NTNU

Norwegian University of
Science and Technology

Triangulated categories and localization

Karin Marie Jacobsen

Master of Science in Physics and Mathematics

Problem Description

The student should give a description of the following topics:

- abstract triangulated categories

- localisation with respect to arbitrary sets of morphisms

- localisation as used in the construction of derived categories of module categories

Moreover, the student should give detailed examples of derived categories for finite dimensional algebras and other concrete triangulated categories.

Abstract

We study Gabriel-Zisman localization, localization by a multiplicative system and by a null system. We define the triangulated category and the derived category. Finally we describe a scheme for localization from a triangulated category to a module category.

Preface

This thesis was written for the course TMA4900 - Mathematics, Master Thesis.

I first learned about category theory during my exchange year at the University of Helsinki, and fell in love with the abstract grace and versatility of the material. My specialization project during the spring of 2011 allowed me to study it in the context of abelian categories, and in this thesis I have studied subjects connected to the concept of localization, i.e. the creation of inverses in categories. I was particularly happy to be able to include some current research by my advisor in the final chapter.

I would like to thank my advisor, Aslak Bakke Buan. He has been reassuring, helpful and supportive, whilst also being a master hunter of typos and ambiguities in the manuscript Second, I must thank my parents, who have always been encouraging and always there for me, and my "almost godmother" professor Gunilla Borgefors who is a constant source of intellectual inspiration. Finally I want to thank two people who over the past year have listened patiently to me panic, complain and enthuse about category theory: My roommate Anna and my beloved Petter.

<div align="center">
Karin Marie Jacobsen

Trondheim, January 2012
</div>

Contents

1 Introduction

This thesis will consider a concept in category theory called *localization*. We will take a category $\mathcal{C}$ and a class of morphisms S in the category, and construct a new category $\mathcal{C}_S$ in which the morphisms of S have become isomorphisms. Moreover, we equip $\mathcal{C}_S$ with a suitable universal property, so that we are able to call it unique.

After an introductory study of the Ore condition in ring theory we will consider the very general Gabriel-Zisman localization. We will define a multiplicative system, a triangulated category and a null system. All this will set the stage for defining the derived category. The construction of the derived category moves us from an abelian category to a triangulated category. As the final part of the thesis, we will study a recent article by Aslak Bakke Buan and Robert J. Marsh [3], in which a triangulated category is turned into an abelian category through the process of localization.

In some ways localization is an old idea. The concept is used in a different guise in set theory when constructing the integers $\mathbb{Z}$; take the set of natural numbers $\mathbb{N}$, and consider the set of functions f_n, where $f_n(m) = m + n$. When all these functions are turned into isomorphisms, that is when $\mathbb{N}$ is localized with respect to the set $\{f_n | n \in \mathbb{N}\}$, we can identify $n = f_n(0)$ and $-n = f_n^{-1}(0)$, and thus we have defined $\mathbb{Z}$. Similarly, the construction of the rational numbers from the integers can also be described as a localization. We will study the latter in an example in the following subsection.

1.1 The Ore condition

We first consider a form of localization that is accomplished without the use of category theory. This section follows [2, ch. 12.2].

We want to take a possibly non-commutative domain R (a ring with no zero divisors), and show it to be embedded in a division ring Q. For this to be feasible, we need R to fulfill the following condition, called the *Ore condition*.

Definition 1.1. Let R be a domain. We call R a *right Ore domain* if for any non-zero $a, b \in R$ there exist nonzero $x, y \in R$ so that $ax = by$.

Note that we only consider rings with multiplicative identity. An equivalent definition is given by the following lemma; we will need both definitions.

Lemma 1.2. *A domain R is a right Ore domain if and only if the intersection of any two nonzero right ideals is nonzero.*

Proof. Suppose R is a right Ore domain and let I and J be two nonzero right ideals in R. Let $a \in I$ and $b \in J$ be nonzero; then we know that there exist nonzero $x, y \in R$ so that $ax = by$. As R is a domain, we know that $ax \neq 0$. However, as I and J are right ideals, it follows that $ax \in I$ and $by \in J$, and thus $ax = by \in I \cap J$, which must be nonzero.

Conversely, suppose that the domain R is such that no two nonzero right ideals have a zero intersection. Let $a, b \in R$ be nonzero. There exist nonzero ideals aR and bR [1].

---

[1] $aR = \{ar | r \in R\}$

Their intersection must be nonzero, so there are elements $x, y \in R$ such that $ax = by$. $\square$

For the remainder of this section, suppose R is a right Ore domain. Let $\mathbf{C}$ be the set of nonzero right ideals in R; by the above lemma $\mathbf{C}$ is closed under intersections.

A map f defined on a right ideal R' of R is called R-linear if $f(ab) = f(a)b$ and $f(a + b) = f(a) + f(b)$ for all $a, b \in R'$. Suppose that $I \in \mathbf{C}$, that $f : I \to R$ is an R-linear mapping and let $X \in \mathbf{C}$. We take note of some facts about f:

- $f^{-1}(X)$ is a nonempty set as $f(0) = 0 \in X$.

- If $a, b \in f^{-1}(X)$, then $f(a - b) = f(a) - f(b) \in X$, so $a - b \in f^{-1}(X)$.

- If $x \in f^{-1}(X)$ and $r \in R$, then $f(rx) = rf(x) \in X$, so $rx \in f^{-1}(X)$.

- By the above $f^{-1}(X)$ is an ideal. If $f(I) = \{0\}$ for some ideal I; then $f(I) \subset X$, so $I \in f^{-1}(X)$, which is non-zero and thus an element of $\mathbf{C}$.

  On the other hand, suppose $f(I) \neq \{0\}$. As $f(I)$ must be a nonzero right ideal, we know that $f(I) \cap X$ is nonzero; thus there is some non-zero $x \in I$ so that $f(x) \in X$. It follows that $x \in f^{-1}(X)$, which then must be nonzero and $f^{-1}(X) \in \mathbf{C}$.

For two ideals I and J we define $\mathrm{Hom}_R(I, J)$ to be the set of all R-linear maps between I and J. Define the set $H = \bigcup_{I \in \mathbf{C}} \mathrm{Hom}_R(I, R)$, and define a relation $\sim$ on H by setting $f \sim g$ if and only if $f = g$ when restricted to some ideal $I \in \mathbf{C}$. The relation is clearly reflexive and symmetric. If $f = g$ on the right ideal I and $g = h$ on the right ideal J, then $f = h$ on the ideal $I \cap J \in \mathbf{C}$ and so the relation is transitive. Thus $\sim$ is an equivalence relation on H and we can form the set $Q = H/\sim$ with elements $[f] = \{g \in H | g \sim f\}$.

