mathbb{R}$ and $z \notin U$, we have

$$u(x) \leq c + a|x - z| \; \forall \; x \in U \text{ if it holds for } x \in \partial U.$$

The reformulation for comparison with cones from below is obvious.

With these definitions in hand, we turn to the task of deriving the $\infty$-Laplace equation from comparison with cones.

**Proposition 5.2.** *$u$ is $\infty$-harmonic in $\Omega$ if and only if $u$ enjoys comparison with cones.*

*Proof.* First, assume that $u$ satisfies $\Delta_\infty \geq 0$ in the viscosity sense, that is if $x \in U$ and $\phi \in C^2$ is so that $u - \psi$ has a maximum at $x$, then $\Delta_\infty \geq 0$. If $G$ is a radial function of $x$, i.e $x \mapsto G(|x - a|)$, we have that

$$\Delta_\infty G(|x - a|) = G''(|x - a|)G'(|x - a|)^2 \quad \text{when } x \neq a.$$

Choosing $\phi(x) = a|x - z| - \gamma|x - z|^2$, we see that

$$\Delta_\infty \phi(x) = -2\gamma(a - 2\gamma|x - z|)^2 < 0$$

for all $x \in \Omega, x \neq z$ for $\gamma$ small enough. But since $\Delta_\infty u \geq 0$, we have that $u - \phi$ cannot have a maximum in $U \subset \Omega$ at a point different from $z$, since $u$ is a viscosity solution. Thus, we have, for $z \notin U$,

$$u(x) - (a|x - z| - \gamma|x - z|^2) \leq \max_{w \in \partial\Omega}(u(w) - (a|w - z| - \gamma|w - z|^2)).$$

Letting $\gamma \to 0^+$, we have

$$u(x) - a|x - z| \leq \max_{w \in \partial\Omega}(u(w) - a|w - z|),$$

which proves that $u$ enjoys comparison with cones from above.

For the other implication, assume that $u$ enjoys comparison with cones from above. In particular, if

$$C(x) = u(y) + \max_{w : |w - y| = r} \left( \frac{u(w) - u(y)}{r} \right) |x - y|, \tag{5.3}$$

we have that $u(x) \leq C(x)$ in $B_r(y)$, since it holds trivially on $\partial(B_r(y)) \setminus \{y\})$. We can rewrite (5.3) as

$$u(x) - u(y) \leq \max_{w:|w-y|=r} (u(w) - u(x)) \frac{|x-y|}{r - |x-y|}. \tag{5.4}$$

We want to prove that if (5.4) holds, then, if $u - \phi$ has a maximum at $x$ for some $\phi \in C^2$, then $\Delta_\infty \phi(x) = \langle D^2\phi(x) D\phi(x), \phi(x) \rangle \geq 0$.

If $x$ is a maximum of $u - \phi$, we have that

$$u(x) - u(y) \geq \phi(x) - \phi(y) \text{ and } u(w) - u(x) \leq \phi(w) - \phi(x)$$

for $w, y$ in a neighborhood of $x$. This implies that we can consider $\phi$ instead of $u$ in (5.4). Since we $\phi$ is twice differentiable, we know that there exists $p \in \mathbb{R}^n$ and $X \in \mathbb{S}^n$ such that

$$\phi(z) = \phi(x) + \langle p, z - x \rangle + \frac{1}{2} \langle X(z-x), z - x \rangle + o(|z-x|^2)$$

Where $p = D\phi(x)$ and $X = D^2\phi(x)$.

First, replace $u$ by $\phi$ in (5.4), then choose $z = y = x - \lambda p$ and expand the left hand side of the equation, that is, $\phi(x) - \phi(y)$. This gives

$$\phi(x) - \phi(x - \lambda p) = \lambda |p|^2 + \frac{\lambda^2}{2} \langle Xp, p \rangle + o(\lambda^2).$$

Now, let $w_{r,\lambda}$ be a point where the maximum on the right hand side of (5.4) is attained. Again, expanding $\phi(w_{r,\lambda}) - \phi(x)$ and get

$$\lambda|p|^2 + \frac{\lambda^2}{2}\langle Xp, p \rangle + o(\lambda^2) \leq \Bigg( \langle p, w_{r,\lambda} - x \rangle$$
$$+ \frac{1}{2} \langle X(w_{r,\lambda} - x), w_{r,\lambda} - x \rangle + o((r+\lambda)^2) \Bigg) \frac{\lambda|p|}{r - \lambda|p|}$$

