NTNU – Trondheim
Norwegian University of
Science and Technology

On the Hunter-Saxton equation

Anders Samuelsen Nordli

Master of Science in Physics and Mathematics
Submission date: June 2012
Supervisor: Helge Holden, MATH

Norwegian University of Science and Technology
Department of Mathematical Sciences

Problem description

The Hunter-Saxton equation, $(u_t+uu_x)_x = \frac{1}{2}u_x^2$, has been widely studied since it was introduced by Hunter and Saxton as a model of a liquid crystals [10]. Later, a generalization known as the Hunter-Saxton system, or the two-component Hunter-Saxton, has been studied in a periodic setting [15]. The problem is to prove global existence of conservative and dissipative weak solutions of the Hunter-Saxton system on $\mathbb{R} \times [0, \infty)$.

Abstract

The Cauchy problem for a two-component Hunter-Saxton equation,

$$\begin{aligned}(u_t + uu_x)_x &= \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2,\\ \rho_t + (u\rho)_x &= 0,\end{aligned}$$

on $\mathbb{R} \times [0, \infty)$ is studied. Conservative and dissipative weak solutions are defined and shown to exist globally. This is done by explicitly solving systems of ordinary differential equation in the Lagrangian coordinates, and using these solutions to construct semigroups of conservative and dissipative solutions.

Sammendrag

Cauchyproblemet for en tokomponents Hunter-Saxton-likning

$$\begin{aligned}(u_t + uu_x)_x &= \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2,\\ \rho_t + (u\rho)_x) &= 0,\end{aligned}$$

på $\mathbb{R} \times [0, \infty)$ ble studert. Konservative og dissipative svake løsninger ble definert, og global eksistens av slike løsninger bevist. Dette ble gjort ved å eksplisitt løse systemer av ordinære differensiallikninger i Lagrangekoordinater, og disse løsningene ble brukt til å konstruere semigrupper av konservative og dissipative løsninger

Preface

This is a master's thesis in industrial mathematics written at NTNU, Trondheim. I would like to thank my supervisor Professor Helge Holden for suggesting the fascinating Hunter-Saxton equation as topic for my master's thesis, and for excellent supervision. I would also like to express my gratitude towards Espen Birger Nilsen, Katrin Grunert and Henriette Rogstad for reading the thesis, suggesting numerous improvements, and pointing out mistakes.

Contents

Chapter 1

Introduction

The Hunter-Saxton equation

$$\big(u_t(x,t) + u(x,t)u_x(x,t)\big)_x = \frac{1}{2}u_x(x,t)^2, \tag{1.1}$$

where subscript means differentiation with respect to the subscripted variable, was introduced by Hunter and Saxton [10] as a model of the dynamics of a nematic liquid crystal. Liquid crystals consist of long molecules in fluid phase, and each molecule has an orientation. The orientation is described by a unit vector n. For a nematic liquid crystal it does not matter whether one use n or $-n$. If we assume that the liquid crystal is one dimensional and that the only freedom molecules have is orientation in the plane. Then the orientation is given by $n(x,t) = \big(\cos u(x,t), \sin u(x,t)\big)$, where x is a space variable moving with some predetermined velocity, t is a slow time, and u is determined by the Hunter-Saxton equation [10]. The initial value problem

$$\begin{aligned} (u_t + uu_x)_x &= \frac{1}{2}u_x^2, \\ u|_{t=0} &= u_0, \end{aligned} \tag{1.2}$$

has been widely studied after its introduction by Hunter and Saxton. Equation (1.2) exhibits interesting properties such as wave breaking in finite time [10], it is completely integrable, bi-variational, and it has a bi-Hamiltonian structure [11]. The solution can be extended past wave breaking to a weak solution in several ways [10]. One possibility is to conserve the energy, $\int u_x^2 \, \mathrm{d}x$, which gives conservative solutions, another is to lose all energy at wave breaking, henceforth known as blow-up, and get dissipative solutions. A special class of solutions are multipeakons related to the multipeakon solutions of the Camassa-Holm equation [4]. For each t these solutions are continuous piecewise linear functions, and they can be computed exactly [12, 13]. Existence of general dissipative and conservative solutions was shown by approximation by multipeakons and passing to the limit using the theory of Young measures and Friedrich's mollifiers [17, 18].

The Hunter-Saxton equation was generalized to a two-component equation and studied by Wunsch [15] in a periodic setting. The two-component equation in a periodic setting has received some attention lately [15, 16, 19]. In this work we study the initial value problem

$$\begin{aligned} (u_t + uu_x)_x &= \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2, \\ \rho_t + (u\rho)_x &= 0, \\ u|_{t=0} &= u_0, \\ \rho|_{t=0} &= \rho_0. \end{aligned} \tag{1.3}$$

The system (1.3) arises in the study of the dynamics of non-dissipative dark matter [14].

We are going to define conservative and dissipative weak solutions of (1.3) and show global existence of such solutions in $\mathbb{R} \times [0, \infty)$. This is achieved by transforming the problem from a system of partial differential equations to a system of ordinary differential equations in t. The ordinary differential equations can be solved explicitly and the solution operators, S_t and S_t^d in the conservative and dissipative case, respectively, constitutes semigroups. Much of the work is devoted to be able to return to the original setting in such a way that the operators advancing the solutions, T_t and T_t^d, of the partial differential equations in t, constitute semigroups. To do so we find that there is some redundancy in the space of solutions of the ordinary differential equations in the sense that several solutions is transformed to one solution of the partial differential equations. Then redundant solutions are identified. In the conservative case the semigroup turn out to be continuous with respect to the natural metrics in the transformed setting. We treat multipeakon solutions as examples to get some intuition

More specifically, Chapter 2 deals with the conservative solutions. First some properties of smooth solutions are investigated, and these leads to characteristic equations. The characteristic equations are then solved, and it is shown that one can return to the original variables in a nice way, and that one get a weak solution. The chapter ends with treating multipeakons as an example and discussing the limit as $\rho_0 \to 0$.

Chapter 3 starts with an other type of multipeakons. They are weak solutions, but lose energy and are thus named dissipative multipeakons. Dissipative multipeakons satisfy a set of characteristic equations of some sort, and these equations can be solved. It is then shown, as in Chapter 2, that one can go from the original setting to characteristic equations, solve the characteristic equations, and then return to the original setting with a weak solution.

The reader is assumed to be familiar with characteristic equations for partial differential equations, Lebesgue spaces, weak derivatives, distributions, Sobolev spaces and Radon measures. The reader is directed to Evans [5] for an introduction to these concepts. There are several ways to integrate (1.3), but here we will use the skew-symmetric

$$D^{-1} = \frac{1}{2}\left(\int_{-\infty}^{x} - \int_{x}^{\infty}\right). \tag{1.4}$$

To reduce the length of equations and formulas we will omit one or several of the variables as often as possible when writing functions. We use $\mathbf{1}_E$ to denote the characteristic function on the set E.

Chapter 2

Global conservative solutions

2.1 Characteristic equations

To construct global solutions we derive characteristic equations from (1.3). First we note that any smooth solution (u, ρ) must satisfy a transport equation.

Proposition 2.1. *Let* (u, ρ) *be a smooth solution of* (1.3)*. Then*

$$(u_x^2 + \rho^2)_t + (uu_x^2 + u\rho^2)_x = 0. \tag{2.1}$$

Proof. Let (u, ρ) be classical smooth solutions of (1.3). Evaluating the derivatives involved yield

$$\begin{aligned}(u_x^2 + \rho^2)_t + (uu_x^2 + u\rho^2)_x &= 2(u_x u_{xt} + \rho\rho_t) + u_x^3 + 2uu_x u_{xx} + u_x\rho^2 + 2u\rho\rho_x \\ &= 2u_x\big((u_t + uu_x)_x - \frac{1}{2}(u_x^2 + \rho^2)\big) + 2\rho(\rho_t + (u\rho)_x) \\ &= 0,\end{aligned}$$

where (1.3) is used in order to get the last equality. □

Formally the system (1.3) can be written in complex variables as

$$\zeta_t + u\zeta_x = \frac{i}{2}\zeta^2, \tag{2.2}$$

$$
\begin{aligned}
\zeta &= \rho + iu_x, \\
\zeta|_{t=0} &= \rho_0 + iu_{0x}, \\
u|_{t=0} &= u_0,
\end{aligned}
$$

and this will be used when we solve the characteristic equations. To derive characteristic equations we define the Lagrangian coordinates.

Definition 2.2. *Let (u, ρ) be a solution of (1.3), for each $\xi \in \mathbb{R}$ define the functions q, z, v, H, η by*

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} q(\xi, t) &= u(q(\xi, t), t), \\
z(\xi, t) &= u(q(\xi, t), t), \\
v(\xi, t) &= u_x(q(\xi, t), t), \\
H(\xi, t) &= \int_{-\infty}^{q(\xi,t)} \left(u_x(y, t)^2 + \rho(y, t)^2\right) \mathrm{d}y, \\
\eta(\xi, t) &= \rho(q(\xi, t), t).
\end{aligned}
$$

We call it Lagrangian coordinates because we look at the solutions as they are seen by a particle traveling the curve q.

Proposition 2.3. *Assume that (u, ρ) is a smooth solution of (1.3) such that u and ρ are compactly supported. Then the quantities defined in Definition 2.2 are determined by a system of ordinary differential equations*

$$\dot{q} = z, \tag{2.3a}$$

$$\dot{z} = \frac{1}{2} H - \frac{1}{4} H_{tot}, \tag{2.3b}$$

$$\dot{H} = 0, \tag{2.3c}$$

$$\dot{\eta} = -v\eta, \tag{2.3d}$$

$$\dot{v} = \frac{1}{2}(\eta^2 - v^2), \tag{2.3e}$$

where $H_{tot} = \int_{-\infty}^{\infty} \left(u_x(y, t)^2 + \rho(y, t)^2\right) \mathrm{d}y$.

Proof. Use (1.3), $u_t(-\infty) = -u_t(\infty)$, and (1.4). □

Remark 2.4. *One could use some other condition on the growth at plus and minus infinity. For example* $(1-\alpha)u_t(-\infty) = -\alpha u_t(\infty)$ *leads to* $\dot{z} = \frac{1}{2}(H - \alpha H_{tot})$ *for* $0 \leq \alpha \leq 1$. *Several choices of* α *are used in the litterature, in [12, 18] the choice* $\alpha = 0$ *was used and in [2, 3, 13] there was* $\alpha = \frac{1}{2}$. *The selection of* α *corresponds to selection of antiderivative of* $u_x^2 + \rho^2$ *when integrating* (1.3)

$$D^{-1}f(x) = \frac{1}{2}\left((1-\alpha)\int_{-\infty}^{x} f(y)\,\mathrm{d}y - \alpha\int_{x}^{\infty} f(y)\,\mathrm{d}y\right).$$

We consider $\alpha = \frac{1}{2}$ *only.*

The system in Proposition 2.3 can be solved explicitly.

Proposition 2.5. *The solution of the system of differential equations* (2.3) *is*

$$q(\xi,t) = \frac{1}{4}\big(H(\xi,0) - \frac{1}{2}H_{tot}\big)t^2 + z_0(\xi)t + q_0(\xi), \tag{2.4a}$$

$$z(\xi,t) = \frac{1}{2}\big(H_0(\xi) - \frac{1}{2}H_{tot}\big)t + z_0(\xi), \tag{2.4b}$$

$$H(\xi,t) = \int_{-\infty}^{\xi} \big(\frac{z_{0\xi}(y)^2}{q_{0\xi}(y)} + \eta_0(y)^2 q_{0\xi}(y)\big)\,\mathrm{d}y, \tag{2.4c}$$

$$\eta(\xi,t) = \frac{\eta_0(\xi)}{\big(1+\frac{1}{2}v_0(\xi)t\big)^2 + \big(\frac{1}{2}\eta_0(\xi)t\big)^2}, \tag{2.4d}$$

$$v(\xi,t) = \frac{v_0(\xi) + \frac{1}{2}\big(\eta_0(\xi)^2 + v_0(\xi)^2\big)t}{\big(1+\frac{1}{2}v_0(\xi)t\big)^2 + \big(\frac{1}{2}\eta_0(\xi)t\big)^2}. \tag{2.4e}$$

Proof. The three first quantities H, z and q follow directly by integrating the equations. To derive the expressions for η and v let $\zeta = \eta + iv \in \mathbb{C}$ and observe that the characteristic equations for η and v reduces to $\dot{\zeta} = \frac{i}{2}\zeta^2$, which is separable. The transformation to complex numbers is similar to the transformation of (1.3) to (2.2). □

2.2 Continuous semigroup

We will now construct a continuous semigroup of solutions. First we note that there is some redundancy in the characteristic equations, i.e. the chain rule implies that the solution must satisfy $vq_\xi = z_\xi$. As a consequence we will henceforth say that (q, z, H, η) is a solution of the characteristic equations if $(q, z, H, \eta, \frac{z_\xi}{q_\xi})$ is. We have found the quantity $r = \eta q_\xi$ more useful than η, mainly because of the property

$$\dot{r} = 0, \tag{2.5}$$

which follows directly from the explicit solutions in Proposition 2.5. We will define the correct space for the solution in Lagrangian coordinates. To do so we will need the Banach spaces in the next definition.

Definition 2.6. *Let E_1, E_2 be the Banach spaces defined by*

$$E_1 = \{f \in L^\infty(\mathbb{R}) \mid f' \in L^2(\mathbb{R}) \text{ such that } \lim_{x\to-\infty} f(x) = 0\},$$
$$E_2 = \{f \in L^\infty(\mathbb{R}) \mid f' \in L^2(\mathbb{R})\}$$

equipped with the norm $\|f\| = \|f\|_{L^\infty(\mathbb{R})} + \|f'\|_{L^2(\mathbb{R})}$. Define $B = E_2 \times E_2 \times E_1 \times L^2(\mathbb{R})$. The notation $L^p(\mathbb{R})$ where $1 \le p \le \infty$ is used for the Lebesgue spaces on $\mathbb{R}$.

The Banach space B is far too big. We need to restrict it so that it contains the solutions, but not more.

Definition 2.7. *Let $\mathcal{F}$ consist of the elements $(\zeta, z, H, r) \in B$ such that*

(*i*) $\zeta, z, H \in W^{1,\infty}(\mathbb{R}), \zeta + \mathrm{id} = q$,

(*ii*) $q_\xi \ge 0, H_\xi \ge 0,\ q_\xi + H_\xi \ge c > 0$ *almost everywhere (a.e.),*

(*iii*) $q_\xi H_\xi = z_\xi^2 + r^2$ *a.e.*

We will frequently write $(q, z, H, r) \in \mathcal{F}$ for $(\zeta, z, H, r) \in \mathcal{F}$. Let

$$\mathcal{F}_0 = \{(q, z, H, r) \in \mathcal{F} \mid q + H = \mathrm{id}\}.$$

Here id *is the identity function and the Sobolev space*

$$W^{1,\infty}(\mathbb{R}) = \{f \in L^\infty(\mathbb{R}) \mid f' \in L^\infty(\mathbb{R})\},$$

with the norm $\|f\|_{W^{1,\infty}(\mathbb{R})} = \|f\|_{L^\infty(\mathbb{R})} + \|f'\|_{L^\infty(\mathbb{R})}$.

Note that condition (*iii*) and (*i*) implies that we have $r \in L^\infty(\mathbb{R})$ for free. The space $\mathcal{F}$ is a metric space with the metric inherited from B. The next theorem is the foundation for the rest of the thesis. It is remarkable that a seemingly difficult system of partial differential equations as (1.3) can be reduced to a relatively simple system of ordinary differential equations (2.3).