Define an addition operation on Q by setting $[f] + [g] = [f + g]$. It can be checked that this operation is well-defined, and that $(Q, +)$ is an abelian group. Next, define multiplication by setting $[f][g] = [fg]$. That it is well-defined follows from the properties of $\mathbf{C}$. It is not hard to see that with these operations Q becomes a ring. Finally we check that Q is a division ring by showing that every nonzero element of Q has an inverse.

Let $[f] \in Q$ be non-zero. As $f = 0$ when restricted to $\mathrm{Ker}(f)$, we must have $\mathrm{Ker}(f) \notin \mathbf{C}$. Thus $\mathrm{Ker}(f) = \{0\}$ and f is one-to-one. Let $g : \mathrm{Im}(f) \to R$ be defined by $g(f(x)) = x$, this is well-defined because f is one-to-one, and is also R-linear. On the domain of f it holds that $gf = 1$, and so $[gf] = [1]$. Similarly $fg = 1$ on $\mathrm{Im}(f)$, so $[f]^{-1} = [g]$. Thus Q is a division ring.

Theorem 1.3. *A domain R is a right Ore domain if and only if there exists a division ring Q such that:*

(i) R is a subring of Q.

(ii) All objects in Q are of the form ab^{-1} for $a, b \in R$.

Proof. Suppose R is a right Ore domain, and let Q be defined as above.

(i) Consider the map $\phi : R \to Q$ defined by setting $\phi(a) = [f_a]$, where $f_a(x) = ax$. Note that f_a is R-linear. We check that ϕ is a ring homomorphism:

- From $\phi(1_R) = \text{id}_R$ we see that ϕ maps the unity in R to the unity in Q.

- We see that $f_{a+b}(x) = (a + b)x = ax + bx = f_a(x) + f_b(x)$, thus $\phi(a + b) = \phi(a) + \phi(b)$.

- Similarly, as $f_{ab}(x) = (ab)x = a(bx) = f_a f_b(x)$ we get that $\phi(ab) = \phi(a)\phi(b)$.

Since ϕ is a ring homomorphism, $\text{Im}(\phi)$ is a subring of Q. Of course, ϕ is onto its image; to see that it is one-to-one, suppose $\phi(a) = 0$. This means that for some nonzero ideal I, we have $aI = 0$. Thus, there is some nonzero $x \in I$ so that $ax = 0$; since R is a domain this means that $a = 0$. Thus R is isomorphic to a subring of Q, and can itself be considered a subring of Q. Due to this identification we will denote f_a simply as a when there is no risk of confusion.

(ii) We note that if $q \in Q$, where $q : I \to R$, then for $a, b \in R$ it holds that $(qa)(b) = (qf_a)(b) = q(ab) = q(a)b$, so we see that $q(a) = qa$ in Q. Now assume $q : I \to R$ is in Q, that $b \in I$ is non-zero, and set $a = q(b)$. Then $qb = a$ and as Q is a division ring, $q = ab^{-1}$.

To see the reverse implication, suppose both *(i)* and *(ii)* hold and let $a, b \in R$. Then $b^{-1}a \in Q$, and so there exist $x, y \in R$ so that $b^{-1}a = xy^{-1}$. However, this implies that $ax = by$. Thus R is a right Ore domain. $\qquad\square$

Example 1.4. Consider the set of integers, $\mathbb{Z}$. This is a commutative domain; it is easy to see that it satisfies the Ore condition. Then by theorem 1.3 we know that there exists a division ring Q where every element can be written on the form xy^{-1}, where $x, y \in \mathbb{Z}$ and $y \neq 0$. Moreover, for any two $x, y \in \mathbb{Z}$, with $y \neq 0$, we have $xy^{-1} \in Q$. It is by now obvious that $Q = \mathbb{Q}$, the field of rational numbers.

2   Localization of a general category

Let $\mathcal{C}$ be a category and let S be a class of morphisms in $\mathcal{C}$. The goal of *localization* is to construct a category $\mathcal{C}_S$ where the morphisms in S have inverses. We formalize this idea as follows:

Definition 2.1. The localization of the category $\mathcal{C}$ by a class S of morphisms in $\mathcal{C}$ is a category $\mathcal{C}_S$ with a functor $F : \mathcal{C} \to \mathcal{C}_S$ so that:

- For any morphism $s \in S$, the image $F(s)$ is an isomorphism.

- The functor $F : \mathcal{C} \to \mathcal{C}_S$ is universal; if there exists another category $\mathcal{D}$ with a functor $G : \mathcal{C} \to \mathcal{D}$ taking elements of S to isomorphisms in $\mathcal{D}$, then there exists a

unique functor $H : \mathcal{C}_S \to \mathcal{D}$ so that the following diagram commutes.

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\;G\;} & \mathcal{D} \\
{\scriptstyle F}\downarrow & \nearrow {\scriptstyle H} & \\
\mathcal{C}_S & &
\end{array}
$$

2.1   A simple proof of existence

We start by showing the existence of localization in one type of categories, known as *Gabriel-Zisman* localization. This section follows [5, ch. III.2.2].

Suppose that $\mathcal{C}$ is a small category, that is to say that $\mathrm{Ob}(\mathcal{C})$ is a set. Then

$$
\mathrm{Mor}(\mathcal{C}) = \bigcup \{ \mathrm{Hom}_{\mathcal{C}}(X, Y) | X, Y \in \mathrm{Ob}(\mathcal{C}) \}
$$

is a set, and so is any class S of morphisms in $\mathcal{C}$. To find the localization of $\mathcal{C}$ by S we start by constructing a graph Γ. The vertices of Γ are the objects of $\mathcal{C}$. For all $X, Y \in \mathrm{Ob}(\mathcal{C})$, let there be an edge from X to Y in the graph Γ for each morphism in $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. Moreover, for each morphism $X \xrightarrow{s} Y \in S$, add an arrow $Y \xrightarrow{x_s} X$. This arrow is called the *formal inverse* of s.

