AUSLANDER–GORENSTEIN ALGEBRAS AND PRECLUSTER TILTING

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To the memory of Maurice Auslander.

Abstract. We generalize the notions of n-cluster tilting subcategories and τ-selfinjective algebras into n-precluster tilting subcategories and τ_n-selfinjective algebras, where we show that a subcategory naturally associated to n-precluster tilting subcategories has a higher Auslander–Reiten theory. Furthermore, we give a bijection between n-precluster tilting subcategories and n-minimal Auslander–Gorenstein algebras, which is a higher dimensional analog of Auslander–Solberg correspondence (Auslander–Solberg, 1993) as well as a Gorenstein analog of n-Auslander correspondence (Iyama, 2007). The Auslander–Reiten theory associated to an n-precluster tilting subcategory is used to classify the n-minimal Auslander–Gorenstein algebras into four disjoint classes. Our method is based on relative homological algebra due to Auslander–Solberg.

Contents

1. Introduction

Higher Auslander–Reiten theory was introduced in [21] by looking at n-cluster tilting subcategories instead of the whole module category. It is known that any n-cluster tilting subcategory has n-almost split sequences, and that finite n-cluster

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tilting subcategories correspond to n-Auslander algebras [22]. Moreover Artin algebras Λ which have n-cluster tilting modules are called n-representation-finite [13] and have been studied. (We do not assume $\operatorname{gldim}\Lambda \leq n$ in this paper in contrast with several earlier papers [18, 24, 25].) See for example [1, 19, 20, 26, 28, 29, 30, 31, 34, 36] for further results in higher Auslander–Reiten theory.

In this paper we introduce the weaker notions of n-precluster tilting subcategories (Definition 3.2) and τ_n-selfinjective algebras (Definition 3.4). These notions generalize and unify two seemingly different concepts, namely (finite) n-cluster tilting subcategories [21] and τ-selfinjective algebras (or $D\operatorname{Tr}$-selfinjective algebras) [11]. Both of these concepts are generalizations of algebras of finite representation type and they are linked through the Wedderburn correspondence introduced in [2]. The first class corresponds to *n-Auslander algebras* Γ, that is, Artin algebras Γ satisfying

$$\operatorname{domdim}\Gamma \geq n+1 \geq \operatorname{gldim}\Gamma.$$

The latter class corresponds to algebras Γ that are a Gorenstein analog of Auslander algebras which satisfy

$$\operatorname{domdim}\Gamma \geq 2 \geq \operatorname{id}_\Gamma \Gamma.$$

Here domdim, gldim and id denote dominant dimension, global dimension and injective dimension, respectively. A central notion in higher dimensional Auslander–Reiten theory is n-cluster tilting subcategories, which are the whole module categories in the classical case $n = 1$. As in the process of going from classical Auslander–Reiten theory to higher Auslander–Reiten theory, we demonstrate that the number 2 is quite symbolic, as we show that n-precluster-tilting subcategories correspond to the following class of Artin algebras.

Definition 1.1. We call an Artin algebra Γ *an n-minimal Auslander–Gorenstein algebra* if it satisfies

$$\operatorname{domdim}\Gamma \geq n+1 \geq \operatorname{id}_\Gamma \Gamma.$$

We will observe in Proposition 4.1(b) that this condition is left-right symmetric. Recall that an Artin algebra Γ is called *Gorenstein* (or *Iwanaga–Gorenstein* more precisely) if $\operatorname{id}_\Gamma \Gamma$ and $\operatorname{id}\Gamma_\Gamma$ are finite [15], and a Gorenstein algebra is called *Auslander–Gorenstein* if the minimal injective coresolution

$$0 \to \Gamma \to I^0 \to I^1 \to \cdots$$

of the Γ-module Γ satisfies $\operatorname{pd}_\Gamma I^i \leq i$ holds for every $i \geq 0$ [16, 5, 6]. Thus our n-minimal Auslander–Gorenstein algebras can be regarded as the most basic class among Auslander–Gorenstein algebras since $\operatorname{pd}_\Gamma I^i$ has the 'minimal' value 0 for $0 \leq i \leq n$.

We have the following diagrams of generalizations of Auslander algebras and representation-finite algebras:

$$\begin{array}{ccc}
\text{Auslander algebras} & \dashrightarrow & \text{Gorenstein algebras } \Gamma \text{ with } \operatorname{domdim}\Gamma \geq 2 \geq \operatorname{id}_\Gamma \Gamma \\
\downarrow & & \downarrow \\
n\text{-Auslander algebras} & \dashrightarrow & n\text{-minimal Auslander–Gorenstein algebras}
\end{array}$$

$$\begin{array}{ccc}
\text{representation-finite algebras} & \dashrightarrow & \tau\text{-selfinjective algebras} \\
\downarrow & & \downarrow \\
n\text{-representation-finite algebras} & \dashrightarrow & \tau_n\text{-selfinjective algebras}
\end{array}$$

We introduce the concept of n-precluster tilting subcategories to describe these algebras, where such a subcategory $\mathcal{C}$ is defined to be a functorially finite generating-cogenerating subcategory stable under the n-th Auslander–Reiten translate and selforthogonal in the interval $[1, n-1]$ (that is, $\operatorname{Ext}^i(\mathcal{C}, \mathcal{C}) = 0$ for $i = 1, 2, \dots, n-1$). More precisely, we show the following results.

Theorem 4.5. *Fix $n \geq 1$. There is a bijection between Morita-equivalence classes of n-minimal Auslander–Gorenstein algebras and equivalence classes of finite n-precluster tilting subcategories $\mathcal{C}$ of Artin algebras, where the correspondences are given in* Propositions 4.3 *and* 4.4.

For a Gorenstein algebra Γ, we denote by

$$\operatorname{CM}\Gamma = \{X \in \operatorname{mod}\Gamma \mid \operatorname{Ext}^i_\Gamma(X, \Gamma) = 0 \text{ for } i > 0\}$$

the category of *maximal Cohen–Macaulay* Γ-modules. We denote by $\underline{\operatorname{CM}}\,\Gamma$ the stable category of $\operatorname{CM}\Gamma$, that is, $\operatorname{CM}\Gamma$ modulo the ideal in $\operatorname{CM}\Gamma$ generated by $\operatorname{add}\Gamma$. It is basic that $\underline{\operatorname{CM}}\,\Gamma$ forms a triangulated category. On the other hand, for an n-precluster tilting subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$, let

$$\begin{aligned} \mathcal{Z}(\mathcal{C}) &= \{X \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(\mathcal{C}, X) = 0 \text{ for } i = 1, 2, \dots, n-1\}, \\ \mathcal{U}(\mathcal{C}) &= \mathcal{Z}(\mathcal{C})/[\mathcal{C}], \end{aligned}$$

where $\mathcal{Z}(\mathcal{C})/[\mathcal{C}]$ denotes the category $\mathcal{Z}(\mathcal{C})$ modulo the ideal $[\mathcal{C}]$ in $\mathcal{Z}(\mathcal{C})$ generated by the subcategory $\mathcal{C}$. Note that $\mathcal{Z}(\mathcal{C}) = \operatorname{mod}\Lambda$ for the case $n = 1$. We show that the category $\mathcal{Z}(\mathcal{C})$ is a Frobenius category whose stable category is $\mathcal{U}(\mathcal{C})$ (Proposition 3.12). Hence $\mathcal{U}(\mathcal{C})$ forms a triangulated category. In fact it is an analog of Calabi–Yau reduction of triangulated categories [27].

Theorem 4.7. *Given an Artin algebra Λ with a finite n-precluster tilting subcategory $\mathcal{C} = \operatorname{add} M$, let $\Gamma = \operatorname{End}_\Lambda(M)$ be the corresponding n-minimal Auslander–Gorenstein algebra (see* Theorem 4.5*). Then $\mathcal{Z}(\mathcal{C})$ and $\operatorname{CM}\Gamma$ are dual categories via the functors $\operatorname{Hom}_\Lambda(-, M)\colon \mathcal{Z}(\mathcal{C}) \to \operatorname{CM}\Gamma$ and $\operatorname{Hom}_\Gamma(-, M)\colon \operatorname{CM}\Gamma \to \mathcal{Z}(\mathcal{C})$. Moreover these functors induce triangle equivalences between $\mathcal{U}(\mathcal{C})$ and $(\underline{\operatorname{CM}}\,\Gamma)^{\mathrm{op}}$.*

Furthermore we show that there is a higher Auslander–Reiten theory also for n-precluster tilting subcategories, though with some differences. The first difference is that one cannot define n-fold almost split sequences as for n-cluster tilting subcategories, but one is forced to introduce n-fold almost split extensions (see Definition 5.1). This is because in this more general setting there does not exist a unique exact sequence representing this extension. Namely, we have the following results, where we denote by $\mathcal{P}(\Lambda)$ (respectively $\mathcal{I}(\Lambda)$) the category of finitely generated projective (respectively injective) Λ-modules.

Theorem 5.10. *Let $\mathcal{C}$ be an n-precluster tilting subcategory of $\operatorname{mod}\Lambda$, X an indecomposable module in $\mathcal{Z}(\mathcal{C}) \setminus \mathcal{P}(\Lambda)$, and $Y := \tau_n(X)$ the corresponding indecomposable module in $\mathcal{Z}(\mathcal{C}) \setminus \mathcal{I}(\Lambda)$.*

(a) *For each $0 \leq i \leq n-1$, an n-fold almost split extension in $\operatorname{Ext}^n_\Lambda(X, Y)$ can be represented as*

$$0 \to Y \to C_{n-1} \to \cdots \to C_{i+1} \to Z_i \to C_{i-1} \to \cdots \to C_0 \to X \to 0$$

with Z_i in $\mathcal{Z}(\mathcal{C})$ and C_j in $\mathcal{C}$ for each $j \neq i$.

(b) *The following sequences are exact.*

$$0 \to \mathrm{Hom}_\Lambda(\mathcal{C}, Y) \to \mathrm{Hom}_\Lambda(\mathcal{C}, C_{n-1}) \to \cdots \to \mathrm{Hom}_\Lambda(\mathcal{C}, C_{i+1}) \to \mathrm{Hom}_\Lambda(\mathcal{C}, Z_i)$$
$$\to \mathrm{Hom}_\Lambda(\mathcal{C}, C_{i-1}) \to \cdots \to \mathrm{Hom}_\Lambda(\mathcal{C}, C_0) \to \mathrm{rad}_\Lambda(\mathcal{C}, X) \to 0,$$
$$0 \to \mathrm{Hom}_\Lambda(X, \mathcal{C}) \to \mathrm{Hom}_\Lambda(C_0, \mathcal{C}) \to \cdots \to \mathrm{Hom}_\Lambda(C_{i-1}, \mathcal{C}) \to \mathrm{Hom}_\Lambda(Z_i, \mathcal{C})$$
$$\to \mathrm{Hom}_\Lambda(C_{i+1}, \mathcal{C}) \to \cdots \to \mathrm{Hom}_\Lambda(C_{n-1}, \mathcal{C}) \to \mathrm{rad}_\Lambda(Y, \mathcal{C}) \to 0.$$

(c) *If X and Y do not belong to $\mathcal{C}$, then the n-fold almost split extension in* (a) *can be given as a Yoneda product of a minimal projective resolution of X in $\mathcal{Z}(\mathcal{C})$*

$$0 \to \Omega^i_{\mathcal{Z}(\mathcal{C})}(X) \to C_{i-1} \to \cdots \to C_0 \to X \to 0,$$

an almost split sequence in $\mathcal{Z}(\mathcal{C})$

$$0 \to \Omega^{-(n-i-1)}_{\mathcal{Z}(\mathcal{C})}(Y) \to Z_i \to \Omega^i_{\mathcal{Z}(\mathcal{C})}(X) \to 0,$$

and a minimal injective coresolution of Y in $\mathcal{Z}(\mathcal{C})$

$$0 \to Y \to C_{n-1} \to \cdots \to C_{i+1} \to \Omega^{-(n-i-1)}_{\mathcal{Z}(\mathcal{C})}(Y) \to 0.$$

We use this Auslander–Reiten theory to classify the n-minimal Auslander–Gorenstein algebras into four disjoint classes, as was done for the case $n = 1$ in [11].

We refer to related results. In [12], the authors studied (n, m)-ortho-symmetric modules, where our n-precluster tilting modules are precisely $(n-1, 0)$-ortho-symmetric. In [32], the author proved Theorem 4.5 independently.

The paper is organized as follows. In the second section we recall the relative homological algebra over Artin algebras we need (see [8, 9, 10]) and some unpublished results of Maurice Auslander and the second author. In Section 3 the notions of n-precluster tilting subcategories and τ_n-selfinjective algebras are introduced and their basic properties are discussed. We show in the next section that there is a one-to-one correspondence between finite n-precluster subcategories and n-minimal Auslander–Gorenstein Artin algebras, where the n-Auslander algebras are characterized within this class. In the fifth section we show that there is a meaningful higher Auslander–Reiten theory in n-precluster tilting subcategories also. This theory is transferred in the next section over to the subcategory of maximal Cohen–Macaulay modules over the n-minimal Auslander–Gorenstein Artin algebras. In the final section we use higher Auslander–Reiten theory to classify the n-minimal Auslander–Gorenstein Artin algebras into four disjoint classes.

Notations. Throughout the paper, all modules are left modules. The composition of morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ is denoted by $gf\colon X \to Z$.

Let R be a commutative Artinian ring and Λ an Artin R-algebra. We denote by $\mathrm{mod}\,\Lambda$ the category of finitely generated Λ-modules, by $\mathcal{P}(\Lambda)$ (respectively $\mathcal{I}(\Lambda)$) the category of finitely generated projective (respectively injective) Λ-modules, and by $D\colon \mathrm{mod}\,\Lambda \leftrightarrow \mathrm{mod}\,\Lambda^{\mathrm{op}}$ the duality $\mathrm{Hom}_R(-, E)$, where E is the injective hull of the R-module $R/\mathrm{rad}\,R$. We denote by $\tau\colon \underline{\mathrm{mod}}\,\Lambda \to \overline{\mathrm{mod}}\,\Lambda$ the Auslander–Reiten translation.

Let $\mathcal{X}$ be a full subcategory of $\mathrm{mod}\,\Lambda$. We call $\mathcal{X}$ a *generator* (respectively *cogenerator*) if $\Lambda \in \mathcal{X}$ (respectively $D\Lambda \in \mathcal{X}$). A *right $\mathcal{X}$-approximation* of $A \in \mathrm{mod}\,\Lambda$ is a morphism $f\colon X \to A$ with $X \in \mathcal{X}$ such that any morphism $g\colon Y \to A$ with $Y \in \mathcal{X}$ factors through f. We call $\mathcal{X}$ *contravariantly finite* if any $A \in \mathrm{mod}\,\Lambda$ has a right $\mathcal{X}$-approximation. Dually, we define a *left $\mathcal{X}$-approximation* and a *covariantly finite* subcategory. A contravariantly and covariantly finite subcategory is called *functorially finite*.

2. Preliminaries on relative homological algebra

A systematic study of relative homological algebra over Artin algebras was carried out in [8, 9, 10]. We recall the relevant background and results, and in addition we give some unpublished results of Maurice Auslander and the second author.

2.1. **Relative homological algebra.** We start with the setup for relative homological algebra, where we assume throughout that Λ is an Artin algebra. Relative homological algebra for us begins with defining a set of exact sequences, and this is done through giving an additive sub-bifunctor of $\operatorname{Ext}^1_\Lambda(-,-)$ (see [8] for further details). Let

$$F \subseteq \operatorname{Ext}^1_\Lambda(-,-)\colon (\operatorname{mod}\Lambda)^{\mathrm{op}} \times \operatorname{mod}\Lambda \to \mathrm{Ab}$$

be an additive sub-bifunctor. Such an additive sub-bifunctor is nothing else than, for each pair of Λ-modules C and A, a chosen set of short exact sequences, $F(C,A)$, starting in A and ending in C, which is closed under pullbacks, pushouts and Baer sums (or direct sums of short exact sequences).

Definition 2.1. An exact sequence $\eta\colon 0 \to A \to B \to C \to 0$ is said to be *F-exact* if η is in $F(C,A)$.

In the rest, we fix a subcategory $\mathcal{X}$ of $\operatorname{mod}\Lambda$. We consider the following collection $F_{\mathcal{X}}(C,A)$ of short exact sequences given for a pair of modules A and C in $\operatorname{mod}\Lambda$:

$$F_{\mathcal{X}}(C,A) = \{0 \to A \to B \to C \to 0 \in \operatorname{Ext}^1_\Lambda(C,A) \mid \operatorname{Hom}_\Lambda(X,B) \to \operatorname{Hom}_\Lambda(X,C) \to 0 \text{ exact for all } X \text{ in } \mathcal{X}\}.$$

Dually one defines $F^{\mathcal{X}}$. By [8, Proposition 1.7] these collections induce additive sub-bifunctors $F_{\mathcal{X}}$ and $F^{\mathcal{X}}$ of $\operatorname{Ext}^1_\Lambda(-,-)$, and

$$F_{\mathcal{X}} = F^{\tau\mathcal{X}} \quad\text{and}\quad F_{\tau^-\mathcal{X}} = F^{\mathcal{X}} \tag{2.1}$$

by [8, Proposition 1.8]. If $\mathcal{X} = \operatorname{add} X$ holds for some X in $\operatorname{mod}\Lambda$, we denote $F_{\mathcal{X}}$ and $F^{\mathcal{X}}$ by F_X and F^X respectively.

We can endow $\operatorname{mod}\Lambda$ with a new exact structure by the following result, where we call a full subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ *F-extension closed* if, for every F-exact sequence $0 \to A \to B \to C \to 0$ such that A and C are in $\mathcal{C}$, B is in $\mathcal{C}$.

Proposition 2.2 ([14])**.** *Let $\mathcal{X}$ be a full subcategory of $\operatorname{mod}\Lambda$ and $F = F^{\mathcal{X}}$ (respectively $F = F_{\mathcal{X}}$). Then $\operatorname{mod}\Lambda$ has a structure of an exact category whose short exact sequences are precisely the F-exact sequences. More generally, any F-extension closed subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ has a structure of an exact category whose short exact sequences are precisely the F-exact sequences contained in $\mathcal{C}$.*

Proof. The first assertion follows from [14, Propositions 1.4 and 1.7]. The second assertion is a general property of extension closed subcategories of an exact category. □

We denote by $(\operatorname{mod}\Lambda, F)$ and $(\mathcal{C}, F)$ the exact categories given in Proposition 2.2.

