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Applications of p-adic Numbers to well understood Quantum Mechanics

With a focus on Weyl Systems and the
Harmonic Oscillator

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Abstract

In this thesis we look at how it is possible to construct models in quantum mechanics by using p-adic numbers.

First we look closely at different quantum mechanical models using the real numbers, as it is necessary to understand them well before moving on to p-adic numbers. The most promising model, where Weyl systems are used, is studied in detail. Here time translation is not generated by the Hamiltonian, but constructed directly as an operator possessing some fundamental structure in relation to the classical dynamics.

Then we develop the relevant theory of the field of p-adic numbers $\mathbb{Q}_p$, with a focus on the properties of $\mathbb{Q}_p$ as a locally compact abelian group. Here we present alternative proofs to those found in the literature. In particular, we give an independent proof of the selfduality of $\mathbb{Q}_p$.

In the last chapters we look at some models using $\mathbb{Q}_p$. We generalize the idea of Weyl systems from real to p-adic numbers, and we discuss the physical implications. When using Weyl systems, time is p-adic.

We also produce MatLab algorithms for numerical computations in connection with approximations of p-adic models by finite models.

Sammendrag (abstract in Norwegian)

I denne masteroppgaven ser vi på mulighetene for å konstruere modeller i kvantemekanikk ved bruk av p-adiske tall.

Først ser vi på forskjellige kvantemekaniske modeller som bruker de reelle tall, da det er viktig å forstå disse godt før vi prøver å generalisere dem til det p-adiske tilfellet. Den mest lovende modellen, hvor Weyl-system benyttes, blir studert i detalj. Her er ikke tidstranslasjon generert av Hamiltonoperatoren, men konstruert direkte som en operator som oppfyller likninger som knytter den til klassisk dynamikk.

Deretter diskuterer vi relevant teori for de p-adiske tallene $\mathbb{Q}_p$, med fokus på egenskapene til $\mathbb{Q}_p$ som en lokalkompakt abelsk gruppe. Her presenterer vi et uavhengig bevis for selvdualitet av $\mathbb{Q}_p$.

I de siste kapitlene ser vi på noen konkrete kvantemekaniske modeller som tar i bruk $\mathbb{Q}_p$. Vi generaliserer Weyl-systemet til p-adiske tall, og vi diskuterer de fysiske konsekvensene av modellene; her er tiden p-adisk.

I tillegg produserer vi MatLab-algoritmer for numeriske beregninger.

Preface

My education is a combination of physics and mathematics, which clearly shows in this thesis. So for physicists it will be a bit terse (especially the chapter on p-adic analysis), and for mathematicians the chapter on quantum mechanics may contain a bit of hand-waving. The chapters on real quantum mechanics and on p-adic analysis can be read separately, and not all parts are strictly needed for the remaining chapters; some sections are included to give us a broader understanding. Think of this thesis more as presenting the possibilities than leading up to a final conclusion.[1]

I apologize in advance for some of the huge calculations, but they are essential and I was not able to find them in the literature.

I would like to thank professor Trond Digernes for his excellent supervision, and for teaching me operator algebra which was essential in developing a clear and intuitive understanding of quantum mechanics. I would also like to thank the rest of the department of mathematics, and the department of physics, for teaching me all the wonderful things they did, and for the opportunities they gave me for passing on knowledge to other students. In the end, a master degree would have been boring without my friends. So thank you all for the wonderful times, the discussions and the laughter (and the dinners); and for making me more of a Leonard than a Sheldon.

[1] The title of this thesis is: Applications of p-adic numbers to well understood quantum mechanics.

Contents

Chapter 1

Introduction

1.1 When all you have is a hammer, everything looks like a nail

Let us first try to tell you what we are trying to accomplish. We want to make models in quantum mechanics using the p-adic numbers. Why? Because it might work.[1] There are lots of speculations on this subject, we refer the interested reader to the review [1] by Dragovich et. al. from 2008. In this thesis we will take a broad perspective, and try to follow several different approaches. To do that we need several different ways of understanding quantum mechanics; and we have devoted an entire chapter to this. We will also need quite a bit of mathematical background, and we will look at analysis in $\mathbb{Q}_p$. The presentation will also include numerical schemes for calculations.

But what kind of problems in quantum mechanics do we want to solve? None, our primary focus is to reinvent the wheel, to reproduce known results in the well understood parts of quantum mechanics. The benefit will come later, when applying p-adic numbers to unresolved problems.[2]

There are several papers and books focused on developing quantum theories over the p-adic numbers and the adeles, as in [2], [3] and [4]. Here quantum mechanics over the p-adic numbers means that the wavefunctions have as domain the p-adic numbers and as codomain the complex numbers. Whether time is real or p-adic varies. However, we have not found anyone showing how this can be used to compute the results we already know from doing quantum mechanics over the real numbers.

[1] But even though it does not seem unreasonable that p-adic numbers will be used in modeling physical systems sooner or later, there must be some reason for us to promote this way of making models, enough of a reason to spend time studying them. In some mathematical sense, p-adic numbers is the second most natural field to study after $\mathbb{R}$ (and $\mathbb{C}$). This is because of the classification theorem for locally compact fields which says that any completion of $\mathbb{Q}$ is either $\mathbb{R}$, or $\mathbb{Q}_p$ for some prime p. This theorem is also known as Ostrowski's theorem.

[2] One thing we can hope to accomplish in the long run is gaining insight into the bad convergence properties, like renormalization.

1.2 Mathematical background

In this section we will give a short review of the mathematical background we need. We assume that the reader is familiar with (the definitions and basic theorems of) groups, rings, fields, vector spaces, metric spaces, measure theory, probability theory, Hilbert spaces and distributions.[3] In addition it would be helpful to have a familiarity with p-adic numbers. For p-adic numbers we refer to Gouvea's book [5] (with a focus on number theory), and Vladimirov's book [2] (with a focus on integration theory and mathematical physics). In addition, the book [6] by Kochubei is very readable, and deals with a slightly generalized version of the p-adic numbers (he considers any locally compact field).

1.2.1 The p-adic numbers

Let us take a quick recap of p-adic numbers. First we define the p-adic norm $|-|_p$. For $x \in \mathbb{Z}$ we have $|x|_p = p^{-l}$ if p^l divides x, but p^{l+1} does not divide x. We can extend this norm to $\mathbb{Q}$ by $|\frac{a}{b}|_p = \frac{|a|_p}{|b|_p}$. This norm gives a metric, and we call the completion of $\mathbb{Q}$ under this metric for $\mathbb{Q}_p$. We will often write $|x| = |x|_p$ when $x \in \mathbb{Q}_p$. It is well known that there is a unique representation of $x \in \mathbb{Q}_p$ of the form

$$x = \sum_{v(x)}^{\infty} x_i p^i, \tag{1.1}$$

where $0 \leq x_i < p$ and $x_{v(x)} \neq 0$ and $v(x) \in \mathbb{Z}$ is a number depending on x. This also defines the valuation $v : \mathbb{Q}_p \to \mathbb{Z}$.[4]

1.2.2 Haar measure μ and Fourier transform $\mathcal{F}$ on an abelian group G

We will need integration theory and Fourier transforms on $\mathbb{R}, \mathbb{C}, \mathbb{T}, \mathbb{Q}_p$ and $\mathbb{Z}_p$. To present this in a unified way, we will do it more generally. So let $(G, +)$ be a locally compact abelian group (for example an abelian group with a metric where addition and inverse are continuous in said metric). First we will define a Haar measure[5] on G, then we will give the Fourier transform.

A Borel measure μ is a function on the sigma-algebra generated by the open sets in G, with the properties:

[3] Without this knowledge the proofs will be difficult to understand, but we will strive to make the discussion accessible to those with an intuitive understanding of these concepts, with a special eye to graduate level students in physics and/or mathematics.

[4] There are many ways to introduce the p-adic numbers, some start with the valuation and then get a metric, while some use the algebraic inverse limit.

[5] The concept of measure generalizes the concepts of length, area etc. The Lebesgue measure λ on $\mathbb{R}$ can be written as

$$\lambda(A) = \int_{-\infty}^{\infty} \chi_A(x)\, \mathrm{d}x, \tag{1.2}$$

where $\chi_A(x) = 1$ for $x \in A$ and 0 otherwise.

- $\mu(A) \in \mathbb{R}^+ \cup \{\infty\}$ for all A.
- $\mu(\emptyset) = 0$.
- Given a sequence[6] of pairwise disjoint sets A_i then

$$\mu\left(\bigcup_i A_i\right) = \sum_i \mu(A_i). \tag{1.3}$$

Definition 1.1. The measure μ is a Radon measure on G if:

- μ is a Borel measure.
- For any compact set K we have $\mu(K) < \infty$.
- Every Borel set E is outer regular, $\mu(E) = \inf\{\mu(U) : E \subseteq U,\ U \text{ open}\}$.
- Every open set $\mathcal{O}$ is inner regular, $\mu(\mathcal{O}) = \sup\{\mu(K) : K \subseteq \mathcal{O},\ K \text{ compact}\}$.

Definition 1.2. The measure μ is a Haar measure on G if:

- μ is a Radon measure.
- μ is translation invariant, i.e. $\mu(A + x) = \mu(A)$.

Theorem 1.3. *For any abelian locally compact group G there exist a Haar measure μ which is unique up to multiplication with a constant.*

Proof. See theorem 2.10 (page 37) in [7], and note that we are talking about an abelian group, so that the notions of left and right Haar measure are the same as our definition. □

To get uniqueness of the Haar measure, what we normally do is specify a subset H where $\mu(H)$ is not 0 or ∞ (for any Haar measure), and require $\mu(H) = 1$. For $\mathbb{R}$ we choose $H = [0, 1]$ and get the well known Lebesgue measure. For $\mathbb{Z}$ we choose $H = \{0\}$, and for $\mathbb{Q}_p$ we choose $H = \mathbb{Z}_p$. For compact groups we often choose $\mu(G) = 1$.

Given a group G and a Haar measure μ we can define integration[7], and we will write

$$\int_G f(x)\ \mathrm{d}\mu(x) = \int f(x)\ \mathrm{d}x. \tag{1.4}$$

The Fourier transform is an essential tool. To define the Fourier transform, we need the concept of dual group. We have already assumed that G is a locally compact abelian group. Let us also assume that the topology is second-countable, i.e. the topology has a countable basis.[8]

Let $\mathbb{T}$ denote the set of complex numbers with absolute value 1 under multiplication.

[6] This sequence may be finite or countable.

[7] For measurable functions and integration theory we refer you to McDonald and Weiss [8].

[8] We could define the Fourier transform for any locally compact (and Hausdorff) group, but this will not be useful to us. Assuming G is abelian, and that the topology is second countable makes the notation a lot simpler.

Definition 1.4. The dual of G is the set $\widehat{G}$ of continuous homomorphisms from G to $\mathbb{T}$, i.e. the collection of functions $\gamma : G \to \mathbb{T}$ with

$$\gamma(x+y) = \gamma(x)\gamma(y)$$
$$\gamma(\lim_n x_n) = \lim_n \gamma(x_n)$$

The following proposition is given without proof; see the beginning of chapter 4.1 in [7].

Proposition 1.5. The dual $\widehat{G}$ can be given an abelian structure by

$$(\gamma_1 + \gamma_2)(x) = \gamma_1(x)\gamma_2(x). \tag{1.5}$$

Furthermore, $\widehat{G}$ with the topology of uniform convergence on compacts is a locally compact group.

This means that we can take the dual of $\widehat{G}$. Before proceeding to some examples it is natural to state the Pontryagin duality theorem.

Theorem 1.6. *The natural evalulate-functorial is a group-isomorphism and a homeomorphism between G and $\widehat{\widehat{G}}$, by*

$$x \mapsto eval_x = [\gamma \mapsto \gamma(x)]. \tag{1.6}$$

Proof. See [7] section 4.3. □

Let us now look at some of the dual groups we will use. We will come back to the dual of, and Fourier transform over, $\mathbb{Q}_p$ later (in chapter 3).

Proposition 1.7. We have $\Psi : \mathbb{R} \simeq \widehat{\mathbb{R}}$ by

$$\Psi : x \mapsto \gamma_x = [y \mapsto e^{ixy}], \tag{1.7}$$

and $\Phi : \mathbb{Z} \simeq \widehat{\mathbb{T}}$ by

$$\Phi : n \mapsto \gamma_n = [z \mapsto z^n], \tag{1.8}$$

Proof. See [7] theorem 4.5. □

We note that the dual of $\mathbb{T}$ has the discrete topology. The following proposition is proposition 4.4 in [7].

Proposition 1.8. If G is compact we get that $\widehat{G}$ has the discrete topology.

Let $L^2(G)$ be the set of square integrable functions defined μ almost everywhere.

Definition 1.9. Given G and μ, the Fourier transform of $f \in L^2(G)$ gives $\mathcal{F}f \in L^2(\widehat{G})$ by

$$\tilde{f}(\gamma) = \mathcal{F}f(\gamma) = \int_G \gamma(x)^* f(x)\ \mathrm{d}\mu(x), \tag{1.9}$$

where z^* is complex conjugation.

Definition 1.10. The dual measure $\widehat{\gamma}$ on $\widehat{G}$ of μ on G, is the measure that makes the Fourier transform

$$\mathcal{F} : L^2(G, \mu) \to L^2(\widehat{G}, \widehat{\mu}) \tag{1.10}$$

unitary.

Theorem 1.11. *The inverse Fourier transform of $\tilde{f} \in L^2(\widehat{G})$ is given by*

$$\mathcal{F}^{-1}\tilde{f}(x) = \int_{\widehat{G}} \gamma(x)\tilde{f}(\gamma)\ \mathrm{d}\widehat{\mu}, \tag{1.11}$$

where $\widehat{\mu}$ is the dual of the Haar measure μ. If, $\tilde{f}$ is the Fourier transform of f, then $f(x) = \mathcal{F}^{-1}\tilde{f}(x) \in L^2$.

In applications of the Fourier transform, there are several choices to be made. Firstly there is a choice in the identification $\widehat{G}_1 \simeq G_2$, secondly there is a choice in the Haar measure (as this is unique up to a constant), and thirdly we can put a constant in front of the integral (which would correspond to changing the measure).

We will use unitary Fourier transforms, but there is still choices to be made. To clarify notation, we will now present the definitions we will use when writing $\mathcal{F}$. On $\mathbb{R}$ we use the normal Lebesgue measure.

Definition 1.12. The Fourier transform on $\mathbb{R}$ is

$$(\mathcal{F}f)(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixy} f(x)\ \mathrm{d}x, \tag{1.12}$$

with inverse

$$(\mathcal{F}^{-1}\tilde{f})(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ixy} \tilde{f}(y)\ \mathrm{d}y. \tag{1.13}$$

On the circle $\mathbb{T}$ we choose $\mu(\mathbb{T}) = 1$, or, said in another way, we have the measure-theoretical identification $t \mapsto e^{2\pi i t}$ from $[0, 1)$ onto $\mathbb{T}$, and we let the measure carry over from $[0, 1)$.

Definition 1.13. The Fourier transform on $\mathbb{T}$ is

$$\mathcal{F}f(n) = \int_{\mathbb{T}} z^{-n} f(z)\ \mathrm{d}z, \tag{1.14}$$

with inverse

$$\mathcal{F}^{-1}\tilde{f}(z) = \sum_{n=-\infty}^{\infty} z^n \tilde{f}(n). \tag{1.15}$$

Sometimes we want to consider $(\mathbb{T}, +) = \mathbb{R}/\mathbb{Z}$ (that is the interval $[0, 1)$ with operations modulo 1), which is naturally isomorphic to the torus. To avoid confusion we will use $\mathbb{T}$ for the complex numbers under multiplication, and $(\mathbb{T}, +)$ for the periodic interval.

Definition 1.14. The Fourier transform on $(\mathbb{T}, +)$ is

$$\mathcal{F}f(n) = \int_0^1 e^{-2\pi i n x} f(x)\ \mathrm{d}x, \tag{1.16}$$

with inverse

$$\mathcal{F}^{-1}\tilde{f}(x) = \sum_{n=-\infty}^{\infty} e^{2\pi i n x} \tilde{f}(n). \tag{1.17}$$

We have natural isomorphisms between $\epsilon\mathbb{Z}$ and $\mathbb{Z}$, and between $\epsilon\mathbb{Z}_n$ and $\mathbb{Z}_n$ (for any $\epsilon \in \mathbb{R}$). The Fourier transform can be transported by using these isomorphisms.

Definition 1.15. The Fourier transform on $\epsilon\mathbb{Z}_n$ ($\epsilon \in \mathbb{R}$) is

$$(\mathcal{F}f)(\epsilon j) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{-2\pi i \frac{k}{n} j} f(\epsilon k), \tag{1.18}$$

with inverse

$$(\mathcal{F}^{-1}\tilde{f})(\epsilon j) = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{2\pi i \frac{k}{n} j} \tilde{f}(\epsilon k). \tag{1.19}$$

Let us take a look at the general behavior when we have something akin to Fourier series; when we have summation instead of integration in the inverse Fourier transform.

Proposition 1.16. Assume G is compact, and $\mu(G) = 1$. Then $\widehat{G}$ is a basis for $L^2(G)$.

Proof. By proposition 1.8 we see that equation 1.11 becomes

$$f = \mathcal{F}^{-1}\tilde{f} = \sum_{\gamma \in \widehat{G}} \tilde{f}(\gamma)\gamma. \tag{1.20}$$

The Fourier transform is invertible, hence every $f \in L^2(G)$ can be written uniquely in this form. Hence $\widehat{G}$ is a complete set.

Orthonormality follows from the fact that $\mathcal{F}$ is an isometry, so that

$$\begin{aligned}
\langle \gamma_1 | \gamma_2 \rangle &= \langle \mathcal{F}\gamma_1 | \mathcal{F}\gamma_2 \rangle \\
&= \langle \begin{cases} 1 & \gamma = \gamma_1 \\ 0 & \text{else} \end{cases} | \begin{cases} 1 & \gamma = \gamma_2 \\ 0 & \text{else} \end{cases} \rangle \\
&= \sum_{\gamma \in \widehat{G}} \begin{cases} 1 & \gamma = \gamma_1 \\ 0 & \text{else} \end{cases} \cdot \begin{cases} 1 & \gamma = \gamma_2 \\ 0 & \text{else} \end{cases} \\
&= \delta(\gamma_1 - \gamma_2)
\end{aligned}$$

□

Proposition 1.17. Let $f : \mathbb{R} \to \mathbb{C}$, and $\mathcal{F}$ the Fourier transform on $\mathbb{R}$. Then

$$\mathcal{F}\frac{d^n}{dx^n}f = (iy)^n \mathcal{F}f \tag{1.21}$$

for any f in the Schwartz space.[9]

Proof.

$$\begin{aligned}
(\mathcal{F}f)(y) &= \frac{1}{\sqrt{2\pi}} \int e^{-ixy} f(x) \, \mathrm{d}x \\
(\mathcal{F}\frac{d^n}{dx^n}f)(y) &= \frac{1}{\sqrt{2\pi}} \int e^{-ixy} \frac{d^n}{dx^n} f(x) \, \mathrm{d}x \\
&= (-1)^n \frac{1}{\sqrt{2\pi}} \int f(x) \frac{d^n}{dx^n} e^{-ixy} \, \mathrm{d}x \\
&= (-1)^n \frac{1}{\sqrt{2\pi}} \int (-iy)^n e^{-ixy} f(x) \, \mathrm{d}x \\
&= (iy)^n \frac{1}{\sqrt{2\pi}} \int e^{-ixy} f(x) \, \mathrm{d}x \\
&= (iy)^n (\mathcal{F}f)(y)
\end{aligned}$$

□

1.3 Unitary representations of groups in Hilbert spaces

Definition 1.18. A strongly continuous unitary representation, also known as a unitary representation, of the locally compact group G on the Hilbert space $\mathcal{H}$ is a homomorphism

$$\pi : G \to \mathcal{U}(\mathcal{H}) = \{U : \mathcal{H} \to \mathcal{H} | U^{-1} = U^*\}, \tag{1.22}$$

where the group operation on $\mathcal{U}(\mathcal{H})$ is composition, and where the function $g \mapsto \pi(g)f$ is continuous for any $f \in \mathcal{H}$. The group G is *represented* as the image of π.