Theorem 2.8. *The solution of the system* (2.3) *in Proposition 2.3 constitutes a semigroup* S_t *in* $\mathcal{F}$ *which is countinuous with respect to the* B*-norm. Thus* $X(t) = \big(q(t), z(t), H(t), r(t)\big) = S_t(X_0)$ *denotes the solution at time* t *with initial data* X_0*.*

Proof. First we see that we do not need to require $\dot{r} = 0$ in addition to $(i)-(iii)$ in Definition 2.7, it follows from differentiating (iii) with respect to t. This implies that $r(\xi, t) = r(\xi, 0) \in L^2(\mathbb{R})$. Furthermore the explicit solutions in Proposition 2.5 are in B for each t. We need to check the semigroup property $S_t S_s = S_{t+s}$ and that the map $S_t : \mathcal{F} \to \mathcal{F}$ is continuous with respect to the B-norm. First the semigroup property

$$
\begin{aligned}
S_t S_s(r_0) =& r_0 = S_{t+s}(r_0), && (2.6)\\
S_t S_s(H_0) =& H_0 = S_{t+s}(H_0), && (2.7)\\
S_t S_s(z_0) =& \frac{1}{2}(H_0 - \frac{1}{2}H_{tot})t + z(s)\\
=& \frac{1}{2}(H_0 - \frac{1}{2}H_{tot})t + \frac{1}{2}(H_0 - \frac{1}{2}H_{tot})s + z_0\\
=& S_{t+s}(z_0), && (2.8)\\
S_t S_s(q_0) =& \frac{1}{4}(H_0 - \frac{1}{2}H_{tot})t^2 + z(s)t + q(s)\\
=& \frac{1}{4}(H_0 - \frac{1}{2}H_{tot})t^2 + \frac{1}{2}(H_0 - \frac{1}{2}H_{tot})st + z_0 t\\
&+ \frac{1}{4}(H_0 - \frac{1}{2}H_{tot})s^2 + z_0 + q_0\\
=& S_{t+s}(q_0). && (2.9)
\end{aligned}
$$

Next fix $t \in [0, \infty)$. Then

$$
\begin{aligned}
\|r(t)\|_{L^2(\mathbb{R})} &= \|r(0)\|_{L^2(\mathbb{R})},\\
\|H(t)\|_{L^\infty(\mathbb{R})} &= \|H(0)\|_{L^\infty(\mathbb{R})},\\
\|H_\xi(t)\|_{L^2(\mathbb{R})} &= \|H_\xi(0)\|_{L^2(\mathbb{R})},\\
\|z(t)\|_{L^\infty(\mathbb{R})} &\leq \frac{1}{4}t\|H(0)\|_{L^\infty(\mathbb{R})} + \|z(0)\|_{L^\infty(\mathbb{R})},\\
\|z_\xi(t)\|_{L^2(\mathbb{R})} &\leq \frac{1}{2}t\|H_\xi(0)\|_{L^2(\mathbb{R})} + \|z_\xi(0)\|_{L^2(\mathbb{R})},\\
\|q(t) - \mathrm{id}\|_{L^\infty(\mathbb{R})} &\leq \frac{1}{8}t^2\|H(0)\|_{L^\infty(\mathbb{R})} + t\|z(0)\|_{L^\infty(\mathbb{R})},
\end{aligned}
$$

$$\|q_\xi(t) - 1\|_{L^2(\mathbb{R})} \leq \frac{1}{4}t^2\|H_\xi(0)\|_{L^2(\mathbb{R})} + t\|z_\xi(0)\|_{L^2(\mathbb{R})}.$$

Summing we get that $\|S_t(X)\|_B \leq (t^2+t+1)\|X\|_B$. A similar calculation yields that $\|S_t(X)-S_t(Y)\|_B \leq (\frac{1}{2}t^2+t+1)\|X-Y\|_B$ and we have Lipschitz stability. Furthermore, $S_t(X)$ is in the space $\big(W^{1,\infty}(\mathbb{R})\big)^3 \times L^\infty(\mathbb{R})$ as $q(t), z(t), H(t), r(t)$ are linear combinations of q, z, H, r for each t. By differentiating (2.3) with respect to ξ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(q_\xi H_\xi - z_\xi^2 - r^2) = 0. \tag{2.10}$$

Hence the relation $q_\xi H_\xi = z_\xi^2 + r^2$ holds. We need to show that $q_\xi + H_\xi \geq c(t) > 0$. This is proved in the same way as in [3, Theorem 2.3]. The initial data $(q, z, H, r)|_{t=0} \in \mathcal{F}$ implies that $(q_\xi + H_\xi)|_{t=0} \geq c$. Continuity in t implies that it holds, perhaps with another $c > 0$, in a vicinity of $t = 0$. Let $[0, T)$ be the largest interval for which it holds. Then for $t \in [0, T)$ we have $q_\xi \geq 0, H_\xi \geq 0$ and

$$|z_\xi| \leq \frac{1}{2}(q_\xi + H_\xi). \tag{2.11}$$

Taking the t-derivative of $\frac{1}{q_\xi+H_\xi}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{q_\xi + H_\xi}) = -\frac{z_\xi}{(q_\xi + H_\xi)^2} \leq \frac{1}{2(q_\xi + H_\xi)}.$$

The Gronwall lemma gives that

$$\frac{1}{q_\xi + H_\xi}(\xi, t) \leq \frac{1}{q_\xi + H_\xi}(\xi, 0)e^{\frac{t}{2}},$$

and $q_\xi + H_\xi \geq c(t) > 0$ for $t \in [0, \infty)$ and $(q, z, H, r) \in \mathcal{F}$ for all t. □

Having solved the problem in Lagrangian coordinates, we want to return to the original variables. This requires q to be invertible.

Theorem 2.9. *Let $X_0 \in \mathcal{F}$ and $\big(q(t), z(t), H(t), r(t)\big) = S_t(X_0)$. Then the function q is invertible for almost every t and satisfies $q_\xi(\xi, t) > 0$ for almost every $\xi \in \mathbb{R}$.*

Proof. The proof is from [9, Lemma 2.7]. Define the set

$$\mathcal{N} = \{(\xi, t) \in \mathbb{R} \times [0, T] | q_\xi(\xi, t) = 0\}. \tag{2.12}$$

Fubini's theorem gives

$$\mathrm{m}(\mathcal{N}) = \int_{\mathbb{R}} \mathrm{m}(\mathcal{N}_{[\xi]})\,\mathrm{d}\xi = \int_{[0,T]} \mathrm{m}(\mathcal{N}^{[t]})\,\mathrm{d}t, \tag{2.13}$$

where $\mathcal{N}_{[\xi]}$ and $\mathcal{N}^{[t]}$ are the ξ-section and t-section respectively and m the Lebesgue measure. For each $n \in \mathbb{N}$ let

$$\mathcal{N}^n_{[\xi]} = \{t \in [0,T] | q_\xi(\xi,t) = 0 \text{ and } q_\xi(\xi,\tau) > 0 \text{ for all } \tau \in [t - \frac{1}{n}, t + \frac{1}{n}] - \{t\}\}.$$

For $t \in \mathcal{N}_{[\xi]}$ we have that $q_\xi(\xi,t) = 0$, $q_{\xi t} = z_\xi = 0$ by the identity $q_\xi H_\xi = z_\xi^2 + r^2$ and $q_{\xi tt} = z_{\xi t} = \frac{1}{2} H_\xi > 0$ by the definition of $\mathcal{F}$, namely $q_\xi + H_\xi > 0$. This implies that, for a small neighborhood of t with t removed, q_ξ is strictly positive. Hence $t \in \mathcal{N}^n_{[\xi]}$ for some n and $\mathcal{N}_{[\xi]} = \bigcup_{n=1}^{\infty} \mathcal{N}^n_{[\xi]}$. The sets $\mathcal{N}^n_{[\xi]}$ consist by definition of countable isolated points as the distance between two points is at least $\frac{1}{n}$. Then $\mathrm{m}(\mathcal{N}_{[\xi]}) \leq \sum_{n=1}^{\infty} \mathrm{m}(\mathcal{N}^n_{[\xi]}) = 0$. From (2.13) it follows that $\mathrm{m}(\mathcal{N}^{[t]}) = 0$ for almost every $t \in [0,T]$. Which again implies that q is strictly increasing and invertible for almost every $t > 0$. □

We want to study solutions that conserve "energy", that is, $\int (u_x^2 + \rho^2)\,\mathrm{d}x$ should be constant in t. We need to include a measure μ where information on energy density is stored across blow-up. This motivates the following definition.

Definition 2.10. *Let $\mathcal{D}$ be the set of triples (u, ρ, μ) such that*

- *(i)* $u \in E_2, \rho \in L^2(\mathbb{R}), \mu$ *is a positive finite Radon measure,*
- *(ii)* *the function* $\mu(-\infty, x) \in E_1$,
- *(iii)* $\mu_{ac} = (u_x^2 + \rho^2)\mathrm{d}x$, *where* μ_{ac} *is the absolutely continuous part of* μ.

We want to be able to construct a solution of the characteristic equations from initial values in $\mathcal{D}$. First some concepts we will need in order to define maps between $\mathcal{F}$ and $\mathcal{D}$. To construct a Radon measure from the Lagrangian coordinates we need to define the push-forward of measures by measurable functions.

Definition 2.11. *A continuous function f is said to be proper if $f^{-1}(K)$ is compact whenever K is compact. Let g be a measurable function and ν a measure. Then we define the push-forward of ν by g by, $g_\#(\nu)(A) = \nu\big(g^{-1}(A)\big)$.*

Remark 2.12. *If f is continuous and proper and μ is a Radon measure, then $f_{\#}(\mu)$ is a Radon measure. See [1, Remark 1.71].*

First we define a function mapping the initial values to Lagrangian coordinates.

Definition 2.13. *For any $(u, \rho, \mu) \in \mathcal{D}$ let*

$$q(\xi) = \sup\{x | \mu((-\infty, x)) + x < \xi\}, \tag{2.14a}$$

$$H(\xi) = \xi - q(\xi), \tag{2.14b}$$

$$z(\xi) = u \circ q(\xi), \tag{2.14c}$$

$$r(\xi) = (\rho \circ q(\xi)) q_\xi(\xi). \tag{2.14d}$$

Then $X = (q, z, H, r) \in \mathcal{F}_0$ and we denote $L : \mathcal{D} \to \mathcal{F}$ the mapping defined above.

The defintion gives a solution of the system in Proposition 2.3 for each $L(u_0, \rho_0, \mu_0) \in \mathcal{F}$. We need a way to go back to the original variables from a solution of the characteristic equations.

Definition 2.14. *We define a mapping $M : \mathcal{F} \to \mathcal{D}$ as follows. Given $X = (q, z, H, r) \in \mathcal{F}$ let $M(X) = (u, \rho, \mu)$ where*

$$u(x) = z(\xi), \tag{2.15a}$$

$$\rho(x) = \begin{cases} \frac{1}{q_\xi(\xi)} r(\xi), q_\xi(\xi) \neq 0 \\ 0, q_\xi(\xi) = 0, \end{cases} \tag{2.15b}$$

$$\mu = q_{\#}(H_\xi \mathrm{d}\xi), \tag{2.15c}$$

where $x = q(\xi)$.

It is not at all clear that the two foregoing definitions are consistent.

Proposition 2.15. *The mappings L and M are well defined.*

Proof. The proof is presented in [9, Theorem 3.8 and 3.11]. The parts on ρ and r is similar to the proof of a similar proposition in [8, Theorem 4.9 and 4.10]. The proof differs from the proofs in the references where we prove that $q_\xi H_\xi = z_\xi^2 + r^2$ holds.
Step 1, M: Let $X = (q, z, H, r) \in \mathcal{F}$ and $(u, \rho, \mu) = M(X)$. We need to show that (u, ρ, μ) is well defined and in $\mathcal{D}$. From $q - \mathrm{id} \in W^{1,\infty}(\mathbb{R})$ and

$q_\xi \geq 0$ we have that q is surjective and increasing. Thus for any x there exists ξ such that $x = q(\xi)$. As q is increasing we have that, if there are ξ_1, ξ_2 such that $x = q(\xi_1) = q(\xi_2)$, then $q_\xi(\xi) = 0$ for all $\xi \in [\xi_1, \xi_2]$. This gives that $z_\xi = 0$ in the same interval and u is well defined. To show that $u \in E_2$ note that $z \in E_2$ and that $q - \mathrm{id} \in W^{1,\infty}(\mathbb{R}), c \geq q_\xi \geq 0$ for some c. Then $\|u\|_{L^\infty(\mathbb{R})} = \|z\|_{L^\infty(\mathbb{R})}$ and $\|u_x\|^2_{L^2(\mathbb{R})} = \int\limits_{\{\xi\in\mathbb{R} | q_\xi(\xi)>0\}} z_\xi^2 q_\xi^2 \,\mathrm{d}\xi \leq c^2 \|z\|_{E_2}$. For r we have that $q_\xi(\xi) = 0$ implies that $r(\xi) = 0$. The definition of $\mathcal{F}$ states that $q_\xi H_\xi \geq r^2$, hence $\rho(x) = \frac{r}{q_\xi} \leq \sqrt{\frac{H_\xi}{q_\xi}}$. We then have that

$$\int\limits_{\mathbb{R}} \rho(x)^2 \,\mathrm{d}x = \int\limits_{\{\xi\in\mathbb{R} | q_\xi(\xi)>0\}} \rho(q(\xi))^2 q_\xi(\xi) \,\mathrm{d}\xi \leq \int\limits_{\mathbb{R}} H_\xi \,\mathrm{d}\xi < \infty,$$

and $\rho \in L^2(\mathbb{R})$. As $H_\xi \,\mathrm{d}\xi$ is a Radon measure and q is continuous and proper μ is a Radon measure, and $\mu(\mathbb{R}) = \int\limits_{q^{-1}(\mathbb{R})} H_\xi(\xi) \,\mathrm{d}\xi = H(\infty) < \infty$.

Step 2, L: To prove well definedness of L, let $(u, \rho, \mu) \in \mathcal{D}$ and define $X = (q, z, H, r) = L(u, \rho, \mu)$. First we note that q is increasing as the supremum is taken over larger and larger sets. Furthermore $\lim\limits_{\xi\to\pm\infty} q(\xi) = \pm\infty$ and for any $z > q(\xi)$ we have $\xi \leq z + \mu(-\infty, z)$. The measure is finite, $\mu_{ac} = (u_x^2 + \rho^2)\,\mathrm{d}x$ and $u_x, \rho \in L^2$ and $\xi - z \leq \mu(\mathbb{R})$. We choose z close to $q(\xi)$ and get that

$$\xi - q(\xi) \leq \mu(\mathbb{R}). \tag{2.16}$$

From the fact that $\mu(-\infty, y) \geq 0$ we deduce that $\xi \geq q(\xi)$. Thus $\mathrm{id} - q \in L^\infty(\mathbb{R})$. A function f is Lipschitz continuous if $\sup\limits_{y\neq x} \frac{|f(x)-f(y)|}{|x-y|} = C < \infty$, the value C is called the Lipschitz constant of f. We prove that q is Lipschitz continuous with Lipschitz constant less than or equal to one. Let $\xi < \xi'$, x_i' be an increasing sequence converging to $q(\xi')$ and x_i a decreasing sequence converging to $q(\xi)$. Then $\mu\big((-\infty, x_i)\big) + x_i \geq \xi$ and $\mu\big((-\infty, x_i')\big) + x_i' < \xi'$. Subtracting the inequalities gives

$$\mu\big((-\infty, x_i')\big) + x_i' - \mu\big((-\infty, x_i)\big) - x_i < \xi' - \xi. \tag{2.17}$$

For i large enough $x_i' > x_i$ and $q(\xi') - q(\xi) < \xi' - \xi$ follows by letting i go to infinity. By Rademacher's theorem, see for example [5], q is differentiable almost everywhere. We decompose μ into its absolutely continuous part, singular continuous part and singular part denoted μ_{ac}, μ_{sc} and μ_s, respectively, see for example [6]. Since $(u, \rho, \mu) \in \mathcal{D}$ we have $\mu_{ac} = (u_x^2 + \rho^2)\,\mathrm{d}x$. The support of μ_s is countable and $F(x) = \mu\big((-\infty, x)\big)$ is lower semi-continuous with its points of discontinuity being the support of μ_s, see [6]. Let $A = \big(q^{-1}(\mathrm{supp}(\mu_s))\big)^c$, i.e. A consists of the points where $F \circ q$ is continuous, then

$$\mu\big((-\infty, q(\xi))\big) + q(\xi) = \xi. \tag{2.18}$$

Indeed from the definition of $q(\xi)$ there exists an increasing sequence x_i which converges to $q(\xi)$ such that $F(x_i) + x_i < \xi$. Since F is lower semi-continuous we can put the limit inside $\lim_{i\to\infty} F(x_i) = F\big(q(\xi)\big)$ and thus $F(q(\xi)) + q(\xi) \leq \xi$. We assume that there is a ξ such that $F(q(\xi)) + q(\xi) < \xi$ and aim for a contradiction. From the definition of A we get that $q(\xi)$ is a point of continuity for F, thus there exists $x > q(\xi)$ such that $F(x) + x < \xi$, but this is a contradiction to the definition of q in Definition 2.13 and proves (2.18). We want to show that the equation $q_\xi H_\xi = z_\xi^2 + r^2$ holds almost everywhere. First we note that A is of full measure. And for ξ in A equation (2.18) holds and $H(\xi) = \xi - \big(\xi - F \circ q(\xi)\big) = F \circ q(\xi)$. We then decompose $\mu = \mu_{ac} + \nu$ where ν is singular with respect to the Lebesgue measure. Then the derivative of F exists and $\frac{\mathrm{d}}{\mathrm{d}x} F(x) = u_x^2(x) + \rho^2(x)$ almost everywhere [6, Theorem 3.22]. As q is differentiable almost everywhere we have by (2.18),

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(F \circ q(\xi) + q(\xi)\right) = q_\xi(\xi)\left(u_x^2 \circ q(\xi) + \rho^2 \circ q(\xi)\right) + q_\xi(\xi) = 1. \tag{2.19}$$

Using (2.19) and the definition of $H = \mathrm{id} - q$, z and r we get that

$$q_\xi H_\xi = \frac{1}{u_x^2 \circ q + \rho^2 \circ q + 1} \frac{u_x^2 \circ q + \rho^2 \circ q}{u_x^2 \circ q + \rho^2 \circ q + 1} = (u_x^2 \circ q + \rho^2 \circ q) q_\xi^2 = z_\xi^2 + r^2,$$

holds almost everywhere. We have to prove that $X \in B$ and $\lim_{\xi\to-\infty} H(\xi) = 0$. We have already shown that $H(\xi) = F(q(\xi))$ almost everywhere. That $H \in W^{1,\infty}(\mathbb{R})$ follows from $\|H_\xi\|_{L^\infty(\mathbb{R})} = \|1 - q_\xi\|_{L^\infty(\mathbb{R})} \leq 1 + \|q_\xi\|_{L^\infty(\mathbb{R})}$. We can find a sequence $\xi_i \in A$ such that $\lim_{i\to\infty} \xi_i = -\infty$, and we have that $\lim_{i\to\infty} H(\xi_i) = 0$. Since H is monotone this implies that $\lim_{\xi\to-\infty} H(\xi) = 0$. From (2.16) we conclude that $\|H\|_{L^\infty(\mathbb{R})} \leq \mu(\mathbb{R})$ and as $H_\xi \geq 0$,

$$\|H_\xi\|_{L^2(\mathbb{R})}^2 \leq \|H_\xi\|_{L^\infty(\mathbb{R})} \|H_\xi\|_{L^1(\mathbb{R})} \leq \|H\|_{L^\infty(\mathbb{R})}^2 \leq \mu(\mathbb{R})^2,$$

and $H \in E_1$. We have $\zeta = q - \mathrm{id} = -H \in E_2$. From Definition 2.7, (iii), we have

$$\|z_\xi\|_{L^2(\mathbb{R})}^2 \leq \|q_\xi H_\xi\|_{L^1(\mathbb{R})} \leq \|H\|_{L^\infty(\mathbb{R})},$$

$$\|r\|^2_{L^2(\mathbb{R})} \le \|q_\xi H_\xi\|_{L^1(\mathbb{R})} \le \|H\|_{L^\infty(\mathbb{R})},$$

and we are done. □

We have now transformed the problem, solved it and transformed back. This is best illustrated with an example.