Recall the following definitions from [8], which coincide with the corresponding notions in the exact category $(\operatorname{mod}\Lambda, F)$.

Definition 2.3. (i) A Λ-module P is said to be *F-projective* if all F-exact sequences $0 \to A \to B \to P \to 0$ split. The full subcategory of $\operatorname{mod}\Lambda$ consisting of all F-projective modules is denoted by $\mathcal{P}(F)$.

(ii) A Λ-module I is said to be *F-injective* if all F-exact sequences $0 \to I \to B \to C \to 0$ split. The full subcategory of $\operatorname{mod}\Lambda$ consisting of all F-injective modules is denoted by $\mathcal{I}(F)$.

(iii) F has *enough F-projectives* (respectively *F-injectives*) if for each C (respectively A) in $\operatorname{mod}\Lambda$ there exists an F-exact sequence

$$0 \to C' \to P \to C \to 0$$

with P in $\mathcal{P}(F)$ (respectively $0 \to A \to I \to A' \to 0$ with I in $\mathcal{I}(F)$).

The $F_{\mathcal{X}}$-projectives and $F_{\mathcal{X}}$-injectives are given as

$$\mathcal{P}(F_{\mathcal{X}}) = \operatorname{add}\{\mathcal{X}, \mathcal{P}(\Lambda)\} \quad \text{and} \quad \mathcal{I}(F_{\mathcal{X}}) = \operatorname{add}\{\tau\mathcal{X}, \mathcal{I}(\Lambda)\} \tag{2.2}$$

by [8, Proposition 1.10]. For example, we have $\mathcal{P}(F_\Lambda) = \mathcal{P}(\operatorname{Ext}^1_\Lambda(-,-)) = \mathcal{P}(\Lambda)$ and $\mathcal{I}(F^{D\Lambda}) = \mathcal{I}(\operatorname{Ext}^1_\Lambda(-,-)) = \mathcal{I}(\Lambda)$. Furthermore an additive sub-bifunctor F of $\operatorname{Ext}^1_\Lambda(-,-)$ has enough F-projectives and F-injectives if and only if $\mathcal{P}(F)$ is functorially finite in $\operatorname{mod}\Lambda$ and $F = F_{\mathcal{P}(F)}$ (see [8, Corollary 1.13]). In this case we denote by $\Omega^1_F(X)$ the kernel of the F-projective cover of X. The Λ-module $\Omega^{-1}_F(Y)$ is defined dually.

Assume from now on that F is an additive sub-bifunctor of $\operatorname{Ext}^1_\Lambda(-,-)$ with enough F-projectives and F-injectives. Recall that an exact sequence

$$\cdots \to C_{i+1} \xrightarrow{f_{i+1}} C_i \xrightarrow{f_i} C_{i-1} \to \cdots$$

is called *F-exact* if all the short exact sequences $0 \to \operatorname{Im} f_{i+1} \to C_i \to \operatorname{Im} f_i \to 0$ are F-exact. Given two modules A and C in $\operatorname{mod}\Lambda$, there exist F-exact sequences

$$\mathbb{P}\colon\ \cdots \to P_2 \to P_1 \to P_0 \to C \to 0$$

with P_i in $\mathcal{P}(F)$ and

$$\mathbb{I}\colon 0 \to A \to I^0 \to I^1 \to I^2 \to \cdots$$

with I^j in $\mathcal{I}(F)$. We call these exact sequences an *F-projective resolution* and an *F-injective coresolution* of C and A, respectively. For $i \geq 1$, the i-th homologies of the complexes $\operatorname{Hom}_\Lambda(\mathbb{P}, A)$ and $\operatorname{Hom}_\Lambda(C, \mathbb{I})$ are isomorphic, and denoted by $\operatorname{Ext}^i_F(C, A)$ and called the i-th F-relative extension group (see [8, Section 2]). Then $\operatorname{Ext}^1_F(-,-)$ is naturally identified with F. Using $\operatorname{Ext}^i_F(-,-)$, we define F-relative projective dimension, F-relative injective dimension and F-relative global dimension as in the absolute setting and the basic properties are the same in the F-relative setting. On the other hand, we denote by

$$\underline{\operatorname{Hom}}_F(A, C) \quad (\text{respectively } \overline{\operatorname{Hom}}_F(A, C))$$

$\operatorname{Hom}_\Lambda(A, C)$ modulo all the homomorphisms factoring through an F-projective (respectively F-injective) module. The *F-stable category*

$$\underline{\operatorname{mod}}_F \Lambda \quad (\text{respectively } \overline{\operatorname{mod}}_F \Lambda)$$

has the same objects as $\operatorname{mod}\Lambda$, and the morphism sets are given by $\underline{\operatorname{Hom}}_F(A, C)$ (respectively $\overline{\operatorname{Hom}}_F(A, C)$). For $F = F_{\mathcal{X}}$, the Auslander–Reiten translation $\tau\colon\ \underline{\operatorname{mod}}\,\Lambda \simeq \overline{\operatorname{mod}}\,\Lambda$ induces the F-relative Auslander–Reiten translation

$$\tau\colon\ \underline{\operatorname{mod}}_F \Lambda \simeq \overline{\operatorname{mod}}_F \Lambda. \tag{2.3}$$

Another central result that has an analog in the F-relative setting is the Auslander–Reiten formula, which we recall next.

Proposition 2.4 ([8, Proposition 2.3]). *Let F be an additive sub-bifunctor of $\operatorname{Ext}^1_\Lambda(-,-)$ with enough F-projectives (and F-injectives). Then for all modules A and C in $\operatorname{mod}\Lambda$ we have an isomorphism*

$$\operatorname{Ext}^1_F(C,\tau A) \simeq D\,\underline{\operatorname{Hom}}_F(A,C).$$

In general the higher F-relative extension groups $\operatorname{Ext}^i_F(C,A)$ are not necessarily related to the higher absolute extension groups $\operatorname{Ext}^i_\Lambda(C,A)$. However, in some situations one can compute the absolute extensions by F-relative ones, as described in the next result.

Proposition 2.5 ([33, Proposition 1.3]). *Let $\mathcal{X}$ be a functorially finite subcategory of $\operatorname{mod}\Lambda$.*

(a) *A module C in $\operatorname{mod}\Lambda$ satisfies $\operatorname{Ext}^i_\Lambda(C,\mathcal{X}) = 0$ for $0 < i < n$ if and only if $\operatorname{Ext}^i_{F^{\mathcal{X}}}(C,A) = \operatorname{Ext}^i_\Lambda(C,A)$ holds for $0 < i < n$ and for all A in $\operatorname{mod}\Lambda$.*
(b) *A module A in $\operatorname{mod}\Lambda$ satisfies $\operatorname{Ext}^i_\Lambda(\mathcal{X},A) = 0$ for $0 < i < n$ if and only if $\operatorname{Ext}^i_{F_{\mathcal{X}}}(C,A) = \operatorname{Ext}^i_\Lambda(C,A)$ for $0 < i < n$ and for all C in $\operatorname{mod}\Lambda$.*

Proof. We only show (a). This is proved in [33, Proposition 1.3] under the assumption that $\mathcal{X}$ is a cogenerator. We can drop it by considering $\mathcal{X}' = \operatorname{add}\{\mathcal{X},\mathcal{I}(\Lambda)\}$ and using $F^{\mathcal{X}} = F^{\mathcal{X}'}$. □

2.2. **The Auslander–Reiten translation revisited.** Next we recall some unpublished results of Maurice Auslander and the second author that we need later. We show that the Auslander–Reiten translation τ gives a bijection between $\operatorname{Ext}^1_F(C,A)$ and $\operatorname{Ext}^1_G(\tau(C),\tau(A))$ for certain F and G, and moreover τ commutes with relative syzygies.

The transpose is a duality $\operatorname{Tr}\colon \underline{\operatorname{mod}}\,\Lambda \to \underline{\operatorname{mod}}\,\Lambda^{\mathrm{op}}$ [3]. However, from an exact sequence

$$0 \to A \to B \to C \to 0$$

in $\operatorname{mod}\Lambda$, there is not necessarily a naturally associated exact sequence

$$0 \to \operatorname{Tr} C \to \operatorname{Tr} B \oplus X \to \operatorname{Tr} A \to 0$$

in $\operatorname{mod}\Lambda^{\mathrm{op}}$. We show that when restricting to appropriate classes of exact sequences we have such a natural correspondence.

Proposition 2.6. *Let $\mathcal{X}$ be a functorially finite generator-cogenerator in $\operatorname{mod}\Lambda$, and let $F = F^{\mathcal{X}}$, $G^{\mathrm{op}} = F^{D(\mathcal{X})}$ and $G = F_{\mathcal{X}}$.*

(a) *If $0 \to A \to B \to C \to 0$ is F-exact in $\operatorname{mod}\Lambda$, then there are a G^{op}-exact sequence*

$$0 \to \operatorname{Tr} C \to \operatorname{Tr} B \oplus P \to \operatorname{Tr} A \to 0$$

in $\operatorname{mod}\Lambda^{\mathrm{op}}$ for some projective Λ^{op}-module P and a G-exact sequence

$$0 \to \tau(A) \to \tau(B) \oplus I \to \tau(C) \to 0$$

in $\operatorname{mod}\Lambda$ for some injective Λ-module I.
(b) *For all A and C in $\operatorname{mod}\Lambda$, we have functorial isomorphisms*

$$F(C,A) \simeq G^{\mathrm{op}}(\operatorname{Tr} A, \operatorname{Tr} C) \simeq G(\tau(C),\tau(A)).$$

Proof. (a) Let $\eta\colon 0 \to A \to B \xrightarrow{f} C \to 0$ be an F-exact sequence. By the Horseshoe Lemma we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_1 & \longrightarrow & P_1 \oplus Q_1 & \longrightarrow & Q_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_0 & \longrightarrow & P_0 \oplus Q_0 & \longrightarrow & Q_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with P_i and Q_i projective Λ-modules for $i = 0, 1$. Let $(-)^* = \operatorname{Hom}_\Lambda(-, \Lambda)$. Then this induces the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^* & \longrightarrow & B^* & \longrightarrow & A^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q_0^* & \longrightarrow & P_0^* \oplus Q_0^* & \longrightarrow & P_0^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q_1^* & \longrightarrow & P_1^* \oplus Q_1^* & \longrightarrow & P_1^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\eta'\colon 0 & \longrightarrow & \operatorname{Tr} C & \longrightarrow & \operatorname{Tr} B \oplus P & \xrightarrow{g} & \operatorname{Tr} A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where P is a projective Λ^{op}-module. The columns are exact by the definition of Tr, and the top row is exact because η is F-exact and $\mathcal{X}$ contains Λ. Since the second and the third rows are split exact, the Snake Lemma shows that the bottom row η' is also exact. It remains to show that η' is G^{op}-exact. Then the dual of this sequence is G-exact.

Since η is F-exact, the map $f\colon \underline{\operatorname{Hom}}_\Lambda(B, \mathcal{X}) \to \underline{\operatorname{Hom}}_\Lambda(A, \mathcal{X})$ is surjective. Since Tr is a duality, the map $g = \operatorname{Tr} f\colon \underline{\operatorname{Hom}}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr} \mathcal{X}, \operatorname{Tr} B) \to \underline{\operatorname{Hom}}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr} \mathcal{X}, \operatorname{Tr} A)$ is surjective. By a standard argument, the map $g\colon \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr} \mathcal{X}, \operatorname{Tr} B \oplus P) \to \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr} \mathcal{X}, \operatorname{Tr} A)$ is also surjective. Since $\tau \operatorname{Tr} \mathcal{X} = D(\mathcal{X})$ and hence $F_{\operatorname{Tr} \mathcal{X}} = G^{\mathrm{op}}$ holds, η' is G^{op}-exact.

(b) By (a), we have a morphism $F(C, A) \to G^{\mathrm{op}}(\operatorname{Tr} A, \operatorname{Tr} C)$ of bifunctors. Since the same argument gives the inverse morphism $G^{\mathrm{op}}(\operatorname{Tr} A, \operatorname{Tr} C) \to F(C, A)$ of bifunctors, we have the first desired isomorphism. The second isomorphism follows immediately by applying the duality D. □

We end this subsection by showing how this induces isomorphisms on relative extension groups and relative stable homomorphism sets.

Theorem 2.7. [i)] *Let $\mathcal{X} \subseteq \operatorname{mod} \Lambda$ be a functorially finite generator-cogenerator, and let $F = F^{\mathcal{X}}$ and $G = F_{\mathcal{X}}$ be the corresponding additive sub-bifunctors of $\operatorname{Ext}^1_\Lambda(-,-)$. Then the following is true.*

(a) *τ gives an equivalence $\underline{\operatorname{mod}}_F \Lambda \simeq \overline{\operatorname{mod}}_G \Lambda$.*

[i)] Theorem 2.7 was obtained by Maurice Auslander and the second author, but they never got to be published in printed form before now. However, the results were presented at a seminar at the University of Bielefeld, Germany.

(b) *The following diagrams commute up to isomorphisms of functors.*

$$\begin{array}{ccc} \underline{\mathrm{mod}}_F\,\Lambda & \xrightarrow{\tau} & \underline{\mathrm{mod}}_G\,\Lambda \\ \downarrow{\scriptstyle \Omega_F} & & \downarrow{\scriptstyle \Omega_G} \\ \underline{\mathrm{mod}}_F\,\Lambda & \xrightarrow{\tau} & \underline{\mathrm{mod}}_G\,\Lambda \end{array} \qquad \begin{array}{ccc} \overline{\mathrm{mod}}_F\,\Lambda & \xleftarrow{\tau^-} & \overline{\mathrm{mod}}_G\,\Lambda \\ \uparrow{\scriptstyle \Omega_F^-} & & \uparrow{\scriptstyle \Omega_G^-} \\ \overline{\mathrm{mod}}_F\,\Lambda & \xleftarrow{\tau^-} & \overline{\mathrm{mod}}_G\,\Lambda \end{array}$$

(c) τ *induces a functorial isomorphism in both variables*

$$\varphi_n = \varphi_{C,A,n}\colon \mathrm{Ext}_F^n(C,A) \simeq \mathrm{Ext}_G^n(\tau(C),\tau(A))$$

for all pairs of A and C in $\mathrm{mod}\,\Lambda$ *and* $n \geq 1$.

Proof. (a) We have an equivalence $\tau\colon \underline{\mathrm{mod}}_F\,\Lambda \simeq \overline{\mathrm{mod}}_F\,\Lambda$ in (2.3). Since $\mathcal{X}$ is a generator-cogenerator, $\mathcal{I}(F) = \mathcal{P}(G)$ holds by (2.2). Thus $\overline{\mathrm{mod}}_F\,\Lambda = \underline{\mathrm{mod}}_G\,\Lambda$ holds, and the assertion follows.

(b) We only prove commutativity of the left diagram. Let

$$0 \to \Omega_F(A) \to P \to A \to 0$$

be F-exact with P in $\mathcal{P}(F)$. By Proposition 2.6, we have a G-exact sequence

$$0 \to \tau\Omega_F(A) \to \tau(P) \oplus I \to \tau(A) \to 0$$

with I in $\mathcal{I}(\Lambda)$. Since $\tau(P) \oplus I$ is in $\mathcal{P}(G)$ by (2.2), we have an isomorphism $\tau\Omega_F \simeq \Omega_G\tau$ of functors.

(c) We have the following functorial isomorphisms

$$\begin{aligned} \mathrm{Ext}_F^n(C,A) &\simeq \mathrm{Ext}_F^1(\Omega_F^{n-1}(C),A) \\ &\simeq \mathrm{Ext}_G^1(\tau\Omega_F^{n-1}(C),\tau(A)) \\ &\simeq \mathrm{Ext}_G^1(\Omega_G^{n-1}\tau(C),\tau(A)) \\ &\simeq \mathrm{Ext}_G^n(\tau(C),\tau(A)) \end{aligned}$$

by dimension shift, Proposition 2.6(b) and (b), where all the involved isomorphisms are functorial. The claim follows. □

2.3. **Relative tilting theory.** Tilting theory is an important topic in representation theory of Artin algebras and elsewhere. It also has a relative version, which we recall from [9, 10]. Here we always assume that our additive sub-bifunctor F has enough F-projectives and enough F-injectives. For a subcategory $\mathcal{C}$ in $\mathrm{mod}\,\Lambda$, let

$$\begin{aligned} \mathcal{C}^{\perp_F} &= \{M \in \mathrm{mod}\,\Lambda \mid \mathrm{Ext}_F^i(\mathcal{C},M) = 0 \text{ for all } i > 0\}, \\ {}^{\perp_F}\mathcal{C} &= \{M \in \mathrm{mod}\,\Lambda \mid \mathrm{Ext}_F^i(M,\mathcal{C}) = 0 \text{ for all } i > 0\}. \end{aligned}$$

When $F = \mathrm{Ext}_\Lambda^1(-,-)$, we simply denote $\mathcal{C}^{\perp_F}$ and ${}^{\perp_F}\mathcal{C}$ by $\mathcal{C}^\perp$ and ${}^\perp\mathcal{C}$ respectively.

Definition 2.8. We call T in $\mathrm{mod}\,\Lambda$ an *F-cotilting module* if

(i) $\mathrm{id}_F\,T < \infty$,
(ii) $\mathrm{Ext}_F^i(T,T) = 0$ for $i > 0$,
(iii) for all I in $\mathcal{I}(F)$ there exists an F-exact sequence

$$0 \to T_n \to T_{n-1} \to \cdots \to T_1 \to T_0 \to I \to 0$$

with T_i in $\mathrm{add}\,T$.

It is shown in [9, Corollary 3.14] that if there exists an F-cotilting module T, then $\mathcal{P}(F)$ and hence $\mathcal{I}(F)$ are of finite type. In fact, they contain the same number of non-isomorphic indecomposable objects as $\operatorname{add} T$.

Next we collect the basic results on relative cotilting modules that we need later.

Theorem 2.9 ([9])**.** *Let X be a generator in $\operatorname{mod}\Lambda$ and $F = F_X$. Let T be an F-cotilting Λ-module and $\Gamma = \operatorname{End}_\Lambda(T)$. Then we have the following.*

(a) $\Lambda \simeq \operatorname{End}_\Gamma(T)$.