1.4 Classical physics

Before we end this chapter, let us review some classical physics. Newton's three laws is the first formulation of physics seen by a student. In addition we have the principle of least action together with the Lagrangian, and we have the Hamilton formulation.

In this section let $V(x, p, t)$ denote the potential energy of the mechanical system, dependent on the position $x \in \mathbb{R}^3$, the momentum $p \in \mathbb{R}^3$, and the time $t \in \mathbb{R}$. Further, $T(x, p, t)$ is the kinetic energy.

[9] We use the Schwartz space $\mathcal{S}$ when physicists write 'sufficiently nice'. This is the space of infinitely differentiable functions decaying faster than any x^{-2k}.

Newton's equation now gives a second order differential equation where we have initial values $[x(t_1), p(t_1)]$ or boundary values $[x(t_1), x(t_2)]$ (for $t_1 \neq t_2$), and which can be solved both forwards and backwards in time.

A different approach is via the Lagrangian and the action.

Definition 1.19. The Lagrangian of a physical system is

$$L = T - V, \tag{1.23}$$

and depends on a parameter x, and its derivative $\dot{x}$. The action of a path $x(t)$ (from $t = t_1$ to $t = t_2$) is

$$A = \int_{t_1}^{t_2} L \, \mathrm{d}t. \tag{1.24}$$

Understanding the action is difficult, but what we need is the fact that the action is minimal in any legal classical path. The *principle of least action* is that given endpoints $x(t_1)$ and $x(t_2)$ the path $x(t)$ taken by the system is the one with a stationary action. If you look at the space of (differentiable) paths ($f : \mathbb{R} \to \mathbb{R}^3$), and A as a function on this space, the system chooses the path where the Gâteaux derivative of A (in all directions) is zero.

It is important to note that the configuration x of the system need not be represented by using the Cartesian coordinates of the position, but one can use any set of parameters that completely describes the system.

Proposition 1.20. Writing x_1, x_2, x_3 for the parameters describing the system the path of least action satisfies the equations

$$\frac{\partial L}{\partial x_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_i} = 0. \tag{1.25}$$

Hamiltonian mechanics use the Hamiltonian, which is the energy of the system. We will only do these mechanics on a closed system, that is a system without external forces and where no energy is added to the system.

Definition 1.21. The Hamiltonian of a closed system is

$$H = T + V, \tag{1.26}$$

which depends on the (generalized) position $[q_1, q_2, q_3]$ and momentum $[p_1, p_2, p_3]$, but it is independent of time t. The equations of motion are

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{1.27}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}. \tag{1.28}$$

This can be generalized slightly through the use of Poisson brackets, as follows.

Proposition 1.22. The time evolution of an observable $A = A(p, q, t)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}A = \{A, H\} + \frac{\partial A}{\partial t} = \sum_i \frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial A}{\partial t}. \tag{1.29}$$

Note how the Lagrangian formulation gives a system of n second order equations, while the Hamiltonian formulation gives a system of $2n$ first order equations (and requires initial values for $[q(t_1), p(t_1)]$). Note also that we don't need Newton's formulation when we have Hamilton's. Using the Newtonian equations one would have to do experiments to determine the forces, like gravity and spring constants. When using the Hamiltonian equations one needs to determine the Hamiltonian by doing experiments.

Let us now give our main examples. These will be used throughout the text.

Example 1.23. For a free 1-dimensional particle we get the following equations.

$$\begin{aligned} T &= \frac{1}{2}m\dot{x}^2 \\ V &= 0 \\ L &= \frac{1}{2}m\dot{x}^2 \\ \frac{\mathrm{d}}{\mathrm{d}t}m\dot{x} &= 0, \end{aligned}$$

where the last equation comes from equation 1.25, and tells us that the velocity $\dot{x}$ is constant.

From the Hamiltonian equations we get

$$\begin{aligned} H &= \frac{1}{2m}p^2 \\ \dot{p} &= -0 \\ \dot{x} &= \frac{p}{m}, \end{aligned}$$

giving the solution

$$\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} x(t_1) + \int_{t_1}^{t} \frac{p}{m}\ \mathrm{d}t \\ p(t_1) \end{pmatrix} = \begin{pmatrix} x(t_1) + (t - t_1)\frac{p(t_1)}{m} \\ p(t_1) \end{pmatrix}. \tag{1.30}$$

Example 1.24. For a harmonic oscillator (mass-spring system) we get the following equations, where k is the spring constant and $w^2 = \frac{k}{m}$.

$$\begin{aligned} T &= \frac{1}{2}m\dot{x}^2 \\ V &= \frac{1}{2}kx^2 \\ L &= \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \\ -kx - \frac{\mathrm{d}}{\mathrm{d}t}m\dot{x} &= 0 \\ \ddot{x} &= -\frac{k}{m}x \\ x(t) &= \frac{\dot{x}(0)}{w}\sin(wt) + x(0)\cos(wt), \end{aligned}$$

where the third last equation comes from equation 1.25.

From the Hamiltonian equations we get

$$\begin{aligned} H &= \frac{1}{2m}p^2 + \frac{1}{2}kx^2 \\ \dot{p} &= -kx \\ \dot{x} &= \frac{p}{m}, \end{aligned}$$

giving the solution

$$\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} \frac{p(0)}{mw}\sin(wt) + x(0)\cos(wt) \\ p(0)\cos(wt) - x(0)mw\sin(wt) \end{pmatrix}. \tag{1.31}$$

1.5 Classical mechanics and generators of time-translations

We want to emphasize a different view of the classical mechanics, which will be useful when understanding Weyl-systems.

To begin, note that

$$(e^{sD}v)(t) = v(t+s), \tag{1.32}$$

where v can be a vector, as long as $Dv(t) = \dot{v}(t)$ is the differential operator, and we use exponentiation of matrices. To see this, write out the Taylor expansion. This can be used to calculate $x(t)$ and $p(t)$ by

$$\begin{pmatrix} x \\ p \end{pmatrix} = e^{tD}\begin{pmatrix} x(0) \\ p(0) \end{pmatrix}. \tag{1.33}$$

For example, take the harmonic oscillator,

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} p/m \\ -kx \end{pmatrix} = \begin{pmatrix} 0 & 1/m \\ -k & 0 \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix}, \tag{1.34}$$

so that

$$D = \begin{pmatrix} 0 & 1/m \\ -k & 0 \end{pmatrix}. \tag{1.35}$$

Then we can compute

$$e^{tD} = \begin{pmatrix} \cos(\sqrt{k/m}t) & \frac{\sin(\sqrt{k/m}t)}{\sqrt{km}} \\ -\sqrt{km}\sin(\sqrt{k/m}t) & \cos(\sqrt{k/m}t) \end{pmatrix}, \tag{1.36}$$

which is a different way of presenting the solution to the problem; here we have given the generator of time-translations (the operator that sends the position and momentum a time t into the future). It is common to write $T_t = e^{tD}$ for this operator.

Chapter 2

Formulations of quantum mechanics

In quantum mechanics we model systems with a Hilbert space $\mathcal{H}$. A vector $\psi \in \mathcal{H}$ is called a wave-function. The name is sometimes reserved for a normalized vector, or the 1-dimensional subspace spanned by ψ. We can only get experimental information about subspaces, so in a 1-dimensional subspace any vector is a good representative. But since the integral of a probability distribution is 1 and the probability distribution is $P(x) = |\psi(x)|^2$ we use unit vectors (with respect to $||-||_2$).

Here we feel the compulsion to clarify what the probability distribution means. It is not a result of an uncertainty on our part, not of flawed measurements, but something far deeper.[1] Let us think of a particle (like an electron), and a physical description (wave) ψ such that the probability of finding the electron in the interval $[x, x+dx]$ is $|\psi(x)|^2\, dx$. Then this is the result of theoretically perfect experiments. The probabilities come from the fact that *the electron does not know where it is*, the uncertainty is a basic property of the electron. Even worse, a probability with phase $e^{i\theta}$ will cancel out a probability with the phase $e^{-i\theta}$. Even though it may seem counter-intuitive, this model is extremely successful at predicting behavior at the Planck scale.

We want to note that the Hilbert space we use is always L^2 with complex valued functions, but it is not necessarily $L^2(\mathbb{R})$. We will use $L^2(G)$ for any (locally compact Hausdorff second countable) group G, and the theory is stated for any separable Hilbert space.

Let us first look at the physical implications when there is no time evolution. Let us say that we know the state of the system to be $\psi \in \mathcal{H}$. What happens if

[1] The notion of probability comes from modeling everyday experiences with too many inputs. For example, when we throw a die we say there is a 1/6 chance of getting a six. But this uncertainty is not in the nature of the die, as we could calculate exactly where the die would land (given sufficiently precise input information). So everyday probability and uncertainty is in our maps and models, and not in reality itself. In quantum mechanics, however, the uncertainty is the nature of reality itself.

we do a measurement asking whether the state is $\phi \in \mathcal{H}$? We get the answer 'yes' with probability

$$P_y = |\langle\phi|\psi\rangle|^2, \tag{2.1}$$

and 'no' with probability

$$P_n = |\langle\phi^\perp|\psi\rangle|^2 = \sum_{\mu\in\phi^\perp} |\langle\mu|\psi\rangle|^2, \tag{2.2}$$

where the last sum is over an orthonormal basis of $\phi^\perp$. We see that $P_y + P_n = 1$, and note that the answer to the experiment can have a nonzero probability of being 'yes' even though $\phi \neq \psi$.[2]

In this chapter we will first look at the basics of quantum mechanics, then we will look at different formulations. All of this will be done without considering the p-adic numbers. Later we will draw inspiration from the different formulations when trying to define quantum mechanics over $\mathbb{Q}_p$.

2.1 Schrödinger equation as a differential equation

Let us now consider the time-evolution of a physical system by using the Scrödinger picture, i.e. we let the wave-functions depend on time. Given $\psi(t = 0)$ we find $\psi(t)$ by solving the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi = \widehat{H}\psi, \tag{2.3}$$

where $\widehat{H}$ is the energy observable. The operator $\widehat{H}$ will depend on the system under consideration, we will assume it is independent of time. In classical mechanics we use $\widehat{H} = T(\widehat{p}) + V(\widehat{x})$, so an analog can be constructed given that $\widehat{p}(\psi)$ and $\widehat{x}(\psi)$ are understood. Given that ψ is the position representation of the system, so that $|\psi(x)|^2dx$ is the probability of observing the particle in $[x, x + dx]$, we define $\widehat{x}\psi(x) = x \cdot \psi(x)$. Note that this is natural because, given a ψ with support in $[a - \epsilon, a + \epsilon]$ (where ϵ small),

$$\langle\psi|x|\psi\rangle \approx a. \tag{2.4}$$

But what about $\widehat{p}\psi$? There are several ways to define $\widehat{p}\psi$, and we will come back to this, but the two most common are

$$\widehat{p} = -i\hbar\frac{\partial}{\partial x} \tag{2.5}$$

$$\widehat{p} = \mathcal{F}\widehat{x}\mathcal{F}^{-1}, \tag{2.6}$$

where the latter is much easier to generalize than the former. We also have the formula

$$[\widehat{x}, \widehat{p}] = \widehat{x}\widehat{p} - \widehat{p}\widehat{x} = i\hbar. \tag{2.7}$$

A different way to define $\widehat{x}$, and $\widehat{p}$ is by viewing them as generators of 1-parameter unitary groups. We will come back to this in the section on Weyl formulation.

[2] To see how this gives experimental results, look at the double-slit experiment for electrons.

Example 2.1. The classical Hamiltonian for the free particle (example 1.23) is

$$H = \frac{1}{2m}p^2 \tag{2.8}$$

Using equation 2.5 we get

$$\widehat{H}f(x) = \frac{-\hbar^2}{2m}\left(\frac{\partial}{\partial x}\right)^2 f(x). \tag{2.9}$$

Example 2.2. The classical Hamiltonian for the harmonic oscillator (example 1.24) is

$$H = \frac{1}{2m}p^2 + \frac{1}{2}kx^2 \tag{2.10}$$

Using equation 2.5 we get

$$\widehat{H}f(x) = \frac{-\hbar^2}{2m}\left(\frac{\partial}{\partial x}\right)^2 f(x) + \frac{1}{2}kx^2 f(x). \tag{2.11}$$

Here one usually takes $\omega^2 = k/m$ and gets

$$\widehat{H}f(x) = \frac{-\hbar^2}{2m}\left(\frac{\partial}{\partial x}\right)^2 f(x) + \frac{1}{2}m\omega^2 x^2 f(x), \tag{2.12}$$

with eigenfunctions

$$\psi_n(x) = (2^n n!)^{-1/2}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/(2\hbar)} P_n(x), \tag{2.13}$$

where P_n are the Hermite polynomials

$$P_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}\left(e^{-x^2}\right). \tag{2.14}$$

Example 2.3. When writing about the harmonic oscillator it is common to consider $\widehat{H} = \widehat{p}^2/2 + \widehat{x}^2/2$. Let us look at how redefining the units gives this equation.[3]

Let now

$$\begin{aligned} X &= \left(\frac{\hbar}{m\omega}\right)^{1/2} \\ E &= \hbar\omega \\ x &= Xx' \\ H &= EH', \end{aligned}$$

then the equation becomes

$$\widehat{H}'f(x) = (\hbar\omega)^{-1}\left(\frac{\hbar}{m\omega}\right)^{-1}\frac{-\hbar^2}{2m}\left(\frac{\partial}{\partial x'}\right)^2 f(x) + (\hbar\omega)^{-1}\left(\frac{\hbar}{m\omega}\right)\frac{1}{2}m\omega^2 x'^2 f(x), \tag{2.15}$$

[3]Physicists often just set $m = 1$ or $\omega = 1$ or even $\hbar = 1$ (natural units), but this might lead to confusion as the units do not really disappear.

where most of the stuff cancels and we get

$$\widehat{H}f(X^{-1}x) = \frac{1}{2}\left(\frac{\partial}{\partial x}\right)^2 f(X^{-1}x) + \frac{1}{2}x^2 f(X^{-1}x), \tag{2.16}$$

after renaming. It is important to note that Hamilton's equations (equation 1.27 and 1.28) do not change when multiplying the energy by a constant, as it is independent of the choice of units. The function f is not given directly as a function of the space variable, but as a function of the space variable divided by X. With different masses m and angular frequencies ω we get different arguments in the function f, that is, we get different Hilbert spaces.

2.1.1 Time evolution

We did assume that $\widehat{H}$ is independent of time.[4] This gives us a simple time evolution process, as equation 2.3 becomes separable. Let $|n\rangle$ be an eigenvector of $\widehat{H}$ with eigenvalue E_n, then

$$i\hbar\frac{\partial}{\partial t}\psi = E_n\psi, \tag{2.17}$$

with the initial value $\psi(0) = |n\rangle$. This has solution

$$\psi(t) = e^{-itE_n/\hbar}|n\rangle. \tag{2.18}$$

In general $\psi(0) = |\phi\rangle$ would give, if we have a complete ortonormal set of eigenvectors $|n\rangle$,

$$\psi(t) = \sum_n \langle n|\phi\rangle e^{-itE_n/\hbar}|n\rangle. \tag{2.19}$$

2.2 Operator algebra - how to construct observables

The Hamilton operator we just described is an example of an *observable*, an entity that can be determined by doing experiments. Now we want to show how any observable can be constructed from yes/no questions, and that we end up with self-adjoint (or as physicists say: Hermetian) operators on the Hilbert space.

Let us first take the simplest form of observables, yes/no questions about whether ψ is in the closed subspace $E \subseteq \mathcal{H}$, where 'yes' has value 1, and 'no' has value 0 (and your answer in the state ψ can only be 0 or 1). What self-adjoint projection is the observable to this question?

Proposition 2.4. There is a 1-1 correspondence between the set of self-adjoint projections and the set of all closed subspaces of $\mathcal{H}$. Let $\mathcal{C} : P \mapsto Im(P)$ be this correspondence and observe that:

[4] We do this for simplicity, as we are looking for simple applications in this thesis.

- $\mathcal{C}(0) = 0$
- $\mathcal{C}(I - P) = \mathcal{C}(P)^{\perp}$
- $\mathcal{C}(P_1 P_2) = \mathcal{C}(P_1) \cap \mathcal{C}(P_2)$
- $P_1 P_2 = 0 \Rightarrow \mathcal{C}(P_1 + P_2) = \mathcal{C}(P_1) \oplus \mathcal{C}(P_2)$

The answer to the question is: "The self-adjoint projection with image E".

Let us take a more complicated example, working on $L^2((\mathbb{T}, +))$, (as before $(\mathbb{T}, +)$ denotes $[0, 1) = \mathbb{R}/\mathbb{Z}$).

Example 2.5. We want to combine three observations, giving them one value from $\mathbb{R}$ for a 'yes'-answer, and 0 for a 'no' answer.

The first observation is whether the particle is in $[0, \frac{1}{2})$ and 'yes' has value 1, the second observation is whether ψ is $\sin(x)$ and 'yes' has value $\sqrt{2}$, the third observation is whether ψ is constant (almost everywhere) and 'yes' has value π.

First we will create one projection operator for each observation

$$\begin{aligned} P_1\psi &= \chi_{[0,\frac{1}{2})}\psi \\ P_2\psi &= \langle \sin(x)|\psi\rangle \cdot |\sin(x)\rangle \\ P_3\psi &= \int_0^1 \psi(x)\ \mathrm{d}x = \langle 1|\psi\rangle|1\rangle, \end{aligned}$$

but these projections do not have the correct values associated with them. Per definition, projections are self-adjoint operators, and so is any linear combination with real valued coefficients. Let us first scale (here $P_i\psi_i = \psi_i$):

$$\begin{aligned} \langle\psi_1|P_1'|\psi_1\rangle = 1 &\Rightarrow P_1\psi = \chi_{[0,\frac{1}{2})}\psi \\ \langle\psi_2|P_2'|\psi_2\rangle = \sqrt{2} &\Rightarrow P_2\psi = 4\sqrt{2}\cdot\langle\sin(x)|\psi\rangle\cdot|\sin(x)\rangle \\ \langle\psi_3|P_3'|\psi_3\rangle = \pi &\Rightarrow P_3\psi = \pi\int_0^1 \psi(x)\ \mathrm{d}x. \end{aligned}$$

In total this gives us the observable corresponding to the self-adjoint operator $A : \mathcal{H} \to \mathcal{H}$ where

$$A\psi = \sum_i P_i'\psi = \chi_{[0,\frac{1}{2})}\psi + 4\sqrt{2}\pi\int_0^1 \sin(x)\psi(x)\ \mathrm{d}x|\sin(x)\rangle + \pi\int_0^1 \psi(x)\ \mathrm{d}x \quad (2.20)$$

What is the general picture of the situation in this example? We want to assign subspaces of $\mathcal{H}$ to real numbers (experimental results). This is exactly the definition of a projection valued measure, a measure on the set of reals giving projection operators on $\mathcal{H}$.