Dividing by $\lambda$ and letting $\lambda \to 0^+$, we get

$$|p|^2 \leq \left( \langle p, \frac{w_r - x}{r} \rangle + \frac{1}{2} \langle X \frac{w_r - x}{r}, w_r - x \rangle \right) |p| + |p| o(r) \tag{5.5}$$

$$\leq |p|^2 + \frac{1}{2} \left\langle X \frac{w_r - x}{r}, w_r - x \right\rangle |p| + |p| o(r). \tag{5.6}$$

where $w_r$ is a limit point of $w_{r,\lambda}$, and this implies that $w_r \in \partial B_r(x)$. This again implies that $(w_r - x)/r$ is a unit vector. Since $|w_r - x| = r$, we have

$$\langle X \frac{w_r - x}{r}, w_r - x \rangle \to 0$$

as $r \to 0^+$. From this and (5.5), we see that $(w_r - x)/r \to p/|p|$. Re-arranging (5.6), dividing by $r$ and letting $r \to 0^+$, we arrive at

$$0 \leq \lim_{r \to 0^+} \left\langle X \frac{w_r - x}{r}, \frac{w_r - x}{r} \right\rangle = \langle Xp, p \rangle.$$

This concludes the proof, as we have shown that if $u$ enjoys comparison with cones from above, then $\Delta_\infty u \geq 0$. $\square$

The perceptive reader will have noticed that we have only proved the equivalences for subsolutions and comparison with cones from above. The proof for supersolutions and comparison with cones from below are analogous.

With these tools in hand, we can show directly that a function that enjoys comparison with cones is an absolute minimizer for the functional $\mathcal{L}(u, \Omega)$, as defined in (1.5). This gives the following proposition:

**Proposition 5.3.** *A function $u$ enjoys comparison with cones if and only if $u$ is an absolute minimizer for $\mathcal{L}(u, \Omega)$.*

*Proof.* First assume that $u$ is an absolute minimizer, that is

$$\mathcal{L}(u, U) \leq \mathcal{L}(v, U) \text{ whenever } U \subset \Omega \text{ and } u = v \text{ on } \partial U.$$

Let $C(x) = a|x - z|$ be a cone. For $z \notin U$, define the set

$$A = \{x \in U \mid u(x) - a|x - z| > \max_{w \in \partial U}(u(w) - a|w - z|)\}. \qquad (5.7)$$

It is clear that if $u$ enjoys comparison with cones, then $A = \emptyset$. So we assume that $A$ is not empty, and argue by contradiction. Now, for $x \in \partial A$ we have

$$u(x) = a|x - z| + \max_{w \in \partial U}(u(w) - a|w - z|).$$

We now have that $u(x) = C(x)$ on $\partial A$, and furthermore, since $u$ is absolutely minimizing:

$$\mathcal{L}(u, A) = \mathcal{L}(C, \partial A) = |a|.$$

Let now $x_0$ be a point in $A$, and let $\gamma(t) = z + t(x_0 - z)$ be a ray of $C$ through $x_0$. A segment of this ray is contained in $A$, and it meets $\partial\Omega$ at its endpoints. We see that $C(\gamma(t)) = at|x_0 - z|$ since $C$ is linear on this segment. Furthermore, $u(\gamma(t))$ has Lipschitz constant $a|x_0 - z|$ and is equal to $C$ on the endpoints of the segment.

Note that if a function $u$ has Lipschitz constant

$$\frac{|u(z) - u(w)|}{|z - w|}$$

along a line segment from $z$ to $w$, then

$$u(w + t(z - w)) = u(w) + t(u(z) - u(w))$$

for $t \in [0, 1]$. This implies that $u(\gamma) = C(\gamma)$, but since $x_0$ is on the line segment, we have that $C(x_0) = u(x_0)$, contradicting the fact that $x_0 \in A$.

Now assume that $u$ enjoys comparison with cones. We want to show that this implies that $u$ is absolutely Lipschitz minimizing. For any $y \in \partial U$ and $x \in U$, we have that

$$u(y) - \mathcal{L}(u, \partial U)|x - y| \leq u(x) \leq u(y) + \mathcal{L}(u, \partial U)|x - y|. \tag{5.8}$$

Notice that $C(x) = u(y) + \mathcal{L}(u, \partial U)|x - y|$ is a cone function. Since the above inequality holds for $x \in \partial U$, and $u$ enjoys comparison with cones, we have that (5.8) holds for $x \in U$ as well. This implies that

$$\mathcal{L}(u, \partial(U \setminus \{x\}) = \mathcal{L}(u, \partial U).$$

Repeating this, we get that for $x, y \in U$, we have

$$\mathcal{L}(u, \partial(U \setminus \{x, y\}) = \mathcal{L}(u, \partial U).$$

This shows that

$$|u(x) - u(y)| \leq \mathcal{L}(u, \partial U)|x - y|$$

for any $x, y \in U$, and hence $\mathcal{L}(u, \partial U) = \mathcal{L}(u, U)$ and $u$ is absolutely Lipschitz minimizing. $\square$