Example 2.16. *Let $u_0 = 0$, $\rho_0 = \mathbf{1}_{(-1,1)}$, and $\mu_0 = \rho_0^2 \,\mathrm{d}x + \delta_0$, and let $(q_0, z_0, H_0, r_0) = M(u_0, \rho_0, \mu_0)$ and $(q, z, H, r) = S_t(q_0, z_0, H_0, r_0)$. Then*

$$r(\xi, t) = \begin{cases} 0, & \xi < -1, \\ \frac{1}{2}, & -1 < \xi < 1, \\ 0, & 1 < \xi < 2, \\ \frac{1}{2}, & 2 < \xi < 4, \\ 0, & 4 < \xi, \end{cases}$$

$$H(\xi, t) = \begin{cases} 0, & \xi < -1, \\ \frac{\xi+1}{2}, & -1 < \xi < 1, \\ \xi, & 1 < \xi < 2, \\ \frac{\xi}{2} + 1, & 2 < \xi < 4, \\ 3, & 4 < \xi, \end{cases}$$

$$z(\xi, t) = \begin{cases} -\frac{3}{4}t, & \xi < -1, \\ \frac{\xi-2}{4}t, & -1 < \xi < 1, \\ \frac{1}{2}\left(\xi - \frac{3}{2}\right)t, & 1 < \xi < 2, \\ \frac{\xi-1}{4}t, & 2 < \xi < 4, \\ \frac{3}{4}t, & 4 < \xi, \end{cases}$$

$$q(\xi, t) = \begin{cases} -\frac{3}{8}t^2 + \xi, & \xi < -1, \\ \frac{\xi-2}{8}t^2 + \frac{\xi-1}{2}, & -1 < \xi < 1, \\ \frac{1}{4}\left(\xi - \frac{3}{2}\right)t^2, & 1 < \xi < 2, \\ \frac{\xi-1}{8}t^2 + \frac{\xi}{2} - 1, & 2 < \xi < 4, \\ \frac{3}{8}t^2 + \xi - 3, & 4 < \xi. \end{cases}$$

First we remark that q is invertible for all $t > 0$. We apply L to (q, z, H, r)

when $t > 0$ and obtain

$$u(x,t) = \begin{cases} -\frac{3}{4}t, & x < -\frac{3}{8}t^2 - 1, \\ \frac{2x-1}{t^2+4}t, & -\frac{3}{8}t^2 - 1 < x < -\frac{1}{8}t^2, \\ \frac{2x}{t}, & -\frac{1}{8}t^2 < x < \frac{1}{8}t^2, \\ \frac{2x+1}{t^2+4}t, & \frac{1}{8}t^2 < x < \frac{3}{8}t^2 + 1, \\ \frac{3}{4}t, & \frac{3}{8}t^2 + 1 < x, \end{cases}$$

$$\rho(x,t) = \begin{cases} 0, & x < -\frac{3}{8}t^2 - 1, \\ \frac{1}{\frac{1}{4}t^2+1}, & -\frac{3}{8}t^2 - 1 < x < -\frac{1}{8}t^2, \\ 0, & -\frac{1}{8}t^2 < x < \frac{1}{8}t^2, \\ \frac{1}{\frac{1}{4}t^2+1}, & \frac{1}{8}t^2 < x < \frac{3}{8}t^2 + 1, \\ 0, & \frac{3}{8}t^2 + 1 < x, \end{cases}$$

$$\mu = (\rho^2 + u_x^2)\,\mathrm{d}x.$$

But we want more. We want to the solution operator to be a semigroup, and we would prefer it to be a continuous map for each t. We note that the mapping M is invariant under a certain group action on $\mathcal{F}$.

Definition 2.17. *Let G be the group of homeomorphisms $f : \mathbb{R} \to \mathbb{R}$ such that both $f - \mathrm{id} \in W^{1,\infty}(\mathbb{R})$, $f^{-1} - \mathrm{id} \in W^{1,\infty}(\mathbb{R})$, and $f_\xi - 1 \in L^2(\mathbb{R})$. Define a group action $A : \mathcal{F} \times G \to \mathcal{F}$ by $(X, f) \mapsto (q \circ f, z \circ f, H \circ f, (r \circ f) \cdot f')$. We will also need the closed subsets $G_\alpha = \{f \in G | \|f - \mathrm{id}\|_{W^{1,\infty}(\mathbb{R})} + \|f^{-1} - \mathrm{id}\|_{W^{1,\infty}(\mathbb{R})} \leq \alpha\}$.*

We prove that the group action is well defined.

Proposition 2.18. *The group action $A : \mathcal{F} \times G \to \mathcal{F}$, defined in Definition 2.17, is well defined.*

Proof. Let $f \in G$ and $X \in \mathcal{F}$. We show that $A(X, f) = (\bar{q}, \bar{z}, \bar{H}, \bar{r}) \in \mathcal{F}$. First $A(X, f) \in B$. Then

$$\bar{q} - \mathrm{id}| = |q \circ f - q + q - \mathrm{id}| \leq \|q_\xi\|_{L^\infty(\mathbb{R})} \|f - \mathrm{id}\|_{L^\infty(\mathbb{R})} + \|q - \mathrm{id}\|_{L^\infty(\mathbb{R})}, \quad (2.20)$$

and z and H satisfies $|\bar{z}| \leq \|z\|_{L^\infty(\mathbb{R})}$, and $|\bar{H}| \leq \|H\|_{L^\infty(\mathbb{R})}$. Furthermore, by substitution,

$$\int_{\mathbb{R}} |\bar{r}|^2 \,\mathrm{d}\xi \leq \|f_\xi\|_{L^\infty(\mathbb{R})} \|r\|^2_{L^2(\mathbb{R})}, \quad (2.21)$$

and by the same argument $(\bar{H})_\xi$ and $(\bar{z})_\xi$ is square integrable. The square integrability of $(\bar{q})_\xi - 1$ is proved by

$$\left(\int\limits_{\mathbb{R}} |(\bar{q})_\xi - 1|^2 \,\mathrm{d}\xi\right)^{\frac{1}{2}} \leq \|f_\xi\|_{L^\infty(\mathbb{R})}^{\frac{1}{2}} \|q_\xi - 1\|_{L^2(\mathbb{R})} + \|f_\xi - 1\|_{L^2(\mathbb{R})}. \tag{2.22}$$

As f_ξ is bounded we get that $(\bar{H})_\xi$ and $(\bar{z})_\xi$ is bounded. Boundedness of $\bar{q}_\xi$ follows from

$$|\bar{q}_\xi| = |(q \circ f - f - \mathrm{id})_\xi| \leq \|q_\xi\|_{L^\infty(\mathbb{R})} \|f_\xi\|_{L^\infty(\mathbb{R})} + \|f_\xi - 1\|_{L^\infty(\mathbb{R})}. \tag{2.23}$$

The condition $f^{-1} - \mathrm{id} \in W^{1,\infty}(\mathbb{R})$ ensures that $0 < m \leq f_\xi \leq M < \infty$ for some m and M. Thus $(\bar{q})_\xi$ and $(\bar{H})_\xi$ are non-negative and $(\bar{q})_\xi + (\bar{H})_\xi \geq cm > 0$ where c is from condition (ii) in Definition 2.7. Condition (iii) in Definition 2.7 holds by the chain rule. □

The group action induces an equivalence relation on $\mathcal{F}$, defined by $X \sim Y$ if there exists $f \in G$ such that $A(X, f) = Y$. We use this equivalence relation to define the quotient $\mathcal{F}/G$.

Proposition 2.19. *If $X \sim Y$, then $M(X) = M(Y)$.*

Proof. Let $X \sim Y$, $A(X, f) = Y$ and $M(X) = (u, \rho, \mu)$. Then $M(Y) = (\tilde{u}, \tilde{\rho}, \tilde{\mu})$ will be given by

$$\begin{aligned}
\tilde{u} &= z \circ f \circ f^{-1} \circ q^{-1} = u, \\
\tilde{\rho} &= \big(\frac{1}{(q_\xi \circ f) f'} (r \circ f) f'\big) \circ f^{-1} \circ q^{-1} = \rho, \\
\tilde{\mu}(A) &= (q \circ f)_\# \big((H \circ f)_\xi \,\mathrm{d}\xi\big)(A) = \int\limits_{f^{-1} \circ q^{-1}(A)} (H_\xi \circ f)(\xi) f'(\xi) \,\mathrm{d}\xi \\
&= \int\limits_{q^{-1}(A)} H_\xi \,\mathrm{d}\xi = \mu(A),
\end{aligned}$$

for each measurable $A \subseteq \mathbb{R}$. □

Note that $q + H \in G$ as $C > (q + H)_\xi \geq c > 0$, so we could try to calculate $Y = A(X, (q + H)^{-1})$ for $X \in \mathcal{F}$. We have that $Y \in \mathcal{F}$, furthermore $q_Y + H_Y = q \circ (q + H)^{-1} + H \circ (q + H)^{-1} = \mathrm{id}$. Thus $Y \in \mathcal{F}_0$ and we make the following definition.

Definition 2.20. *Let $\Pi : \mathcal{F} \to \mathcal{F}_0$ be given by $\Pi(X) = A(X, (q+H)^{-1})$.*

From the above we can conclude that for each $X \in \mathcal{F}$ there exists $Y \in \mathcal{F}_0$ such that $X \sim Y$. The Y is in fact unique.

Proposition 2.21. *The function $X_0 \mapsto [X_0] \in \mathcal{F}/G$ is a bijection from $\mathcal{F}_0$ to $\mathcal{F}/G$. Furthermore for $X, Y \in \mathcal{F}$ we have that $X \sim Y$ if and only if $\Pi(X) = \Pi(Y)$.*

Proof. We have already shown that the function $X_0 \mapsto [X_0] \in \mathcal{F}/G$ is onto. Let $X, Y \in \mathcal{F}_0$ be equivalent. Then $q_Y = q_X \circ f$ and $H_Y = H_X \circ f$ and thus $\mathrm{id} = q_Y + H_Y = (q_X + H_X) \circ f = \mathrm{id} \circ f = f$ and $X = Y$. Thus the function is a bijection. Let $X, Y \in \mathcal{F}$ such that $X \sim Y$. Then $Y = A(X, f)$ and $\Pi(Y) = A(A(X, f), f^{-1} \circ (q+H)^{-1}) = A(X, (q+H)^{-1}) = \Pi(X)$ as $(f^{-1})' = \frac{1}{f'}$. □

The functions M and L are bijections and inverses of each other, thus $\mathcal{D}$ and $\mathcal{F}_0$ are in bijection. The idea is to try to project the solution down to $\mathcal{F}_0$ after S_t has been used, and use M and L to get a semigroup in $\mathcal{D}$.

Proposition 2.22. *The functions L and M defined in Definition 2.13 and 2.14 respectivly, satisfy*

$$\begin{aligned} M \circ L &= \mathrm{id}_{\mathcal{D}}, \\ L \circ M &= \mathrm{id}_{\mathcal{F}_0}. \end{aligned} \tag{2.24}$$

Proof. Let $v = (u, \rho, \mu) \in \mathcal{D}$, then $L(v)$ is given by

$$\begin{aligned} q(\xi) &= \sup\{x | \mu\big((-\infty, x)\big) + x < \xi\}, \\ z(\xi) &= u \circ q(\xi), \\ H(\xi) &= \xi - q(\xi), \\ r(\xi) &= \big(\rho \circ q(\xi)\big) q_\xi(\xi) \end{aligned}$$

Well definedness of L gives that $q_\xi H_\xi = z_\xi^2 + r^2$ almost everywhere, hence $r = 0$ when $q_\xi = 0$. As q is increasing we can abuse notation and write q^{-1} when we apply M to $L(v)$. Then $M(L(v))$ is

$$\begin{aligned} \tilde{u} &= u \circ q \circ q^{-1} = u, \\ \tilde{\rho} &= \frac{1}{q_\xi} q_\xi (\rho \circ q \circ q^{-1}) = \rho, \end{aligned}$$

$$\begin{aligned}\tilde{\mu}(A) = q_{\#}(H_\xi\, \mathrm{d}\xi)(A) &= \int\limits_{q^{-1}(A)} H_\xi(\xi)\, \mathrm{d}\xi = \int\limits_{q^{-1}(A)} \big(\xi - q(\xi)\big)_\xi\, \mathrm{d}\xi \\ &= \mu(A)\end{aligned}$$

for each measurable $A \subseteq \mathbb{R}$. Thus the first equality is proved. To prove the other let $X = (q, z, H, r) \in \mathcal{F}_0$. Note that for $X \in \mathcal{F}_0$ $q_\xi = 0$ implies that both z_ξ and r equals 0. Thus, we can still abuse notation as before and $M(X)$ is given by

$$\begin{aligned} u &= z \circ q^{-1}, \\ \rho &= \frac{1}{q_\xi} r \circ q^{-1}, \\ \mu &= q_{\#}(H_\xi\, \mathrm{d}\xi), \end{aligned}$$

and $L(M(X)) = (\tilde{q}, \tilde{z}, \tilde{H}, \tilde{r})$ by

$$\begin{aligned} \tilde{q}(\xi) &= \sup\{x | q_{\#}(H_\xi\, \mathrm{d}\xi)\big((-\infty, x)\big) + x < \xi\} \\ &= \sup\{x | \int\limits_{-\infty}^{q^{-1}(x)} H_\xi(\xi)\, \mathrm{d}\xi + x < \xi\} \\ &= \sup\{x | \int\limits_{-\infty}^{q^{-1}(x)} \big(\xi - q(\xi)\big)_\xi\, \mathrm{d}\xi + x < \xi\} \\ &= q(\xi), \\ \tilde{z} &= u \circ q = z \circ q \circ \tilde{q}^{-1} = z \circ q \circ q^{-1} = z, \\ \tilde{H} &= \mathrm{id} - \tilde{q} = \mathrm{id} - q = H, \\ \tilde{r} &= \big((\frac{1}{q_\xi} r \circ q^{-1}) \circ \tilde{q}\big)\tilde{q}_\xi = r. \end{aligned}$$

And the result is proved. □

We then arrive at the following propositions which states that we can define a semigroup of solutions on $\mathcal{F}/G$, and that this semigroup has a nice representation in $\mathcal{F}_0$.

Proposition 2.23. *For any $X \in \mathcal{F}$ and $f \in G$ the mapping S_t satisfies*

$$S_t(A(X, f)) = A(S_t(X), f). \tag{2.25}$$

This implies that

$$\Pi \circ S_t \circ \Pi = \Pi \circ S_t. \tag{2.26}$$

Hence we can define a semigroup of solutions on $\mathcal{F}/G$. It corresponds to the mapping $\tilde{S}_t$ from $\mathcal{F}_0$ to $\mathcal{F}_0$ given by

$$\tilde{S}_t = \Pi \circ S_t, \tag{2.27}$$

which defines a semigroup on $\mathcal{F}_0$.

Proof. Let $X_0 = (q_0, z_0, H_0, r_0) \in \mathcal{F}$ and $f \in G$, then $A(X, f) = \big(q_0 \circ f, z_0 \circ f, H_0 \circ f, (r_0 \circ f)f'\big)$. By Proposition 2.5 we have

$$\begin{aligned} S_t\big(A(X_0, f)\big) =& S_t\big(q_0 \circ f, z_0 \circ f, H_0 \circ f, (r_0 \circ f)f'\big) \\ =& \big(\frac{1}{4}(H_0 \circ f - \frac{1}{2}H_{tot})t^2 + tz_0 \circ f + q_0 \circ f, \\ & \frac{1}{2}(H_0 \circ f - \frac{1}{2}H_{tot}t + z_0 \circ f, H_0 \circ f, (r_0 \circ f)f'\big) \\ =& A(S_t(X_0), f). \end{aligned}$$

□

The only part missing from our plan is the continuity. The problem is that Π is not continuous. This is handled by showing that if the initial data is in $\mathcal{F}_0$, then the solution at each t is not arbitrary in $\mathcal{F}$. We are then able to shrink the domain of Π somewhat, and this makes Π continuous.

Definition 2.24. *Given $\alpha \geq 0$ let $\mathcal{F}_\alpha$ be the sets*

$$\mathcal{F}_\alpha = \{X \in \mathcal{F} \mid q + H \in G_\alpha\}, \tag{2.28}$$

where G_α is defined as in Definition 2.17.

The idea of the previous definition is that we can write $\mathcal{F}$ as a union of sets that are easier to control.

Proposition 2.25. *The equality*

$$\mathcal{F} = \bigcup_{\alpha \geq 0} \mathcal{F}_\alpha \tag{2.29}$$

holds.

Proof. As $q_\xi + H_\xi > 0$, $H_\xi \in L^\infty(\mathbb{R})$ and $q_\xi - 1 \in L^\infty(\mathbb{R})$ we have that $q + H \in G$. And every $g \in G$ is in G_α for some $\alpha \geq 0$ by the definition of G. □

We will need three lemmas whose proofs are taken from [9, Lemma 3.2, 3.3, 3.5].