A path in Γ is a finite sequence of arrows such that each arrow starts in the vertex where the previous arrow in the sequence ended. Let two paths be equivalent if one can be transformed into the other by applying the following elementary equivalences a finite number of times:

- If $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$, then the path $X \xrightarrow{f} Y \xrightarrow{g} Z$ is equivalent to (can be replaced by) $X \xrightarrow{gf} Z$.

- If $X \xrightarrow{s} Y \in S$, then $X \xrightarrow{s} Y \xrightarrow{x_s} X$ is equivalent to $X \xrightarrow{\mathrm{id}_X} X$ and $Y \xrightarrow{x_s} X \xrightarrow{s} Y$ is equivalent to $Y \xrightarrow{\mathrm{id}_Y} Y$.

This relation is clearly reflexive, symmetric and transitive; thus it is an equivalence relation.

The category $\mathcal{C}_S$ is defined by setting $\mathrm{Ob}(\mathcal{C}_S) = \mathrm{Ob}(\mathcal{C})$ and for $X, Y \in \mathrm{Ob}(\mathcal{C})$ setting $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ to be the set of equivalence classes of paths from X to Y. Composition of morphisms follows from the joining of paths; this can be shown to be independent of the choices of representatives of the equivalence classes of paths.

The functor $F : \mathcal{C} \to \mathcal{C}_S$ is defined by setting $F(X) = X$ for $X \in \mathrm{Ob}(\mathcal{C})$ and letting $F(f)$ be the equivalence class of the path $X \xrightarrow{f} Y$ for $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. We see that if $s \in S$, then $F(s) = X \xrightarrow{s} Y$ has an inverse, namely its formal inverse $Y \xrightarrow{x_s} X$.

To show the universality of $\mathcal{C}_S$, we assume the existence of a category $\mathcal{D}$ with a functor $G : \mathcal{C} \to \mathcal{D}$ so that $G(s)$ has an inverse for any $s \in S$. We construct the required functor $H : \mathcal{C}_S \to \mathcal{D}$ as follows:

- For $X \in \mathrm{Ob}(\mathcal{C}_S)$, set $H(X) = G(X)$ (remember that $\mathrm{Ob}(\mathcal{C}_S) = \mathrm{Ob}(\mathcal{C})$).

- For any $f \in \mathrm{Mor}(\mathcal{C})$, set $H(X \xrightarrow{f} Y) = G(X)$.

- For any $s \in S$, set $H(Y \xrightarrow{x_s} X) = G(s)^{-1}$.

- For any other path ϕ, the composition requirement in the definition of a functor gives $H(\phi)$:

  Suppose $\phi = \phi_1 \ldots \phi_n$, where each ϕ_i denotes an arrow in Γ. Then $H(\phi) = H(\phi_1) \ldots H(\phi_n)$.

To check that H is well-defined, consider two equivalent paths $\alpha = (X \xrightarrow{a_0} A_1 \to \ldots \to A_n \xrightarrow{a_n} Y)$ and $\beta = (X \xrightarrow{b_0} B_1 \to \ldots \to B_m \xrightarrow{b_m} Y)$. The functor H will map these paths to respectively $H(\alpha) = H(a_n)H(a_{n-1})\ldots H(a_0)$ and $H(\beta) = H(b_m)H(b_{m-1})\ldots H(b_0)$. We know that by a finite number of elementary operations we can turn α into β. However, each of these elementary operations can be translated to an equality in $\mathcal{D}$. When

$$X \xrightarrow{f} Y \xrightarrow{g} Z = X \xrightarrow{gf} Z$$

in $\mathcal{C}_S$ it also holds that

$$F(g)F(f) = F(gf)$$

in $\mathcal{D}$, and similarly when

$$X \xrightarrow{s} Y \xrightarrow{x_s} X = X \xrightarrow{\mathrm{id}_X} X$$

it also holds that

$$H(x_s x) = F(x)^{-1}F(x) = \mathrm{id}_{F(x)} = F(\mathrm{id}_X) = H(\mathrm{id}_X).$$

For each step of the process of turning α into β by the elementary equivalences we can find an equality in $\mathcal{D}$, so $H(\alpha) = H(\beta)$. Thus H is well-defined. Moreover as any other functor H' satisfying $H'F = G$ must satisfy the very requirements set forth in the definition of H we have that $H' = H$ and H is unique.

2.2 Skeletally small categories

The proof of existence in the above section works only for small categories. In general the categories we encounter are not small; examples of non-small categories include the category of sets, of rings and of modules over a ring. Luckily we can extend the proof to be valid for a wider range of categories. We start with a definition.

Definition 2.2. A category $\mathcal{C}$ is *skeletally small* if the family of isomorphism classes of $\mathrm{Ob}\,\mathcal{C}$ is a set.

An important example of a skeletally small category is the category of modules over a ring.

Suppose that $\mathcal{C}$ is skeletally small, and that S is a class of morphisms in $\mathcal{C}$. For each equivalence class, use the strong axiom of choice to pick an object X to represent it. We denote the set of representing objects R. For each object $X \in \mathcal{C}$, choose an isomorphism $c_X : X \to X'$, where X' is the representing object of the isomorphism class containing X. If $X = X'$, set $c_X = \mathrm{id}_X$. We construct the graph Γ as follows:

- The vertices of Γ are the objects in R.

- For each $f \in \text{Hom}_{\mathcal{C}}(X, Y)$ where $X, Y \in R$, add an arrow $X \xrightarrow{f} Y$.

- For each $s : X \to Y \in S$, add an arrow $Y' \xrightarrow{(c_Y^{-1} s c_X)_S} X'$.

We define the equivalence classes of paths as before, and are ready to define the category $\mathcal{C}_S$. We set $\text{Ob}(\mathcal{C}_S) = \text{Ob}(\mathcal{C})$. For two objects $X, Y \in \text{Ob}(\mathcal{C}_S)$, we set

$$\text{Hom}_{\mathcal{C}_S}(X, Y) = \{X \xrightarrow{c_X} X' \xrightarrow{\phi} Y' \xrightarrow{c_Y^{-1}} Y \,|\, \phi \text{ is an equivalence class of paths } X' \to Y'\}.$$

The composition rules defined on the paths is used in $\mathcal{C}_S$ as well.

The functor $F : \mathcal{C} \to \mathcal{C}_S$ is defined as follows:

- $F(X) = X$ for $X \in \text{Ob}(\mathcal{C})$.

- $F(f) = X \xrightarrow{c_X} X' \xrightarrow{c_Y f c_X^{-1}} Y' \xrightarrow{c_Y^{-1}} Y$ for $f \in \text{Hom}_{\mathcal{C}}(X, Y)$.