(b) *Any C in ${}^{\perp_F}T$ has an F-exact sequence*

$$0 \to C \to T_0 \xrightarrow{f_0} T_1 \xrightarrow{f_1} T_2 \xrightarrow{f_2} T_3 \to \cdots$$

with T_i in $\operatorname{add} T$ and $\operatorname{Im} f_i$ in ${}^{\perp_F}T$ for all $i \geq 0$.

(c) $\operatorname{Ext}^i_F(C, A) \simeq \operatorname{Ext}^i_\Gamma(\operatorname{Hom}_\Lambda(A, T), \operatorname{Hom}_\Lambda(C, T))$ *for all modules A and C in ${}^{\perp_F}T$ and $i \geq 0$.*

(d) *The module $U = \operatorname{Hom}_\Lambda(X, T)$ is a cotilting Γ-module with $\operatorname{id}_F T \leq \operatorname{id}_\Gamma U \leq \operatorname{id}_F T + 2$* [ii)].

(e) $\operatorname{Hom}_\Lambda(-, T)\colon \operatorname{mod}\Lambda \to \operatorname{mod}\Gamma$ *and* $\operatorname{Hom}_\Gamma(-, T)\colon \operatorname{mod}\Gamma \to \operatorname{mod}\Lambda$ *induce quasi-inverse dualities* $\operatorname{Hom}_\Lambda(-, T)\colon {}^{\perp_F}T \to {}^{\perp}U$ *and* $\operatorname{Hom}_\Gamma(-, T)\colon {}^{\perp}U \to {}^{\perp_F}T$.

Proof. (a) is [9, Corollary 3.4], (b) is [9, Theorem 3.2(a)], (c) is [9, Proposition 3.7], (d) is [9, Theorem 3.13(d)], and (e) is [9, Corollary 3.6(a), Proposition 3.8(b)]. □

Now let F be an additive sub-bifunctor of $\operatorname{Ext}^1_\Lambda(-,-)$, and $\mathcal{X}$ a generator of $\operatorname{mod}\Lambda$. Then $\mathcal{X}\text{-resdim}_F M$ is defined to be the infimum of n such that there exists an F-exact sequence

$$0 \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to M \to 0,$$

where X_i is in $\mathcal{X}$ for all $i \geq 0$. We denote by $\widehat{\mathcal{X}}$ the full subcategory of $\operatorname{mod}\Lambda$ consisting of all M in $\operatorname{mod}\Lambda$ with $\mathcal{X}\text{-resdim}_F M < \infty$. Let $\mathcal{X}\text{-resdim}_F(\operatorname{mod}\Lambda) := \sup\{\mathcal{X}\text{-resdim}_F M \mid M \in \operatorname{mod}\Lambda\}$.

The following result is taken from [10] (with the exception of (c)), which connects special direct summands of absolute cotilting modules with relative cotilting modules. Let Γ be an Artin algebra and X in $\operatorname{mod}\Gamma$. Recall that a direct summand Y of X is said to be a *dualizing summand of X* if there exists an exact sequence

$$0 \to X \xrightarrow{f} Y^0 \to Y^1$$

with f a left $(\operatorname{add} Y)$-approximation and Y^i in $\operatorname{add} Y$. This is shown to be equivalent to the natural homomorphism

$$X \to \operatorname{Hom}_\Lambda(\operatorname{Hom}_\Gamma(X, Y), Y)$$

being an isomorphism, where $\Lambda = \operatorname{End}_\Gamma(Y)$ [10, Proposition 2.1].

Theorem 2.10. *Let Γ be an Artin algebra and U a cotilting Γ-module. For a dualizing summand T of U, let $\Lambda = \operatorname{End}_\Gamma(T)$, $X = \operatorname{Hom}_\Gamma(U, T)$ and $F = F_X$ an additive sub-bifunctor of $\operatorname{Ext}^1_\Lambda(-,-)$. Then the following assertions hold.*

(a) $\Gamma \simeq \operatorname{End}_\Lambda(T)$.

(b) *T is an F-cotilting Λ-module with* $\operatorname{id}_F T \leq \max\{\operatorname{id}_\Gamma U, 2\}$.

(c) *If T is injective as a Γ-module, then* $\operatorname{id}_F T \leq \max\{\operatorname{id}_\Gamma U - 2, 0\}$.

[ii)] This inequality is a corrected version of [9, Theorem 3.13(d)] due to an error in the statement of [9, Proposition 3.11], where the bound should be $\operatorname{id}_F T + 2$, instead of $\max\{\operatorname{id}_F T, 2\}$.

Proof. (a) is [10, Proposition 2.4(b)], and (b) is [10, Proposition 2.7(c)].

(c) Let $r = \operatorname{id}_\Gamma U$ and $t = \max\{r-2, 0\}$. We prove that $({}^{\perp_F}T)$-$\operatorname{resdim}_F(\operatorname{mod}\Lambda) \leq t$, as this implies that $\operatorname{id}_F T \leq t$. This is done by showing that $\Omega^t_F(C)$ is in ${}^{\perp_F}T$ for all C in $\operatorname{mod}\Lambda$.

Assume that ${}_\Gamma T$ is injective. Then by [11, Lemma 2.4] the module ${}_\Lambda T$ is a cogenerator in $\operatorname{mod}\Lambda$, and therefore

$$C \simeq \operatorname{Hom}_\Gamma(\operatorname{Hom}_\Lambda(C, T), T)$$

for all modules C in $\operatorname{mod}\Lambda$. In particular, $C \simeq \operatorname{Hom}_\Gamma(B, T)$ for some Γ-module B. Let

$$\mathbb{P}\colon B \xrightarrow{f^0} U^0 \xrightarrow{f^1} U^1 \xrightarrow{f^2} U^2 \xrightarrow{f^3} \cdots,$$

be a sequence of minimal left $(\operatorname{add} U)$-approximations of B, and let $B^0 = B$ and $B^j = \operatorname{Coker} f^{j-1}$ for $j \geq 1$. Then an F-projective resolution of C is given by

$$\operatorname{Hom}_\Gamma(\mathbb{P}, T)\colon \cdots \to \operatorname{Hom}_\Gamma(U^2, T) \to \operatorname{Hom}_\Gamma(U^1, T) \to \operatorname{Hom}_\Gamma(U^0, T) \to C \to 0,$$

and therefore we have, for every $j \geq 0$,

$$\Omega^j_F(C) = \operatorname{Hom}_\Gamma(B^j, T).$$

Let $\Sigma = \operatorname{End}_\Gamma(U)$. Then U is a cotilting Σ-module with $\operatorname{id}_\Sigma U = \operatorname{id}_\Gamma U$. The above complex $\mathbb{P}$ gives rise to a projective resolution of the Σ-module $\operatorname{Hom}_\Gamma(B, U)$,

$$\cdots \to \operatorname{Hom}_\Gamma(U^2, U) \to \operatorname{Hom}_\Gamma(U^1, U) \to \operatorname{Hom}_\Gamma(U^0, U) \to \operatorname{Hom}_\Gamma(B, U) \to 0.$$

Given a projective presentation $F_1 \to F_0 \to B \to 0$ of the Γ-module B, it induces an exact sequence

$$0 \to \operatorname{Hom}_\Gamma(B, U) \to \operatorname{Hom}_\Gamma(F_0, U) \to \operatorname{Hom}_\Gamma(F_1, U) \to B' \to 0$$

of Σ-modules with $\operatorname{Hom}_\Gamma(F_0, U)$ and $\operatorname{Hom}_\Gamma(F_1, U)$ in $\operatorname{add}_\Sigma U$. Then $\Omega^{j+2}_\Sigma(B') = \operatorname{Hom}_\Gamma(B^j, U)$ holds for every $j \geq 0$.

Now we show $\operatorname{Hom}_\Gamma(B^t, U)$ is in ${}^{\perp}_{\Sigma}U$. If $r \leq 2$, then $\operatorname{Hom}_\Gamma(B^t, U) = \operatorname{Hom}_\Gamma(B, U) = \Omega^2_\Sigma(B')$ is in ${}^{\perp}_{\Sigma}U$ since $\operatorname{id}_\Gamma U = r \leq 2$. If $r > 2$, then $\operatorname{Hom}_\Gamma(B^t, U) = \operatorname{Hom}_\Gamma(B^{r-2}, U) = \Omega^r_\Sigma(B')$ is in ${}^{\perp}_{\Sigma}U$ since $\operatorname{id}_\Gamma U = r$.

By (a), (b) and Theorem 2.9(e), we have dualities

$${}^{\perp}_{\Sigma}U \xleftarrow{\operatorname{Hom}_\Gamma(-,U)} {}^{\perp}_{\Gamma}U \xrightarrow{\operatorname{Hom}_\Gamma(-,T)} {}^{\perp_F}T.$$

We take B'' in ${}^{\perp}_{\Gamma}U$ such that $\operatorname{Hom}_\Gamma(B'', U) \simeq \operatorname{Hom}_\Gamma(B^t, U)$ as Σ-modules. Then

$$\Omega^t_F(C) = \operatorname{Hom}_\Gamma(B^t, T) \simeq \operatorname{Hom}_\Gamma(B'', T) \in {}^{\perp_F}T$$

as Λ-modules. Thus the claim holds. □

Let F be an additive sub-bifunctor of $\operatorname{Ext}^1_\Lambda(-,-)$. A full subcategory $\mathcal{X}$ of $\operatorname{mod}\Lambda$ is *F-resolving* (respectively *F-coresolving*) if

(i) $\mathcal{X}$ is F-extension closed,
(ii) $\mathcal{P}(F)$ (respectively $\mathcal{I}(F)$) is contained in $\mathcal{X}$,
(iii) if $0 \to A \to B \to C \to 0$ is F-exact and B and C are in $\mathcal{X}$ (respectively A and B are in $\mathcal{X}$), then A (respectively C) is in $\mathcal{X}$.

We need the following preparation from Auslander–Buchweitz theory.

Proposition 2.11 ([9, Theorems 2.4, 2.5, Proposition 2.2])**.** *Let F be an additive sub-bifunctor of* $\operatorname{Ext}^1_\Lambda(-,-)$*, and $\mathcal{X}$ an F-resolving subcategory of* $\operatorname{mod}\Lambda$*. Assume that the exact category $(\mathcal{X},F)$ given in* Proposition 2.2 *has enough F-injectives and* $\widehat{\mathcal{X}}=\operatorname{mod}\Lambda$*. Then the following assertions hold.*

(a) *$\mathcal{X}$ is a contravariantly finite subcategory of* $\operatorname{mod}\Lambda$ *and $\mathcal{Y}:=\mathcal{X}^{\perp_F}$ is a covariantly finite subcategory of* $\operatorname{mod}\Lambda$.
(b) $\mathcal{X}\text{-resdim}_F(\operatorname{mod}\Lambda)=\operatorname{id}_F(\mathcal{X}^{\perp_F})$ *holds.*

3. Elementary properties of n-precluster tilting subcategories

Recall that for $n\geq 1$ a subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ is called *n-cluster tilting* if $\mathcal{C}$ is functorially finite and

$$\mathcal{C}={}^{\perp_{n-1}}\mathcal{C}=\mathcal{C}^{\perp_{n-1}},$$

where ${}^{\perp_{n-1}}\mathcal{C}$ is the full subcategory of $\operatorname{mod}\Lambda$ given by the modules

$${}^{\perp_{n-1}}\mathcal{C}=\{X\in\operatorname{mod}\Lambda\mid \operatorname{Ext}^i_\Lambda(X,\mathcal{C})=0 \text{ for } 0<i<n\}.$$

The full subcategory $\mathcal{C}^{\perp_{n-1}}$ is defined dually. In particular, it follows immediately from the definition that $\mathcal{C}$ is a generator-cogenerator for $\operatorname{mod}\Lambda$ and $\operatorname{Ext}^i_\Lambda(\mathcal{C},\mathcal{C})=0$ for $0<i<n$. In [21] the functors

$$\tau_n=\tau\Omega^{n-1}_\Lambda\colon \underline{\operatorname{mod}}\,\Lambda\to\overline{\operatorname{mod}}\,\Lambda \quad\text{and}\quad \tau_n^-=\tau^-\Omega^{-(n-1)}_\Lambda\colon \overline{\operatorname{mod}}\,\Lambda\to\underline{\operatorname{mod}}\,\Lambda$$

are defined as the *n-Auslander–Reiten translations.* By [22, Theorem 2.3.1], the pair (τ_n^-,τ_n) forms an adjunction. By [21, Theorem 1.4.1], they induce equivalences

$$\tau_n\colon \underline{{}^{\perp_{n-1}}\Lambda}\to\overline{D\Lambda^{\perp_{n-1}}} \quad\text{and}\quad \tau_n^-\colon \overline{D\Lambda^{\perp_{n-1}}}\to\underline{{}^{\perp_{n-1}}\Lambda}. \tag{3.1}$$

In particular τ_n and τ_n^- give bijections between indecomposable non-projective modules in ${}^{\perp_{n-1}}\Lambda$ and indecomposable non-injective modules in $D\Lambda^{\perp_{n-1}}$, while they do not preserve indecomposability for arbitrary modules. Moreover, for any n-cluster tilting subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$, they restrict to equivalences

$$\tau_n\colon\underline{\mathcal{C}}\to\overline{\mathcal{C}} \quad\text{and}\quad \tau_n^-\colon\overline{\mathcal{C}}\to\underline{\mathcal{C}}.$$

The next result gives a higher analog of Auslander–Reiten duality.

Lemma 3.1 ([21, Theorem 1.5])**.** *We have the following.*

(a) $\underline{\operatorname{Hom}}_\Lambda(C,A)\simeq D\operatorname{Ext}^n_\Lambda(A,\tau_n(C))$ *and* $\operatorname{Ext}^i_\Lambda(C,A)\simeq D\operatorname{Ext}^{n-i}_\Lambda(A,\tau_n(C))$ *for $0<i<n$, for all modules C in ${}^{\perp_{n-1}}\Lambda$ and all modules A in* $\operatorname{mod}\Lambda$.
(b) $\overline{\operatorname{Hom}}_\Lambda(C,A)\simeq D\operatorname{Ext}^n_\Lambda(\tau_n^-(A),C)$ *and* $\operatorname{Ext}^i_\Lambda(C,A)\simeq D\operatorname{Ext}^{n-i}_\Lambda(\tau_n^-(A),C)$ *for $0<i<n$, for all modules C in* $\operatorname{mod}\Lambda$ *and all modules A in $D\Lambda^{\perp_{n-1}}$.*

Proof. We give a proof of the second isomorphisms in our language of relative homological algebra.

(a) Let $F=F^\Lambda$ be an additive sub-bifunctor of $\operatorname{Ext}^1_\Lambda(-,-)$. Let C be in ${}^{\perp_{n-1}}\Lambda$, and let

$$\eta\colon 0\to\Omega^{n-1}_\Lambda(C)\to P_{n-2}\to\cdots\to P_0\to C\to 0$$

be a minimal projective resolution of C. Since C is in ${}^{\perp_{n-1}}\Lambda$, we have that the exact sequence η is F-exact. Using this we have for $0<i<n$ and an arbitrary

Λ-module A that

$$\begin{aligned} D\operatorname{Ext}^i_F(C,A) &\simeq D\operatorname{Ext}^1_F(\Omega^{i-1}_\Lambda(C),A) \\ &\simeq D\,\overline{\operatorname{Hom}}_F(\Omega^i_\Lambda(C),A) \\ &\simeq \operatorname{Ext}^1_F(\tau^-(A),\Omega^i_\Lambda(C)) \\ &\simeq \operatorname{Ext}^{n-i}_F(\tau^-(A),\Omega^{n-1}_\Lambda(C)) \\ &\simeq \operatorname{Ext}^{n-i}_\Lambda(A,\tau_n(C)), \end{aligned}$$

where the second isomorphism uses that P_i is in $\mathcal{I}(F)$, the third is the Auslander–Reiten formula, the first and the fourth use dimension shift and the last one is given by Theorem 2.7 and $G = F_\Lambda = \operatorname{Ext}^1_\Lambda(-,-)$.

Since C is in ${}^{\perp_{n-1}}\Lambda$, we have that $\operatorname{Ext}^i_\Lambda(C,A) \simeq \operatorname{Ext}^i_F(C,A)$ for $0 < i < n$ and for all modules A in $\operatorname{mod}\Lambda$ by Proposition 2.5(a). Thus the assertion follows.

(b) Similar proof as in (a). □

We introduce the notion of an n-precluster tilting subcategory by relaxing the conditions for an n-cluster tilting subcategory.

Definition 3.2. A subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ is called an *n-precluster tilting subcategory* if it satisfies the following conditions.

(i) $\mathcal{C}$ is a generator-cogenerator for $\operatorname{mod}\Lambda$,
(ii) $\tau_n(\mathcal{C}) \subseteq \mathcal{C}$ and $\tau_n^-(\mathcal{C}) \subseteq \mathcal{C}$,
(iii) $\operatorname{Ext}^i_\Lambda(\mathcal{C},\mathcal{C}) = 0$ for $0 < i < n$,
(iv) $\mathcal{C}$ is a functorially finite subcategory of $\operatorname{mod}\Lambda$.

If moreover $\mathcal{C}$ admits an additive generator M, we say that M is an *n-precluster tilting module.*

Using τ_n and τ_n^-, we define the subcategories

$$\mathcal{P}_n = \operatorname{add}\{\tau_n^{-i}(\Lambda)\}_{i=0}^\infty \quad\text{and}\quad \mathcal{I}_n = \operatorname{add}\{\tau_n^{i}(D(\Lambda_\Lambda))\}_{i=0}^\infty.$$

For any n-precluster tilting subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$, we have

$$\mathcal{P}_n \vee \mathcal{I}_n \subseteq \mathcal{C} \quad\text{and}\quad \mathcal{C} \subseteq D\Lambda^{\perp_{n-1}} \cap {}^{\perp_{n-1}}\Lambda$$

by (i), (ii) and (i), (iii) respectively.

Recall from [11] that an Artin algebra Λ is called *τ-selfinjective* if $\mathcal{P}_1$ is of finite type, which is shown to be equivalent to that $\mathcal{P}_1$ is equal to $\mathcal{I}_1$. We show next that this is equivalent to the existence of a 1-precluster tilting Λ-module.

Example 3.3. An Artin algebra Λ is τ-selfinjective if and only if Λ has a 1-precluster tilting module.