From these equivalences, we easily get the following corollary which relates solutions of the "Eikonal equation" to $\infty$-harmonic functions:

**Corollary 5.4.** *Let $u$ be everywhere differentiable in $\Omega$, and let $u$ solve the eikonal equation in $\Omega$, i.e*

$$|Du| = b \text{ for some constant } b. \tag{5.9}$$

*Then $u$ is $\infty$-harmonic in $\Omega$*

*Proof.* Assume that $u(x) \leq C(x) = a + b|x - z|$ on $\partial U$, $z \notin \partial U$, $U \subset \Omega$. If $u(x) - C(x)$ has a maximum at $x_0$ in $U \setminus \{z\}$, then $Du(x_0) = DC(x_0)$. Differentiating, we have that

$$Du(x_0) = b\frac{x_0 - z}{|x_0 - z|}, \;\; |Du(x_0)| = b.$$

Now, if $b = 0$, we see that $|Du| = 0$, and $u$ is a constant, and also $\infty$-harmonic. If $b \neq 0$, then $u$ and $C$ are viscosity solutions of (5.9) in $U$, and $u \leq C$ on $\partial U$. It follows from Theorem 3.5 in [Koi04, p. 29] that $u \leq C$ in $U$, and thus $u$ enjoys comparison with cones from above in $U$. By the previous proposition, we have that $u$ is $\infty$-subharmonic.

The other inequality is analogous, and the corollary is proved. $\square$

## 5.2 Existence

The technique of comparison with cones allows us to prove the existence of certain $\infty$-harmonic functions on a domain of $\mathbb{R}^n$. Until now, we have only proved the existence of solutions on bounded domains, but restricting our boundary function $f$ we can prove the following:

**Proposition 5.5.** *Let $\Omega$ be a (possibly unbounded) open subset of $\mathbb{R}^n$, and assume $0 \in \partial\Omega$. Let $A^{\pm}, B^{\pm} \in \mathbb{R}$, $A^+ \geq A^-$. Assume that $f : \partial\Omega \to \mathbb{R}$ satisfies*

$$A^-|x| + B^- \leq f(x) \leq A^+|x| + B^+ \text{ for } x \in \partial\Omega.$$

*Then there exists a continuous solution $u$ to the $\infty$-Laplace equation, such that $u = f$ on $\partial\Omega$, and such that*

$$A^-|x| + B^- \leq u(x) \leq A^+|x| + B^+ \text{ for } x \in \Omega.$$

The proof runs by Perron's method. We start by defining functions $\overline{h}(x), \underline{h}(x) : \mathbb{R}^n \to R$ by

$$\overline{h}(x) = \inf_{\overline{C} \in \mathcal{U}} \overline{C}(x),$$
$$\mathcal{U} = \{\overline{C}(x) : \overline{C}(x) = c + a|x - z|, a > A^+, \overline{C}(x) \geq f \text{ on } \partial\Omega\},$$

$$\underline{h}(x) = \sup_{\underline{C} \in \mathcal{L}} \underline{C}(x),$$
$$\mathcal{L} = \{\underline{C}(x) : \underline{C}(x) = c + a|x - z|, a < A^-, \underline{C}(x) \leq f \text{ on } \partial\Omega\},$$

where $z \in \partial\Omega$ and $c \in \mathbb{R}$. We then have the following:

**Lemma 5.6.** *The functions $\overline{h}(x), \underline{h}(x)$ satisfy*

1. $\overline{h}(x), \underline{h}(x)$ *are well-defined,*
2. $\overline{h}(x) = \underline{h}(x) = f$ *on* $\partial\Omega$,
3. $\overline{h}(x) \leq A^+|x| + B^+$, $\underline{h}(x) \geq A^-|x| + B^-$,
4. $\overline{h}(x) \geq \underline{h}(x)$,
5. $\overline{h}(x)$ *is upper semi-continuous, and* $\underline{h}(x)$ *is lower semi-continuous,*
6. $\overline{h}(x)$ *is a viscosity subsolution, and* $\underline{h}(x)$ *is a viscosity supersolution in* $\Omega$,