Lemma 2.26. *Let $\alpha \geq 0$. If $f \in G_\alpha$, then $\frac{1}{1+\alpha} \leq f_\xi \leq 1+\alpha$ almost everywhere. Conversely, if f is absolutely continuous, $f - \mathrm{id} \in L^\infty(\mathbb{R})$ and there exists $c > 1$ such that $\frac{1}{c} \leq f_\xi \leq c$ almost everywhere, then $f \in G_\alpha$ for some α depending only on c and $\|f - \mathrm{id}\|_{L^\infty(\mathbb{R})}$.*

Proof. The proof is taken from [9, Lemma 3.2]. Given $f \in G_\alpha$ we have that f is Lipschitz continuous and hence differentiable almost everywhere, and we have the basic formula $f_\xi^{-1}\big(f(\xi)\big) = \frac{1}{f_\xi(\xi)}$. This implies that $f_\xi(\xi) \geq \frac{1}{\|(f^{-1})_\xi\|_{L^\infty(\mathbb{R})}} \geq \frac{1}{1+\alpha}$ where the last inequality follows from the triangle inequality and definition of G_α. Note that $f_\xi \geq 0$ as it is a homeomorphism and close to the identity map. This holds almost everywhere as f^{-1} is one-to-one and Lipschitz continuous. The other inequality, $f_\xi \leq 1 + \|f_\xi - 1\|_{L^\infty(\mathbb{R})} \leq 1 + \alpha$, is proved by the triangle inequality and the definition of G_α. Assume now that f is absolutely continuous, $f - \mathrm{id} \in L^\infty(\mathbb{R})$ and $\frac{1}{c} \leq f_\xi \leq c$ almost everywhere for some $c \geq 1$. Since f_ξ is bounded, f and $f - \mathrm{id}$ are Lipschitz and $f - \mathrm{id} \in W^{1,\infty}(\mathbb{R})$. Moreover $f_\xi \geq \frac{1}{c} > 0$, so f is strictly increasing and continuous ad thus invertible and we have

$$\begin{aligned} f^{-1}(\xi_2) - f^{-1}(\xi_1) &= \int_{\xi_1}^{\xi_2} (f^{-1})_\xi(\xi)\,\mathrm{d}\xi \\ &= \int_{f^{-1}(\xi_1)}^{f^{-1}(\xi_2)} (f^{-1})_x(f(x))\,\mathrm{d}f(x) \\ &= \int_{f^{-1}(\xi_1)}^{f^{-1}(\xi_2)} \frac{f_x(x)}{f_x(x)}\,\mathrm{d}x \\ &\leq \int_{f^{-1}(\xi_1)}^{f^{-1}(\xi_2)} c f_x(x)\,\mathrm{d}x \\ &= c(\xi_2 - \xi_1). \end{aligned} \tag{2.30}$$

Hence f^{-1} is Lipschitz and $(f^{-1})_\xi \le c$. We have $f^{-1}(\xi') - \xi' = \xi - f(\xi)$ for $\xi' = f(\xi)$ and $\|f - \mathrm{id}\|_{L^\infty(\mathbb{R})} = \|f^{-1} - \mathrm{id}\|_{L^\infty(\mathbb{R})}$ which implies that $f \in G_\alpha$ for some $\alpha \ge 0$. □

The next lemma shows that S_t plays nicely with $q + H$, that is, the norms $\|q + H - \mathrm{id}\|_{L^\infty(\mathbb{R})}$ and $\|(q+H)^{-1} - \mathrm{id}\|_{L^\infty(\mathbb{R})}$, do not grow uncontrollably as t progresses.

Lemma 2.27. *Given $\alpha, T \ge 0$ and $X \in \mathcal{F}_\alpha$ we have $S_t(X) \in \mathcal{F}_\beta$ for all $t \in [0,T]$ where β depends on T, α and $\|X\|_B$.*

Proof. The proof is taken from [9, Lemma 3.3]. Let $X = (q, z, H, r) \in \mathcal{F}_\alpha$ and $X(t) = \big(q(t), z(t), H(t), r(t)\big) = S_t(X)$. By definition we have that $q(\xi, 0) + H(\xi, 0) \in G_\alpha$ and thus by Lemma 2.26, $\frac{1}{c} \le q_\xi(\xi, 0) + H_\xi(\xi, 0) \le c$ almost everywhere for some c which depends on α only. Consider a fixed ξ and apply the Gronwall lemma backward in time to the three first equations of (2.3) differentiated with respect to ξ to obtain

$$|q_\xi(0)| + |z_\xi(0)| + |H_\xi(0)| \le e^{CT}\big(|q_\xi(t)| + |z_\xi(t)| + |H_\xi(t)|\big), \tag{2.31}$$

for some constant C which depends on $\|X(t)\|_{C([0,T],B)}$, which in turn depends on $\|X\|_B$ and T only. From Definition 2.7 $q_\xi(t), H_\xi(t) \ge 0$. We use (2.11) to reduce (2.31) to

$$\frac{1}{c} \le q_\xi(0) + H_\xi(0) \le \frac{3}{2} e^{CT}\big(q_\xi(t) + H_\xi(t)\big),$$

and $q_\xi(t) + H_\xi(t) \ge \frac{2}{3c} e^{-CT}$. By applying Gronwall's lemma forward in time we obtain $q_\xi(t) + H_\xi(t) \le \frac{3c}{2} e^{CT}$. Hence by Lemma 2.26 we have that $q + H \in G_\beta$ where β only depends on T, α and $\|X\|_B$ and thus $S_t(X) \in \mathcal{F}_\beta$. □

We arrive at the lemma on continuity of Π.

Lemma 2.28. *The restriction of Π to $\mathcal{F}_\alpha$ is continuous.*

Proof. The proof without the r part is taken from [9, Lemma 3.5]. We have made a slight change as $z_n, z \in H^1(\mathbb{R})$ in [9], while here $z, z_n \in E_2$. Let $X_n = (q_n, z_n, H_n, r_n) \in \mathcal{F}_\alpha$ converge to $X = (q, z, H, r) \in \mathcal{F}_\alpha$ in the topology induced by $\|-\|_B$. We denote $\bar{X} = (\bar{q}, \bar{z}, \bar{H}, \bar{r}) = \Pi(X)$ and similarly $\bar{X}_n = \Pi(X_n)$. First we prove that $\bar{H}_n$ tends to $\bar{H}$ in $L^\infty(\mathbb{R})$. Let $f_n = q_n + H_n$, $f = q + H$ and we have by construction $f, f_n \in G_\alpha$. Thus $\bar{H}_n - \bar{H} = (H_n - H) \circ f_n^{-1} + H \circ f_n^{-1} - H \circ f^{-1}$ and we have

$$\|\bar{H}_n - \bar{H}\|_{L^\infty(\mathbb{R})} \le \|H - H_n\|_{L^\infty(\mathbb{R})} + \|\bar{H} \circ f - \bar{H} \circ f_n\|_{L^\infty(\mathbb{R})}. \tag{2.32}$$

From the definition of $\mathcal{F}_0$ we have that $\bar{H}$ is Lipschitz with Lipschitz constant less than or equal to one. The definition of the B-norm gives that f_n and H_n converges to f and H respectively in $L^\infty(\mathbb{R})$. By (2.32) we get that $\bar{H}_n \to \bar{H}$ in $L^\infty(\mathbb{R})$. Let us now prove that $(\bar{H}_n)_\xi$ converges to $(\bar{H})_\xi$ in $L^2(\mathbb{R})$. We have by the chain rule $(\bar{H}_n)_\xi - \bar{H}_\xi = \frac{(H_n)_\xi}{(f_n)_\xi} \circ f_n^{-1} - \frac{H_\xi}{f_\xi} \circ f^{-1}$ which can be written as

$$(\bar{H}_n)_\xi - \bar{H}_\xi = \frac{(H_n)_\xi - H_\xi}{(f_n)_\xi} \circ f_n^{-1} + \frac{H_\xi}{(f_n)_\xi} \circ f_n^{-1} - \frac{H_\xi}{f_\xi} \circ f^{-1}. \tag{2.33}$$

Since $f_n \in G_\alpha$, there exists by Lemma 2.26 a constant $c > 1$ such that $\frac{1}{c} \leq (f_n)_\xi \leq c$ almost everywhere for all n. We have

$$\left\| \frac{(H_n)_\xi - H_\xi}{(f_n)_\xi} \circ f_n^{-1} \right\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \big((H_n)_\xi - H_\xi\big)^2 \frac{1}{(f_n)_\xi} \, d\xi \leq c \|(H_n)_\xi - H_\xi\|^2_{L^2(\mathbb{R})}, \tag{2.34}$$

where we made the change of variables $\xi' = f_n^{-1}(\xi)$. We proceed to the next term in (2.33),

$$\frac{H_\xi}{(f_n)_\xi} \circ f_n^{-1} = (\bar{H}_\xi \circ g_n) \cdot (g_n)_\xi,$$

where $g_n = f \circ f_n^{-1}$. We will prove that $\lim_{n\to\infty} \|(g_n)_\xi - 1\|_{L^2(\mathbb{R})} = 0$. After change of variables we get

$$\|(g_n)_\xi - 1\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \left(\frac{f_\xi}{(f_n)_\xi} \circ f_n^{-1} - 1 \right)^2 d\xi \leq c \|f_\xi - (f_n)_\xi\|^2_{L^2(\mathbb{R})},$$

which proves that the limit as $n \to \infty$ is zero. We have

$$\begin{aligned} \|(\bar{H}_\xi \circ g_n)(g_n)_\xi - \bar{H}_\xi\|_{L^2(\mathbb{R})} \leq \|\bar{H}_\xi \circ g_n\|_{L^\infty(\mathbb{R})} \|(g_n)_\xi - 1\|_{L^2(\mathbb{R})} \\ + \|\bar{H}_\xi \circ g_n - \bar{H}_\xi\|_{L^2(\mathbb{R})}, \end{aligned} \tag{2.35}$$

where $\|\bar{H}_\xi \circ g_n\|_{L^\infty(\mathbb{R})} < 1$ from the Lipschitz property of $\bar{H}$. If we can control the second term we have shown that $(\bar{H}_n)_\xi$ converges to $\bar{H}_\xi$. Let $h \in C_0^\infty(\mathbb{R})$ such that $\|h - \bar{H}_\xi\|_{L^2(\mathbb{R})} < \frac{\epsilon}{3}$ and observe that $\frac{1}{c} \leq (g_n)_\xi \leq c$ almost everywhere. Then one can prove that $\|\bar{H}_\xi \circ g_n - h \circ g_n\|_{L^2(\mathbb{R})} \leq c \|\bar{H}_\xi - h\|_{L^2(\mathbb{R})}$. Thus $f_n \to f$ in $L^\infty(\mathbb{R})$ implies that $g_n \to \mathrm{id}$ in $L^\infty(\mathbb{R})$ and there exists a compact K independent of n such that $\mathrm{supp}(h \circ g_n) \subseteq K$. Then by the Lebesgue dominated convergence theorem we obtain $h \circ g_n \to h$ in $L^2(\mathbb{R})$. Summarizing, this together with (2.33), (2.34) and (2.35), gives that $(\bar{H}_n)_\xi \to \bar{H}_\xi$ in $L^2(\mathbb{R})$. It follows that $\bar{q_n}_\xi - 1 \to \bar{q}_\xi - 1$ in $L^2(\mathbb{R})$ and, similarly, one proves that $\lim_{n\to\infty} \|(\bar{z}_n)_\xi - \bar{z}_\xi\|_{L^2(\mathbb{R})}$ and $\lim_{n\to\infty} \|(\bar{r}_n) - \bar{r}\|_{L^2(\mathbb{R})}$. It remains to prove that $\bar{z}_n \to \bar{z}$ in $L^\infty(\mathbb{R})$. We write

$$\bar{z}_n - \bar{z} = (z_n - z) \circ f_n^{-1} + z \circ f_n^{-1} - z \circ f^{-1}, \tag{2.36}$$

and need to prove that $\|z \circ f_n^{-1} - z \circ f^{-1}\|_{L^\infty(\mathbb{R})} \to 0$. But as z by construction lies in $W^{1,\infty}(\mathbb{R})$ and $f_n \to f$ in $L^\infty(\mathbb{R})$ we have that $|z \circ f_n^{-1} - z \circ f^{-1}| \leq \|z_\xi\|_{L^\infty(\mathbb{R})} \|f_n^{-1} - f^{-1}\|_{L^\infty(\mathbb{R})}$. We use the invertibility and Lipschitz property of f^{-1}, f_n^{-1} as stated in equation (2.30) to get

$$\begin{aligned}
|f_n^{-1}(\xi) - f^{-1}(\xi)| &= |x - f^{-1}\big(f_n(x)\big)| \\
&= |f^{-1}\big(f(x)\big) - f^{-1}\big(f_n(x)\big)| \\
&\leq C\|f - f_n\|_{L^\infty(\mathbb{R})}.
\end{aligned}$$

□

The above lemmas imply continuity of $\tilde{S}_t$. The next theorem is the last result in the section on the solution in Lagrangian coordinates.

Theorem 2.29. *The mapping $\tilde{S}_t$ is continuous.*

Proof. The semigroup $\tilde{S}_t = \Pi \circ S_t$, where S_t is continuous. By Lemma 2.27 we have that for $X \in \mathcal{F}_0$ $S_t(X) \in \mathcal{F}_\alpha$, and by Lemma 2.28 the map $\Pi : \mathcal{F}_\alpha \to \mathcal{F}_0$ is continuous and hence the composition is continuous. □

2.3 Global existence of conservative solutions

We begin by defining conservative weak solutions.

Definition 2.30. *Assume that $u : \mathbb{R} \times [0,\infty) \to \mathbb{R}$, $\rho : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ and $\mu : [0,\infty) \to \mathcal{M}^+(\mathbb{R})$ satisfy*
(i) $u \in C\big([0,\infty), E_2\big)$, $\rho \in C\big([0,\infty), L^2(\mathbb{R})\big)$, $\mu(t)$ finite,
(ii) the equations

$$\iint\limits_{\mathbb{R}\times[0,\infty)} \big(u\phi_{tx} + \frac{1}{2}u^2\phi_{xx} - \frac{1}{2}u_x^2\phi - \frac{1}{2}\rho^2\phi\big)\,\mathrm{d}x\mathrm{d}t = -\int\limits_{\mathbb{R}} (u\phi_x)|_{t=0}\,\mathrm{d}x, \tag{2.37}$$

and

$$\iint\limits_{\mathbb{R}\times[0,\infty)} \big(\rho\phi_t + (u\rho)\phi_x\big)\,\mathrm{d}x\mathrm{d}t = -\int\limits_{\mathbb{R}} (\rho\phi)|_{t=0}\,\mathrm{d}x, \tag{2.38}$$

for all $\phi \in C_0^\infty\big(\mathbb{R}\times[0,\infty)\big)$*. Then* (u,ρ) *is a weak solution of the two-component Hunter-Saxton equation* (1.3)*. If* (u,ρ,μ) *in addition satisfies*

$$\iint\limits_{\mathbb{R}\times[0,\infty)} \big((u_x^2+\rho^2)\phi_t + (uu_x^2 + u\rho^2)\phi_x\big)\,\mathrm{d}x\mathrm{d}t = \int\limits_{\mathbb{R}} \phi|_{t=0}\,\mathrm{d}\mu(0), \tag{2.39}$$

we say that (u,ρ,μ) *is a conservative weak solution.*

Define T_t as

$$T_t = M\circ \tilde{S}_t \circ L,$$

and the metric $d_{\mathcal{D}}$ as

$$d_{\mathcal{D}}\big((u_1,\rho_1,\mu_1),(u_2,\rho_2,\mu_2)\big) = d_{\mathcal{F}_0}\big(L(u_1,\rho_1,\mu_1), L(u_2,\rho_2,\mu_2)\big).$$

The following existence theorem is the main result in this section.

Theorem 2.31. *The mapping* T_t *is a continuous semigroup of solutions with respect to the metric* $d_{\mathcal{D}}$*. For any initial data* $(u_0,\rho_0,\mu_0)\in\mathcal{D}$ *let the solution be denoted* $\big(u(t),\rho(t),\mu(t)\big) = T_t(u_0,\rho_0,\mu_0)$*. Then* (u,ρ,μ) *is a conservative weak solution of* (1.3) *in the sense of Definition 2.30. For almost all* t*,* $\mu = (u_x^2+\rho^2)\,\mathrm{d}x$.