Definition 2.6. Let $(\Omega, \mathcal{M})$ be a measurable space, and $\mathcal{H}$ a Hilbert space. The map $E : \mathcal{M} \to B(\mathcal{H})$ is a resolution of identity, also known as a projection-valued measure, on $(\Omega, \mathcal{M})$ if the following six items hold

- $E(\emptyset) = 0$
- $E(\Omega) = 1_{B(\mathcal{H})}$
- $E(w)$ is a self-adjoint projection for all measurable w.
- $E(w \cap w') = E(w)E(w')$
- $w \cap w' = \emptyset \Rightarrow E(w \cup w') = E(w) + E(w')$
- For all $x, y \in \mathcal{H}$ the function $E_{x,y} : w \mapsto \langle E(w)x, y\rangle$ is a complex measure.

Remark: In the last item, only countable additivity of disjoint sets needs to be shown.

We know that there is a one to one correspondence between projection-valued measures and self-adjoint operators on the Hilbert space (from the spectral theorem).[5] So when we are talking about an observable, we will often think of it as a projection-valued measure.

This definition also clears up the hassle about the eigenvectors of the position operator. In physics we often write

$$I = \sum_i |v_i\rangle\langle v_i| \tag{2.21}$$

given an operator A with eigenvalues λ_i and eigenvectors v_i, where $(v_i)_i$ is an orthonormal basis for $\mathcal{H}$. This is close to the definition of projection-valued measure, and shows why it is also called resolution of identity. There is nothing wrong with this equation, but let us consider the position operator x in the Hilbert space over $(\mathbb{T}, +)$. Then physicists often write

$$I = \int dk|k\rangle\langle k|, \tag{2.22}$$

where $\langle k|k'\rangle$ is the delta distribution $\delta_k(k')$.

How does this look from the mathematical point of view? Given the well known measure space $([0,1),\mathcal{M})$, the resolution of identity $E : \mathcal{M} \to \mathcal{H}$ is defined by $E(A)(\psi) = \mathbf{1}_A\psi$, where $\mathbf{1}_A$ is the characteristic function of A. Note that $E([0,1)) = I$ with this definition. What happens here hints to the general case. If we have a continuous spectrum (no eigenvectors), so that $\dim(E(\{\lambda\})) = 0$, we cannot write the identity as a sum of projections. Instead we have to use a projection-valued measure.

Even though we did not give an example of this, we can have an eigenvalue λ with a degenerated eigenspace, so that $\dim(E(\{\lambda\})) > 1$

[5] The reason we have not included the spectral theorem is that it would require several pages to get through all the definitions and preliminaries, like the Gelfand-Naimark transform.

Before we leave this topic, let us comment on unbounded operators. quantum mechanics in $L^2(\mathbb{R})$ requires the unbounded operators $\widehat{q}$ and $\widehat{p}$ (unbounded because $\mathbb{R}$ is not compact), which have a domain that is dense in $\mathcal{H}$. This makes the theory a lot more technical, but the results are similar.

2.2.1 Doing experiments

An observable has its name because it can be observed, by doing experiments. Given an observable T, we can set up an experiment that measures T (at least in theory). What kind of experimental values will we get?

Let us prepare the system in a state $\psi \in \mathcal{H}$, and observe the self-adjoint operator T. Let $v_{i,k}$ be a basis for $\mathcal{H}$ of eigenvectors of T with eigenvalue λ_i where k is the index for having degenerate eigenvalues (yes, this requires some assumptions on T, but within physical precision we can do this). Then the probability of our experimental machine printing a is 0 if a is not an eigenvalue λ_i. If, on the other hand $a = \lambda_b$ for some b then the probability of measuring a is

$$\sum_k |\langle v_{b,k}|\psi\rangle|^2. \tag{2.23}$$

Confusion can arise when people talk about the experimental value of A in state ψ as the expected value. We will come back to this, but please note that it may not at all be possible to get the expected value as a result when doing a single experiment.

2.2.2 Functions on the spectrum

To construct $\widehat{H}$ we need to understand what $f(T)$ means (for example with $f : x \mapsto x^2$). When we view the operator T as partitioning the Hilbert space into closed subspaces (resolution of identity), so that $\mathcal{H} = \sum_i E_i$ where E_i has eigenvalue λ_i, then $f(T)$ is the operator partitioning the Hilbert space into $\mathcal{H} = \sum_i E_i$ where E_i has eigenvalue $f(\lambda_i)$. Let $\bar{T}$ denote this correspondence between E_i and λ_i (the inverse of the resolution of identity) for the operator T. Then

$$\begin{array}{ccccc} & \bar{T} & & f & \\ E_1 & \longrightarrow & \lambda_1 & \longrightarrow & f(\lambda_1) \\ E_2 & \longrightarrow & \lambda_2 & \longrightarrow & f(\lambda_2) \end{array} \tag{2.24}$$

and so on. For this to work when the spectrum is more complicated (not just eigenvectors) we need that the function f is Borel measurable. Note that this can be used to take the square root of a positive operator, the logarithm of some unitary operators and more. For precise formulation and proofs we recommend chapter 1 in Folland's book on abstract harmonic analysis [7]. We will end with a simple example.

Example 2.7. Why is the square of the observable T actually $T \circ T$?

Let ψ for simplicity be an eigenvector of T with eigenvalue λ. Then the square of T should give the experimental result λ^2. Fortunately

$$T \circ T\psi = T\lambda\psi = \lambda T\psi = \lambda\lambda\psi = \lambda^2\psi. \tag{2.25}$$

So what we call T^2 is actually the observable with the property of being the square of the observable T.

2.2.3 Combining Hilbert spaces (or underlying groups)

The Hilbert space in consideration depends on what we want to model. For example one could model an electron in a Coulomb-potential. If you only care about the radial part, you can use $L^2(\mathbb{R})$, but if you also need the angular momentum you would work in $L^2(\mathbb{R}^3)$. If you also desire to know the spin of the electron, you would need to add on a Hilbert space describing the spin. To describe the spin we use a finite group (the group of Pauli spin matrices).

To see how this works in general we give the following definition and proposition using the tensor product.

Definition 2.8. The tensor product of $L^2(G_1)$ and $L^2(G_2)$, $L^2(G_1) \otimes L^2(G_2)$, is the dense linear span in $L^2(G_1 \oplus G_2)$ of

$$\{h | h(x,y) = f(x)g(y) \text{ with } f \in L^2(G_2) \text{ and } g \in L^2(G_1)\}. \tag{2.26}$$

Proposition 2.9. We have that

$$L^2(G_1 \oplus G_2) = L^2(G_1) \otimes L^2(G_2). \tag{2.27}$$

2.3 Uncertainty principle and commutators

The commutators and the uncertainty relation are fundamental to understanding the consequences of quantum mechanics. If two observables commute, it is theoretically possible to design an experiment in which they are measured simultaneously. When developing a p-adic model it is essential that we end up with the same physical implications. So it is natural to take a short review of how this works.

2.3.1 Expectation and variance

Given an observable A the expected value for an observation when preparing the system in state ψ is

$$\langle A \rangle_\psi = \langle \psi | A | \psi \rangle = \int_{\mathbb{R}} \psi^* A(\psi) \, \mathrm{d}x, \tag{2.28}$$

where the last equality is only for $L^2(\mathbb{R})$. When it is clear which state is considered, we often write $\langle A \rangle$.

Experimentally, given an infinite sequence of experimental values a_i (to get this sequence you have to do an infinite number of identical and independent experiments measuring A in the state ψ), we get

$$\langle A\rangle_\psi = \lim_n \frac{\sum_{i=1}^n a_i}{n}, \tag{2.29}$$

from the law of large numbers.

2.3.2 Deriving the uncertainty relation

Let $\mathcal{H}$ be any separable Hilbert space.

Definition 2.10. The variance $\sigma_A \in \mathbb{R}$ of a symmetric operator A given a state $\psi \in \mathcal{H}$ is

$$\sigma_A = \sqrt{\langle A^2\rangle - \langle A\rangle^2} = \langle A - \langle A\rangle\rangle \tag{2.30}$$

Note that we do not write ψ when it is understood from the context.

Definition 2.11. The commutator of A and B is

$$[A, B] = AB - BA, \tag{2.31}$$

where the multiplication is composition.

The domain of the commutator can be much smaller than the domain of A or B, but this will not be a problem in the quantum mechanical applications as the resulting domain usually is dense in $\mathcal{H}$.

Theorem 2.12. *Assume A and B are symmetric operators on $\mathcal{H}$. Then the Robinson uncertainty relation,*

$$\sigma_A\sigma_B \geq \left|\frac{1}{2i}\langle[A, B]\rangle\right|, \tag{2.32}$$

holds for any $\psi \in \mathcal{H}$ whenever both sides of the inequality makes sense.

Proof. The Cauchy-Schwartz inequality for $|(A - \langle A\rangle)\psi\rangle$ and $|(B - \langle B\rangle)\psi\rangle$ gives

$$\begin{aligned}\sigma_A^2\sigma_B^2 &= \langle\psi|(A - \langle A\rangle)^2|\psi\rangle\langle\psi|(B - \langle B\rangle)^2|\psi\rangle \\ &\geq |\langle\psi|(A - \langle A\rangle)(B - \langle B\rangle)|\psi\rangle|^2 \\ &= |z|^2,\end{aligned}$$

defining $z = \langle\psi|(A - \langle A\rangle)(B - \langle B\rangle)|\psi\rangle \in \mathbb{C}$. Further

$$\begin{aligned}z - z^* &= \langle\psi|(A - \langle A\rangle)(B - \langle B\rangle)|\psi\rangle - \langle\psi|(B - \langle B\rangle)(A - \langle A\rangle)|\psi\rangle \\ &= \langle\psi|AB - A\langle B\rangle - B\langle A\rangle + \langle A\rangle\langle B\rangle|\psi\rangle - \langle\psi|BA - A\langle B\rangle - B\langle A\rangle + \langle A\rangle\langle B\rangle|\psi\rangle \\ &= \langle\psi|AB - BA|\psi\rangle \\ &= \langle[A, B]\rangle.\end{aligned}$$

Now, $|z| \geq |Im(z)| = |(z - z^*)/2i|$ gives us

$$\begin{aligned}\sigma_A \sigma_B &\geq |z| \geq |Im(z)| = |(z - z^*)/2i| \\ &\geq |\frac{1}{2i}\langle [A, B] \rangle|.\end{aligned}$$

□

It is very interesting that this last theorem (and proof) has nothing to do with the domain of the function ψ, so that it will work equally well for the Hilbert space $L^2(\mathbb{Q}_p)$ using the p-adic numbers.

Example 2.13. Let us use the last theorem on $\widehat{x}$ and $\widehat{p}$ on $L^2(\mathbb{R})$.

$$\sigma_x \sigma_p \geq |\frac{1}{2i}\langle [\widehat{x}, \widehat{p}] \rangle| = |\frac{1}{2i}\langle i\hbar \rangle| = \frac{\hbar}{2}, \tag{2.33}$$

which is a well known fact.[6]

Summing up, if we can make a model that has $[\widehat{q}, \widehat{p}] = kiI$ we automatically get an uncertainty relation for $\widehat{q}$ and $\widehat{p}$.

2.4 Generator of translations

A common term for the momentum operator $\widehat{p}$ is *the generator of translations*. In this section we will look at how the operators $\widehat{q}$ and $\widehat{p}$ can be viewed as generators of groups of unitary operators. This will lead to the formulation of a Weyl system, which is essentially a product of the U and V in the first definition.

Definition 2.14. Define $U, V : \mathbb{R} \to \mathcal{U}(\mathcal{H})$ by $U(t) = U_t$, $V(s) = V_s$, and

$$\begin{aligned}U_t f(x) &= e^{itx} f(x) \\ V_s f(x) &= f(x + s)\end{aligned}$$

Proposition 2.15. We have

$$\begin{aligned}U_t V_s &= e^{its} V_s U_t \\ \lim_{t \to 0} \frac{U_t - I}{t} \psi &= ix\psi = i\widehat{x}\psi \\ \lim_{s \to 0} \frac{V_s - I}{s} \psi &= D\psi = i\widehat{p}\psi \\ V_s &= e^{sD},\end{aligned}$$

where $Df(x) = f'(x)$, for ψ in a dense subspace of $\mathcal{H}$.

[6] This is sometimes known as Heisenberg's uncertainty relation, and is usually interpreted as "you cannot both know the position and the momentum of your particle".

Proposition 2.16. The Fourier transform of U_t is V_t, and the Fourier transform of $\widehat{p}$ is $\widehat{x}$, that is

$$\mathcal{F}V_t = U_t\mathcal{F} \tag{2.34}$$
$$\mathcal{F}\widehat{p} = \widehat{x}\mathcal{F}. \tag{2.35}$$

Proof. Let all integrals be from minus infinity to infinity, and see that

$$\begin{aligned}(\mathcal{F}V_t f)(y) &= \frac{1}{\sqrt{2\pi}}\int e^{-ixy} f(x+t)\ \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}}\int e^{-i(x-t)y} f(x)\ \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} e^{ity}\int e^{-ixy} f(x)\ \mathrm{d}x \\ &= (U_t\mathcal{F}f)(y).\end{aligned}$$

□

The Stone-von Neumann theorem says something about how this is the only way to do quantum mechanics, and that there is a correspondence between U, V -systems and pairs of operators satisfying $[\widehat{q}, \widehat{p}] = kiI$.

2.5 Weyl formulation

A Weyl system is essentially a choice of coordinates in our Hilbert space. In this section we give a definition of Weyl systems over $\mathbb{R}$, and we give a very nice choice in $L^2(\mathbb{R})$.

Definition 2.17. A Weyl system[7] on $L^2(\mathbb{R})$ is a function $W : \mathbb{R} \times \mathbb{R} \to \mathcal{U}(\mathcal{H})$, so that $W(q,p)$ is a unitary operator, with the property that

$$W(z+z') = B(z,z')W(z)W(z'), \tag{2.36}$$

where $B : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{T}$, and $z = (q,p) \in \mathbb{R}^2$.

Proposition 2.18. Let $\chi(x) = e^{aix}$ for some real a,[8] then

$$W(q,p)\psi(x) = \chi\left(\frac{qp}{2} + qx\right)\psi(x+p) \tag{2.37}$$

is a Weyl system on $\mathcal{H} = L^2(\mathbb{R})$, where

$$B((q,p),(q',p')) = \chi\left(\frac{q'p - p'q}{2}\right). \tag{2.38}$$

[7] What we call a Weyl system is more commonly called a multiplicative projection, or a projective representation. In the literature one finds that the U_t and V_s from the previous section is called a Weyl system. However, the example W we present in the next proposition is actually just a combination of U_t and V_s.

[8] We do not want to specify which character in the dual group to use, because the good choice depends on the physical system under consideration.

Proof. To see that $W(q,p)$ is a unitary operator, note that $W(q,p) = \chi\left(\frac{qp}{2}\right) U_{at} V_p$, is a combination of unitary operators.

Next we want to show that W satisfies equation 2.36: Note that

$$W(p,q)^{-1} = \chi\left(\frac{qp}{2} - qx\right)\psi(x-p), \tag{2.39}$$

and calculate B by

$$\begin{aligned}
W(z)W(z')W(z+z')^{-1}\psi &= W(q,p)W(q',p')W(q+q',p+p')^{-1}\psi \\
&= W(q,p)W(q',p') \\
&\cdot \chi\left(\frac{(q+q')(p+p')}{2} - (q+q')x\right)\psi(x-(p+p')) \\
&= W(q,p)\chi\left(\frac{q'p'}{2} + q'x\right) \\
&\cdot \chi\left(\frac{(q+q')(p+p')}{2} - (q+q')(x+p')\right)\psi(x-p) \\
&= W(q,p)\chi\left(\frac{qp+q'p-qp'}{2}\right)\chi(-qx)\,\psi(x-p) \\
&= \chi\left(\frac{qp+q'p-qp'}{2}\right) \\
&\cdot \chi(-q(x+p))\,\chi\left(\frac{qp}{2}+qx\right)\psi(x) \\
&= \chi\left(\frac{q'p-qp'}{2}\right)\psi(x) \\
&= B(z,z')\psi(x),
\end{aligned}$$

where B has the required form. □

2.6 Time evolution by propagator

Weyl systems are useful in general, but time evolution depends on the physical systems we want to describe. If you have a bad choice of coordinates (i.e. Weyl system) then the time evolution operator is very difficult to use.

If we know the Hamiltonian, we get a unitary group

$$U_t = e^{itH}, \tag{2.40}$$

defining time evolution of a wave ψ, so that

$$U_t\psi(x,s) = \psi(x,s+t). \tag{2.41}$$

This can be reformulated as

$$(U_t\psi)(x) = \int K_t(x,y)\psi(y)\,\mathrm{d}y, \tag{2.42}$$

for some propagator $K_t(x, y)$. This can be calculated without knowing H.[9]

For the nondimensionalized Harmonic oscillator ($H = \hat{p}^2/2 + \hat{q}^2/2$) we get

$$K(x, y, t) = K_t(x, y) = \left(\frac{1 - i \ \mathtt{sign}(\sin t)}{(2|\sin t|)^{1/2}}\right) \exp\left(2\pi i \frac{x^2 + y^2}{2\tan t} - 2\pi i \frac{xy}{\sin t}\right) \quad (2.43)$$

from Dragovich [3]. We will not use the Hamiltonian or the Feynman-Kac formula to get this. What we will do is take a guess, and prove that it satisfies certain important conditions.[10] We will not use the equation for the nondimensionalized system, because nondimensionalization might not make sense for wavefunctions in $L^2(\mathbb{Q}_p)$ as $\mu(37A) = \mu(A)$ (for $p \neq 37$), and the order is not necessarily preserved when multiplying with a scalar.

We claim that the propagator for the full Harmonic oscillator is

$$K(x, y, t) = \Big(\frac{m\omega}{2\pi i\hbar \sin \omega t}\Big)^{1/2} \exp\left[\Big(\frac{m\omega}{2i\hbar}\Big)\left(-\frac{x^2}{\tan \omega t} - \frac{y^2}{\tan \omega t} + \frac{2xy}{\sin \omega t}\right)\right]. \quad (2.44)$$

In the following, we often use the formula ($a, b, c \in \mathbb{C}$)

$$\int_{\mathbb{R}} \exp(-ax^2 \pm bx - c) = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2 - 4ac}{4a}\right), \quad (2.45)$$

where the real part of a is greater than 0, and where $i^{-1/2}$ is defined so that the formula holds.

Example 2.19. The lowest eigenstate of the harmonic oscillator

$$\psi_0(x) = \Big(\frac{m\omega}{\pi\hbar}\Big)^{1/4} e^{-m\omega x^2/(2\hbar)}. \quad (2.46)$$

We want to show that the two different time evolutions of this state gives the same result. When doing this we can drop the constants in front of the exponential, because what we are doing is really a time evolution of the 1-dimensional subspace spanned by ψ.

Using equation 2.18 we get the following time evolution

$$\psi_0(x, t) = e^{-it(\hbar\omega/2)/\hbar}\psi_0(x) = e^{-it\omega/2}\psi_0(x). \quad (2.47)$$

Using the propagator in equation 2.44, time evolution of ψ gives

$$\begin{aligned}\psi_0(x, t) &= (U(t)\psi)(x) \\ &= \int K_t(x, y)\psi(y) \ \mathrm{d}y \\ &= \Big(\frac{m\omega}{2\pi i\hbar \sin \omega t}\Big)^{1/2} \sqrt{\frac{\pi}{a}} \exp\left[\frac{b^2}{4a} - c\right],\end{aligned}$$

[9] This is sometimes known as the Feynman-Kac formula. For more information, see wikipedia's page 'propagator'.

