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The Ungraded Derived Category

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Abstract. By means of the ungraded derived category we prove that the orbit category of the bounded derived category of an iterated tilted algebra with respect to translation is triangulated in such a way that the canonical functor from the bounded derived category to the orbit category becomes a triangle functor.

Sammendrag. Ved bruk av den ugraderte deriverte kategorien viser vi at banekategorien til den begrensede deriverte kategorien av en iterert tiltet algebra med hensyn på translasjon er triangulert slik at den kanoniske funktoren fra den begrensede deriverte kategorien til banekategorien blir en triangelfunktor.

Foreword

This thesis was written under the supervision of Associate Professor Steffen Oppermann in the field of homological algebra. It marks the conclusion of my time as a student for the degree of Master of Science in Mathematics at NTNU.

I owe my deepest gratitude to my supervisor, without whom this thesis would have never come into existence. For teaching me beautiful mathematics, for suggesting the subject of this thesis and for always being able and willing to enlighten me, I thank you.

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Introduction

Triangulated categories were introduced in the early sixties and have been growing increasingly important ever since. In homological algebra, a basic fact is that derived categories are triangulated. These categories have been intensely studied the last decades and are in many cases well understood. Consequently, showing that a given category carries a triangulated structure that is in some way connected to the triangulated structure of a derived category might reveal a lot of information about the category in question.

Purpose

Given a category $\mathcal{C}$ and an equivalence $F : \mathcal{C} \to \mathcal{C}$, the orbit category $\mathcal{C}/F$ has the objects of $\mathcal{C}$ and morphism spaces given by $\mathrm{Hom}_{\mathcal{C}/F}(A, B) := \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(A, F^n B)$. The goal of this thesis is to show that if Λ is an iterated tilted algebra then $D^b(\mathrm{Mod}\,\Lambda)/(-[1])$ is triangulated in such a way that the canonical functor $D^b(\mathrm{Mod}\,\Lambda) \to D^b(\mathrm{Mod}\,\Lambda)/(-[1])$ is a triangle functor. This result also follows from the work of Bernhard Keller in [Kel05], but we give a proof avoiding the use of dg-categories. Along the way we will learn a lot about the so-called ungraded derived category, as it is a key ingredient in our proof.

Overview

Chapter 1 is dedicated to introducing (or, for some of us, recalling) notions from category theory that will be employed in the thesis. We try to keep it short, and for the reader not familiar with additive and abelian categories there are probably other places to start that are more suitable. The reader well acquainted with the concepts introduced might feel the chapter is superfluous, but we prefer this to risking ambiguity[1].

In Chapter 2 we look at the connection between an exact category $\mathcal{C}$ and its augmentation $\mathcal{C}[\epsilon]$, and in particular how the projectives (injectives) of the former completely determine the projectives (injectives) of the latter under the assumption of idempotent completeness. Our motivation is that, in the following chapter, this will help us find a Frobenius category as a special case by imposing a non-canonical exact structure on a module category.

In Chapter 3 we introduce the ungraded derived category and show how it becomes triangulated. This happens through an exposition packed with parallels to the 'ordinary' setup of complexes. Indeed, we translate the concept of a mapping cone to the ungraded setup and show how this gives essentially all triangles, even though the triangulated structure originates from the stable category of a Frobenius category.

The merit of Chapter 4 is providing sufficient conditions on $\mathcal{A}$ for the objects of $D_{\mathrm{ung}}(\mathcal{A})$ to 'admit a grading'. We show that $\mathcal{A}$ being hereditary is sufficient before, using heavily the existence of standard equivalences, we show that also $\mathcal{A}$ being the module category of

[1]A typical example justifying this approach is given by the concept of exact categories. Although most readers probably have some intuitive feel for it, we are perhaps being unreasonable if we expect that he or she can actually write down the correct definition.

an iterated tilted algebra is enough.

In CHAPTER 5 the analogy to the graded setup is again striking. Here we describe the subcategory of the ungraded homotopy category consisting of homotopically projectives, and go on to show that this category is triangle equivalent to the ungraded derived category. This is important as it will allow us to do calculations in the homotopy category rather than in its localization, which is a tremendous advantage.

Our main objective is obtained in CHAPTER 6. After showing that the orbit category embeds in the ungraded derived category, giving a proof of our main theorem reduces to merely assembling some of our previous results.

The purpose of CHAPTER 7 is to briefly discuss the possibility of a converse of our main result. To justify why a further investigation is meaningful and might lead to an affirmative answer we invoke the work of Ringel in [Rin98] and Happel and Zacharia in [HZ08].

Terminology and Conventions

We call a category **additive** if it has finite products and each Hom-set is an abelian group such that composition of morphisms is bilinear over $\mathbb{Z}$. By the term **algebra** we mean an algebra over a field. For an algebra Λ we denote by $\operatorname{Mod}\Lambda$ ($\operatorname{Proj}\Lambda$, $\operatorname{Inj}\Lambda$) the category of left Λ-modules (projective, injective). By $\operatorname{mod}\Lambda$ ($\operatorname{proj}\Lambda$, $\operatorname{inj}\Lambda$) we denote the full subcategory of $\operatorname{Mod}\Lambda$ ($\operatorname{Proj}\Lambda$, $\operatorname{Inj}\Lambda$) of finitely generated modules.

The Intended Reader

Even though we make an effort to keep the thesis self-contained, it is intended for the reader that already knows a bit of homological algebra. Throughout, having seen derived categories would help. For understanding what goes on in Chapter 4, familiarity with derived functors and the existence of standard equivalences would be an advantage.

CHAPTER 1

Preliminaries

The aim of this chapter is to present some necessary preliminaries from category theory. For the reader not accustomed to the concepts of additive and abelian categories, a reasonable starting point may be [HJ10].

1.1 Triangulated Categories

We start by introducing some terminology. A **category with translation** $(\mathcal{T}, \Sigma)$ is a category $\mathcal{T}$ together with an equivalence

$$\Sigma : \mathcal{T} \to \mathcal{T}$$

called the **translation functor**. In a category with translation, a **triangle** is a sequence of objects and morphisms of the form

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

and a **morphism of triangles** is a triple (α, β, γ) of morphisms in $\mathcal{T}$ such that each square in

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r, "h"] \arrow[d, "\gamma"] & \Sigma A \arrow[d, "\Sigma\alpha"] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & \Sigma A' \end{tikzcd}$$

is commutative. Such a morphism is called an **isomorphism of triangles** if α, β and γ are isomorphisms.

The following definition is due to Verdier (a slightly edited version of his PhD thesis can be found in [Ver96]).

DEFINITION. A **triangulated category** is an additive category with translation $(\mathcal{T}, \Sigma)$ together with a collection of distinguished triangles satisfying the following axioms.

Tr1 The class of distinguished triangles is closed under isomorphism of triangles and each morphism $f : A \to B$ embeds in a distinguished triangle

$$A \xrightarrow{f} B \to C_f \to \Sigma A.$$

C_f is called a **cone** of f. In particular, for each $A \in \mathcal{T}$,

$$A \xrightarrow{1} A \to 0 \to \Sigma A$$

is distinguished.

Tr2 If

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

is distinguished, then so is

$$B \xrightarrow{g} C \xrightarrow{h} \Sigma A \xrightarrow{-\Sigma f} \Sigma B.$$

Tr3 Any diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] & \Sigma A \\
A \arrow[r] & B \arrow[r] & C \arrow[r] & \Sigma A
\end{tikzcd}$$

whose rows are distinguished triangles and whose square is commutative embeds in a morphism of triangles.

Tr4 Given three distinguished triangles

$$A \xrightarrow{f_1} B \xrightarrow{g_1} C' \xrightarrow{h_1} \Sigma A$$
$$B \xrightarrow{f_2} C \xrightarrow{g_2} A' \xrightarrow{h_2} \Sigma B$$
$$A \xrightarrow{f_3} C \xrightarrow{g_3} B' \xrightarrow{h_3} \Sigma A$$

with $f_3 = f_2 f_1$, there is a fourth distinguished triangle

$$C' \xrightarrow{f_4} B' \xrightarrow{g_4} A' \xrightarrow{h_4} \Sigma C'$$

such that all is commutative in

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, "1"] & B \arrow[r, "g_1"] \arrow[d, "f_2"] & C' \arrow[r, "h_1"] \arrow[d, "f_4"] & \Sigma A \arrow[d, "1"] \\
A \arrow[r, "f_3"] & C \arrow[r, "g_3"] \arrow[d, "g_2"] & B' \arrow[r, "h_3"] \arrow[d, "g_4"] & \Sigma A \arrow[d, "\Sigma f_1"] \\
& A' \arrow[r, "1"] \arrow[d, "h_2"] & A' \arrow[r, "h_2"] \arrow[d, "h_4"] & \Sigma B \\
& \Sigma B \arrow[r, "\Sigma g_1"] & \Sigma C' &
\end{tikzcd}$$

Remark. May showed in [May01] that Tr3 can be derived from the other axioms. Further, Tr2 can be strengthened to an 'if and only if' statement (i.e. we may rotate in both directions). □

Now that we know what a triangulated category is, let us look at some basic properties and related concepts. An important notion is the following.

Definition. Let $\mathcal{T}$ be triangulated and $\mathcal{A}$ abelian. An additive (covariant) functor $H : \mathcal{T} \to \mathcal{A}$ is called **homological** if any distinguished triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

in $\mathcal{T}$ yields a long exact sequence

$$\ldots \xrightarrow{H\Sigma^{i-1}h} H(\Sigma^i A) \xrightarrow{H\Sigma^i f} H(\Sigma^i B) \xrightarrow{H\Sigma^i g} H(\Sigma^i C) \xrightarrow{H\Sigma^i h} H(\Sigma^{i+1} A) \xrightarrow{H\Sigma^{i+1} f} \ldots$$

in $\mathcal{A}$. Dually, a contravariant functor taking distinguished triangles to long exact sequences is called **cohomological**.

Dealing with triangulated categories, the slogan is often 'Hom-functors are homological'. The following two lemmas explain why.

1.1 Lemma. *In a triangulated category, the composition of two consecutive maps in a distinguished triangle vanishes.*

Proof. Take a distinguished triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \to \Sigma A.$$

By Tr2 it suffices to show that f and g compose to zero. This follows readily, as

$$\begin{tikzcd}
A \arrow[r, "1"] \arrow[d, "1"] & A \arrow[r] \arrow[d, "f"] & 0 \arrow[r] & \Sigma A \\
A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & \Sigma A
\end{tikzcd}$$

can be completed to a morphism of triangles, i.e. $gf = 0$. □

1.2 Lemma. *Let $\mathcal{T}$ be a triangulated category and take any $X \in \mathcal{T}$. Then* $\mathrm{Hom}_{\mathcal{T}}(X, -)$ *is a homological functor.*

Proof. Let

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

be distinguished. By Tr2 it suffices to show that

$$\mathrm{Hom}_{\mathcal{T}}(X,A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{T}}(X,B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{T}}(X,C)$$

is exact. We saw in Lemma 1.1 that $gf = 0$, so the only implication we need to worry about is $\mathrm{Ker}\, g_* \subseteq \mathrm{Im}\, f_*$. Take $\alpha \in \mathrm{Ker}\, g_*$ and consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "1"] & X \arrow[r] \arrow[d, "\alpha"] & 0 \arrow[d] \\
\Sigma^{-1}C \arrow[r, "-\Sigma^{-1}h"] & A \arrow[r, "f"] & B \arrow[r, "g"] & C
\end{tikzcd}$$

whose rows are distinguished and whose rightmost square is commutative by assumption. This yields the existence of a morphism $\beta : X \to A$ such that $f\beta = \alpha$, i.e. $\alpha \in \mathrm{Im}\, f_*$. $\square$

REMARK. A similar argument shows that $\mathrm{Hom}_{\mathcal{T}}(-,X)$ is a cohomological functor. $\square$

1.3 LEMMA. *Let $\mathcal{T}$ be triangulated and let*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

be distinguished. Then f is split mono if and only if g is split epi if and only if $h = 0$.

PROOF. Assume f is split mono, say by $\hat{f} : B \to A$. Then

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "1"] & B \arrow[r, "g"] \arrow[d, "\hat{f}"] & C \arrow[r, "h"] & \Sigma A \\
A \arrow[r, "1"] & A \arrow[r] & 0 \arrow[r] & \Sigma A
\end{tikzcd}$$

embeds in a morphism of triangles, forcing $h = 0$.

Conversely, assume $h = 0$. Then

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r, "0"] \arrow[d] & \Sigma A \arrow[d, "1"] \\
A \arrow[r, "1"] & A \arrow[r] & 0 \arrow[r] & \Sigma A
\end{tikzcd}$$

embeds in a morphism of triangles, yielding the existence of $\hat{f} : B \to A$ such that $\hat{f}f = 1_A$.

The equivalence of $h = 0$ and g being split epi is obtained similarly. $\square$

The next result is known as the 'triangulated five lemma'.

1.4 Lemma. *Let $\mathcal{T}$ be triangulated and let*

$$\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \Sigma A \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\Sigma\alpha} \\
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & \Sigma A'
\end{array}$$

be a morphism of distinguished triangles. If two out of α, β and γ are isomorphisms, then so is the third.

Proof. By Tr2 it suffices to prove that γ is an isomorphism provided that α and β are. In this case applying $\mathrm{Hom}_{\mathcal{T}}(C', -) = \mathrm{Hom}(C', -)$ gives the diagram

$$\begin{array}{ccccccccc}
\mathrm{Hom}(C', A) & \xrightarrow{f_*} & \mathrm{Hom}(C', B) & \xrightarrow{g_*} & \mathrm{Hom}(C', C) & \xrightarrow{h_*} & \mathrm{Hom}(C', \Sigma A) & \xrightarrow{\Sigma f_*} & \mathrm{Hom}(C', \Sigma B) \\
\downarrow{\scriptstyle\alpha_*} & & \downarrow{\scriptstyle\beta_*} & & \downarrow{\scriptstyle\gamma_*} & & \downarrow{\scriptstyle\Sigma\alpha_*} & & \downarrow{\scriptstyle\Sigma\beta_*} \\
\mathrm{Hom}(C', A') & \xrightarrow{f'_*} & \mathrm{Hom}(C', B') & \xrightarrow{g'_*} & \mathrm{Hom}(C', C') & \xrightarrow{h'_*} & \mathrm{Hom}(C', \Sigma A') & \xrightarrow{\Sigma f'_*} & \mathrm{Hom}(C', \Sigma B')
\end{array}$$

whose rows are exact and whose squares are commutative. Now γ_* is an isomorphism by the (ordinary) five lemma. Hence there is some $\hat{\gamma} : C' \to C$ such that $\gamma\hat{\gamma} = 1_{C'}$. Similarly, applying $\mathrm{Hom}_{\mathcal{T}}(-, C)$ yields a left inverse of γ. $\square$

Remark. By Tr3 and Lemma 1.4 it follows that the cone of a morphism is unique up to isomorphism. $\square$

Definition. Let $\mathcal{T}$ be triangulated. A full, additive subcategory $\mathcal{C} \subseteq \mathcal{T}$ is a **triangulated subcategory** of $\mathcal{T}$ if it is closed under isomorphism and translation and if moreover $A, C \in \mathcal{C}$ implies $B \in \mathcal{C}$ whenever there is a distinguished triangle

$$A \to B \to C \to \Sigma A$$

in $\mathcal{T}$.

Remark. A triangulated subcategory inherits a canonical triangulated structure. $\square$

In order to compare triangulated categories we need functors that respect the triangulated structures. The following definition is the natural notion.

Definition. A **triangle functor** is an additive functor $F : (\mathcal{T}, \Sigma) \to (\mathcal{T}', \Sigma')$ between triangulated categories together with a natural isomorphism

$$\phi : F\Sigma_{\mathcal{T}} \to \Sigma_{\mathcal{T}'} F$$

such that

$$FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \xrightarrow{\phi_A Fh} \Sigma_{\mathcal{T}'} FA$$

is distinguished in $\mathcal{T}'$ whenever

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma_{\mathcal{T}} A$$

is distinguished in $\mathcal{T}$.

Remark. One easily verifies that the composition of two triangle functors is again a triangle functor. Also, if $\mathcal{C} \subseteq \mathcal{T}$ is a triangulated subcategory then the canonical functor $\mathcal{C} \to \mathcal{T}$ is a triangle functor. □

1.2 Exact Categories

The concept of exact categories is due to Quillen ([Qui73]). It captures essential properties of short exact sequences, but does not require the presence of an abelian category. Recall that in an additive category, a pair of composable morphisms

$$A \xrightarrow{i} B \xrightarrow{p} C$$

is **exact** if i is a kernel of p and p is a cokernel of i. A **morphism of exact pairs** is what one should expect, namely a triple of morphisms making both squares commutative in

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & B \arrow[r, "p"] \arrow[d] & C \arrow[d] \\ A' \arrow[r, "i'"] & B' \arrow[r, "p'"] & C' \end{tikzcd}$$

The above is an **isomorphism of exact pairs** if the vertical morphisms are all isomorphisms. Keller showed in [Kel90] that the original axioms for exact categories are not minimal, and the following definition is due to him.

Definition. An **exact category** $(\mathcal{C}, \mathscr{E})$ is an additive category $\mathcal{C}$ with a class $\mathscr{E}$ of distinguished exact pairs (we shall call $(i,p) \in \mathscr{E}$ a **conflation** consisting of the **inflation** i and the **deflation** p), such that the following axioms hold.

E1 $\mathscr{E}$ is closed under isomorphisms of exact pairs.

E2 The identity morphism on the 0-object is a deflation and the composition of two deflations is again a deflation.

E3 For each $f : C' \to C$ and each deflation $p : B \to C$ there is a pullback diagram

$$\begin{tikzcd} B' \arrow[r, "p'"] \arrow[d] & C' \arrow[d, "f"] \\ B \arrow[r, "p"] & C \end{tikzcd}$$

in which p' is a deflation.

E3$^{\text{op}}$ For each $f : A \to A'$ and each inflation $i : A \to B$ there is a pushout diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"] & B \arrow[d] \\ A' \arrow[r, "i'"] & B' \end{tikzcd}$$

in which i' is an inflation.

One important feature of exact categories is that there is a natural way of defining exact functors between them. Since any abelian category can be viewed as an exact category in the obvious way, the following definition generalizes the classical notion of exact functors.

Definition. A functor $(\mathcal{C}, \mathscr{E}) \to (\mathcal{C}', \mathscr{E}')$ between exact categories is called **exact** if it takes members of $\mathscr{E}$ to $\mathscr{E}'$.

Consequently, exact categories allow equivalent formulations of the concepts of injective and projective objects that we know from module categories.

Definition. Let **Ab** be the category of abelian groups with the canonical exact structure. An object P in an exact category $\mathcal{C}$ is called **projective** if $\operatorname{Hom}_{\mathcal{C}}(P, -) : \mathcal{C} \to \mathbf{Ab}$ is an exact functor. Dually, if $\operatorname{Hom}_{\mathcal{C}}(-, I) : \mathcal{C} \to \mathbf{Ab}$ is exact then I is **injective**. We denote by $\operatorname{Proj} \mathcal{C}$ $(\operatorname{Inj} \mathcal{C})$ the full subcategory of projectives (injectives).

Remark. As one would expect, $P \in \mathcal{C}$ is projective if and only if each conflation ending in P splits. Of course, the dual characterization of injectives also holds. $\square$

1.3 Stable Categories

The concept of a quotient category is similar to that of a quotient module. Given an additive category $\mathcal{C}$ we can impose equivalence relations on Hom-spaces, obtaining a 'version' of $\mathcal{C}$ in which certain objects are annihilated. In this thesis we shall take an interest in this construction when $\mathcal{C}$ admits a notion of projective and injective objects.

Definition. For an exact category $\mathcal{C}$, the corresponding **stable** category $\underline{\mathcal{C}}$ has the same objects as $\mathcal{C}$ while the morphisms are given by, for any $A, B \in \mathcal{C}$,

$$\underline{\operatorname{Hom}}_{\mathcal{C}}(A, B) = \operatorname{Hom}_{\underline{\mathcal{C}}}(A, B) := \operatorname{Hom}_{\mathcal{C}}(A, B)/I(A, B),$$

where $I(A, B)$ is the subgroup of morphisms $A \to B$ that factor through a projective object.

In other words, two morphisms $f, g \in \operatorname{Hom}_{\mathcal{C}}(A, B)$ represent the same morphism in $\underline{\mathcal{C}}$ if there is a commutative diagram

$$\begin{tikzcd} & P \arrow[dr] & \\ A \arrow[ur] \arrow[rr, "f-g"'] & & B \end{tikzcd}$$

in $\mathcal{C}$ with $P \in \operatorname{Proj}\mathcal{C}$. In particular, any projective object in $\mathcal{C}$ will be isomorphic to zero in $\underline{\mathcal{C}}$.

For the purpose of doing homological algebra, a convenient property of module categories is the existence of projective and injective resolutions of any object.

DEFINITION. An exact category $\mathcal{C}$ has **enough projectives** if for any $A \in \mathcal{C}$ there is a deflation $P \to A$ with P projective. Dually, $\mathcal{C}$ has **enough injectives** if for any $A \in \mathcal{C}$ there is an inflation $A \to I$ with I injective.

The following property certainly does not hold for arbitrary module categories. There are however examples where it holds, for instance $\operatorname{mod} kG$ for any finite group G.

DEFINITION. An exact category is **Frobenius** if it has enough projectives and injectives and moreover the projectives coincide with the injectives.

The Frobenius axioms appear quite restrictive, so one could morally expect that they have some marvellous consequence. Indeed they do: Happel showed in [Hap88] that the stable category $\underline{\mathcal{C}}$ becomes triangulated whenever $\mathcal{C}$ is Frobenius. For the sake of self containedness we give a description of the triangulation.

Describing a triangulated structure on a given category amounts to specifying the translation functor Σ and the distinguished triangles. For each object A in $\mathcal{C}$, choose a conflation

$$A \xrightarrow{i_A} I_A \xrightarrow{p_A} C$$

where I_A is injective. Then $\Sigma A := C$. On a morphism $f : A \to B$ the translation is given by the following commutative diagram whose rows are conflations in $\mathcal{C}$.

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[d, "f"'] & I_A \arrow[r, "p_A"] \arrow[d] & \Sigma A \arrow[d, dashed, "\Sigma f"] \\ B \arrow[r, "i_B"] & I_B \arrow[r, "p_A"] & \Sigma B \end{tikzcd}$$

Here, the middle vertical morphism exists by injectivity of I_B while Σf comes from the cokernel property of ΣA. Further, to the morphism f, the associated **standard triangle** is

$$A \xrightarrow{f} B \xrightarrow{c_f} C_f \xrightarrow{\omega_f} \Sigma(A)$$

which is constructed via the pushout of f and i_A

$$\begin{tikzcd}
A \arrow[r, "i_A"] \arrow[d, "f"] & I_A \arrow[d] \arrow[ddr, bend left, "p_A"] & \\
B \arrow[r, "c_f"] \arrow[drr, bend right, "0"'] & C_f \arrow[dr, dashed, "\omega_f"] & \\
 & & \Sigma A
\end{tikzcd}$$

Now the class of distinguished triangles in $\underline{\mathcal{C}}$ is obtained as the closure of the class of standard triangles with respect to isomorphism of triangles.

Remark. There is, of course, no hope of these constructions being well defined in $\mathcal{C}$, as there are choices to be made throughout. A considerable part of showing that the above gives a triangulated structure on $\underline{\mathcal{C}}$ is checking that these choices actually do not matter in the latter category. □

1.4 Localization of Categories

The concept of localizing a category generalises that of localizing a commutative ring R. Indeed, viewing R as the category $\mathcal{R}$ with a single object $\cdot$ and $\mathrm{End}_{\mathcal{R}}(\cdot) = R$, a multiplicatively closed set $S \subseteq R$ translates to what we shall call a 'multiplicative system' of morphisms in $\mathcal{R}$, and viewing $S^{-1}R$ as a category yields precisely the localization $S^{-1}\mathcal{R}$ of $\mathcal{R}$. References for this section are [Wei94, Kra07].

We should start by making a comment to avoid ambiguity. Given a picture

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "G"'] & & \mathcal{E} \\
 & \mathcal{D} &
\end{tikzcd}$$

of categories and functors, we say that G 'factors uniquely' through F if there is a functor $H : \mathcal{E} \to \mathcal{D}$, unique up to natural isomorphism, such that G is naturally isomorphic to HF.

Definition. Let S be a class of morphisms in a category $\mathcal{C}$. The **localization** of $\mathcal{C}$ with respect to S is a category $S^{-1}\mathcal{C}$ together with a functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ satistfying the following.

L1 Qs is an isomorphism for each $s \in S$.

L2 Any functor $\mathcal{C} \to \mathcal{D}$ taking members of S to isomorphisms in $\mathcal{D}$ factors uniquely through Q.

Remark. It follows that localizations are unique up to equivalence. □

In general, localizations are difficult to understand if they even exist. Fortunately, in this thesis we shall only have to consider extremely well behaved ones. A key notion is the following.