*Proof.* To prove 1., we will find a cone satisfying the properties required for $\overline{h}$. Fix $z \in \partial\Omega$. For any $\epsilon > 0$, choose a $\delta$ so that

$$f(x) < f(z) + \epsilon \ \forall \ z \in B_\delta(z)$$

Then choose a positive $a$, $a > A^+$, so that

$$f(z) + \epsilon + a\delta > \max_{x \in B_\delta(z)} (A^+|x| + B^+) \tag{5.10}$$

and, for $z \neq 0$,

$$f(z) + \epsilon + a|z| > B^+. \tag{5.11}$$

We then claim that the cone defined by

$$\overline{C}(x) = f(z) + \epsilon + a|x - z|$$

satisfies $\overline{C}(x) \geq A^+|x| + B^+$ in $\mathbb{R}^n \setminus B_\delta(z)$. Define the set $A$ by

$$A = \{x \in \mathbb{R}^n \setminus \overline{B_\delta(z)} : \overline{C}(x) < A^+|x| + B^+\}.$$

Assume that $A$ is not empty. Then $A \cap B_\delta(z) = \emptyset$, and hence $\overline{C}(x) = A^+|x| + B^+$ on $\partial A$. Because of (5.11), we have that $A$ contains neither of the vertices of the cones $\overline{C}(x)$ or $A^+|x| + B^+$. Since $a > A^+$, this implies that $\overline{C}(x) = A^+|x| + B^+$ in $A$, a contradiction, and hence $A$ is empty. Hence the function $\overline{h}(x)$ is well defined.

From the definition of $\overline{C}(x)$, we see that $f(z) = \overline{C}(z) + \epsilon$. Since $z \in \partial\Omega$ and $\epsilon$ were arbitrary, this implies that $f = \overline{h}(x)$ on $\partial\Omega$, and this proves 2.

For 3., given an $\epsilon > 0$, we can define $\overline{C}(x) = (A^+ + \epsilon)|x| + B^+$. Then $\overline{h}(x) \leq A^+|x| + B^+$ by the definition.

Now, let $\overline{C}(x) = \overline{a}|x - \overline{z}| + \overline{b}$ and $\underline{C}(x) = \underline{a}|x - \underline{z}| + \underline{b}$ be any two cones from the definition of $\overline{h}(x)$ and $\underline{h}(x)$, respectively. As in the proof of 1., define

$$A = \{x \in \mathbb{R}^n \setminus \overline{B_\delta(z)} : \overline{C}(x) < \underline{C}(x)\}$$

Again, if $A \neq \emptyset$, then $\overline{C}(x) = \underline{C}(x)$, on $\partial A$, and $A$ does not contain either of the vertices. Hence $\overline{C}(x) = \underline{C}(x)$ in $A$, a contradiction, and $A$ is empty. We conclude that $\overline{C}(x) \geq \underline{C}(x)$, and therefore $\overline{h}(x) \geq \underline{h}(x)$. We have thus proved 4.

Since $\overline{h}(x)$ is the supermum of continuous functions, we immediately have that $\overline{h}(x)$ is upper semi-continuous. The same reasoning shows that $\underline{h}(x)$ is lower semi-continuous.

For the proof of 6., we notice that $\mathcal{U}$ and $\mathcal{L}$ are non-empty sets containing viscosity solutions of $\Delta_\infty u = 0$. Furthermore, both $\overline{h}(x)$ and $\underline{h}(x)$ are bounded on compact subsets of $\mathbb{R}^n$ By Theorem 4.2 in [Koi04], we have that $\overline{h}(x)$ and $\underline{h}(x)$ are a viscosity subsolution and supersolution, respectively. □

*Proof of Proposition 5.5.* Define

$$u(x) = \sup_{v \in \mathcal{A}} v(x) \tag{5.12}$$

where

$$\mathcal{A} = \{v(x) : \underline{h} \leq v \leq \overline{h},\ v \text{ enjoys comparison with cones from above}\}$$

We now have that $u$ clearly satisfies $u = f$ on $\partial\Omega$ by 2. in 5.6. By 3. in the same lemma, we see that

$$A^-|x| + B^- \leq u(x) \leq A^+|x| + B^+ \text{ for } x \in \Omega$$

Furthermore, by the definition of $\mathcal{A}$, and the fact that $\overline{h}(x)$, $\underline{h}(x)$ are upper and -lower semi continuous, Theorem 4.3 in [Koi04] gives directly that $u$ is a viscosity solution of the $\infty$-Laplace equation, since the fact that $v$ enjoys comparison with cones from above implies that $v$ is a viscosity subsolution. Thus we have verified all the desired properties of $u$. $\square$