[10] In the same way that you can guess the solution of a differential equation, and then show that it satisfies your equation.

using equation 2.45, with

$$\begin{aligned} a &= \frac{m\omega}{2i\hbar}\left(\frac{1}{\tan\omega t} - i\right) \\ b &= \frac{m\omega}{2i\hbar}\frac{2x}{\sin\omega t} \\ c &= \frac{m\omega}{2i\hbar}\frac{x^2}{\tan\omega t}. \end{aligned}$$

Then we get

$$\begin{aligned} \psi_0(x,t) &= \left(\frac{m\omega\pi}{2\pi i\hbar\sin\omega t\left(\frac{m\omega}{2i\hbar}\left(\frac{1}{tan\omega t} - i\right)\right)}\right)^{1/2} \exp\left[\frac{b^2}{4a} - c\right] \\ &= \left(\frac{1}{\cos\omega t - i\sin\omega t}\right)^{1/2} \exp\left[\left(\frac{m\omega}{2i\hbar}\right)\left(\frac{x^2(\sin\omega t)^{-2}}{\frac{1}{tan\omega t} - i} - \frac{x^2}{\tan\omega t}\right)\right] \\ &= \left(e^{-i\omega t}\right)^{-1/2} \exp\left[\left(\frac{m\omega x^2}{2i\hbar\sin\omega t}\right)\left(\frac{1}{\cos\omega t - i\sin\omega t} - \cos\omega t\right)\right] \\ &= e^{-i\omega t/2} \exp\left[\left(\frac{m\omega x^2}{2i\hbar\sin\omega t}\right)(i\sin\omega t)\right] \\ &= e^{-i\omega t/2} \exp\left(\frac{m\omega x^2}{2\hbar}\right) = e^{-i\omega t/2}\psi(x), \end{aligned}$$

which is the same as before, so the propagator gives time evolution for the vacuum state ψ_0.

As we mentioned before, we do not want to calculate the propagator, but what is the alternative? We look for some propagator K satisfying the equations in the following proposition.

Proposition 2.20. The operator U_t defined by equation 2.42 and 2.44 satisfies

$$U(t)U(t') = U(t+t'), \tag{2.48}$$

and

$$U(t)W(z)U(t)^{-1} = W(T_t z), \tag{2.49}$$

where T_t comes from the classical time evolution (equation 1.36) and is

$$T_t(q,p) = \begin{pmatrix} \cos(\omega t) & \frac{\sin(\omega t)}{m\omega} \\ -m\omega\sin(\omega t) & \cos(\omega t) \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}. \tag{2.50}$$

Proof.

$$\begin{aligned}
U(t)U(t')\psi(x) &= \int K_t(x,y)\int K_{t'}(y,z)\psi(z)\ \mathrm{d}y\ \mathrm{d}z \\
&= \int \Big(\frac{m\omega}{2\pi i\hbar\sin\omega t}\Big)^{1/2}\exp\left[\Big(\frac{m\omega}{2i\hbar}\Big)\left(-\frac{x^2}{\tan\omega t}-\frac{y^2}{\tan\omega t}+\frac{2xy}{\sin\omega t}\right)\right] \\
&\cdot\Big(\frac{m\omega}{2\pi i\hbar\sin\omega t'}\Big)^{1/2}\exp\left[\Big(\frac{m\omega}{2i\hbar}\Big)\left(-\frac{y^2}{\tan\omega t'}-\frac{z^2}{\tan\omega t'}+\frac{2yz}{\sin\omega t'}\right)\right]\psi(z)\ \mathrm{d}y\ \mathrm{d}z \\
&= \int \frac{m\omega}{2\pi i\hbar}(\sin s\sin p)^{-1/2}\sqrt{\frac{\pi}{a}}\exp\left[\frac{b^2}{4a}-c\right]\psi(z)\ \mathrm{d}y\ \mathrm{d}z,
\end{aligned}$$

with

$$\begin{aligned}
s &= \omega t \\
p &= \omega t' \\
a &= \frac{m\omega}{2i\hbar}(\cot s+\cot p) \\
&= \frac{m\omega}{2i\hbar}(\sin p\sin s)^{-1}\sin(p+s) \\
b &= \frac{m\omega}{2i\hbar}\left(\frac{x}{\sin s}+\frac{z}{\sin p}\right) \\
c &= \frac{m\omega}{2i\hbar}\left(\frac{x^2}{\tan s}+\frac{z^2}{\tan p}\right),
\end{aligned}$$

so that

$$\begin{aligned}
\frac{b^2}{4a}-c &= \frac{m\omega}{2i\hbar}\left[\left(\frac{x}{\sin s}+\frac{z}{\sin p}\right)^2\sin p\sin s(\sin(p+s))^{-1}-\frac{x^2}{\tan s}+\frac{z^2}{\tan p}\right] \\
&= \frac{m\omega}{2i\hbar}\left[x^2\left(\frac{\sin p}{\sin s\sin(p+s)}-\cot s\right)+z^2\left(\frac{\sin s}{\sin p\sin(p+s)}-\cot p\right)-\frac{2xz}{\sin(p+s)}\right] \\
&= \frac{m\omega}{2i\hbar}\left[-\frac{x^2}{\tan(p+s)}-\frac{z^2}{\tan(p+s)}+\frac{2xz}{\sin(p+s)}\right].
\end{aligned}$$

For the part of the integrand in front of the exponential we get

$$\begin{aligned}
\frac{m\omega}{2\pi i\hbar}(\sin s\sin p)^{-1}\sqrt{\frac{\pi}{a}} &= \frac{m\omega}{2\sqrt{\pi}i\hbar}(\sin s\sin p)^{-1/2}\left(\frac{m\omega}{2i\hbar}(\cot s+\cot p)\right)^{-1/2} \\
&= \Big(\frac{m\omega}{2\pi i\hbar}\Big)^{1/2}(\sin p\cos s+\sin s\cos p)^{-1/2} \\
&= \Big(\frac{m\omega}{2\pi i\hbar}\Big)^{1/2}(\sin(p+s))^{-1/2}.
\end{aligned}$$

In total this gives that

$$\begin{aligned} U(t)U(t')\psi(x) &= \int \left(\frac{m\omega}{2\pi i\hbar}\right)^{1/2} (\sin(p+s))^{-1/2} \\ &\quad \cdot \exp\left[\frac{m\omega}{2i\hbar}\left(-\frac{x^2}{\tan(p+s)} - \frac{z^2}{\tan(p+s)} + \frac{2xz}{\sin(p+s)}\right)\right]\psi(z)\ \mathrm{d}z \\ &= \int K_{t+t'}(x,z)\psi(z)\ \mathrm{d}z \\ &= U(t+t'). \end{aligned}$$

Showing the second equation is a bit difficult to do directly, so what we will do is first to show it for $p = 0$, and then for $q = 0$. This will be sufficient as

$$\begin{aligned} W(T_t z) &= W(T_t(q,0) + T_t(0,p)) \\ &= B(T_t(q,0), T_t(0,p))W(T_t(q,0))W(T_t(0,p)) \\ &= B(T_t(q,0), T_t(0,p))U(t)W(q,0)U(t)^{-1}U(t)W(0,p)U(t)^{-1} \\ &= U(t)B(T_t(q,0), T_t(0,p))W(q,0)W(0,p)U(t)^{-1} \\ &= U(t)W(q,p)U(t)^{-1}, \end{aligned}$$

because

$$\begin{aligned} B(T_t(q,0), T_t(0,p)) &= B((q\cos\omega t, -qm\omega\sin\omega t), (p(m\omega)^{-1}\sin\omega t), p\cos\omega t)) \\ &= \chi\left(\frac{-pq\sin^2\omega t - pq\cos^2\omega t}{2}\right) \\ &= \chi\left(\frac{-pq}{2}\right) \\ &= B((q,0),(0,p)). \end{aligned}$$

Let us now show equation 2.49 for $z = (q, 0)$. Since

$$W(T_t(q,0))\psi(x) = \chi\left(-\frac{1}{2}m\omega q^2\cos\omega t\sin\omega t + q\cos\omega t x\right)\psi(x - qm\omega\sin\omega t),$$

what we need to show is that

$$K_t(x,y)\chi(qy) = K_t(x - qm\omega\sin\omega t, y)\chi\left(-\frac{1}{2}m\omega q^2\cos\omega t\sin\omega t + qx\cos\omega t\right).$$

Given that

$$a = \frac{m^2\omega^2}{\hbar}, \quad \chi(x) = e^{-2m\omega\frac{m\omega}{2i\hbar}x}, \tag{2.51}$$

we need to show that

$$\begin{aligned} \left(-\frac{x^2}{\tan\omega t} - \frac{y^2}{\tan\omega t} + \frac{2xy}{\sin\omega t}\right) - 2qym\omega &= \left(-\frac{(x - qm\omega\sin\omega t)^2}{\tan\omega t} - \frac{y^2}{\tan\omega t} + \frac{2(x - qm\omega\sin\omega t)y}{\sin\omega t}\right) \\ &\quad + m^2\omega^2q^2\cos\omega t\sin\omega t - 2qxm\omega\cos\omega t, \end{aligned}$$

which is equivalent to

$$-2qym\omega = (2xqm\omega\cos\omega t - q^2m^2\omega^2\sin\omega t\cos\omega t - 2qm\omega y) + m^2\omega^2q^2\cos\omega t\sin\omega t - 2qxm\omega\cos\omega t,$$

which is true (for all x and y) if and only if all

$$\begin{aligned} 0 &= -q^2m^2\omega^2\sin\omega t\cos\omega t + q^2m^2\omega^2\sin\omega t\cos\omega t \\ -2qm\omega &= -2qm\omega \\ 0 &= 2qm\omega\cos\omega t - 2qm\omega\cos\omega t \end{aligned}$$

are true (the first equation is for the constant, the second for y, the third for x). It is easy to see that these three equations hold.

Let us now show equation 2.49 for $z = (0, p)$. Since

$$W(T_t(0,p))\psi(x) = \chi\left((2m\omega)^{-1}p^2\cos\omega t\sin\omega t + (m\omega)^{-1}xp\sin\omega t\right)\psi(x + p\cos\omega t),$$

what we need to show is that

$$K_t(x, y-p) = K_t(x + p\cos\omega t, y)\chi\left((2m\omega)^{-1}p^2\cos\omega t\sin\omega t + (m\omega)^{-1}xp\sin\omega t\right).$$

Given that a is the same as above, we need to show that

$$\begin{aligned} \frac{2yp\cos\omega t}{\sin\omega t} - \frac{p^2\cos\omega t}{\sin\omega t} - \frac{2xp}{\sin\omega t} = &-\frac{2xp\cos^2\omega t}{\sin\omega t} - \frac{p^2\cos^3\omega t}{\sin\omega t} \\ &+ 2yp\frac{\cos\omega t}{\sin\omega t} - p^2\cos\omega t\sin\omega t - 2xp\sin\omega t \end{aligned}$$

which is true (for all x and y) if and only if all

$$\begin{aligned} -\frac{p^2\cos\omega t}{\sin\omega t} &= -\frac{p^2\cos^3\omega t}{\sin\omega t} - p^2\cos\omega t\sin\omega t \\ \frac{2p\cos\omega t}{\sin\omega t} &= 2p\frac{\cos\omega t}{\sin\omega t} \\ -\frac{2p}{\sin\omega t} &= -\frac{2p\cos^2\omega t}{\sin\omega t} - 2p\sin\omega t \end{aligned}$$

are true (the first equation is for the constant, the second for y, the third for x). Showing these equations is easy. □

2.7 Finite approximations in $L^2(\mathbb{R})$

One of the main problems with $\widehat{x}$ and $\widehat{p}$ is that their spectrum is continuous, so that we don't get an orthonormal basis of $\mathcal{H}$ from the eigenvectors. (Note: $\widehat{x}$ is the same operator as $\widehat{q}$, just emphasizing the coordinate description $\phi(x)$.) We can consider finite approximations of $\widehat{x}$ and $\widehat{p}$.

Given $\eta > 0$ (η some big precision) we can define a position operator $\widehat{x}_\eta$ as multiplication with the η-stair function

$$\mathrm{stair}_\eta(x) = \frac{1}{\eta}\lfloor\eta x\rfloor = \sum_i i\eta^{-1}\ \chi_{[i\eta^{-1},(i+1)\eta^{-1})} \tag{2.52}$$

This approximates the position operator $\widehat{x}$.

Proposition 2.21. The η-stair function converges uniformly to $f(x) = x$ as $\eta \to \infty$. Furthermore $\widehat{x}_\eta \to \widehat{x}$ in the operator norm.

Proof. First see that

$$|x - \text{stair}_\eta(x)| = \frac{1}{\eta}(\eta x - \lfloor \eta x \rfloor) < \frac{1}{\eta} \to 0, \tag{2.53}$$

then note that for any $\psi \in D_{\widehat{x}} = \{f \in L^2(\mathbb{R}) : xf(x) \in L^2(\mathbb{R})\}$ we have

$$||(\widehat{x}_\eta - \widehat{x})(\psi)||^2 = \int_{\mathbb{R}} |(\text{stair}_\eta(x) - x)|^2 |\psi(x)|^2 \, \mathrm{d}x < \frac{1}{\eta^2}||\psi||^2. \tag{2.54}$$

□

Note that this also holds in $L^2(\mathbb{T})$.

2.7.1 Schwinger approximations

Here we will look at the Schwinger approximation, which approximates $\mathcal{H} = L^2(\mathbb{R})$ by $\mathcal{H}_2 = L^2(G)$ with a finite group G using periodic boundary conditions.

Here we will consider only approximations over the reals, a similar approach will be seen later for the p-adic numbers. The following presentation owes heavily to [9], and a complete proof of convergence can be found in [10].

The numerical idea is to avoid a discretization scheme for the derivatives ($\widehat{p}$ and $\widehat{p}^2$), and use the Fourier transform instead. To be able to use the finite Fourier transform we need to have periodic boundary conditions. This is a good approximation if the original problem is periodic (like for a free particle on a periodic interval), or if the wave-functions have decayed to some very small value at the boundary (which will be the case for the harmonic oscillator).

In order to have a scheme that will converge, we choose to increase the length of the interval at the same time as we decrease the step-size. Given n we model approximately the interval from $-\sqrt{n}$ to $+\sqrt{n}$ with step-size approximately $1/\sqrt{n}$.

Definition 2.22. Given a Hamiltonian H on $L^2(\mathbb{R})$, and an integer n, we define a grid G_n and a step size ϵ_n, by

$$\begin{aligned}
\epsilon_n &= \sqrt{\frac{2\pi}{n}} \\
\mathbf{M}_n &= [-(n-1)/2], -(n-3)/2, -(n-5)/2, ..., (n-3)/2, (n-1)/2] \\
\mathbf{G}_n &= \epsilon_n \mathbf{M}_n.
\end{aligned}$$

To get a finite cyclic group we define addition on G_n modulo $\epsilon_n n$. Then we want an inclusion $L^2(G_n) \subseteq L^2(\mathbb{R})$.

Definition 2.23. Define the real Schwinger inclusion to be the following unitary linear injection:

$$\mathcal{I}_{(S,\mathbb{R},n)} : L^2(G_n) \to L^2(\mathbb{R}) \tag{2.55}$$

Table 2.1: Results for the Schwinger approximation for the harmonic oscillator in $L^2(\mathbb{R})$. For $N = 81$ the error may be due to machine precision or the precision of the MatLab eigenvalue solver.

Analytical	$n = 12$	$n = 81$
0.5	0.500000065379456	0.500000000000050
1.5	1.499997524117474	1.499999999999974
2.5	2.500038664576120	2.499999999999966
3.5	3.499548442356141	3.499999999999972
4.5	4.502914300652827	4.499999999999973
5.5	5.481199032508385	5.500000000000019
6.5	6.561903192170472	6.500000000000021
7.5	7.270244513931606	7.499999999999971
8.5	8.750364859037072	8.499999999999986
9.5	9.078011293650551	9.499999999999957
10.5	10.935958083327678	10.499999999999948
11.5	14.294444938848978	11.499999999999970

by

$$\mathcal{I}_{(S,\mathbb{R},n)}(f) = \epsilon_n^{-1/2} \sum_{i \in G_n} f(n)\ \chi_{[G_n(i),G_n(i+1))}. \tag{2.56}$$

Definition 2.24. The position operator on $L^2(G_n)$ is

$$\widehat{x}f(a) = af(a), \tag{2.57}$$

where a takes values in $G_n \subseteq \mathbb{R}$.

When we look at this operator in $\mathcal{I}_{(S,\mathbb{R},n)}(L^2(G_n))$, it is the $\widehat{x}_\eta$ operator from equation 2.52.

2.7.2 Results

We used MatLab to implement the ideas for Schwinger approximation on the harmonic oscillator. The files are attached in the appendix, and it is possible to copy paste them from this thesis into MatLab. For the results with $n = 12$ and $n = 81$, together with the well known analytical results, see table 2.1. For the corresponding plots see figure B.1 and B.2 in the appendix.

Chapter 3

Analysis on $\mathbb{Q}_p$

As we mentioned in the introduction, any p-adic number x can be written uniquely as

$$x = \sum_{i=v(x)}^{\infty} x_i p^i, \tag{3.1}$$

(with $x_{v(x)} > 0$, and $0 \leq x_i < p$ for all i) where addition and multiplication works in the obvious way. The metric is given by

$$d(x, y) = |x - y| = p^{-v(x-y)}, \tag{3.2}$$

and we have the strong triangle inequality making d an ultrametric:

$$|\sum_n x_n| \leq \max_n |x_n|. \tag{3.3}$$

There is a very nice reformulation of the open ball with centre x and radius r

$$\mathcal{O}(x, r) = \{y \in \mathbb{Q}_p : |x - y| < r\} = \{y \in \mathbb{Q}_p : x_i = y_i \ \forall i \leq -\log_p(r)\}, \tag{3.4}$$

which clearly gives that any point in an open ball is a center of that ball. Furthermore, since the function v takes values in $\mathbb{Z}$, we see that the open ball of radius r is the same as the closed ball of radius $r - \epsilon$ (for a sufficiently small ϵ depending on r) which is compact.

3.1 Integration theory on $\mathbb{Q}_p$

To get integration we need a measure. From the introduction we know that $\mathbb{Q}_p$ has a unique Haar measure μ (as we have chosen $\mu(\mathbb{Z}_p) = 1$).[1]

Now we want to look at some properties of the measure μ, and then some properties of the integral. First let us establish some notation.

[1] This is because $\mathbb{Q}_p$ is a group where the topology is a separable metric space. Remember that $\mathbb{Z}_p = \{z \in \mathbb{Q}_p : v(z) \geq 0\}$.

Let us write

$$B_k(x) = \{y \in \mathbb{Q}_p : |x - y| \leq p^k\} = x + p^k\mathbb{Z}_p = \mathcal{O}(x, p^{k+1}), \tag{3.5}$$

and

$$B_k = B_k(0), \tag{3.6}$$

for the ball of radius p^k. From equation 3.4 we see that

$$B_k(x) = \bigcup_{i=0}^{p-1} B_{k-1}(x + ip^k), \tag{3.7}$$

where the union is disjoint. The sphere $S_k(x)$ around x is

$$S_k(x) = \{y \in \mathbb{Q}_p : |x - y| = p^k\}, \tag{3.8}$$

and we get

$$S_k(x) = B_k(x) \setminus B_{k-1}(x), \tag{3.9}$$

Let also

$$S_k = S_k(0). \tag{3.10}$$

What is then μ on these sets?

Proposition 3.1. We have that

$$\mu(B_k(x)) = p^k, \tag{3.11}$$

and

$$\mu(S_k(x)) = (p-1)p^{k-1}, \tag{3.12}$$

Note that this is all we need to know about the measure μ (in addition to the measure axioms) as it determines μ uniquely (the set of balls generates the sigma-algebra).