Definition. A class S of morphisms in $\mathcal{C}$ is a **multiplicative system** if

M1 S is closed under compositions and contains each identity morphism.

M2 (Ore condition) If $s : A \to B$ belongs to S, then any pair of morphisms $B' \to B$ and $A \to A''$ can be completed to a pair of commutative diagrams

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "s'"] & A \arrow[d, "s"] \\ B' \arrow[r] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d, "s"] & A'' \arrow[d, "s''"] \\ B \arrow[r] & B'' \end{tikzcd}$$

in which s' and s'' belong to S.

M3 For any two parallel morphisms $f, g : A \to B$, the following are equivalent.

· $sf = sg$ for some $s \in S$.

· $fs' = gs'$ for some $s' \in S$.

When S is multiplicative, $S^{-1}\mathcal{C}$ can be described as follows, due to [GZ67]. The objects are those of $\mathcal{C}$ while the morphisms $A \to B$ are equivalence classes of 'roofs' of morphisms

$$\begin{tikzcd} & X_1 \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & \\ A & & B \end{tikzcd}$$

where $s_1 \in S$. The above roof is denoted by (s_1, f_1) and is said to be equivalent to (s_2, f_2) if there is a third roof (s_3, f_3) such that both squares in

$$\begin{tikzcd} & & X_3 \arrow[dl, "s_3"'] \arrow[dr, "f_3"] & & \\ & X_1 \arrow[dl, "s_1"'] \arrow[drrr, "f_1" near end] & & X_2 \arrow[dr, "f_2"] \arrow[dlll, "s_2"' near end] & \\ A & & & & B \end{tikzcd}$$

are commutative. If (s, f) and (t, g) are composable, using the Ore condition, we let their composition be the roof (st', gf')

$$\begin{tikzcd}
 & & Z \arrow[dl, "t'"'] \arrow[dr, "f'"] & & \\
 & X \arrow[dl, "s"'] \arrow[dr, "f"] & & Y \arrow[dl, "t"'] \arrow[dr, "g"] & \\
A & & B & & C
\end{tikzcd}$$

whose square is commutative. For a proof of the fact that the above relation is an equivalence relation and that the composition of roofs is well defined, see for instance [GM03]. The localization functor $Q : \mathcal{C} \to S^{-1}\mathcal{C}$ is the identity on objects and takes a morphism $f : A \to B$ to the roof $(1_A, f)$.

Since we are interested in triangulated categories, it is only natural to ask what happens if one localizes a triangulated category $\mathcal{T}$. In particular, when does $S^{-1}\mathcal{T}$ carry a triangulated structure in such a way that $Q : \mathcal{T} \to S^{-1}\mathcal{T}$ is a triangle functor? As Lemma 1.5 will show, the following is a sufficient condition on the multiplicative system.

Definition. A multiplicative system S in $\mathcal{T}$ is **compatible with the triangulation** if

M4 S is closed under Σ and Σ^{-1}.

M5 Any commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "s"] & B \arrow[r] \arrow[d, "s'"] & C \arrow[r] & \Sigma A \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & \Sigma A'
\end{tikzcd}$$

where $s, s' \in S$ and the rows are distinguished triangles can be completed to a morphism of triangles by $s'' : C \to C'$ in S.

1.5 Lemma. *Let S be a multiplicative system compatible with the triangulation in $\mathcal{T}$. Then $S^{-1}\mathcal{T}$ is triangulated in such a way that $Q : \mathcal{T} \to S^{-1}\mathcal{T}$ is a triangle functor.*

Proof. Since S is closed under Σ and Σ^{-1} (M4), the functors $Q\Sigma$ and $Q\Sigma^{-1}$ both make the morphisms in S invertible. Hence, by the universal property of Q, there are unique functors $\widetilde{\Sigma}$ and $\widetilde{\Sigma^{-1}}$ making both the following diagrams commutative.

$$\begin{tikzcd}
\mathcal{T} \arrow[r, "\Sigma"] \arrow[d, "Q"] & \mathcal{T} \arrow[d, "Q"] \\
S^{-1}\mathcal{T} \arrow[r, "\widetilde{\Sigma}"] & S^{-1}\mathcal{T}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{T} \arrow[r, "\Sigma^{-1}"] \arrow[d, "Q"] & \mathcal{T} \arrow[d, "Q"] \\
S^{-1}\mathcal{T} \arrow[r, "\widetilde{\Sigma^{-1}}"] & S^{-1}\mathcal{T}
\end{tikzcd}$$

Now it is easy to see that $\widetilde{\Sigma}$ and $\widetilde{\Sigma^{-1}}$ are mutually inverse equivalences. Indeed, consider the commutative diagram

$$\begin{tikzcd}
\mathcal{T} \arrow[r, "\Sigma\Sigma^{-1} = 1 = \Sigma^{-1}\Sigma"] \arrow[d, "Q"] & \mathcal{T} \arrow[d, "Q"] \\
S^{-1}\mathcal{T} \arrow[r, "1"] & S^{-1}\mathcal{T}
\end{tikzcd}$$

which is also completed by both $\widetilde{\Sigma}\widetilde{\Sigma^{-1}}$ and $\widetilde{\Sigma^{-1}}\widetilde{\Sigma}$. The universal property of Q implies that both these functors are the identity on $S^{-1}\mathcal{T}$.

Since there is a natural isomorphism $\phi : Q\Sigma \to \widetilde{\Sigma}Q$, it is only natural to let $\widetilde{\Sigma}$ be the translation functor on $S^{-1}\mathcal{T}$. This leads us to taking as distinguished triangles in $S^{-1}\mathcal{T}$ the triangles isomorphic to one of the form

$$A \xrightarrow{Qf} B \xrightarrow{Qg} C \xrightarrow{\phi_A Qh} \widetilde{\Sigma}A$$

where

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

is distinguished in $\mathcal{T}$. The canonical reference for the verification of axioms Tr1-Tr4 is [Ver96, II.2.2.6]. It is clear that Q will be a triangle functor by construction. □

Lemma 1.5 would be a very useful result if we knew how to get our hands on the particular classes of morphisms it requires. The following result tells us how we can find such a class whenever there is a homological functor around.

1.6 LEMMA. *Let $H : \mathcal{T} \to \mathcal{A}$ be a homological functor and take S to be the class of morphisms s in $\mathcal{T}$ such that $H\Sigma^n s$ is an isomorphism for each $n \in \mathbb{Z}$. Then S is a multiplicative system compatible with the triangulation.*

PROOF. M1 and M4 are immediate, while M5 follows from the five lemma. For M2, take $s : A \to B$ in S and an arbitrary $f : B' \to B$. Then, since M5 holds, the diagram

$$\begin{tikzcd}
A' \arrow[r] & A \arrow[r, "c_f s"] \arrow[d, "s"] & C_f \arrow[r] \arrow[d, "1"] & \Sigma A' \\
B' \arrow[r, "f"] & B \arrow[r, "c_f"] & C_f \arrow[r] & \Sigma B'
\end{tikzcd}$$

can be completed to a morphism of triangles by some $s' : A' \to B'$ in S. The remaining half of M2 is shown in a similar manner. Lastly, to show M3, take $f, g : A \to B$ to be arbitrary and $s : A' \to A$ in S such that $fs = gs$. Then

$$\begin{tikzcd}
A' \arrow[r, "s"] \arrow[d] & A \arrow[r, "c_s"] \arrow[d, "f-g"] & C_s \arrow[r] & \Sigma A' \\
0 \arrow[r] & B \arrow[r, "1"] & B \arrow[r] & 0
\end{tikzcd}$$

can be completed to a morphism of triangles, say by $\psi : C_s \to B$. Consider the standard triangle

$$C_s \xrightarrow{\psi} B \xrightarrow{r} C_\psi \to \Sigma C_s$$

associated to ψ. For once, $rf - rg = r(f - g) = r\psi c_s = 0$ by Lemma 1.1. Further, since $s \in S$, $H\Sigma^i C_s$ vanishes for each $i \in \mathbb{Z}$. Indeed,

$$H\Sigma^i A' \xrightarrow{\cong} H\Sigma^i A \to H\Sigma^i C_s \to H\Sigma^{i+1} A' \xrightarrow{\cong} H\Sigma^{i+1} A$$

is exact. Therefore, the $H\Sigma^i r$ must all be isomorphisms since also

$$H\Sigma^i C_s \to H\Sigma^i B \xrightarrow{H\Sigma^i r} H\Sigma^i C_\psi \to H\Sigma^{i+1} C_s$$

is exact. This means $r \in S$. The other implication in M3 is shown similarly. □

If $\mathcal{S}$ is a triangulated subcategory of $\mathcal{T}$, a natural construction called the **Verdier localization** $\mathcal{T}/\mathcal{S}$ arises. Let $S(\mathcal{S})$ denote the class of morphisms in $\mathcal{T}$ whose cone belongs to $\mathcal{S}$. The following lemma is [Ver96, II.2.1.8].

1.7 Lemma. *Let $\mathcal{S}$ be a triangulated subcategory of $\mathcal{T}$. Then $S(\mathcal{S})$ is a multiplicative system compatible with the triangulation in $\mathcal{T}$.* □

In light of this we define

$$\mathcal{T}/\mathcal{S} := S(\mathcal{S})^{-1}\mathcal{T}.$$

It follows that $\mathcal{T}/\mathcal{S}$ carries a triangulated structure in such a way that the localization functor $Q : \mathcal{T} \to \mathcal{T}/\mathcal{S}$ is a triangle functor. It is easy to check that each object of $\mathcal{S}$ is annihilated by Q. If $\mathcal{S}$ is thick in $\mathcal{T}$, then also the reverse inclusion holds, i.e. each object annihilated by Q belongs to $\mathcal{S}$.

CHAPTER 2

The Categories $\mathcal{C}$ and $\mathcal{C}[\epsilon]$

Given an additive category $\mathcal{C}$, one obtains another additive category $\mathcal{C}[\epsilon]$ in the following way. An object in $\mathcal{C}[\epsilon]$ is a pair (A, ϵ_A) where A is an object in $\mathcal{C}$ and $\epsilon_A \in \operatorname{End}_{\mathcal{C}}(A)$ has the property $\epsilon_A^2 = 0$ (and will be called the **differential** of A). A morphism $f \in \operatorname{Hom}_{\mathcal{C}[\epsilon]}(A, B)$ is what one might expect, namely a morphism $f \in \operatorname{Hom}_{\mathcal{C}}(A, B)$ that commutes with the differentials involved. I.e. f is a morphism in $\mathcal{C}[\epsilon]$ if the following diagram is commutative in $\mathcal{C}$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\epsilon_A"] & B \arrow[d, "\epsilon_B"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

The additive structure of $\mathcal{C}[\epsilon]$ should also not be a surprise. Given objects A and B in $\mathcal{C}[\epsilon]$, their sum is the underlying object $A \oplus B$ with differential $\left(\begin{smallmatrix} \epsilon_A & 0 \\ 0 & \epsilon_B \end{smallmatrix}\right)$.

Remark. Any object in $\mathcal{C}$ may be viewed as an object in $\mathcal{C}[\epsilon]$ by equipping it with the zero differential. $\square$

The aim of this chapter is to investigate the relationship between $\mathcal{C}$ and $\mathcal{C}[\epsilon]$. More explicitly, we shall see that there is a natural way of obtaining an exact structure on the latter from one on the former. The projective and injective objects of $\mathcal{C}[\epsilon]$ will therefore be completely determined by those of $\mathcal{C}$.

2.1 Two-Sided Adjoints

The above setup comes with two functors we wish to examine. One is the forgetful functor $F : \mathcal{C}[\epsilon] \to \mathcal{C}$ which is the identity both on underlying objects and on morphisms (i.e. it does nothing, but forgets about differentials).[1] The other is the augmenting functor $-[\epsilon] : \mathcal{C} \to \mathcal{C}[\epsilon]$. This takes an object A in $\mathcal{C}$ to the object $A[\epsilon]$ in $\mathcal{C}[\epsilon]$, defined as

$$A[\epsilon] := A \oplus A \qquad \text{with differential} \qquad \epsilon_{A[\epsilon]} := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

[1] It is often convenient to write A instead of FA when A is an object in $\mathcal{C}[\epsilon]$ and f instead of Ff when f is a morphism in $\mathcal{C}[\epsilon]$. Throughout, the reader will notice that we are not consistently using neither the sometimes overstated FA and Ff nor the potentially ambiguous A and f. Instead, we try to choose the notation best suited for each scenario we face.

For a morphism $f : A \to B$ in $\mathcal{C}$ we define $f[\epsilon] : A[\epsilon] \to B[\epsilon]$ as the matrix $\left(\begin{smallmatrix} f & 0 \\ 0 & f \end{smallmatrix}\right)$. It is trivial that $f[\epsilon]\epsilon_{A[\epsilon]} = \epsilon_{B[\epsilon]}f[\epsilon]$, i.e. $f[\epsilon] \in \mathrm{Hom}_{\mathcal{C}[\epsilon]}(A[\epsilon], B[\epsilon])$, but this should still be noted; we ought to get used to checking that maps commute with differentials before accepting them as morphisms in $\mathcal{C}[\epsilon]$.

The following proposition reveals a key relationship between these functors which we shall exploit several times in this chapter. The result might be surprising, as forgetful functors often have left adjoints but fail to appear as left adjoints themselves.

2.1 PROPOSITION. *The functors*

$$\mathcal{C} \underset{F}{\overset{-[\epsilon]}{\rightleftarrows}} \mathcal{C}[\epsilon]$$

are two-sided adjoints, i.e. both $(F, -[\epsilon])$ and $(-[\epsilon], F)$ are adjoint pairs.

PROOF. Fix $C \in \mathcal{C}$ and $(A, \epsilon_A) \in \mathcal{C}[\epsilon]$.

To show why $(F, -[\epsilon])$ is an adjoint pair we will produce an isomorphism

$$\phi_{A,C} : \mathrm{Hom}_{\mathcal{C}}(FA, C) \to \mathrm{Hom}_{\mathcal{C}[\epsilon]}(A, C[\epsilon])$$

which is natural in both A and C. We will omit the indices and write ϕ. Let

$$f \overset{\phi}{\longmapsto} \begin{pmatrix} f\epsilon_A \\ f \end{pmatrix}.$$

Routine calculations show that $\phi(f)$ commutes with the differentials involved, so ϕ is well defined. Further, let

$$\phi^{-1} : \mathrm{Hom}_{\mathcal{C}[\epsilon]}(A, C[\epsilon]) \to \mathrm{Hom}_{\mathcal{C}}(FA, C)$$

be given by $g = \left(\begin{smallmatrix} g_1 \\ g_2 \end{smallmatrix}\right) \mapsto g_2$. It is clear that $\phi^{-1}\phi = 1$. On the other hand,

$$\phi\phi^{-1}(g) = \phi(g_2) = \begin{pmatrix} g_2\epsilon_A \\ g_2 \end{pmatrix}.$$

But since g is a morphism in $\mathcal{C}[\epsilon]$,

$$\begin{pmatrix} 0 \\ g_1 \end{pmatrix} = \epsilon_{C[\epsilon]}g = g\epsilon_A = \begin{pmatrix} g_1\epsilon_A \\ g_2\epsilon_A \end{pmatrix}.$$

This yields $g_1 = g_2\epsilon_A$, i.e. $\phi\phi^{-1}(g) = g$, so ϕ is indeed an isomorphism. To check naturality of ϕ take a morphism $\alpha : A \to A'$ in $\mathcal{C}[\epsilon]$ and a morphism $\beta : C \to C'$ in $\mathcal{C}$. For any $\gamma \in \mathrm{Hom}_{\mathcal{C}}(FA', C)$ we have

$$\phi((F\alpha)^*(\gamma)) = \phi(\gamma F\alpha) = \phi(\gamma\alpha) = \begin{pmatrix} \gamma\alpha\epsilon_A \\ \gamma\alpha \end{pmatrix} = \begin{pmatrix} \gamma\epsilon_{A'}\alpha \\ \gamma\alpha \end{pmatrix} = \begin{pmatrix} \gamma\epsilon_{A'} \\ \gamma \end{pmatrix} \alpha = \phi(\gamma)\alpha$$

while also

$$\alpha^*\phi(\gamma) = \phi(\gamma)\alpha.$$

Moreover, if $\gamma' \in \mathrm{Hom}_{\mathcal{C}}(FA, C)$ then

$$\phi(\beta_*(\gamma')) = \phi(\beta\gamma')$$

while

$$\beta[\epsilon]_*\phi(\gamma') = \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} \gamma'\epsilon_A \\ \gamma' \end{pmatrix} = \begin{pmatrix} \beta\gamma'\epsilon_A \\ \beta\gamma' \end{pmatrix} = \phi(\beta\gamma').$$

This shows that both squares in the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(FA', C) \arrow[r, "(F\alpha)^*"] \arrow[d, "\phi"] & \mathrm{Hom}_{\mathcal{C}}(FA, C) \arrow[r, "\beta_*"] \arrow[d, "\phi"] & \mathrm{Hom}_{\mathcal{C}}(FA, C') \arrow[d, "\phi"] \\
\mathrm{Hom}_{\mathcal{C}[\epsilon]}(A', C[\epsilon]) \arrow[r, "\alpha^*"] & \mathrm{Hom}_{\mathcal{C}[\epsilon]}(A, C[\epsilon]) \arrow[r, "\beta[\epsilon]_*"] & \mathrm{Hom}_{\mathcal{C}[\epsilon]}(A, C'[\epsilon])
\end{tikzcd}$$

are commutative, so $(F, -[\epsilon])$ is indeed an adjoint pair.

Let us turn our attention to $(-[\epsilon], F)$. We will find an isomorphism

$$\psi_{A,C} : \mathrm{Hom}_{\mathcal{C}[\epsilon]}(C[\epsilon], A) \to \mathrm{Hom}_{\mathcal{C}}(C, FA)$$

which is natural in each argument. To ease the notation we will write ψ with no indices. For $f = \begin{pmatrix} f_1 & f_2 \end{pmatrix} \in \mathrm{Hom}_{\mathcal{C}[\epsilon]}(C[\epsilon], A)$ define $\psi(f) := f_1$. Further, let

$$\psi^{-1} : \mathrm{Hom}_{\mathcal{C}}(C, FA) \to \mathrm{Hom}_{\mathcal{C}[\epsilon]}(C[\epsilon], A)$$

be given by $\psi^{-1}(g) := \begin{pmatrix} g & \epsilon_A g \end{pmatrix}$. One easily sees that $\psi^{-1}(g)$ commutes with the differentials of $C[\epsilon]$ and A, i.e. ψ^{-1} is well defined. It is now immediate that $\psi\psi^{-1} = 1$. Also,

$$\psi^{-1}\psi(f) = \psi^{-1}(f_1) = \begin{pmatrix} f_1 & \epsilon_A f_1 \end{pmatrix} = f.$$

The last equality holds because f being a morphism in $\mathcal{C}[\epsilon]$ means

$$\begin{pmatrix} f_2 & 0 \end{pmatrix} = f\epsilon_{C[\epsilon]} = \epsilon_A f = \begin{pmatrix} \epsilon_A f_1 & f_2 \end{pmatrix}$$

i.e. $\epsilon_A f_1 = f_2$. This establishes the fact that ψ is an isomorphism. Checking naturality of ψ is done by showing commutativity of both squares in

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}[\epsilon]}(C'[\epsilon], A) \arrow[r, "\beta[\epsilon]^*"] \arrow[d, "\psi"] & \mathrm{Hom}_{\mathcal{C}[\epsilon]}(C[\epsilon], A) \arrow[r, "\alpha_*"] \arrow[d, "\psi"] & \mathrm{Hom}_{\mathcal{C}[\epsilon]}(C[\epsilon], A') \arrow[d, "\psi"] \\
\mathrm{Hom}_{\mathcal{C}}(C', FA) \arrow[r, "\beta^*"] & \mathrm{Hom}_{\mathcal{C}}(C, FA) \arrow[r, "F\alpha_*"] & \mathrm{Hom}_{\mathcal{C}}(C, FA')
\end{tikzcd}$$

where $\alpha : A \to A'$ is some morphism in $\mathcal{C}[\epsilon]$ and $\beta : C \to C'$ is some morphism in $\mathcal{C}$. The left hand square is commutative because for any $\gamma = \begin{pmatrix}\gamma_1 & \gamma_2\end{pmatrix} \in \mathrm{Hom}_{\mathcal{C}[\epsilon]}(C'[\epsilon], A)$ we have

$$\psi(\beta[\epsilon])^*(\gamma) = \psi(\begin{pmatrix}\gamma_1\beta & \gamma_2\beta\end{pmatrix}) = \gamma_1\beta$$

while

$$\beta^*\psi(\gamma) = \beta^*(\gamma_1) = \gamma_1\beta.$$

The right hand square is commutative because for any $\gamma' = \begin{pmatrix}\gamma_1' & \gamma_2'\end{pmatrix} \in \mathrm{Hom}_{\mathcal{C}[\epsilon]}(C[\epsilon], A)$,

$$\psi\alpha_*(\gamma') = \psi(\begin{pmatrix}\alpha\gamma_1' & \alpha\gamma_2'\end{pmatrix}) = \alpha\gamma_1'$$

while

$$(F\alpha)_*\psi(\gamma') = F\alpha\gamma_1' = \alpha\gamma_1'.$$

□

2.2 An Induced Exact Structure

As promised, we shall study the interplay between $\mathcal{C}$ and $\mathcal{C}[\epsilon]$ when the former possesses an exact structure, in which case such a structure is induced on the latter in the most natural of ways.

2.2 LEMMA. *Let $\mathcal{C}$ be an exact category, and let $\mathscr{E}$ be the class of pairs of composable morphisms in $\mathcal{C}[\epsilon]$ that become conflations in $\mathcal{C}$ via the forgetful functor. Then $(\mathcal{C}[\epsilon], \mathscr{E})$ is an exact category.*

PROOF. We must first of all make sure that $\mathscr{E}$ is a class of exact pairs. Assume that $A \xrightarrow{i} B \xrightarrow{p} C$ is a pair of morphisms in $\mathcal{C}[\epsilon]$ such that $(i, p) \in \mathscr{E}$. Then $pi = 0$ in $\mathcal{C}$, hence also in $\mathcal{C}[\epsilon]$. Further, let $p' \in \mathrm{Hom}_{\mathcal{C}[\epsilon]}(B, C')$ have the property $p'i = 0$. Since p is a cokernel of i in $\mathcal{C}$, there is a unique $\phi \in \mathrm{Hom}_{\mathcal{C}}(C, C')$ with the property $\phi p = p'$.

$$\begin{tikzcd} A \arrow[r, "i"] & B \arrow[r, "p"] \arrow[dr, "p'"'] & C \arrow[d, "(\exists!)\phi"] \\ & & C' \end{tikzcd}$$

So p will be a cokernel of i also in $\mathcal{C}[\epsilon]$ if we can show that ϕ commutes with differentials. To see why it does, we use the fact that p and $p' = \phi p$ do commute with differentials, i.e. $p\epsilon_B = \epsilon_C p$ and $\phi p \epsilon_B = \epsilon_{C'}\phi p$. Combining these gives $\epsilon_{C'}\phi p = \phi\epsilon_C p$, which yields $\epsilon_{C'}\phi = \phi\epsilon_C$ since p is a cokernel and hence right cancellable. The fact that i is a kernel of p follows from the dual argument.

Let us show E1. Let $(i, p) \in \mathscr{E}$ and assume there is an isomorphism of pairs

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\cong"] & B \arrow[r, "p"] \arrow[d, "\cong"] & C \arrow[d, "\cong"] \\ A' \arrow[r, "i'"] & B' \arrow[r, "p'"] & C' \end{tikzcd}$$

in $\mathcal{C}[\epsilon]$. This must be an isomorphism also in $\mathcal{C}$, which means (i', p') is a conflation in $\mathcal{C}$, so $(i', p') \in \mathscr{E}$.

To show E2, start by noting that since the identity on 0 is a deflation in $\mathcal{C}$, $0 \to 0 \to 0$ must be a conflation in $\mathcal{C}$. This obviously means that the identity on 0 is a deflation also in $\mathcal{C}[\epsilon]$. Further, assume $p_1 : B \to M$ and $p_2 : M \to C$ are deflations in $\mathcal{C}[\epsilon]$. We wish to show that $p := p_2 p_1$ is also a deflation. By assumption, p is a deflation in $\mathcal{C}$, i.e. it is part of some conflation $K \xrightarrow{i} B \xrightarrow{p} C$. It suffices to equip K with a differential such that $i : K \to B$ becomes a morphism in $\mathcal{C}[\epsilon]$. To see why such a differential exists, consider the diagram

$$\begin{tikzcd} K \arrow[r, "i"] & B \arrow[r, "p"] \arrow[d, "\epsilon_B"] & C \arrow[d, "\epsilon_C"] \\ K \arrow[r, "i"] & B \arrow[r, "p"] & C \end{tikzcd}$$

in which the square is commutative. Hence $p\epsilon_Y i = \epsilon_Z p i = 0$. Since i is a kernel of p, this means that $\epsilon_Y i$ must factor (uniquely) through i, i.e. there is some morphism ϵ_K making

$$\begin{tikzcd} K \arrow[r, "i"] \arrow[d, "\epsilon_K"] & B \arrow[d, "\epsilon_B"] \\ K \arrow[r, "i"] & B \end{tikzcd}$$

commutative. Thus, if ϵ_K squares to zero, then it is the desired differential of K. Commutativity of the latter diagram ensures $i\epsilon_K^2 = \epsilon_Y^2 i = 0$, which yields $\epsilon_K^2 = 0$ since i is a kernel and hence left cancellable.