Chapter 6

# Concluding Remarks

## 6.1 Some Comments Regarding Regularity

A central question regarding the $\infty$-Laplace equation is: How regular are the solutions? In Chapters 2 and 3 we derived the equation from a limit procedure, and showed that if $u$ solves the Dirichlet problem

$$\begin{aligned}\Delta_\infty u &= 0 \text{ in } \Omega\\ u &= f \text{ on } \partial\Omega\end{aligned}$$

then $u$ satisfies

$$\|u\|_{\text{Lip}} \leq \|f\|_{\text{Lip}}$$

From this we conclude that $\infty$-harmonic functions are Lipschitz-continuous, and by Rademacher's Theorem differentiable *almost everywhere* in $\Omega$. On the other hand, from Corollary 5.4 we see that the solutions can in general be no better than merely differentiable solutions of the eikonal equation.

An example of an $\infty$-harmonic function in $\mathbb{R}^2$ is

$$u(x, y) = x^{4/3} - y^{4/3}.$$

From this, we see that $u \in C^{1,\frac{1}{3}}$, but the second derivatives does not exist along the coordinate axes. It is natural to ask if every $\infty$-harmonic function has this Hölder exponent, but the question remains undecided.

Savin proved in [Sav05] that every $\infty$-harmonic function in $\mathbb{R}^2$ is continuously differentiable. However, for higher dimensions, Smart and Evans proved that the solutions are differentiable, see [ES11]. The question regarding continuity of the differential is still open in this case.

## 6.2 Applications

Several interesting applications of the $\infty$-Laplace equation have been discovered since Aronssons work. We present two such applications below.

The $\infty$-Laplace equation arises in the field of image processing. If we consider an open, bounded $\Omega \subset \mathbb{R}^2$, and assume that an interpolating function $u(x)$ is $C^2(\Omega)$ at $x \in \Omega$. If we then demand that $u$ satisfies

$$u(x) = \frac{1}{2}(u(x + hDu) + u(x - hDu)) + o(h^2),$$

we can use the Taylor expansion of $u$ near $x$ to see that as $h \to 0$, we must have that $u$ satisfies (1) in $\Omega$.

V. Caselles and J. M Morel proved in [CMS98] that, in fact, *any* interpolating function $u$ that is able to interpolate data given on both curves and points while at the same time having bounded gradient, must be a viscosity solution of the $\infty$-Laplace equation in $\Omega$.

The $\infty$-Laplace equation also has interesting applications to game theory and stochastic calculus. In [PSSW11], Peres et. al considered the following *tug-of-war*-game between two players: Given an open, bounded domain $\Omega$ of $\mathbb{R}^n$ and a continuous function $f$ on $\partial\Omega$, the game proceeds as follows:

- Fix an $x_0 \in \Omega$, and a step-size $\epsilon > 0$.
- At each turn, the players toss a fair coin.
- At the k'th turn, the winner of the toss chooses an $x_k$ such that $|x_{k-1} - x_k| < \epsilon$.
- We assume that the players always choose the optimal move at each turn

- The game ends when $x_k \in \partial\Omega$, and Player 1 wins $f(x_k)$.

Then the *value function* $u_\epsilon$ for Player 1 is such that

$$u_\epsilon \to u \text{ as } \epsilon \to 0$$

uniformly, where $u$ solves the initial value problem (1), with $u = f$ on $\partial\Omega$. This can be compared to the case of the Laplace equation, where we know that the value function $u$ for the usual Brownian motion satisfies $\Delta u = 0$.

Appendix A

# Viscosity Theory

## A.1 Definition and Examples

Here we will formally state the definition of *viscosity solutions* of an elliptic partial differential equation, and provide some examples.

**Definition A.1.** *We say that a partial differential equation $F$ is (degenerate) elliptic, if*

$$F(x, r, p, X) \leq F(x, r, p, Y) \tag{A.1}$$

*for all $x \in \Omega$, $r \in \mathbb{R}$, $p \in \mathbb{R}$, $X, Y \in \mathbb{S}^n$, provided $X \geq Y$ in the usual ordering of symmetric matrices.*

If $F$ does not depend upon $X$ (i.e the equation is first order), $F$ is automatically elliptic.

**Example.** If we let $F(x, r, p, X) = -\langle Xp, p\rangle$, We see that if $X \geq Y$, then

$$-\langle Xp, p\rangle \leq -\langle Yp, p\rangle \ \forall\ p \in \mathbb{R}^n$$

by the usual ordering on $\mathbb{S}^n$. Hence the negative $\infty$-Laplace equation is an elliptic PDE. This is why one often sees $-\Delta_\infty$ in the literature.

We now turn to the definition of what it means for a function $u$ to be a viscosity solution of a general PDE. Notice that we do not pose any regularity conditions on $u$.