Proof. By translation invariance $\mu(B_k(x)) = \mu(B_k)$. For $k = 0$ we have $\mu(B_k) = \mu(\mathbb{Z}_p) = 1$ by definition. Using equation 3.7 together with translation invariance we get

$$\begin{aligned}\mu(B_k) &= \mu\left(\bigcup_{i=0}^{p-1} B_{k-1}(ip^k)\right) \\ &= \sum_{i=0}^{p-1} \mu(B_{k-1}(ip^k)) \\ &= p\mu(B_{k-1}),\end{aligned}$$

hence

$$\mu(B_k) = p^m\mu(B_{k-m})$$

for any $m \in \mathbb{Z}$. By choosing $m = k$ we get

$$\mu(B_k) = p^k\mu(B_0) = p^k\mu(\mathbb{Z}_p) = p^k.$$

The second equation follows from additivity of the measure. □

There is another useful equation, with a rich theoretical background.[2] We will now show that

$$\mu(xA) = |x|_p \mu(A), \tag{3.13}$$

for any $x \in \mathbb{Q}_p$ and measurable set A.

Lemma 3.2. *For any open ball $U = b + p^n\mathbb{Z}_p$ and any $a \in \mathbb{Q}_p$ we have $\mu(aU) = |a|_p\mu(U)$*

Proof. Write $a = p^m u$ where $|u| = 1$. Then $aU = ab + p^{m+n}\mathbb{Z}_p$. By proposition 3.1 $\mu(U) = p^{-n}$ and $\mu(aU) = p^{-m-n}$. By remembering that $|a|_p = p^{-m}$ we get the required equality. □

Theorem 3.3. *For any measurable set E we have $\mu(aE) = |a|_p\mu(E)$*

Proof. By now, we know that $\mu(a\mathbb{Z}_p) = |a|\mu(\mathbb{Z}_p) = |a|$. Let us define a new measure

$$\nu_a(E) = \mu(aE) \tag{3.14}$$

for any $a \in \mathbb{Q}_p$. Clearly this is a Haar measure, so there exist a constant c with the property that

$$\nu_a = c\mu. \tag{3.15}$$

This c can be determined by $c = c\mu(\mathbb{Z}_p) = \nu_a(\mathbb{Z}_p) = \mu(a\mathbb{Z}_p) = |a|$. So for any measurable E, we get

$$\mu(aE) = \nu_a(E) = c\mu(E) = |a|_p\mu(E). \tag{3.16}$$

□

We end this section with another similarity between $\mathbb{Q}_p$ and $\mathbb{R}$.

Proposition 3.4. Any open set in $\mathbb{Q}_p$ can be written as a (countable) disjoint union of open balls.

Proof. Let U be an open set, then $U = \bigcup_{i\in I} W_i$ with W_i open balls and I countable (as $\mathbb{Q}_p$ is a separable metric space). Given any two balls with nonempty intersection, $B_r(x)$ and $B_s(y)$, and z an element in the intersection, we can write them as $B_r(z)$ and $B_s(z)$, so that one is contained in the other.

Hence for any overlapping W_i and W_j we can remove one of them from the union, while still covering U. In this way we get an index-set $J \subseteq I$ such that $U = \bigcup_{i\in J} W_i$ is a disjoint union. □

[2] This equation can be used to define the norm given that you have a locally compact field. This gives a very different way to develop the theory.

3.2 Fourier transform on $\mathbb{Q}_p$

3.2.1 The rational part

As a preparation we need the following function on $\mathbb{Q}_p$.

Definition 3.5. The rational part function is

$$x = \sum_{v(x)}^{\infty} x_i p^i \mapsto \{x\} = \sum_{v(x)}^{-1} x_i p^i, \tag{3.17}$$

and can be considered a function from $\mathbb{Q}_p$ into $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Q}_p$.

A locally constant function f on $\mathbb{Q}_p$ has the property that there is some B_k so that $f(x) = f(x+h)$ for $h \in B_k$.

Proposition 3.6. The rational part function is locally constant, and satisfies

$$\{x+y\} = \{x\} + \{y\} + N_{x,y}, \tag{3.18}$$

where $N_{x,y}$ is either 0 or -1.

Proof. Clearly $\{z\} = 0$ for $z \in \mathbb{Z}_p$, and because carry in $\mathbb{Q}_p$ is to the right, $\{x+z\} = \{x\}$ for $z \in \mathbb{Z}_p$.

The equation comes from the fact that carry may only influence $\{-\}$ if we carry from $(x+y)_{-1}$ to $(x+y)_0$, in which case, the difference between $\{x+y\}$ and $\{x\}+\{y\}$ is 1. □

For the rest of this thesis we will write

$$\gamma_x(y) = e^{2\pi i\{xy\}}, \tag{3.19}$$

for any $x \in \mathbb{Q}_p$

Proposition 3.7. The function γ_x is a continuous homomorphism from $\mathbb{Q}_p$ to $\mathbb{T}$.

Proof. The continuity follows from that the rational part function is locally constant. The homomorphism property follows from

$$e^{2\pi i\{x+y\}} = e^{2\pi i(\{x\}+\{y\}+N_{x,y})} = e^{2\pi i\{x\}} e^{2\pi i\{y\}}. \tag{3.20}$$

□

3.2.2 Characters on $\mathbb{Q}_p$

In this section we present an independent proof that $\mathbb{Q}_p$ is self-dual. Let us first recall the definition of a character.

Definition 3.8. A character, or additive character, on $\mathbb{Q}_p$ is a continuous group homomorphism $\gamma : (\mathbb{Q}_p, +) \to (\mathbb{T}, \cdot)$. Here $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Denote the set of all characters on $\mathbb{Q}_p$ by $\widehat{\mathbb{Q}}_p$.

We already have a continuum of examples by proposition 3.7. We want to show that these are all the possibilities.

Proposition 3.9. For a character $\gamma \in \widehat{\mathbb{Q}}_p$ we have for any $n \in \mathbb{Z}$

$$\gamma(nx) = \gamma(x)^n \tag{3.21}$$

Proof. This follows easily from the definition. □

Proposition 3.10. Assume $\gamma(p^N) = 1$, then $\gamma(p^N\mathbb{Z}_p) = 1$.[3]

Proof. We have that $p^N\mathbb{Z}$ is dense in $p^N\mathbb{Z}_p$. By continuity of γ we need only show that for any $x = p^N z$ ($z \in \mathbb{Z}$) we have $\gamma(x) = 1$. But this is trivial as $\gamma(x) = \gamma(p^N)^z = 1$. □

Lemma 3.11. *For any character γ on $\mathbb{Q}_p$ or $p^{-K}\mathbb{Z}_p$ there is some N such that $\gamma(p^N) = 1$.*

Proof.

$$1 = \gamma(0) = \lim_{n\to\infty} \gamma(p^n) = \lim_{n\to\infty} \gamma(1)^{p^n} \in \mathbb{T} \tag{3.22}$$

We see that this sequence must be eventually constant, i.e. there is some N such that $\gamma(p^N) = \gamma(1)^{p^N} = 1$.

Why is this?[4] Let $x_n = \gamma(1)^{p^n} = e^{-i\theta_n}$, with $\theta_n \in (-\pi, \pi]$. As $x_n \to 1$ we can assume $\theta_n \in (-p^{-1}, p^{-1})$ by considering $n \geq M$ for some big M. Now, if $\theta_M \neq 0$, then for some k we must have $p^{-1} \leq |p^k\theta_M| \leq 1$, so that $\theta_{M+k} = p^k\theta_M \notin (-p^{-1}, p^{-1})$ (even modulo 2π), which is a contradiction. Hence $\theta_M = 0$ and $\gamma(1)^{p^M} = 1$. □

Lemma 3.12. *Given any $x \in \mathbb{Q}_p$, the function $\gamma_x : \mathbb{Q}_p \to \mathbb{T}$ defined by*

$$\gamma_x(y) = e^{2\pi i\{xy\}} \tag{3.23}$$

is a character on $\mathbb{Q}_p$ and on $p^{-K}\mathbb{Z}_p$ (for all K).

Proof. The first part follows from proposition 3.7. Finally, $p^{-K}\mathbb{Z}_p$ is an open subgroup of $\mathbb{Q}_p$. □

Lemma 3.13. *Any character on $p^{-K}\mathbb{Z}_p$ is on the form γ_x where $x = \sum_{i=-N}^{K-1} x_i p^i$.*

Proof. Given γ there exist N such that $\gamma(p^N) = 1$. Note that γ is determined by $\gamma(p^{-K})$. The equation $\gamma(p^{-K})^{p^{N+K}} = 1$ has at most[5] p^{N+K} solutions in $\mathbb{C}$, so there are at most p^{N+K} different characters (given N and K).

All γ_x where $x = \sum_{i=-N}^{K-1} x_i p^i$ are unique characters on $p^{-K}\mathbb{Z}_p$ with the property that $\gamma_x(p^N) = 1$. Since there are p^{N+K} of them, we get that γ must be one of them. □

[3] Here γ of a set A is the set $\{\gamma(a) : a \in A\}$, and 1 denotes the set $\{1\}$.

[4] This can be made into a general statement in $\mathbb{C}$: Let $z \in \mathbb{C}$, if $z^n \to 1$ as $n \to \infty$ then $z^N = 1$ for some $N \in \mathbb{N}$.

[5] Well, it's 'exactly' that many solutions, but we only need 'at most' in our proof.

Theorem 3.14. *Any character on $\mathbb{Q}_p$ is on the form γ_x where $x = \sum_{i=-N}^{\infty} x_i p^i$.*

Proof. First we see that γ is a character on $p^{-K}\mathbb{Z}_p$ for any K. So on all these sets there exist $a_K \in \mathbb{Q}_p$ such that $\gamma = \gamma_{a_K}$ on the set. For any K and L with $K < L$, both a_K and a_L are characters on $p^{-K}\mathbb{Z}_p$, hence $a_K = a_L \mod p^K$. Hence we can define

$$a = \lim_{K\to\infty} a_K \in \mathbb{Q}_p. \tag{3.24}$$

Since $\gamma = \gamma_a$ on all the sets $p^{-K}\mathbb{Z}_p$, we get $\gamma = \gamma_a$ on $\mathbb{Q}_p$. □

Corollary 3.15. *We have that $\mathbb{Q}_p$ is self-dual, i.e. The function*

$$\Psi : x \mapsto \gamma_x \tag{3.25}$$

is an isomorphism and a homeomorphism between $\mathbb{Q}_p$ and $\widehat{\mathbb{Q}}_p$.

Proof. First, because of the previous theorem it is easy to see that Ψ is a bijection.

Second, notice that

$$\begin{aligned}\gamma_{x+z}(y) &= e^{2\pi i\{(x+z)y\}} = e^{2\pi i(\{xy\}+\{zy\}+N_{x,y,z}} \\ &= e^{2\pi i\{xy\}} e^{2\pi i\{zy\}} = \gamma_x(y)\gamma_z(y).\end{aligned}$$

To show continuity, let $x_n \to x \in \mathbb{Q}_p$. Choose any compact set K in $\mathbb{Q}_p$. Then $K \subseteq B_r$ for some $r \in \mathbb{Z}$. Take N so large that $x_n - x \in B_{-r}$ for $n > N$. Then $\gamma_{x_n}/\gamma_x = \gamma_{x_n - x}$, and for all $y \in K$ we have $y(x_n - x) \in B_0 = \mathbb{Z}_p$ so that $\gamma_{x_n - x} = 1$ on K. Hence for any compact K there exist N such that for $n > N$ we have $\gamma_{x_n} = \gamma_x$ on K. It follows that $\gamma_{x_n} \to \gamma_x \in \widehat{\mathbb{Q}}_p$ (where the topology is uniform convergence on compacts).

To show that the inverse is continuous, assume that $\gamma_{x_n} \to \gamma_x \in \widehat{\mathbb{Q}}_p$. Consider the compacts $p^{-K}\mathbb{Z}_p$. First we will establish that

$$\gamma_x(p^M) = 1 \Rightarrow \gamma_{x_n}(p^M) = 1, \tag{3.26}$$

for $n > N$. So let M be so that $\gamma(p^M) = 1$, and take ϵ so small that $|\omega - 1| > \epsilon$ when $\omega^p = 1$ (and $\omega \neq 1$). By convergence in $\widehat{\mathbb{Q}}_p$ we have some N (depending on K and ϵ) such that $|\gamma_x - \gamma_{x_n}| < \epsilon$ on $p^{-K}\mathbb{Z}_p$ (when $n > N$). Let l_n be the biggest number in $\mathbb{Z}$ so that $\gamma_{x_n}(p^{l_n}) \neq 1$.[6] Then

$$\gamma_{x_n}(p^{l_n+1}) = (\gamma_{x_n}(p^{l_n}))^p = 1, \tag{3.27}$$

and so $\gamma_{x_n}(p^{l_n})$ is a p-th root of unity (what we called ω). By our choice of ϵ, when $n > N$ we get that $\gamma_x(p^{l_n}) \neq 1$, hence

$$l_n < M. \tag{3.28}$$

Let us now drop the first N terms of the sequence by re-indexing (we can do that without changing any convergence properties of the sequence), and we have a sequence with the property in equation 3.26 for all n.

[6] Do not think of l_n as a sequence, only as a number depending on the n you are considering.

Both γ_x and γ_{x_n} are characters on $p^{-K}\mathbb{Z}_p$, and the values of γ_{x_n} converge uniformly on $p^{-K}\mathbb{Z}_p$ to the values of γ_x. According to lemma 3.13 there is a finite number of different characters on $p^{-K}\mathbb{Z}_p$ with the property that $\gamma(p^M) = 1$, hence for some big N_K we get

$$\gamma_x(y) = \gamma_{x_n}(y) \text{ for } y \in p^{-K}\mathbb{Z}_p \text{ and for } n > N_K. \tag{3.29}$$

In other words, $\gamma_{x-x_n}(y) = 1$ for $y = p^{-K}$ and $n > N_K$, hence $y(x - x_n) \in \mathbb{Z}_p$ and $x - x_n \in p^K\mathbb{Z}_p$. For any $\epsilon = p^{-K}$ we get N_{K+1} with the property that $|x_n - x| < \epsilon$ for $n > N_{K+1}$. This is the same as $x_n \to x \in \mathbb{Q}_p$. □

3.2.3 Onwards to the Fourier transform

The Fourier transform on $\mathbb{Q}_p$ is

$$(\mathcal{F}f)(y) = \int_{\mathbb{Q}_p} e^{-2\pi i\{xy\}} f(x)\ \mathrm{d}x, \tag{3.30}$$

with inverse

$$(\mathcal{F}^{-1}f)(x) = \int_{\mathbb{Q}_p} e^{2\pi i\{xy\}} f(y)\ \mathrm{d}y. \tag{3.31}$$

3.2.4 Properties of the Fourier transform on $L^2(\mathbb{Q}_p)$

For the real Fourier transform there are several invariant subspaces, like the space spanned by Gaussian functions ($e^{x^2/2}$). Are there any invariant subspaces in $L^2(\mathbb{Q}_p)$? Yes, let us look at some particularly interesting subspaces.

Definition 3.16. Define the following subspaces of $L^2(\mathbb{Q}_p)$:

$$\begin{aligned} C_k &= \{f \in L^2(\mathbb{Q}_p) : \text{support}(f) \subseteq B_k\}. \\ S_k &= \{f \in L^2(\mathbb{Q}_p) : f(x + a) = f(x)\ \forall a \in B_{-k}\}. \\ W_k &= C_k \cap S_k. \end{aligned}$$

Proposition 3.17. We have that

$$\begin{aligned} C_k &\subseteq C_{k+1} \\ S_k &\subseteq S_{k+1} \\ W_k &\subseteq W_{k+1} \\ \mathcal{F}(C_k) &= S_k \\ \mathcal{F}(S_k) &= C_k \\ \mathcal{F}(W_k) &= W_k \end{aligned}$$

Proof. Given f with support$(f) \subseteq B_k$, and $a \in B_{-k}$ we have

$$\begin{aligned}(\mathcal{F}f)(y+a) &= \int_{\mathbb{Q}_p} e^{-2\pi i\{x(y+a)\}} f(x)\ \mathrm{d}x \\ &= \int_{B_k} e^{-2\pi i\{xy\}} e^{-2\pi i\{xa\}} f(x)\ \mathrm{d}x \\ &= \int_{B_k} e^{-2\pi i\{xy\}} e^{-2\pi i\cdot 0} f(x)\ \mathrm{d}x \\ &= \int_{B_k} e^{-2\pi i\{xy\}} f(x)\ \mathrm{d}x \\ &= (\mathcal{F}f)(y).\end{aligned}$$

Similarly, given f with $f(x+a) = f(x)\ \forall a \in B_{-k}$, and $y \notin B_k$

$$\begin{aligned}(\mathcal{F}f)(y) &= \int_{B_k} e^{-2\pi i\{xy\}} f(x)\ \mathrm{d}x \\ &= \int_{B_k} e^{-2\pi i\{xy\}} f(x-p^{-k})\ \mathrm{d}x \\ &= \int_{B_k} e^{-2\pi i\{(x+p^{-k})y\}} f(x)\ \mathrm{d}x \\ &= \int_{B_k} e^{-2\pi i\{xy\}} e^{-2\pi i\{p^{-k}y\}} f(x)\ \mathrm{d}x \\ &= e^{-2\pi i\{p^{-k}y\}} (\mathcal{F}f)(y).\end{aligned}$$

Since $p^{-k}y \notin \mathbb{Z}_p$ we get $0 < \{p^{-k}y\} < 1$, and $e^{-2\pi i\{p^{-k}y\}} \neq 1$. Hence $(\mathcal{F}f)(y) = 0$.

Finally

$$\mathcal{F}(W_k) = \mathcal{F}(C_k \cap S_k) = \mathcal{F}(C_k) \cap \mathcal{F}(S_k) = S_k \cap C_k = W_k.$$

□

3.3 Ordering and intervals on $\mathbb{Q}_p$

One of the arguments that we can't have p-adic time is because time must be ordered for it to make any sense to us. Apart from using t as a real parameter on our complex Hilbert space, we can use p-adic time with the following lexicographic ordering. These ideas can be found in [11].

In this section, indexing will be used to denote the canonical expansion in equation 3.1 without comment.

Definition 3.18. We say that $x < y$ for $x, y \in \mathbb{Q}_p$ if

- $v(x) < v(y)$, or
- $v(x) = v(y)$ and $x_i = y_i$ for $v(x) \leq i < k$ and $x_k < y_k$ (for some k).

Of course, $x \leq y$ if $x < y$ or $x = y$.

Note that this gives a total ordering, and that $0 < x$ for all $x \neq 0 \in \mathbb{Q}_p$.

We can define

$$[a, b] = \{x \in \mathbb{Q}_p : a \leq x \leq b\}, \tag{3.32}$$

but we get

$$[1, 2] - 2 \neq [1 - 2, 2 - 2] = [-1, 0], \tag{3.33}$$

as the last set is empty.

This is really bad, but can we remedy it? For example by restricting the z we add with to be from $\mathbb{Z}^+$?

Proposition 3.19. For any $a < b \in \mathbb{Q}_p$ there is some $z \in p^k\mathbb{Z}^+$ so that

$$b + z < a + z, \tag{3.34}$$

where we can choose $k = \min\{v(a), v(b)\} - 1$.

Proof. Assume $v(a) = v(b)$. Then there is a least i such that $a(i) < b(i)$. Let $z = (p - b_i)p^i$. Then $(a + z)_i \neq 0$, and $(b + z)_i = 0$, while $(a + z)_k = (b + z)_k$ for $k < i$. Hence $b + z < a + z$.

If $v(a) \neq v(b)$ we can add $p^{\min\{v(a),v(b)\}-1}$ to both a and b. After this, if $b > a$ we are done. If a is still smaller than b we can do the same procedure as above. □

Assuming $a, b \in p\mathbb{Z}_p$ (which is the domain of the p-adic sine, and therefore will be the domain of our p-adic time - all of this will come later, in chapter 6), we get some $z \in \mathbb{Z}^+$ with the property that translating with z changes the time-order of a and b.

We rephrase this result and get the following proposition.