Proof. The proof is similar to the proof of Theorem 5.2 in [7]. We prove that T_t is a semigroup,

$$T_tT_s = M\tilde{S}_tLM\tilde{S}_sL = M\tilde{S}_t\tilde{S}_sL = T_{t+s},$$

where it is used that $\tilde{S}_t$ is a semigroup in $\mathcal{F}_0$. That T_t is continuous with respect to the $d_{\mathcal{D}}$ metric is a direct result of the facts that $\tilde{S}_t$ is continuous with respect to the $d_{\mathcal{F}_0}$ metric and that $L\circ M = \mathrm{id}$. Let $(u_0,\rho_0,\mu_0)\in\mathcal{D}$, $(u(t),\rho(t),\mu(t)) = T_t(u_0,\rho,\mu_0)$, and $\phi\in C_0^\infty\big(\mathbb{R}\times[0,\infty)\big)$. To be able to evaluate the integrals in Definition 2.30 we do the change of variables $x = q(\xi)$ where q is determined by the mapping L defined in Definition 2.13. Then we get

$$\iint\limits_{\mathbb{R}\times[0,\infty)} \big(u\phi_{tx} + \frac{1}{2}u^2\phi_{xx} - \frac{1}{2}u_x^2\phi - \frac{1}{2}\rho^2\phi\big)\,\mathrm{d}x\mathrm{d}t$$

$$
\begin{aligned}
&= \iint\limits_{\mathbb{R}\times[0,\infty)} \big(z(\phi_{tx}\circ q) + \frac{1}{2}z^2(\phi_{xx}\circ q) - \frac{1}{2}\frac{z_\xi^2+r^2}{q_\xi^2}(\phi\circ q)\big)q_\xi \,\mathrm{d}\xi\mathrm{d}t \\
&= \iint\limits_{\mathbb{R}\times[0,\infty)} z\frac{\mathrm{d}}{\mathrm{d}t}(\phi_x\circ q)q_\xi \,\mathrm{d}\xi\mathrm{d}t - \frac{1}{2}\iint\limits_{\mathbb{R}\times[0,\infty)} zq_t(\phi_{xx}\circ q)q_\xi\,\mathrm{d}\xi\mathrm{d}t \\
&\quad - \frac{1}{2}\iint\limits_{\mathbb{R}\times[0,\infty)} H_\xi(\phi\circ q)\,\mathrm{d}\xi\mathrm{d}t \\
&= -\int\limits_{\mathbb{R}} \big(zq_\xi(\phi_x\circ q)|_{t=0}\big)\,\mathrm{d}\xi - \iint\limits_{\mathbb{R}\times[0,\infty)} \frac{\mathrm{d}}{\mathrm{d}t}(zq_\xi)(\phi_x\circ q)\,\mathrm{d}\xi\mathrm{d}t \\
&\quad - \frac{1}{2}\iint\limits_{\mathbb{R}\times[0,\infty)} zq_t(\phi_{xx}\circ q)q_\xi\,\mathrm{d}\xi\mathrm{d}t - \frac{1}{2}\iint\limits_{\mathbb{R}\times[0,\infty)} H_\xi(\phi\circ q)\,\mathrm{d}\xi\mathrm{d}t \\
&= -\int\limits_{\mathbb{R}} \big(u\phi_x|_{t=0}\big)\,\mathrm{d}x - \iint\limits_{\mathbb{R}\times[0,\infty)} z_tq_\xi(\phi_x\circ q)\,\mathrm{d}\xi\mathrm{d}t - \frac{1}{2}\iint\limits_{\mathbb{R}\times[0,\infty)} (z^2)_\xi(\phi_x\circ q)\,\mathrm{d}\xi\mathrm{d}t \\
&\quad - \frac{1}{2}\iint\limits_{\mathbb{R}\times[0,\infty)} z^2(\phi_{xx}\circ q)q_\xi\,\mathrm{d}\xi\mathrm{d}t - \frac{1}{2}\iint\limits_{\mathbb{R}\times[0,\infty)} H_\xi(\phi\circ q)\,\mathrm{d}\xi\mathrm{d}t \\
&= -\int\limits_{\mathbb{R}} \big(u\phi_x|_{t=0}\big)\,\mathrm{d}x + \iint\limits_{\mathbb{R}\times[0,\infty)} z_{t\xi}(\phi\circ q)\,\mathrm{d}\xi\mathrm{d}t - \frac{1}{2}\iint\limits_{\mathbb{R}\times[0,\infty)} (z^2)_\xi(\phi_x\circ q)\,\mathrm{d}\xi\mathrm{d}t \\
&\quad + \frac{1}{2}\iint\limits_{\mathbb{R}\times[0,\infty)} (z^2)_\xi(\phi_x\circ q)\,\mathrm{d}\xi\mathrm{d}t - \frac{1}{2}\iint\limits_{\mathbb{R}\times[0,\infty)} H_\xi(\phi\circ q)\,\mathrm{d}\xi\mathrm{d}t \\
&= -\int\limits_{\mathbb{R}} \big(u\phi_x|_{t=0}\big)\,\mathrm{d}x,
\end{aligned} \tag{2.40}
$$

where we have used the characteristic equations (2.3) extensively. The calculation for ρ is quite similar

$$
\begin{aligned}
&\iint\limits_{\mathbb{R}\times[0,\infty)} \big(\rho\phi_t + \rho u\phi_x\big)\,\mathrm{d}x\mathrm{d}t \\
&= \iint\limits_{\mathbb{R}\times[0,\infty)} \big(\frac{r}{q_\xi}(\phi_t\circ q) + \frac{r}{q_\xi}z(\phi_x\circ q)\big)q_\xi\,\mathrm{d}\xi\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&= \iint\limits_{\mathbb{R}\times[0,\infty)} r\frac{\mathrm{d}}{\mathrm{d}t}(\phi\circ q)\,\mathrm{d}\xi\mathrm{d}t \\
&= \iint\limits_{\mathbb{R}\times[0,\infty)} \frac{\mathrm{d}}{\mathrm{d}t}\big(r(\phi\circ q)\big)\,\mathrm{d}\xi\mathrm{d}t \\
&= -\int\limits_{\mathbb{R}} r(\phi\circ q)|_{t=0}\,\mathrm{d}\xi \\
&= -\int\limits_{\mathbb{R}} (\rho\phi)|_{t=0}\,\mathrm{d}x. \qquad (2.41)
\end{aligned}
$$

Where we have used that $\dot{r} = 0$. We prove that the weak solution we have constructed is conservative. The calculation is essentially the same as the two calculations above. The details are

$$
\begin{aligned}
&\iint\limits_{\mathbb{R}\times[0,\infty)} \big((u_x^2+\rho^2)\phi_t + u(u_x^2+\rho^2)\phi_x\big)\,\mathrm{d}x\mathrm{d}t \\
&= \iint\limits_{\mathbb{R}\times[0,\infty)} \big(H_\xi(\phi_t\circ q) + zH_\xi(\phi_x\circ q)\big)\,\mathrm{d}\xi\mathrm{d}t \\
&= \iint\limits_{\mathbb{R}\times[0,\infty)} H_\xi\frac{\mathrm{d}}{\mathrm{d}t}(\phi\circ q)\,\mathrm{d}\xi\mathrm{d}t \\
&= \iint\limits_{\mathbb{R}\times[0,\infty)} \frac{\mathrm{d}}{\mathrm{d}t}\big(H_\xi(\phi\circ q)\big)\,\mathrm{d}\xi\mathrm{d}t \\
&= \int\limits_{\mathbb{R}} \phi|_{t=0}\,\mathrm{d}\mu(0). \qquad (2.42)
\end{aligned}
$$

We have from L and M that $\mu = (u_x^2 + \rho^2)\,\mathrm{d}x$ whenever q is invertible. By Theorem 2.9, q is invertible for almost every t. □

We end this section with some examples. First general multipeakons, then a couple of concrete examples of multipeakons.

Example 2.32. *Let $\{x_i\}_{i=1}^n$ be a strictly increasing sequence in $\mathbb{R}$ and the initial* *data be given by*

$$u_0(x) = \begin{cases} c_0, & x < x_1, \\ p_i(x - x_i) + c_i, & x_i \le x < x_{i+1}, \\ c_n, & x_n < x, \end{cases}$$

$$\rho_0(x) = \begin{cases} 0, & x < x_1, \\ \rho_i, & x_i \le x < x_{i+1}, \\ 0, & x_n < x, \end{cases}$$

$$\mu_0 = (u_{0x}^2 + \rho_0^2)\,\mathrm{d}x,$$

where the p_i*'s and* x_i*'s are arbitrary and the* c_i*'s are chosen such that* u_0 *is continuous. Then* $T_t(u_0, \rho_0, \mu_0)$ *is given by the formulas*

$$u(x,t) = \begin{cases} -\frac{1}{4}\mu_0(\mathbb{R})t + c_0, & x < x_1(t), \\ \frac{p_i + \frac{1}{2}(p_i^2+\rho_i^2)t}{(1+\frac{1}{2}p_it)^2+(\frac{1}{2}\rho_it)^2}\big(x - x_i(t)\big) \\ \qquad + \frac{1}{2}\big(\mu_0\big((-\infty, x_i]\big) - \frac{1}{2}\mu_0(\mathbb{R})\big)t + c_i & x_i(t) \le x < x_{i+1}(t), \\ \frac{1}{4}\mu_0(\mathbb{R})t + c_n, & x_n(t) < x, \end{cases}$$

$$\rho(x,t) = \begin{cases} 0, & x < x_1(t), \\ \frac{\rho_i}{(1+\frac{1}{2}p_it)^2+(\frac{1}{2}\rho_it)^2}, & x_i(t) \le x < x_{i+1}(t), \\ 0, & x_n(t) < x, \end{cases}$$

$$\mu(t) = (u_x^2 + \rho^2)\,\mathrm{d}x,$$

where

$$x_i(t) = \frac{1}{4}\big(\mu_0\big((-\infty, x_i]\big) - \frac{1}{2}\mu_0(\mathbb{R})\big)t^2 + c_it + x_i,$$

or equivalently,

$$x_1(t) = -\frac{1}{8}\mu_0(\mathbb{R})t^2 + c_0t + x_1,$$

$$x_{i+1}(t) = x_i(t) + (x_{i+1} - x_i)\big((1 + \frac{1}{2}p_it)^2 + (\frac{1}{2}\rho_it)^2\big).$$

A solution of this form is called a conservative multipeakon solution of (1.3).

First an example where ρ_0 differs from zero when u_0 does.

Example 2.33. *If we let*

$$u_0(x) = \begin{cases} 0, & x \le -1, \\ x+1, & -1 < x < 0, \\ 1-x, & 0 < x < 1, \\ 0, & 1 \le x, \end{cases}$$

$$\rho_0(x) = \begin{cases} 0, & x \le -1, \\ 1, & -1 < x < 1, \\ 0, & 1 \le x, \end{cases}$$

then the conservative multipeakon solution is given by

$$u(x,t) = \begin{cases} -t, x < x_1(t), \\ \frac{1+t}{(1+\frac{1}{2}t)^2+(\frac{1}{2}t)^2}\big(x - x_1(t)\big) - t, x_1(t) \le x < x_2(t), \\ \frac{-1+t}{(1-\frac{1}{2}t)^2+(\frac{1}{2}t)^2}\big(x - x_2(t)\big) + 1, x_2(t) \le x < x_3(t), \\ t, x_3(t) \le x, \end{cases}$$

$$\rho(x,t) = \begin{cases} 0, x < x_1(t), \\ \frac{1}{(1+\frac{1}{2}t)^2+(\frac{1}{2}t)^2}, x_1(t) < x < x_2(t), \\ \frac{1}{(1-\frac{1}{2}t)^2+(\frac{1}{2}t)^2}, x_2(t) < x < x_3(t), \\ 0, x_3(t) < x, \end{cases}$$

where

$$x_1(t) = -\frac{1}{2}t^2 - 1,$$
$$x_2(t) = t,$$
$$x_3(t) = \frac{1}{2}t^2 + 1,$$

as shown in Figure 2.1.

We compute an example when $\rho_0 = 0$. This to highlight the differences between conservative and dissipative solutions.

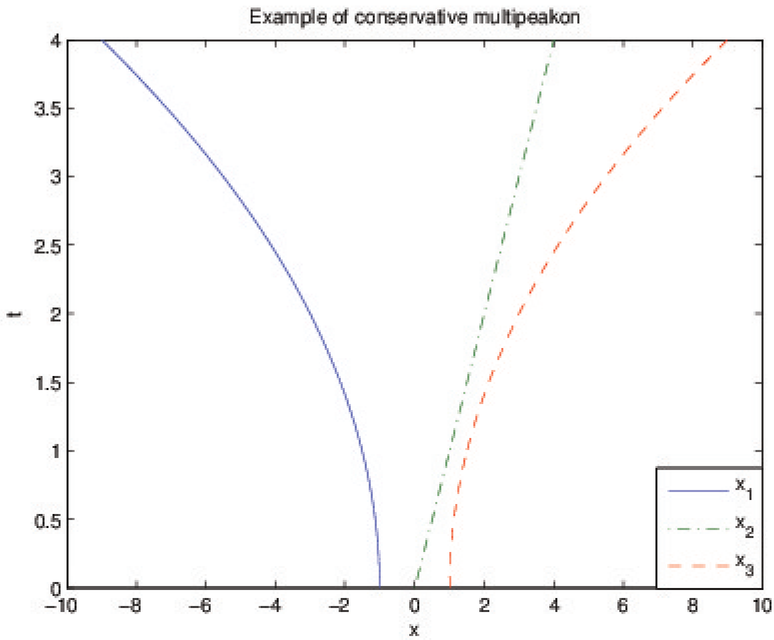

Figure 2.1: Plot of $x_i, i = 1, 2, 3$ in Example 2.33

Example 2.34. *If we let*

$$u_0(x) = \begin{cases} 0, & x \leq -1, \\ x+1, & -1 < x < 0, \\ 1-x, & 0 < x < 1, \\ 0, & 1 \leq x, \end{cases}$$
$$\rho_0(x) = 0$$

then the conservative multipeakon solution is given by

$$u(x,t) = \begin{cases} -t, x < x_1(t), \\ \frac{1}{1+\frac{1}{2}t}\big(x - x_1(t)\big) - t, x_1(t) \leq x < x_2(t), \\ \frac{1}{1-\frac{1}{2}t}\big(x - x_2(t)\big) + 1, x_2(t) \leq x < x_3(t), \\ t, x_3(t) \leq x, \end{cases}$$
$$\rho(x,t) = 0$$

where

$$x_1(t) = -\frac{1}{4}t^2 - 1,$$
$$x_2(t) = t,$$
$$x_3(t) = \frac{1}{4}t^2 + 1,$$

as shown in Figure 2.2.

2.4 The solution when ρ_0 vanishes

If we let $\rho = 0$ in (1.3) we are left with the standard Hunter-Saxton equation. It is interesting to see what happens when we let $\rho_0 \to 0$ in some sense. One would expect that one recovers the conservative solutions of the Hunter-Saxton equation and thus has another way to define these solutions. This would create a nice symmetry between conservative and dissipative solutions as the latter can be viewed as vanishing viscosity solutions [13]. First we prove that $\rho_0 = 0$ implies $\rho = 0$, and that the two-component equation is a generalization of the Hunter-Saxton equation (1.2).

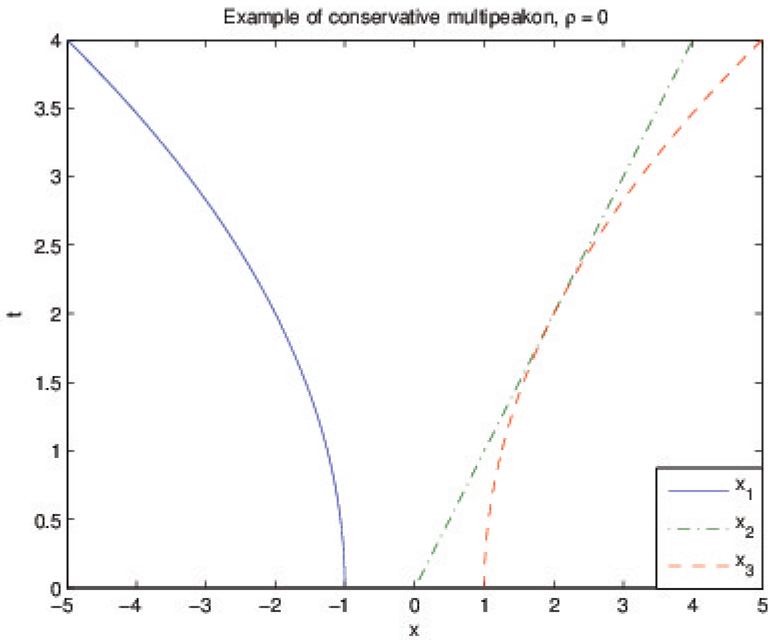

Figure 2.2: Plot of $x_i, i = 1, 2, 3$ in Example 2.34

Lemma 2.35. *If $\rho_0 = 0$ then $(u, \rho, \mu) = T_t(u_0, \rho_0, \mu_0)$ satisfies $\rho = 0$ for all t.*

Proof. If $\rho_0 = 0$ then $r(0) = (\rho_0 \circ q(0))q_\xi(0) = 0$ and as $r_t = 0$ we have that $r(t) = 0$ for all t. This implies, by transforming back, that $\rho = 0$ for almost every t. □

The next proposition states that the Hunter-Saxton system (1.3) indeed is a generalization of the Hunter-Saxton equation (1.2).

Proposition 2.36. *The solution generated by T_t and the initial data $(u_0, 0, \mu_0)$ is a conservative solution of the Hunter-Saxton equation* (1.2).

Proof. This follows from the proof of Theorem 2.31 and the previous lemma. Insert $\rho = 0$ everywhere and the definition of conservative weak solutions of the Hunter-Saxton equation is satisfied. Note that the definition of conservative weak solutions of the Hunter-Saxton equation is the same as Definition 2.30 with $\rho = 0$. This coincides with the definition of conservative weak solution of the Hunter-Saxton equation in [3]. □

We have that ρ does not explode on us. That is, if $\rho_0 \to 0$, then $\rho \to 0$ for each t as the next lemma states.

Lemma 2.37. *Let $(u_0, 0, \mu_0) \in \mathcal{D}$ and $\rho_0^n \to 0$ in $L^2(\mathbb{R})$. Then $\big(u_0, \rho_0^n, \mu_0 + (\rho_0^n)^2\,\mathrm{d}x\big) \to (u_0, 0, \mu_0)$ in $\mathcal{D}$ with respect to the $d_{\mathcal{D}}$ metric.*

Proof. We have to calculate $\|L(u_0, 0, \mu_0) - L(u_0, \rho_0^n, \mu_0 + (\rho_0^n)^2\,\mathrm{d}x)\|_B$. Let $(q, z, H, 0) = L(u_0, 0, \mu_0) \in \mathcal{F}_0$ and $(q_n, z_n, H_n, r_n) = L(u_0, \rho_0^n, \mu_0 + (\rho_0^n)^2\,\mathrm{d}x) \in \mathcal{F}_0$. Note that $\mu\left((-\infty, x)\right) \leq \mu\left((-\infty, x)\right) + \int_{-\infty}^{x} (\rho_0^n)^2 \mathrm{d}y$. By taking the difference we obtain

$$\begin{aligned}
0 \leq q(\xi) - q_n(\xi) &= \sup\{x | \mu_0\big((-\infty, x)\big) + x < \xi\} \\
&\quad - \sup\{x | \mu_0\big((-\infty, x)\big) + \int_{-\infty}^{x} (\rho_0^n)^2(y)\,\mathrm{d}y + x < \xi\} \\
&\leq \sup\{x | \mu_0\big((-\infty, x)\big) + x < \xi\} \\
&\quad - \sup\{x | \mu_0\big((-\infty, x)\big) + \int_{-\infty}^{\infty} (\rho_0^n)^2(y)\,\mathrm{d}y + x < \xi\} \\
&= q(\xi) - q(\xi - \|\rho_0^n\|_2^2), \qquad (2.43)
\end{aligned}$$

and hence $q_n - \mathrm{id} \to q - \mathrm{id}$ in E_2 as the asymptotics are converging and translation is continuous in $H^1(\mathbb{R})$. Moreover the Lipschitz property of q, (2.17), gives that $\|q - q_n\|_\infty \leq \|\rho_0^n\|_2^2$. For z_n, H_n and r_n the following calculation shows

$$\begin{aligned} z(\xi) - z_n(\xi) &= u_0 \circ q(\xi) - u_0 \circ q_n(\xi), \\ H(\xi) - H_n(\xi) &= \big(\xi - q(\xi)\big) - \big(\xi - q_n(\xi)\big) = q_n(\xi) - q(\xi), \\ r_n(\xi) &= \rho_0^n \circ q_n(\xi) \cdot (q_n)_\xi(\xi). \end{aligned} \tag{2.44}$$

That translation by $f_n(\xi) \in L^\infty(\mathbb{R}), f_n \to 0$ is continuous in $L^2(\mathbb{R})$ has the same proof as when f_n is constant (by considering g in C_0^∞ together with a density argument). That $r_n \to 0$ follows from $(q_n)_\xi \leq 1$ which is a result of $(q_n)_\xi + (H_n)_\xi = 1$. □

Finally, we have the theorem on convergence as $\rho_0 \to 0$.