**Definition A.2.** *We say that a function $u : \Omega \to \mathbb{R}$ is a viscosity subsolution of $F(x, r, p, X) = 0$ if, for any $\phi \in C^2(\Omega)$*

$$F(x, \phi(x), D\phi(x), D^2\phi(x)) \leq 0$$

*provided $u - \phi$ attains it maximum at $x \in \Omega$. Similarly, $u$ is a viscosity supersolution if*

$$F(x, \phi(x), D\phi(x), D^2\phi(x)) \geq 0$$

*provided $u - \phi$ attains it minimum at $x \in \Omega$. A function that is both a viscosity sub and supersolution is simply said to be a viscosity solution.*

Notice that to check if $u$ is a viscosity solution, we do not plug $u$ into $F$ and and see if the equation is satisfied. Instead, we test with twice differentiable functions that touches $u$ from below and above. This definition of a solution might seem somewhat outlandish, but the next proposition assures us that for elliptic PDE's, at least twice differentiable classical solutions are viscosity solutions.

**Proposition A.3.** *Assume that $F$ is elliptic. A function $u$ is a classical solution if and only if $u \in C^2(\Omega)$ and $u$ is a viscosity solution.*

*Proof.* We will only prove the proposition for subsolutions, the proof for supersolutions is symmetric. First, assume that $u \in C^2(\Omega)$ and that $u$ is a classical subsolution. Pick any $\phi \in C^2(\Omega)$. If $u - \phi$ attains its maximum at any point $x_0 \in \Omega$, we have

$$\begin{aligned} D(u - \phi)(x_0) &= 0 \\ D^2(u - \phi)(x_0) &\leq 0 \end{aligned}$$

Hence $Du = D\phi$ and $D^2u \leq D^2\phi$ at $x_0$, and because of (A.1):

$$0 \geq F(x_0, u(x_0), Du(x_0), D^2(x_0)) \geq F(x, \phi(x_0), D\phi(x_0), D^2\phi(x_0))$$

On the other hand, assume that $u \in C^2(\Omega)$ is a viscosity solution. Choosing $\phi = u$, we have that $u - \phi$ attains its maximum at any point in $\Omega$, and hence

$$0 \geq F(x, u(x), Du(x), D^2(x))$$

for any $x \in \Omega$, by the definition of viscosity solution. $\square$

We illustrate this concept with a simple example.

**The Eikonal Equation** Consider the following Dirichlet problem on $\Omega = (-1, 1)$:

$$|u'(x)|^2 = 1, \; u(\pm 1) = 0 \tag{A.2}$$

We expect the solution to be $u(x) = 1 - |x|$ for $x \in [-1, 1]$. However, this solution is not differentiable at $x = 0$, and in fact (A.2) cannot have *any* $C^1$ solution. To see this, notice that by the Fundamental Theorem of Calculus, we have

$$0 = u(1) - u(-1) = \int_{-1}^{1} u'(y) \,\mathrm{d}y$$

If $u'$ is continuous, by the mean value theorem, there must exist a point $a \in [-1, 1]$ such that $u'(a) = 0$, contradicting (A.2). This implies that the solution $1 - |x|$ must be interpreted in a more relaxed sense. We will show that $u(x) = 1 - |x|$ is a solution in the viscosity sense. To this end, assume that $\phi \in C^2([-1, 1])$ is such that $u - \phi$ attains its minimum at some $x_0$. We see that if $x_0 \neq 0$, then $u$ is differentiable, and hence a viscosity solution by A.1. So assume that $u - \phi$ attains its minimum at 0, and that $\phi(0) = u(0)$. Then we have, for $x$ close to 0:

$$u(0) - \phi(0) \leq u(x) - \phi(x)$$

This implies

$$|x| \leq \phi(0) - \phi(x),$$

which gives

$$\begin{aligned} 1 &\leq \frac{\phi(0) - \phi(x)}{x} \quad \text{for } x > 0, \\ -1 &\geq \frac{\phi(0) - \phi(x)}{x} \quad \text{for } x < 0. \end{aligned}$$

Letting $x \to 0$, and remembering that $\phi$ is differentiable, we conclude that $|\phi'(0)| \geq 1$. The case if $u - \phi$ attains its maximum at $x = 0$ is similar, and hence we conclude that $u(x) = 1 - |x|$ is a viscosity solution of (A.2).