Proof. If Λ is τ-selfinjective, then clearly the additive generator of $\mathcal{P}_1$ is a 1-precluster tilting module since $D\Lambda \in \mathcal{I}_1 = \mathcal{P}_1$. If Λ has a finite 1-precluster subcategory $\mathcal{C}$, then it is clear from the definition that $\mathcal{P}_1$ is contained in $\mathcal{C}$. Hence $\mathcal{P}_1$ is of finite type and Λ is τ-selfinjective. □

This observation leads us to the following definition.

Definition 3.4. An Artin algebra Λ is called *τ_n-selfinjective* if Λ admits an n-precluster tilting module.

This is a common generalization of selfinjective algebras and n-representation-finite algebras. We continue by asking and giving one answer to the natural question: When is an Artin algebra τ_n-selfinjective?

Proposition 3.5. *Let Λ be an Artin algebra and $n \geq 1$. Then the following conditions are equivalent.*

(i) *Λ is τ_n-selfinjective.*
(ii) *$\mathcal{P}_n \vee \mathcal{I}_n$ is of finite type and $\mathrm{Ext}^i_\Lambda(\mathcal{P}_n \vee \mathcal{I}_n, \mathcal{P}_n \vee \mathcal{I}_n) = 0$ for $0 < i < n$.*
(iii) *$\mathcal{I}_n$ is of finite type, $\mathcal{I}_n \subset {}^{\perp_{n-1}}\Lambda$ and $\mathrm{Ext}^i_\Lambda(\mathcal{I}_n, \mathcal{I}_n) = 0$ for $0 < i < n$.*
(iv) *$\Lambda \in \mathcal{I}_n$ and $\mathrm{Ext}^i_\Lambda(\mathcal{I}_n, \mathcal{I}_n) = 0$ for $0 < i < n$.*
(v) *$\mathcal{P}_n$ is of finite type, $\mathcal{P}_n \subset D\Lambda^{\perp_{n-1}}$ and $\mathrm{Ext}^i_\Lambda(\mathcal{P}_n, \mathcal{P}_n) = 0$ for $0 < i < n$.*
(vi) *$D\Lambda \in \mathcal{P}_n$ and $\mathrm{Ext}^i_\Lambda(\mathcal{P}_n, \mathcal{P}_n) = 0$ for $0 < i < n$.*

Moreover, if these conditions are satisfied, then every additive generator of $\mathcal{P}_n \vee \mathcal{I}_n$ is an n-precluster tilting Λ-module.

Proof. (i) is equivalent to (ii): Assume that there exists an n-precluster tilting module M in $\mathrm{mod}\,\Lambda$. Since $\mathcal{P}_n \vee \mathcal{I}_n \subset \mathrm{add}\,M$, it is immediate that $\mathcal{P}_n \vee \mathcal{I}_n$ is of finite type and satisfies $\mathrm{Ext}^i_\Lambda(\mathcal{P}_n \vee \mathcal{I}_n, \mathcal{P}_n \vee \mathcal{I}_n) = 0$ for $0 < i < n$. Conversely, if $\mathcal{P}_n \vee \mathcal{I}_n$ satisfies (ii), then an additive generator of $\mathcal{P}_n \vee \mathcal{I}_n$ is an n-precluster tilting Λ-module.

(ii) implies (iii): This is immediate.

(iii) implies (iv): For each indecomposable non-projective module X in ${}^{\perp_{n-1}}\Lambda$, we know that $\tau_n(X)$ is indecomposable again by the equivalence (3.1). Since $\mathcal{I}_n$ is of finite type, then $\tau_n^l(I) \neq 0$ is projective for some $l \geq 0$ for all indecomposable injective modules I. Since the number of indecomposable projectives and of indecomposable injectives coincide, all indecomposable projective modules must occur in this way, hence Λ is in $\mathcal{I}_n$ and (iv) is satisfied.

(iv) implies (ii): For each indecomposable projective modules P there exists an indecomposable injective module I such that $P \simeq \tau_n^l(I)$. Since $\tau_n^{-i}(P) \simeq \tau_n^{l-i}(I)$ for $0 \leq i \leq l$ and $\tau_n^{-(l+1)}(P) = 0$ hold, we have $\mathcal{P}_n \vee \mathcal{I}_n = \mathcal{I}_n$. Thus (ii) is satisfied.

The equivalences of (ii), (v) and (vi) are shown dually. □

Using the above we have the following consequence for $n = 2$ thanks to a general property of $\mathcal{I}_2$.

Proposition 3.6 (cf. [23, Proposition 1.7])**.** *Let Λ be an Artin algebra. Then Λ is τ_2-selfinjective if and only if $\Lambda \in \mathcal{I}_2$.*

Proof. By [23, Proposition 2.5] the subcategory $\mathcal{I}_2$ satisfies $\mathrm{Ext}^1_\Lambda(\mathcal{I}_2, \mathcal{I}_2) = 0$. The claim then follows immediately from Proposition 3.5(iv)⇒(i). □

The next results give analog of properties of higher Auslander–Reiten translation for n-cluster tilting subcategories.

Proposition 3.7. *Let Λ be an Artin algebra and $\mathcal{C}$ an n-precluster tilting subcategory with $n \geq 1$. Then we have the following.*

(a) *We have mutually quasi-inverse equivalences $\tau_n \colon \underline{\mathcal{C}} \to \overline{\mathcal{C}}$ and $\tau_n^- \colon \overline{\mathcal{C}} \to \underline{\mathcal{C}}$.*
(b) *τ_n and τ_n^- give bijections between indecomposable non-projective modules in $\mathcal{C}$ and indecomposable non-injective modules in $\mathcal{C}$.*
(c) *We have $\mathrm{add}\{\tau_n^-(\mathcal{C}), \Lambda\} = \mathcal{C} = \mathrm{add}\{\tau_n(\mathcal{C}), D\Lambda\}$.*
(d) *There exists a full subcategory $\mathcal{D}$ of $\mathrm{mod}\,\Lambda$ such that $\mathcal{C} = \mathrm{add}\{\mathcal{P}_n \vee \mathcal{I}_n, \mathcal{D}\}$, $(\mathcal{P}_n \vee \mathcal{I}_n) \cap \mathcal{D} = \{0\}$ and $\tau_n(\mathcal{D}) = \mathcal{D} = \tau_n^-(\mathcal{D})$.*

Proof. (a) We have $\mathcal{C} \subseteq D\Lambda^{\perp_{n-1}} \cap {}^{\perp_{n-1}}\Lambda$. Since $\tau_n(\mathcal{C}) \subseteq \mathcal{C}$ and $\tau_n^-(\mathcal{C}) \subseteq \mathcal{C}$ hold, the claim follows from the equivalences (3.1).

(b)(c)(d) These follow immediately from (a). □

An n-cluster tilting subcategory $\mathcal{C}$ satisfies by definition the equalities $\mathcal{C} = {}^{\perp_{n-1}}\mathcal{C} = \mathcal{C}^{\perp_{n-1}}$. The same is not true for an n-precluster tilting subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ (e.g. Λ is non-semisimple selfinjective and $\mathcal{C} = \operatorname{add}\Lambda$). But the equality ${}^{\perp_{n-1}}\mathcal{C} = \mathcal{C}^{\perp_{n-1}}$ is shown still to be true.

Proposition 3.8. *Let $\mathcal{C}$ be a subcategory of* $\operatorname{mod}\Lambda$ *satisfying the conditions* (i), (iii) *and* (iv) *in* Definition 3.2.

(a) *We have $\Omega^-_{F^{\mathcal{C}}}({}^{\perp_{n-1}}\mathcal{C}) \subseteq {}^{\perp_{n-1}}\mathcal{C}$ and $\Omega_{F_{\mathcal{C}}}(\mathcal{C}^{\perp_{n-1}}) \subseteq \mathcal{C}^{\perp_{n-1}}$.*
(b) *Assume $n > 1$. Then $\mathcal{C}$ is an n-precluster tilting subcategory of* $\operatorname{mod}\Lambda$ *if and only if $\mathcal{C}^{\perp_{n-1}} = {}^{\perp_{n-1}}\mathcal{C}$.*

Proof. (a) We only prove the first inclusion since the other one is dual.

Let X be in ${}^{\perp_{n-1}}\mathcal{C}$, and $\eta\colon 0 \to X \xrightarrow{f} C^0 \to Y \to 0$ be an exact sequence with a minimal left $\mathcal{C}$-approximation f of X. Applying $\operatorname{Hom}_\Lambda(-,\mathcal{C})$, one easily shows that Y is in ${}^{\perp_{n-1}}\mathcal{C}$.

(b) It suffices to show that $\tau_n(\mathcal{C}) \subseteq \mathcal{C}$ and $\tau_n^-(\mathcal{C}) \subseteq \mathcal{C}$ hold if and only if $\mathcal{C}^{\perp_{n-1}} = {}^{\perp_{n-1}}\mathcal{C}$ holds.

Assume $\tau_n(\mathcal{C}) \subseteq \mathcal{C}$ and $\tau_n^-(\mathcal{C}) \subseteq \mathcal{C}$. Since $\mathcal{C} \subset {}^{\perp_{n-1}}\Lambda$, we have

$$\operatorname{Ext}^i_\Lambda(\mathcal{C}, {}^{\perp_{n-1}}\mathcal{C}) \simeq D\operatorname{Ext}^{n-i}_\Lambda({}^{\perp_{n-1}}\mathcal{C}, \tau_n(\mathcal{C})) = 0$$

for all $0 < i < n$ by Lemma 3.1. Thus ${}^{\perp_{n-1}}\mathcal{C} \subseteq \mathcal{C}^{\perp_{n-1}}$ holds. Similarly $\mathcal{C}^{\perp_{n-1}} \subseteq {}^{\perp_{n-1}}\mathcal{C}$ holds. Consequently $\mathcal{C}^{\perp_{n-1}} = {}^{\perp_{n-1}}\mathcal{C}$.

Assume that $\mathcal{C}^{\perp_{n-1}} = {}^{\perp_{n-1}}\mathcal{C}$. By Lemma 3.1, we have

$$\operatorname{Ext}^i_\Lambda({}^{\perp_{n-1}}\mathcal{C}, \tau_n(\mathcal{C})) \simeq D\operatorname{Ext}^{n-i}_\Lambda(\mathcal{C}, {}^{\perp_{n-1}}\mathcal{C}) = D\operatorname{Ext}^{n-i}_\Lambda(\mathcal{C}, \mathcal{C}^{\perp_{n-1}}) = 0$$

for all $0 < i < n$. Thus $\tau_n(\mathcal{C}) \subseteq ({}^{\perp_{n-1}}\mathcal{C})^{\perp_{n-1}}$. We show $({}^{\perp_{n-1}}\mathcal{C})^{\perp_{n-1}} \subseteq \mathcal{C}$. Since $\mathcal{C} \subseteq {}^{\perp_{n-1}}\mathcal{C}$, we have ${}^{\perp_{n-1}}\mathcal{C} = \mathcal{C}^{\perp_{n-1}} \supseteq ({}^{\perp_{n-1}}\mathcal{C})^{\perp_{n-1}}$. For any X in $({}^{\perp_{n-1}}\mathcal{C})^{\perp_{n-1}}$, there exists an exact sequence $\eta\colon 0 \to X \to C^0 \to Y \to 0$ with $C^0 \in \mathcal{C}$ and $Y \in {}^{\perp_{n-1}}\mathcal{C}$ by (a). This splits by the assumption $n > 1$, and X is a direct summand of C^0. Hence X is in $\mathcal{C}$.

Similarly we prove that $\tau_n^-(\mathcal{C}) \subseteq \mathcal{C}$. □

Now we introduce the following category $\mathcal{U}(\mathcal{C})$, which is an analog of Calabi–Yau reduction of triangulated categories [27].

Definition 3.9. For an n-precluster tilting subcategory $\mathcal{C}$ in $\operatorname{mod}\Lambda$, let

$$\mathcal{Z}(\mathcal{C}) = \mathcal{C}^{\perp_{n-1}} = {}^{\perp_{n-1}}\mathcal{C} \quad\text{and}\quad \mathcal{U}(\mathcal{C}) = \mathcal{Z}(\mathcal{C})/[\mathcal{C}].$$

Note that when $\mathcal{C}$ is a 1-precluster tilting subcategory, then $\mathcal{Z}(\mathcal{C}) = \operatorname{mod}\Lambda$, since the orthogonality condition is void.

The next result gives basic properties of $\mathcal{Z}(\mathcal{C})$ which generalize those of n-cluster tilting subcategories [21, Theorems 2.3, 2.3.1, 2.2.3]. In particular it gives higher Auslander–Reiten translation for $\mathcal{Z}(\mathcal{C})$ extending Proposition 3.7.

Theorem 3.10. *Let Λ be an Artin algebra and $\mathcal{C}$ an n-precluster tilting subcategory of* $\operatorname{mod}\Lambda$ *with $n \geq 1$. Then we have the following.*

(a) *We have equivalences $\tau_n\colon \underline{\mathcal{Z}(\mathcal{C})} \to \overline{\mathcal{Z}(\mathcal{C})}$ and $\tau_n^-\colon \overline{\mathcal{Z}(\mathcal{C})} \to \underline{\mathcal{Z}(\mathcal{C})}$.*

(b) *τ_n and τ_n^- give bijections between indecomposable non-projective modules in $\mathcal{Z}(\mathcal{C})$ and indecomposable non-injective modules in $\mathcal{Z}(\mathcal{C})$.*
(c) *We have $\operatorname{add}\{\tau_n(\mathcal{Z}(\mathcal{C})), D\Lambda\} = \mathcal{Z}(\mathcal{C}) = \operatorname{add}\{\tau_n^-(\mathcal{Z}(\mathcal{C})), \Lambda\}$.*
(d) *For every X and Y in $\mathcal{Z}(\mathcal{C})$ and $0 < i < n$, we have functorial isomorphisms*

$$\underline{\operatorname{Hom}}_\Lambda(X,Y) \simeq D\operatorname{Ext}^n_\Lambda(Y,\tau_n(X)), \qquad \operatorname{Ext}^i_\Lambda(X,Y) \simeq D\operatorname{Ext}^{n-i}_\Lambda(Y,\tau_n(X))$$
$$\overline{\operatorname{Hom}}_\Lambda(X,Y) \simeq D\operatorname{Ext}^n_\Lambda(\tau_n^-(Y),X), \qquad \operatorname{Ext}^i_\Lambda(X,Y) \simeq D\operatorname{Ext}^{n-i}_\Lambda(\tau_n^-(Y),X).$$

(e) *For every X in $\operatorname{mod}\Lambda$, there exists an $F_\mathcal{C}$-exact sequence*

$$0 \to Z_{n-1} \to C_{n-2} \to \cdots \to C_0 \to X \to 0$$

with C_i in $\mathcal{C}$ for every i and Z_{n-1} in $\mathcal{Z}(\mathcal{C})$.

Proof. (a) Thanks to the equivalences (3.1), it suffices to show $\tau_n(\mathcal{Z}(\mathcal{C})) \subseteq \mathcal{Z}(\mathcal{C})$ and $\tau_n^-(\mathcal{Z}(\mathcal{C})) \subseteq \mathcal{Z}(\mathcal{C})$. Using Lemma 3.1, we have

$$0 = D\operatorname{Ext}^i_\Lambda(\mathcal{Z}(\mathcal{C}),\mathcal{C}) \simeq \operatorname{Ext}^{n-i}_\Lambda(\mathcal{C},\tau_n(\mathcal{Z}(\mathcal{C})))$$
$$0 = D\operatorname{Ext}^i_\Lambda(\mathcal{C},\mathcal{Z}(\mathcal{C})) \simeq \operatorname{Ext}^{n-i}_\Lambda(\tau_n^-(\mathcal{Z}(\mathcal{C})),\mathcal{C})$$

for $0 < i < n$. Thus the assertion follows.

(b)(c) Immediate from (a).

(d) This follows from Lemma 3.1.

(e) Let X be in $\operatorname{mod}\Lambda$, and let $0 \to \Omega_{F_\mathcal{C}}(X) \to C_0 \to X \to 0$ be an $F_\mathcal{C}$-exact sequence given by an $F_\mathcal{C}$-projective cover. Then $\operatorname{Ext}^1_\Lambda(\mathcal{C},\Omega_{F_\mathcal{C}}(X)) = 0$ holds. Taking an $F_\mathcal{C}$-exact sequence $0 \to \Omega^2_{F_\mathcal{C}}(X) \to C_1 \to \Omega_{F_\mathcal{C}}(X) \to 0$ given by an $F_\mathcal{C}$-projective cover, it follows that $\operatorname{Ext}^i_\Lambda(\mathcal{C},\Omega^2_{F_\mathcal{C}}(X)) = 0$ for $i = 1,2$. Continuing this process we obtain that $\operatorname{Ext}^i_\Lambda(\mathcal{C},\Omega^{n-1}_{F_\mathcal{C}}(X)) = 0$ for $0 < i < n$, and hence $\Omega^{n-1}_{F_\mathcal{C}}(X)$ is in $\mathcal{Z}(\mathcal{C})$. □

The following easy property below is useful.

Lemma 3.11. *Let $\mathcal{C}$ be an n-precluster tilting subcategory of $\operatorname{mod}\Lambda$ with $n \geq 1$.*
(a) *We have $F_\mathcal{C}|_{\mathcal{Z}(\mathcal{C})\times\mathcal{Z}(\mathcal{C})} = F^\mathcal{C}|_{\mathcal{Z}(\mathcal{C})\times\mathcal{Z}(\mathcal{C})}$.*
(b) *For every $0 < i < n$, we have $\operatorname{Ext}^i_{F_\mathcal{C}}(-,-)|_{\mathcal{Z}(\mathcal{C})\times\mathcal{Z}(\mathcal{C})} = \operatorname{Ext}^i_\Lambda(-,-)|_{\mathcal{Z}(\mathcal{C})\times\mathcal{Z}(\mathcal{C})} = \operatorname{Ext}^i_{F^\mathcal{C}}(-,-)|_{\mathcal{Z}(\mathcal{C})\times\mathcal{Z}(\mathcal{C})}$.*
(c) *The exact categories $(\mathcal{Z}(\mathcal{C}), F_\mathcal{C})$ and $(\mathcal{Z}(\mathcal{C}), F^\mathcal{C})$ are the same. It coincides with $(\mathcal{Z}(\mathcal{C}), \operatorname{Ext}^1_\Lambda(-,-))$ if $n \geq 2$.*

Proof. (a) (c) For $n = 1$, we have $F_\mathcal{C} = F^{\tau\mathcal{C}} = F^\mathcal{C}$ by (2.1) since $\mathcal{C}$ is 1-precluster tilting, and for $n \geq 2$, we have $F_\mathcal{C}|_{\mathcal{Z}(\mathcal{C})\times\mathcal{Z}(\mathcal{C})} = \operatorname{Ext}^1_\Lambda(-,-)|_{\mathcal{Z}(\mathcal{C})\times\mathcal{Z}(\mathcal{C})} = F^\mathcal{C}|_{\mathcal{Z}(\mathcal{C})\times\mathcal{Z}(\mathcal{C})}$.