Theorem 2.38. *Let $u_0 \in E_2$ and $(\rho_n)_0 \in L^2(\mathbb{R})$ such that $(\rho_n)_0 \to 0$ in $L^2(\mathbb{R})$. Denote by (u_n, ρ_n, μ_n) the conservative weak solution of* (1.3) *with initial data $\big(u_0, (\rho_n)_0, (\mu_n)_0\big)$, where $(\mu_n)_0 = \mu_0 + (\rho_n)_0^2\,\mathrm{d}x$. Then $u_n \to u$ for some u in E_2 and $\mu_n \to \mu$ for some $\mu \in \mathcal{M}^+(\mathbb{R})$ for each t and (u, μ) is a conservative weak solution of* (1.2) *with initial data (u_0, μ_0). The convergence is in the metric $d_{\mathcal{D}}$.*

Proof. The result follows from Lemma 2.37 and the continuous semigroup property in Theorem 2.31. □

Chapter 3

Global dissipative solutions

Motivated by the existence of conservative and dissipative weak solutions of (1.2) [12], we try to find similar solutions of the system (1.3).

3.1 Dissipative multipeakons

We will first define a class of weak solutions, dissipative multipeakon solutions, and use these as motivating examples and illustrations. Conservative multipeakons are defined in Example 2.32. Note that there will be blow up if and only if $\rho_i = 0, p_i < 0$, so we would expect that if this condition is not met dissipative and conservative solutions will coincide. We try to define dissipative solutions by removing the part that blew up.

Definition 3.1. *Let $\{x_i\}_{i=1}^n$ be a strictly increasing sequence in $\mathbb{R}$ and the initial data be given by*

$$u_0(x) = \begin{cases} c_0, & x < x_1, \\ p_i(x - x_i) + c_i, & x_i \leq x < x_{i+1}, \\ c_n, & x_n < x, \end{cases}$$

$$\rho_0(x) = \begin{cases} 0, & x < x_1, \\ \rho_i, & x_i \leq x < x_{i+1}, \\ 0, & x_n < x, \end{cases}$$

where the p_i's and x_i's can be chosen freely and c_i's are chosen such that u_0 is continuous. Then the dissipative multipeakon solution is defined by the formulas

$$u(x,t) = \begin{cases} -\frac{1}{4}H(t)t + \frac{1}{2}\sum\limits_{i=1}^{n} p_i(x_{i+1}-x_i)\mathbf{1}_i^B(t) + c_0, & x \le x_1(t), \\ \frac{p_i + \frac{1}{2}(p_i^2+\rho_i^2)t}{(1+\frac{1}{2}p_it)^2+(\frac{1}{2}\rho_i t)^2}\big(x - x_i(t)\big)\mathbf{1}_i^A(t) & \\ \qquad +u\big(x_i(t),t\big), & x_i(t) \le x \le x_{i+1}(t), \\ \frac{1}{4}H(t)t + \frac{1}{2}\sum\limits_{i=1}^{n} p_i(x_{i+1}-x_i)\mathbf{1}_i^B + c_n, & x_n(t) \le x, \end{cases}$$

$$\rho(x,t) = \begin{cases} 0, & x < x_1(t), \\ \frac{\rho_i}{(1+\frac{1}{2}p_it)^2+(\frac{1}{2}\rho_i t)^2}, & x_i(t) \le x < x_{i+1}(t), \\ 0, & x_n(t) < x, \end{cases}$$

$$H(t) = \sum_{i=1}^{n-1} \left(p_i^2+\rho_i^2\right)\left(x_{i+1}-x_i\right)\mathbf{1}_i^A(t)),$$

where

$$\begin{aligned} x_1(t) =& x_1 + \frac{1}{2}\sum_{i=1}^{n}(x_{i+1}-x_i)\mathbf{1}_i^B(t) + c_0 t + \frac{1}{2}\sum_{i=1}^{n} p_i(x_{i+1}-x_i)\mathbf{1}_i^B(t)t \\ & - \frac{1}{8}\sum_{i=1}^{n}\left(p_i^2+\rho_i^2\right)(x_{i+1}-x_i)\mathbf{1}_i^A(t), \\ x_{i+1}(t) =& x_i(t) + (x_{i+1}-x_i)\big((1+\frac{1}{2}p_it)^2 + (\frac{1}{2}\rho_i t)^2\big)\mathbf{1}_i^A(t), \\ \mathbf{1}_i^A(t) =& \mathbf{1}_{\{1+\frac{1}{2}p_it\ge 0\}\cup\{\rho_i\ne 0\}}(t), \\ \mathbf{1}_i^B(t) =& 1 - \mathbf{1}_i^A(t). \end{aligned}$$

Where $\mathbf{1}_E$ denotes the characteristic function on the set E.

We do two examples of dissipative multipeakons. The initial data is the same as in Example 2.33 and Example 2.34, respectively.

Example 3.2. *If we let*

$$u_0(x) = \begin{cases} 0, & x \le -1, \\ x+1, & -1 < x < 0, \\ 1-x, & 0 < x < 1, \\ 0, & 1 \le x, \end{cases}$$

$$\rho_0(x) = \begin{cases} 0, & x \leq -1, \\ 1, & -1 < x < 1, \\ 0, & 1 \leq x, \end{cases}$$

then the dissipative multipeakon solution is identical to the conservative multipeakon solution with the same initial data in Example 2.33.

The next example is more interesting.

Example 3.3. *The initial data*

$$u_0(x) = \begin{cases} 0, & x \leq -1, \\ x+1, & -1 < x < 0, \\ 1-x, & 0 < x < 1, \\ 0, & 1 \leq x, \end{cases}$$
$$\rho_0(x) = 0,$$

gives the dissipative multipeakon solution

$$u(x,t) = \begin{cases} -\frac{1}{2}t\mathbf{1}_{\{t\leq 2\}} - \left(\frac{1}{4}t+\frac{1}{2}\right)\mathbf{1}_{\{t>2\}}, & x \leq x_1(t) \\ \frac{1}{1+\frac{1}{2}t}\big(x-x_1(t)\big) - \frac{1}{2}t\mathbf{1}_{\{t\leq 2\}} - \left(\frac{1}{4}t+\frac{1}{2}\right)\mathbf{1}_{\{t>2\}}, & x_1(t) \leq x \leq x_2(t), \\ \frac{1}{1-\frac{1}{2}t}\big(x-x_2(t)\big)\mathbf{1}_{\{t\leq 2\}} + u(x_2(t),t), & x_2(t) \leq x \leq x_3(t), \\ \frac{1}{2}t\mathbf{1}_{\{t\leq 2\}} + \left(\frac{1}{4}t+\frac{1}{2}\right)\mathbf{1}_{\{t>2\}}, & x_3(t) \leq x. \end{cases}$$

$$\begin{aligned}
\rho(x,t) =&0, \\
H(t) =& \begin{cases} 2, & t \leq 2, \\ 1, & t > 2, \end{cases} \\
x_1(t) =& x_1 + \frac{1}{2}\mathbf{1}_{\{t>2\}} + c_0 t - \frac{1}{2}\mathbf{1}_{\{t>2\}} - \frac{1}{8}H(t)t^2, \\
x_2(t) =& x_1(t) + \left(1+\frac{1}{2}t\right)^2, \\
x_3(t) =& x_2(t) + \left(1-\frac{1}{2}t\right)^2 \mathbf{1}_{\{t\leq 2\}}.
\end{aligned}$$

It is different from the conservative multipeakon with the same initial data in Example 2.34. The plot of x_i's in Figure 3.1 illustrate this fact.

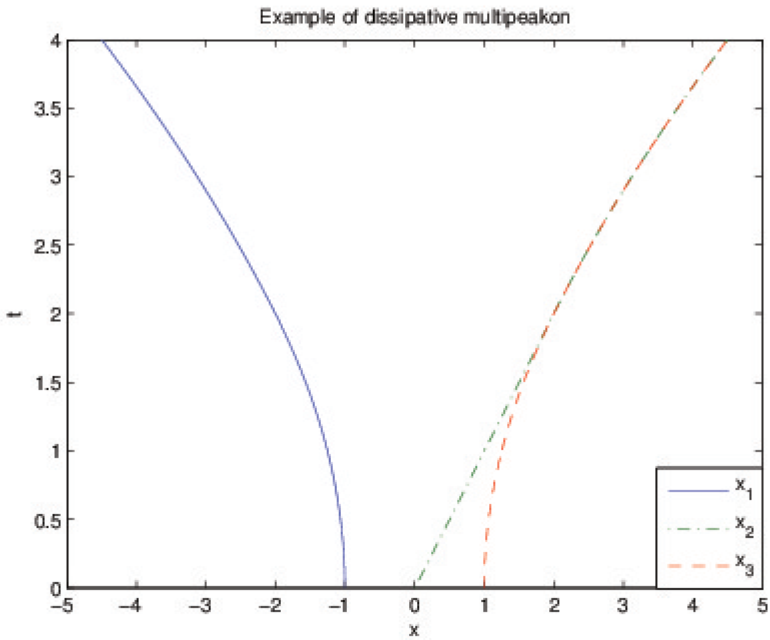

Figure 3.1: Plot of $x_i, i = 1, 2, 3$ in Example 3.3

The definition of dissipative multipeakon solutions is not very useful unless such solutions are weak solutions of (1.3) in the sense of Defintion 2.30. But this is indeed the case, as shown in the next lemma.

Lemma 3.4. *The dissipative multipeakon solutions defined in Definition 3.1 are weak solutions of* (1.3) *in the sense of Definition 2.30.*

Proof. We calculate the required integrals.

$$\begin{aligned}
&\iint\limits_{\mathbb{R}\times[0,\infty)} \big(u\phi_{tx} + \frac{1}{2}u^2\phi_{xx} - \frac{1}{2}u_x^2\phi - \frac{1}{2}\rho^2\phi\big)\,\mathrm{d}x\mathrm{d}t \\
&= \int\limits_0^\infty \bigg(\int_{-\infty}^{x_1} \big(u(x_1)\phi_{xt} + \frac{1}{2}u(x_1)^2\phi_{xx}\big)\,\mathrm{d}x \\
&\quad + \int_{x_n}^{\infty} \big(u(x_n)\phi_{xt} + \frac{1}{2}u(x_n)^2\phi_{xx}\big)\,\mathrm{d}x \\
&\quad + \sum_{i=1}^{n-1} \int\limits_{x_i}^{x_{i+1}} \big(u(x)\phi_{xt} + \frac{1}{2}u(x)^2\phi_{xx} - \frac{1}{2}(u_x^2+\rho^2)\phi\big)\,\mathrm{d}x \bigg)\,\mathrm{d}t \\
&= \int\limits_0^\infty \bigg[\sum_{i=1}^{n-1} \int\limits_{x_i}^{x_{i+1}} \bigg(-\frac{p_i + \frac{1}{2}(p_i^2+\rho_i^2)t}{(1+\frac{1}{2}p_it)^2 + (\frac{1}{2}\rho_it)^2}\phi_t + \bigg(\frac{p_i + \frac{1}{2}(p_i^2+\rho_i^2)t}{(1+\frac{1}{2}p_it)^2 + (\frac{1}{2}\rho_it)^2}\bigg)^2 \phi \\
&\quad - \frac{1}{2}\bigg(\frac{\rho_i^2 + p_i^2}{(1+\frac{1}{2}p_it)^2 + (\frac{1}{2}\rho_it)^2}\bigg)\phi\bigg)\,\mathrm{d}x \\
&\quad - \frac{p_i + \frac{1}{2}(p_i^2+\rho_i^2)t}{(1+\frac{1}{2}p_it)^2 + (\frac{1}{2}\rho_it)^2}\big(\dot{x}_{i+1}\phi(x_{i+1}) - \dot{x}_i\phi(x_i)\big)\bigg]\,\mathrm{d}t \\
&= \int\limits_0^\infty \frac{\partial}{\partial t} \int\limits_{-\infty}^{\infty} u\phi_x\,\mathrm{d}x\mathrm{d}t = -\int\limits_{-\infty}^{\infty} (u\phi_x)|_{t=0}\,\mathrm{d}x.
\end{aligned} \tag{3.1}$$

The identity $u(x_i) = \dot{x}_i$, which can be proved by induction on i, has been used. The second integral reads

$$\iint\limits_{\mathbb{R}\times[0,\infty)} \big(\rho\phi_t + (u\rho)\phi_x\big)\,\mathrm{d}x\mathrm{d}t$$

$$
\begin{aligned}
&= \int_0^\infty \sum_{i=1}^{n-1} \int_{x_i}^{x_{i+1}} \Big(\frac{\rho_i}{(1+\frac{1}{2}p_i t)^2 + (\frac{1}{2}\rho_i t)^2} \phi_t + \frac{\rho_i}{(1+\frac{1}{2}p_i t)^2 + (\frac{1}{2}\rho_i t)^2} u(x)\phi_x \Big) \, \mathrm{d}x\mathrm{d}t \\
&= \int_0^\infty \sum_{i=1}^{n-1} \frac{\rho_i}{(1+\frac{1}{2}p_i t)^2 + (\frac{1}{2}\rho_i t)^2} \bigg(\int_{x_i}^{x_{i+1}} (\phi_t - \frac{p_i + \frac{1}{2}(p_i^2+\rho_i^2)t}{(1+\frac{1}{2}p_i t)^2 + (\frac{1}{2}\rho_i t)^2} \phi) \, \mathrm{d}x \\
&\quad + u(x_{i+1})\phi(x_{i+1}) - u(x_i)\phi(x_i) \bigg) \, \mathrm{d}t \\
&= \int_0^\infty \frac{\partial}{\partial t} \int_{-\infty}^\infty \rho\phi \, \mathrm{d}x\mathrm{d}t = -\int_{-\infty}^\infty (\rho\phi)|_{t=0} \, \mathrm{d}x. \qquad (3.2)
\end{aligned}
$$

□

A number of interesting properties of the dissipative multipeakon solutions are collected in the following proposition.

Proposition 3.5. *The quantities*

$$
\begin{aligned}
q(\xi,t) &= x_i(t) + (\xi - x_i)\big((1+\frac{1}{2}u_{0x}(\xi)t)^2 + (\frac{1}{2}\rho_0(\xi)t)^2\big)\mathbf{1}_{\{1+\frac{1}{2}u_{0x}t\geq 0\}\cup\{\rho_0\neq 0\}}, \\
z(\xi,t) &= u\big(q(\xi,t),t\big), \\
H(\xi,t) &= \int_{-\infty}^{q(\xi,t)} \big(u_x(x,t)^2 + \rho(x,t)^2\big) \, \mathrm{d}x, \\
r(\xi,t) &= \rho\big(q(\xi,t),t\big)q_\xi(\xi,t),
\end{aligned}
$$

satisfies

$$
\begin{aligned}
\dot{q} &= z, && (3.3a) \\
\dot{H}_\xi &= -H_\xi(\xi,0)\delta_{\{1+\frac{1}{2}u_{0x}(\xi)t=0,\rho_0(\xi)=0\}}(t), && (3.3b) \\
\dot{z} &= \frac{1}{2}H - \frac{1}{4}H_{tot}, && (3.3c) \\
\dot{r} &= 0. && (3.3d)
\end{aligned}
$$

We will now generalize the concept of dissipative solutions by letting (3.3) generate the flow in a space $\mathcal{F}^d$ similar to $\mathcal{F}$ defined in Definition 2.7. Our aim is to construct a semigroup of dissipative solutions similar to what we did for conservative solutions. This semigroup will unfortunately not be continuous in the $d_{\mathcal{D}}$-metric. We shall see that this is related to the fact that if $\rho_0 \neq 0, \mu_0 = (u_x^2 + \rho^2)\,\mathrm{d}x$ then the conservative and dissipative solutions are identical.

3.2 Characteristic system in the dissipative case

First we define a "characteristic system" which will generate the flow in $\mathcal{F}^d$. Let

$$\dot{q} = z, \tag{3.4a}$$

$$\dot{z} = \frac{1}{2}H - \frac{1}{4}H_{tot}, \tag{3.4b}$$

$$\dot{H}_\xi = -H_{0\xi}\delta_{\{1+\frac{1}{2}\frac{z_{0\xi}}{q_{0\xi}}t=0, r_0=0\}}(t), \tag{3.4c}$$

$$\dot{r} = 0, \tag{3.4d}$$

where δ_a is the Dirac measure, $\delta_a(A) = 1$ if $a \in A$, $\delta_a(A) = 0$ otherwise. The system (3.4) can be solved explicitly. We will be interested in the initial value problem $H_0(\xi) = \int_{-\infty}^{\xi} \frac{z_{0\xi}(y)^2 + r_0(y)^2}{q_{0\xi}(y)}\,\mathrm{d}y$, $q_0 + H_0 = \mathrm{id}$.