For $s \in S$, the inverse of $F(s) = X \xrightarrow{c_X} X' \xrightarrow{c_Y s c_X^{-1}} Y' \xrightarrow{c_Y^{-1}} Y$ is the morphism $Y \xrightarrow{c_Y} Y' \xrightarrow{(c_Y s c_X^{-1})_S} X' \xrightarrow{c_X^{-1}} X$.

It remains to prove universality for the category $\mathcal{C}_S$. Suppose that $\mathcal{D}$ is a category with a functor $G : \mathcal{C} \to \mathcal{D}$, so that $G(s)$ has an inverse for each $s \in S$. Construct the functor H as follows:

- $H(X) = G(X)$ for $X \in \text{Ob}(\mathcal{C}_S) = \text{Ob}(\mathcal{C})$.

- $H(G(f)) = F(f)$ for $f \in \text{Mor}(\mathcal{C})$.

- $H(G(x)^{-1}) = F(x)^{-1}$.

By a proof analogous to that in the previous section, this makes H well-defined and unique.

2.3   Multiplicative systems and roofs

The two previous sections have shown procedures of localization that are restricted by the type of category they work in. We now give a method of localization where the restriction is put on the class of morphisms S and not on the category $\mathcal{C}$. The conditions on the morphisms will turn out to be highly similar to the Ore condition, and so will the description of the elements in the localization. Multiplicative systems are widely described, for example in [5], [9] and [10].

Definition 2.3. A *multiplicative system* in a category $\mathcal{C}$ is a class S of morphisms satisfying the following restrictions:

S1 $\text{id}_X \in S$ for any $X \in \text{Ob}(\mathcal{C})$.

S2 For any pair of morphisms $f, g \in S$ such that the composition fg exists, we have $fg \in S$.

S3 For $Y \xrightarrow{f} Z$ and $X \xrightarrow{s} Z \in S$, there exist morphisms $U \xrightarrow{t} Y \in S$ and $U \xrightarrow{u} X$ so that the following diagram commutes:

$$
\begin{array}{ccc}
U & \xrightarrow{\;t\;} & Y \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle f} \\
X & \xrightarrow{\;s\;} & Z
\end{array}
$$

The same holds when all arrows are reversed.

S4 Suppose $f, g \in \operatorname{Hom}_{\mathcal{C}}(X, Y)$. The following are equivalent:

- There exist $t : Y \to Y' \in S$ such that $tf = tg$.
- There exist $s : X' \to X \in S$ such that $fs = gs$.

We note that the first two axioms have little effect on localization; as id_X is an isomorphism, it will automatically have an inverse in the localized category $\mathcal{C}_S$, so adding it to S makes no difference to the localization. Similarly, if f and g are in S they will have inverses f^{-1} and g^{-1} in $\mathcal{C}_S$. Then $g^{-1}f^{-1}$ will be an inverse to fg, so we might as well add fg to S. The third axiom is practically the same as the Ore condition. Along the same lines, in a domain R with $s, a, b \in R$ non-zero it holds that $sa = sb \neq 0$ if and only if $a = b$, and in this case any $r \in R$ makes $ar = br$. The last axiom thus makes the morphisms of S act like elements of a domain (to some extent; after all categories are not assumed to be additive).

As we saw above, a morphism in the category $\mathcal{C}_S$ is given by a chain

$$
X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} X_n \tag{2.1}
$$

where $\alpha_i \in \operatorname{Hom}_{\mathcal{C}}(X_i, X_{i+1})$ or $\alpha_i = x_s$ with $s \in S \cap \operatorname{Hom}_{\mathcal{C}}(X_{i+1}, X_i)$. If S is multiplicative, it simplifies the description of morphisms a great deal:

- Let $s : X \to Y$ and $t : Y \to Z$ be morphisms so that $s, t \in S$; then we know that $ts \in S$ as well. We get that $x_{ts}ts = \operatorname{id}_Z$ and consequently $x_{ts} = x_s x_t$. Thus multiplicative systems allow us to compose the formal inverses in $\mathcal{C}_S$ just as we compose the morphisms directly from $\mathcal{C}$.

- Suppose we have a morphism (path) in $\mathcal{C}_S$ given by $X_1 \xrightarrow{f} X_2 \xrightarrow{x_s} X_3$. We know that there exist morphisms g and t in $\mathcal{C}$ with $t \in S$ such that the following square commutes:

$$
\begin{array}{ccc}
X_2' & \xrightarrow{\;t\;} & X_1 \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\
X_3 & \xrightarrow{\;s\;} & X_2
\end{array}
$$

As $ft = sg$ we get, by multiplying with the formal inverses of s and t, that $x_s f = g x_t$.

With these properties, and remembering that a morphism $X_1 \xrightarrow{f} X_2 = X_1 \xrightarrow{\mathrm{id}_{X_1}} X_1 \xrightarrow{f} X_2$, we see that the general morphism in (2.1) is equivalent to a short chain $X_1 \xrightarrow{x_s} Y \xrightarrow{f} X_n$ where $s \in S$. This short chain is also known as a left fraction[2]. We will use this to give a third definition of $\mathcal{C}_S$.

Example 2.4. Let R be a domain, and define the category $\mathcal{R}$ as the category consisting of a single object $\bullet$, morphisms given by $\mathrm{Hom}_{\mathcal{R}}(\bullet, \bullet) = R$ with the composition in being multiplication in R. There is also an additive structure on $\mathcal{R}$, following from the addition of elements in R. Note that $\mathcal{R}$ is a small category, so we can use Gabriel-Zisman localization on it.
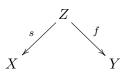
Define S as the set of morphisms not equal to 0. We consider the axioms of the multiplicative system, and see that **S1** is fulfilled, as we only consider rings with identities. **S2** is fulfilled because R is a domain, and so is **S4**: if $s \in S$, then since $sa = sb$ implies that $s(a - b)$ and s is not equal to zero, it must hold that $a = b$. Finally, we see that **S3** is fulfilled if R is a right and left Ore domain, so suppose R is an Ore domain

We know that there exists a category $\mathcal{R}_S$ and a functor $F : \mathcal{R} \to \mathcal{R}_S$ so that every non-zero element in S is turned into an isomorphism by F. Moreover, we know that any morphism in $\mathcal{R}_S$ is of the form rs^{-1} with $s \in S$ and $r \in R$ (and any element of this form is a morphism in $\mathcal{R}_S$). We can add any two elements as^{-1} and bt^{-1} by first using the fact that we can find two elements $p, q \in S$ so that $sp = tq$. We then define $as^{-1} + bt^{-1} = app^{-1}s^{-1} + bqq^{-1}t^{-1} = (ap + bq)(sp)^{-1}$; we can check that with this as the addition operation, $\mathrm{Hom}_{\mathcal{R}_S}(\bullet, \bullet)$ becomes a division ring. In other words we have shown the same thing as we did in theorem 1.3.