## A.2 Semi-jets

**Definition A.4** (Semi-jets)**.** *Let $u : \Omega \to \mathbb{R}$ be a function. The semi-jets of $u$ at $x$ is*

$$J^{2,+}u(x) = \left\{ (p, X) \in \mathbb{R}^n \times \mathbb{S}^n : \begin{array}{r} u(y) \le u(x) + \langle p, y - x \rangle \\ + \frac{1}{2}\langle X(y - x), y - x \rangle \\ + o(|x - y|^2) \text{ as } y \to x \end{array} \right\}, \tag{A.3}$$

*and*

$$J^{2,-}u(x) = \left\{ (p, X) \in \mathbb{R}^n \times \mathbb{S}^n : \begin{array}{r} u(y) \ge u(x) + \langle p, y - x \rangle \\ + \frac{1}{2}\langle X(y - x), y - x \rangle \\ + o(|x - y|^2) \text{ as } y \to x \end{array} . \right\} \tag{A.4}$$

Notice that if $J^{2,+}u(x) \cap J^{2,-}u(x) \neq \emptyset$, then

$$J^{2,+}u(x) \cap J^{2,-}u(x) = (Du(x), D^2u(x))$$

from the definition. We also have the closure of the semi-jets:

**Definition A.5.**

$$\overline{J}^{2,\pm}u(x) = \left\{ (p, X) \in \mathbb{R}^n \times \mathbb{S}^n : \begin{array}{r} \exists\ x_k \in \Omega \text{ and } \exists (p_k, X_k) \in J^{2,\pm}u(x_k) \\ \text{such that } (x_k, u(x_k), p_k, X_k) \to \\ (x, u(x), p, X) \text{ as} k \to \infty \end{array} \right\}.$$

These semi-jets will give us an important characterization of viscosity solutions, as described by the following proposition:

**Proposition A.6.** *The following are equivalent.*

1. *$u$ is a viscosity subsolution (resp. supersolution).*

2. *For $x \in \Omega$ and $(p, X) \in J^{2,+}u(x)$ (resp. $J^{2,-}u(x)$), we have $F(x, u(x), p, X) \le 0$ (resp. $\ge 0$).*

3. *For $x \in \Omega$ and $(p, X) \in \overline{J}^{2,+}u(x)$ (resp. $\overline{J}^{2,-}u(x)$), we have $F(x, u(x), p, X) \leq 0$ (resp. $\geq 0$).*

The proof of these equivalences can be found in [Koi04, p. 19].

**Remark.** From these equivalences, we see that we know the semi-jets from their graphs. Indeed, we have

$$J^{2,+}u(x) = \left\{ (p, X) \in \mathbb{R}^n \times \mathbb{S}^n : \begin{array}{c} \exists\, \phi \in C^2(\Omega) \text{ so that } u - \phi \\ \text{attains its maximum at x} \end{array} \right\}.$$

Likewise,

$$J^{2,-}u(x) = \left\{ (p, X) \in \mathbb{R}^n \times \mathbb{S}^n : \begin{array}{c} \exists\, \phi \in C^2(\Omega) \text{ so that } u - \phi \\ \text{attains its minimum at x} \end{array} \right\}.$$

**Example.** In light of the remark above, we will investigate the semi-jets of the solution of the eikonal equation found in (A.2), namely $u(x) = 1 - |x|$ for $-1 \leq x \leq 1$. Since $u$ is smooth at every point except 0, we have that

$$J^{2,+}u(x) = J^{2,-}u(x) = (1, 0) \quad \text{for } x \neq 0$$

by the remark following A.4.

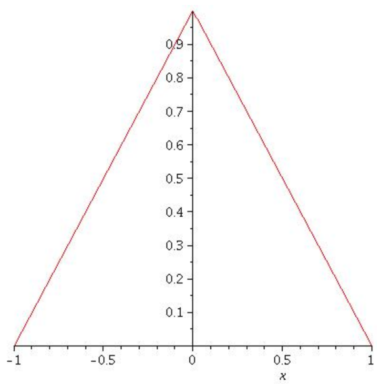


**Figure A.1:** Eikonal solution

However, from Figure A.1 we see that $J^{2,-}u(0) = \emptyset$ since there can be no smooth function $\phi$ such that $u - \phi$ attains its minimum at

$x = 0$. On the other hand, any function $\phi$ such that $u - \phi$ attains its maximum at $x = 0$ that have derivative 1 or $-1$ at 0 must have non-negative second derivative. Furthermore, a $\phi$ that has derivative strictly between $-1$ and 1 can have any second derivative at $x = 0$. Collecting this, we have that

$$J^{2,-}u(0) = (\{1\} \times [0, \infty)) \cup (\{-1\} \times [0, \infty)) \cup ((-1, 1) \times \mathbb{R}).$$