Proposition 3.20. For any $a < b \in \mathbb{Q}_p$ there is some $z \in p^k\mathbb{Z}$

$$[a, b] + z \neq [a + z, b + z]. \tag{3.35}$$

Proof. By the previous proposition, we can find a z so that $[a + z, b + z] = \emptyset$. □

3.4 Important maps between $\mathbb{R}$ and $\mathbb{Q}_p$

In this section we will focus on three interesting maps, the first we saw in [11] and can be used to transfer the Haar measure between $\mathbb{R}$ and $\mathbb{Q}_p$, but is not order-preserving. The second was presented in [2] and does preserve the ordering, but maps the p-adic numbers to a Cantor-like (nowhere dense and measure zero) subset of $\mathbb{R}$. The third is a slight generalization of the rational part function.

Definition 3.21. The simple almost-inclusion of $\mathbb{Q}_p$ into $\mathbb{R}$ is

$$P_s : \sum_{i=v(x)}^{\infty} x_i p^i \mapsto \sum_{i=v(x)}^{\infty} x_i p^{-i}. \tag{3.36}$$

Definition 3.22. The Vladimirov inclusion of $\mathbb{Q}_p$ into $\mathbb{R}$ is

$$P_V : \sum_{i=v(x)}^{\infty} x_i p^i \mapsto \sum_{i=v(x)}^{\infty} x_i p^{-2i}. \tag{3.37}$$

Definition 3.23. The k-rational part function from $\mathbb{Q}_p$ to $\mathbb{Q}$ is

$$P_{R,k} : \sum_{i=v(x)}^{\infty} x_i p^i \mapsto p^k \{p^{-k} x\} = \sum_{i=v(x)}^{k-1} x_i p^i. \tag{3.38}$$

Proposition 3.24. The function P_s is

- not injective,
- injective almost everywhere,
- surjective onto the positive real numbers
- continuous,
- unable to preserve the ordering.

Proof. Note that

$$\sum_{i=0}^{\infty} (p-1)p^i \tag{3.39}$$

is mapped to

$$\sum_{i=0}^{\infty} (p-1)p^{-i} = \frac{p-1}{1-1/p} = p, \tag{3.40}$$

and

$$p^{-1} \tag{3.41}$$

is also mapped to p. This makes P_s not injective, but these counterexamples requires the ambiguity in the notation of the real numbers (i.e. $1 = 0.99999...$), and the set of these ambiguous numbers is countable. The preimage under P is also countable, hence of measure zero.

For surjectivity, it is enough to note that every positive real number can be written as an infinite positive expansion with base p (where p does not need to be a prime, e.g. the decimal system).

Let us now show continuity. Let $|P(x) - P(y)| < \epsilon = p^{-N}$, choose $\delta = p^{-N-1}$,

then

$$\begin{aligned}
|x-y|_p < \delta \Rightarrow x_i = y_i \ \forall_{i<N+1} \Rightarrow \\
|f(x)-f(y)| &= |\sum_{i=v(x)}^{\infty} x_i p^{-i} - \sum_{i=v(y)}^{\infty} y_i p^{-i}| \\
&= |\sum_{i=\min\{v(x),v(y)\}}^{\infty} (x_i - y_i) p^{-i}| \\
&= |\sum_{i=N+1}^{\infty} (x_i - y_i) p^{-i}| \\
&\leq \sum_{i=N+1}^{\infty} |x_i - y_i| p^{-i} \\
&\leq \sum_{i=N+1}^{\infty} 2p p^{-i} \\
&\leq \frac{2p}{p-1} p^{-N-1} \leq 2p^{-N} \leq 2\epsilon
\end{aligned}$$

This concludes the proof of continuity.

To see that it does not preserve the ordering, the fact that it is not injective is sufficient. □

Proposition 3.25. The function P_s preserves the measure of a set, that is

$$\mu_{\mathbb{R}}(P(B)) = p\mu_{\mathbb{Q}_p}(B) \tag{3.42}$$

Proof. This is the result by Minggen et. al in [11], where they define the measure $\mu_{\mathbb{Q}_p}$ using the above equation (after dividing out a set of measure zero) in their definition 2.1. Then they prove that it actually is a Haar measure in Theorem 2.3. Since it is a Haar measure with $\mu_{\mathbb{Q}_p}(\mathbb{Z}_p) = 1$ it is the same as our measure. □

Proposition 3.26. The inclusion P_V is

- injective,
- not surjective,
- maps onto a Cantor-set (nowhere dense and measure zero),
- continuous,
- able to preserve the ordering.

Proof. See chapter I.6 in [2]. □

Proposition 3.27. The k-rational part function from $\mathbb{Q}_p$ to $\mathbb{R}$

- can be considered a function onto $\mathbb{Q}_p \diagup (p^k \mathbb{Z}_p)$

- satisfies $\lim_k P_{R,k}(x) \to x \in \mathbb{Q}_p$ (where the limit is in $|-|_p$)
- locally constant.

Proof. The first three points are easy to see. The last point follows from proposition 3.6. □

3.5 Differentiation and anti-integration

In this section we want to point out the many notions of differentiation in $\mathbb{Q}_p$. The fact that we have several ways to generalize differentiation is a problem, both when trying to define position and momentum operators, and when describing the generator of a unitary representation.

3.5.1 Banach differentiation

With $f : \mathbb{Q}_p \to \mathbb{C}$ we can define derivative between the two spaces in a similar way to what we do between two Banach spaces. For some f and some points $x \in \mathbb{Q}_p$, define $(D_B f)(x)$ whenever

$$\lim_{h \to 0} \frac{|f(x+h) - f(x) - (D_B f)(x)|}{|h|} = 0, \tag{3.43}$$

where $h \in \mathbb{Q}_p$, the numerator uses complex absolute value and the denominator p-adic absolute value.

3.5.2 Fourier transform of the p-norm

By looking at proposition 2.16 we get the idea for the following definition.

Definition 3.28. The Fourier-Norm (FN) derivative D_{FN} of f is

$$D_{FN}(f) = (\mathcal{F}^{-1}|x|_p\mathcal{F})(f). \tag{3.44}$$

3.5.3 Fourier transform of the fractional part

By looking at proposition 2.16 we get the idea for the following definition.

Definition 3.29. The Fourier-Character (FC) derivative D_{FC} of f is

$$D_{FC}(f) = (\mathcal{F}^{-1}\{x\}\mathcal{F})(f). \tag{3.45}$$

We can also use the k-rational part from definition 3.23

$$D_{FC,k}(f) = (\mathcal{F}^{-1}p^k\{p^{-k}x\}\mathcal{F})(f), \tag{3.46}$$

or perhaps the limit as $k \to \infty$ in some sense.

3.5.4 The anti-integration

From [11] we have a notion of integration from 0 to x, so we can define differentiation to be the opposite of integration.

Definition 3.30. The Anti-Integration (AI) derivative of $f : \mathbb{Q}_p \to \mathbb{C}$ (when it exists) is the function $D_{AI}(f)$ defined almost everywhere satisfying

$$\int_0^x D_{AI}(f)(x')\ \mathrm{d}x' = f(x). \tag{3.47}$$

3.6 Finite approximations in $L^2(\mathbb{Q}_p)$

When we want to use a compact subgroup of $\mathbb{R}$ we use addition modulo 1 on $[0, 1)$, but in $\mathbb{Q}_p$ the compact set $\mathbb{Z}_p$ is already a group. To discretize (in $\mathbb{R}$) this compact group into a finite group, we can use the set $A_n = \{\frac{i}{n} | 0 \leq i < n\} \subseteq [0, 1)$ with addition modulo 1. Similarly we can use

$$G_n = p^{-n}\mathbb{Z}_p/(p^n\mathbb{Z}_p) \tag{3.48}$$

in the p-adic case.

3.6.1 Approximating H

Similarly to what we did in the real case, we define a position operator $\widehat{x}$ on $\mathbb{Q}_p$. Then we take the momentum operator to be the Fourier transform of $\widehat{x}$. Both of these operators can also be considered operators on G_n. The Fourier transform factors nicely, i.e.

$$\widehat{p} = \mathcal{F}_n \widehat{x} \mathcal{F}_n^{-1} \text{ on } G_n, \tag{3.49}$$

where $\mathcal{F}_n$ is the finite Fourier transform on G_n.

We have constructed a MatLab algorithm that can be found in the appendix. In table 3.1 and 3.2 you can find the numerical results, together with the analythical results from chapter X.9 and X.10 in [2].

For $p = 3$ the multiplicity is calculated as follows. According to [2] (page 181 - 182) we get that

$$\lambda_N^l = 3^{2N} + 3^{2l-2N}, \quad N \in \mathbb{Z}, \quad l = 2, 3, ... \tag{3.50}$$

has multiplicity at least

$$4 \sum_{1 \leq i \leq d} 3^{y_i - 2}, \tag{3.51}$$

where $\{(x_i, y_i) : i = 1, 2, ..., d\}$ are the solutions for $x \in \mathbb{Z}$ and $y = 2, 3, ...$ of

$$\lambda_N^l = 3^{2x} + 3^{2y-2x}. \tag{3.52}$$

Table 3.1: Results for the approximation for the harmonic oscillator on $L^2(\mathbb{Q}_p)$ with $p = 3$, and with the position operator chosen as $\widehat{x}f(x) = |x|_p f(x)$ (the momentum operator is the Fourier transform of the position operator).

Analytical	$n = 1$	$n = 2$
< 4.5	0.681874953487972	0.669253240146968
-	4.703703703703697	4.692719683556841
-	4.725532453919437	4.716187293920393
5	4.999999999999997	4.999999999999983
5	5.000000000000003	5.000000000000011
9	8.999999999999991	8.999999999999979
9	8.999999999999995	8.999999999999980
9	8.999999999999998	9.000000000000018
9	9.000000000000000	9.000000000000025
-	-	40.522643828102950
-	-	40.522652744396467
$40 + \frac{10}{18}$	-	40.555555555555586
$40 + \frac{10}{18}$	-	40.555555555555770
41	-	40.999999999999865
41	-	41.000000000000000
41	-	41.000000000000057
41	-	41.000000000000078
41	-	41.000000000000163
41	-	41.000000000000171
41	-	41.000000000000185

By the first lemma in chapter X.10 in [2] ($\alpha = 2$) we get that the only solutions of the previous equation are $x = N$ and $y = l - N$, or $x = l - N$ and $y = N$ so that the multiplicity of λ_N^l is no less than

$$4\left(3^{l-N-2} + 3^{N-2}\right). \tag{3.53}$$

Further we are given $p - 2 = 1$ eigenvectors with eigenvalue

$$\lambda_N^1 = 3^{2N} + 3^{2-2N}, \tag{3.54}$$

so that the multiplicity here is at least 1.

In the numerical calculations we used $H = \frac{1}{2}\widehat{x}^2 + \frac{1}{2}\widehat{p}^2$, so that our eigenvalues are

$$\frac{\lambda_N^l}{2}. \tag{3.55}$$

Table 3.2: Results for harmonic oscillator using $n = 4$, for $p = 3$, $\widehat{x}f(x) = |x|_p f(x)$. L. mult. stands for least multiplicity, and e.v. for eigenvalue.

Analytical e.v.	L. mult.	Numerical e.v.	Mult.
<4.5	-	0.6688	1
-	-	4.6923	1
-	-	4.7158	1
5	1	5.0000	2
9	3	9.0000	4
-	-	40.5214	2
-	-	40.5556	2
41	5	41.0000	8
45	6	45.0000	24
81	8	81.0000	36
-	-	364.5000	2
-	-	364.5100	2
-	-	364.5600	3
365	13	365.0000	24
-	-	369.0000	72
-	-	405.0000	216
-	-	729	324
-	-	3000<x<3700	2916
-	-	6165	>2900

Chapter 4

Classical mechanics and Weyl quantization in $\mathbb{Q}_p$

In this chapter we want to generalize classical mechanics to the p-adic case, and to look at a generalization of the Weyl system usable over $\mathbb{Q}_p$.

4.1 Classical mechanics

Let us first use p-adic numbers in classical mechanics. Using the same Hamiltonian as in the real case, but with p-adic coordinates and time; according to [12] (equation 3.4-3.5) we get

$$\begin{aligned} H &= \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2 \Rightarrow \\ \dot{p} &= -m\omega^2 q,\ \dot{q} = \frac{1}{m}p;\ p(0) = p,\ q(0) = q, \end{aligned}$$

with the analytic solution

$$\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{m\omega}p\sin\omega t + q\cos\omega t \\ p\cos\omega t - qm\omega\sin\omega t \end{pmatrix}, \tag{4.1}$$

where

$$\begin{aligned} \sin x &= \sum_{n=1}^{\infty}(-1)^n \frac{x^{2n-1}}{(2n-1)!},\ |x|_p < 1,\ |x|_2 < \frac{1}{2} \\ \cos x &= \sum_{n=1}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!},\ |x|_p < 1,\ |x|_2 < \frac{1}{2}. \end{aligned}$$

Several questions come naturally. The first is, 'Does this give the same physical results as in the real case?'. One of the main motivations for using $\mathbb{Q}_p$ is that we need the rationals, because any result of a physical measurement can be interpreted in $\mathbb{Q}$. So let us look at this solution from that point of view.

Proposition 4.1. We see that equation 4.1 looks the same but is not the same as the solution in example 1.24.

Proof. The expression in equation 4.1, is the same as in 1.31, even the expressions for the definition of sine and cosine are the same in the real and the p-adic case. However, the sum to infinity has a different interpretation in the real numbers (convergence in $|-|_\infty$) and the p-adic (convergence in $|-|_p$). E.g.

$$\sin\left(\frac{p}{p^2+1}\right) = \sum_{n=1}^{\infty}(-1)^n \frac{1}{(2n-1)!}\left(\frac{p}{p^2+1}\right)^{2n-1}, \tag{4.2}$$

which clearly converges both in $|-|_\infty$ and in $|-|_p$. As long as none of these are in $\mathbb{Q}$, they live in completely different number fields. □

4.2 Weyl systems in the general case

Assume that G is a locally compact abelian group, which is also self-dual. Definition 2.17 can be generalized to the following definition.

Definition 4.2. A Weyl system on $\mathcal{H} = L^2(G)$ is a function $W : G \times G \to \mathcal{U}(\mathcal{H})$, so that $W(q,p)$ is a unitary operator, with the property that

$$W(z+z') = B(z,z')W(z)W(z'), \tag{4.3}$$

where $B : G^2 \times G^2 \to \mathbb{T}$, and $z = (q,p) \in G^2$.

Proposition 4.3. Let $\chi(x) = e^{2\pi i\{ax\}}$ for some $a \in \mathbb{Q}_p$,[1] then

$$W(q,p)\psi(x) = \chi\left(\frac{qp}{2} + qx\right)\psi(x+p) \tag{4.4}$$

is a Weyl system on $\mathcal{H} = L^2(\mathbb{Q}_p)$, where

$$B((q,p),(q',p')) = \chi\left(\frac{q'p - p'q}{2}\right). \tag{4.5}$$

Proof. The proof is exactly the same as the proof of proposition 2.18. □

The following theorem shows how the system we just defined is the best choice; all other choices are essentially the same as this, possibly with redundancy.

Theorem 4.4. *There is only one irreducible Weyl-system over $\mathbb{Q}_p$ up to unitary isomorphisms, namely $W(q,p)$. All Weyl systems over $\mathbb{Q}_p$ can be written as a direct sum of copies of this.*

[1] This χ is what we called the character γ_a in chapter 3. This a will become $-m\omega$ in proposition 6.1.

Proof. Using Theorem 3, and corollaries 1 and 2 on pages 247-249 in Vladimirov [2], we only need to show that $W(q,p)$ has a one-dimensional vacuum space, i.e. that there is only one invariant wavefunction in $L^2(\mathbb{Q}_p)$ (up to multiplication with a constant). The a we write in this proof is the a in the definition of W (see the previous proposition).

So let ψ be so that

$$W(q,p)\psi(x) = \chi\left(\frac{qp}{2} + qx\right)\psi(x+p) = \psi(x), \tag{4.6}$$

where $(q,p) \in a^{-1}B_0 \times B_0$. Then we get

$$\begin{aligned}
\psi(x+p) &= \psi(x)\ \forall_{p\in B_0} \\
\chi(qx)\,\psi(x) &= \psi(x)\ \forall_{q\in a^{-1}B_0} \\
\text{support}\ (\psi) &\subseteq \{x : \chi(qx) = 1\}\ \forall_{q\in a^{-1}B_0}
\end{aligned}$$

We choose $q = a^{-1} \in a^{-1}B_0$, and get

$$\text{support}\ (\psi) \subseteq \{x : e^{2\pi i\{x\}} = 1\} = B_0 \tag{4.7}$$

As $\psi(x) = \psi(0)$ for $x \in B_0$ with $\psi(x) = 0$ for $x \notin B_0$ we have that ψ is constant. Hence the vacuum space is one-dimensional.

Let us comment why Vladimirov use $(q,p) \in V_0 = B_0 \times B_0$, but we use $(q,p) \in a^{-1}B_0 \times B_0$. If you look at their symplectic form, it is the same as ours if you scale our q by a^{-1} to cancel the a hidden in our symplectic form B (the notation differs considerably, and they write the symplectic form on the other side of their equations). □

4.3 Weyl quantization

Weyl quantization[2] is another way to interpret $f(Q,P)$ for operators Q and P (the first way is to use f as a function on the spectrum of the operators, as mentioned in section 2.2.2). Intuitively, given $f : \mathbb{C}^2 \to \mathbb{C}$

$$\begin{aligned}
f(Q,P) &= \int\int f(q,p)\delta_Q(q)\delta_P(p)\ \mathrm{d}q\ \mathrm{d}p \\
&= \int\int f(q,p) \int\int \left(e^{2\pi ia(Q-q)}e^{2\pi ib(P-p)}\right)\ \mathrm{d}a\ \mathrm{d}b\ \mathrm{d}q\ \mathrm{d}p \\
&= \int\int\int\int f(q,p)\left(e^{2\pi ia(Q-q)}e^{2\pi ib(P-p)}\right)\ \mathrm{d}q\ \mathrm{d}p\ \mathrm{d}a\ \mathrm{d}b,
\end{aligned}$$

[2] Also known as phase-space quantization.

which can be made rigorous over $L^2(\mathbb{R})$. To be able to use this over $L^2(\mathbb{Q}_p)$ we continue the calculations

$$\begin{aligned} f(Q,P) &= \int\int\int\int f(q,p)\left(e^{-2\pi iaq}e^{-2\pi ip)}U_aV_b\right)\,\mathrm{d}q\,\mathrm{d}p\,\mathrm{d}a\,\mathrm{d}b \\ &= \int\int U_aV_b\int\int f(q,p)\left(e^{-2\pi iaq}e^{-2\pi ip)}\right)\,\mathrm{d}q\,\mathrm{d}p\,\mathrm{d}a\,\mathrm{d}b \\ &= \int\int \tilde{f}(q,p)e^{-2\pi iab/2}W_{a,b}\,\mathrm{d}a\,\mathrm{d}b, \end{aligned}$$

where $W_{ab} = e^{2\pi iab/2}U_aV_b$ is the Weyl system with character $\chi(x) = e^{2\pi ix}$, and $\tilde{f}$ is the two dimensional Fourier transform. This can be generalized to $\mathbb{Q}_p$.

Definition 4.5. Given $f : \mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{C}$, the quantization of (the classical function) f is

$$A_f = \int_{\mathbb{Q}_p}\int_{\mathbb{Q}_p} \tilde{f}\chi_p(-qp/2)W_{q,p}\,\mathrm{d}s\,\mathrm{d}t, \tag{4.8}$$

where $\tilde{f} = \mathcal{F}_{\mathbb{Q}_p} \otimes \mathcal{F}_{\mathbb{Q}_p}(f)$, and the character in the Weyl system is $\chi_p(x) = e^{2\pi i\{x\}}$.