Verifying E3 and its dual now remains. Given any $f \in \mathrm{Hom}_{\mathcal{C}[\epsilon]}(C', C)$ and a deflation $p \in \mathrm{Hom}_{\mathcal{C}[\epsilon]}(B, C)$ there does exist a pullback diagram

$$\begin{tikzcd} B' \arrow[r, "p'"] \arrow[d, "f'"] & C' \arrow[d, "f"] \\ B \arrow[r, "p"] & C \end{tikzcd} \tag{2.1}$$

in $\mathcal{C}$ such that p' is a deflation. The first step towards showing that this is a pullback also in $\mathcal{C}[\epsilon]$ is to equip B' with a differential. (2.1) is commutative, hence $f\epsilon_{C'}p' = \epsilon_C f p' = \epsilon_C p f' = p\epsilon_B f'$, since f and p are morphisms in $\mathcal{C}[\epsilon]$. So by the universal property of the pullback in $\mathcal{C}$, there is a unique morphism $\epsilon_{B'} : B' \to B'$ such that all is commutative in

$$\begin{tikzcd}
B' \arrow[dr, dashed, "\epsilon_{B'}"] \arrow[drr, bend left, "\epsilon_{C'}p'"] \arrow[ddr, bend right, "\epsilon_B f'"'] & & \\
& B' \arrow[r, "p'"] \arrow[d, "f'"] & C' \arrow[d, "f"] \\
& B \arrow[r, "p"] & C
\end{tikzcd}$$

Before proceeding, we should note that $\epsilon_{B'}$ squares to zero. Consider the diagram

$$\begin{tikzcd}
B' \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "0"'] & & \\
& B' \arrow[r, "p'"] \arrow[d, "f'"] & C' \arrow[d, "f"] \\
& B \arrow[r, "p"] & C
\end{tikzcd}$$

in $\mathcal{C}$. Of course, the zero morphism makes all commutative, but so does $\epsilon_{B'}^2$. Indeed, $f'\epsilon_{B'}^2 = \epsilon_B f'\epsilon_{B'} = \epsilon_B^2 f' = 0$ and $p'\epsilon_{B'}^2 = \epsilon_{C'}p'\epsilon_{B'} = \epsilon_{C'}^2 p' = 0$. This forces $\epsilon_{B'}^2 = 0$. So by the construction of $\epsilon_{B'}$ it is clear that f' and p' are morphisms in $\mathcal{C}[\epsilon]$. Since p' is a deflation in $\mathcal{C}$, the dual construction of the one used in showing E2 will ensure that p' is a deflation also in $\mathcal{C}[\epsilon]$. The only piece missing is showing that diagram (2.1) enjoys the appropriate universal property also in $\mathcal{C}[\epsilon]$. Assume there is a diagram

$$\begin{tikzcd}
X \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "\alpha"'] & & \\
& B' \arrow[r, "p'"] \arrow[d, "f'"] & C' \arrow[d, "f"] \\
& B \arrow[r, "p"] & C
\end{tikzcd}$$

in $\mathcal{C}[\epsilon]$ in which $f\beta = p\alpha$. The diagram can be completed uniquely in $\mathcal{C}$ by $\phi : X \to B'$ and it suffices to show that ϕ is a morphism in $\mathcal{C}[\epsilon]$. This amounts to showing that $\phi\epsilon_X - \epsilon_{B'}\phi = 0$. To see why this equality holds, invoke the following diagram in $\mathcal{C}$.

$$\begin{tikzcd}
B' \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "0"'] & & \\
& B' \arrow[r, "p'"] \arrow[d, "f'"] & C' \arrow[d, "f"] \\
& B \arrow[r, "p"] & C
\end{tikzcd}$$

The zero morphism completes this, of course, but so does $\phi\epsilon_X - \epsilon_{B'}\phi$:

$$f'(\phi\epsilon_X - \epsilon_{B'}\phi) = f'\phi\epsilon_X - f'\epsilon_{B'}\phi = f'\phi\epsilon_X - \epsilon_B f'\phi = \alpha\epsilon_X - \epsilon_B\alpha = 0$$

and

$$p'(\phi\epsilon_X - \epsilon_{B'}\phi) = p'\phi\epsilon_X - p'\epsilon_{B'}\phi = p'\phi\epsilon_X - \epsilon_{C'}p'\phi = \beta\epsilon_X - \epsilon_{C'}\beta = 0.$$

This forces the conclusion $\phi\epsilon_X - \epsilon_{B'}\phi = 0$, completing the argument.

A dual argument to the previous one applies to prove E3$^{\mathrm{op}}$. □

In homological algebra it is a standard fact that any functor between abelian categories which has a left adjoint commutes with limits, and consequently is left exact. Similarly, a functor with a right adjoint commutes with colimits, making it right exact. One could hope that this translates neatly to the setup where the categories involved are merely exact. In particular, from what we know about F and $-[\epsilon]$, the following lemma might not come as a surprise.

2.3 Lemma. *The functors $F : \mathcal{C}[\epsilon] \to \mathcal{C}$ and $-[\epsilon] : \mathcal{C} \to \mathcal{C}[\epsilon]$ are both exact.*

Proof. The exactness of F is immediate from the definition of exact functors and the way we imposed the exact structure on $\mathcal{C}[\epsilon]$ in Lemma 2.2.

Given a conflation $A \xrightarrow{i} B \xrightarrow{p} C$ in $\mathcal{C}$, its image under $-[\epsilon]$ is of course

$$A \oplus A \xrightarrow{\left(\begin{smallmatrix} i & 0 \\ 0 & i \end{smallmatrix}\right)} B \oplus B \xrightarrow{\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)} C \oplus C.$$

Being the direct sum of two conflations when restricted to $\mathcal{C}$, it is itself a conflation therein, hence also in $\mathcal{C}[\epsilon]$. □

2.3 Projective and Injective Objects in $\mathcal{C}[\epsilon]$

In this section we shall assume that $\mathcal{C}$ is idempotent complete. This will enable us to reveal an appealing relationship between $\mathcal{C}$ and $\mathcal{C}[\epsilon]$, namely that the projectives (injectives) of $\mathcal{C}[\epsilon]$ are completely determined by the projectives (injectives) of $\mathcal{C}$. Intuitively, this is hardly a surprise, as projectives and injectives are determined by exact structure, and the exact structure of $\mathcal{C}[\epsilon]$ is determined by that of $\mathcal{C}$. Let us start by recalling the definition.

Definition. An additive category $\mathcal{C}$ is **idempotent complete** if idempotents split, i.e. when $e = e^2 \in \mathrm{End}_{\mathcal{C}}(X)$, then there is an object $Y \in \mathcal{C}$ and morphisms $X \xrightarrow{\pi} Y$ and $Y \xrightarrow{\mu} X$ such that $\pi\mu = 1_Y$ and $\mu\pi = e$.

An immediate consequence of the assumption on $\mathcal{C}$ is the following.

2.4 Lemma. *$\mathcal{C}[\epsilon]$ is idempotent complete.*

PROOF. Take $e = e^2 \in \mathrm{End}_{\mathcal{C}[\epsilon]}(X)$. Restricting to $\mathcal{C}$, there is an object Y and morphisms $X \xrightarrow{\pi} Y$ and $Y \xrightarrow{\mu} X$ such that $\pi\mu = 1_Y$ and $\mu\pi = e$. Hence, it suffices to equip Y with a differential such that π and μ become morphisms in $\mathcal{C}[\epsilon]$. To this end let $\epsilon_Y := \pi\epsilon_X\mu$. Note that ϵ_Y is a differential, as

$$\epsilon_Y^2 = \pi\epsilon_X\mu\pi\epsilon_X\mu = \pi\epsilon_X e\epsilon_X\mu = \pi e\epsilon_X\epsilon_X\mu = 0.$$

It is easy to check that π and μ become morphisms in $\mathcal{C}[\epsilon]$:

$$\pi\epsilon_X = \pi\mu\pi\epsilon_X = \pi e\epsilon_X = \pi\epsilon_X e = \pi\epsilon_X\mu\pi = \epsilon_Y\pi$$

and

$$\epsilon_X\mu = \epsilon_X\mu\pi\mu = \epsilon_X e\mu = e\epsilon_X\mu = \mu\pi\epsilon_X\mu = \mu\epsilon_Y.$$

□

We employ a notation where $(\mathcal{S})[\epsilon]$ denotes the 'essential image' of the subcategory $\mathcal{S}$ of $\mathcal{C}$ under the functor $-[\epsilon]$. I.e. we take the closure of the image of $\mathcal{S}$ with respect to isomorphisms. In particular

$$(\mathrm{Proj}\,\mathcal{C})[\epsilon] := \{M \in \mathcal{C}[\epsilon] : M \cong P[\epsilon] \text{ for some } P \in \mathrm{Proj}\,\mathcal{C}\}$$

and

$$(\mathrm{Inj}\,\mathcal{C})[\epsilon] := \{M \in \mathcal{C}[\epsilon] : M \cong I[\epsilon] \text{ for some } I \in \mathrm{Inj}\,\mathcal{C}\}.$$

In the process of showing the envisioned relationship between projectives we will need that the functor $-[\epsilon]$ preserves thickness, i.e. that $(\mathcal{S})[\epsilon]$ is closed under taking direct summands in $\mathcal{C}[\epsilon]$ whenever $\mathcal{S}$ enjoys the same property in $\mathcal{C}$. This is established in the following two lemmas.

2.5 LEMMA. *For any $X \in \mathcal{C}$, the direct summands of X are in 1-1 correspondence with the direct summands of $X[\epsilon]$ up to conjugation.*

PROOF. Because of Lemma 2.4 it suffices to show that the idempotents of $\mathrm{End}_{\mathcal{C}}(X)$ are in 1-1 correspondence (up to conjugation) with the idempotents of $\mathrm{End}_{\mathcal{C}[\epsilon]}(X[\epsilon])$. Clearly, there is an injective map $\mathrm{End}_{\mathcal{C}}(X) \to \mathrm{End}_{\mathcal{C}[\epsilon]}(X[\epsilon])$ given by

$$h \mapsto \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}$$

taking idempotents to idempotents. So to prove the lemma it suffices to show that any idempotent in $\mathrm{End}_{\mathcal{C}[\epsilon]}(X[\epsilon])$ can be brought by conjugation to the form $\left(\begin{smallmatrix} e & 0 \\ 0 & e \end{smallmatrix}\right)$ with e an idempotent in $\mathrm{End}_{\mathcal{C}}(X)$. For $f = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{End}_{\mathcal{C}[\epsilon]}(X[\epsilon])$, the requirement $f\epsilon_{X[\epsilon]} = \epsilon_{X[\epsilon]}f$ translates to

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & 0 \\ d & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

i.e. $b = 0$ and $a = d$. If we further impose the requirement that f should be idempotent, then also

$$\begin{pmatrix} a & 0 \\ c & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & a \end{pmatrix}^2 = \begin{pmatrix} a^2 & 0 \\ ac + ca & a^2 \end{pmatrix}.$$

must hold.

An arbitrary idempotent $f \in \mathrm{End}_{\mathcal{C}[\epsilon]}(X[\epsilon])$ must therefore be of the form $f = \left(\begin{smallmatrix} e & 0 \\ \phi & e \end{smallmatrix}\right)$ where $e^2 = e$ and $\phi = e\phi + \phi e$. The proof will be complete if we can find an automorphism g of $X[\epsilon]$ satisfying $gfg^{-1} = \left(\begin{smallmatrix} e & 0 \\ 0 & e \end{smallmatrix}\right)$. Before proceeding, note that the criteria $e^2 = e$ and $\phi = e\phi + \phi e$ imply the vanishing of a certain product which will be crucial in the following calculation, namely $e\phi e = 0$. This follows from

$$\begin{aligned} e\phi e &= e(\phi e + e\phi)e \\ &= e\phi e + e\phi e. \end{aligned}$$

As it turns out, letting

$$g := \begin{pmatrix} 1 & 0 \\ e\phi - \phi e & 1 \end{pmatrix}$$

does the job. g is obviously invertible, with

$$g^{-1} = \begin{pmatrix} 1 & 0 \\ \phi e - e\phi & 1 \end{pmatrix}$$

Hence,

$$\begin{aligned} gfg^{-1} &= \begin{pmatrix} 1 & 0 \\ e\phi - \phi e & 1 \end{pmatrix} \begin{pmatrix} e & 0 \\ \phi & e \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \phi e - e\phi & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ e\phi - \phi e & 1 \end{pmatrix} \begin{pmatrix} e & 0 \\ \phi + e\phi e - e^2\phi & e \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ e\phi - \phi e & 1 \end{pmatrix} \begin{pmatrix} e & 0 \\ \phi - e\phi & e \end{pmatrix} \\ &= \begin{pmatrix} e & 0 \\ e\phi e - \phi e^2 + \phi - e\phi & e \end{pmatrix} \\ &= \begin{pmatrix} e & 0 \\ \phi - (e\phi + \phi e) & e \end{pmatrix} \\ &= \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} \end{aligned}$$

□

2.6 Lemma. *If $\mathcal{S}$ is thick in $\mathcal{C}$, then $(\mathcal{S})[\epsilon]$ is thick in $\mathcal{C}[\epsilon]$.*

Proof. Let $X \in \mathcal{S}$ and suppose A is a direct summand of $X[\epsilon]$. It suffices to show that $A \in (\mathcal{S})[\epsilon]$. Let $\overline{e} \in \mathrm{End}_{\mathcal{C}[\epsilon]}$ be the idempotent corresponding to A and note that we may assume $\overline{e} = \left(\begin{smallmatrix} e & 0 \\ 0 & e \end{smallmatrix}\right)$ by Lemma 2.5. Since $e \in \mathrm{End}_{\mathcal{C}}(X)$ is an idempotent, there is some $Y \in \mathcal{C}$ with morphisms $X \xrightarrow{\pi} Y$ and $Y \xrightarrow{\mu} X$ such that $\pi\mu = 1$ and $\mu\pi = e$. This means Y is a summand of X, so $Y \in \mathcal{S}$ by thickness. It follows that $A = Y[\epsilon]$, which belongs to $(\mathcal{S})[\epsilon]$. $\square$

Together with Proposition 2.1, the following result constitutes the main contribution of this chapter.

2.7 PROPOSITION. *In $\mathcal{C}[\epsilon]$, we have*

i) $\mathrm{Proj}\,\mathcal{C}[\epsilon] = (\mathrm{Proj}\,\mathcal{C})[\epsilon]$

ii) $\mathrm{Inj}\,\mathcal{C}[\epsilon] = (\mathrm{Inj}\,\mathcal{C})[\epsilon]$

PROOF. *i)* To show the inclusion $(\mathrm{Proj}\,\mathcal{C})[\epsilon] \subseteq \mathrm{Proj}\,\mathcal{C}[\epsilon]$, take $P \in \mathrm{Proj}\,\mathcal{C}$. By the adjointness of $-[\epsilon]$ and F we have

$$\mathrm{Hom}_{\mathcal{C}[\epsilon]}(P[\epsilon], -) = \mathrm{Hom}_{\mathcal{C}}(P, F(-)) = \mathrm{Hom}_{\mathcal{C}}(P, -) \circ F,$$

which is an exact functor. Indeed, $\mathrm{Hom}_{\mathcal{C}}(P, -)$ is exact by assumption, so it is the composition of two exact functors. This shows $P[\epsilon] \in \mathrm{Proj}\,\mathcal{C}[\epsilon]$, hence $(\mathrm{Proj}\,\mathcal{C})[\epsilon] \subseteq \mathrm{Proj}\,\mathcal{C}[\epsilon]$.

For the reverse inclusion, let $Q \in \mathrm{Proj}\,\mathcal{C}[\epsilon]$. We will show that $Q \in (\mathrm{Proj}\,\mathcal{C})[\epsilon]$ by finding a deflation $X \to Q$ in $\mathcal{C}[\epsilon]$ with $X \in (\mathrm{Proj}\,\mathcal{C})[\epsilon]$. This will suffice since the deflation must split by projectivity of Q and $(\mathrm{Proj}\,\mathcal{C})[\epsilon]$ is thick in $\mathcal{C}[\epsilon]$ by Lemma 2.6. The natural candidate for X is $FQ[\epsilon]$, so let us see if it works. First of all, FQ is certainly in $\mathrm{Proj}\,\mathcal{C}$, since the appropriate Hom-functor

$$\mathrm{Hom}_{\mathcal{C}}(FQ, -) = \mathrm{Hom}_{\mathcal{C}[\epsilon]}(Q, -[\epsilon]) = \mathrm{Hom}_{\mathcal{C}[\epsilon]}(Q, -) \circ -[\epsilon]$$

is exact. Indeed, $\mathrm{Hom}_{\mathcal{C}[\epsilon]}(Q, -)$ is exact by assumption, so the above is the composition of two exact functors. Next, consider

$$\sigma : FQ[\epsilon] = Q \oplus Q \xrightarrow{(\,1 \;\, \epsilon_Q\,)} Q$$

which commutes with the differentials involved by a routine calculation. Showing that σ is a deflation in $\mathcal{C}[\epsilon]$ will suffice. To see why it is, invoke the canonical exact pair

$$Q \xrightarrow{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} Q \oplus Q \xrightarrow{(\,0 \;\, 1\,)} Q$$

in $\mathcal{C}$, which is allways a conflation. By including a kernel of σ we get the exact pair

$$\hat{Q} \xrightarrow{\left(\begin{smallmatrix} -\epsilon_Q \\ 1 \end{smallmatrix}\right)} Q \oplus Q \xrightarrow{(\,1 \;\, \epsilon_Q\,)} Q$$

of morphisms in $\mathcal{C}[\epsilon]$ which we would like to be a conflation. Here, $\hat{Q}$ is the underlying object Q equipped with the differential $\epsilon_{\hat{Q}} := -\epsilon_Q$.[2] Passing to $\mathcal{C}$ the differentials are forgotten and, since the class of conflations is closed under isomorphism, it suffices to note that the diagram

$$\begin{tikzcd}
Q \arrow[r, "\begin{pmatrix} -\epsilon_Q \\ 1 \end{pmatrix}"] \arrow[d, "1"] & Q \oplus Q \arrow[r, "\begin{pmatrix} 1 & \epsilon_Q \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 0 & 1 \\ 1 & \epsilon_Q \end{pmatrix}"] & Q \arrow[d, "1"] \\
Q \arrow[r, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] & Q \oplus Q \arrow[r, "\begin{pmatrix} 0 & 1 \end{pmatrix}"] & Q
\end{tikzcd}$$

is an isomorphism of exact pairs in $\mathcal{C}$.

ii) We turn our attention to the injectives. The argument for the second part of the proposition is dual to that of the first one, but we wish to give a complete proof. Assume $I \in \operatorname{Inj}\mathcal{C}$. By adjointness of F and $-[\epsilon]$ we have

$$\operatorname{Hom}_{\mathcal{C}[\epsilon]}(-, I[\epsilon]) = \operatorname{Hom}_{\mathcal{C}}(F-, I) = \operatorname{Hom}_{\mathcal{C}}(-, I) \circ F,$$

which is exact because $\operatorname{Hom}_{\mathcal{C}}(-, I)$ is exact by assumption. This ensures $\mathcal{C}[\epsilon] \in \operatorname{Inj}\mathcal{C}[\epsilon]$, i.e. $(\operatorname{Inj}\mathcal{C})[\epsilon] \subseteq \operatorname{Inj}\mathcal{C}[\epsilon]$.

For the remaining inclusion, take $J \in \operatorname{Inj}\mathcal{C}[\epsilon]$. To prove that $J \in (\operatorname{Inj}\mathcal{C})[\epsilon]$, we will find an inflation $J \to Y$ in $\mathcal{C}[\epsilon]$ such that $Y \in (\operatorname{Inj}\mathcal{C})[\epsilon]$. The injectivity of J will make the inflation split, so the proof will be complete by thickness of $(\operatorname{Inj}\mathcal{C})[\epsilon]$ in $\mathcal{C}[\epsilon]$ (Lemma 2.6). To construct such an inflation, start by observing that $FJ \in \operatorname{Inj}\mathcal{C}$. This follows from exactness of the functor

$$\operatorname{Hom}_{\mathcal{C}}(-, FJ) = \operatorname{Hom}_{\mathcal{C}[\epsilon]}(-[\epsilon], J) = \operatorname{Hom}_{\mathcal{C}[\epsilon]}(-, J) \circ -[\epsilon],$$

where $\operatorname{Hom}_{\mathcal{C}[\epsilon]}(-, J)$ is exact by assumption. Define

$$\gamma : J \xrightarrow{\begin{pmatrix} \epsilon_J \\ 1 \end{pmatrix}} J \oplus J = FJ[\epsilon].$$

A routine calculation shows that γ commutes with the differentials involved, so the only piece missing from a complete proof is a demonstration of why γ is in fact an inflation in $\mathcal{C}[\epsilon]$. To this end, turn to the canonical conflation

$$J \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} J \oplus J \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} J$$

in $\mathcal{C}$. Defining $\hat{J}$ as the underlying object J equipped with the differential $\epsilon_{\hat{J}} := -\epsilon_J$,[3] we obtain

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[2] We introduce $\hat{Q}$ because $\hat{Q} \xrightarrow{\begin{pmatrix} -\epsilon_Q \\ 1 \end{pmatrix}} Q \oplus Q$ is a morphism in $\mathcal{C}[\epsilon]$ while $Q \xrightarrow{\begin{pmatrix} -\epsilon_Q \\ 1 \end{pmatrix}} Q \oplus Q$ is not.

[3] We introduce $\hat{J}$ because $J \oplus J \xrightarrow{\begin{pmatrix} -1 & \epsilon_J \end{pmatrix}} \hat{J}$ is a morphism in $\mathcal{C}[\epsilon]$ while $J \oplus J \xrightarrow{\begin{pmatrix} -1 & \epsilon_J \end{pmatrix}} J$ is not.

$$J \xrightarrow{\left(\begin{smallmatrix}\epsilon_J\\ 1\end{smallmatrix}\right)} J \oplus J \xrightarrow{\left(\begin{smallmatrix}-1 & \epsilon_J\end{smallmatrix}\right)} \hat{J}$$

which is an exact pair in $\mathcal{C}[\epsilon]$. Again, differentials are forgotten when we pass to $\mathcal{C}$, so the latter pair is even a conflation since the diagram

$$\begin{tikzcd}
J \arrow[r, "\left(\begin{smallmatrix}1\\ 0\end{smallmatrix}\right)"] \arrow[d, "1"] & J \oplus J \arrow[r, "\left(\begin{smallmatrix}0 & 1\end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix}\epsilon_J & -1\\ 1 & 0\end{smallmatrix}\right)"] & J \arrow[d, "1"] \\
J \arrow[r, "\left(\begin{smallmatrix}\epsilon_J\\ 1\end{smallmatrix}\right)"] & J \oplus J \arrow[r, "\left(\begin{smallmatrix}-1 & \epsilon_J\end{smallmatrix}\right)"] & J
\end{tikzcd}$$

is an isomorphism of exact pairs in $\mathcal{C}$. □

We end the chapter with showing that $\mathcal{C}[\epsilon]$ is Frobenius whenever $\mathcal{C}$ is. This should not be surprising as the Frobenius properties are defined in terms of exact structure.

2.8 Lemma. *If $\mathcal{C}$ has enough projectives (injectives), then also $\mathcal{C}[\epsilon]$ has enough projectives (injectives).*

Proof. We only deal with the projectives, as the injectives can be handled dually. So take $(M, \epsilon_M) \in \mathcal{C}[\epsilon]$ and consider

$$FM[\epsilon] \xrightarrow{\left(\begin{smallmatrix}1 & \epsilon_M\end{smallmatrix}\right)} M$$

which is a deflation in $\mathcal{C}[\epsilon]$ by the proof of Proposition 2.7. Since $\mathcal{C}$ has enough injectives there is a deflation $p : P \to FM$ with $P \in \operatorname{Proj}\mathcal{C}$. But by Proposition 2.7 again, $P[\epsilon]$ belongs to $\operatorname{Proj}\mathcal{C}[\epsilon]$. Since the composition

$$P[\epsilon] \xrightarrow{p[\epsilon]} FM[\epsilon] \xrightarrow{\left(\begin{smallmatrix}1 & \epsilon_M\end{smallmatrix}\right)} M$$

is a deflation, $\mathcal{C}[\epsilon]$ has enough projectives. □

2.9 Corollary. *If $\mathcal{C}$ is Frobenius, then so is $\mathcal{C}[\epsilon]$.*

Proof. If $\operatorname{Proj}\mathcal{C} = \operatorname{Inj}\mathcal{C}$ then $\operatorname{Proj}\mathcal{C}[\epsilon] = \operatorname{Inj}\mathcal{C}[\epsilon]$ by Proposition 2.7. The remaining Frobenius property is Lemma 2.8. □

CHAPTER 3

The Ungraded Derived Category

In this chapter we shall introduce and try to understand the ungraded derived category of an algebra. In particular, we will see that this is a triangulated category. Throughout our presentation we will encounter parallels to the graded setup. Ungraded derived categories will appear frequently also in chapters to follow.