Note that while Gabriel-Zisman localization required $\mathcal{C}$ to be a small category, the below localization sets no such restriction.

Definition 2.5. Let $\mathcal{C}$ be a category and S a multiplicative system of morphisms in $\mathcal{C}$. The localization $\mathcal{C}_S$ of $\mathcal{C}$ by s is defined as follows:

- $\mathrm{Ob}(\mathcal{C}_S) = \mathrm{Ob}(\mathcal{C})$.

- The morphisms of $\phi \in \mathrm{Hom}_{\mathcal{C}_S}(X, Y)$ are equivalence classes of triples $\phi = (Z, s, f)$, where $Z \in \mathrm{Ob}(\mathcal{C}_S)$, $s : X' \to X$, $f : X' \to Y$ and $s \in S$, usually drawn as a "roof":

$$
\begin{array}{ccc}
 & Z & \\
{\scriptstyle s}\swarrow & & \searrow{\scriptstyle f} \\
X & & Y
\end{array}
$$

  Two morphisms $\phi = (Z, s, f)$ and $\phi' = (Z', s', f')$ between X and Y are set equivalent if there exists a commutative diagram.

---

[2]It is also equivalent to a short chain $X_1 \xrightarrow{g} Y \xrightarrow{x_t} X_n$ with $t \in S$, known as a right fraction.

(2.2)

We require that $(Z'', t, g) \in \mathrm{Hom}_{\mathcal{C}_S}(Z, Z')$. We will denote this relation as

$$(Z, s, f) \sim (Z', s', f').$$

Composition of a pair of morphisms $(U, s, f) \in \mathrm{Hom}_{\mathcal{C}_S}(X, Y)$ and $(V, t, g) \in \mathrm{Hom}_{\mathcal{C}_S}(Y, Z)$ is defined to be a morphism (W, sr, gh), given by the following commutative diagram:

We obtain W, r and h by the completion property (**S3**) of multiplicative systems.

We will show that $\mathcal{C}_S$ is a well-defined category, and then show that it gives a localization of $\mathcal{C}$ by S.

Lemma 2.6. *The relation $\sim$ is an equivalence relation.*

Proof. The relation is obviously reflexive. To see that it is symmetric, suppose that $(Z, s, f) \sim (Z', s', f')$ as shown in diagram (2.2). There exists a completion of

We see that $s'u = str = s'gr$, so by **S4** there exists a morphism $x : W' \to W$ so that $x \in S$ and $ux = grx$. Then we know that the following diagram commutes:

As $ux \in S$, we know that (W', ux, trx) is a morphism, so $(Z', s', f') \sim (Z, s, f)$.

Finally, suppose $(Z, s, f) \sim (Z', s', f')$ and $(Z', s', f') \sim (Z'', s'', f'')$; in other words that the following diagram commutes (with $r, p \in S$):

We are aiming to prove that $(Z, s, f) \sim (Z'', s'', f'')$ in order to show that the relation is transitive. Consider the completion of the diagram

We know that $s'hu = sru = s'pk$, and so there must exist a morphism $x : U' \to U$ such that $x \in S$ and $hux = pkx$. Setting $t = rux$ and $g = ikx$ we see that

$$st = srux = s'pkx = s''ikx = s''g$$
$$\text{and } ft = frux = f'hux = f'pkx = f''ikx = f''g.$$

This means that the following diagram commutes

and $(Z, s, f) \sim (Z'', s'', f'')$.   □

Lemma 2.7. *The composition of morphisms in $\mathcal{C}_S$ is well-defined with respect to the equivalence relation $\sim$.*

Proof. We divide this proof into two parts. First we show that two different compositions of the same two morphisms must be equivalent. Then we show that if the factors of two compositions are different but pairwise equivalent, then the compositions are equivalent as well.

First, suppose that we have two morphisms

$$
\begin{array}{ccc}
& U & \\
s \swarrow & & \searrow f \\
X & & Y
\end{array}
\qquad
\begin{array}{ccc}
& V & \\
t \swarrow & & \searrow g \\
Y & & Z
\end{array}
$$

Furthermore, suppose that the two following are both valid compositions of the morphisms:

We know that there exists an object T and morphisms x, y so that the following diagram commutes

$$
\begin{array}{ccc}
T & \xrightarrow{y} & W' \\
\downarrow x & & \downarrow r' \\
W & \xrightarrow{r} & V
\end{array}
$$

Both r and r' are in S, so we can select either $x \in S$ or $y \in S$. We choose the former. Since $rx = r'y$ we have that $thx = frx = fr'y = th'y$. Using axiom **S4** we see that there exists a morphism $w : T \to T'$, $w \in S$ so that $hxw = h'yw$. Setting $u = xw$ and $v = yw$ we see that $u \in S$ and that the following diagram commutes.

Therefore the two compositions are equivalent.

Next we want to show that if

$$
\begin{array}{ccc}
& U & \\
s \swarrow & & \searrow f \\
X & & Y
\end{array}
\qquad \sim \qquad
\begin{array}{ccc}
& U' & \\
s' \swarrow & & \searrow f' \\
X & & Y
\end{array}
$$

and

$$
\begin{array}{ccc}
& V & & & V' \\
t \swarrow & & \searrow g & \qquad \sim \qquad & t' \swarrow & & \searrow g' \\
Y & & Z & & Y & & Z
\end{array}
$$

then it holds that

$$(V, t, g)(U, s, f) = (W, sr, gh) \sim (W', s'r', g'h') = (V', t', g')(U', s', f').$$

To simplify this task we look at one factor of the composition at a time. Assume $V = V'$, $t = t'$ and $g = g'$. By $(U, s, f) \sim (U', s', f')$ we know that there must exist a morphism (A, a, a') so that the following diagram commutes:

$$
\begin{array}{c}
A \\
a \swarrow \quad \searrow a' \\
U \qquad\qquad U' \\
s \swarrow \quad s' \searrow \quad f \swarrow \quad \searrow f' \\
X \qquad\qquad\qquad\qquad Y
\end{array}
$$

Using axiom **S3** several times we arrive at the following commutative diagram:

$$
\begin{array}{c}
B \xrightarrow{\;x \in S\;} W \\
\end{array}
$$

Chasing the arrows, we get that $srxy = s'r'x'y'$ and $ghxy = gh'x'y'$. Thus we have confirmed the equivalence $(W, sr, gh) \sim (W', s'r', gh')$.