Chapter 5

Quantum mechanics with real time using Schrödinger-like operators

In this chapter we will look at what Vladimirov et. al. calls Schrödinger-like operators. Let

$$Q^\alpha(f)(x) = |x|_p^\alpha f(x) \tag{5.1}$$

on $L^2(\mathbb{Q}_p)$. Further $P^\alpha = \tilde{Q}^\alpha = \mathcal{F}Q^\alpha\mathcal{F}^{-1}$. Then we can define an operator that seems to correspond to the Schrödinger operator for a harmonic oscillator,

$$H = Q^2 + P^2. \tag{5.2}$$

5.1 Generators of time translation

How can operators like H be used to generate time translation on $L^2(\mathbb{Q}_p)$? We can define

$$U_t = e^{itH}, \tag{5.3}$$

or use $\chi(tH)$ for any character χ on $\mathbb{R}$. If we regard $L^2(\mathbb{Q}_p)$ as a complex (separable) Hilbert space, there is only one operator generating time translations (up to multiples of I, and unitary transformations), namely the one we already know from $L^2(\mathbb{R})$. We can hope to find nicer descriptions of some quantum systems in this way (especially those with fractal properties), but merely using the isomorphism between $L^2(\mathbb{R})$ and $L^2(\mathbb{Q}_p)$ as separable Hilbert spaces to find the required H seems artificial at best.

5.2 Basis for $L^2(\mathbb{Q}_p)$ of eigenvectors of P^α

Let us find a basis of eigenvectors for D^α and for $|x|_p^\alpha$ that is also invariant under the Fourier transform. The material in this section is taken from [2] (chapter IX.5

and onwards) and [6] (Chapter 3).[1]

Proposition 5.1. *H is self-adjoint and densely defined.*

It is natural to look for locally constant compactly supported functions. So let them be supported on a sphere S_k.

Given that $l \geq 2, N \in \mathbb{Z}, k \in \{1, 2, ..., p-1\}$, and $\epsilon_{(l)}$ (given an l) can be written uniquely as $\epsilon_{(l)} = \epsilon_0 + \epsilon_1 p^1 + ... + \epsilon_{l-2} p^{l-2}$.

$$\phi_{l,N,k,\epsilon_{(l)}}(x) = p^{\frac{N+1-l}{2}} \delta(|x| - p^{l-N}) \delta(x_0 - k) \chi_p(\epsilon_{(l)} p^{l-2N} x^2) \tag{5.4}$$

where $x_0 = (x - \{x\}_p) \bmod p$. Define also

$$\phi_{1,N,k}(x) = p^{\frac{N-1}{2}} \Omega(|x|_p - p^{1-N}) \chi_p(k p^{-N} x). \tag{5.5}$$

Then the Fourier transforms are

$$\tilde{\phi}_{l,N,k,\epsilon_{(l)}}(x) = \rho p^{-\frac{N-1}{2}} \delta(|x| - p^N) \delta(x_0 + 2\epsilon_0 k) \chi_p \left(\frac{1}{\epsilon_{(l)}} p^{2N-l} x^2 \right) \tag{5.6}$$

where ρ is a constant (not important for us - it can be found in Kochubei chapter 3), and

$$\tilde{\phi}_{1,N,k}(x) = p^{-\frac{N-1}{2}} \delta(|x|_p - p^N) \delta(x_0 - k). \tag{5.7}$$

Proposition 5.2. *The set $\{\phi\}$ is a basis for $L^2(\mathbb{Q}_p)$ of eigenvectors of P^α.*

Proof. According to Vladimirov (page 163-165) This is a complete orthornormal set in $L^2(\mathbb{Q}_p)$, i.e. a basis. It is easy to see that any function multiplied with $\delta(|x| - p^N)$ is an eigenfunction of Q^α with eigenvalue $p^{\alpha N}$. Since all the $\tilde{\phi}$ look like this, we get that all ϕ are eigenfunctions of D^α.[2] □

5.3 Commutator of Q and P

For all the functions

$$\phi_{l,N}(x) \quad l > 1, \tag{5.8}$$

from the previous section, we argued why they are eigenfunctions of P^α with eigenvalues $p^{\alpha N}$. As they contain $\delta(|x| - p^{l-N})$, they are eigenfunctions of Q^α with eigenvalue $p^{\alpha(l-N)}$.

Proposition 5.3. *For all the functions*

$$\phi_{l,N}(x) \quad l > 1 \tag{5.9}$$

the operators Q and P commute, i.e. $QP\phi = PQ\phi$.

[1] The book by Vladimirov is frequently cited, but the book by Kochubei is more clearly written.
[2] This is also theorem 3.1 in Volovich.

Proof. We have

$$\begin{aligned}[Q,P]\phi_{l,N} &= QP\phi_{l,N} - PQ\phi_{l,N} \\ &= p^{(l-N)}p^N\phi_{l,N} - p^N p^{(l-N)}\phi_{l,N} \\ &= 0\end{aligned}$$

□

This proposition states that the commutator is 0 for an infinite dimensional subspace, and its closed span. There is no way to repair this, and get a true commutator for most of $L^2(\mathbb{Q}_p)$, which is what we would expect from a theory in quantum mechanics with position operator Q and momentum operator P. To sum up, we claim that this choice of H is not only bad for the Harmonic oscillator, but for any quantum mechanical system.

Chapter 6

Quantum mechanics with p-adic time using Weyl systems

Whether we should use real or p-adic time is not evident. There are two strong arguments for using p-adic time. The first is that relativity theory suggests that space and time should be easily comparable. The second is that the time evolution corresponding with the Weyl system and the classical physics is easily generalized. This is the first thing we will look at in this chapter.

There is also a strong argument against using p-adic time, namely that there is no good total ordering on $\mathbb{Q}_p$; we will also look at this.

6.1 Time evolution using the Weyl system

In this section we use the character $\chi = e^{2\pi i\{x\}}$. Let us define, as before, the time evolution to be

$$(U_t\psi)(x) = \int_{\mathbb{Q}_p} K_t(x,y)\ \mathrm{d}y. \tag{6.1}$$

For the full Harmonic oscillator we will take the following propagator as our guess. (We found it in Dragovich [3])

$$K_t(x,y) = \lambda_p(2\omega t)|\omega t|^{-1/2}\chi\left(\frac{xy}{\sin\omega t} - \frac{x^2+y^2}{2\tan\omega t}\right), \tag{6.2}$$

with (write $a = p^{v(a)}(a_0 + a_1p^1 + a_2p^2 + ...)$)

$$\lambda_p(a) = \begin{cases} 1,\ v(a) = 2k \\ \left(\frac{a_0}{p}\right),\ v(a) = 2k+1,\ p \bmod 4 = 3 \\ i\left(\frac{a_0}{p}\right),\ v(a) = 2k+1,\ p \bmod 4 = 3 \end{cases}, \tag{6.3}$$

for $p \neq 2$, and

$$\lambda_2(a) = \begin{cases} 2^{-1/2}(1+(-1)_1^a i), \ v(a) = 2k \\ (-1)^{a_1+a_2} 2^{-1/2}(1+(-1)_1^a i), \ v(a) = 2k+1 \end{cases} . \tag{6.4}$$

Here

$$\left(\frac{a}{p}\right) = a^{(p-1)/2} \bmod \mathrm{p} \ = \begin{cases} 1, \ \sqrt{a} \in F_p \\ 0, \ a \bmod \mathrm{p} \ = 0 \\ -1, \ \text{else} \end{cases} \tag{6.5}$$

is the Legendre symbol.

Similarly to proposition 2.20 we now show the following.

Proposition 6.1. The operator U_t defined by equation 6.1 and 6.2 satisfies

$$U(t)U(t') = U(t+t'), \tag{6.6}$$

and

$$U(t)W(z)U(t)^{-1} = W(T_t z), \tag{6.7}$$

where T_t comes from the classical time evolution (equation 4.1) and is

$$T_t(q,p) = \begin{pmatrix} \cos(\omega t) & \frac{\sin(\omega t)}{m\omega} \\ -m\omega \sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}. \tag{6.8}$$

Here we choose the Weyl-system

$$W(q,p) = \chi\left(-m\omega\left(\frac{qp}{2} + qx\right)\right)\psi(x+p). \tag{6.9}$$

Proof. First we show that $K_{t/\omega + t'/\omega}(x,y) = \int K_{t/\omega}(x,z) K_{t'/\omega}(z,y)$ (since ω is fixed, this is equivalent to what we want to get). First observe that

$$\begin{aligned} |\sin(t)| &= |t| \\ |\cos(t)| &= 1 \\ |\tan(t+t')| &= \frac{|\sin(t+t')|}{|\cos(t+t')|} = |t+t'| \end{aligned}$$

for all t, t' where sin is defined ($|t| < 1$). If we write

$$\frac{t+t'}{2tt'} = p^{-A}(a_0 + a_1 p + a_2 p^2 + ...) \tag{6.10}$$

and

$$\frac{\tan(t) + \tan(t')}{2\tan(t)\tan(t')} = p^{-B}(b_0 + b_1 p + b_2 p^2 + ...) \tag{6.11}$$

we see that $A = B$ and $a_0 = b_0$, hence

$$\lambda_p\left(\frac{t+t'}{2tt'}\right)^{-1} = \lambda_p\left(-\frac{t+t'}{2tt'}\right) = \lambda_p\left(-\frac{\tan(t)+\tan(t')}{2\tan(t)\tan(t')}\right). \tag{6.12}$$

Consider now

$$\begin{aligned}\int K_{t/\omega}(x,z)K_{t'/\omega}(z,y) &= \int \lambda_p(2t)\lambda_p(2t')|t|^{-1/2}|t'|^{-1/2} \\ &\quad \cdot \chi\left(\frac{xz}{\sin t} - \frac{x^2+z^2}{2\tan t} + \frac{zy}{\sin t'} - \frac{z^2+y^2}{2\tan t'}\right)\,\mathrm{d}z \\ &= \lambda_p(2t)\lambda_p(2t')|tt'|^{-1/2}\lambda_p(\alpha)|2\alpha|^{-1/2} \\ &\quad \cdot \chi\left(\frac{-\beta^2}{2\alpha} - \frac{x^2}{2\tan t} - \frac{y^2}{2\tan t'}\right)\,\mathrm{d}z,\end{aligned}$$

where

$$\begin{aligned}\alpha &= \frac{-1}{2}(\cot t + \cot t') = \frac{-1}{2}\frac{\sin(t+t')}{\sin t \sin t'} \\ \beta &= \frac{x}{\sin t} + \frac{y}{\sin t'}.\end{aligned}$$

Then

$$\begin{aligned}\int K_{t/\omega}(x,z)K_{t'/\omega}(z,y) &= \lambda_p(2t)\lambda_p(2t')|tt'|^{-1/2}\lambda_p(\frac{-1}{2}(\cot t + \cot t'))\left|\frac{\tan t + \tan t'}{\tan t \tan t'}\right|^{-1/2} \\ &\quad \cdot \chi\left(\frac{-\beta^2}{4\alpha} - \frac{x^2}{2\tan t} - \frac{y^2}{2\tan t'}\right)\,\mathrm{d}z \\ &= \lambda_p(2t+2t')\lambda_p(1/2t+1/2t')|t+t'|^{-1/2}\lambda_p(\frac{-1}{2}(\cot t + \cot t')) \\ &\quad \cdot \chi\left(\frac{-\beta^2}{4\alpha} - \frac{x^2}{2\tan t} - \frac{y^2}{2\tan t'}\right)\,\mathrm{d}z \\ &= \lambda_p(2t+2t')|t+t'|^{-1/2}\chi\left(\frac{-\beta^2}{4\alpha} - \frac{x^2}{2\tan t} - \frac{y^2}{2\tan t'}\right)\,\mathrm{d}z.\end{aligned}$$

This equals $K_{t/\omega+t'/\omega}(x,y)$ if and only if

$$\frac{-\beta^2}{4\alpha} - \frac{x^2}{2\tan t} - \frac{y^2}{2\tan t'} = \frac{xy}{\sin(t+t')} - \frac{x^2+y^2}{\tan(t+t')}, \tag{6.13}$$

where the left side is

$$\left(\frac{x}{\sin t} + \frac{y}{\sin t'}\right)^2 \left(\frac{\sin(t+t')}{\sin t \sin t'}\right)^{-1}\frac{1}{2} - \frac{x^2}{2\tan t} - \frac{y^2}{2\tan t'}$$

which equals

$$\frac{xy}{\sin(t+t')} + \frac{x^2}{2}\left(\frac{-1}{\tan t} + \frac{\sin t'}{\sin t \sin(t+t')}\right) + \frac{y^2}{2}\left(\frac{-1}{\tan t'} + \frac{\sin t}{\sin t' \sin(t+t')}\right)$$

which equals the right side of equation 6.13.

Now we want to show that

$$U_t W(q,0) = W(T_t(q,0))U_t, \tag{6.14}$$

and similarly for $(0,p)$ before we use the same argument as in the real case. This is the same as showing

$$\int K_t(x,y)\chi(-m\omega qy)\psi(y)\ \mathrm{d}y = W(q\cos\omega t, -qm\omega\sin\omega t)\int K_t(x,y)\psi(y)\ \mathrm{d}y \tag{6.15}$$

which is the same as

$$\int K_t(x,y)\chi(-m\omega qy)\psi(y)\ \mathrm{d}y = \chi(-m\omega qx\cos\omega t)\chi\left(-m\omega\frac{-q^2m\omega\cos\omega t\sin\omega t}{2}\right) \cdot\int K_t(x-qm\omega\sin\omega t,y)\psi(y)\ \mathrm{d}y$$

which is true if

$$K_t(x,y)\chi(-m\omega qy) = \chi(qx\cos\omega t)\chi\left(\frac{q^2m^2\omega^2\cos\omega t\sin\omega t}{2}\right)K_t(x-qm\omega\sin\omega t,y). \tag{6.16}$$

Looking at the definition of K_t we see that this is true if

$$\chi\left(\frac{xy}{\sin\omega t}-\frac{x^2+y^2}{2\tan\omega t}\right)\chi(-m\omega qy) = \chi(-m\omega qx\cos\omega t)\chi\left(\frac{q^2m^2\omega^2\cos\omega t\sin\omega t}{2}\right) \cdot\chi\left(\frac{(x-qm\omega\sin\omega t)y}{\sin t}-\frac{(x-qm\omega\sin\omega t)^2+y^2}{2\tan\omega t}\right)$$

true if

$$\frac{xy}{\sin\omega t}-\frac{x^2+y^2}{2\tan\omega t}-m\omega qy = -m\omega qx\cos\omega t+\frac{q^2m^2\omega^2\cos\omega t\sin\omega t}{2}-qm\omega y+\frac{xy}{\sin\omega t} -\frac{x^2+y^2}{2\tan\omega t}+xqm\omega\cos\omega t-\frac{q^2m^2\omega^2\sin\omega t\cos\omega t}{2}$$

which clearly holds.

Let us now show that

$$U_t W(0,p) = W(T_t(0,p))U_t, \tag{6.17}$$

This is the same as showing

$$\int K_t(x,y)\psi(y+p)\ \mathrm{d}y = W((m\omega)^{-1}p\sin\omega t, -p\cos\omega t)\int K_t(x,y)\psi(y)\ \mathrm{d}y \tag{6.18}$$

which follows if

$$K_t(x,y-p) = \chi(-m\omega(m\omega)^{-1}px\sin\omega t) \cdot\chi\left(-m\omega\frac{(m\omega)^{-1}p\sin\omega t\cdot(p\cos\omega t)}{2}\right)K_t(x+p\cos\omega t,y),$$

true if

$$\chi\left(\frac{x(y-p)}{\sin\omega t}-\frac{x^2+(y-p)^2}{2\tan\omega t}\right)=\chi(-px\sin\omega t)\chi\left(\frac{-p^2\sin\omega t\cdot\cos\omega t}{2}\right)\cdot\chi\left(\frac{(x+p\cos\omega t)y}{\sin t}-\frac{(x+p\cos\omega t)^2+y^2}{2\tan\omega t}\right),$$

true if (equation also multiplied with 2)

$$\frac{-2xp}{\sin\omega t}-\frac{p^2-2yp}{\tan\omega t}=-2px\sin\omega t-p^2\sin\omega t\cdot\cos\omega t+\\+\frac{2py\cos\omega t-2xp\cos^2\omega t-p^2\cos^3\omega t}{\sin\omega t},$$

which is true (for all x and y) if and only if all

$$\begin{aligned}-\frac{p^2\cos\omega t}{\sin\omega t}&=-\frac{p^2\cos^3\omega t}{\sin\omega t}-p^2\cos\omega t\sin\omega t\\ \frac{2p\cos\omega t}{\sin\omega t}&=2p\frac{\cos\omega t}{\sin\omega t}\\ -\frac{2p}{\sin\omega t}&=-\frac{2p\cos^2\omega t}{\sin\omega t}-2p\sin\omega t\end{aligned}$$

are true (the first equation is for the constant, the second for y, the third for x). Showing these equations is easy.

Now we can do exactly the same as we did in the proof of proposition 2.20 to show that

$$W(T_tz)=U(t)W(q,p)U(t)^{-1}. \tag{6.19}$$

This concludes our proof. □

Theorem 6.2. *Given W and T_t, there is only one time evolution U_t up to scalar functions $c(t):\mathbb{Q}_p\to\mathbb{T}$.*

Proof. Let U_t and V_t be two different time evolutions, i.e.

$$\begin{aligned}U_tW(z)U_{-t}&=W(T_tz)\\ V_tW(z)V_{-t}&=W(T_tz),\end{aligned}$$

so that

$$\begin{aligned}U_tW(z)U_{-t}&=V_tW(z)V_{-t}\\ W(z)&=U_{-t}V_tW(z)V_{-t}U_t\\ &=(U_{-t}V_t)W(z)(U_{-t}V_t)^{-1}.\end{aligned}$$

Since W is irreducible by theorem 4.4, the only operators commuting with W are the scalars, hence

$$\begin{aligned}(U_{-t}V_t)&=c(t)I\\ V_t&=c(t)U_t,\end{aligned}$$

where $c(t):\mathbb{Q}_p\to\mathbb{T}$. □

6.2 Ordering, p-adic time, and stroboscopic measuring

Before and after are important concepts referring to time, so we would like there to be an ordering on $p\mathbb{Z}_p$, the domain of p-adic sine and cosine. The only ordering we have seen in the literature is the one we presented in section 3.3. This is not preserved by time translation, as seen in proposition 3.19, and the following comment. We do not know how to make sense of this, perhaps we need to consider just $p\mathbb{Z}$ as the domain of sine (this is dense in $p\mathbb{Z}_p$).

Another problem we face is how to understand why we only have the domain $p\mathbb{Z}_p$ for time (assuming $\omega = 1$), and not all of $\mathbb{Q}_p$. In the article [4] by Freund and Olson they write about stroboscopic measurements, with a finite possible precision. This could lead to a physical understanding of our theory.

Chapter 7

Further ideas

7.1 How many different quantum mechanics are there?

Ordinary quantum mechanics is a triple $(\mathcal{H}, W, \mathcal{U})$, where $\mathcal{H}$ is a Hilbert space, W_{qp} the Weyl system, and U_t the time evolution. Assuming the Hilbert spaces to be separable and complex, we have an isomorphism between any two Hilbert spaces. This unitary isomorphism can carry the Weyl system and the time evolution with them (Weyl system specifies the coordinates we choose, time evolution specifies the physical system we consider). It would be interesting to study how many different descriptions we can have of the same system, and to consider how we use the underlying group structure when $\mathcal{H} = L^2(G)$.

7.2 Finding good Hamiltonians in real time

Using the ideas of chapter 5 we can look for different Hamilonians, and different physical systems to find applications. Here it would be natural to use the algorithms we have developed.