Remark. Replacing Mod ($\underline{\text{Mod}}$, Proj, Inj) by mod ($\underline{\text{mod}}$, proj, inj) throughout this chapter, each result remains valid. $\square$

3.1 A Rudimentary Definition

Although we shall later restrict to module categories, we define the ungraded derived category of an arbitrary abelian category $\mathcal{A}$. Consider the augmentation $\mathcal{A}[\epsilon]$ as described in Chapter 2. Since the objects in the latter category are equipped with differentials, they admit a notion of homology. Indeed, since any differential ϵ squares to zero there is a canonical monomorphism $\operatorname{Im}\epsilon \to \operatorname{Ker}\epsilon$ in $\mathcal{A}$.

Definition. Let $(A, \epsilon_A) \in \mathcal{A}[\epsilon]$. The **homology** $H(A) \in \mathcal{A}$ of A is the cokernel of the canonical monomorphism $\operatorname{Im}\epsilon_A \to \operatorname{Ker}\epsilon_A$. If $H(A) = 0$ we say that A is **acyclic**.

Homology even defines a functor $H : \mathcal{A}[\epsilon] \to \mathcal{A}$ as one would expect. Given $f : A \to B$ in $\mathcal{A}[\epsilon]$, the square is commutative in the diagram

$$\begin{array}{ccccc} \operatorname{Im}\epsilon_A & \longrightarrow & \operatorname{Ker}\epsilon_A & \longrightarrow & H(A) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \\ \operatorname{Im}\epsilon_B & \longrightarrow & \operatorname{Ker}\epsilon_B & \longrightarrow & H(B) \end{array}$$

in $\mathcal{A}$, where we by abuse of notation have written f instead of its restriction. By the cokernel property of $H(A)$ there is a unique morphism $H(A) \to H(B)$ completing the diagram, which we define to be Hf.

Definition. A morphism f in $\mathcal{A}[\epsilon]$ is called a **quasi-isomorphism** if Hf is an isomorphism.

Denote by S the class of quasi-isomorphisms in $\mathcal{A}[\epsilon]$. The **ungraded derived category** of $\mathcal{A}$ is

$$D_{\text{ung}}(\mathcal{A}) := S^{-1}\mathcal{A}[\epsilon].$$

The analogy between the ungraded and the 'ordinary' derived category should be clear. Recall that in the ordinary setup, the derived category is the localization of the category of complexes with respect to quasi-isomorphisms, but may also be viewed as a localization of the homotopy category. One often prefers the latter approach, since the homotopy category itself is triangulated and therefore reveals information about a triangulated structure on the derived category. As we shall see, also this phenomenon has an analogue in the ungraded setup.

So what we seem to need is a homotopy category with a triangulated structure. Our strategy will be to construct a Frobenius category and consider its stable category. Then we show that this stable category conicides with what should morally be the notion of a homotopy category in the ungraded setup.

REMARK. At this point the alert reader will raise the question of whether $D_{\text{ung}}(\mathcal{A})$ is even a category. It is not immediate that $\operatorname{Hom}_{D_{\text{ung}}(\mathcal{A})}(A, B)$ is a set. At least in the case of $\mathcal{A}$ being a module category we will rid ourselves of this concern in Chapter 5, when we show that $D_{\text{ung}}(\operatorname{Mod}\Lambda)$ is equivalent to a particular category. □

3.2 Obtaining a Frobenius Category

$\operatorname{Mod}\Lambda$, being an abelian category, comes with a canonical exact structure. Other exact structures may be imposed, though, for instance the trivial one, where the conflations are the split exact pairs. I.e. the sequence

$$0 \to A \to B \to C \to 0$$

of Λ-modules and -homomorphisms is to be considered exact only if it splits.

3.1 LEMMA. *The trivial structure on* $\operatorname{Mod}\Lambda$ *is an exact structure.*

PROOF. Each axiom in the definition of exact structures is easily verified. □

At this point we should note that the augmentation $(\operatorname{Mod}\Lambda)[\epsilon]$ coincides with the category $\operatorname{Mod}\Lambda[\epsilon]$ of modules over the ring of dual numbers $\Lambda[\epsilon] = \Lambda[X]/(X^2)$. Indeed, to an object $(M, \epsilon_M) \in (\operatorname{Mod}\Lambda)[\epsilon]$, associate the $\Lambda[\epsilon]$-module M with $(\lambda_1 + \lambda_2\epsilon)m := \lambda_1 m + \lambda_2 \epsilon_M(m)$. Conversely, given a $\Lambda[\epsilon]$-module M we let the associated object in $(\operatorname{Mod}\Lambda)[\epsilon]$ be (M, ϵ_M) where $\epsilon_M(m) := \epsilon m$. These constructions are clearly mutually inverse.

By Lemma 2.2 the trivial exact structure on $\operatorname{Mod}\Lambda$ induces an exact structure on $\operatorname{Mod}\Lambda[\epsilon]$. This is given by letting the sequence

$$0 \to A \to B \to C \to 0$$

in $\operatorname{Mod}\Lambda[\epsilon]$ be a conflation if and only if it is split exact when restricted to a sequence in $\operatorname{Mod}\Lambda$. From now on, this will be the exact structure we equip $\operatorname{Mod}\Lambda[\epsilon]$ with. As the next proposition shows, there are severe ramifications.

3.2 PROPOSITION. $\operatorname{Mod}\Lambda[\epsilon]$ *is a Frobenius category.*

Proof. By Corollary 2.9 it suffices to show that $\operatorname{Mod}\Lambda$ with the trivial exact structure is Frobenius. But since every conflation in $\operatorname{Mod}\Lambda$ splits, each Λ-module is both projective and injective. $\square$

Hence, by [Hap88], the stable category $\underline{\operatorname{Mod}}\,\Lambda[\epsilon]$ is triangulated. We are able to give an explicit description of its translation functor.

3.3 Lemma. *Let $M = (M, \epsilon_M) \in \underline{\operatorname{Mod}}\,\Lambda[\epsilon]$. Then $\Sigma M = (M, -\epsilon_M)$. For a morphism $f : M \to N$ in $\underline{\operatorname{Mod}}\,\Lambda[\epsilon]$, we have $\Sigma f = f$.*

Proof. We know from Chapter 2 that

$$i : M \to M[\epsilon]$$

given by $m \mapsto (\epsilon_M(m), m)$ is an inflation in $\operatorname{Mod}\Lambda[\epsilon]$ where $M[\epsilon] = FM[\epsilon]$ is injective. If we let $\Sigma M = (M, -\epsilon_M)$ then the sequence

$$0 \to M \xrightarrow{i} M[\epsilon] \xrightarrow{p} \Sigma M \to 0$$

is easily seen to be a conflation for $p : (m_1, m_2) \mapsto m_1 - \epsilon_M(m_2)$. By the definition of the translation functor in $\underline{\operatorname{Mod}}\,\Lambda[\epsilon]$ this proves the first statement of the lemma.

For $f : M \to N$ recall that Σf is given by the cokernel morphism in

$$\begin{tikzcd} M \arrow[r, "i_M"] \arrow[d, "f"] & M[\epsilon] \arrow[r, "p_M"] \arrow[d, "\tilde{f}"] & \Sigma M \arrow[d, "\Sigma f"] \\ N \arrow[r, "i_N"] & N[\epsilon] \arrow[r, "p_N"] & \Sigma N \end{tikzcd}$$

where $\tilde{f} : M[\epsilon] \to N[\epsilon]$ is any $\Lambda[\epsilon]$-morphism making the left hand square commutative. It is straight forward to verify that such an $\tilde{f}$ is given by the matrix

$$\begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}.$$

Now, for any $m \in \Sigma M$, we know that $\Sigma f(m) = p_N \tilde{f} p_M^{-1}(m)$ where $p_M^{-1}(m)$ is any choice of a pre-image of m, for instance $p_M^{-1}(m) = (m, 0)$. Hence

$$\Sigma f(m) = p_N \tilde{f}((m, 0)) = p_N((f(m), 0)) = f(m),$$

completing the proof. $\square$

Remark. It immediately follows that Σ^2 is the identity on $\underline{\operatorname{Mod}}\,\Lambda[\epsilon]$, i.e. $\Sigma^{-1} = \Sigma$. $\square$

REMARK. An equally evident consequence is that translation does not affect homology. More precisely, $H(M) = H(\Sigma M)$ for each $M \in \underline{\mathrm{Mod}}\, \Lambda[\epsilon]$ and $Hf = H\Sigma f$ for any $\Lambda[\epsilon]$-homomorphism f. □

Before we move on to ungraded homotopy, let us see why the seemingly self-evident statement 'homology is a homological functor' is true in $\underline{\mathrm{Mod}}\, \Lambda[\epsilon]$.

3.4 LEMMA. *$H : \underline{\mathrm{Mod}}\, \Lambda[\epsilon] \to \mathrm{Mod}\, \Lambda$ is a homological functor.*

PROOF. The strategy is to show that $H(-) \cong \underline{\mathrm{Hom}}_{\Lambda[\epsilon]}(\Lambda, -)$. This will suffice, since Hom-functors are allways homological (Lemma 1.2). So let M be any $\Lambda[\epsilon]$-module.

First, we claim that $\mathrm{Hom}_{\Lambda[\epsilon]}(\Lambda, M)$ can be identified with the kernel of ϵ_M. To see why this is true, note that if $m \in \mathrm{Ker}\, \epsilon_M$ then the associated $f : \Lambda \to M$ determined by $1 \mapsto m$ belongs to $\mathrm{Hom}_{\Lambda[\epsilon]}(\Lambda, M)$. Indeed, for any $\lambda \in \Lambda$, $f\epsilon_\Lambda(\lambda) = 0$ since Λ is endowed with the trivial differential when we view it as a $\Lambda[\epsilon]$-module, while $\epsilon_M f(\lambda) = \epsilon_M \lambda f(1) = \lambda \epsilon_M(m) = 0$ by assumption. Conversely, for any $g \in \mathrm{Hom}_{\Lambda[\epsilon]}(\Lambda, M)$, the associated $m' := g(1)$ is annihilated by ϵ_M since $\epsilon_M(m') = \epsilon_M g(1) = g(\epsilon_\Lambda \cdot 1) = g(0) = 0$. We have established

$$\mathrm{Hom}_{\Lambda[\epsilon]}(\Lambda, M) = \mathrm{Ker}\, \epsilon_M,$$

proving the claim.

We proceed to observe that a morphism $f : \Lambda \to M$ in $\mathrm{Mod}\, \Lambda[\epsilon]$ factors through an injective object if and only if it factors through the injective hull of Λ, i.e.

$$\phi : \Lambda \to \Lambda[\epsilon]$$

given by $\lambda \mapsto (0, \lambda)$. Obviously, only one implication needs to be shown. So assume f factors as $\Lambda \xrightarrow{\alpha} I \to M$. By injectivity of I there is some $\gamma : \Lambda[\epsilon] \to I$ making

$$\begin{tikzcd}
0 \arrow[r] & \Lambda \arrow[r, "\phi"] \arrow[d, "\alpha"'] & \Lambda[\epsilon] \arrow[dl, "\gamma"] \\
 & I &
\end{tikzcd}$$

commutative, which means that f factors through ϕ. This observation enables yet another description of the morphisms $\Lambda \to M$ factoring through an injective, namely as the image of ϵ_M. To verify this assertion, start with some $f : \Lambda \to M$ factoring through an injective. Then it factors through ϕ, i.e. there is a commutative diagram

$$\begin{tikzcd}
\Lambda \arrow[rr, "f"] \arrow[dr, "\phi"'] & & M \\
 & \Lambda[\epsilon] \arrow[ur, "g"'] &
\end{tikzcd}$$

in $\operatorname{Mod}\Lambda[\epsilon]$. It now follows that $m := f(1)$ belongs to $\operatorname{Im}\epsilon_M$ since $m = f(1) = g\phi(1) = g(0,1) = g\epsilon_{\Lambda[\epsilon]}(1,0) = \epsilon_M g(1,0)$. On the other hand, take some arbitrary $\epsilon_M(x) \in \operatorname{Im}\epsilon_M$. Then the $\Lambda[\epsilon]$-homomorphism (easily verified) $f : \Lambda \to M$ determined by $1 \mapsto \epsilon_M(x)$ certainly factors through ϕ. Indeed, consider the $\Lambda[\epsilon]$-linear (also easy to check)

$$\psi : \Lambda[\epsilon] \to M$$

given by $(\lambda_1, \lambda_2) \mapsto \lambda_1 x + \lambda_1\epsilon_M(x)$. This has the property $f = \psi\phi$, furnishing a proof of the fact that

$$\{f : \Lambda \to M\colon\ f \text{ factors through an injective}\} = \operatorname{Im}\epsilon_M.$$

In total we have shown

$$\begin{aligned}\underline{\operatorname{Hom}}_{\Lambda[\epsilon]}(\Lambda, M) &:= \frac{\operatorname{Hom}_{\Lambda[\epsilon]}(\Lambda, M)}{\{f : \Lambda \to M\colon\ f \text{ factors through an injective}\}} \\ &= \operatorname{Ker}\epsilon_M / \operatorname{Im}\epsilon_M \\ &= H(M)\end{aligned}$$

as sought. □

3.3 Comparing The Stable Category to The Homotopy Category

As one would expect, if $\mathcal{C}$ is an additive category then the notion of homotopy in $\mathcal{C}[\epsilon]$ is given by the following definition.

Definition. A morphism $f : A \to B$ in $\mathcal{C}[\epsilon]$ is **nullhomotopic** if there is a morphism $s : A \to B$ in $\mathcal{C}$ such that $f = \epsilon_B s + s\epsilon_A$. Two parallel morphisms $f, g : A \to B$ in $\mathcal{C}[\epsilon]$ are called **homotopic**, denoted by $f \sim g$, if $f - g$ is nullhomotopic.

The following lemma should not come as a surprise.

3.5 Lemma. *Given a pair of composable morphisms*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in $\mathcal{C}[\epsilon]$, if either $f \sim 0$ or $g \sim 0$ then also the composition $gf \sim 0$.

Proof. Assume $f \sim 0$, so there is some $s : A \to B$ in $\mathcal{C}$ with the property $f = \epsilon_B s + s\epsilon_A$. Letting $\tilde{s} := gs$ we get $gf = g(\epsilon_B s + s\epsilon_A) = g\epsilon_B s + gs\epsilon_A = \epsilon_C gs + gs\epsilon_A = \epsilon_C\tilde{s} + \tilde{s}\epsilon_A$, i.e. $gf \sim 0$.

Assume now that $g \sim 0$, meaning $g = \epsilon_C r + r\epsilon_B$ for some $r : B \to C$ in $\mathcal{C}$. Then $\tilde{r} := rf$ does the job as $gf = (\epsilon_C r + r\epsilon_B)f = \epsilon_C rf + r\epsilon_B f = \epsilon_C rf + rf\epsilon_A = \epsilon_C\tilde{r} + \tilde{r}\epsilon_A$. This means $gf \sim 0$. □

Lemma 3.5 ensures that the following construction actually gives a category.

DEFINITION. The **homotopy category** $K_{\text{ung}}(\mathcal{C})$ of $\mathcal{C}[\epsilon]$ has the objects of $\mathcal{C}[\epsilon]$ and morphism spaces given by

$$\operatorname{Hom}_{K_{\text{ung}}(\mathcal{C})}(A,B) := \operatorname{Hom}_{\mathcal{C}[\epsilon]}(A,B)/\sim\ .$$

Recall that we are equipping $\operatorname{Mod}\Lambda[\epsilon]$ with the exact structure induced by the trivial structure on $\operatorname{Mod}\Lambda$. This makes the following lemma possible.

3.6 LEMMA. *Let $(A,\epsilon_A) \in \operatorname{Mod}\Lambda[\epsilon]$. Then $1_A \sim 0$ if and only if A is projective.*

PROOF. Assume there is some $s : A \to A$ such that $1_A = \epsilon_A s + s\epsilon_A$. Take the Λ-module $Y := \operatorname{Im}\epsilon_A$ and define

$$\phi : A \to Y[\epsilon]$$

by $a \mapsto (\epsilon_A(a), \epsilon_A s(a))$. Note that ϕ is a morphism in $\operatorname{Mod}\Lambda[\epsilon]$ since

$$\phi\epsilon_A(a) = (0, \epsilon_A s\epsilon_A(a)) = (0, \epsilon_A(1_A - \epsilon_A s)(a)) = (0, \epsilon_A(a)) = \epsilon_{Y[\epsilon]}\phi(a).$$

First, assume $\phi(a) = 0$. This means $\epsilon_A(a) = 0 = \epsilon_A s(a)$, which yields $a = 1_A(a) = \epsilon_A s(a) + s\epsilon_A(a) = 0$, i.e. ϕ is a monomorphism. Next, take any $b \in Y$. Then $b = \epsilon_A(a)$ for some $a \in A$, hence

$$b = (\epsilon_A s + s\epsilon_A)\epsilon_A(a) = \epsilon_A s\epsilon_A(a) = \epsilon_A s(b)$$

while

$$\epsilon_A ss(b) = \epsilon_A ss\epsilon_A(a) = (1_A - s\epsilon_A)(1_A - \epsilon_A s)(a) = (1_A - (s\epsilon_A + \epsilon_A s)(a) = 0.$$

This means

$$\phi(b) = (\epsilon_A(b), \epsilon_A s(b)) = (0, b)$$

and

$$\phi(s(b)) = (\epsilon_A s(b), \epsilon_A ss(b)) = (b, 0),$$

i.e. ϕ is an epimorphism, hence an isomorphism, establishing projectivity of A (since $\operatorname{Proj}\Lambda[\epsilon] = (\operatorname{Mod}\Lambda)[\epsilon]$ by the proof of Proposition 3.2).

For the remaining implication, assume A is projective. Consider the morphism

$$p : A[\epsilon] \to A$$

defined by $(a_1, a_2) \mapsto a_1 + \epsilon_A(a_2)$, which we now know is a deflation in $\mathrm{Mod}\,\Lambda[\epsilon]$. By projectivity of A, p splits (this time over $\Lambda[\epsilon]$). Let q be the $\Lambda[\epsilon]$-linear splitting of p. Of course, we may write

$$q = \begin{pmatrix} t \\ s \end{pmatrix}$$

for $t, s \in \mathrm{End}_\Lambda(A)$. We claim that s gives the desired homotopy $1_A \sim 0$. To see why this is the case we exploit the fact that q commutes with the differentials involved to obtain, for any $a \in A$,

$$(t\epsilon_A(a), s\epsilon_A(a)) = q\epsilon_A(a) = \epsilon_{A[\epsilon]}q(a) = (0, t(a))$$

implying $t(a) = s\epsilon_A(a)$. Combined with the property $pq = 1_A$ this yields

$$a = pq(a) = t(a) + \epsilon_A s(a) = s\epsilon_A(a) + \epsilon_A s(a)$$

as required. □

In Section 3.1 we alluded to the fact that the stable category $\underline{\mathrm{Mod}}\,\Lambda[\epsilon]$ and the homotopy category $K_{\mathrm{ung}}(\mathrm{Mod}\,\Lambda)$ are the same. We are now up to the task of giving a proof.

3.7 Proposition. *The categories* $\underline{\mathrm{Mod}}\,\Lambda[\epsilon]$ *and* $K_{\mathrm{ung}}(\mathrm{Mod}\,\Lambda)$ *coincide.*

Proof. Both categories in question coincide with $\mathrm{Mod}\,\Lambda[\epsilon]$ on objects, so it is sufficient to show that a morphism $f : A \to B$ in $\mathrm{Mod}\,\Lambda[\epsilon]$ is nullhomotopic if and only if it factors through a projective $\Lambda[\epsilon]$-module.

For the 'if' part, assume the diagram

$$\begin{tikzcd} & P \arrow[dr, "\beta"] & \\ A \arrow[ur, "\alpha"] \arrow[rr, "f"] & & B \end{tikzcd}$$

is commutative with $P \in \mathrm{Proj}\,\Lambda[\epsilon]$. By Lemma 3.6 1_P is nullhomotopic. By Lemma 3.5 this means that also $\alpha = 1_P\alpha$ is nullhomotopic. Yet another application of Lemma 3.5 gives that $f = \beta\alpha$ is nullhomotopic.

Conversely, assume there is some Λ-linear $s : A \to B$ such that $f = \epsilon_B s + s\epsilon_A$. The natural candidate for f to factor through is the projective $\Lambda[\epsilon]$-module $B[\epsilon] = FB[\epsilon]$ along with

$$p : B[\epsilon] \to B$$

given by $(b_1, b_2) \mapsto b_1 + \epsilon_B(b_2)$. The proof will be complete if we can find a morphism $\gamma : A \to B[\epsilon]$ in $\mathrm{Mod}\,\Lambda[\epsilon]$ making

$$\begin{tikzcd} & B[\epsilon] \arrow[dr, "p"] & \\ A \arrow[ur, "\gamma"] \arrow[rr, "f"] & & B \end{tikzcd}$$

commutative. It is a straight forward calculation to verify that

$$\gamma := \begin{pmatrix} s\epsilon_A \\ s \end{pmatrix}$$

is $\Lambda[\epsilon]$-linear. Further, γ does make the above triangle commutative. Indeed, for any $a \in A$,

$$p\gamma(a) = s\epsilon_A(a) + \epsilon_B s(a) = f(a)$$

which completes the proof. □

3.4 Computing Cones

In the graded setup there is the concept of the mapping cone of a morphism of complexes, yielding essentially all distinguished triangles in the homotopy category. The first result of this section shows how this translates neatly to the ungraded setup. The natural notion of an ungraded mapping cone is the following.

Definition. Let $f : A \to B$ be a morphism of $\Lambda[\epsilon]$-modules. The **mapping cone** of f is the $\Lambda[\epsilon]$-module

$$M_f := (A \oplus B, \left(\begin{smallmatrix} -\epsilon_A & 0 \\ f & \epsilon_B \end{smallmatrix}\right)).$$

Recall that cones in $K_{\text{ung}}(\operatorname{Mod}\Lambda)$ are given by finding an inflation and constructing a pushout. This seems like a bit of work, at least compared to the simple construction of a mapping cone. The following lemma makes life easier.

3.8 Lemma. *The cone of a $\Lambda[\epsilon]$-morphism $f : A \to B$ is given by its mapping cone. In particular, the standard triangle associated to f is*

$$A \xrightarrow{f} B \xrightarrow{\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)} M_f \xrightarrow{(1\ 0)} \Sigma A.$$

Proof. To show that M_f is the cone of f it suffices to verify that

$$
\begin{tikzcd}
A \arrow[r, "\left(\begin{smallmatrix}\epsilon_A \\ 1\end{smallmatrix}\right)"] \arrow[d, "f"] & A[\epsilon] \arrow[d, "\hat{f} := \left(\begin{smallmatrix}1 & -\epsilon_A \\ 0 & f\end{smallmatrix}\right)"] \\
B \arrow[r, "\left(\begin{smallmatrix}0 \\ 1\end{smallmatrix}\right)"] & M_f
\end{tikzcd}
$$

is a pushout. A straight forward calculation shows that the 2×2-matrix is a $\Lambda[\epsilon]$-morphism, and another one shows that the square is commutative. We are left to check the universal property. So take some X with $\Lambda[\epsilon]$-linear $\begin{pmatrix}\alpha_1 & \alpha_2\end{pmatrix} : A[\epsilon] \to X$ and $\beta : B \to X$ such that $\alpha_1 \epsilon_A + \alpha_2 = \beta f$.

$$
\begin{tikzcd}
A \arrow[r, "\left(\begin{smallmatrix}\epsilon_A \\ 1\end{smallmatrix}\right)"] \arrow[d, "f"] & A[\epsilon] \arrow[d, "\hat{f}"] \arrow[ddr, bend left, "\left(\begin{smallmatrix}\alpha_1 & \alpha_2\end{smallmatrix}\right)"] & \\
B \arrow[r, "\left(\begin{smallmatrix}0 \\ 1\end{smallmatrix}\right)"] \arrow[drr, bend right, "\beta"'] & M_f & \\
& & X
\end{tikzcd}
$$

One easily checks that $\omega := \begin{pmatrix}\alpha_1 & \beta\end{pmatrix} : M_f \to X$ is $\Lambda[\epsilon]$-linear. Also, ω clearly makes all commutative and is unique with this property.