Proposition 3.6. *The solution of the system* (3.4) *with initial data*

$$q|_{t=0} = q_0, \tag{3.5a}$$

$$z|_{t=0} = z_0, \tag{3.5b}$$

$$r|_{t=0} = r_0, \tag{3.5c}$$

$$H|_{t=0} = \int_{-\infty}^{\mathrm{id}} \frac{z_{0\xi}(\xi)^2 + r_0(\xi)^2}{q_{0\xi}(\xi)}\,\mathrm{d}\xi, \tag{3.5d}$$

is given by

$$\begin{aligned} q(\xi,t) = &\frac{1}{4}\Bigg(\int_{-\infty}^{\xi} \frac{z_{0\xi}(y)^2 + r_0(y)^2}{q_{0\xi}(y)}\mathbf{1}^A(y,t)\,\mathrm{d}y \\ &- \frac{1}{2}\int_{-\infty}^{\infty} \frac{z_{0\xi}(y)^2 + r_0(y)^2}{q_{0\xi}(y)}\mathbf{1}^A(y,t)\,\mathrm{d}y\Bigg)t^2 \end{aligned}$$

$$
\begin{aligned}
&- \Big(\int_{-\infty}^{\xi} z_{0\xi}(y) \mathbf{1}^B(y,t)\,\mathrm{d}y - \frac{1}{2} \int_{-\infty}^{\infty} z_{0\xi}(y) \mathbf{1}^B(y,t)\,\mathrm{d}y \Big) t + z_0(\xi) t \\
&- \Big(\int_{-\infty}^{\xi} q_{0\xi} \mathbf{1}^B(t)\,\mathrm{d}y - \frac{1}{2} \int_{-\infty}^{\infty} q_{0\xi} \mathbf{1}^B(t)\,\mathrm{d}y \Big) + q_0(\xi), \qquad (3.6)
\end{aligned}
$$

$$
\begin{aligned}
z(\xi,t) =& \frac{1}{2} \Big(\int_{-\infty}^{\xi} \frac{z_{0\xi}(y)^2 + r_0(y)^2}{q_{0\xi}(y)} \mathbf{1}^A(y,t)\,\mathrm{d}y \\
&- \frac{1}{2} \int_{-\infty}^{\infty} \frac{z_{0\xi}(y)^2 + r_0(y)^2}{q_{0\xi}(y)} \mathbf{1}^A(y,t)\,\mathrm{d}y \Big) t \\
&- \Big(\int_{-\infty}^{\xi} z_{0\xi}(y) \mathbf{1}^B(y,t)\,\mathrm{d}y - \frac{1}{2} \int_{-\infty}^{\infty} z_{0\xi}(y) \mathbf{1}^B(y,t)\,\mathrm{d}y \Big) \\
&+ z_0(\xi), \qquad (3.7) \\
H(\xi,t) =& \int_{-\infty}^{\xi} \frac{z_{0\xi}(y)^2 + r_0(y)^2}{q_{0\xi}(y)} \mathbf{1}^A(y,t)\,\mathrm{d}y, \qquad (3.8) \\
r(\xi,t) =& r_0(\xi). \qquad (3.9)
\end{aligned}
$$

The characteristic functions $\mathbf{1}^A$ and $\mathbf{1}^B$ are given by

$$
\mathbf{1}^A(\xi,t) = \begin{cases} 0 & \text{if } r_0(\xi) = 0, z_{0\xi}(\xi) < 0, t > -\frac{2q_{0\xi}(\xi)}{z_{0\xi}(\xi)}, \\ 1 & \text{else}, \end{cases}
$$

$$
\mathbf{1}^B(\xi,t) = \begin{cases} 1 & \text{if } r_0(\xi) = 0, z_{0\xi}(\xi) < 0, t > -\frac{2q_{0\xi}(\xi)}{z_{0\xi}(\xi)}, \\ 0 & \text{else}. \end{cases}
$$

Note that $\mathbf{1}^A + \mathbf{1}^B = 1$.

Proof. The formula for r and H is obtained by integrating. To get the formula for z we integrate again and change the order of integration by the Tonelli's theorem as the integrand in H is non-negative. The formula is reached by

$$\int_0^t \mathbf{1}^A(s)\,\mathrm{d}s = t\mathbf{1}^A(t) - \frac{2q_{0\xi}}{z_{0\xi}}\mathbf{1}^B(t). \tag{3.10}$$

To go from z to q we integrate one more time. The integral $\int z_{0\xi}\mathbf{1}^B\,\mathrm{d}\xi$ is finite. As $z_0 \in W^{1,\infty}(\mathbb{R})$ it suffices to show that the Lebesgue measure of supp $\mathbf{1}^B(t)$ is finite for each t. The definition of $\mathbf{1}^B(t)$ gives

$$\begin{aligned} \mathrm{m}\big(\operatorname{supp}\mathbf{1}^B(t)\big) &= \mathrm{m}\Big(\big\{\xi \mid t > -\frac{2q_{0\xi}}{z_{0\xi}}, z_{0\xi} < 0, r_0 = 0\big\}\Big) \\ &\le \mathrm{m}\Big(\big\{\xi \mid |z_{0\xi}|^2 > \frac{4c^2}{t^2}\big\}\Big) < \infty, \end{aligned} \tag{3.11}$$

as $z_{0\xi} \in L^2(\mathbb{R})$ and $0 \le q_{0\xi} < c$ for some $c > 0$. □

From the proposition above we have that once q_ξ becomes zero it remains zero for all times, and $H_\xi = 0$ whenever $q_\xi = 0$. Thus we can not use the space $\mathcal{F}$ for dissipative solutions.

Definition 3.7. *The set $\mathcal{F}^d$ consists of $X = (q, z, H, r) \in B$ such that*

$$\begin{aligned} &(i) \quad \zeta, z, H \in W^{1,\infty}(\mathbb{R}), \zeta + \mathrm{id} = q, \\ &(ii) \quad q_\xi \ge 0, H_\xi \ge 0, (q+H)_\xi \ge c > 0 \ \text{or}\ (q+H)_\xi = 0 \ \text{a.e.} \\ &(iii) \quad q_\xi H_\xi = z_\xi^2 + r^2 \quad \text{a.e.} \end{aligned}$$

Note that $\mathcal{F} \subseteq \mathcal{F}^d$. The next lemma shows that $\mathcal{F}^d$ is the correct space to look for solutions.

Lemma 3.8. *If $X_0 = (q_0, z_0, H_0, r_0) \in \mathcal{F}^d$ then the solution X_t of (3.4) at t with initial data X_0 is in B.*

Proof. We compute using the explicit solutions in Proposition 3.6. For each t we have that H is bounded by H_0, z by $\frac{1}{4}Ht + 2z_0$ and q by $\frac{1}{8}Ht^2 + zt + 2q_0$. The derivatives and r are bounded by

$$\begin{aligned} r(t) &= r_0, \\ 0 \le H_\xi(t) &\le H_{0\xi}, \\ |z_\xi| &\le \frac{1}{2}H_{0\xi}t + |z_{0\xi}|, \end{aligned}$$

$$|q_\xi| \leq \frac{1}{4}H_{0\xi}t^2 + |z_{0\xi}|t + q_{0\xi}, \tag{3.12}$$

which are square integrable. □

3.3 Global existence of dissipative solutions

The strategy is similar to the one in the conservative case. The main difference being that here we do not attempt to establish continuity. There are other differences of a more "tactical" nature, for example a different approach to the projection to $\mathcal{F}_0$ and more difficult calculations in the Lagrangian coordinates.

Theorem 3.9. *The solution operators $S_t^d : \mathcal{F}^d \to \mathcal{F}^d$ constitutes a semigroup. The operator S_t^d is not continuous with respect to the B-norm.*

Proof. Let $X_0 = (q_0, H_0, z_0, r_0)$ denote the initial data and $S_t^d(X_0) = X(t) = \big(q(t), H(t), z(t), r(t)\big)$ the solution at t. We need to show that the solution is in $\mathcal{F}^d$ and that it is unique. To that end, consider

$$\begin{aligned} z_\xi^2 =& \frac{1}{4}\Big(\frac{z_{0\xi}^2 + r_0^2}{q_{0\xi}}\Big)^2 \mathbf{1}^A t^2 + z_{0\xi}\Big(\frac{z_{0\xi}^2 + r_0^2}{q_{0\xi}}\Big)\mathbf{1}^A t + z_{0\xi}^2 \mathbf{1}^A, \\ r^2 =& r_0^2 = r_0^2 \mathbf{1}^A, \\ q_\xi H_\xi =& \frac{1}{4}\Big(\frac{z_{0\xi}^2 + r_0^2}{q_{0\xi}}\Big)^2 \mathbf{1}^A t^2 + z_{0\xi}\Big(\frac{z_{0\xi}^2 + r_0^2}{q_{0\xi}}\Big)\mathbf{1}^A t + z_{0\xi}^2 \mathbf{1}^A + r_0^2 \mathbf{1}^A, \end{aligned} \tag{3.13}$$

which proves that (iii) in Definition 2.7 holds. Furthermore, $q - \mathrm{id}, z, H \in W^{1,\infty}(\mathbb{R})$ as this holds for the initial data, and for each t the solutions are linear combinations of the initial data. Non-negativity of H_ξ follows from $q_{0\xi}$ being non-negative, and thus q_ξ has to be non-negative due to (iii). That $(q + H)_\xi \geq c > 0$ or $(q + H)_\xi = 0$ almost everywhere holds is proven in much the same way as in the conservative case in the proof of Theorem 2.8. The only difference is that we do not have continuity in t across the blow-up time. We show the semigroup proporty $S_t^d \circ S_s^d = S_{t+s}^d$ next. The notation $S_t^d(f_0) = f(t)$ for $f = q, z, H, r, \mathbf{1}^A$ with $f_0 = q_0, z_0, H_0, r_0, \mathbf{1}^A(0)$, respectively, is used. The key identity is

$$1 + \frac{1}{2}\frac{z_\xi(y,s)}{q_\xi(y,s)}t = 1 + \frac{1}{2}\frac{z_{0\xi}(y)}{q_{0\xi}(y)}(t+s), \tag{3.14}$$

whenever $\mathbf{1}^A(y,s) = 1$. This implies that $S_t^d \circ S_s^d(\mathbf{1}^A) = S_{t+s}^d(\mathbf{1}^A)$, and thus

$$S_t^d S_s^d(r_0) = r_0 = S_{t+s}^d(r_0), \tag{3.15}$$

$$\begin{aligned}
S_t^d S_s^d(H_0) &= \int_{-\infty}^{\xi} \frac{z_\xi(y,s)^2 + r(y,s)^2}{q_\xi(y,s)} \mathbf{1}_s^A(y,t)\,\mathrm{d}y \\
&= \int_{-\infty}^{\xi} \frac{\frac{1}{4}(\frac{z_{0\xi}^2+r_0^2}{q_{0\xi}})^2 s^2 + z_{0\xi}\frac{z_{0\xi}^2+r_0^2}{q_{0\xi}} s + z_{0\xi}^2 + r_0(y)^2}{\frac{1}{4}(\frac{z_{0\xi}^2+r_0^2}{q_{0\xi}})\mathbf{1}^A s^2 + z_{0\xi}\mathbf{1}^A s + q_{0\xi}} \mathbf{1}^A(y,s)\mathbf{1}_s^A(y,t)\,\mathrm{d}y \\
&= \int_{-\infty}^{\xi} \frac{z_{0\xi}^2 + r_0^2}{q_{0\xi}} \mathbf{1}^A(y,t+s)\,\mathrm{d}y = S_{t+s}^d(H_0), \qquad (3.16)
\end{aligned}$$

$$\begin{aligned}
S_t^d S_s^d(z_0) =& \frac{1}{2}\Big(\int_{-\infty}^{\xi} \frac{z_\xi(s)^2 + r(s)^2}{q_\xi(s)} \mathbf{1}_s^A(t)\,\mathrm{d}y - \frac{1}{2}\int_{-\infty}^{\infty} \frac{z_\xi(s)^2 + r(s)^2}{q_\xi(s)} \mathbf{1}_s^A(t)\,\mathrm{d}y\Big) t \\
&- \Big(\int_{-\infty}^{\xi} z_\xi(s)\mathbf{1}_s^B(t)\,\mathrm{d}y - \frac{1}{2}\int_{-\infty}^{\infty} z_\xi(s)\mathbf{1}_s^B(t)\,\mathrm{d}y\Big) \\
&+ \frac{1}{2}\Big(\int_{-\infty}^{\xi} \frac{z_{0\xi}^2 + r_0^2}{q_{0\xi}} \mathbf{1}^A(s)\,\mathrm{d}y - \frac{1}{2}\int_{-\infty}^{\infty} \frac{z_{0\xi}^2 + r_0^2}{q_{0\xi}} \mathbf{1}^A(s)\,\mathrm{d}y\Big) s \\
&- \Big(\int_{-\infty}^{\xi} z_{0\xi}\mathbf{1}^B(s)\,\mathrm{d}y - \frac{1}{2}\int_{-\infty}^{\infty} z_{0\xi}\mathbf{1}^B(s)\,\mathrm{d}y\Big) \\
&+ z_0(\xi) \\
=& \frac{1}{2}\Big(\int_{-\infty}^{\xi} \frac{z_{0\xi}^2 + r_0^2}{q_{0\xi}} \mathbf{1}^A(t+s)\,\mathrm{d}y - \frac{1}{2}\int_{-\infty}^{\infty} \frac{z_{0\xi}^2 + r_0^2}{q_{0\xi}} \mathbf{1}^A(t+s)\,\mathrm{d}y\Big) t \\
&- \Big(\int_{-\infty}^{\xi} z_{0\xi}\mathbf{1}^A(s)\mathbf{1}^B(t+s)\,\mathrm{d}y - \frac{1}{2}\int_{-\infty}^{\infty} z_{0\xi}\mathbf{1}^A(s)\mathbf{1}^B(t+s)\,\mathrm{d}y\Big) \\
&- \frac{1}{2}\Big(\int_{-\infty}^{\xi} \frac{z_{0\xi}^2 + r_0^2}{q_{0\xi}} \mathbf{1}^A(s)\mathbf{1}^B(t+s)\,\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2}\int_{-\infty}^{\infty} \frac{z_{0\xi}^2 + r_0^2}{q_{0\xi}} \mathbf{1}^A(s)\mathbf{1}^B(t+s)\,\mathrm{d}y\Big)s \\
&+ \frac{1}{2}\Big(\int_{-\infty}^{\xi} \frac{z_{0\xi}^2 + r_0^2}{q_{0\xi}} \mathbf{1}^A(s)\,\mathrm{d}y - \frac{1}{2}\int_{-\infty}^{\infty} \frac{z_{0\xi}^2 + r_0^2}{q_{0\xi}} \mathbf{1}^A(s)\,\mathrm{d}y\Big)s \\
&- \Big(\int_{-\infty}^{\xi} z_{0\xi}\mathbf{1}^B(s)\,\mathrm{d}y - \frac{1}{2}\int_{-\infty}^{\infty} z_{0\xi}\mathbf{1}^B(s)\,\mathrm{d}y\Big) \\
&+ z_0(\xi) \\
=& \frac{1}{2}\Big(\int_{-\infty}^{\xi} \frac{z_{0\xi}^2 + r_0^2}{q_{0\xi}} \mathbf{1}^A(t+s)\,\mathrm{d}y - \frac{1}{2}\int_{-\infty}^{\infty} \frac{z_{0\xi}^2 + r_0^2}{q_{0\xi}} \mathbf{1}^A(t+s)\,\mathrm{d}y\Big)(t+s) \\
&- \Big(\int_{-\infty}^{\xi} z_{0\xi}\mathbf{1}^B(t+s)\,\mathrm{d}y - \frac{1}{2}\int_{-\infty}^{\infty} z_{0\xi}\mathbf{1}^B(t+s)\,\mathrm{d}y\Big) + z_0(\xi) \\
=& S^d_{t+s}(z_0) \qquad (3.17)
\end{aligned}
$$

$$
\begin{aligned}
S_t^d S_s^d(q_0) =& \frac{1}{4}\big(H(\xi, t+s) - \frac{1}{2}H_{tot}(t+s)\big)t^2 \\
&- \Big(\int_{-\infty}^{\xi} \left(\frac{1}{2}\frac{z_{0\xi}^2 + r_0^2}{q_{0\xi}} s + z_{0\xi}\right) \mathbf{1}^A(s)\mathbf{1}^B(t+s)\,\mathrm{d}y \\
&- \frac{1}{2}\int_{-\infty}^{\infty} \left(\frac{1}{2}\frac{z_{0\xi}^2 + r_0^2}{q_{0\xi}} s + z_{0\xi}\right) \mathbf{1}^A(s)\mathbf{1}^B(t+s)\,\mathrm{d}y\Big)t \\
&+ \frac{1}{2}\big(H(\xi, s) - \frac{1}{2}H_{tot}(s)\big)t \\
&- \Big(\int_{-\infty}^{\xi} z_{0\xi}\mathbf{1}^B(s)\,\mathrm{d}y - \frac{1}{2}\int_{-\infty}^{\infty} z_{0\xi}\mathbf{1}^B(s)\,\mathrm{d}y\Big)t + z_0(\xi)t \\
&- \Big(\int_{-\infty}^{\xi} \Big[\big(1 + \frac{1}{2}\frac{z_{0\xi}}{q_{0\xi}} s\big)^2 + \big(\frac{1}{2}\frac{r_0}{q_{0\xi}} s\big)^2\Big] q_{0\xi}\mathbf{1}^A(s)\mathbf{1}^B(t+s)\,\mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{2}\int_{-\infty}^{\infty} \Big[\big(1+\frac{1}{2}\frac{z_{0\xi}}{q_{0\xi}}s\big)^2 + \big(\frac{1}{2}\frac{r_0}{q_{0\xi}}s\big)^2\Big] q_{0\xi}\mathbf{1}^A(s)\mathbf{1}^B(t+s)\,\mathrm{d}y\Big) \\
&+ \frac{1}{4}\big(H(\xi,s) - \frac{1}{2}H_{tot}(s)\big)s^2 \\
&- \Big(\int_{-\infty}^{\xi} z_{0\xi}\mathbf{1}^B(s)\,\mathrm{d}y - \frac{1}{2}\int_{-\infty}^{\xi} z_{0\xi}\mathbf{1}^B(s)\,\mathrm{d}y\Big)s + z_0(\xi)s \\
&- \Big(\int_{-\infty}^{\xi} q_{0\xi}\mathbf{1}^B(s)\,\mathrm{d}y - \frac{1}{2}\int_{-\infty}^{\xi} q_{0\xi}\mathbf{1}^B(s)\,\mathrm{d}y\Big) + q_0(\xi) \\
=&\frac{1}{4}\big(H(\xi,t+s) - \frac{1}{2}H_{tot}(t+s)\big)(t^2+2ts+s^2) \\
&- \Big(\int_{-\infty}^{\xi} z_{0\xi}\mathbf{1}^B(t+s)\,\mathrm{d}y - \frac{1}{2}\int_{-\infty}^{\infty} z_{0\xi}\mathbf{1}^B(t+s)\,\mathrm{d}y\Big)(t+s) \\
&- \Big(\int_{-\infty}^{\xi} q_{0\xi}\mathbf{1}^B(t+s)\,\mathrm{d}y - \frac{1}{2}\int_{-\infty}^{\infty} q_{0\xi}\mathbf{1}^B(t+s)\,\mathrm{d}y\Big) \\
&+ z_0(\xi)(t+s) + q_0(\xi) \\
=&S^d_{t+s}(q_0).
\end{aligned} \tag{3.18}
$$