(b) This is immediate from Proposition 2.5. □

The category $\mathcal{Z}(\mathcal{C})$ enjoys the following remarkable properties.

Proposition 3.12. *Let $\mathcal{C}$ be an n-precluster tilting subcategory of $\operatorname{mod}\Lambda$ for some $n \geq 1$.*
(a) *$\mathcal{Z}(\mathcal{C})$ is extension closed.*
(b) *$\mathcal{Z}(\mathcal{C})$ has a structure of a Frobenius category whose short exact sequences are precisely $F_\mathcal{C}$-exact sequences, and projective-injective objects are precisely $\mathcal{C}$.*
(c) *$\mathcal{U}(\mathcal{C})$ has a structure of a triangulated category with the suspension functor $[1] = \Omega^{-1}_{F^\mathcal{C}}$.*

Proof. (a) This is easily checked by using the long exact sequence of Ext's.

(b) By Proposition 2.2, we have an exact category $(\mathcal{Z}(\mathcal{C}), F_{\mathcal{C}})$, which coincides with $(\mathcal{Z}(\mathcal{C}), F^{\mathcal{C}})$ by Lemma 3.11(c). Thus any object in $\mathcal{C}$ is projective-injective in $\mathcal{Z}(\mathcal{C})$. By Proposition 3.8(a), $\mathcal{Z}(\mathcal{C})$ has enough projectives and enough injectives. Therefore the projective and the injective objects coincide, and they are equal to $\mathcal{C}$. Thus the assertion follows.

(c) This is a general property of Frobenius categories [17]. □

We show that the triangulated category $\mathcal{U}(\mathcal{C})$ admits a Serre functor, which is an analog of [27, Theorem 4.7].

Theorem 3.13. *Let $\mathcal{C}$ be an n-precluster tilting subcategory of* $\operatorname{mod}\Lambda$ *with $n \geq 1$.*

(a) *The triangulated category $\mathcal{U}(\mathcal{C})$ admits a Serre functor S given by $S = [n] \circ \tau_n$.*
(b) *The triangulated category $\mathcal{U}(\mathcal{C})$ has almost split triangles, i.e. any indecomposable object X in $\mathcal{U}(\mathcal{C})$ has almost split triangles in $\mathcal{U}(\mathcal{C})$:*

$$SX[-1] \to E \to X \to SX \quad \textit{and} \quad X \to E' \to S^{-1}X[1] \to X[1].$$

(c) *The Frobenius category $\mathcal{Z}(\mathcal{C})$ has almost split sequences, i.e. any indecomposable module X in $\mathcal{Z}(\mathcal{C}) \setminus \mathcal{C}$ has almost split sequences in $\mathcal{Z}(\mathcal{C})$*

$$0 \to \tau_{\mathcal{Z}(\mathcal{C})}(X) \to E \to X \to 0 \quad \textit{and} \quad 0 \to X \to E' \to \tau^-_{\mathcal{Z}(\mathcal{C})}(X) \to 0,$$

where $\tau_{\mathcal{Z}(\mathcal{C})} := \Omega^{-(n-1)}_{\mathcal{Z}(\mathcal{C})}\tau_n$ and $\tau^-_{\mathcal{Z}(\mathcal{C})} := \Omega^{n-1}_{\mathcal{Z}(\mathcal{C})}\tau^-_n$.

Proof. (a) Let $F = F^{\mathcal{C}}$ and $G = F_{\mathcal{C}}$. Then $\mathcal{I}(F) = \mathcal{P}(G) = \mathcal{C}$ holds. For all X in $\mathcal{Z}(\mathcal{C})$, take an F-injective coresolution

$$0 \to \tau_n(X) \to C^0 \to \cdots \to C^{n-1} \to \Omega_F^{-n}(\tau_n(X)) \to 0,$$

which gives a G-projective resolution since $\mathcal{Z}(\mathcal{C})$ is Frobenius by Proposition 3.12. Thus, for every Y in $\operatorname{mod}\Lambda$, we obtain functorial isomorphisms

$$\begin{aligned}\operatorname{Ext}^n_G(Y, \tau_n(X)) &\simeq \operatorname{Coker}(\operatorname{Hom}_\Lambda(Y, C^{n-1}) \to \operatorname{Hom}_\Lambda(Y, \Omega_F^{-n}(\tau_n(X))) \\ &\simeq \underline{\operatorname{Hom}}_G(Y, \Omega_F^{-n}(\tau_n(X))).\end{aligned}$$

Moreover we have functorial isomorphisms

$$\begin{aligned}\operatorname{Ext}^n_G(Y, \tau_n(X)) &\simeq \operatorname{Ext}^n_F(\tau^-(Y), \Omega^{n-1}_\Lambda(X)) \\ &\simeq \operatorname{Ext}^1_F(\tau^-(Y), X) \\ &\simeq D\,\overline{\operatorname{Hom}}_F(X, Y),\end{aligned}$$

where the first isomorphism is given by Theorem 2.7, the second follows from $\mathcal{P}(\Lambda) \subset \mathcal{I}(F)$, and the third is the Auslander–Reiten duality (Proposition 2.4). Combining them, for every Y in $\mathcal{Z}(\mathcal{C})$, we have functorial isomorphisms

$$\begin{aligned}\operatorname{Hom}_{\mathcal{U}(\mathcal{C})}(Y, \Omega_F^{-n}(\tau_n(X))) &= \underline{\operatorname{Hom}}_G(Y, \Omega_F^{-n}(\tau_n(X))) \\ &\simeq D\,\overline{\operatorname{Hom}}_F(X, Y) = D(\operatorname{Hom}_{\mathcal{U}(\mathcal{C})}(X, Y)).\end{aligned} \tag{3.2}$$

Thus $\mathcal{U}(\mathcal{C})$ has $S = \Omega_F^{-n}\tau_n = [n] \circ \tau_n$ as a Serre functor.

(b) Since $\mathcal{U}(\mathcal{C})$ has a Serre functor, it has almost split triangles [37].

(c) Immediate from (b). □

The next bijective correspondence is an analog of a property of Calabi–Yau reduction of triangulated categories [27, Theorem 4.9]. One of the consequences is that, if $\mathcal{P}_n \vee \mathcal{I}_n$ is a functorially finite subcategory of $\operatorname{mod}\Lambda$, then the classification problem of n-cluster tilting subcategories in $\operatorname{mod}\Lambda$ can be reduced to the same problem in the triangulated category $\mathcal{U}(\mathcal{P}_n \vee \mathcal{I}_n)$.

Theorem 3.14. *Let $\mathcal{C}$ be an n-precluster tilting subcategory of $\operatorname{mod}\Lambda$ with $n \geq 1$. Then there exists a bijection between n-cluster tilting subcategories of $\operatorname{mod}\Lambda$ containing $\mathcal{C}$ and n-cluster tilting subcategories of $\mathcal{U}(\mathcal{C})$ given by $\mathcal{C}' \mapsto \mathcal{C}'/[\mathcal{C}]$.*

Proof. Any n-cluster tilting subcategory of $\operatorname{mod}\Lambda$ containing $\mathcal{C}$ is clearly contained in $\mathcal{Z}(\mathcal{C})$. On the other hand, let $\mathcal{C}'$ be a subcategory of $\operatorname{mod}\Lambda$ containing $\mathcal{C}$. It follows from Lemma 3.11(b) that $\mathcal{C}'$ is an n-cluster tilting subcategory of $\operatorname{mod}\Lambda$ if and only if $\mathcal{C}'/[\mathcal{C}]$ is an n-cluster tilting subcategory of $\mathcal{U}(\mathcal{C})$. Thus the assertion follows. $\square$

Next we give another proof of existence of almost split sequences in $\mathcal{Z}(\mathcal{C})$ by showing that $\mathcal{Z}(\mathcal{C})$ is functorially finite.

Theorem 3.15. *Let $\mathcal{C}$ be an n-precluster tilting subcategory of $\operatorname{mod}\Lambda$ with $n \geq 1$.*

(a) *$\mathcal{Z}(\mathcal{C})$ is $F_{\mathcal{C}}$-resolving and $F^{\mathcal{C}}$-coresolving in $\operatorname{mod}\Lambda$.*
(b) *$\mathcal{Z}(\mathcal{C})$ is functorially finite in $\operatorname{mod}\Lambda$ with $\mathcal{Z}(\mathcal{C})\text{-}\operatorname{resdim}_{F_{\mathcal{C}}}(\operatorname{mod}\Lambda) \leq n-1$ and $\operatorname{id}_{F_{\mathcal{C}}}\mathcal{C} \leq n-1$.*
(c) *$\mathcal{Z}(\mathcal{C})$ has almost split sequences.*

Proof. Since the case $n = 1$ is clear, we assume $n > 1$ in the rest.

(a) This is easily checked by using the long exact sequence of Ext's.

(b) By Theorem 3.10(e) we have $\mathcal{Z}(\mathcal{C})\text{-}\operatorname{resdim}_{F_{\mathcal{C}}}(\operatorname{mod}\Lambda) \leq n-1$ and hence $\widehat{\mathcal{Z}(\mathcal{C})} = \operatorname{mod}\Lambda$. Moreover $\mathcal{Z}(\mathcal{C})$ is $F_{\mathcal{C}}$-resolving by (a), and the exact category $(\mathcal{Z}(\mathcal{C}), F_{\mathcal{C}})$ has enough injectives $\mathcal{C}$ by Proposition 3.12(b). Therefore, by Proposition 2.11, $\mathcal{Z}(\mathcal{C})$ is contravariantly finite in $\operatorname{mod}\Lambda$ and $\operatorname{id}_{F_{\mathcal{C}}}\mathcal{C} \leq n-1$ holds. Dual arguments show that $\mathcal{Z}(\mathcal{C})$ is covariantly finite, hence it is functorially finite.

(c) Since $\mathcal{Z}(\mathcal{C})$ is extension closed and functorially finite, it has almost split sequences [7]. $\square$

We end this section with the following observation on Auslander–Buchweitz type approximations by $\mathcal{Z}(\mathcal{C})$.

Corollary 3.16. *Let $\mathcal{C}$ be an n-precluster tilting subcategory of Λ with $n \geq 1$ and X in $\operatorname{mod}\Lambda$.*

(a) *For every $0 \leq i \leq n-1$, there exists an $F_{\mathcal{C}}$-exact sequence*

$$0 \to C_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_{i+2}} C_{i+1} \xrightarrow{f_{i+1}} Z_i \xrightarrow{f_i} C_{i-1} \xrightarrow{f_{i-1}} \cdots \xrightarrow{f_1} C_0 \xrightarrow{f_0} X \to 0 \tag{3.3}$$

with Z_i in $\mathcal{Z}(\mathcal{C})$ and C_j in $\mathcal{C}$ for every j. Moreover $\operatorname{Im} f_j$ is in $\mathcal{C}^{\perp_j}$ for every j.

(b) *For every $0 \leq i \leq n-1$, there exists an $F^{\mathcal{C}}$-exact sequence*

$$0 \to X \xrightarrow{f^0} C^0 \xrightarrow{f^1} \cdots \xrightarrow{f^{i-1}} C^{i-1} \xrightarrow{f^i} Z^i \xrightarrow{f^{i+1}} C^{i+1} \xrightarrow{f^{i+2}} \cdots \xrightarrow{f^{n-1}} C^{n-1} \to 0$$

with Z^i in $\mathcal{Z}(\mathcal{C})$ and C^j in $\mathcal{C}$ for every j. Moreover $\operatorname{Im} f^j$ is in ${}^{\perp_j}\mathcal{C}$ for every j.

Proof. (a) The case $i = n-1$ was shown in Theorem 3.10(e).

Assume that the sequence (3.3) exists. Take an exact sequence $0 \to Z_i \to C_i \to Z'_{i-1} \to 0$ with C_i in $\mathcal{C}$ and Z'_{i-1} in $\mathcal{Z}(\mathcal{C})$, and consider the pushout diagram.

$$\begin{array}{ccccccccccccccccc}
 & & & & & & 0 & & 0 & & & & & & & & \\
 & & & & & & \downarrow & & \downarrow & & & & & & & & \\
0 & \to & C_{n-1} & \to & \cdots & \to C_{i+1} & \to Z_i & \to & C_{i-1} & \to & C_{i-2} & \to & \cdots & \to & C_0 & \to X & \to 0 \\
 & & \| & & & \| & \downarrow & & \downarrow & & \| & & & & \| & \| & \\
0 & \to & C_{n-1} & \to & \cdots & \to C_{i+1} & \to C_i & \to & Z_{i-1} & \to & C_{i-2} & \to & \cdots & \to & C_0 & \to X & \to 0 \\
 & & & & & & \downarrow & & \downarrow & & & & & & & & \\
 & & & & & & Z'_{i-1} & = & Z'_{i-1} & & & & & & & & \\
 & & & & & & \downarrow & & \downarrow & & & & & & & & \\
 & & & & & & 0 & & 0 & & & & & & & &
\end{array}$$

Since $Z_{i-1} \simeq Z'_{i-1} \oplus C_{i-1}$ belongs to $\mathcal{Z}(\mathcal{C})$, the lower sequence gives the desired sequence for $i-1$. Thus the assertion follows inductively.

(b) This is dual to (a). □

4. Auslander–Solberg correspondence

In [22], an Artin algebra Γ is called an *n-Auslander algebra* if $\operatorname{domdim}\Gamma \geq n+1 \geq \operatorname{gldim}\Gamma$. In addition, there exists a bijection between the set of equivalence classes of n-cluster tilting subcategories with additive generators and Morita-equivalence classes of n-Auslander algebras. A slightly more general class of Artin algebras were considered in [11] for $n = 1$, namely Artin Gorenstein algebras Γ with $\operatorname{domdim}\Gamma = 2 = \operatorname{id}_{\Gamma}\Gamma$. These algebras were classified in [11] in terms of τ-selfinjective algebras Λ. If one considers the weaker condition $\operatorname{domdim}\Gamma \geq 2 \geq \operatorname{id}_{\Gamma}\Gamma$, then the difference is precisely the class of selfinjective algebras (Proposition 4.1).

This section is devoted to extending the above result by exhibiting a bijection between the Morita-equivalence classes of Artin Gorenstein algebras Γ with $\operatorname{domdim}\Gamma \geq n+1 \geq \operatorname{id}_{\Gamma}\Gamma$ for $n \geq 1$ and equivalence classes of finite n-precluster tilting subcategories. This class of Gorenstein algebras we call *n-minimal Auslander–Gorenstein algebras*.

First we give some general properties of the class of n-minimal Auslander–Gorenstein algebras.

Proposition 4.1. (a) *If Γ is an n-minimal Auslander–Gorenstein algebra, then either* $\operatorname{domdim}\Gamma = n+1 = \operatorname{id}_{\Gamma}\Gamma$ *holds or Γ is selfinjective. In the latter case Γ is an m-Auslander–Gorenstein algebra for all $m \geq 0$.*

(b) *An algebra Γ is n-minimal Auslander–Gorenstein if and only if Γ^{op} is n-minimal Auslander–Gorenstein.*

Proof. (a) If Γ is not selfinjective, then the last term of the minimal injective coresolution of Γ is not projective. Thus the assertion follows from the definition of an n-minimal Auslander–Gorenstein algebra.

(b) Although the assertion follows from well-known results ([38, Theorem 7.7] and [5, Corollary 5.5]), we give a direct argument here. Assume that Γ is n-minimal Auslander–Gorenstein. The minimal injective coresolution

$$0 \to \Gamma \to I^0 \to \cdots \to I^n \to I^{n+1} \to 0 \tag{4.1}$$

of Γ gives a projective resolution of the injective Γ-module I^{n+1}. Thus Ω^{n+1} and $\Omega^{-(n+1)}$ give mutually inverse bijections between indecomposable non-projective injective Γ-modules and indecomposable non-injective projective Γ-modules since their numbers are the same. In particular, all indecomposable non-projective injective Γ-modules appear in I^{n+1} as a direct summand. Applying D to (4.1), we have an exact sequence

$$0 \to DI^{n+1} \to DI^n \to \cdots \to DI^0 \to D\Gamma \to 0$$

of Γ^{op}-modules with projective-injective Γ^{op}-modules DI^i with $0 \leq i \leq n$. Since all indecomposable non-injective projective Γ^{op}-modules appear in DI^{n+1} as a direct summand, Γ^{op} is also n-minimal Auslander–Gorenstein. $\square$

Let us recall the following simple observation.

Lemma 4.2 ([35, Lemma 3], [22, $d = m = 0$ in Theorem 4.2.1])**.** *Let Λ be an Artin algebra, M a generator-cogenerator of Λ, $\Gamma = \mathrm{End}_\Lambda(M)$ and $n \geq 1$. Then* $\mathrm{domdim}\,\Gamma \geq n+1$ *holds if and only if* $\mathrm{Ext}^i_\Lambda(M,M) = 0$ *for* $0 < i < n$.

The following result shows that a finite n-precluster tilting subcategory gives rise to an n-minimal Auslander–Gorenstein algebra.

Proposition 4.3. *Let Λ be an Artin algebra and M an n-precluster tilting Λ-module with $n \geq 1$.*

(a) $\Gamma = \mathrm{End}_\Lambda(M)$ *is an n-minimal Auslander–Gorenstein algebra and the Γ-module* $I = {}_\Gamma M$ *satisfies* $\mathcal{P}(\Gamma) \cap \mathcal{I}(\Gamma) = \mathrm{add}\, I$ *and* $\Lambda \simeq \mathrm{End}_\Gamma(I)$.

(b) M *is an F_M-cotilting Λ-module with* $\mathrm{id}_{F_M} M \leq n-1$.