The second claim of the lemma is now immediate. Indeed, the standard triangle associated to f is by definition

$$A \xrightarrow{f} B \xrightarrow{\left(\begin{smallmatrix}0 \\ 1\end{smallmatrix}\right)} M_f \xrightarrow{\omega} \Sigma A$$

where ω is the unique morphism making all commutative in

$$
\begin{tikzcd}
A \arrow[r, "\left(\begin{smallmatrix}\epsilon_A \\ 1\end{smallmatrix}\right)"] \arrow[d, "f"] & A[\epsilon] \arrow[d, "\hat{f}"] \arrow[ddr, bend left, "\left(\begin{smallmatrix}1 & -\epsilon_A\end{smallmatrix}\right)"] & \\
B \arrow[r, "\left(\begin{smallmatrix}0 \\ 1\end{smallmatrix}\right)"] \arrow[drr, bend right, "0"'] & M_f & \\
& & \Sigma A
\end{tikzcd}
$$

By the above, this is given by $\omega = \begin{pmatrix}1 & 0\end{pmatrix}$. □

Recall that in the setup of complexes any short exact sequence that is split in each degree (i.e. the splitting maps need not constitute maps of complexes) embeds in a distinguished triangle in the homotopy category. The next result, which is a rather easy consequence of Lemma 3.8, shows that also this translates in a natural way to the ungraded setup. Indeed, the concept of 'split in each degree' should correspond to 'split when restricted to Λ'.

3.9 Corollary. *Each conflation in* $\operatorname{Mod}\Lambda[\epsilon]$ *embeds in a distinguished triangle in the homotopy category* $K_{\text{ung}}(\operatorname{Mod}\Lambda)$.

Proof. Any conflation in $\operatorname{Mod}\Lambda[\epsilon]$ is a split exact sequence when restricted to $\operatorname{Mod}\Lambda$, so it is of the form

$$0 \to A \xrightarrow{\binom{0}{1}} C \oplus A \xrightarrow{(1\ 0)} C \to 0.$$

In order for the morphisms appearing to be $\Lambda[\epsilon]$-linear, one easily checks that the differential of $C \oplus A$ must be a matrix of the form $\begin{pmatrix} \epsilon_C & 0 \\ f & \epsilon_A \end{pmatrix}$ where $f : C \to A$ is some Λ-homomorphism. Since the matrix must square to zero we get the additional requirement $f\epsilon_C + \epsilon_A f = 0$. This means $f : \Sigma C \to A$ is even a $\Lambda[\epsilon]$-homomorphism, as $f\epsilon_{\Sigma C} = -f\epsilon_C = \epsilon_A f$ by Lemma 3.3. Since $\Sigma^2 C = C$ the standard triangle associated to f is

$$\Sigma C \xrightarrow{f} A \xrightarrow{\binom{0}{1}} M_f \xrightarrow{(1\ 0)} C$$

where M_f is the underlying Λ-module $\Sigma C \oplus A$ with differential $\begin{pmatrix} -\epsilon_{\Sigma C} & 0 \\ f & \epsilon_A \end{pmatrix}$. Of course, since $-\epsilon_{\Sigma C} = \epsilon_C$, M_f is precisely the $\Lambda[\epsilon]$-module $C \oplus A$ appearing in the above conflation, i.e.

$$\Sigma C \xrightarrow{f} A \xrightarrow{\binom{0}{1}} C \oplus A \xrightarrow{(1\ 0)} C$$

is a distinguished triangle, completing the proof. □

3.5 The Triangulated Structure of $D_{\text{ung}}(\operatorname{Mod}\Lambda)$

Recall that we took $D_{\text{ung}}(\operatorname{Mod}\Lambda)$ as $S^{-1}\operatorname{Mod}\Lambda[\epsilon]$ where S denotes the class of quasi-isomorphisms in $\operatorname{Mod}\Lambda[\epsilon]$, but hinted that there might be a 'better' way of defining the ungraded derived category. The most important reason why the initial definition is sub optimal, is that it gives no reason for $D_{\text{ung}}(\operatorname{Mod}\Lambda)$ to be triangulated. The improvement we have in mind is localizing with respect to the class T of quasi-isomorphisms in $\underline{\operatorname{Mod}}\,\Lambda[\epsilon]$ instead.

Our first concern should be whether $S^{-1}\operatorname{Mod}\Lambda[\epsilon]$ and $T^{-1}\,\underline{\operatorname{Mod}}\,\Lambda[\epsilon]$ are even linked. As the reader would suspect, the answer is a resounding 'yes'. Indeed, denoting by Q_S and Q_T the respective localization functors we have the following.

3.10 Lemma. *The canonical projection* $P : \operatorname{Mod}\Lambda[\epsilon] \to \underline{\operatorname{Mod}}\,\Lambda[\epsilon]$ *induces a unique equivalence* $\tilde{P}$ *making the following diagram commutative.*

$$\begin{tikzcd}
\mathrm{Mod}\,\Lambda[\epsilon] \arrow[r, "Q_S"] \arrow[d, "P"] & S^{-1}\,\mathrm{Mod}\,\Lambda[\epsilon] \arrow[d, "\tilde{P}"] \\
\underline{\mathrm{Mod}}\,\mathit{\Lambda}[\epsilon] \arrow[r, "Q_T"] & T^{-1}\,\underline{\mathrm{Mod}}\,\mathit{\Lambda}[\epsilon]
\end{tikzcd}$$

Proof. It is clear that $S = P^{-1}(T)$. Consequently, $Q_T P$ makes each $s \in S$ invertible and hence factors uniquely through Q_T via $\tilde{P} : S^{-1}\,\mathrm{Mod}\,\Lambda[\epsilon] \to T^{-1}\,\underline{\mathrm{Mod}}\,\mathit{\Lambda}[\epsilon]$. It is easy to check that homotopic $\Lambda[\epsilon]$-morphisms induce the same map on homology modules. Since $\underline{\mathrm{Mod}}\,\mathit{\Lambda}[\epsilon] = K_{\mathrm{ung}}(\mathrm{Mod}\,\Lambda)$ this means that Q_S factors through P via some functor $G : \underline{\mathrm{Mod}}\,\mathit{\Lambda}[\epsilon] \to S^{-1}\,\mathrm{Mod}\,\Lambda[\epsilon]$. Now G makes each $t \in T$ invertible and hence factors uniquely through Q_T via

$$\tilde{G} : T^{-1}\,\underline{\mathrm{Mod}}\,\mathit{\Lambda}[\epsilon] \to S^{-1}\,\mathrm{Mod}\,\Lambda[\epsilon].$$

We claim that $\tilde{P}$ and $\tilde{G}$ are mutually inverse. First, notice that $\tilde{P}\tilde{G}Q_T = \tilde{P}G = Q_T$. This means Q_T factors through both the identity functor and $\tilde{P}\tilde{G}$, so $\tilde{P}\tilde{G} = \mathrm{id}$ by the universal property. Second, $\tilde{G}\tilde{P}Q_S = \tilde{G}Q_T P = GP = Q_S$. So, using universality again, Q_S factoring through both the identity functor and $\tilde{G}\tilde{P}$ implies $\tilde{G}\tilde{P} = \mathrm{id}$. □

Hence, we could just as well take $D_{\mathrm{ung}}(\mathrm{Mod}\,\Lambda) := T^{-1}\,\underline{\mathrm{Mod}}\,\mathit{\Lambda}[\epsilon]$.

3.11 Lemma. *T is closed under Σ^i for each $i \in \mathbb{Z}$.*

Proof. Take $A \xrightarrow{t} B$ in T. Since $\Sigma^{-1} = \Sigma$ by Lemma 3.3, it suffices to show $\Sigma t \in T$. But this is immediate from the same lemma. □

So Lemma 1.6 tells us that T is a multiplicative system compatible with the triangulation in $\underline{\mathrm{Mod}}\,\mathit{\Lambda}[\epsilon]$, and we suddenly know a whole lot about $D_{\mathrm{ung}}(\mathrm{Mod}\,\Lambda)$. Indeed, the following corollary is now an immediate consequence of Lemma 1.5 and its proof.

3.12 Corollary. *$D_{\mathrm{ung}}(\mathrm{Mod}\,\Lambda)$ is a triangulated category. Its distinguished triangles are those isomorphic to the image of some distinguished triangle in $\underline{\mathrm{Mod}}\,\mathit{\Lambda}[\epsilon]$ under the localization functor Q_T, which becomes a triangle functor. Our results on the translation functor and cones in $\underline{\mathrm{Mod}}\,\mathit{\Lambda}[\epsilon]$ carry over to $D_{\mathrm{ung}}(\mathrm{Mod}\,\Lambda)$.* □

Further, by [GZ67], $D_{\mathrm{ung}}(\mathrm{Mod}\,\Lambda)$ admits a calculus of roofs.

Remark. Equivalently, $D_{\mathrm{ung}}(\mathrm{Mod}\,\Lambda)$ is the Verdier localization $\underline{\mathrm{Mod}}\,\mathit{\Lambda}[\epsilon]\,/\mathcal{S}$ where $\mathcal{S}$ denotes the thick triangulated subcategory of acyclic $\Lambda[\epsilon]$-modules. □

CHAPTER 4

The Density of Certain Functors

Given an additive category $\mathcal{C}$ with countable coproducts, there is a natural functor from the category of complexes of objects in $\mathcal{C}$ to $\mathcal{C}[\epsilon]$ which we shall call the **$\oplus$-functor**. Given a complex

$$A = \cdots \to A_{i+1} \xrightarrow{d_{i+1}} A_i \xrightarrow{d_i} A_{i-1} \to \cdots$$

we let $\oplus A$ be the underlying object $\bigoplus\limits_{i\in\mathbb{Z}} A_i$ with differential ϵ_A given by

$$A_i \xrightarrow{d_i} A_{i-1} \hookrightarrow \oplus A.$$

on summands of $\oplus A$. To a morphism $f = \big(f_i\big) : A \to B$ of complexes we assign the obvious morphism $\oplus f : \oplus A \to \oplus B$ in $\mathcal{C}[\epsilon]$ whose components are

$$A_i \xrightarrow{f_i} B_i \hookrightarrow \oplus B.$$

It is easy to verify that the same construction gives a functor

$$\oplus : K(\mathcal{C}) \to K_{\mathrm{ung}}(\mathcal{C})$$

and even, given an abelian category $\mathcal{A}$, a functor

$$\oplus : D(\mathcal{A}) \to D_{\mathrm{ung}}(\mathcal{A}).$$

The result from this chapter that will be used later in the thesis is Corollary 4.8, stating that the restriction of the latter functor to $D^b(\operatorname{Mod}\Lambda)$ is dense whenever Λ is an iterated tilted algebra.

We are however able to prove more than this. Roughly speaking, we shall see that the existence of a standard equivalence between derived categories implies the existence of an equivalence on the level of ungraded derived categories making a certain diagram commutative (Proposition 4.3 and, by applying the very same argument, Corollary 4.7). This combined with an appealing structural property of the ungraded derived category of a hereditary abelian category (see Section 4.1) will give Corollary 4.8 as an immediate consequence.

4.1 $D_{\mathrm{ung}}(\mathcal{H})$ for Hereditary $\mathcal{H}$

Recall the following well known fact. If $\mathcal{H}$ is a hereditary abelian category then any object in $D(\mathcal{H})$ can be represented by a direct sum of shifts of objects in $\mathcal{H}$. To be more precise, in $D(\mathcal{H})$ there is an isomorphism

$$X \cong \bigoplus_n H^n(X)[n]$$

for each X. We shall see that a similar statement is true about $D_{\mathrm{ung}}(\mathcal{H})$.

4.1 Lemma. *Let $\mathcal{H}$ be a hereditary abelian category. Then any $(M, \epsilon_M) \in \mathcal{H}[\epsilon]$ is isomorphic in $D_{\mathrm{ung}}(\mathcal{H})$ to its homology $(H(M), 0)$.*

Proof. The lemma will be shown by constructing a third object in $\mathcal{H}[\epsilon]$ which is quasi-isomorphic to both M and $H(M)$.

The assumption on $\mathcal{H}$ means $\mathrm{Ext}^1_{\mathcal{H}}(X, -)$ is right exact for each $X \in \mathcal{H}$. So the exactness of

$$M \xrightarrow{\epsilon_M} \mathrm{Im}\,\epsilon_M \to 0$$

implies exactness of

$$\mathrm{Ext}^1_{\mathcal{H}}(H(M), M) \xrightarrow{\epsilon_{M*}} \mathrm{Ext}^1_{\mathcal{H}}(H(M), \mathrm{Im}\,\epsilon_M) \to 0.$$

This means that any extension of $\mathrm{Im}\,\epsilon_M$ by $H(M)$ is equivalent to the image under ϵ_{M*} of some extension of M by $H(M)$. In particular, there is a commutative diagram in $\mathcal{H}$ in which the rows are exact:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "i"] \arrow[d, "\epsilon_M"] & E \arrow[r, "p"] \arrow[d, "s"] & H(M) \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & \mathrm{Im}\,\epsilon_M \arrow[r] & \mathrm{Ker}\,\epsilon_M \arrow[r] & H(M) \arrow[r] & 0
\end{tikzcd} \tag{4.1}$$

The object in $\mathcal{H}[\epsilon]$ that saves the day is $(M \oplus E, \epsilon_\oplus := \left(\begin{smallmatrix} 0 & 0 \\ i & 0 \end{smallmatrix}\right))$.

To see why $M \oplus E$ and M are quasi-isomorphic, consider

$$\phi : M \oplus E \xrightarrow{(1\ s)} M$$

which is a morphism in $\mathcal{H}[\epsilon]$ by commutativity of the left hand square in diagram (4.1). Clearly, $0 \oplus \mathrm{Im}\,i$ is an image and $0 \oplus E$ is a kernel of $\epsilon_\oplus$. So $H\phi$ is given by the diagram

$$\begin{tikzcd}
0 \arrow[r] & 0 \oplus \mathrm{Im}\,i \arrow[r] \arrow[d, "(0\ s)"] & 0 \oplus E \arrow[r] \arrow[d, "(0\ s)"] & H(M \oplus E) \arrow[r] \arrow[d, "H\phi"] & 0 \\
0 \arrow[r] & \mathrm{Im}\,\epsilon_M \arrow[r] & \mathrm{Ker}\,\epsilon_M \arrow[r] & H(M) \arrow[r] & 0
\end{tikzcd}$$

However, this is just diagram (4.1) in disguise, which means that $H\phi$ is indeed an isomorphism.

Showing that $M \oplus E$ and $H(M)$ are quasi-isomorphic is perhaps easier, as both 'admit a grading'. I.e. $M \oplus E$ is the image of the complex

$$\cdots \to 0 \to M \xrightarrow{i} E \to 0 \to \cdots$$

under the $\oplus$-functor while $H(M)$ is the image of

$$\cdots \to 0 \to H(M) \to 0 \to \cdots .$$

There is an obvious quasi-isomorphism if complexes

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & M \arrow[r, "i"] \arrow[d] & E \arrow[r] \arrow[d, "p"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & H(M) \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

which means that their images under $\oplus$ are quasi-isomorphic via $M \oplus E \xrightarrow{(\,0\ p\,)} H(M)$. $\square$

We have essentially shown what was the aim of this section. For later reference, we give an explicit statement of an immediate consequence.

4.2 Proposition. *If $\mathcal{H}$ is a hereditary abelian category, then*

$$\oplus : D^b(\mathcal{H}) \to D_{\mathrm{ung}}(\mathcal{H})$$

is dense.

Proof. Take $(M, \epsilon_M) \in \mathcal{H}[\epsilon]$. By Lemma 4.1 any stalk complex in $D^b(\mathcal{H})$ with $H(M)$ in its non-vanishing degree will be sent by $\oplus$ to an object isomorphic in $D_{\mathrm{ung}}(\mathcal{H})$ to M. $\square$

4.2 Using Standard Equivalences

This section is devoted to proving the following.

4.3 Proposition. *Let Λ and Γ be algebras and T a complex of Γ-Λ-bimodules such that*

$$T \otimes^{\mathbf{L}}_{\Lambda} - : D(\operatorname{Mod}\Lambda) \to D(\operatorname{Mod}\Gamma)$$

is an equivalence. Then there is an equivalence

$$T\widetilde{\otimes}_\Lambda - : D_{\mathrm{ung}}(\operatorname{Mod}\Lambda) \to D_{\mathrm{ung}}(\operatorname{Mod}\Gamma)$$

making the diagram

$$\begin{tikzcd} D(\operatorname{Mod}\Lambda) \arrow[r, "T\otimes^{\mathbf{L}}_\Lambda -"] \arrow[d, "\oplus"] & D(\operatorname{Mod}\Gamma) \arrow[d, "\oplus"] \\ D_{\mathrm{ung}}(\operatorname{Mod}\Lambda) \arrow[r, "T\widetilde{\otimes}_\Lambda -"] & D_{\mathrm{ung}}(\operatorname{Mod}\Gamma) \end{tikzcd} \tag{4.2}$$

commutative.

The default strategy in the setup of bounded derived categories would be to first replace T by a projective resoultion. Essentially the same strategy works also when we face unbounded complexes, but we need the following notion.

Definition. A complex P of Λ-modules is called **homotopically projective** if

$$\operatorname{Hom}_{K(\operatorname{Mod}\Lambda)}(P,K) = 0$$

for each acyclic complex K.

The concept of homotopically projective resolutions extends that of (bounded) projective resolutions and hence provides a way for us to calculate total derived functors. By the work of Keller in [Kel98] there is a homotopically projective complex $\mathbf{p}\,T$ quasi-isomorphic to T such that

$$\operatorname{Hom}_{D(\operatorname{Mod}\Lambda)}(T,M) \cong \operatorname{Hom}_{K(\operatorname{Mod}\Lambda)}(\mathbf{p}\,T,M)$$

for each complex M, which means we can think of $\mathbf{p}\,T$ as a resolution of T.

In particular, we are justified in assuming that T is the homotopically projective complex

$$\cdots \to T_{i+1} \xrightarrow{d^T_{i+1}} T_i \xrightarrow{d^T_i} T_{i-1} \to \cdots .$$

Naturally, the first step towards proving Proposition 4.3 is defining the functor $T\widetilde{\otimes}_\Lambda -$. To this end, given some $(M,\epsilon_M) \in \operatorname{Mod}\Lambda[\epsilon]$, consider the diagram

$$\begin{tikzcd} \cdots \arrow[r] & T_{i+1}\otimes_\Lambda M \arrow[r, "d^T_{i+1}\otimes 1"] \arrow[d, "1\otimes\epsilon_M"] & T_i\otimes_\Lambda M \arrow[r, "d^T_i\otimes 1"] \arrow[d, "1\otimes\epsilon_M"] & T_{i-1}\otimes_\Lambda M \arrow[r] \arrow[d, "1\otimes\epsilon_M"] & \cdots \\ \cdots \arrow[r] & T_{i+1}\otimes_\Lambda M \arrow[r, "d^T_{i+1}\otimes 1"] & T_i\otimes_\Lambda M \arrow[r, "d^T_i\otimes 1"] & T_{i-1}\otimes_\Lambda M \arrow[r] & \cdots \end{tikzcd} \tag{4.3}$$

whose squares are clearly commutative. This leads us to defining

$$T\widetilde{\otimes}_\Lambda M = (T\widetilde{\otimes}_\Lambda -)(M) := \bigoplus_{i\in\mathbb{Z}}(T_i \otimes_\Lambda M)$$

with differential $\epsilon_{T\widetilde{\otimes}_\Lambda M}$ given on simple tensors by

$$t_i \otimes m \mapsto d_i^T(t_i) \otimes m + (-1)^i t_i \otimes \epsilon_M(m).$$

An easy calculation shows that $\epsilon^2_{T\widetilde{\otimes}_\Lambda M}$ vanishes because of commutativity of the squares in diagram (4.3). For a morphism $f : (M, \epsilon_M) \to (N, \epsilon_N)$ in $\operatorname{Mod}\Lambda[\epsilon]$ we let

$$1\widetilde{\otimes}_\Lambda f := (T\widetilde{\otimes}_\Lambda -)(f) : T\widetilde{\otimes}_\Lambda M \to T\widetilde{\otimes}_\Lambda N$$

be given by $\big(t_i \otimes m\big) \mapsto \big(t_i \otimes f(m)\big)$ which is clearly $\Gamma[\epsilon]$-linear. To show that this construction gives a well defined functor $D_{\mathrm{ung}}(\operatorname{Mod}\Lambda) \to D_{\mathrm{ung}}(\operatorname{Mod}\Gamma)$ it suffices to observe that $1\widetilde{\otimes}_\Lambda f$ is a quasi-isomorphism whenever f is.

The following lemma settles one part on Proposition 4.3.

4.4 Lemma. *The functor $T\widetilde{\otimes}_\Lambda - : D_{\mathrm{ung}}(\operatorname{Mod}\Lambda) \to D_{\mathrm{ung}}(\operatorname{Mod}\Gamma)$ makes diagram (4.2) commutative.*

Proof. Take some complex

$$X = \cdots \to X_{i+1} \xrightarrow{d^X_{i+1}} X_i \xrightarrow{d^X_i} X_{i-1} \to \cdots$$

in $D(\operatorname{Mod}\Lambda)$. Applying $T \otimes^{\mathbf{L}}_\Lambda -$ yields the complex

$$P = \cdots \to P_{i+1} \xrightarrow{d^P_{i+1}} P_i \xrightarrow{d^P_i} P_{i-1} \to \cdots$$

where $P_k = \bigoplus_{i+j=k} T_i \otimes_\Lambda X_j$ for each $k \in \mathbb{Z}$. Its differentials are given by

$$t_i \otimes x_j \mapsto d_i^T(t_i) \otimes x_j + (-1)^i t_i \otimes d_j^X(x_j).$$

Passing to $D_{\mathrm{ung}}(\operatorname{Mod}\Gamma)$ yields the $\Gamma[\epsilon]$-module

$$\oplus P = \bigoplus_{k\in\mathbb{Z}}\Big[\bigoplus_{i+j=k} T_i \otimes_\Lambda X_j\Big],$$

i.e. the module consisting of precisely one copy of $T_i \otimes_\Lambda X_j$ for each $(i, j) \in \mathbb{Z} \times \mathbb{Z}$. Of course, the action of the differential of $\oplus P$ on the simple tensors $t_i \otimes x_j$ is precisely the action of each differential of P on the simple tensors of P_k.

Going the other way around diagram (4.2) yields the $\Gamma[\epsilon]$-module

$$\bigoplus_{i\in\mathbb{Z}} \big[T_i \otimes_\Lambda (\oplus X)\big].$$

Of course, as a Γ-module, this is the same as

$$\bigoplus_{i\in\mathbb{Z}} \big[\bigoplus_{j\in\mathbb{Z}} T_i \otimes_\Lambda X_j\big]$$

which is also precisely one copy of $T_i \otimes_\Lambda X_j$ for each $(i,j) \in \mathbb{Z}\times\mathbb{Z}$. And if one is not intimidated by the notation, then it is a simple matter to read off that the differential acts on the simple tensors of each summand as

$$t_i \otimes x_j \mapsto d_i^T(t_i) \otimes x_j + (-1)^i t_i \otimes d_j^X(x_j).$$

□

To complete the proof of Proposition 4.3 we will find an inverse functor of $T\widetilde{\otimes}_\Lambda -$. Certainly, on the level of ordinary derived categories there is an inverse functor of $T \otimes^{\mathbf{L}}_\Lambda -$ given by

$$\mathbf{R}\mathrm{Hom}_\Gamma(T,-) : D(\mathrm{Mod}\,\Gamma) \to D(\mathrm{Mod}\,\Lambda).$$

If the inverse of $T \otimes^{\mathbf{L}}_\Lambda -$ was also some derived tensor product, say $S \otimes^{\mathbf{L}}_\Gamma -$, then we could repeat the above construction to obtain a functor

$$S\widetilde{\otimes}_\Gamma - : D_{\mathrm{ung}}(\mathrm{Mod}\,\Gamma) \to D_{\mathrm{ung}}(\mathrm{Mod}\,\Lambda).$$

Morally, it would not be unreasonable to hope for the latter to be an inverse of $T\widetilde{\otimes}_\Lambda -$. The next lemma shows why we actually have derived tensor products in both directions on the level of ordinary derived categories.