That S_t^d is not continuous follows from the explicit solutions in Proposition 3.6, where one can observe that H does not depend continuously on the B-norm of r. Uniqueness follows by uniqueness, up to a constant, of the antiderivative in the space of distributions. □

Define the function $\Lambda : \mathcal{D} \to \mathcal{D}$ by

$$\Lambda(u,\rho,\mu) = \big(u,\rho,(u_x^2+\rho^2)\,\mathrm{d}x\big)\,, \tag{3.19}$$

and the map $L^d : \mathcal{D} \to \mathcal{F}^d$ by

$$L^d = L \circ \Lambda. \tag{3.20}$$

Since both L and Λ are well defined, L^d is well defined as well. To go back to the original variables we extend M to a map M^d in the natural way, i.e. the domain is extended from $\mathcal{F}$ to $\mathcal{F}^d$ and the formulas remain the same. As in the conservative case there is some redundancy corresponding to relabeling.

Definition 3.10. *Let ϕ be such that $(q+H)\circ\phi = \mathrm{id}$. Define $\Pi^d : \mathcal{F}^d \to \mathcal{F}_0$ by*

$$\Pi^d\big(q,z,H,r\big) = \left(q\circ\phi, z\circ\phi, H\circ\phi, (r\circ\phi)\phi_\xi\right). \tag{3.21}$$

In order for Π^d to be well defined we need that it is independent on the choice of ϕ and that the range is $\mathcal{F}_0$.

Proposition 3.11. *The function Π^d defined in Definition 3.10 is well defined.*

Proof. We will show that Π^d is a function, that the range is $\mathcal{F}_0$ and that it is independent of the choice of right inverse. First note that ϕ is strictly increasing since ϕ is one-to-one and increasing. This implies the existence of ϕ_ξ almost everywhere [6, Theorem 3.23]. We will show that $\phi_\xi - 1 \in L^2(\mathbb{R})$. Let $A = \{\xi \mid \phi_\xi \text{ exists}\}$. Then

$$\int\limits_A |\phi_\xi - 1|^2 \mathrm{d}\xi = \int\limits_A |1-(q_\xi+H_\xi)\circ\phi|^2\phi_\xi^2 \mathrm{d}\xi \leq \|\phi_\xi\|_{L^\infty(\mathbb{R})} \int\limits_{\mathbb{R}} |q_\xi + H_{xi} - 1|^2 \mathrm{d}\xi < \infty.$$

Condition (ii) in the definition of $\mathcal{F}^d$, Definition 3.7, implies that

$$\frac{1}{\|q_\xi + H_\xi\|_{L^\infty(\mathbb{R})}} \leq \phi_\xi \leq \frac{1}{c} \qquad \text{almost everywhere,} \tag{3.22}$$

for some $c > 0$. The points of discontinuity is precisely the points where $(q + H)_\xi = 0$. This implies that the range of Π^d is B and $W^{1,\infty}(\mathbb{R})^3 \times L^\infty(\mathbb{R})$, for details see the proof of Proposition 2.18. Let $X = (q,z,H,r) \in \mathcal{F}^d$ and $\Pi^d(X) = (\bar{q},\bar{z},\bar{H},\bar{r})$. We have by (iii) in Definition 3.7 that $q_\xi > 0$ whenever $r \neq 0$, thus ϕ_ξ exists whenever $r \neq 0$ and Π^d is well defined if we define $(r\circ\phi)\phi_\xi = 0$ when $r\circ\phi = 0$. Let ψ be another right inverse of $(q+H)$. Then $\psi = \phi$ whenever $(q+H)_\xi \neq 0$, and when $(q+H)_\xi = 0$ we have $q_\xi = z_\xi = r = H_\xi = 0$ and thus Π^d is independent of choice of right inverse. To see that the range is $\mathcal{F}_0$ we observe that $\bar{q} + \bar{h} = (q+H)\circ\phi = \mathrm{id}$, and as q, h are increasing so are $\bar{q}, \bar{H}$. Furthermore,

$$\bar{q}_\xi \bar{H}_\xi = q_\xi H_\xi \phi_\xi^2 = (z_\xi^2 + r^2)\phi_\xi^2 = \bar{z}_\xi^2 + \bar{r}^2. \tag{3.23}$$

The boundedness of $\bar{z}$, $\bar{H}$ and $\mathrm{id} - \bar{q}$ follows directly from the boundedness of z, H and $q - \mathrm{id}$. The same argument holds for the boundedness of $\bar{q}_\xi$, $\bar{z}_\xi$ and $\bar{H}_\xi$. The square integrability follows. □

That ϕ is one-to-one and thus invertible on all "interesting" sets, that is all sets where $q_\xi \neq 0$, yields the next proposition.

Proposition 3.12. *If $X \in \mathcal{F}^d$, then*

$$M^d(X) = M \circ \Pi^d(X). \tag{3.24}$$

Proof. Define $(u, \rho, \mu) = M \circ \Pi^d(X)$ and $(\bar{u}, \bar{\rho}, \bar{\mu})$. Let ξ be given and $x = q \circ \phi(\xi)$. The one-to-one property of ϕ and that $q \circ \phi$ is onto guarantees the existence of a function f such that $(q \circ \phi) \circ (\phi^{-1} \circ f) = \text{id}$. Thus

$$z \circ \phi(\phi^{-1} \circ f) = z \circ f \tag{3.25}$$

and $u(x) = \bar{u}(x)$. Likewise

$$\frac{\phi_\xi(r \circ \phi)}{\phi_\xi(q_\xi \circ \phi)}(\phi^{-1} \circ f) = \frac{r}{q_\xi} \circ f \tag{3.26}$$

and $\rho(x) = \bar{\rho}(x)$. It remains to show that $\mu = \bar{\mu}$, but this follows from

$$\int\limits_{(q\circ\phi)^{-1}(A)} H_\xi \circ \phi \phi_\xi \, \mathrm{d}\xi = \int\limits_{q^{-1}(A)} H_\xi \, \mathrm{d}\xi. \tag{3.27}$$

□

We can then define the mapping $\tilde{S}_t^d : \mathcal{F}_0 \to \mathcal{F}_0$ by $\tilde{S}_t^d = \Pi^d \circ S_t^d$.

Proposition 3.13. *The mappings Π^d and S_t^d satisfies*

$$\Pi^d \circ S_t^d \circ \Pi^d = \Pi^d \circ S_t^d. \tag{3.28}$$

In other words, $\tilde{S}_t^d$ is a semigroup in $\mathcal{F}_0$.

Proof. Let $X = (q, z, H, r) \in \mathcal{F}^d$ and $X(t) = \big(q(t), z(t), H(t), r(t)\big) = S_t^d(X)$. Then $S_t^d \circ \Pi^d(X) = \big(\bar{q}(t), \bar{z}(t), \bar{H}(t), \bar{r}(t)\big)$ is given by

$$\begin{aligned}
\bar{r}(t) =& \bar{r}(0) = r \circ \phi_0 = r(t) \circ \phi_0, \\
\bar{H}(t) =& \int\limits_{-\infty}^{\xi} \bar{H}_\xi(0) \mathbf{1}^A(t) \, \mathrm{d}\xi \\
=& \int\limits_{-\infty}^{\xi} H_\xi \circ \phi_0(\xi) \phi_{0\xi}(\xi) \, \mathrm{d}\xi
\end{aligned}$$

$$=H(t) \circ \phi_0,$$

which implies that

$$\begin{aligned} \bar{z}(t) =& z(t) \circ \phi_0 \\ \bar{q}(t) =& q(t) \circ \phi_0. \end{aligned}$$

We apply Π^d to $S_t^d \circ \Pi^d(X)$, realize that $\phi_t = \phi_0^{-1} \circ \phi$ where ϕ is a right inverse of $q(t) + H(t)$ and get

$$\begin{aligned} \bar{r}(t) \circ \phi_t &= r \circ \phi, \\ \bar{H}(t) \circ \phi_t &= H \circ \phi, \\ \bar{z}(t) \circ \phi_t &= z \circ \phi, \\ \bar{q}(t) \circ \phi_t &= q \circ \phi, \end{aligned}$$

which equals $\Pi \circ S_t^d(X)$. □

We define the map $T_t^d : \mathcal{D} \to \mathcal{D}$ by

$$T_t^d = M^d \circ \tilde{S}_t^d \circ L^d. \tag{3.29}$$

Definition 3.14. *A weak solution* (u, ρ, μ) *of* (1.3) *in the sense of Definition 2.30 is said to be a dissipative weak solution if it satisfies the entropy criterion*

$$\rho(x,t) = 0 \quad \Rightarrow \quad u_x(x,t) \leq \frac{1}{t} \text{ almost everywhere,} \tag{3.30}$$

in addition to

$$\int \left(u_x(x,t)^2 + \rho(x,t)^2\right) \,\mathrm{d}x \leq \int \left(u_{0x}(x)^2 + \rho_0(x)^2\right) \,\mathrm{d}x. \tag{3.31}$$

And the final theorem that concludes the chapter on dissipative solutions.

Theorem 3.15. *The map* T_t^d *constitutes a semigroup in* $\mathcal{D}$ *of dissipative weak solutions of* (1.3) *in the sense of Definition 3.14. The operator* T_t^d *is not continuous with respect to the metric* $d_{\mathcal{D}}$.

Proof. The proof that it is a weak solution is the same as in the conservative case, the point being that $\dot{q} = z$ and $\dot{z} = \frac{1}{2}H - \frac{1}{4}H_{tot}$ still holds. To see that the solution is dissipative note that $\rho(x,t) = 0$ implies that $r(\xi, s) = 0$ for all s and $q(\xi) = x$. Then by solving the characteristic equations (3.4) one finds that $u_x \leq \frac{1}{t}$. To prove the semigroup property it suffices to show that $\mu(t) = (u_x^2 + \rho^2)\mathrm{d}x$. That it holds for $t = 0$ follows directly from Proposition 2.22 and the definition of Λ. For $t > 0$ note that $q_\xi(\xi) = 0$ implies that $H_\xi(\xi) = 0$. This implies that

$$\mu(t)\big(A\big) = \int\limits_{q^{-1}(A)} H_\xi \, \mathrm{d}\xi = \int\limits_{q^{-1}(A \cap \{q_\xi > 0\})} H_\xi \, \mathrm{d}\xi, \tag{3.32}$$

and for every right inverse f of q we have

$$\mu(t)\big((-\infty, x]\big) = \int\limits_{-\infty}^{f(x)} H_\xi \, \mathrm{d}\xi. \tag{3.33}$$

As H_ξ is a measurable function, $\mu(t)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. The formula $\mu(t) = (u_x^2 + \rho^2)\,\mathrm{d}x$ is proved by substitution in the above integral. □

Remark 3.16. *The singular part of μ_0 is simply ignored, so in the dissipative case we can restrict our attention to $\mathcal{D}^d = E_2 \times L^2(\mathbb{R})$ without any loss of generality.*

Remark 3.17. *If $\mathcal{D}$ is restricted to $\rho = 0$ the dissipative semigroup is continuous with respect to some other metric [2]. The difficulty in extending the metric lies in the fact that if $\rho_0 \neq 0$ and $\mu_0 = (u_{0x}^2 + \rho_0^2)\,\mathrm{d}x$, then the dissipative and conservative solutions are identical. Thus if the initial data converges, then the solutions converges to the conservative solution in $d_{\mathcal{D}}$.*

Chapter 4

Conclusions and future studies

4.1 Future studies

One would want to try to establish the results for the two-component Camassa-Holm equation and the standard Hunter-Saxton in our setting. The first questions that come to mind are:

- Can one construct a metric on $\mathcal{D}$ such that the semigroup of dissipative solutions is continuous? If so, can it be made Lipschitz continuous? This is done for the Hunter-Saxton equation in [2].
- Is the semigroup of conservative solutions Lipschitz continuous, as in the Hunter-Saxton case [3]?

4.2 Conclusions

A global continuous semigroup of conservative solutions of the Hunter-Saxton system (1.3), defined in Definition 2.30, was shown to exists in Theorem 2.31. Furthermore, if $\rho \to 0 \in L^2(\mathbb{R})$, then the solution constructed by the semigroup approaches the conservative solution of the Hunter-Saxton equation (1.2). Conservative multipeakons were demonstrated to exist in Example 2.32.

Similarly, a global semigroup of dissipative solutions of the Hunter-Saxton system (1.3), defined in Definition 3.14, was shown to exists in Theorem 3.15.

Dissipative multipeakons exist as demonstrated in Definition 3.1 and Lemma 3.4.

References

[1] Luigi Ambrosio, Nicola Fusco, and Diego Pallara. *Functions of Bounded Variation and free Discontinuity Problems.* Oxford University Press, 2000.

[2] Alberto Bressan and Adrian Constantin. Global solutions of the Hunter–Saxton equation. *SIAM Journal on Mathematical Analysis*, 37(3):996–1026, 2005.

[3] Alberto Bressan, Helge Holden, and Xavier Raynaud. Lipschitz metric for the Hunter–Saxton equation. *Journal de Mathématiques Pures et Appliquées*, 94(1):68–92, 2010.

[4] Roberto Camassa and Darryl D. Holm. An integrable shallow water equation with peaked solitons. *Phys. Rev. Lett.*, 71:1661–1664, Sep 1993.

[5] Lawrence C. Evans. *Partial Differential Equations.* American Mathematical Society, 2 edition, 2010.

[6] Gerald B. Folland. *Real Analysis.* John Wiley & Sons Inc., New York, 1984.

[7] Katrin Grunert, Helge Holden, and Xavier Raynaud. Global conservative solutions of the Camassa-Holm equation for initial data nonvanishing asymptotics. *Discrete and Continuous Dynamical Systems*, (to appear).

[8] Katrin Grunert, Helge Holden, and Xavier Raynaud. Global solutions for the two-component Camassa–Holm system. *Communications in Partial Differential Equations*, (to appear).

[9] Helge Holden and Xavier Raynaud. Global conservative solutions of the Camassa–Holm equation–A Lagrangian point of view. *Communications in Partial Differential Equations*, 32(10):1511–1549, 2007.

[10] John K. Hunter and Ralph Saxton. Dynamics of director fields. *SIAM Journal on Applied Mathematics*, 51(6):1498–1521, December 1991.

[11] John K. Hunter and Yuxi Zheng. On a completely integrable nonlinear hyperbolic variational equation. *Physica D: Nonlinear Phenomena*, 79(2–4):361–386, 1994.

[12] John K. Hunter and Yuxi Zheng. On a nonlinear hyperbolic variational equation: I. Global existence of weak solutions. *Archive for Rational Mechanics and Analysis*, 129:305–353, 1995.

[13] John K. Hunter and Yuxi Zheng. On a nonlinear hyperbolic variational equation: II. The zero-viscosity and dispersion limits. *Archive for Rational Mechanics and Analysis*, 129:355–383, 1995.

[14] Maxim V Pavlov. The Gurevich–Zybin system. *Journal of Physics A: Mathematical and General*, 38(17):3823–3841, 2005.

[15] Marcus Wunsch. On the Hunter–Saxton system. *Discrete and Continuous Dynamical Systems Series B*, 12(3):647–656, October 2009.

[16] Marcus Wunsch. The generalized Hunter–Saxton system. *SIAM Journal on Mathematical Analysis*, 42(3):1286–1304, 2010.

[17] Ping Zhang and Yuxi Zheng. On oscillations of an asymptotic equation of a nonlinear variational wave equation. *Asymptotic Analysis*, 18(3):307–327, 1998.

[18] Ping Zhang and Yuxi Zheng. Existence and uniqueness of solutions of an asymptotic equation arising from a variational wave equation with general data. *Archive for Rational Mechanics and Analysis*, 155:49–83, 2000.

[19] Dafeng Zuo. A two-component μ-Hunter–Saxton equation. *Inverse Problems*, 26(8):085003, 2010.