Next suppose instead that $U = U'$, $s = s'$ and $f = f'$. By $(V, t, g) \sim (V', t', g')$, we know that there exists a morphism (A, a, a') so that the following diagram commutes:

$$
\begin{array}{c}
A \\
a \swarrow \quad \searrow a' \\
V \qquad\qquad V' \\
t \swarrow \quad t' \searrow \quad g \swarrow \quad \searrow g' \\
X \qquad\qquad\qquad\qquad Y
\end{array}
$$

As $t'a' = ta \in S$, we know that there must exist an object B'' with morphisms so that the following diagram commutes:

$$
\begin{array}{ccc}
B'' & \overset{u}{\longrightarrow} & A \\
\downarrow{\scriptstyle v} & & \downarrow{\scriptstyle t'a'} \\
W' & \underset{t'h'}{\longrightarrow} & Y
\end{array}
$$

Note that $v \in S$. As $t'a'u = t'h'v$ and $t \in S$, we can find a morphism $B' \overset{w}{\to} B''$ in S so that $a'uw = h'vw$. For the sake of clarity we set $x' = vw$ and $b' = uw$. Let B be the completion

$$
\begin{array}{ccc}
B & \overset{b}{\longrightarrow} & A \\
\downarrow{\scriptstyle x} & & \downarrow{\scriptstyle a} \\
W' & \underset{h}{\longrightarrow} & Y
\end{array}
$$

where $x \in S$ because $a \in S$. Let C' be the completion

$$
\begin{array}{ccc}
C' & \overset{u}{\longrightarrow} & B \\
\downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle rx} \\
B' & \underset{r'x'}{\longrightarrow} & U
\end{array}
$$

We see that

$$
tabu = thxu = frxu = fr'x'u' = th'x'u = t'a'b'u'.
$$

Again, as $t'a' = ta \in S$ we get that there exists a morphism $C \overset{v}{\to} C' \in S$ so that $buv = b'u'v$. Setting $y = uv$ and $y' = u'v'$ we get the following commutative diagram:

We see that $srxy = s'r'x'y'$, and also $ghxy = g'h'x'y'$. Thus $(W, sr, gh) \sim (W, sr', g'h')$.

Thus it follows that for two pairs of equivalent (and compatible) morphisms so that $(U, s, f) \sim (U', s', f')$ and $(V, t, g) \sim (V', t', g')$ it holds that

$$
(W, sr, gh) = (V, t, g)(U, s, f) \sim (V, t, g)(U', s', f')
$$
$$
\sim (V', t', g')(U', s', f') = (W', s'r', g'h').
$$

Thus the composition of two roofs is well-defined with respect to the equivalence relation on $\mathcal{C}_S$. □

Theorem 2.8. *In definition 2.5, the given $\mathcal{C}_S$ is a category, and there exists a functor $\mathcal{C} \to \mathcal{C}_S$ so that the two form a localization of $\mathcal{C}$ with respect to S.*

Proof. To see that $\mathcal{C}_S$ is a category, we really only need to check the axioms on the morphisms. That identities exist in $\mathcal{C}_S$ is obvious; for $X \in \mathrm{Ob}(\mathcal{C}_S)$ the morphism $(X, \mathrm{id}_X, \mathrm{id}_X)$ is the identity.

Next we consider associativity. Suppose that we have three morphisms $(U, r, f) : X \to Y$, $(V, s, g) : Y \to Z$ and $(W, t, h) : Z \to Z'$. We need to show that

$$((W, t, h)(V, s, g))(U, r, f) = (W, t, h)((V, s, g)(U, r, f)),$$

which means that for the two diagrams

and

which give the two possible compositions of the morphisms, it holds that

$$(B, rab, hb') \sim (D, rd, hc'd').$$

As both (D, rd, gcd') and (A, ra, ga') are compositions of (U, r, f) and (V, s, g), we see that $(D, rd, gcd') \sim (A, ra, ga')$. Thus there must exist a morphism $(E, e, e') : A \to D$ so that $ga'e = gcd'e' = tc'd'e'$. Using axiom **S3** on the morphisms $B \xrightarrow{ab} U$ and $D \xrightarrow{d} U$, we

see that there exists an object X and morphisms u and u' so that the following diagram commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{\;u'\;} & D \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle d} \\
B & \xrightarrow[ab]{} & U
\end{array}
$$

We choose $u \in S$ (as both ab and d are in S, we are free to chose). We see that

$$abu = du'$$
$$fabu = fdu'$$
$$sa'bu = scd'u'$$
$$a'bv = cdv'$$
$$ga'bv = gcdv$$
$$tb'v = tc'd'v'$$
$$b'w = c'd'w'.$$

The morphisms v, v', w and w' are defined using axiom **S4**. We see that $rabw = rdw'$ and $hb'w = hc'd'w'$. Thus the morphisms (B, rab, hb') and $(D, rd, hc'd')$ are equivalent, composition of morphisms is an associative operation and $\mathcal{C}_S$ is a category.

We define the functor $F : \mathcal{C} \to \mathcal{C}_S$ on objects by letting $F(X) = X$ for $X \in \mathrm{Ob}(\mathcal{C})$. For a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ we set $F(f) = (X, \mathrm{id}_X, f)$. Then

$$F(\mathrm{id}_X) = (X, \mathrm{id}_X, \mathrm{id}_X),$$

which is the identity on $F(X)$ and for two morphisms $f : X \to Y$ and $g : Y \to Z$ it holds that

$$F(gf) = (X, \mathrm{id}_X, fg) = (Y, \mathrm{id}_X, g)(X, \mathrm{id}_X, f) = F(g)F(f).$$

Thus F is a functor.

Finally, we need to verify that $\mathcal{C}_S$ and F is a localization of $\mathcal{C}$ with respect to S. If $X \xrightarrow{s} Y \in S$, then the morphism (X, s, id_X) is a two-sided inverse of $F(s) = (X, \mathrm{id}_X, s)$, so $F(s)$ is an isomorphism. It remains to check that F is universal.