4.5 Lemma. *The functors* $\mathbf{R}\mathrm{Hom}_\Gamma(T,-)$ *and* $\mathrm{Hom}_\Gamma(T,\Gamma) \otimes^{\mathbf{L}}_\Gamma -$ *are naturally isomorphic.*

Proof. It is sufficient to show that the functors

$$\mathrm{Hom}_\Gamma(T,-)\,,\ \mathrm{Hom}_\Gamma(T,\Gamma) \otimes_\Gamma - : \mathrm{Mod}\,\Gamma \to \mathrm{Mod}\,\Lambda$$

coincide on $\mathrm{Proj}\,\Gamma$, as this means they will give rise to the same derived functor. But this follows readily, as the functors in question agree on Γ via the natural isomorphism

$$\mathrm{Hom}_\Gamma(T,\Gamma) \cong \mathrm{Hom}_\Gamma(T,\Gamma) \otimes_\Gamma \Gamma.$$

Naturality means that idempotents are preserved, which again implies that direct summands of Γ are preserved. Thus, the functors in question coincide on $\mathrm{Proj}\,\Gamma = \mathrm{Add}\,\Gamma$. □

Hence there is the diagram

$$\begin{tikzcd}
D(\operatorname{Mod}\Lambda) \arrow[r, bend left, "T\otimes^{\mathbf{L}}_{\Lambda}-"] \arrow[d, "\oplus"] & D(\operatorname{Mod}\Gamma) \arrow[l, bend left, "\operatorname{Hom}_{\Gamma}(T,\Gamma)\otimes^{\mathbf{L}}_{\Gamma}-"] \arrow[d, "\oplus"] \\
D_{\mathrm{ung}}(\operatorname{Mod}\Lambda) \arrow[r, bend left, "T\widetilde{\otimes}_{\Lambda}-"] & D_{\mathrm{ung}}(\operatorname{Mod}\Gamma) \arrow[l, bend left, "\operatorname{Hom}_{\Gamma}(T,\Gamma)\widetilde{\otimes}_{\Gamma}-"]
\end{tikzcd}$$

which is commutative in two ways (i.e. in both the 'top left $\to$ bottom right' and the 'top right $\to$ bottom left' sense). If we can show that the $\widetilde{\otimes}$-functors on the level of ungraded derived categories compose to the respective identity functors then the proof of Propopsition 4.3 is complete. By the symmetry of the situation it suffices to handle one of these compositions.

4.6 Lemma. *There is an isomorphism of functors*

$$\operatorname{Hom}_{\Gamma}(T,\Gamma)\widetilde{\otimes}_{\Gamma}(T\widetilde{\otimes}_{\Lambda}-) \cong \mathrm{id}_{D_{\mathrm{ung}}(\operatorname{Mod}\Lambda)}\,.$$

Proof. We start by claiming that the proof reduces to showing commutativity of

$$\begin{tikzcd}
D_{\mathrm{ung}}(\operatorname{Mod}\Lambda) \arrow[rr, "(\operatorname{Hom}_{\Gamma}(T,\Gamma)\otimes^{\mathbf{L}}_{\Gamma}T)\widetilde{\otimes}_{\Lambda}-"] \arrow[dr, "T\widetilde{\otimes}_{\Lambda}-"'] & & D_{\mathrm{ung}}(\operatorname{Mod}\Lambda) \\
& D_{\mathrm{ung}}(\operatorname{Mod}\Gamma) \arrow[ur, "\operatorname{Hom}_{\Gamma}(T,\Gamma)\widetilde{\otimes}_{\Gamma}-"'] &
\end{tikzcd} \tag{4.4}$$

To see why this does suffice, observe that

$$\operatorname{Hom}_{\Gamma}(T,\Gamma)\otimes^{\mathbf{L}}_{\Gamma}T \cong \mathbf{R}\operatorname{Hom}_{\Gamma}(T,T) \cong \operatorname{Hom}_{\Gamma}(T,T) = \Lambda$$

in $D(\operatorname{Mod}\Lambda)$ by Lemma 4.5. Combined with the natural isomorphism

$$\Lambda\widetilde{\otimes}_{\Lambda}- \cong \mathrm{id}_{D_{\mathrm{ung}}(\operatorname{Mod}\Lambda)}$$

this proves the claim made. Here we used that quasi-isomorphic complexes give naturally isomorphic $\widetilde{\otimes}$-functor, which is easily checked. Indeed, if $\big(f_i\big) : A \to B$ is a quasi-isomorphism of complexes then, for any $M \in \operatorname{Mod}\Lambda[\epsilon]$, there is a quasi-isomorphism

$$A\widetilde{\otimes}_{\Lambda}M \to B\widetilde{\otimes}_{\Lambda}M$$

given on simple tensors by $a_i \otimes m \mapsto f_i(a_i) \otimes m$ which clearly commutes with induced maps.

So let us check commutativity of diagram (4.4). Take $(M, \epsilon_M) \in D_{\mathrm{ung}}(\mathrm{Mod}\,\Lambda)$ and recall that

$$T \widetilde{\otimes}_\Lambda M = \bigoplus_{i \in \mathbb{Z}} (T_i \otimes_\Lambda M)$$

with differential given by

$$t_i \otimes m \mapsto d_i^T(t_i) \otimes m + (-1)^i t_i \otimes \epsilon_M(m).$$

For the sake of notation we write $A_i := \mathrm{Hom}_\Gamma(T_i, \Gamma)$. Then $\mathrm{Hom}_\Gamma(T, \Gamma)$ is the complex

$$A = \cdots \to A_{i+1} \xrightarrow{d_{i+1}^A} A_i \xrightarrow{d_i^A} A_{i-1} \to \cdots$$

where $d_i^A = (d_i^T)^*$. So going via $D_{\mathrm{ung}}(\mathrm{Mod}\,\Gamma)$ in diagram (4.4) yields the underlying Λ-module

$$A \widetilde{\otimes}_\Gamma (T \widetilde{\otimes}_\Lambda M) = \bigoplus_{j \in \mathbb{Z}} A_j \otimes_\Gamma (T \widetilde{\otimes}_\Lambda M) = \bigoplus_{(j,i) \in \mathbb{Z} \times \mathbb{Z}} A_j \otimes_\Gamma T_i \otimes_\Lambda M$$

with differential acting on each simple tensor of the form $a_j \otimes t_i \otimes m$ as

$$\begin{aligned} a_j \otimes t_i \otimes m \mapsto\ & d_j^A(a_j) \otimes t_i \otimes m + (-1)^j a_j \otimes d_{T \widetilde{\otimes}_\Lambda M}(t_i \otimes m) \\ &= d_j^A(a_j) \otimes t_i \otimes m + (-1)^j a_j \otimes \left[d_i^T(t_i) \otimes m + (-1)^i t_i \otimes \epsilon_M(m) \right] \\ &= \left[d_j^A(a_j) \otimes t_i + (-1)^j a_j \otimes d_i^T(t_i) \right] \otimes m + (-1)^{j+i} a_j \otimes t_i \otimes \epsilon_M(m). \end{aligned}$$

To see what happens if we go 'straight across' diagram (4.4), start by observing that $A \otimes_\Gamma^{\mathbf{L}} T$ is the complex

$$\cdots \to B_{p+1} \xrightarrow{d_{p+1}^B} B_p \xrightarrow{d_p^B} B_{p-1} \to \cdots$$

in which $B_p = \bigoplus_{j+i=p} A_j \otimes_\Gamma T_i$ and

$$a_j \otimes t_i \overset{d_p^B}{\longmapsto} d_j^A(a_j) \otimes t_i + (-1)^j a_j \otimes d_i^T(t_i).$$

Hence

$$(A \otimes_\Gamma^{\mathbf{L}} T) \widetilde{\otimes}_\Lambda M = \bigoplus_{p \in \mathbb{Z}} B_p \otimes_\Lambda M$$

with differential given by

$$b_p \otimes m \mapsto d_p^B(b_p) \otimes m + (-1)^p b_p \otimes \epsilon_M(m).$$

An equally valid way of writing the latter is of course as the Λ-module

$$\bigoplus_{p\in\mathbb{Z}}\Big[\bigoplus_{j+i=p} A_j \otimes_\Gamma T_i \otimes_\Lambda M\Big] = \bigoplus_{(j,i)\in\mathbb{Z}\times\mathbb{Z}} A_j \otimes_\Gamma T_i \otimes_\Lambda M$$

with differential

$$a_j \otimes t_i \otimes m \mapsto \big[d_j^A(a_j)\otimes t_i + (-1)^j a_j \otimes d_i^T(t_i)\big]\otimes m + (-1)^{j+i} a_j \otimes t_i \otimes \epsilon_M(m)$$

(since $j+i=p$). This means diagram (4.4) is indeed commutative. □

4.3 Density of The ⊕-Functor for Iterated Tilted Algebras

Observe that the validity of the results in the previous section depends neither on the derived categories being unbounded nor on the fact that modules are allowed to be infinitely generated. We may therefore conclude the following.

4.7 Corollary. *Replacing the derived categories in Proposition 4.3 by bounded derived categories, the statement remains true. Further replacing the categories of modules by their full subcategories of finitely generated modules, the statement remains true.* □

Using the theory of tilting complexes, in particular [Kel98, Kön98], the particular result mentioned at the start of this chapter now follows readily.

4.8 Corollary. *If Γ is an iterated tilted algebra, then $\oplus : D^b(\operatorname{Mod}\Gamma) \to D_{\text{ung}}(\operatorname{Mod}\Gamma)$ is dense.*

Proof. Γ being iterated tilted means there is a hereditary algebra Λ such that the categories $D(\operatorname{Mod}\Lambda)$ and $D(\operatorname{Mod}\Gamma)$ are equivalent. This holds only if the categories $D^b(\operatorname{Mod}\Lambda)$ and $D^b(\operatorname{Mod}\Gamma)$ admit a standard equivalence, in which case we are precisely in the setup of Corollary 4.7. Thus, there is a complex T and a commutative diagram

$$\begin{tikzcd} D^b(\operatorname{Mod}\Lambda) \arrow[r, "T\otimes^{\mathbf{L}}_\Lambda -"] \arrow[d, "\oplus"] & D^b(\operatorname{Mod}\Gamma) \arrow[d, "\oplus"] \\ D_{\text{ung}}(\operatorname{Mod}\Lambda) \arrow[r, "T\widetilde{\otimes}_\Lambda -"] & D_{\text{ung}}(\operatorname{Mod}\Gamma) \end{tikzcd}$$

in which both the horizontal functors are equivalences. Moreover, Propopsition 4.2 says that $\oplus : D^b(\operatorname{Mod}\Lambda) \to D_{\text{ung}}(\operatorname{Mod}\Lambda)$ is dense. This clearly implies density of the right hand ⊕-functor. □

REMARK. Γ is iterated tilted if and only if $D^b(\operatorname{mod}\Gamma) \cong D^b(\operatorname{mod}\Lambda)$ for a hereditary algebra Λ (see for instance [Kel98]). Since equivalence of these categories also implies the existence of a standard equivalence, the proof of Corollary 4.8 will adapt to show that $\oplus : D^b(\operatorname{mod}\Gamma) \to D_{\text{ung}}(\operatorname{mod}\Gamma)$ is dense under the additional assumption that the algebras are noetherian[1]. $\square$

[1] Of course, any assumption implying that the categories of finitiely generated modules are abelian will do.

CHAPTER 5

A New Description of $D_{\mathrm{ung}}(\operatorname{Mod}\Lambda)$

The aim of this chapter is to give a different description of the triangulated category $D_{\mathrm{ung}}(\operatorname{Mod}\Lambda)$. To be more precise, we shall find a triangle equivalence

$$D_{\mathrm{ung}}(\operatorname{Mod}\Lambda) \cong \mathcal{H}_p(\underline{\operatorname{Mod}}\,\Lambda[\epsilon])$$

where $\mathcal{H}_p$ denotes the restriction to the full subcategory of 'homotopically projective' $\Lambda[\epsilon]$-modules. The reason why this equivalence is useful should be evident. Even though we formally know what $D_{\mathrm{ung}}(\operatorname{Mod}\Lambda)$ looks like, the fact that it is a localization can make computations difficult. This has its analogue in the setup of complexes, where we often prefer working in $K^{b,+}(\operatorname{Proj}\Lambda)$ rather than in $D^b(\operatorname{Mod}\Lambda)$. A further advantage of this description of $D_{\mathrm{ung}}(\operatorname{Mod}\Lambda)$ is that we in some sense know all the homotopically projective $\Lambda[\epsilon]$-modules, as we shall see in Section 5.2.

Remark. The results of this chapter do not carry over if we replace Mod by mod. In particular Proposition 5.1, which is necessary for the rest of the discussion to make sense, fails if we only allow finitely generated modules. □

5.1 A Projective Resoultion

This section is dedicated to a construction that will be key later in this chapter.

5.1 Proposition. *For any $M = (M, \epsilon_M) \in \operatorname{Mod}\Lambda[\epsilon]$ there is an exact sequence*

$$\cdots \to (P_2, \hat{\epsilon}_2) \xrightarrow{d_2} (P_1, \hat{\epsilon}_1) \xrightarrow{d_1} (P_0, \hat{\epsilon}_0) \xrightarrow{d_0} (M, \epsilon_M) \to 0$$

in $\operatorname{Mod}\Lambda[\epsilon]$ satisfying the following.

i) Each P_i restricts to $\operatorname{Proj}\Lambda$.

ii) We have a decomposition $P_i = P_i' \oplus P_i''$ such that $\epsilon_i := (-1)^i\hat{\epsilon}_i$ is given by a matrix of the form $\left(\begin{smallmatrix} 0 & \overline{\epsilon_i} \\ 0 & 0 \end{smallmatrix}\right)$ for each $i \geq 0$.

iii) The sum of underlying Λ-modules $pM := \bigoplus_{i\geq 0} P_i$ admits a differential ϵ_{pM} making it quasi-isomorphic to M.

Proof. In $\operatorname{Mod}\Lambda$, fix epimorphisms

$$X \xrightarrow{f} \operatorname{Im}\epsilon_M \qquad \text{and} \qquad Y \xrightarrow{g} H(M)$$

with X and Y in $\operatorname{Proj}\Lambda$. Our construction starts with combining X and Y into a larger projective module with an epimorphism onto M by using extensions. Consider the following diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "i_1"] \arrow[d, "f"] & X \oplus Y \arrow[r, "p_1"] \arrow[d, "h"] & Y \arrow[r] \arrow[d, "g"] \arrow[dl, dashed, "\sigma"'] & 0 \\
0 \arrow[r] & \operatorname{Im}\epsilon_M \arrow[r, "i_2"] & \operatorname{Ker}\epsilon_M \arrow[r, "p_2"] & H(M) \arrow[r] & 0
\end{tikzcd}$$

i_1 and p_1 are the canonical split monomorphism and epimorphism, respectively, while σ has the property $p_2\sigma = g$ (such a σ can be found by projectivity of Y). h is defined by $h(x,y) := i_2 f(x) + \sigma(y)$ and clearly makes both squares commutative. Further, h must be an epimorphism by the five lemma. Now the following diagram also has exact rows, where $P_0 := X \oplus Y \oplus X$.

$$\begin{tikzcd}
0 \arrow[r] & X \oplus Y \arrow[r, "i_3"] \arrow[d, "h"] & P_0 \arrow[r, "p_3"] \arrow[d, "d_0"] & X \arrow[r] \arrow[d, "f"] \arrow[dl, dashed, "\gamma"'] & 0 \\
0 \arrow[r] & \operatorname{Ker}\epsilon_M \arrow[r, "i_4"] & M \arrow[r, "\epsilon_M"] & \operatorname{Im}\epsilon_M \arrow[r] & 0
\end{tikzcd}$$

Here $i_3(x,y) := (x,y,0)$ and $p_3(x_1,y,x_2) := x_2$, while γ has the property $\epsilon_M\gamma = f$ by projectivity of X. As one should expect by now, $d_0(x_1,y,x_2) := i_4 h(x_1,y) + \gamma(x_2)$ and makes both squares commutative. By the five lemma again, d_0 is an epimorphism, which is what we were aiming for.

The next step in our construction is equipping P_0 with a differential $\hat{\epsilon}_0$ in such a way that d_0 becomes a morphism in $\operatorname{Mod}\Lambda[\epsilon]$. A natural candidate seems to be $\hat{\epsilon}_0 := i_3 i_1 p_3$, which is clearly a differential as $p_3 i_3 = 0$. Further, we have commutativity of each square in

$$\begin{tikzcd}
P_0 \arrow[r, "p_3"] \arrow[d, "d_0"] & X \arrow[r, "i_1"] \arrow[d, "f"] & X \oplus Y \arrow[r, "i_3"] \arrow[d, "h"] & P_0 \arrow[d, "d_0"] \\
M \arrow[r, "\epsilon_M"] & \operatorname{Im}\epsilon_M \arrow[r, "i_2"] & \operatorname{Ker}\epsilon_M \arrow[r, "i_4"] & M
\end{tikzcd}$$

by the above, so d_0 has the desired property of commuting with the differentials of P_0 and M. Note that letting $P_0' := X$ and $P_0'' := Y \oplus X$ yields $P_0 = P_0' \oplus P_0''$. Also, the differential $\hat{\epsilon}_0$ is given by the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : (x_1, y, x_2) \mapsto (x_2, 0, 0).$$

Since $\hat{\epsilon}_0$ restricts as

$$\hat{\epsilon}_0|_{\operatorname{Ker} d_0} : \operatorname{Ker} d_0 \to \operatorname{Ker} d_0$$

a $\Lambda[\epsilon]$-structure is imposed on $\operatorname{Ker} d_0$. Thus the above process may be repeated, replacing M by $\operatorname{Ker} d_0$. This produces P_1 made from Λ-projective summands with differential $\hat{\epsilon}_1$, accompanied by a $\Lambda[\epsilon]$-linear map $P_1 \to \operatorname{Ker} d_0$. So we obtain a $\Lambda[\epsilon]$-homomorphism $d_1 : P_1 \to P_0$ as the composition

$$\begin{tikzcd}
P_1 \arrow[rr, "d_1"] \arrow[dr] & & P_0 \\
& \operatorname{Ker} d_0 \arrow[ur, hook] &
\end{tikzcd}$$

Continuing this process results in the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d, "\hat{\epsilon}_2"] & P_1 \arrow[r, "d_1"] \arrow[d, "\hat{\epsilon}_1"] & P_0 \arrow[r, "d_0"] \arrow[d, "\hat{\epsilon}_0"] & M \arrow[r] \arrow[d, "\epsilon_M"] & 0 \\
\cdots \arrow[r] & P_2 \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "d_0"] & M \arrow[r] & 0
\end{tikzcd}$$

in which each square is commutative and the rows are exact. This shows $i)$ of the proposition.

Since each P_i was constructed using the recipe that gave P_0, we have a decomposition $P_i = P_i' \oplus P_i''$ such that $\hat{\epsilon}_i$ is given by the same elementary matrix as $\hat{\epsilon}_0$. This obviously means that $\epsilon_i := (-1)^i \hat{\epsilon}_i$ is given by a matrix of the form $\left(\begin{smallmatrix} 0 & \overline{\epsilon_i} \\ 0 & 0 \end{smallmatrix}\right)$, showing $ii)$ of the proposition.

The important consequence of altering the signs of the differentials is the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d, "\epsilon_2"] & P_1 \arrow[r, "d_1"] \arrow[d, "\epsilon_1"] & P_0 \arrow[r, "d_0"] \arrow[d, "\epsilon_0"] & M \arrow[r] \arrow[d, "\epsilon_M"] & 0 \\
\cdots \arrow[r] & P_2 \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "d_0"] & M \arrow[r] & 0
\end{tikzcd} \tag{5.1}$$

with exact rows and each square anticommutative, except for the rightmost one which is still commutative. Let

$$pM := \bigoplus_{i \geq 0} P_i \in \operatorname{Mod} \Lambda.$$ [1]

Completing the proof amounts to equipping pM with a differential making it quasi-isomorphic to M. Clearly, for this purpose the differential induced on pM by taking direct sums in $\operatorname{Mod} \Lambda[\epsilon]$ does not work. Instead, for $\big(a_i\big) = \big(a_i\big)_{i \in \mathbb{N}} \in pM$ define

$$\big(a_i\big) \xmapsto{\epsilon_{pM}} \big(d_{i+1}(a_{i+1}) + \epsilon_i(a_i)\big).$$

[1] This is the point where the argument would fail if we tried to translate it to $\operatorname{mod} \Lambda$. Indeed, there is no reason why the projective resolution we constructed should terminate after finitely many steps. So regardless of whether or not the P_i are finitely generated, it is not true that $pM \in \operatorname{mod} \Lambda$.

It is clear that this is a differential:

$$\begin{aligned}\big(a_i\big) &\overset{\epsilon_{pM}}{\longmapsto} \big(d_{i+1}(a_{i+1}) + \epsilon_i(a_i)\big) \\ &\overset{\epsilon_{pM}}{\longmapsto} \big(d_{i+1}(d_{i+2}(a_{i+2}) + \epsilon_{i+1}(a_{i+1})) + \epsilon_i(d_{i+1}(a_{i+1}) + \epsilon_i(a_i))\big) \\ &= \big(d_{i+1}\epsilon_{i+1}(a_{i+1}) + \epsilon_i d_{i+1}(a_{i+1})\big) \\ &= \big(0\big)\end{aligned}$$

where the last equality holds because of the 'sign trick' pulled above.

Consider

$$\phi : (pM, \epsilon_{pM}) \to (M, \epsilon_M)$$

given by $\big(a_i\big) \mapsto d_0(a_0)$. The fact that ϕ is a $\Lambda[\epsilon]$-homomorphism is simply restating commutativity of the rightmost square in diagram (5.1). We claim that ϕ is a quasi-isomorphism, i.e. that

$$H\phi : H(pM) \to H(M)$$

given by $\big(a_i\big) + \operatorname{Im}\epsilon_{pM} \mapsto d_0(a_0) + \operatorname{Im}\epsilon_M$ is an isomorphism.

Let us show that $H\phi$ is surjective. Take $m \in \operatorname{Ker}\epsilon_M$ and pick some $a_0 \in P_0$ such that $d_0(a_0) = m$. Then $0 = \epsilon_M(m) = \epsilon_M d_0(a_0) = d_0\epsilon_0(a_0)$, i.e. $\epsilon_0(a_0) \in \operatorname{Ker} d_0 = \operatorname{Im} d_1$. Therefore, we can choose $a_1 \in P_1$ such that $d_1(a_1) = \epsilon_0(a_0)$. Now $d_1\epsilon_1(a_1) = -\epsilon_0 d_1(a_1) = 0$, hence $\epsilon_1(a_1) \in \operatorname{Ker} d_1 = \operatorname{Im} d_2$. This continues, of course, yielding for each $i \geq 1$ an $a_i \in P_i$ satisfying

$$d_i(a_i) = \epsilon_{i-1}(a_{i-1}).$$

This leads us to considering

$$a := \big((-1)^i a_i\big).$$

Note that a belongs to $\operatorname{Ker}\epsilon_{pM}$:

$$\begin{aligned}\epsilon_{pM}(a) &= \big(d_{i+1}((-1)^{i+1}a_{i+1}) + \epsilon_i((-1)^i a_i)\big) \\ &= \big((-1)^{i+1}d_{i+1}(a_{i+1}) + (-1)^i\epsilon_i(a_i)\big) \\ &= \big((-1)^{i+1}\epsilon_i(a_i) + (-1)^i\epsilon_i(a_i)\big) \\ &= \big(0\big).\end{aligned}$$

Surjectivity of $H\phi$ now follows, as $H\phi(a + \operatorname{Im}\epsilon_{pM}) = m + \operatorname{Ker}\epsilon_M$.

For injectivity of $H\phi$ take $a = \big(a_i\big) \in \operatorname{Ker}\epsilon_{pM}$ and assume $H\phi(a + \operatorname{Im}\epsilon_{pM}) = 0$. We must show that $a \in \operatorname{Im}\epsilon_{pM}$. The assumptions mean

$$d_{i+1}(a_{i+1}) + \epsilon_i(a_i) = 0$$

for each $i \geq 0$ and also that $d_0(a_0) \in \operatorname{Im} \epsilon_M$. So let $m \in M$ be such that $\epsilon_M(m) = d_0(a_0)$ and pick $b_0 \in P_0$ satisfying $d_0(b_0) = m$. Then $0 = d_0(a_0) - \epsilon_M d_0(b_0) = d_0(a_0) - d_0\epsilon_0(b_0)$, i.e. $a_0 - \epsilon_0(b_0) \in \operatorname{Ker} d_0 = \operatorname{Im} d_1$. Now we may pick $b_1 \in P_1$ such that $d_1(b_1) = a_0 - \epsilon_0(b_0)$, i.e.

$$a_0 = d_1(b_1) + \epsilon_0(b_0).$$

Further, $d_1(a_1) = -\epsilon_0(a_0) = -\epsilon_0(d_1(b_1) + \epsilon_0(b_1)) = -\epsilon_0 d_1(b_1) = d_1\epsilon_1(b_1)$ which means $a_1 - \epsilon_1(b_1) \in \operatorname{Ker} d_1 = \operatorname{Im} d_2$. So we can pick $b_2 \in P_2$ such that $d_2(b_2) = a_1 - \epsilon_1(b_1)$, i.e.

$$a_1 = d_2(b_2) + \epsilon_1(b_1).$$

Continuing this we obtain $b := \big(b_i\big)$ with the property

$$a_i = d_{i+1}(b_{i+1}) + \epsilon_i(b_i)$$

for each $i \geq 0$. Thus $a = \epsilon_{pM}(b)$. $\square$

Remark. An important consequence of *ii*) is that $\operatorname{Im} \epsilon_i \subset P_i' \subset \operatorname{Ker} \epsilon_i$ (if we identify P_i' with the submodule $P_i' \oplus 0$ of P_i), since

$$(p_i', p_i'') \stackrel{\epsilon_i}{\longmapsto} (\overline{\epsilon_i}(p_i''), 0).$$

By the above proof we even have $\operatorname{Im} \epsilon_i = P_i'$. But since we shall only need an inclusion later on, the statement of Proposition 5.1 is satisfactory. $\square$

5.2 Homotopically Projective $\Lambda[\epsilon]$-modules

Our next aim is introducing the notion of homotopically projective $\Lambda[\epsilon]$-modules and understanding the subcategory of $\underline{\operatorname{Mod}}\, \Lambda[\epsilon]$ that they constitute. Our work in this section is inspired by Keller's results on homotopically projective complexes in [Kel98].