Appendix B

# Some Functional Analysis

## B.1 The Spaces

Let $1 < p < \infty$. We let $L^p(\Omega)$ denote the space of $p$-integrable functions defined on $\Omega$. For $p = 2$, this will be a Hilbert space, and a complete Banach space with the norm

$$\|u\|_p = \left\{ \int_\Omega |u|^p \, \mathrm{d}x \right\}^{1/p}$$

otherwise. It is well-known that the spaces $L^p(\Omega)$ are *reflexive*, that is

$$(L^p(\Omega))^{**} = L^p(\Omega)$$

This is an easy consequence of the fact that the dual of $L^p$ is $L^q$, where $\frac{1}{p} + \frac{1}{q} = 1$.

**Definition B.1.** *We say that a sequence $\{x_n\}_{n=1}^{\infty}$ in a normed space $X$ converges* weakly *to $x \in X$ provided*

$$f(x_n) \to f(x) \tag{B.1}$$

*for all $f \in X^*$, where $X^*$ denotes the dual, that is, the space of all linear bounded functionals on $X$, of $X$. This is often written*

$$x_n \rightharpoonup x$$

We are now ready to give an important characterization of weak convergence in the $L^p$ spaces. We have that any bounded linear functional $L$ on $L^p$ can be written as

$$L(u) = \int_\Omega ug \, \mathrm{d}x \tag{B.2}$$

for some $g \in L^q$. This is Riez' Representation Theorem. (See for example Theorem 9.12 in [WM99].) Thus, $u_k \rightharpoonup u$ in $L^p$ is equivalent to saying that

$$\int_\Omega gu_k \, \mathrm{d}x \to \int_\Omega gu \, \mathrm{d}x \tag{B.3}$$

for every $g \in L^q$.

Another important feature of reflexive Banach spaces, and thus the $L^p$-spaces, is the *weak compactness*.

**Theorem B.2.** *Let $X$ be a reflexive Banach space, and let $\{x_n\}_{n=1}^\infty$ be a bounded sequence in $X$. Then there exists a subsequence, $\{x_{n_j}\}_{j=1}^\infty$ and a $x \in X$ such that*

$$x_{k_j} \rightharpoonup x$$

This is a consequence of the Banach–Alaoglu theorem, see [Rud73, p.68].

## B.2 Weak Derivative and Sobolev Spaces

**Definition B.3.** *Given an locally integrable function $u$, we say that a locally integrable function $v$ is the* weak derivative *of $u$ provided*

$$\int_\Omega u \frac{\partial \phi}{\partial x_i} \, \mathrm{d}x = - \int_\Omega \phi v \, \mathrm{d}x$$

*for every $i, 1 \leq ileqn$ and all $\phi \in C_0^\infty(\Omega)$.*

We then write $v = \frac{\partial u}{\partial x_i}$.

**Definition B.4.** *The* Sobolev space $W^{1,p}(\Omega)$ *consists of all locally integrable functions $u : \Omega \to \mathbb{R}$ such that $Du$ exists (in the weak sense) and $Du \in L^p(\Omega)$.*

**Definition B.5.** *If $u \in W^{1,p}(\Omega)$ then we the define the* Sobolev norm *of $u$ to be*

$$\|u\|_{W^{1,p}} = \begin{cases} \|u\|_{L^p} + \|Du\|_{L^p} & \text{for } 1 \leq p < \infty \\ ess\sup_\Omega |u| + ess\sup_\Omega |Du| & \text{for } p = \infty \end{cases} \tag{B.4}$$

With this norm the spaces $W^{1,p}(\Omega)$ are Banach spaces.

**Remark.** Since $W^{1,\infty}(\Omega)$ consists of the functions with bounded weak derivative, it is natural to assume that functions in $W^{1,\infty}(\Omega)$ are Lipschitz continuous. This is indeed the case, see [Eva98, p.294].

**Definition B.6.** *The space $W_0^{1,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the norm of $W^{1,p}(\Omega)$.*

This means that for any $w \in W_0^{1,p}(\Omega)$ we have that given an $\epsilon > 0$ there exists a function $\phi \in C_0^\infty(\Omega)$ such that

$$\|w - \phi\|_{W_0^{1,p}(\Omega)} < \epsilon.$$

It is clear from the definition that $W^{1,p}(\Omega)$ is closed under weak convergence, but since $W_0^{1,p}(\Omega)$ is a closed, linear subspace of $W^{1,p}(\Omega)$, Mazur's Theorem [Eva98, §D.4] gives the following

**Theorem B.7.** *$W_0^{1,p}(\Omega)$ is closed under weak convergence.*

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