(c) M *is a projective Λ-module if and only if Γ is selfinjective.*

Proof. By Lemma 4.2 we know that $\mathrm{domdim}\,\Gamma \geq n+1$. Let $G = F_M \subseteq \mathrm{Ext}^1_\Lambda(-,-)$. Then clearly $\mathrm{Ext}^i_G(M,M) = 0$ for all $i > 0$. Since $\tau_n^-(M) \in \mathrm{add}\, M$ by our assumption, we have

$$\mathcal{I}(G) = \mathrm{add}\{\tau(M), D\Lambda\} = \mathrm{add}\{\Omega_\Lambda^{-(n-1)}(M), D\Lambda\} \tag{4.2}$$

by Proposition 3.7(a). The start of the minimal injective coresolution

$$0 \to M \to I^0 \to I^1 \to \cdots \to I^{n-2} \to \Omega_\Lambda^{-(n-1)}(M) \to 0$$

of M is G-exact since $\mathrm{Ext}^i_\Lambda(M,M) = 0$ holds for every $0 < i < n$. It follows from (4.2) that $\mathrm{id}_G M \leq n-1$ and $\mathcal{I}(G) \subseteq \widehat{\mathrm{add}\, M}$. Therefore M is a G-cotilting module. By Theorem 2.9(d), $\mathrm{Hom}_\Lambda(M,M) = \Gamma$ is a cotilting Γ-module with $\mathrm{id}\,{}_\Gamma\Gamma \leq \mathrm{id}_G M + 2 \leq n+1$. Thus Γ is an n-minimal Auslander–Gorenstein algebra.

Since $I = \mathrm{Hom}_\Lambda(\Lambda, M)$ belongs to $\mathcal{P}(\Gamma)$ and $DI = \mathrm{Hom}_\Lambda(M, D\Lambda)$ belongs to $\mathcal{P}(\Gamma^{\mathrm{op}})$, we have $I \in \mathcal{P}(\Gamma) \cap \mathcal{I}(\Gamma)$. Taking a projective cover $P \to M \to 0$ in $\mathrm{mod}\,\Lambda$ and applying $\mathrm{Hom}_\Lambda(-,M)$, we have an exact sequence $0 \to \Gamma \to \mathrm{Hom}_\Lambda(P,M)$ in $\mathrm{mod}\,\Gamma$ with $\mathrm{Hom}_\Lambda(P,M) \in \mathrm{add}\, I$. Thus I is an additive generator of $\mathcal{P}(\Gamma) \cap \mathcal{I}(\Gamma)$. In addition, we have $\Lambda \simeq \mathrm{End}_\Gamma(I)$ from Theorem 2.9(a) since $I = M$ is a G-cotilting module.

The last claim is easy and left to the reader. This completes the proof. $\square$

Next we show the converse, namely n-minimal Auslander–Gorenstein algebras Γ give rise to a finite n-precluster tilting subcategory.

Proposition 4.4. *Let Γ be an n-minimal Auslander–Gorenstein algebra for $n \geq 1$. Let I be an additive generator of* $\mathcal{P}(\Gamma) \cap \mathcal{I}(\Gamma)$, $\Lambda = \mathrm{End}_\Gamma(I)$ *and* $M = {}_\Lambda I$.

(a) M *is an n-precluster tilting Λ-module such that* $\mathrm{End}_\Lambda(M) \simeq \Gamma$.

(b) *M is an F_M-cotilting Λ-module with* $\operatorname{id}_{F_M} M \leq n-1$.
(c) Γ *is selfinjective if and only if M is a projective Λ-module.*

Proof. We can assume that I is basic. Since $\operatorname{domdim}\Gamma \geq n+1 \geq 2$, the module I is a dualizing summand of the cotilting Γ-module Γ. Thus the claim $\Gamma \simeq \operatorname{End}_\Lambda(M)$ follows directly from Theorem 2.10(a).

In the rest, we show that M is an n-precluster tilting Λ-module.

(i) We show that M is a generator-cogenerator for Λ.

Since the Γ-module I belongs to $\operatorname{add}\Gamma$, the Λ-module $\Lambda = \operatorname{End}_\Gamma(I)$ belongs to $\operatorname{add}\operatorname{Hom}_\Gamma(\Gamma, I) = \operatorname{add} M$. Thus M is a generator of Λ. Since the Γ-module I belongs to $\operatorname{add} D(\Gamma_\Gamma)$, the Λ^{op}-module $\Lambda = \operatorname{End}_\Gamma(I)$ belongs to $\operatorname{add}\operatorname{Hom}_\Gamma(I, D(\Gamma)) = \operatorname{add} D(M)$. Thus M is a cogenerator of Λ.

(ii) It follows from Lemma 4.2 that $\operatorname{Ext}^i_\Lambda(M,M) = 0$ for $0 < i < n$.

(iii) It remains to show that both $\tau_n^-(M)$ and $\tau_n(M)$ are in $\operatorname{add} M$.

Let $G = F_M \subseteq \operatorname{Ext}^1_\Lambda(-,-)$. Then by Theorem 2.10 we infer that M is a G-cotilting module with $\operatorname{id}_G M \leq \max\{\operatorname{id}_\Gamma \Gamma - 2, 0\} \leq n-1$. By (ii), the injective coresolution

$$0 \to M \to I^0 \to I^1 \to \cdots \to I^{n-2} \to \Omega_\Lambda^{-(n-1)}(M) \to 0$$

of the Λ-module M is G-exact, and gives the start of a G-injective coresolution. Since $\operatorname{id}_G M \leq n-1$, the module $\Omega_\Lambda^{-(n-1)}(M)$ is in $\mathcal{I}(G) = \operatorname{add}\{D\Lambda, \tau(M)\}$. Hence $\tau_n^-(M) = \tau^-(\Omega_\Lambda^{-(n-1)}(M))$ is in $\operatorname{add} M$.

On the other hand, since Γ^{op} is an n-minimal Auslander–Gorenstein algebra such that $\mathcal{P}(\Gamma^{\mathrm{op}}) \cap \mathcal{I}(\Gamma^{\mathrm{op}}) = \operatorname{add} DI$ and $\operatorname{End}_{\Gamma^{\mathrm{op}}}(DI) = \Lambda^{\mathrm{op}}$, the Λ^{op}-module $\tau_n^-(DM)$ is in $\operatorname{add} DM$ by the same argument. Thus the Λ-module $\tau_n(M)$ is in $\operatorname{add}_\Lambda M$. This completes the proof that M is an n-precluster tilting Λ-module.

The last claim is easy and left to the reader. This completes the proof. □

Now we address the bijectivity of the correspondence.

Theorem 4.5. *Fix $n \geq 1$. There is a bijection between Morita-equivalence classes of n-minimal Auslander–Gorenstein algebras and equivalence classes of finite n-precluster tilting subcategories $\mathcal{C}$ of Artin algebras, where the correspondences are given in* Propositions 4.3 *and* 4.4.

Proof. The assertions follow from Propositions 4.3 and 4.4. □

Remark 4.6. Note that the bijection for n-Auslander algebras given in [22] is dual to Theorem 4.5.

The category $\mathcal{Z}(\mathcal{C})$ associated to a finite n-precluster tilting subcategory $\mathcal{C}$ has the following interpretation in terms of the corresponding n-minimal Auslander–Gorenstein algebra.

Theorem 4.7. *Given an Artin algebra Λ with a finite n-precluster tilting subcategory $\mathcal{C} = \operatorname{add} M$, let $\Gamma = \operatorname{End}_\Lambda(M)$ be the corresponding n-minimal Auslander–Gorenstein algebra. Then $\mathcal{Z}(\mathcal{C})$ and $\operatorname{CM}\Gamma$ are dual categories via the functors $\operatorname{Hom}_\Lambda(-,M)\colon \mathcal{Z}(\mathcal{C}) \to \operatorname{CM}\Gamma$ and $\operatorname{Hom}_\Gamma(-,M)\colon \operatorname{CM}\Gamma \to \mathcal{Z}(\mathcal{C})$. Moreover they induce triangle equivalences between $\mathcal{U}(\mathcal{C})$ and $(\underline{\operatorname{CM}}\,\Gamma)^{\mathrm{op}}$.*

Proof. Let $G = F_M$. Then M is a G-cotilting module with $\operatorname{id}_G M \leq n-1$ by the proof of Proposition 4.3(b), and we have a duality

$$\operatorname{Hom}_\Lambda(-,M)\colon {}^{\perp_G}M \to {}^{\perp}\Gamma = \operatorname{CM}\Gamma$$

by Theorem 2.9(e). Let us prove $\mathcal{Z}(\mathcal{C}) = {}^{\perp_G}M$. Since $\mathrm{id}_G M \leq n-1$, it follows that $\mathrm{Ext}^i_G(-,M) = 0$ for all $i \geq n$. Furthermore, since M is in $\mathcal{Z}(\mathcal{C})$, we have that $\mathrm{Ext}^i_G(-,M) = \mathrm{Ext}^i_\Lambda(-,M)$ for $0 < i < n$ by Proposition 2.5(b). Thus $\mathcal{Z}(\mathcal{C}) = {}^{\perp_G}M$ holds. Since $\mathcal{C} = \mathrm{add}\, M$ corresponds to $\mathcal{P}(\Gamma)$ via the duality $\mathrm{Hom}_\Lambda(-,M)$, the last claim follows immediately. □

The class of n-Auslander algebras were introduced in [22] as the Artin algebras Γ with $\mathrm{domdim}\,\Gamma \geq n+1 \geq \mathrm{gldim}\,\Gamma$. Now we characterize this subclass of algebras within the class of n-minimal Auslander–Gorenstein algebras.

Theorem 4.8. *Let Λ be an Artin algebra, $\mathcal{C} = \mathrm{add}\, M$ a finite n-precluster tilting Λ-module with $n \geq 1$ and $\Gamma = \mathrm{End}_\Lambda(M)$. Then the following are equivalent.*

(a) *Γ is an n-Auslander algebra.*
(b) *$\mathrm{gldim}\,\Gamma < \infty$.*
(c) *$\mathrm{CM}\,\Gamma = \mathcal{P}(\Gamma)$.*
(d) *$\mathcal{Z}(\mathcal{C}) = \mathcal{C}$.*

Proof. (a) implies (b): This is obvious.

(b) implies (c): If X in $\mathrm{CM}\,\Gamma$ is non-projective, then $\mathrm{Ext}^m_\Gamma(X,\Gamma) \neq 0$ holds for $m := \mathrm{pd}_\Gamma X > 0$, a contradiction.

(c) implies (a): For any X in $\mathrm{mod}\,\Gamma$, $\Omega^{n+1}_\Gamma(X)$ belongs to $\mathrm{CM}\,\Gamma = \mathcal{P}(\Gamma)$ by using dimension shifting and $\mathrm{id}_\Gamma \Gamma \leq n+1$ in Proposition 4.3(a). Thus $\mathrm{gldim}\,\Gamma \leq n+1$.

(c) is equivalent to (d): This follows from Theorem 4.7. □

For a general Artin Gorenstein algebra Γ, the category of maximal Cohen–Macaulay modules $\mathrm{CM}\,\Gamma = {}^{\perp}\Gamma$ is an extension closed functorially finite subcategory of $\mathrm{mod}\,\Gamma$ [4]. Therefore the category $\mathrm{CM}\,\Gamma$ has minimal left (respectively right) almost split maps and almost split sequences. We denote by $\tau_{\mathrm{CM}\,\Gamma}$ the Auslander–Reiten translation in $\mathrm{CM}\,\Gamma$.

We end this section with the following easy observations, which compare almost split sequences in $\mathcal{Z}(\mathcal{C})$ with those in $\mathrm{CM}\,\Gamma$.

Proposition 4.9. *Let Λ be an Artin algebra, $\mathcal{C} = \mathrm{add}\, M$ a finite n-precluster tilting Λ-module with $n \geq 1$ and $\Gamma = \mathrm{End}_\Lambda(M)$.*

(a) *A morphism $f\colon A \to B$ is (minimal) left almost split (respectively (minimal) right almost split) in $\mathrm{CM}\,\Gamma$ if and only if $\mathrm{Hom}_\Gamma(f,M)\colon \mathrm{Hom}_\Gamma(B,M) \to \mathrm{Hom}_\Gamma(A,M)$ is (minimal) right almost split (respectively (minimal) left almost split) in $\mathcal{Z}(\mathcal{C})$.*
(b) *An exact sequence $0 \to A \to B \to C \to 0$ in $\mathrm{CM}\,\Gamma$ is almost split in $\mathrm{CM}\,\Gamma$ if and only if $0 \to \mathrm{Hom}_\Gamma(C,M) \to \mathrm{Hom}_\Gamma(B,M) \to \mathrm{Hom}_\Gamma(A,M) \to 0$ is almost split in $\mathcal{Z}(\mathcal{C})$.*
(c) *$\tau^-_{\mathcal{Z}(\mathcal{C})}(\mathrm{Hom}_\Gamma(C,M)) \simeq \mathrm{Hom}_\Gamma(\tau_{\mathrm{CM}\,\Gamma}(C),M)$ holds for every indecomposable module C in $\mathrm{CM}\,\Gamma \setminus \mathcal{P}(\Gamma)$, and $\tau^-_{\mathrm{CM}\,\Gamma}(\mathrm{Hom}_\Lambda(X,M)) \simeq \mathrm{Hom}_\Lambda(\tau_{\mathcal{Z}(\mathcal{C})}(X),M)$ holds for every indecomposable module X in $\mathcal{Z}(\mathcal{C}) \setminus \mathcal{C}$.*

Proof. All assertions are immediate from Theorem 4.7. □

5. n-fold almost split extensions

Higher Auslander–Reiten theory on n-cluster tilting subcategories was introduced in [21]. An n-almost split sequence in an n-cluster tilting subcategory $\mathcal{C}$ is an n-fold exact sequence

$$\eta\colon 0 \to Y \to C_n \to \cdots \to C_1 \to X \to 0$$

with indecomposable objects X, Y in $\mathcal{C}$ and objects $\{C_i\}_{i=1}^n$ in $\mathcal{C}$, and the sequences

$$0 \to \mathrm{Hom}_\Lambda(\mathcal{C}, Y) \to \mathrm{Hom}_\Lambda(\mathcal{C}, C_n) \to \cdots \to \mathrm{Hom}_\Lambda(\mathcal{C}, C_1) \to \mathrm{rad}_\Lambda(\mathcal{C}, X) \to 0$$
$$0 \to \mathrm{Hom}_\Lambda(X, \mathcal{C}) \to \mathrm{Hom}_\Lambda(C_1, \mathcal{C}) \to \cdots \to \mathrm{Hom}_\Lambda(C_n, \mathcal{C}) \to \mathrm{rad}_\Lambda(Y, \mathcal{C}) \to 0$$

are exact. This is equivalent to the statement that η represents an element in the socle of $\mathrm{Ext}^n_\Lambda(X, Y)$ as an $\mathrm{End}_\Lambda(X)^{\mathrm{op}}$- or $\mathrm{End}_\Lambda(Y)$-module. These properties play a key role to generalize the notion of n-almost split sequences to more general categories, and we make the following definition.

Definition 5.1. Let Λ be an Artin algebra, X and Y be modules in $\mathrm{mod}\,\Lambda$, and η a non-zero element in $\mathrm{Ext}^n_\Lambda(X, Y)$.

(a) We say that η is a *right n-fold almost split extension of X* if, for every Z in $\mathrm{mod}\,\Lambda$ and a non-zero element ξ in $\mathrm{Ext}^n_\Lambda(X, Z)$, there exists a morphism $f\colon Z \to Y$ such that $\eta = f\xi$, where $f\xi$ stands for $\mathrm{Ext}^n_\Lambda(X, f)(\xi)$.
(b) We say that η is a *left n-fold almost split extension of* Y if, for every Z in $\mathrm{mod}\,\Lambda$ and a non-zero element ξ in $\mathrm{Ext}^n_\Lambda(Z, Y)$, there exists a morphism $g\colon X \to Z$ such that $\eta = \xi g$, where ξg stands for $\mathrm{Ext}^n_\Lambda(g, Y)(\xi)$.
(c) We say that η is an *n-fold almost split extension* if it is a right n-fold almost split extension of X and a left n-fold almost split extension of Y.

We changed the terminology from n-almost split sequence to n-fold almost split extension, since an element in $\mathrm{Ext}^n_\Lambda(X, Y)$ can possibly be represented by several different long exact sequences.

We need the following isomorphisms to study n-fold almost split extensions.

$$\mathrm{Ext}^n_\Lambda(X, -) = \mathrm{Ext}^1_\Lambda(\Omega^{n-1}_\Lambda(X), -) \simeq D\,\overline{\mathrm{Hom}}_\Lambda(-, \tau_n(X)). \tag{5.1}$$

In particular, $\tau_n(X) \neq 0$ implies $\mathrm{Ext}^n_\Lambda(X, \tau_n(X)) \simeq D\overline{\mathrm{End}}_\Lambda(\tau_n(X)) \neq 0$. We have the following characterizations of n-fold almost split extensions.

Lemma 5.2. *Let X and Y be modules in* $\mathrm{mod}\,\Lambda$*, and η a non-zero element in* $\mathrm{Ext}^n_\Lambda(X, Y)$*. Then the following conditions are equivalent.*

(i) *η is a right n-fold almost split extension of X.*
(ii) $\mathrm{Soc}\,\mathrm{Ext}^n_\Lambda(X, -)$ *is a simple functor on* $\mathrm{mod}\,\Lambda$ *and generated by η.*
(iii) *$\tau_n(X)$ is indecomposable, and $f\eta = 0$ holds in* $\mathrm{Ext}^n_\Lambda(X, Z)$ *for every morphism $f\colon Y \to Z$ in the radical of* $\mathrm{mod}\,\Lambda$.

If Y is indecomposable, then the following condition is also equivalent.

(iv) *$Y \simeq \tau_n(X)$ and η is a non-zero element in* $\mathrm{Soc}_{\,\mathrm{End}_\Lambda(Y)}\,\mathrm{Ext}^n_\Lambda(X, Y)$.

Proof. (i) is equivalent to (ii): Clearly (i) is equivalent to that η is contained in every non-zero subfunctor of $\mathrm{Ext}^n_\Lambda(X, -)$. This is equivalent to (ii) since any subfunctor of $\mathrm{Ext}^n_\Lambda(X, -)$ has a non-zero socle by (5.1).