Definition. A $\Lambda[\epsilon]$-module K is **homotopically projective** if

$$\underline{\operatorname{Hom}}_{\Lambda[\epsilon]}(K, N) = 0$$

for each acyclic N. The full subcategory of $\underline{\operatorname{Mod}}\, \Lambda[\epsilon]$ consisting of homotopically projective $\Lambda[\epsilon]$-modules is denoted by $\mathcal{H}_p(\underline{\operatorname{Mod}}\, \Lambda[\epsilon])$.

We start with an easy observation.

5.2 Lemma. *$\mathcal{H}_p(\underline{\mathrm{Mod}}\,\Lambda[\epsilon])$ is a triangulated subcategory of $\underline{\mathrm{Mod}}\,\Lambda[\epsilon]$.*

Proof. $\mathcal{H}_p(\underline{\mathrm{Mod}}\,\Lambda[\epsilon])$ is clearly closed under isomorphisms and translation. Further, assume

$$A \to B \to C \to \Sigma A$$

is distinguished in $\underline{\mathrm{Mod}}\,\Lambda[\epsilon]$ with A and B homotopically projective. Let N be any acyclic $\Lambda[\epsilon]$-module. Applying $\underline{\mathrm{Hom}}_{\Lambda[\epsilon]}(-, N)$ yields exactness of

$$\underline{\mathrm{Hom}}_{\Lambda[\epsilon]}(\Sigma A, N) \to \underline{\mathrm{Hom}}_{\Lambda[\epsilon]}(C, N) \to \underline{\mathrm{Hom}}_{\Lambda[\epsilon]}(B, N).$$

The outer terms vanish by assumption, hence also the middle term is zero, meaning C is homotopically projective. $\square$

We immediately also get the following.

5.3 Lemma. *$\mathcal{H}_p(\underline{\mathrm{Mod}}\,\Lambda[\epsilon])$ is closed under split extensions, i.e. if*

$$0 \to A \to E \to B \to 0$$

is a conflation in $\underline{\mathrm{Mod}}\,\Lambda[\epsilon]$ with A and B homotopically projective, then also E is homotopically projective.

Proof. By Corollary 3.9, there is a distinguished triangle

$$A \to E \to B \to \Sigma A$$

in $\underline{\mathrm{Mod}}\,\Lambda[\epsilon]$. Since $\mathcal{H}_p(\underline{\mathrm{Mod}}\,\Lambda[\epsilon])$ is a triangulated subcategory by Lemma 5.2, this means E is homotopically projective. $\square$

We turn our attention towards describing $\mathcal{H}_p(\underline{\mathrm{Mod}}\,\Lambda[\epsilon])$ more explicitly. The next two results should not be surprising.

5.4 Lemma. *If $P \in \mathrm{Proj}\,\Lambda$, then $(P, 0) \in \mathrm{Mod}\,\Lambda[\epsilon]$ is homotopically projective.*

Proof. Take $f \in \underline{\mathrm{Hom}}_{\Lambda[\epsilon]}(P, N)$ with N acyclic. This means the diagram

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "0"] & N \arrow[d, "\epsilon_N"] \\ P \arrow[r, "f"] & N \end{tikzcd}$$

is commutative in $\operatorname{Mod}\Lambda$, yielding $\operatorname{Im} f \subseteq \operatorname{Ker} \epsilon_N = \operatorname{Im} \epsilon_N$. So by projectivity of P there is a Λ-linear $s : P \to N$ such that

$$\begin{tikzcd} & P \arrow[dl, "s"'] \arrow[d, "f"] & \\ N \arrow[r, "\epsilon_N"] & \operatorname{Im}\epsilon_N \arrow[r] & 0 \end{tikzcd}$$

is commutative. Hence f is nullhomotopic as $f = \epsilon_N s = s\epsilon_P + \epsilon_N s$. $\square$

5.5 Lemma. *Let K be a homotopically projective complex of Λ-modules. Then $\oplus K$ is a homotopically projective $\Lambda[\epsilon]$-module.*

Proof. Let

$$K = \cdots \to K_{i+1} \xrightarrow{d_{i+1}} K_i \xrightarrow{d_i} K_{i-1} \to \cdots$$

and assume $(N, \epsilon_N) \in \operatorname{Mod}\Lambda[\epsilon]$ is acyclic.

Take $f \in \operatorname{Hom}_{\Lambda[\epsilon]}(\oplus K, N)$. Of course, we can write $f = (f_i)$ where f_i is the restriction of f to K_i. f being a $\Lambda[\epsilon]$-homomorphism means that each square in

$$\begin{tikzcd} \cdots \arrow[r] & K_{i+1} \arrow[r, "d_{i+1}"] \arrow[d, "f_{i+1}"] & K_i \arrow[r, "d_i"] \arrow[d, "f_i"] & K_{i-1} \arrow[r] \arrow[d, "f_{i-1}"] & \cdots \\ \cdots \arrow[r] & N \arrow[r, "\epsilon_N"] & N \arrow[r, "\epsilon_N"] & N \arrow[r] & \cdots \end{tikzcd}$$

is commutative. Since K is a homotopically projective complex, this yields the existence of a family of morphisms $s_i : K_i \to N$ satisfying $f_i = \epsilon_N s_i + s_{i-1} d_i$. So the s_i constitute a Λ-linear $s : \oplus K \to N$ such that $f = \epsilon_N s + s\epsilon_K$, meaning f is nullhomotopic. $\square$

What is more, we can produce homotopically projectives using the following recipe. Consider a directed system in $\operatorname{Mod}\Lambda[\epsilon]$

$$0 \to P_0 \xrightarrow{i_0} P_1 \xrightarrow{i_1} P_2 \to \cdots \to P_q \xrightarrow{i_q} P_{q+1} \to \ldots \tag{5.2}$$

in which each i_q is an inflation and moreover each quotient P_{q+1}/P_q has vanishing differential and restricts to $\operatorname{Proj}\Lambda$. It is immediate from Lemma 5.4 that each P_{q+1}/P_q, including P_0, is homotopically projective. Under the assumption that P_q is homotopically projective, also P_{q+1} enjoys this property. Indeed, there is a conflation

$$0 \to P_q \xrightarrow{i_q} P_{q+1} \to P_{q+1}/P_q \to 0$$

which means P_{q+1} is homotopically projective by Lemma 5.3. Thus, the assertion that each P_q is homotopically projective holds by induction. It is the limit of the system, however, that we are really interested in.

5.6 LEMMA. *Let $\{P_q\}$ be the directed system (5.2). Then $\varinjlim P_q$ is homotopically projective.*

PROOF. There is an isomorphism

$$\underline{\operatorname{Hom}}_{\Lambda[\epsilon]}(\varinjlim P_q, N) \cong \varprojlim \underline{\operatorname{Hom}}_{\Lambda[\epsilon]}(P_q, N).$$

If N is acyclic, then each $\underline{\operatorname{Hom}}_{\Lambda[\epsilon]}(P_q, N)$ vanishes, so the limit is zero. □

The next lemma reveals how the concept of (homotopically) projective resolutions of complexes translates to the ungraded setup.

5.7 LEMMA. *Any $(M, \epsilon_M) \in \operatorname{Mod}\Lambda[\epsilon]$ is quasi-isomorphic to a homotopically projective $\Lambda[\epsilon]$-module.*

PROOF. We saw in the proof of Proposition 5.1 how to construct a 'projective resolution'

$$\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$$

and a differential on the Λ-module $pM := \bigoplus_{i\in\mathbb{N}} P_i$ such that pM is quasi-isomorphic to M. We will show that pM is homotopically projective. To this end we will realize pM as the limit of a directed system with the properties of (5.2). Lemma 5.6 says this is sufficient.

For each $n \in \mathbb{N}$ equip $\bigoplus_{i=0}^{n} P_i$ with the differential $\widetilde{\epsilon}_n$ given by

$$\big(p_i\big)_{i=0}^{n} \mapsto \big(\epsilon_n(p_n), d_n(p_n) + \epsilon_{n-1}(p_{n-1}), \ldots, d_1(p_1) + \epsilon_0(a_0)\big)$$

where $\epsilon_i \in \operatorname{End}_\Lambda(P_i)$ is the differential constructed in the proof of Proposition 5.1. Hence we have the directed system

$$0 \subseteq (P_0, \widetilde{\epsilon}_0) \subseteq (P_0 \oplus P_1, \widetilde{\epsilon}_1) \subseteq (P_0 \oplus P_1 \oplus P_2, \widetilde{\epsilon}_2) \subseteq \cdots \tag{5.3}$$

of inclusions in $\operatorname{Mod}\Lambda[\epsilon]$ whose limit is clearly pM. The strategy for completing the proof is to refine system (5.3) into one that satisfies the properties of (5.2), but whose limit also equals pM.

Recall *ii*) of Proposition 5.1. This inspires us to look at

$$0 \subseteq (P_0', \widetilde{\epsilon}_0|) \subseteq (P_0, \widetilde{\epsilon}_0) \subseteq (P_0 \oplus P_1', \widetilde{\epsilon}_1|) \subseteq (P_0 \oplus P_1, \widetilde{\epsilon}_1) \subseteq (P_0 \oplus P_1 \oplus P_2', \widetilde{\epsilon}_2|) \subseteq \ldots \tag{5.4}$$

where $\widetilde{\epsilon}_n|$ denotes the restriction of $\widetilde{\epsilon}_n$ to $P_0 \oplus P_1 \oplus \cdots \oplus P_{n-1} \oplus P'_n$. We claim that the latter directed system is of type (5.2). Clearly, each inclusion is split over Λ and the quotient of any term modulo its immediate predecessor restricts to $\operatorname{Proj}\Lambda$. We are left to prove that these quotients even have vanishing differential. There are two scenarios to consider. One is when the quotient is of the form

$$(P_0 \oplus \cdots \oplus P_{n-1} \oplus P'_n)\Big/(P_0 \oplus \cdots \oplus P_{n-1}).$$

Write $D = P_0 \oplus \cdots \oplus P_{n-1}$. Then the differential of the quotient is given by

$$\begin{aligned}\big(p_0, \ldots, p_{n-1}, p'_n\big) + D \mapsto{} & \big(d_1(p_1) + \epsilon_0(p_0), \ldots, d_n(p'_n) + \epsilon_{n-1}(p_{n-1}), \epsilon_n(p'_n)\big) + D \\ ={} & \big(0, \ldots, 0, \epsilon_n(p'_n)\big) + D \\ ={} & \big(0\big) + D.\end{aligned}$$

The first equality is obvious, while the second one holds because $P'_n \subset \operatorname{Ker}\epsilon_n$ by the remark following Proposition 5.1. The other case is when the quotient is of the form

$$(P_0 \oplus \cdots \oplus P_{n-1} \oplus P_n)\Big/(P_0 \oplus \cdots \oplus P_{n-1} \oplus P'_n).$$

Writing $D' = P_0 \oplus \cdots \oplus P_{n-1} \oplus P'_n$ the differential of the quotient is given by

$$\begin{aligned}\big(p_0, \ldots, p_{n-1}, p_n\big) + D' \mapsto{} & \big(d_1(p_1) + \epsilon_0(p_0), \ldots, d_n(p_n) + \epsilon_{n-1}(p_{n-1}), \epsilon_n(p_n)\big) + D' \\ ={} & \big(0, \ldots, 0, \epsilon_n(p_n)\big) + D' \\ ={} & \big(0\big) + D'.\end{aligned}$$

Again, the first equality is trivial. But since the remark following Proposition 5.1 also gives $\operatorname{Im}\epsilon_n \subset P'_n$, so is the last one. This means that system (5.4) satisfies all the properties of (5.2). The limit of (5.4) is obviously the same as that of (5.3), so the proof is complete. □

Finally, let us prove the main result of this section, which gives a description of all homotopically projective $\Lambda[\epsilon]$-modules.

5.8 Proposition. *Any homotopically projective $\Lambda[\epsilon]$-module is isomorphic in $\underline{\operatorname{Mod}}\,\Lambda[\epsilon]$ to the direct limit of a directed system with the properties of (5.2).*

Proof. Let $K \in \operatorname{Mod}\Lambda[\epsilon]$. By Proposition 5.1 there is a quasi-isomorphism

$$\phi : pK \to K$$

where, by the proof of Lemma 5.7, pK is a limit of the desired form. Under the assumption that K is homotopically projective, we can show that ϕ is an isomorphism already in $\underline{\operatorname{Mod}}\,\Lambda[\epsilon]$.

We claim that ϕ is universal among morphisms $P \to K$ with P homotopically projective (pK is homotopically projective by Lemma 5.6). To see why this is true, consider the distinguished triangle

$$pK \xrightarrow{\phi} K \to C_\phi \to \Sigma pK$$

in which C_ϕ is acyclic. Applying the homological functor $\underline{\mathrm{Hom}}_{\Lambda[\epsilon]}(P,-)$ gives an exact sequence

$$\underline{\mathrm{Hom}}_{\Lambda[\epsilon]}(P,\Sigma^{-1}C_\phi) \to \underline{\mathrm{Hom}}_{\Lambda[\epsilon]}(P,pK) \xrightarrow{\phi_*} \underline{\mathrm{Hom}}_{\Lambda[\epsilon]}(P,K) \to \underline{\mathrm{Hom}}_{\Lambda[\epsilon]}(P,C_\phi).$$

The outer terms vanish, so ϕ_* is an isomorphism. This means the claim holds, as any morphism $P \to K$ must factor uniquely through ϕ. So when K itself is homotopically projective there is a commutative diagram

$$\begin{tikzcd} K \arrow[rr, "1"] \arrow[dr] & & K \\ & pK \arrow[ur, "\phi"'] & \end{tikzcd}$$

Hence K is isomorphic in $\underline{\mathrm{Mod}}\,\Lambda[\epsilon]$ to a direct summand of pK, say $pK \cong K \oplus X$ for some X. But since $H(pK) = H(K)$ we must conclude that X is acyclic. This, however, means $X = 0$. Indeed, otherwise the projection $pK \to X$ would be non-zero and contradict $\underline{\mathrm{Hom}}_{\Lambda\epsilon}(pK,X) = 0$. So $pK \cong K$ in $\underline{\mathrm{Mod}}\,\Lambda[\epsilon]$. □

5.3 Restricting Q_T to an Equivalence

The main objective of this chapter is now within our grasp. Indeed, consider the restriction

$$F : \mathcal{H}_p(\underline{\mathrm{Mod}}\,\Lambda[\epsilon]) \hookrightarrow \underline{\mathrm{Mod}}\,\Lambda[\epsilon] \xrightarrow{Q_T} D_{\mathrm{ung}}(\operatorname{Mod}\Lambda)$$

of the localization functor Q_T. The following is what we have been aiming for.

5.9 Proposition. *The functor $F : \mathcal{H}_p(\underline{\mathrm{Mod}}\,\Lambda[\epsilon]) \to D_{\mathrm{ung}}(\operatorname{Mod}\Lambda)$ is a triangle equivalence.*

Proof. The fact that F is a triangle functor is immediate, as it is the composition of two triangle functors. Density follows from Lemma 5.7 since F is the identity on objects.

For faithfulness, take $f \in \underline{\mathrm{Hom}}_{\Lambda[\epsilon]}(A,B)$ with $A, B \in \mathcal{H}_p(\underline{\mathrm{Mod}}\,\Lambda[\epsilon])$ and assume F maps f to the zero morphism in $D_{\mathrm{ung}}(\operatorname{Mod}\Lambda)$. This means f factors as

$$\begin{tikzcd}
 & N \arrow[dr] & \\
A \arrow[ur] \arrow[rr, "f"] & & B
\end{tikzcd}$$

for some acyclic N, yielding $f = 0$ already in $\mathcal{H}_p(\underline{\mathrm{Mod}}\, \Lambda[\epsilon])$ since $\underline{\mathrm{Hom}}_{\Lambda[\epsilon]}(A, N) = 0$.

Lastly, let us see why F is full. To this end let $A, B \in \mathcal{H}_p(\underline{\mathrm{Mod}}\, \Lambda[\epsilon])$ and take some morphism $A \to B$ in $D_{\mathrm{ung}}(\mathrm{Mod}\, \Lambda)$, i.e. a roof

$$\begin{tikzcd}
 & X \arrow[dl, "q"'] \arrow[dr, "f"] & \\
A & & B
\end{tikzcd}$$

in which q is a quasi-isomorphism. Let the standard triangle associated to q be

$$X \xrightarrow{q} A \xrightarrow{c_q} C_q \to \Sigma X$$

where C_q must be acyclic. This means $c_q = 0$ since A is homotopically projective. So by Lemma 1.3 q is a split epimorphism in $\underline{\mathrm{Mod}}\, \Lambda[\epsilon]$, i.e. $q\hat{q} = 1_A$ for some $\Lambda[\epsilon]$-homomorphism $\hat{q} : A \to X$ which must also be a quasi-isomorphism. Therefore, the diagram

$$\begin{tikzcd}
 & & A \arrow[dl, "\hat{q}"'] \arrow[dr, "1"] & & \\
 & X \arrow[dl, "q"'] \arrow[drrr, "f"] & & A \arrow[dlll, "1"'] \arrow[dr, "f\hat{q}"] & \\
A & & & & B
\end{tikzcd}$$

has two commutative squares and implies that the roofs

$$\begin{tikzcd}
 & X \arrow[dl, "q"'] \arrow[dr, "f"] & & & A \arrow[dl, "1"'] \arrow[dr, "f\hat{q}"] & \\
A & & B & A & & B
\end{tikzcd}$$

are equivalent. Since the right hand roof is the image of $f\hat{q}$ under F, this suffices. $\square$

Remark. It is clear from the above proof that if A is homotopically projective then

$$\operatorname{Hom}_{D_{\mathrm{ung}}(\operatorname{Mod}\Lambda)}(A,B) \cong \operatorname{Hom}_{K_{\mathrm{ung}}(\operatorname{Mod}\Lambda)}(A,B)$$

with no assumption on B. We will use this fact in the proof of Lemma 6.2. $\square$

CHAPTER 6

Our Main Result

Given a category $\mathcal{C}$ and an automorphism $F : \mathcal{C} \to \mathcal{C}$ the associated **orbit category** is the category $\mathcal{C}/F$ whose objects are those of $\mathcal{C}$ and whose morphisms are given by

$$\mathrm{Hom}_{\mathcal{C}/F}(A, B) := \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(A, F^n B).$$

In this chapter we prove the following theorem, where $(-[1])$ denotes the translation functor on the derived category.

6.1 THEOREM. *If Λ is an iterated tilted algebra then $D^b(\mathrm{Mod}\,\Lambda)/(-[1])$ is triangulated in such a way that the canonical projection*

$$\pi : D^b(\mathrm{Mod}\,\Lambda) \to D^b(\mathrm{Mod}\,\Lambda)/(-[1])$$

is a triangle functor.

6.1 An Embedding of The Strong Orbit Category

In this section we prove a result (Proposition 6.2) that will enable us to compare Hom-spaces in the orbit category $D^b(\mathrm{Mod}\,\Lambda)/(-[1])$ to Hom-spaces in $D_{\mathrm{ung}}(\mathrm{Mod}\,\Lambda)$, modulo a slight technical difficulty. In particular it will follow that $D^b(\mathrm{Mod}\,\Lambda)/(-[1])$ embeds in $D_{\mathrm{ung}}(\mathrm{Mod}\,\Lambda)$ whenever Λ is iterated tilted, which will help us prove Theorem 6.1.

To overcome the first technical obstacle we will need a version of the $\oplus$-functor that allows the graded $\Lambda[\epsilon]$-modules it produces to be non-zero in infinitely many degrees.

DEFINITION. The **Π-functor** takes a complex of Λ-modules

$$A = \cdots \to A_{i+1} \xrightarrow{d_{i+1}} A_i \xrightarrow{d_i} A_{i-1} \to \cdots,$$

to the underlying Λ-module $\prod_{i \in \mathbb{Z}} A_i$. The differential of ΠA, which we by abuse of notation[1] denote by ϵ_A, is given by

$$(a_i) \mapsto \big(d_{i+1}(a_{i+1})\big).$$

On morphisms, Π acts as $\oplus$.

[1]This is abusive because ϵ_A also denotes the differential of $\oplus A$.

REMARK. If A is a bounded complex then $\oplus A = \Pi A$. $\square$

6.2 LEMMA. *For each $M, N \in D(\operatorname{Mod}\Lambda)$ there is an isomorphism*

$$\prod_{i\in\mathbb{Z}} \operatorname{Hom}_{D(\operatorname{Mod}\Lambda)}(M, N[i]) \cong \operatorname{Hom}_{D_{\mathrm{ung}}(\operatorname{Mod}\Lambda)}(\oplus M, \Pi N).$$

PROOF. Our first instinct should be that we would like to work with homotopy categories rather than with localizations. This can be obtained by replacing M by a homotopically projective resolution, in which case there is a natural isomorphism

$$\operatorname{Hom}_{D(\operatorname{Mod}\Lambda)}(M, N) \cong \operatorname{Hom}_{K(\operatorname{Mod}\Lambda)}(M, N).$$

Lemma 5.5 says that $\oplus M$ will be a homotopically projective $\Lambda[\epsilon]$-module, so there is a natural isomorphism

$$\operatorname{Hom}_{D_{\mathrm{ung}}(\operatorname{Mod}\Lambda)}(\oplus M, \Pi N) \cong \operatorname{Hom}_{K_{\mathrm{ung}}(\operatorname{Mod}\Lambda)}(\oplus M, \Pi N).$$

by the proof of Theorem 5.9. Therefore, proving the proposition reduces to showing

$$\prod_{i\in\mathbb{Z}} \operatorname{Hom}_{K(\operatorname{Mod}\Lambda)}(M, N[i]) \cong \operatorname{Hom}_{K_{\mathrm{ung}}(\operatorname{Mod}\Lambda)}(\oplus M, \Pi N)$$

which we will manage.

We start by describing a map

$$\Phi : \prod_{i\in\mathbb{Z}} \operatorname{Hom}_{K(\operatorname{Mod}\Lambda)}(M, N[i]) \to \operatorname{Hom}_{K_{\mathrm{ung}}(\operatorname{Mod}\Lambda)}(\oplus M, \Pi N).$$

Let $f = \big(f^i\big) \in \prod_{i\in\mathbb{Z}} \operatorname{Hom}_{K(\operatorname{Mod}\Lambda)}(M, N[i])$ where $f^i : M \to N[i]$ is the morphism of complexes

$$\begin{tikzcd}
\cdots \arrow[r] & M_{j+1} \arrow[r, "d^M_{j+1}"] \arrow[d, "f^i_{j+1}"] & M_j \arrow[r, "d^M_j"] \arrow[d, "f^i_j"] & M_{j-1} \arrow[r] \arrow[d, "f^i_{j-1}"] & \cdots \\
\cdots \arrow[r] & N[i]_{j+1} \arrow[r, "d^{N[i]}_{j+1}"] & N[i]_j \arrow[r, "d^{N[i]}_i"] & N[i]_{j-1} \arrow[r] & \cdots
\end{tikzcd}$$

Define $\Phi(f) : \oplus M \to \Pi N$ as the morphism whose component $M_j \to \Pi N$ is given by

$$m_j \mapsto \big(f^i_j(m_j)\big)_{i\in\mathbb{Z}}.$$ [2]

[2] This is why we use ΠN and not $\oplus N$. Since $\big(f^i_j(m_j)\big)$ will in general not have only finitely many non-zero entries, Φ will not be a map to $\operatorname{Hom}_{K_{\mathrm{ung}}(\operatorname{Mod}\Lambda)}(\oplus M, \oplus N)$.