7.3 Weyl systems with real time

If we are able to find a generator for time evolution in classical physics (over p-adic numbers) parametrized by $t \in \mathbb{R}$, the we could use $U(t)$ also with real time, satisfying

$$U(t)W(z)U(t)^{-1} = W(T_t z). \tag{7.1}$$

7.4 Integration over $\mathbb{R}$ instead of over $\mathbb{Q}_p$

We can find an isomorphism by from $L^2(\mathbb{Q}_p)$ to $L^2(\mathbb{R})$ by using P_s from section 3.4. Define

$$\phi_s : L^2(\mathbb{Q}_p) \to L^2(\mathbb{R}) \tag{7.2}$$

by

$$f_\infty = \phi_s(f_p)(x) = \sqrt{p} f_p(P_s x) \tag{7.3}$$

Claim 1. ϕ_s is an isomorphism onto $L^2(\mathbb{R}^+)$.

We think this can be proved by using that P_s is injective almost everywhere, and surjective on $\mathbb{R}^+$, and proposition 3.25.

Appendix A

Notation

- H and $\widehat{H}$ are operators, $\mathcal{H}$ is Hilbert space
- $\tilde{f}$ is the Fourier transform of the function f
- $\tilde{A} = \mathcal{F}A\mathcal{F}^{-1}$ is the Fourier transform of the operator A
- Eigenstate, eigenfunction and eigenvector are all synonyms

Notation for some sets:

- $a + B = \{a + b : b \in B\}$
- $A + B = \{a + b : a \in A, b \in B\}$
- $aB = \{ab : b \in B\}$
- $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x| \leq 1\}$ is the completion of $\mathbb{Z}$ in $|-|_p$
- $B_k = \{x \in \mathbb{Q}_p : |x| \leq p^k\}$
- $A^+ = \{x \in A : x \geq 0\}$
- $\mathbb{T} = \{x \in \mathbb{C} : |x| = 1\}$
- $(\mathbb{T}, +)$ is the interval $[0, 1]$ where 0 is identified with 1, under addition modulo 1.

Physicist's notation:

- $\psi = |\psi\rangle$ is a vector, called a ket-vector, in a Hilbert space $\mathcal{H}$
- $\langle\psi|$ is a vector, called a bra-vector, in the dual of a Hilbert space
- $\langle\psi|A|\phi\rangle$ is the operator A applied to the vector $|\phi\rangle$ and then the inner product of $|\psi\rangle$ with $A(|\phi\rangle)$

- The inner product $\langle\psi|\phi\rangle$ is conjugate linear in it's first entry (in ψ)

Some functions:

- $\lfloor x \rfloor$ for $x \in \mathbb{R}$ is the floor function (greatest lower bound in $\mathbb{Z}$)
- $\lceil x \rceil$ for $x \in \mathbb{R}$ is the ceiling function (least upper bound in $\mathbb{Z}$)
- $\delta(x)$ is the Dirac delta, $\delta(0) = 1$ while all other values are 0.
- $\delta_y(x)$ is the delta distribution with mass one at y
- $\chi_p(x) = e^{2\pi i\{x\}}$
- $\gamma_a(x) = e^{2\pi i\{ax\}}$
- $\{x\} = \{x\}_p = \sum_{v(x)}^{-1} x_i p^i$ (when $x = \sum_{v(x)}^{\infty} x_i p^i$)
- $|x| = |x|_p$ is the norm of $x \in \mathbb{Q}_p$
- $|x|_\infty$ is the normal absolute value for $x \in \mathbb{Q}$
- $v(x)$ is the valuation ($v : \mathbb{Q}_p \to \mathbb{Z}$) with value given by $|x| = p^{-v(x)}$
- $f(A) = \{f(a) \in C : a \in A\}$ for $f : B \to C$ being any function and $A \subset B$

Appendix B

Figures

Here are the visualization of the Schwinger approximation on $L^2(\mathbb{R})$. The MatLab code generating these figures is in appendix C.

Figure B.1: The first 8 eigenvectors for the harmonic oscillator over $L^2(\mathbb{R})$ using Schwinger approximation with input $n = 12$. The line is the exact solution, while the diamonds represent the approximate solution. There is a small error at the endpoints.

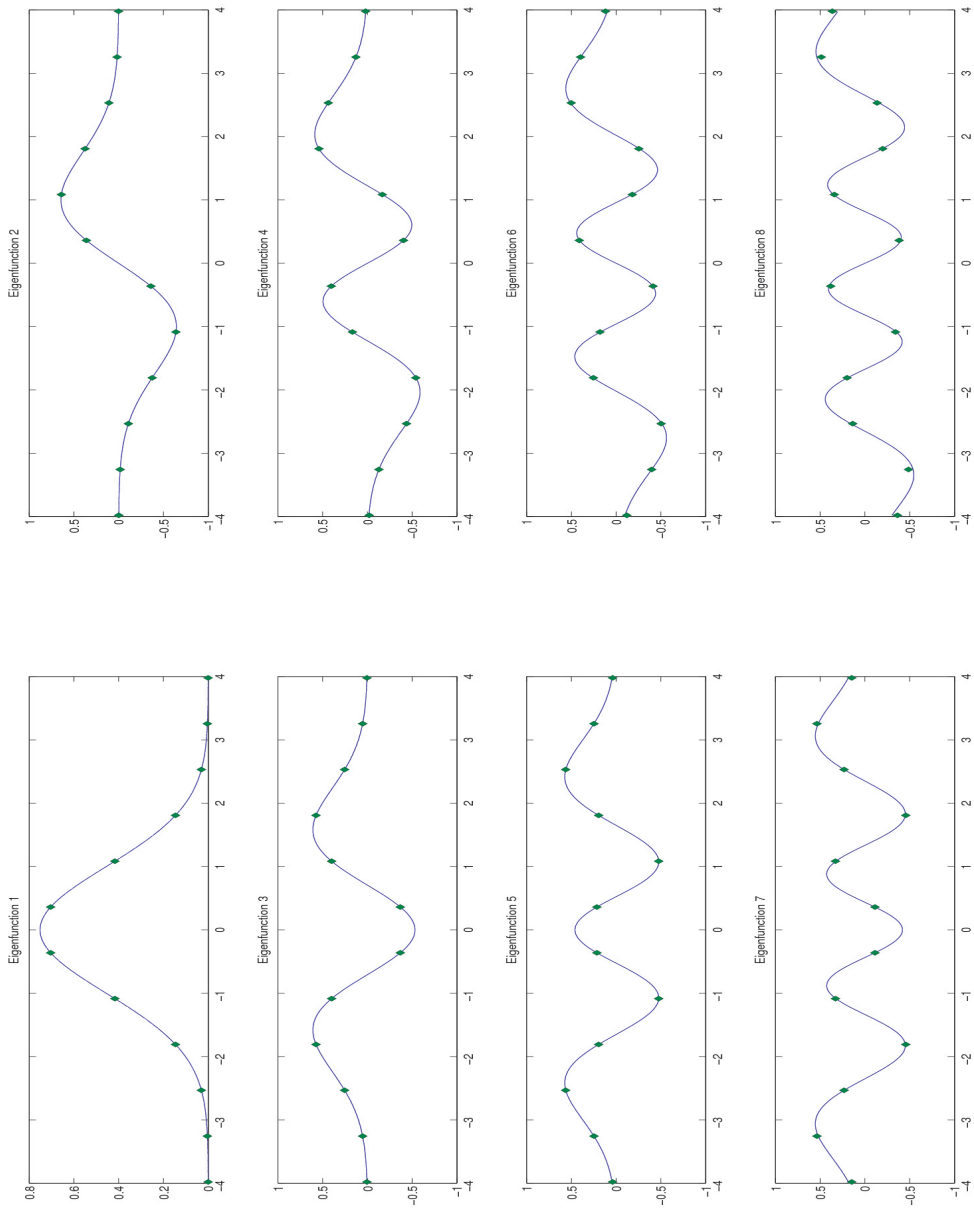
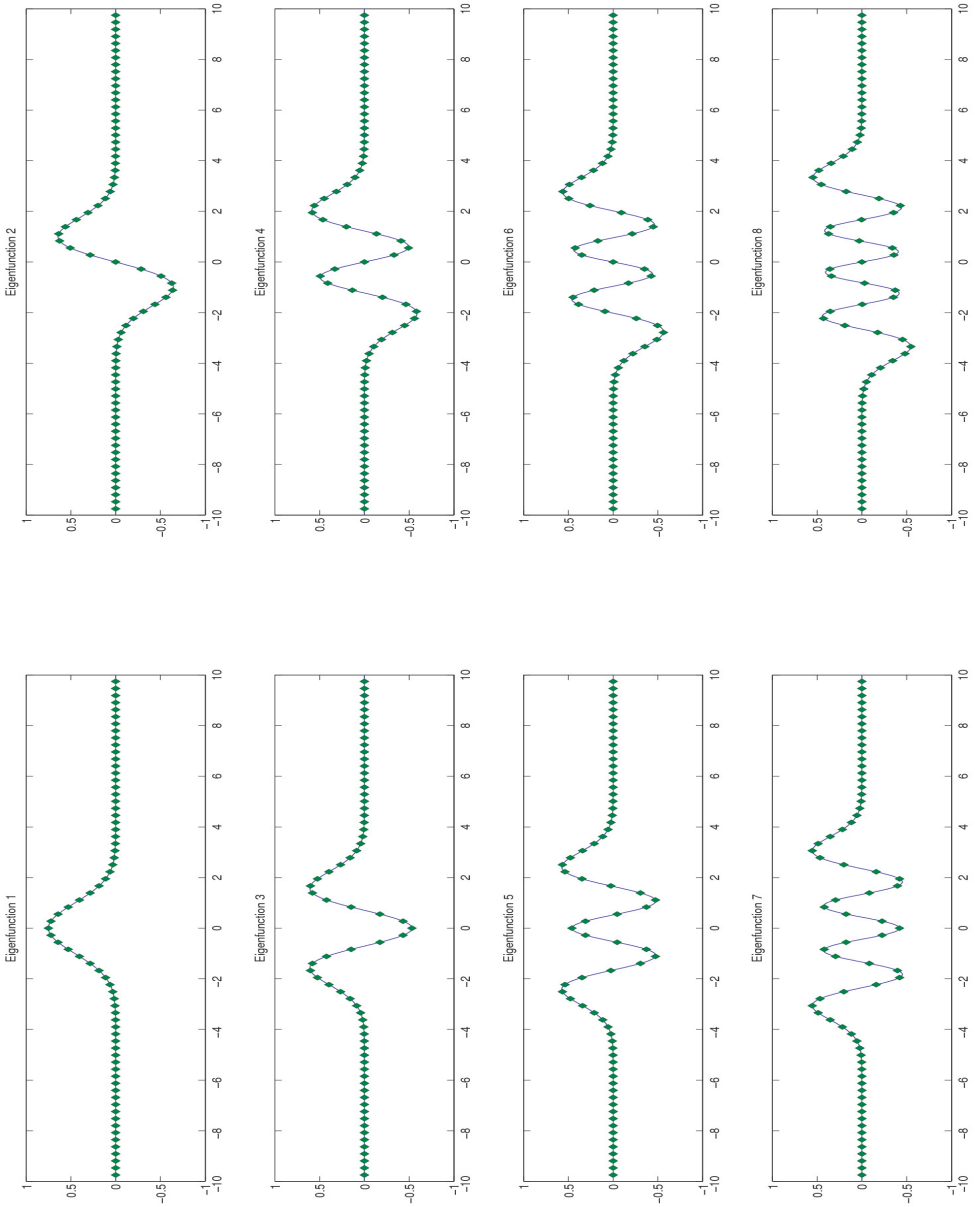


Figure B.2: The first 8 eigenvectors for the harmonic oscillator over $L^2(\mathbb{R})$ using Schwinger approximation with input $n = 81$. The line is the exact solution, while the diamonds represent the approximate solution. There is no discernible error.



Appendix C

Matlab code for Schwinger approximations

Note that you can copy-paste one block at a time from the MatLab-files attached in this appendix into MatLab.

Figure C.1: The MatLab code for the real Schwinger approximation (see chapter 2.7).

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% FiniteApproxReal.m
% Author: Haakon C. Bakka
% Date 21 Feb 2012
%
% This script calculates finite approximations using a Schwinger
% system, where the derivative is the fourier transform of the
% position operator.
%
% Requires: createHamiltonianHarmonicOscillator.m in the same folder.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Clean workspace
clc; clear; close all;

% SECTION: INPUT %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% #Points in the approximation
n = 41;
% Stepsize when plotting the analytic solution
plotStepsize = 0.01;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% SECTION: CONSTANTS %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Hermite polynomials (physicists version)
ph = {  @(x) ones(length(x),1), ...
        @(x) 2*x, ...
        @(x) 4*x.^2-2, ...
```

```
28          @(x) 8*x.^3-12*x, ...
29          @(x) 16*x.^4-48*x.^2+12, ...
30          @(x) 32*x.^5-160*x.^3+120*x, ...
31          @(x) 64*x.^6-480*x.^4 + 720*x.^2 - 120, ...
32          @(x) (128*x.^7 - 1344*x.^5 +3360*x.^3 - 1680*x) };
33  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
34
35  % SECTION: COMPUTATIONS %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
36  % Create the hamiltonian matrix H
37  [ xValues, H ] = createHamiltonianHarmonicOscillator( n, 1 );
38  % Removing errors resulting in imaginary numbers
39  H=real(H);
40  % Solving the problem
41  [eigVectors,temp] = eig(H);
42  eigValues = diag(temp);
43  % Sorting the eigenvalues (increasing) and eigenvectors (dependent)
44  [eigValuesSorted,orginalPositions] = sort(eigValues);
45  eigVectorsSorted = eigVectors(:,orginalPositions);
46  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
47
48  % SECTION: PLOT APPROXIMATION TOGETHER WITH ANALYTHIC SOLUTION %%%%%%
49  % X-coordinates for analythic solution
50  gridXValues = (xValues(1):plotStepsize:-xValues(1))';
51  % A constant used in every for-loop
52  constant = pi^(-1/4) * exp(-gridXValues.^2/2);
53  figure
54  for i = 0:7
55      subplot(4,2,i+1)
56      % Analythic solution y-values
57      y = (2^i * factorial(i))^(-.5)*constant.*ph{i+1}(gridXValues);
58      % Plot analythic solution (blue line)
59      plot(gridXValues, y, 'color', 'blue')
60      hold on
61      % Eigenvector for the eigenvalue number i (0-indexed)
62      approxY = eigVectorsSorted(:, i+1);
63      % Normalization (Imbedding into hilbert space is isometric)
64      tempMax = max(abs(approxY));
65      temp1 = find(abs(approxY)==tempMax,1);
66      temp2 = find(gridXValues >= xValues(temp1), 1);
67      tempSign = sign(approxY(temp1))*sign(y(temp2));
68      approxY = approxY*(n/2/pi)^(1/4)*tempSign;
69      % Plotting finite approximation as green diamonds
70      plot(xValues', approxY, 'color', 'g', ...
71          'LineStyle','none', 'Marker', 'd', 'MarkerSize', 7, ...
72          'MarkerFaceColor', 'g')
73      title(sprintf('Eigenfunction %d', i+1))
74  end
75  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Figure C.2: The MatLab code for creating the Hamiltonian for the harmonic oscillator used in the algorithm in figure C.1.

```
function [ xValues, H ] = createHamiltonianHarmonicOscillator( n, k )
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% createHamiltonianHarmonicOscillator.m
% Author: Haakon C. Bakka
% Date 24 Apr 2012
%
% This function constructs the Hamiltonian of the harmonic oscillator
% for the finite-dimensional approximation. It has not been optimized
% for speed (it is more than fast enough for our purposes).
%
% Input:
%   n - the number of dimensions (or #points)
%   k - a scaling constant for the potential, typically 1 or 0,
%   where 1 gives the harmonic oscillator and 0 a free particle
%
% Output:
%   xValues - the x-coord. at which we approximate the wave-functions
%   H - the Hamiltonian (nxn matrix)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Grid x and scaling constant \epsilon_N
x = (-1/2)*(n-1):(1/2)*(n-1);
epsn = sqrt(2*pi/n);

% Potential V
V = diag(x.^2*epsn^2);

% Discrete fourier transform matrix (unitary)
Fourier = exp (-1i*(x'*x)*epsn^2) / sqrt(n);

% Derivative -(d^2/dx^2)
D = Fourier' * V * Fourier;

% Hamiltonian
H = D/2 + k*V/2;

% X-coordinates (used mainly for plotting the results)
xValues = x*epsn;

end
```

Figure C.3: The MatLab code for the p-adic approximation (see chapter 3.6).

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% FiniteApproxPadic.m
% Author: Haakon C. Bakka
% Date 24 Apr 2012
%
% This script calculates finite approximations, using a Schwinger
% system, for the observable H. (H must be self-adjoint.)
%
% Requires: createPadicHamiltonianHarmOsc.m in the same folder.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Clean workspace
clc; clear; close all
% INPUT
n = 3;
p = 3;

% State problem
[ xValues, H ] = createPadicHamiltonianHarmOsc(p, n , 'norm');

% SECTION: Solve problem %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Remove imaginary errors in H
H = real(H);
% Find eigenvalues (this is by far the most time consuming operation)
ev = real(eig(H));
% Sort eigenvalues from lowest to highest
ev = sort(ev);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% SECTION: Display results %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Display the 20 first eigenvalues
if length(ev)>15
    disp ('The first 15 eigenvalues are')
    disp (ev(1:15))
else
    disp ('The first eigenvalues are')
    disp (ev)
end
% Round off errors (needed for counting)
decimals = 4;
ev = round(ev*10^decimals)*10^(-decimals);
% Counting them using histogram function on a sorted vector
[numberOfDistinct,listOfDistinct] = hist(ev,unique(ev));
display(:,1) = listOfDistinct;
display(:,2) = numberOfDistinct;
disp('The eigenvalues, and the number of occurences of each are')
disp(display)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Figure C.4: The MatLab code for creating the Hamiltonian for the harmonic oscillator used in the algorithm in figure C.3.

```
function [xValues,H]=createPadicHamiltonianHarmOsc(p,n,type)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% createHamiltonianHarmonicOscillator.m
% Author: Haakon C. Bakka
% Date 24 Apr 2012
%
% This function constructs the Hamiltonian of the harmonic oscillator
% for the finite-dimensional p-adic approximation. It has been
% optimized for speed.
%
% Input:
%   p - prime
%   n - the level of precision (p^{2n}) points)
%   k - a scaling constant for the potential, typically 1 or 0,
%   where 1 gives the harmonic oscillator and 0 a free particle
%
% Output:
%   xValues - the x-coord. at which we approximate the wave-functions
%   H - the Hamiltonian (nxn matrix)
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Length (# points) len, grid x and scale \epsilon_N
len = p^(2*n);
x = (0:len-1)';
epsn = p^(-n);

% Potential V
if strcmp(type,'norm')
    % p-adic norm of x
    xnorm = ones(p^(2*n),1);
    for k = 1:(2*n)
        xnorm(find(mod(x,p^k)==0))= p^(-k);
    end
    xepsnnorm = xnorm * p^n;
    V = spdiags(xepsnnorm.^2, 0, len,len);

elseif strcmp(type, 'rationalPart')
    % Rational part \Leftrightarrow (mod 1)
    rationalPart = mod(x*epsn, 1);
    V = spdiags(rationalPart.^2, 0, len,len);

elseif strcmp(type, 'justMultiply')
    V = spdiags(x.^2*epsn^2, 0 , len, len);

else
    disp('error, method not recognized')
    V = zeros(len);
end

% Derivative -(d^2/dx^2)
D = ifft(V*fft(eye(len)));

% Hamiltonian
H = D/2 + V/2;

% xValues (for plotting)
xValues = x*epsn;

end
```

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