Suppose that $G : \mathcal{C} \to \mathcal{D}$ is a functor so that $G(s)$ is an isomorphism for any $s \in S$. We need a functor $H : \mathcal{C}_S \to \mathcal{D}$ so that $G = HF$, and we define it by looking at the requirements set for it.

- For $X \in \mathrm{Ob}(\mathcal{C})$ we must have $G(X) = HF(X) = H(X)$, as $F(X) = X$.

- For $(Z, s, f) \in \mathrm{Hom}_{\mathcal{C}_S}(X, Y)$, we have that

$$(Z, s, f)(Z, \mathrm{id}_Z, s) = (Z, \mathrm{id}_Z, f)$$
$$H((Z, s, f)(Z, \mathrm{id}_Z, s)) = H(Z, \mathrm{id}_Z, f)$$
$$H(Z, s, f)G(s) = H(Z, s, f)HF(s) = HF(f) = G(f)$$

We see that we must have $H(Z, s, f) = G(f)G(s)^{-1}$.

The given H is a well-defined functor from $\mathcal{C}_S$ to $\mathcal{D}$, and we also see that it is unique (the definition of H follows from the requirements on any such functor). It remains to check that H is a functor. That H maps identities to identities is easy to see. Letting $(Y, t, g)(X, s, f) = (Z, sr, gh)$ be a composition of morphism in $\mathcal{C}_S$, we see that

$$
\begin{aligned}
H((Y, t, g)(X, s, f)) &= H(Z, sr, gh) \\
&= G(gh)G(sr)^{-1} \\
&= G(g)G(h)G(r)^{-1}G(s)^{-1} \\
&= G(g)G(t)^{-1}G(t)G(h)G(r)^{-1}G(s)^{-1} \\
&= G(g)G(t)^{-1}G(f)G(r)G(r)^{-1}G(s)^{-1} \\
&= G(g)G(t)^{-1}G(f)G(s)^{-1} \\
&= H(Y, t, g)H(X, s, f).
\end{aligned}
$$

Thus H respects composition, and is a functor.

$\square$

We show that this localization by multiplicative systems preserves addition. This is an important property of localization by a multiplicative systems; in contrast to Gabriel-Zisman localization which generally does not preserve addition.

Theorem 2.9. *[5, ch. III.4.5] Let $\mathcal{C}$ be an additive category and let S be a multiplicative system. Then $\mathcal{C}_S$ is additive, and the localization functor $F : \mathcal{C} \to \mathcal{C}_S$ is an additive functor[3].*

Proof. We prove the theorem by construction; we will first infer a reasonable definition of the addition operation on $\mathcal{C}_S$ and then prove that this operation makes $\mathcal{C}_S$ additive and F an additive functor

Since we want F to be additive, we start by noting that for two morphisms $f, g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ we need

$$
(X, \mathrm{id}_X, f) + (X, \mathrm{id}_X, g) = F(f) + F(g) = F(f + g) = (X, \mathrm{id}_X, f + g).
$$

We extend this for two morphisms $(U, s, f), (U, s, g) \in \mathrm{Hom}_{\mathcal{C}_S}(X, Y)$ by defining $(U, s, f) + (U, s, g) = (U, s, f + g)$. To see that this is well-defined, suppose $(U, s, f) \sim (V, t, f')$ by the roof (W, u, u') and $(U, s, g) \sim (V, t, g')$ by the roof (Z, v, v'). We can find the following square completions

$$
\begin{array}{ccc}
T \xrightarrow{w'} Z \quad & S \xrightarrow{x'} Z \quad & R \xrightarrow{y'} S \\
\downarrow{w} \quad \downarrow{v} \quad & \downarrow{x} \quad \downarrow{v'} \quad & \downarrow{y} \quad \downarrow{x} \\
W \xrightarrow{u} U \quad & W \xrightarrow{u'} U \quad & T \xrightarrow{w} W
\end{array}
$$

---

[3] A functor that acts as a group homomorphism on the Hom-sets; see definition 3.2

where $w, x, y \in S$. Obviously,

$$suwy = tu'wy = tu'xy' = tv'x'y'$$

and moreover,

$$(f + g)uwy = (f + g)uxy' = f'u'xy' + gvx'y = (f' + g')v'x'y'.$$

Thus $(U, s, f + g) \sim (V, t, f' + g')$ and addition is well-defined.

Suppose we have $(U, s, f), (V, r, g) \in \mathrm{Hom}_{\mathcal{C}_S}(X, Y)$, and let (W, p, q) be the roof defined by the following completion:

$$
\begin{array}{ccc}
W & \xrightarrow{\ q\ } & V \\
{\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle t} \\
U & \xrightarrow{\ s\ } & X
\end{array}
$$

Define $r = sp = tq$; then $r \in S$. Moreover, define $f' = fp$ and $g' = gq$. It follows that $(U, s, f) \sim (W, r, f')$ and $(V, t, g) \sim (W, r, g')$, so we can define

$$(U, s, f) + (V, t, g) = (W, r, f') + (W, r, g') = (W, r, f' + g').$$

For $\mathcal{C}_S$ to be additive under this operation, we need to show that $\mathrm{Hom}_{\mathcal{C}_S}(X, Y)$ is an abelian group. We start by showing that addition is associative:

$$
\begin{aligned}
((U, s, f) + (V, t, g)) + (W, r, h) &= ((R, x, f') + (R, x, g')) + (R, x, h') \\
&= (R, x, (f' + g') + h') = (R, x, f' + (g' + h')) \\
&= (U, s, f) + ((V, t, g) + (W, r, h))
\end{aligned}
$$

We see that $(X, \mathrm{id}_X, 0)$ is the zero element for the operation, and that $(U, s, f) \in \mathrm{Hom}_{\mathcal{C}_S}(X, Y)$ has an inverse $(U, s, -f) \in \mathrm{Hom}_{\mathcal{C}_S}(X, Y)$. Commutativity follows from commutativity of $+$ in $\mathcal{C}$. Thus $\mathrm{Hom}_{\mathcal{C}_S}(X, Y)$ is an abelian group.