It is immediate that $\Phi(f)$ is $\Lambda[\epsilon]$-linear. We should also check that Φ is well defined in the sense that it is 'compatible with homotopy'. So assume $f = 0$, which means that each f^i is nullhomotopic. We need to find a Λ-linear $s : \oplus M \to \Pi N$ such that $\Phi(f) = s\epsilon_M + \epsilon_N s$. The assumption means that, for each i, there is a morphism $s^i = \left(s^i_j\right) : M \to N[i]$ of degree 1 (i.e. $s^i_j : M_j \to N[i]_{j+1}$) such that

$$f^i_j = d^{N[i]}_{j+1} s^i_j + s^i_{j-1} d^M_j.$$

It is only natural to choose s as the morphism whose component $M_j \to \Pi N$ is given by

$$m_j \mapsto \left(s^i_j(m_j)\right)_{i\in\mathbb{Z}}.$$

Since the differentials of $\oplus M$ and ΠN restricted to direct summands are by definition the differentials of M and N, respectively, it immediately follows that $\Phi(f) = s\epsilon_M + \epsilon_N s$.

To show that Φ is an isomorphism we will produce an inverse

$$\Psi : \mathrm{Hom}_{K_{\mathrm{ung}}(\mathrm{Mod}\,\Lambda)}(\oplus M, \Pi N) \to \prod_{i\in\mathbb{Z}} \mathrm{Hom}_{K(\mathrm{Mod}\,\Lambda)}(M, N[i]).$$

Let $g : \oplus M \to \Pi N$ be a morphism in $K_{\mathrm{ung}}(\mathrm{Mod}\,\Lambda)$ and denote by g^i_j the composition

$$M_j \xrightarrow{\mu} \oplus M \xrightarrow{g} \Pi N \xrightarrow{\pi} N[i]_j$$

where μ and π are the canonical inclusion and projection, respectively. Fixing i we obtain the following diagram.

$$\begin{tikzcd}
\cdots \arrow[r] & M_{j+1} \arrow[r, "d^M_{j+1}"] \arrow[d, "g^i_{j+1}"] & M_j \arrow[r, "d^M_j"] \arrow[d, "g^i_j"] & M_{j-1} \arrow[r] \arrow[d, "g^i_{j-1}"] & \cdots \\
\cdots \arrow[r] & N[i]_{j+1} \arrow[r, "d^{N[i]}_{j+1}"] & N[i]_j \arrow[r, "d^{N[i]}_i"] & N[i]_{j-1} \arrow[r] & \cdots
\end{tikzcd}$$

This will be denoted by $\left(g^i\right)$ and is a morphism of complexes because g is $\Lambda[\epsilon]$-linear. We let $\Psi(g) := \left(g^i\right)_{i\in\mathbb{Z}}$.

To see why Ψ is compatible with homotopy, assume g is nullhomotopic. We need to show that $\left(g^i\right)$ is nullhomotopic for each i. By assumption there is a Λ-linear $s : \oplus M \to \Pi N$ such that $g = \epsilon_N s + s\epsilon_M$. Denote by s^i_j the composition

$$M_j \xrightarrow{\mu} \oplus M \xrightarrow{s} \Pi N \xrightarrow{\pi} N[i]_{j+1}$$

with the canonical inclusion and projection on the flanks. This gives

$$g^i_j = d^{N[i]}_{j+1} s^i_j + s^i_{j-1} d^M_j$$

for each $j \in \mathbb{Z}$ by the assumption on g. The corresponding diagram making it clear that $\left(g^i\right)$ is nullhomotopic is

$$\begin{tikzcd} & M_j \arrow[r, "d_j^M"] \arrow[dl, "s_j^i"'] \arrow[d, "g_j^i"] & M_{j-1} \arrow[dl, "s_{j-1}^i"] \\ N[i]_{j+1} \arrow[r, "d_{j+1}^{N[i]}"] & N[i]_j & \end{tikzcd}$$

Now we have well defined maps in both directions and what remains is checking that they are mutually inverse. This, however, is now evident. □

For technical reasons again, we employ the following construction.

DEFINITION. Given a category $\mathcal{C}$ and an equivalence $F : \mathcal{C} \to \mathcal{C}$ the associated **strong orbit category** is the category $\mathcal{C}/F_{\text{strong}}$ whose objects are those of $\mathcal{C}$ and whose morphisms are given by

$$\operatorname{Hom}_{\mathcal{C}/F_{\text{strong}}}(A, B) := \prod_{n \in \mathbb{Z}} \operatorname{Hom}_{\mathcal{C}}(A, F^n B).$$

REMARK. If Λ is an iterated tilted algebra then only finitely many $\operatorname{Hom}_{D(\operatorname{Mod}\Lambda)}(M, N[i])$ are non-zero and hence

$$D^b(\operatorname{Mod}\Lambda)/(-[1])_{\text{strong}} = D^b(\operatorname{Mod}\Lambda)/(-[1]).$$

□

We turn our attention towards describing a functor

$$E : D^b(\operatorname{Mod}\Lambda)/(-[1])_{\text{strong}} \to D_{\text{ung}}(\operatorname{Mod}\Lambda).$$

Since $D^b(\operatorname{Mod}\Lambda)/(-[1])_{\text{strong}}$ coincides with $D^b(\operatorname{Mod}\Lambda)$ on objects we take E to be given by $\oplus$ on these. On morphisms we essentially use Φ from the proof of Lemma 6.2. To be more explicit, take $M, N \in D^b(\operatorname{Mod}\Lambda)$ (note that this means $\Pi N = \oplus N$) and some $f \in \prod_{i \in \mathbb{Z}} \operatorname{Hom}_{D^b(\operatorname{Mod}\Lambda)}(M, N[i])$. Identify f with $\hat{f} \in \prod_{i \in \mathbb{Z}} \operatorname{Hom}_{K(\operatorname{Mod}\Lambda)}(M, N[i])$ and map it by Φ into $\operatorname{Hom}_{K_{\text{ung}}(\operatorname{Mod}\Lambda)}(\oplus M, \oplus N) \cong \operatorname{Hom}_{D_{\text{ung}}(\operatorname{Mod}\Lambda)}(\oplus M, \oplus N)$.

6.3 COROLLARY. *E is full and faithful.*

PROOF. Since E is given by Φ on morphisms, the fact that it is full and faithful is just a different way of stating Lemma 6.2. □

6.2 Arriving at Our Main Result

We are almost able to prove Theorem 6.1. The only ingredient missing is provided by the following result.

6.4 Lemma. $\oplus : D(\operatorname{Mod}\Lambda) \to D_{\text{ung}}(\operatorname{Mod}\Lambda)$ *is a triangle functor.*

Proof. Take a morphism $f : A \to B$ of complexes. Recall that the mapping cone of f is the complex $A[1] \oplus B$ with differential given by the matrix $\left(\begin{smallmatrix} -d^A & 0 \\ f & d^B \end{smallmatrix}\right)$ and that this fits into the standard triangle

$$A \xrightarrow{f} B \xrightarrow{\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)} A[1] \oplus B \xrightarrow{(1\ 0)} A[1].$$

It is sufficient to check that this triangle gets mapped by the $\oplus$-functor to a distinguished triangle in $D_{\text{ung}}(\operatorname{Mod}\Lambda)$.

Start by noting that for each complex A we have

$$\oplus(A[1]) = (\bigoplus_{i\in\mathbb{Z}} A_i, -\epsilon_A) = \Sigma(\oplus A).$$

Further, the mapping cone of $\oplus f$ (in the sense of Section 3.4) is the underlying Λ-module $M_{\oplus f} = (\oplus A) \oplus (\oplus B)$ with differential $\left(\begin{smallmatrix} -\epsilon_A & 0 \\ \oplus f & \epsilon_B \end{smallmatrix}\right)$, which clearly coincides with $\oplus(A[1] \oplus B)$. Hence, proving the lemma reduces to showing that

$$\oplus A \xrightarrow{\oplus f} \oplus B \xrightarrow{\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)} M_{\oplus f} \xrightarrow{(1\ 0)} \Sigma(\oplus A).$$

is distinguished. But by Lemma 3.8, this is the standard triangle associated to $\oplus f$. $\square$

Now finishing the chapter is just a matter of combining some of our results.

Proof of Theorem 6.1. Since Λ is iterated tilted $D^b(\operatorname{Mod}\Lambda)/(-[1])$ coincides with $D^b(\operatorname{Mod}\Lambda)/(-[1])_{\text{strong}}$ and there is a commutative diagram

$$\begin{tikzcd} D^b(\operatorname{Mod}\Lambda) \arrow[rr, "\oplus"] \arrow[dr, "\pi"'] & & D_{\text{ung}}(\operatorname{Mod}\Lambda) \\ & D^b(\operatorname{Mod}\Lambda)/(-[1]) \arrow[ur, "E"'] & \end{tikzcd}$$

Together with the density of $\oplus : D^b(\operatorname{Mod}\Lambda) \to D_{\text{ung}}(\operatorname{Mod}\Lambda)$ (Corollary 4.8), this implies that also E must be dense. The latter is therefore an equivalence (it is already full and faithful by Corollary 6.3), so a triangulated structure is imposed by $D_{\text{ung}}(\operatorname{Mod}\Lambda)$ on $D^b(\operatorname{Mod}\Lambda)/(-[1])$. Since $\oplus$ is a triangle functor by Lemma 6.4, this triangulated structure makes π a triangle functor. $\square$

As a final remark we write down an easy consequence.

6.5 Corollary. *Let Λ be noetherian and iterated tilted. Then $D^b(\operatorname{mod}\Lambda)/(-[1])$ is triangulated in such a way that the canonical projection*

$$\pi : D^b(\operatorname{mod}\Lambda) \to D^b(\operatorname{mod}\Lambda)/(-[1])$$

is a triangle functor.

Proof. By the remark following Corollary 4.8, $\oplus : D^b(\operatorname{mod}\Lambda) \to D_{\mathrm{ung}}(\operatorname{mod}\Lambda)$ is dense. Clearly, by the same argument that was used to show Lemma 6.4, this is a triangle functor. Further, if $M, N \in D^b(\operatorname{mod}\Lambda)$ then

$$\prod_{i\in\mathbb{Z}} \operatorname{Hom}_{D^b(\operatorname{mod}\Lambda)}(M, N[i]) \cong \operatorname{Hom}_{D_{\mathrm{ung}}(\operatorname{mod}\Lambda)}(\oplus M, \oplus N)$$

by simply restricting the discussion of the previous section to $D^b(\operatorname{mod}\Lambda)$. So we get an embedding $E' : D^b(\operatorname{mod}\Lambda)/(-[1]) \to D_{\mathrm{ung}}(\operatorname{mod}\Lambda)$ as above, and with it a commutative triangle

$$\begin{tikzcd} D^b(\operatorname{mod}\Lambda) \arrow[rr, "\oplus"] \arrow[dr, "\pi"'] & & D_{\mathrm{ung}}(\operatorname{mod}\Lambda) \\ & D^b(\operatorname{mod}\Lambda)/(-[1]) \arrow[ur, "E'"'] & \end{tikzcd}$$

Now we are in the setup of the proof of Theorem 6.1 and the corollary follows. $\square$

CHAPTER 7

Further Research

This last chapter of the thesis differs from chapters 2 through 6 as we do not present results of our own. The main objective is discussing briefly the possibility of a converse of Theorem 6.1, so a somewhat imprecise style is necessary. The papers [Rin98] by Ringel and [HZ08] by Happel and Zacharia give some reasons why one could expect such a converse to be true. Therefore, we start by giving a short summary of these.

7.1 Piecewise Hereditary Algebras

The aim of the following summary is not to give a complete account of [Rin98] and [HZ08], but rather to build a sketch of the proof of the main result of the latter. Both papers give characterizations of piecewise hereditary algebras, a class of algebras that includes the iterated tilted algebras.

Definition. An algebra Λ is **piecewise hereditary** if there is a hereditary abelian category $\mathcal{H}$ such that $D^b(\operatorname{mod}\Lambda)$ is triangle equivalent to $D^b(\mathcal{H})$.

A key notion will be that of a path in a triangulated category.

Definition. Let $\mathcal{T}$ be a triangulated category with translation $-[1]$. A **path** in $\mathcal{T}$ of length n is a sequence $X_0, \ldots, X_n$ of indecomposable objects such that either $X_i = X_{i-1}[1]$ or $\operatorname{Hom}_{\mathcal{T}}(X_{i-1}, X_i) \neq 0$ for each $1 \leq i \leq n$. The path is **strong** if $\operatorname{Hom}_{\mathcal{T}}(X_{i-1}, X_i) \neq 0$ for each $1 \leq i \leq n$.

Remark. Using only elementary concepts one can show that if Λ is a connected algebra which is not semi-simple then any path in $D^b(\operatorname{mod}\Lambda)$ can be refined to a strong path. $\square$

Ringel gives a characterization of piecewise hereditary algebras in terms of the nonexistence of certain paths. To obtain this he first shows that a well known structural property of $D^b(\mathcal{H})$ for $\mathcal{H}$ hereditary abelian is in fact characteristic. To be precise, if $\mathcal{T}$ is triangulated and $\mathcal{H}$ is a full subcategory such that $\mathcal{T} = \operatorname{add}\left(\bigcup_{i\in\mathbf{Z}} \mathcal{H}[i]\right)$ and $\operatorname{Hom}_{\mathcal{T}}(\mathcal{H}[i], \mathcal{H}[j]) = 0$ for $i > j$, then $\mathcal{H}$ is hereditary abelian and canonically embedded in $\mathcal{T} = D^b(\mathcal{H})$. The non-trivial implication of the below theorem then follows from a somewhat tedious but straight forward argument.

We should note that the original version of the following theorem also includes a homogeneity property. Namely, the existence of a single indecomposable $X \in D^b(\operatorname{mod}\Lambda)$ with no path from $X[1]$ to X implies that this property holds for all indecomposables.

MAIN RESULT OF [RIN98]. *An algebra* Λ *is piecewise hereditary if and only if for each indecomposable* $X \in D^b(\operatorname{mod}\Lambda)$ *there is no path from* $X[1]$ *to* X.

We shall soon see how this paves the way for the main theorem of Happel and Zacharia.

The main technical tool developed in [HZ08] is concerned with shortening strong paths. The proof involves a 'trick' in the form of making a choice that may seem uncalled-for at first glance, and is well worth a read. However, to keep this section from getting too lengthy we do not include it here. The following version is formulated in less generality than the one appearing in the paper, but sufficient for our purpose.

LEMMA FROM [HZ08]. *If* $X_0, \dots, X_n$ *is a strong path in* $K^b(\operatorname{proj}\Lambda)$*, then there is some* $0 \le t \le n-2$*, an indecomposable* Y *in* $K^b(\operatorname{proj}\Lambda)$ *and a strong path* $X_0[t] \to Y \to X_n$.

To state the main result of Happel and Zacharia we must first establish the concept of strong global dimension. In the category $C^b(\operatorname{mod}\Lambda)$ of bounded complexes, one easily checks that the indecomposable projective objects are the complexes

$$\cdots \to 0 \to P \xrightarrow{1} P \to 0 \to \cdots$$

where $P \in \operatorname{mod}\Lambda$ is indecomposable. In analogue to the ungraded setup in which we worked in the previous chapters, the homotopy category $K^b(\operatorname{mod}\Lambda)$ conicides with the stable category of $C^b(\operatorname{mod}\Lambda)$ modulo projectives. Given a complex $X \in K^b(\operatorname{mod}\Lambda)$ we say that its **pre-image** in $C^b(\operatorname{mod}\Lambda)$ is the complex $\hat{X}$ with no projective summands satisfying $X \cong \hat{X}$ in $K^b(\operatorname{mod}\Lambda)$. Note that the pre-image is uniquely determined up to isomorphism in $C^b(\operatorname{mod}\Lambda)$ and hence allows the following definition. If $X \neq 0$ then there are integers $r \geq s$ such that $\hat{X}_r \neq 0 \neq \hat{X}_s$ and $\hat{X}_i = 0$ for each $i \geq r$ and $i \leq s$. The **length** of X is $\ell(X) := r - s$.

DEFINITION. The **strong global dimension** of an algebra Λ is

$$\operatorname{s.gl.dim}\Lambda := \sup\{\ell(X) : X \in K^b(\operatorname{proj}\Lambda) \text{ indecomposable}\}.$$

We are now ready for the final result of this summary. Because it is the important one for the purpose of the next section, and because the argument is elegant and demonstrates the value of the discussion of paths, we include a proof.

MAIN RESULT OF [HZ08]. *An algebra is piecewise hereditary if and only if its strong global dimension is finite.*

PROOF. Assume Λ has finite strong global dimension. This means in particular that the global dimension of Λ is finite, hence $D^b(\operatorname{mod}\Lambda) \cong K^b(\operatorname{proj}\Lambda)$. Let P be an indecomposable projective Λ-module. If Λ is not piecewise hereditary then by [Rin98] there is a path in $K^b(\operatorname{proj}\Lambda)$ from $P[1]$ to P which we may refine to a strong path (we can clearly assume that Λ is connected and different from k). This yields the existence of a strong path from $P[n]$ to P for any $n \geq 1$. By the above lemma there is a positive integer t, an indecomposable $Q^{n,t} \in K^b(\operatorname{proj}\Lambda)$ and a strong path

$$P[n+t] \to Q^{n,t} \to P.$$

This means $Q^{n,t}_{n+t} \neq 0 \neq Q^{n,t}_0$, hence $\ell(Q^{n,t}) \geq n+t$, so there are indecomposable complexes in $K^b(\operatorname{proj}\Lambda)$ of arbitrary length, contradicting $\operatorname{s.gl.dim}\Lambda < \infty$.

Before proceeding, note that if $F : D^b(\operatorname{mod}\Lambda) \to D^b(\mathcal{H})$ is an equivalence with $\mathcal{H}$ hereditary, then we may with no loss of generality assume F is normalized. I.e. there is some $r \geq 0$ such that each indecomposable Λ-module is contained in $\bigcup_{i=0}^{r} \mathcal{H}[i]$ and further there are indecomposable $X, Y \in \operatorname{mod}\Lambda$ such that $FX \in \mathcal{H}[0]$ and $FY \in \mathcal{H}[r]$. It is shown in [Hap88, IV.1] that for each $0 \leq i \leq r$ there is a simple Λ-module S_i such that $FS_i \in \mathcal{H}[i]$. Thus $r \leq \operatorname{rank} K_0(\Lambda) - 1$ where $K_0(\Lambda)$ is the Grothendieck group of Λ.

This sets up the proof of the remaining implication[1] nicely. Let $P \in K^b(\operatorname{proj}\Lambda)$ be indecomposable with $\ell(P) = t$. Up to shifting we can assume $P_0 \neq 0 \neq P_t$ and hence $P_i = 0$ for each $i > t$ and $i < 0$. Thus, there is the diagram

$$\begin{tikzcd}
P_0 = & & & & & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d, "1"] & 0 \arrow[r] \arrow[d] & \cdots \\
P = & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P_t \arrow[r] \arrow[d, "1"] & P_{t-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] & 0 \arrow[r] & \cdots \\
P_t[t] = & \cdots \arrow[r] & 0 \arrow[r] & P_t \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

showing that

$$\operatorname{Hom}_{K^b(\operatorname{mod}\Lambda)}(P_0, P) \neq 0 \neq \operatorname{Hom}_{K^b(\operatorname{mod}\Lambda)}(P, P_t[t])$$

as we can assume P has no projective summands. If Λ is piecewise hereditary there is a normalized equivalence $F : D^b(\operatorname{Mod}\Lambda) \to D^b(\mathcal{H})$, so we have $FP_0 \in \bigcup_{i=0}^{r} \mathcal{H}[i]$ and $FP_t[t] \in \bigcup_{i=t}^{r+t} \mathcal{H}[i]$. Because P is indecomposable we also have $FP \in \mathcal{H}[s]$ for some s. Since

$$\operatorname{Hom}_{D^b(\mathcal{H})}(\mathcal{H}[n], \mathcal{H}[m]) \neq 0 \implies m \in \{n, n+1\}$$

we get $s \leq r+1$ and $t-1 \leq s$. Combining these gives $t \leq r+2$, i.e. $\operatorname{s.gl.dim}\Lambda < \infty$. $\square$

7.2 Connection to Our Work

In [Kel05][2] Keller provides examples of algebras of both infinite and finite global dimension for which the conclusion of Theorem 6.1 does not hold. This raises the natural question

[1] In fact, finiteness of the strong global dimension of a piecewise hereditary algebra was shown already in [KSYZ04]. We give a different proof.

[2] In this paper Keller actually shows a result that implies our Theorem 6.1, but in a different way from us entirely. His approach involves a construction called the 'triangulated hull' of the orbit category using the formalism of dg categories.

of whether Theorem 6.1 can be strengthened to an if and only if statement.

QUESTION. Let Λ be an algebra and assume $D^b(\operatorname{Mod}\Lambda)/(-[1])$ admits a triangulation such that the canonical functor

$$\pi : D^b(\operatorname{Mod}\Lambda) \to D^b(\operatorname{Mod}\Lambda)/(-[1])$$

is a triangle functor. Does it follow that Λ is iterated tilted?

By the main result of [HZ08] the statment 'each indecomposable in $D^b(\operatorname{Mod}\Lambda)$ is a stalk complex' is utterly false unless Λ is piecewise hereditary. Recall that a crucial point in our proof of Theorem 6.1 was showing an ungraded analogue of this statement in the hereditary case, saying that any indecomposable in the ungraded derived category has vanishing differential. Consequently, any $\Lambda[\epsilon]$-module admits a grading as long as Λ is piecewise hereditary. If the analogy between the graded and the ungraded setups extends to the non-hereditary case, then it would essentially read 'there are $\Lambda[\epsilon]$-modules that do not admit a grading', i.e. $\oplus : D^b(\operatorname{Mod}\Lambda) \to D_{\text{ung}}(\operatorname{Mod}\Lambda)$ is not dense. By our proof of Theorem 6.1, this would imply that $D^b(\operatorname{Mod}\Lambda)/(-[1]) \ncong D_{\text{ung}}(\operatorname{Mod}\Lambda)$.

A priori, of course, this does not mean that the conclusion of Theorem 6.1 fails. However, the construction of the triangulated hull by Keller in [Kel05] seems to indicate that $D_{\text{ung}}(\operatorname{Mod}\Lambda)$ is the only candidate among triangulated categories for $D^b(\operatorname{Mod}\Lambda)/(-[1])$ to be equivalent to in order for the projection π to be a triangle functor.

Bibliography

[GM03] Sergei I. Gelfand and Yuri I. Manin, *Methods of homological algebra*, second ed., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003. MR 1950475 (2003m:18001)

[GZ67] P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35, Springer-Verlag New York, Inc., New York, 1967. MR 0210125 (35 #1019)

[Hap88] Dieter Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series, vol. 119, Cambridge University Press, Cambridge, 1988. MR 935124 (89e:16035)

[HJ10] Thorsten Holm and Peter Jørgensen, *Triangulated categories: definitions, properties, and examples*, Triangulated categories, London Math. Soc. Lecture Note Ser., vol. 375, Cambridge Univ. Press, Cambridge, 2010, pp. 1–51. MR 2681706 (2012i:18011)

[HZ08] Dieter Happel and Dan Zacharia, *A homological characterization of piecewise hereditary algebras*, Math. Z. **260** (2008), no. 1, 177–185. MR 2413349 (2009g:16011)

[Kel90] Bernhard Keller, *Chain complexes and stable categories*, Manuscripta Math. **67** (1990), no. 4, 379–417. MR 1052551 (91h:18006)

[Kel98] ———, *On the construction of triangle equivalences*, Derived equivalences for group rings, Lecture Notes in Math., vol. 1685, Springer, Berlin, 1998, pp. 155–176. MR 1649844

[Kel05] ———, *On triangulated orbit categories*, Doc. Math. **10** (2005), 551–581. MR 2184464 (2007c:18006)

[Kön98] Steffen König, *Rickard's fundamental theorem*, Derived equivalences for group rings, Lecture Notes in Math., vol. 1685, Springer, Berlin, 1998, pp. 33–50. MR 1649839

[Kra07] Henning Krause, *Derived categories, resolutions, and Brown representability*, Interactions between homotopy theory and algebra, Contemp. Math., vol. 436, Amer. Math. Soc., Providence, RI, 2007, pp. 101–139. MR 2355771 (2008h:18012a)

[KSYZ04] Otto Kerner, Andrzej Skowroński, Kunio Yamagata, and Dan Zacharia, *Finiteness of the strong global dimension of radical square zero algebras*, Cent. Eur. J. Math. **2** (2004), no. 1, 103–111 (electronic). MR 2041672 (2005g:16012)

[May01] J. P. May, *The additivity of traces in triangulated categories*, Adv. Math. **163** (2001), no. 1, 34–73. MR 1867203 (2002k:18019)

[Qui73] Daniel Quillen, *Higher algebraic K-theory. I*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 85–147. Lecture Notes in Math., Vol. 341. MR 0338129 (49 #2895)

[Rin98] C.M. Ringel, *Hereditary triangulated categories*, Preprint, Sonderforschungsbereich 343, 1998.

[Ver96] Jean-Louis Verdier, *Des catégories dérivées des catégories abéliennes*, Astérisque (1996), no. 239, xii+253 pp. (1997), With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis. MR 1453167 (98c:18007)

[Wei94] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR 1269324 (95f:18001)