(ii) is equivalent to (iii): Since (5.1) holds, $\mathrm{Soc}\,\mathrm{Ext}^n_\Lambda(X, -)$ is simple if and only if $\tau_n(X)$ is indecomposable in $\overline{\mathrm{mod}}\,\Lambda$. In this case, η belongs to the socle if and only if it is annihilated by the radical of $\mathrm{mod}\,\Lambda$. Thus the assertion follows.

(ii) is equivalent to (iv): This is immediate from (5.1). □

Now we have the following existence and uniqueness result of right n-fold almost split extensions.

Proposition 5.3. *Let Λ be an Artin algebra, and $n \geq 1$. A module X in* $\operatorname{mod}\Lambda$ *has a right n-fold almost split extension if and only if $\tau_n(X)$ is indecomposable. If these conditions are satisfied, then the following assertions hold.*

(a) *Let Y be an indecomposable module in* $\operatorname{mod}\Lambda$ *and η an element in* $\operatorname{Ext}^n_\Lambda(X,Y)$. *Then η is a right n-fold almost split extensions of X if and only if $Y \simeq \tau_n(X)$ and η is a non-zero element in* $\operatorname{Soc}_{\operatorname{End}_\Lambda(Y)} \operatorname{Ext}^n_\Lambda(X,Y)$.

(b) *For $i = 1, 2$, assume that η_i in* $\operatorname{Ext}^n_\Lambda(X,Y_i)$ *is a right n-fold almost split extension of X with an indecomposable module Y_i. Then there exists an isomorphism $f\colon Y_1 \to Y_2$ such that $\eta_2 = f\eta_1$.*

Proof. "Only if" part follows from Lemma 5.2(i)⇒(iii), and "if" part follows from Lemma 5.2(iv)⇒(i).

(a) This was shown in Lemma 5.2(i)⇔(iv).

(b) By our assumption, there exists $f\colon Y_1 \to Y_2$ and $g\colon Y_2 \to Y_1$ satisfying $\eta_2 = f\eta_1$ and $\eta_1 = g\eta_2$. Since $(1 - gf)\eta_1 = 0$ holds, $1 - gf$ is not an automorphism of Y_1, and hence gf is an automorphism of Y_1. Similarly fg is an automorphism of Y_2, and hence f and g are isomorphisms. □

We record dual results.

Lemma 5.4. *Let X and Y be modules in* $\operatorname{mod}\Lambda$, *and η a non-zero element in* $\operatorname{Ext}^n_\Lambda(X,Y)$. *Then the following conditions are equivalent.*

(i) *η is a left n-fold almost split extension of Y.*

(ii) $\operatorname{Soc}\operatorname{Ext}^n_\Lambda(-,Y)$ *is a simple functor on* $\operatorname{mod}\Lambda$ *and generated by η.*

(iii) *$\tau^-_n(Y)$ is indecomposable, and $\eta f = 0$ holds in* $\operatorname{Ext}^n_\Lambda(Z,Y)$ *for every morphism $f\colon Z \to X$ in the radical of* $\operatorname{mod}\Lambda$.

If X is indecomposable, then the following condition is also equivalent.

(iv) *$X \simeq \tau^-_n(Y)$ and η is a non-zero element in* $\operatorname{Soc}\operatorname{Ext}^n_\Lambda(X,Y)_{\operatorname{End}_\Lambda(X)}$.

We also have the following dual result for left n-fold almost split extensions.

Proposition 5.5. *Let Λ be an Artin algebra, and $n \geq 1$. A module Y in* $\operatorname{mod}\Lambda$ *has a left n-fold almost split extension if and only if $\tau^-_n(Y)$ is indecomposable. If these conditions are satisfied, then the following assertions hold.*

(a) *Let X be an indecomposable module in* $\operatorname{mod}\Lambda$ *and η an element in* $\operatorname{Ext}^n_\Lambda(X,Y)$. *Then η is a left n-fold almost split extensions of Y if and only if $X \simeq \tau^-_n(Y)$ and η is a non-zero element in* $\operatorname{Soc}\operatorname{Ext}^n_\Lambda(X,Y)_{\operatorname{End}_\Lambda(X)}$.

(b) *For $i = 1, 2$, assume that η_i in* $\operatorname{Ext}^n_\Lambda(X_i,Y)$ *is a left n-fold almost split extension of Y with an indecomposable module X_i. Then there exists an isomorphism $f\colon X_2 \to X_1$ such that $\eta_2 = \eta_1 f$.*

The following easy observation shows that all n-fold almost split extensions can be obtained from those between indecomposable modules.

Lemma 5.6. *Let $X = \bigoplus_{i=1}^m X_i$ and $Y = \bigoplus_{j=1}^{\ell} Y_j$ be modules in* $\operatorname{mod}\Lambda$ *with indecomposable summands X_i and Y_j, and $\eta = (\eta_{ij})_{i,j}$ a non-zero element in* $\operatorname{Ext}^n_\Lambda(X,Y) = \bigoplus_{i,j} \operatorname{Ext}^n_\Lambda(X_i,Y_j)$.

(a) *η is a right n-fold almost split extension of X if and only if the following conditions hold.*

  (i) *There exists $1 \leq i_0 \leq m$ such that* $\operatorname{Ext}^n_\Lambda(X_i,-) = 0$ *for every $i \neq i_0$.*

(ii) *$\eta_{i_0 j}$ is either zero or a right n-fold almost split extension of X_{i_0} for every j.*

(b) *η is a left n-fold almost split extension of Y if and only if the following conditions hold.*

(i) *There exists $1 \leq j_0 \leq \ell$ such that $\operatorname{Ext}^n_\Lambda(-, Y_j) = 0$ for every $j \neq j_0$.*

(ii) *$\eta_{i j_0}$ is either zero or a left n-fold almost split extension of Y_{j_0} for every j.*

(c) *η is an n-fold almost split extension if and only if the following conditions hold.*

(i) *There exist $1 \leq i_0 \leq m$ and $1 \leq j_0 \leq \ell$ such that $\operatorname{Ext}^n_\Lambda(X_i, -) = 0$ for every $i \neq i_0$ and $\operatorname{Ext}^n_\Lambda(-, Y_j) = 0$ for every $j \neq j_0$.*

(ii) *$\eta_{i_0 j_0}$ is an n-fold almost split extension.*

Proof. (a) The "if" part follows directly from definition. We show the "only if" part. It follows from Lemma 5.2(iii) that $\tau_n(X)$ is indecomposable, and hence (i) holds. One can check (ii) easily from definition.

(b) This is dual of (a).

(c) This follows immediately from (a) and (b). □

Now we consider n-fold almost split extensions.

Lemma 5.7. *Let X and Y be indecomposable modules in $\operatorname{mod}\Lambda$ satisfying $Y \simeq \tau_n(X)$ and $X \simeq \tau_n^-(Y)$. Then $\operatorname{Soc}_{\operatorname{End}_\Lambda(Y)} \operatorname{Ext}^n_\Lambda(X, Y) = \operatorname{Soc} \operatorname{Ext}^n_\Lambda(X, Y)_{\operatorname{End}_\Lambda(X)}$ holds.*

Proof. Let $E_X := \operatorname{End}_\Lambda(X)$ and $E_Y := \operatorname{End}_\Lambda(Y)$. Since $\operatorname{Soc}_{E_Y} \operatorname{Ext}^n_\Lambda(X, Y) \simeq D(\operatorname{Top} \overline{\operatorname{Hom}}_\Lambda(Y, \tau_n(X))_{E_Y})$ is simple, it is contained in any non-zero submodule of ${}_{E_Y} \operatorname{Ext}^n_\Lambda(X, Y)$. Since $\operatorname{Soc} \operatorname{Ext}^n_\Lambda(X, Y)_{E_X}$ is also a non-zero submodule of ${}_{E_Y} \operatorname{Ext}^n_\Lambda(X, Y)$, we have $\operatorname{Soc}_{E_Y} \operatorname{Ext}^n_\Lambda(X, Y) \subset \operatorname{Soc} \operatorname{Ext}^n_\Lambda(X, Y)_{E_X}$. Since the reverse inclusion holds by the same argument, the desired equality holds. □

We have the following characterization of n-fold almost split extensions.

Theorem 5.8. *Let Λ be an Artin algebra, and $n \geq 1$.*

(a) *Let X be an indecomposable module in $\operatorname{mod}\Lambda$. There exists an n-fold almost split extension in $\operatorname{Ext}^n_\Lambda(X, Y)$ for some module Y in $\operatorname{mod}\Lambda$ if and only if $\tau_n(X)$ is indecomposable and $\tau_n^- \tau_n(X) \simeq X$. If these conditions are satisfied, then any right n-fold almost split extension η of X in $\operatorname{Ext}^n_\Lambda(X, Y)$ with an indecomposable module Y is an n-fold almost splits extension.*

(b) *Let Y be an indecomposable module in $\operatorname{mod}\Lambda$. There exists an n-fold almost split extension in $\operatorname{Ext}^n_\Lambda(X, Y)$ for some module X in $\operatorname{mod}\Lambda$ if and only if $\tau_n^-(Y)$ is indecomposable and $\tau_n \tau_n^-(Y) \simeq Y$. If these conditions are satisfied, then any left n-fold almost split extension η of Y in $\operatorname{Ext}^n_\Lambda(X, Y)$ with an indecomposable module X is an n-fold almost splits extension.*

Proof. (a) We show the "if" part. Let $Y := \tau_n(X)$. Then $\operatorname{Soc}_{\operatorname{End}_\Lambda(Y)} \operatorname{Ext}^n_\Lambda(X, Y) = \operatorname{Soc} \operatorname{Ext}^n_\Lambda(X, Y)_{\operatorname{End}_\Lambda(X)}$ holds by Lemma 5.7, and any non-zero element is a right n-fold almost split extension of X by Proposition 5.3(a) and left n-fold almost split extension of X by Proposition 5.5(a) .

This argument also gives a proof of the second statement.

We show the "only if" part. By Lemma 5.6, we can assume that Y is indecomposable. By Propositions 5.3(a) and 5.5(a), we have $Y \simeq \tau_n(X)$ and $X \simeq \tau_n^-(Y)$. Thus the assertions hold.

(b) This is dual of (a). □

Applying these general results to n-precluster tilting subcategories, we have the following result.

Theorem 5.9. *Let Λ be an Artin algebra and $\mathcal{C}$ an n-precluster tilting subcategory in $\operatorname{mod}\Lambda$ with $n \geq 1$.*

(a) *Every indecomposable object X in $\mathcal{Z}(\mathcal{C}) \setminus \mathcal{P}(\Lambda)$ has an n-fold almost split extension in $\operatorname{Ext}^n_\Lambda(X, \tau_n(X))$.*
(b) *Every indecomposable object Y in $\mathcal{Z}(\mathcal{C}) \setminus \mathcal{I}(\Lambda)$ has an n-fold almost split extension in $\operatorname{Ext}^n_\Lambda(\tau_n^-(Y), Y)$.*

Proof. Since $\tau_n(X)$ and $\tau_n^-(Y)$ are indecomposable by the equivalences (3.1), the assertions follow immediate from Propositions 5.3 and 5.5. □

Now we consider exact sequences representing a right n-fold almost split extension of a module X in $\operatorname{mod}\Lambda$ such that $\tau_n(X)$ is indecomposable. We consider a projective resolution

$$0 \to \Omega^{n-1}_\Lambda(X) \to P_{n-2} \to \cdots \to P_0 \to X \to 0.$$

Then $\Omega^{n-1}_\Lambda(X) = Z \oplus P$ holds for an indecomposable module Z in $(\operatorname{mod}\Lambda) \setminus \mathcal{P}(\Lambda)$ and a module P in $\mathcal{P}(\Lambda)$. Let

$$0 \to \tau_n(X) \to E \oplus P \to \Omega^{n-1}_\Lambda(X) \to 0$$

be a direct sum of an almost split sequence $0 \to \tau_n(X) \to E \to Z \to 0$ and a split exact sequence $0 \to 0 \to P \to P \to 0$. Taking the Yoneda product of these exact sequences, we have an exact sequence

$$0 \to \tau_n(X) \to E \oplus P \to P_{n-2} \to \cdots \to P_0 \to X \to 0 \tag{5.2}$$

which represents a right n-fold almost split extensions of a module X. In fact, one can apply Lemma 5.2(iii)⇒(i).

Dually, a left n-fold almost split extensions of a module Y in $\operatorname{mod}\Lambda$ such that $\tau_n^-(Y)$ is indecomposable is represented by an exact sequence

$$0 \to Y \to I^0 \to \cdots \to I^{n-2} \to E' \oplus I \to \tau_n^-(Y) \to 0 \tag{5.3}$$

with modules $I^0, \ldots, I^{n-2}$ and I in $\mathcal{I}(\Lambda)$.

These constructions are quite general. However, even if X (respectively Y) in (5.2) (respectively (5.3)) belongs to an n-precluster tilting subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$, the module E (respectively E') does not necessarily belong to even $\mathcal{Z}(\mathcal{C})$.

Under a certain condition on X (respectively Y), the next result gives much nicer representatives, which satisfies similar properties of n-almost split sequences in n-cluster tilting subcategories. This is an analog of [21, Theorem 3.3.1].

Theorem 5.10. *Let $\mathcal{C}$ be an n-precluster tilting subcategory of $\operatorname{mod}\Lambda$, X an indecomposable module in $\mathcal{Z}(\mathcal{C}) \setminus \mathcal{P}(\Lambda)$, and $Y := \tau_n(X)$ the corresponding indecomposable module in $\mathcal{Z}(\mathcal{C}) \setminus \mathcal{I}(\Lambda)$.*

(a) *For each $0 \leq i \leq n-1$, an n-fold almost split extension in $\operatorname{Ext}^n_\Lambda(X, Y)$ can be represented as*

$$0 \to Y \to C_{n-1} \to \cdots \to C_{i+1} \to Z_i \to C_{i-1} \to \cdots \to C_0 \to X \to 0 \tag{5.4}$$

with Z_i in $\mathcal{Z}(\mathcal{C})$ and C_j in $\mathcal{C}$ for each $j \neq i$.

(b) *The following sequences are exact.*

$$\begin{aligned}0 \to \operatorname{Hom}_\Lambda(\mathcal{C}, Y) \to \operatorname{Hom}_\Lambda(\mathcal{C}, C_{n-1}) \to \cdots \to \operatorname{Hom}_\Lambda(\mathcal{C}, C_{i+1}) \to \operatorname{Hom}_\Lambda(\mathcal{C}, Z_i)\\ \to \operatorname{Hom}_\Lambda(\mathcal{C}, C_{i-1}) \to \cdots \to \operatorname{Hom}_\Lambda(\mathcal{C}, C_0) \to \operatorname{rad}_\Lambda(\mathcal{C}, X) \to 0,\\ 0 \to \operatorname{Hom}_\Lambda(X, \mathcal{C}) \to \operatorname{Hom}_\Lambda(C_0, \mathcal{C}) \to \cdots \to \operatorname{Hom}_\Lambda(C_{i-1}, \mathcal{C}) \to \operatorname{Hom}_\Lambda(Z_i, \mathcal{C})\\ \to \operatorname{Hom}_\Lambda(C_{i+1}, \mathcal{C}) \to \cdots \to \operatorname{Hom}_\Lambda(C_{n-1}, \mathcal{C}) \to \operatorname{rad}_\Lambda(Y, \mathcal{C}) \to 0.\end{aligned}$$

(c) *If X and Y do not belong to $\mathcal{C}$, then the n-fold almost split extension in* (a) *can be given as a Yoneda product of a minimal projective resolution of X in $\mathcal{Z}(\mathcal{C})$*

$$0 \to \Omega^i_{\mathcal{Z}(\mathcal{C})}(X) \to C_{i-1} \to \cdots \to C_0 \to X \to 0, \tag{5.5}$$

an almost split sequence in $\mathcal{Z}(\mathcal{C})$

$$0 \to \Omega^{-(n-i-1)}_{\mathcal{Z}(\mathcal{C})}(Y) \to Z_i \to \Omega^i_{\mathcal{Z}(\mathcal{C})}(X) \to 0, \tag{5.6}$$

and a minimal injective coresolution of Y in $\mathcal{Z}(\mathcal{C})$

$$0 \to Y \to C_{n-1} \to \cdots \to C_{i+1} \to \Omega^{-(n-i-1)}_{\mathcal{Z}(\mathcal{C})}(Y) \to 0. \tag{5.7}$$

Proof. (a) By (5.2), an n-fold almost split extension is given by the Yoneda product of $0 \to Y \to E \to \Omega^{n-1}_\Lambda(X) \to 0$ (where $E \oplus P$ is written as E) and $0 \to \Omega^{n-1}_\Lambda(X) \to P_{n-2} \to \cdots \to P_0 \to X \to 0$. Applying Corollary 3.16 to E, we have an exact sequence

$$0 \to E \xrightarrow{f^0} C^0 \xrightarrow{f^1} \cdots \xrightarrow{f^{i-1}} C^{i-1} \xrightarrow{f^i} Z^i \xrightarrow{f^{i+1}} C^{i+1} \xrightarrow{f^{i+2}} \cdots \xrightarrow{f^{n-1}} C^{n-1} \to 0$$

where Z^i is in $\mathcal{Z}(\mathcal{C})$, C^j is in $\mathcal{C}$ and $E^j := \operatorname{Im} f^j$ is in ${}^{\perp_j}\mathcal{C}$. We write $C^i := Z^i$ for simplicity. We have the following pushout diagram.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & E & \longrightarrow & \Omega^{n-1}_\Lambda(X) & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & C^0 & \longrightarrow & W & \longrightarrow & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & E^1 & = & E^1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since $\operatorname{Ext}^1_\Lambda(E^j, P_{n-j}) = 0$ holds for any $j \geq 2$, a Horseshoe Lemma-type argument gives the following commutative diagram of exact sequences.

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \to & \Omega^{n-1}_\Lambda(X) & \longrightarrow & P_{n-2} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & X & \to & 0\\
 & & \downarrow & & \downarrow & & & & \downarrow & & \| & & \\
0 & \longrightarrow & W & \longrightarrow & C^1 \oplus P_{n-2} & \to & \cdots & \to & C^{n-1} \oplus P_0 & \to & X & \to & 0\\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E^1 & \longrightarrow & C^1 & \longrightarrow & \cdots & \longrightarrow & C^{n-1} & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \downarrow & & & & \\
 & & 0 & & 0 & & & & 0 & & & &
\end{array}$$

The Yoneda product $0 \to Y \to C^0 \to C^1 \oplus P_{n-2} \to \cdots \to C^{n-1} \oplus P_0 \to X \to 0$ of the middle sequences of the diagrams above represents the same class in $\operatorname{Ext}^n_\Lambda(X, Y)$ as the sequence (5.2). Thus it represents an n-fold almost split extension.

(b) This is easily checked (see e.g. [21, Lemma 3.2]).

(c) It suffices to show that the Yoneda product of (5.5), (5.6) and (5.7) is an n-fold almost split extension. We have functorial isomorphisms

$$\begin{aligned}\operatorname{Ext}^n_\Lambda(X, \tau_n(X)) &= \operatorname{Hom}_{\mathcal{U}(\mathcal{C})}(X, \tau_n(X)[n]) \simeq \operatorname{Hom}_{\mathcal{U}(\mathcal{C})}(X[-i], \tau_n(X)[n-i]) \\ &= \operatorname{Hom}_{\mathcal{U}(\mathcal{C})}(\Omega^i_{\mathcal{Z}(\mathcal{C})}(X), \Omega^{-(n-i-1)}_{\mathcal{Z}(\mathcal{C})}\tau_n(X)[1])\end{aligned}$$

which induces an isomorphism

$$\begin{aligned} & \operatorname{Soc}_{\operatorname{End}_\Lambda(\tau_n(X))} \operatorname{Ext}^n_\Lambda(X, \tau_n(X)) \\ \simeq\ & \operatorname{Soc}_{\operatorname{End}_{\mathcal{U}(\mathcal{C})}(\Omega^{-(n-i-1)}_{\mathcal{Z}(\mathcal{C})}\tau_n(X))} \operatorname{Hom}_{\mathcal{U}(\mathcal{C})}(\Omega^i_{\mathcal{Z}(\mathcal{C})}(X), \Omega^{-(n-i-1)}_{\mathcal{Z}(\mathcal{C})}\tau_n(X)[1]).\end{aligned}$$

Since the almost split sequence belongs to the right hand side, our Yoneda product belongs to the left hand side. Thus it is an n-fold almost split extension of X by Theorem 5.8. □

Note that the sequences Theorem 5.10(b) are not exact in general if we replace $\mathcal{C}$ by $\mathcal{Z}(\mathcal{C})$. Thus the sequences (5.4) is not necessarily n-exact in $\mathcal{Z}(\mathcal{C})$ in the sense of Jasso [29].

In the rest of this section we let $\mathcal{C} = \operatorname{add} M$ be a finite n-precluster tilting subcategory of $\operatorname{mod}\Lambda$, and let $\Gamma = \operatorname{End}_\Lambda(M)$ be the corresponding n-minimal Auslander–Gorenstein algebra. We show that the category $\operatorname{CM}\Gamma$ of maximal Cohen–Macaulay modules over Γ has n-fold almost split extensions. This is used in the next section to classify the n-minimal Auslander–Gorenstein algebras. We relate the n-fold almost split extensions to the corresponding n-precluster tilting subcategory and transfer properties and constructions, as that of the n-fold Auslander–Reiten translate.

The dualities $\operatorname{Hom}_\Lambda(-, M) \colon \mathcal{Z}(\mathcal{C}) \to \operatorname{CM}\Gamma$ and $\operatorname{Hom}_\Gamma(-, M) \colon \operatorname{CM}\Gamma \to \mathcal{Z}(\mathcal{C})$ induce dualities

$$\underline{\mathcal{Z}(\mathcal{C})} \longleftrightarrow (\operatorname{CM}\Gamma)/[\operatorname{add} I] \quad\text{and}\quad \overline{\mathcal{Z}(\mathcal{C})} \longleftrightarrow (\operatorname{CM}\Gamma)/[\operatorname{add} Q],$$

where $I := {}_\Gamma M$ is an additive generator of $\mathcal{P}(\Gamma) \cap \mathcal{I}(\Gamma)$ and $Q := \operatorname{Hom}_\Lambda(D\Lambda, M) \simeq \operatorname{Hom}_{\Gamma^{\mathrm{op}}}(DI, \Gamma) \in \mathcal{P}(\Gamma)$.

Definition 5.11. Using the equivalences $\tau_n \colon \underline{\mathcal{Z}(\mathcal{C})} \to \overline{\mathcal{Z}(\mathcal{C})}$ and $\tau_n^- \colon \overline{\mathcal{Z}(\mathcal{C})} \to \underline{\mathcal{Z}(\mathcal{C})}$ given in Theorem 3.10, we define equivalences

$$\sigma_n^- \colon (\operatorname{CM}\Gamma)/[\operatorname{add} I] \to (\operatorname{CM}\Gamma)/[\operatorname{add} Q] \quad\text{and}\quad \sigma_n \colon (\operatorname{CM}\Gamma)/[\operatorname{add} Q] \to (\operatorname{CM}\Gamma)/[\operatorname{add} I]$$

making the following diagrams commutative up to isomorphisms of functors.

$$\begin{array}{ccc} \underline{\mathcal{Z}(\mathcal{C})} & \xrightarrow{\operatorname{Hom}_\Lambda(-,M)} & (\operatorname{CM}\Gamma)/[\operatorname{add} I] \\ {\scriptstyle \tau_n}\downarrow & & \downarrow{\scriptstyle \sigma_n^-} \\ \overline{\mathcal{Z}(\mathcal{C})} & \xrightarrow{\operatorname{Hom}_\Lambda(-,M)} & (\operatorname{CM}\Gamma)/[\operatorname{add} Q] \end{array} \qquad \begin{array}{ccc} \underline{\mathcal{Z}(\mathcal{C})} & \xrightarrow{\operatorname{Hom}_\Lambda(-,M)} & (\operatorname{CM}\Gamma)/[\operatorname{add} I] \\ {\scriptstyle \tau_n^-}\uparrow & & \uparrow{\scriptstyle \sigma_n} \\ \overline{\mathcal{Z}(\mathcal{C})} & \xrightarrow{\operatorname{Hom}_\Lambda(-,M)} & (\operatorname{CM}\Gamma)/[\operatorname{add} Q]. \end{array} \tag{5.8}$$

Let Γ be an n-minimal Auslander–Gorenstein Artin algebra for some $n \geq 1$. Then we use Theorem 5.10 to construct n-fold almost split extensions in $\operatorname{CM}\Gamma$.

Corollary 5.12. *Let* Γ *be an* n*-minimal Auslander–Gorenstein Artin algebra with* $n \geq 1$*,* X *an indecomposable module in* $(\mathrm{CM}\,\Gamma) \setminus (\mathrm{add}\, Q)$ *and* $Y := \sigma_n(X)$ *an indecomposable module in* $(\mathrm{CM}\,\Gamma) \setminus (\mathrm{add}\, I)$*.*

(a) *If* X *does not belong to* $\mathcal{P}(\Gamma)$*, then there exists an* n*-fold almost split extension in* $\mathrm{Ext}^n_\Gamma(X, Y)$*. For each* $0 \leq i \leq n-1$*, it is represented as*

$$0 \to Y \to P^0 \to \cdots \to P^{i-1} \to E^i \to P^{i+1} \to \cdots \to P^{n-1} \to X \to 0$$

with P_j *for all* j *in* $\mathcal{P}(\Gamma)$ *and* E_i *in* $\mathrm{CM}\,\Gamma$*.*

(b) *If* X *belongs to* $\mathcal{P}(\Gamma)$*, then there exists an exact sequence*

$$0 \to Y \to P^0 \to \cdots \to P^{i-1} \to E^i \to P^{i+1} \to \cdots \to P^{n-1} \to X \to X/\operatorname{rad} X \to 0,$$

with P_j *for all* j *in* $\mathcal{P}(\Gamma)$ *and* E_i *in* $\mathrm{CM}\,\Gamma$*.*

We call the sequence in (b) a *fundamental sequence*.

Proof. Both assertions follow from Theorem 5.10 and the duality between $\mathrm{CM}\,\Gamma$ and $\mathcal{Z}(\mathcal{C})$ for the corresponding n-precluster tilting subcategory $\mathcal{C}$ (Theorem 4.7). □

6. Four classes of n-minimal Auslander–Gorenstein algebras

1-Auslander–Gorenstein Artin algebras were classified in [11] in the class of selfinjective Artin algebras and three different disjoint classes using the correspondence with the τ-selfinjective Artin algebras. This section is devoted to extending this characterization to n-minimal Auslander–Gorenstein algebras for $n \geq 1$.

Suppose Γ is an n-minimal Auslander–Gorenstein algebra. By Theorem 4.5, there exist an Artin algebra Λ and an n-precluster tilting Λ-module M with $\Gamma \simeq \mathrm{End}_\Lambda(M)$. By Proposition 3.7(d), $\mathrm{add}\, M = \mathrm{add}\{\mathcal{P}_n \vee \mathcal{I}_n, N\}$ holds for a Λ-module N satisfying $\tau_n(N) \simeq N$. Then four distinct cases can occur:

(A) Λ is selfinjective and $N = 0$.
(B) Λ is selfinjective and $N \neq 0$.
(C) Λ is non-selfinjective and $N = 0$.
(D) Λ is non-selfinjective and $N \neq 0$.

This also gives rise to a coarser division, namely (A) and (C) together, or in other words when $N = 0$.

We characterize all these four cases in terms of properties of Γ, and in particular we show that an algebra of type (B) or (D) is constructed from an algebra of type (A) or (C), respectively.

The case (A) is easy, as this occurs if and only if Γ is a selfinjective algebra by Proposition 4.3. So we move on to describe the correspondence given by type (B).

Proposition 6.1. *Let* $n \geq 1$*. The bijection in* Theorem 4.5 *restricts to a bijection between*

$$\{\textit{finite } n\textit{-precluster tilting subcategories of type } (\mathrm{B})\} \quad \textit{and}$$

$$\left\{\begin{array}{l}\textit{non-selfinjective } n\textit{-minimal Auslander–Gorenstein algebras } \Gamma \textit{ with} \\ \textit{an additive generator } I \textit{ of } \mathcal{P}(\Gamma) \cap \mathcal{I}(\Gamma) \textit{ satisfying } \mathrm{add}\,\mathrm{Top}\,{}_\Gamma I = \\ \mathrm{add}\,\mathrm{Soc}\,{}_\Gamma I\end{array}\right\}.$$

Proof. Let Λ be a selfinjective Artin algebra with non-projective module N such that $M = \Lambda \oplus N$ is an n-precluster tilting Λ-module. We want to show that $\Gamma = \mathrm{End}_\Lambda(M)$ is a non-selfinjective n-minimal Auslander–Gorenstein algebra and $\mathrm{add}\,\mathrm{Top}\,{}_\Gamma I = \mathrm{add}\,\mathrm{Soc}\,{}_\Gamma I$, where I is an additive generator of $\mathcal{P}(\Gamma) \cap \mathcal{I}(\Gamma)$.

Let $G = F_M \subseteq \operatorname{Ext}^1_\Lambda(-,-)$. By Proposition 4.3 we have that Γ is a non-selfinjective n-minimal Auslander–Gorenstein algebra, M is a G-cotilting Λ-module, and $I = {}_\Gamma M$ is an additive generator of $\mathcal{P}(\Gamma) \cap \mathcal{I}(\Gamma)$. By [11, Lemma 2.3(b)] we have that

$$\operatorname{Hom}_\Gamma(-,I)\colon \operatorname{Hom}_\Gamma(A,C) \to \operatorname{Hom}_\Lambda(\operatorname{Hom}_\Gamma(C,I), \operatorname{Hom}_\Gamma(A,I))$$

is an isomorphism for all C in $\operatorname{mod}\Gamma$ and all A in $\operatorname{add} I$. In particular for $A = I$ and noting that $\operatorname{Hom}_\Gamma(I,I) \simeq \Lambda$ with Λ selfinjective, we infer that $\operatorname{Hom}_\Gamma(I,C) = 0$ if and only if $\operatorname{Hom}_\Gamma(C,I) = 0$. It follows from this that $\operatorname{add}\operatorname{Top}_\Gamma I = \operatorname{add}\operatorname{Soc}_\Gamma I$.

Conversely, let Γ be a non-selfinjective n-minimal Auslander–Gorenstein algebra, such that $\operatorname{add}\operatorname{Top}_\Gamma I = \operatorname{add}\operatorname{Soc}_\Gamma I$ where I is an additive generator of $\mathcal{P}(\Gamma) \cap \mathcal{I}(\Gamma)$. We want to show that $\Lambda = \operatorname{End}_\Gamma(I)$ is selfinjective, and $M = {}_\Lambda I \simeq \Lambda \oplus N$ with N non-projective such that M is an n-precluster tilting Λ-module.

Let $G = F_M \subseteq \operatorname{Ext}^1_\Lambda(-,-)$. By Proposition 4.4 the module M is an n-precluster tilting Λ-module and $M = {}_\Lambda I$ is a G-cotilting Λ-module. The functor $\operatorname{Hom}_\Gamma(-,I)\colon \operatorname{mod}\Gamma \to \operatorname{mod}\Lambda$ induces a duality between $\mathcal{P}(\Gamma)$ and $\operatorname{add} M$. Since M is a cogenerator, there exists P in $\mathcal{P}(\Gamma)$ such that $\operatorname{Hom}_\Gamma(P,I) \simeq D(\Lambda_\Lambda)$. By [11, Lemma 2.3(b)] we have that

$$\operatorname{Hom}_\Gamma(-,I)\colon \operatorname{Hom}_\Gamma(P,C) \to \operatorname{Hom}_\Lambda(\operatorname{Hom}_\Gamma(C,I), \operatorname{Hom}_\Gamma(P,I))$$

is an isomorphism for all C in $\operatorname{mod}\Gamma$. This implies by the choice of P that $\operatorname{Hom}_\Gamma(P,C) = 0$ if and only if $\operatorname{Hom}_\Gamma(C,I) = 0$. Hence $\operatorname{add}\operatorname{Top}_\Gamma P = \operatorname{add}\operatorname{Soc}_\Gamma I = \operatorname{add}\operatorname{Top}_\Gamma I$, and we obtain that $\operatorname{add} P = \operatorname{add} I$. Therefore $\mathcal{I}(\Lambda) = \operatorname{add}\operatorname{Hom}_\Gamma(P,I) = \mathcal{P}(\Lambda)$ and Λ is selfinjective. □

Remark 6.2. By Proposition 6.1, if Γ is an n-minimal Auslander–Gorenstein algebra of type (B), then there exists a selfinjective algebra Λ and a non-projective n-precluster tilting Λ-module $\Lambda \oplus N$ such that $\tau_n(N) \simeq N$ and $\Gamma \simeq \operatorname{End}_\Lambda(\Lambda \oplus N)$.

Now we complete the classification.

Theorem 6.3. *Let Γ be an n-minimal Auslander–Gorenstein algebra for $n \geq 1$. Denote by I an additive generator of $\mathcal{P}(\Gamma) \cap \mathcal{I}(\Gamma)$, and let $\Lambda = \operatorname{End}_\Gamma(I)$. Then four disjoint cases occur, where σ_n is the functor given in* Definition *5.11:*

(A) Γ *is selfinjective,*
(B) Γ *is not selfinjective and* $\operatorname{add}\operatorname{Top}_\Gamma I = \operatorname{add}\operatorname{Soc}_\Gamma I$,
(C) $\operatorname{add}\operatorname{Top}_\Gamma I \neq \operatorname{add}\operatorname{Soc}_\Gamma I$, *and there exists no indecomposable projective* Γ-*module P such that $(\sigma_n)^t(P) \simeq P$ for some $t > 0$,*
(D) $\operatorname{add}\operatorname{Top}_\Gamma I \neq \operatorname{add}\operatorname{Soc}_\Gamma I$, *and there exists an indecomposable projective* Γ-*module P such that $(\sigma_n)^t(P) \simeq P$ for some $t > 0$.*

In the cases (B), (C) *and* (D) *above, the correspondence given in* Theorem 4.5 *shows that the algebra Γ arises as follows.*

(a) *In case* (B) *there exists a selfinjective algebra Λ with an n-precluster tilting Λ-module M such that $\Gamma \simeq \operatorname{End}_\Lambda(M)$.*
(b) *In case* (C) *there exists an algebra Λ with an n-precluster tilting Λ-module M with no non-zero τ_n-periodic direct summand such that $\Gamma \simeq \operatorname{End}_\Lambda(M)$.*
(c) *In case* (D) *there exists an algebra Λ with an n-precluster tilting Λ-module M with a non-zero τ_n-periodic direct summand such that $\Gamma \simeq \operatorname{End}_\Lambda(M)$.*

Proof. The cases (A) and (B) are already discussed before in Proposition 6.1. The cases (A) and (B) occur if and only if $\operatorname{add}\operatorname{Top}_\Gamma I = \operatorname{add}\operatorname{Soc}_\Gamma I$. Hence the cases

(C) and (D) occur if and only if $\operatorname{add}\operatorname{Top}_\Gamma I \neq \operatorname{add}\operatorname{Soc}_\Gamma I$. Using this and the commutative diagram (5.8), the characterizations of the cases (C) and (D) follow. □

Remark 6.4. As we saw in Remark 6.2, any algebra of type (B) can be constructed from an algebra of type (A). Now we argue that type (D) and type (C) are related in a similar fashion. Let Γ be an n-minimal Auslander–Gorenstein algebra of type (D). Let I denote an additive generator of $\mathcal{P}(\Gamma) \cap \mathcal{I}(\Gamma)$, and let $\Lambda = \operatorname{End}_\Gamma(I)$ and $M = {}_\Lambda I$. Decompose $M = M_0 \oplus N$ with $\operatorname{add} M_0 = \mathcal{P}_n \vee \mathcal{I}_n$ and $\tau_n(N) \simeq N$. Furthermore, $\Gamma_0 = \operatorname{End}_\Gamma(M_0)$ is an n-minimal Auslander–Gorenstein algebra of type (C) since M_0 is also an n-precluster tilting Λ-module. Now we claim that Γ can be constructed from Γ_0.

Since M_0 is a cogenerator for $\operatorname{mod}\Lambda$, the functor

$$\operatorname{Hom}_\Lambda(-, M_0)\colon \operatorname{mod}\Lambda \to \operatorname{mod}\Gamma_0$$

is full and faithful. This implies that

$$\Gamma = \operatorname{Hom}_\Lambda(M, M) \simeq \operatorname{End}_{\Gamma_0}(\operatorname{Hom}_\Lambda(M, M_0)).$$

Hence Γ is constructed from the algebra Γ_0 which is of type (C).

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