That composition is bilinear and that finite direct sums exist follows from the additivity of $\mathcal{C}$ in a similar manner. Thus $\mathcal{C}_S$ is an additive category. Moreover, the function $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}_S}(X, Y)$, obtained from the localization functor F is a group homomorphism, so the localization functor is an additive functor. $\qquad\square$

3 Localization of a triangulated category

Many convenient structures arise when a category is abelian; the most important is perhaps the exact sequence. Sometimes when we are working in an additive category, we can mimic the exact sequence even if our category is not abelian; namely by defining distinguished triangles. We will use it in section 4 when studying the homotopy category in order to find the derived category. More specifically we will use the fact, shown in section 3.2, that the structure of distinguished triangles makes it possible to define a multiplicative system from a collection of objects. More literature on triangulated categories can be found in [9] and [10].

3.1   Triangulated categories

Let $\mathcal{C}$ be a category, and let Σ be a functor $\mathcal{C} \xrightarrow{\Sigma} \mathcal{C}$ which is an equivalence of categories (or simply an "equivalence"). A *triangle* in $\mathcal{C}$ is defined as a sequence of maps

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma(X).$$

A morphism between two triangles $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma(X)$ and $X' \xrightarrow{\alpha'} Y' \xrightarrow{\beta'} Z' \xrightarrow{\gamma'} \Sigma(X')$ is a triple of maps (ϕ_X, ϕ_Y, ϕ_Z) so that the following diagram commutes:

$$
\begin{array}{ccccccc}
X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \xrightarrow{\gamma} & \Sigma(X) \\
\downarrow{\phi_X} & & \downarrow{\phi_Y} & & \downarrow{\phi_Z} & & \downarrow{\Sigma\phi_X} \\
X' & \xrightarrow{\alpha'} & Y' & \xrightarrow{\beta'} & Z' & \xrightarrow{\gamma'} & \Sigma(X')
\end{array}
$$

If all three morphisms ϕ_X, ϕ_Y and ϕ_Z are isomorphisms, this is known as an isomorphism of triangles.

We are now ready to give the definition of a triangulated category.

Definition 3.1. Let $\mathcal{C}$ be an additive category, and let Σ be an equivalence $\mathcal{C} \xrightarrow{\Sigma} \mathcal{C}$. Then $\mathcal{C}$ is know as a *triangulated category* if there exists a family of triangles, called *distinguished triangles*, satisfying the following conditions:

T1 Any triangle isomorphic to a distinguished triangle is distinguished.

T2 For any $X \in \mathrm{Ob}(\mathcal{C})$, the triangle $X \xrightarrow{\mathrm{id}_X} X \to 0 \to \Sigma(X)$ is distinguished.

T3 For any $X, Y \in \mathrm{Ob}(\mathcal{C})$ and any morphism $f \in \mathrm{Hom}(X, Y)$, there exists a distinguished triangle $X \xrightarrow{f} Y \to Z \to \Sigma(X)$.

T4 The triangle $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma(X)$ is distinguished if and only if $Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma(X) \xrightarrow{-\Sigma\alpha} \Sigma(Y)$ is distinguished.

T5 A commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{u} & & \downarrow{v} \\
X' & \xrightarrow{f'} & Y'
\end{array}
$$

can be embedded in a morphism of distinguished triangles as follows[4]:

$$
\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & \Sigma(X) \\
\downarrow{u} & & \downarrow{v} & & \downarrow & & \downarrow{\Sigma(u)} \\
X' & \xrightarrow{f'} & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma(X')
\end{array}
$$

---

[4]The existence of the distinguished triangles themselves is given by axiom **T3**

T6 Given the following distinguished triangles:

$$X \xrightarrow{\ f\ } Y \longrightarrow Z' \longrightarrow \Sigma(X)$$

$$Y \xrightarrow{\ g\ } Z \longrightarrow X' \longrightarrow \Sigma(Y)$$

$$X \xrightarrow{\ gf\ } Z \longrightarrow Y' \longrightarrow \Sigma(X)$$

there exists a distinguished triangle $Z' \to Y' \to X' \to \Sigma(Z')$ so that the following diagram commutes:

$$
\begin{array}{ccccccc}
X & \xrightarrow{\ f\ } & Y & \longrightarrow & Z' & \longrightarrow & \Sigma(X) \\
{\scriptsize \mathrm{id}_X}\downarrow & & {\scriptsize g}\downarrow & & \downarrow & & \downarrow{\scriptsize \mathrm{id}_{\Sigma(X)}} \\
X & \xrightarrow{\ gf\ } & Z & \longrightarrow & Y' & \longrightarrow & \Sigma(X) \\
{\scriptsize f}\downarrow & & {\scriptsize \mathrm{id}_Z}\downarrow & & \downarrow & & \downarrow{\scriptsize \Sigma f} \\
Y & \xrightarrow{\ g\ } & Z & \longrightarrow & X' & \longrightarrow & \Sigma(Y) \\
\downarrow & & \downarrow & & {\scriptsize \mathrm{id}_{X'}}\downarrow & & \downarrow \\
Z' & \longrightarrow & Y' & \longrightarrow & X' & \longrightarrow & \Sigma(Z')
\end{array}
$$

In a triangulated category, the functor Σ is often called the *translation* or *suspension* functor.

In the following discussion, we will need some restrictions on the functors we use. The first one is to preserve additivity.

Definition 3.2. A functor between additive categories, $F : \mathcal{C} \to \mathcal{C}'$ is called *additive* if it acts as a group homomorphism from $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ to $\mathrm{Hom}_{\mathcal{C}'}(F(X), F(Y))$.

One example of an additive functor is the covariant hom-functor $\mathrm{Hom}_{\mathcal{C}}(X, -)$ (the contravariant Hom-functor, $\mathrm{Hom}_{\mathcal{C}}(-, X)$, is additive as well).

Note that if the functor F is additive, then $F(X \oplus Y) = F(X) \oplus F(Y)$.

The second type of functors we define preserves distinguished triangles.

Definition 3.3. Let $\mathcal{C}$ and $\mathcal{C}'$ be two triangulated categories with Σ and Σ' their respective equivalences. An additive functor $F : \mathcal{C} \to \mathcal{C}'$ is called a *functor of triangulated categories* if $F\Sigma = \Sigma'F$ and F maps distinguished triangles to distinguished triangles.

The last type of functors does not preserve triangles, but transforms them into exact sequences. However, as it can be argued that the distinguished triangles in a triangulated category serve a function analogous to that of exact sequences in abelian categories, we could see these functors as preserving this role.

Definition 3.4. Let $\mathcal{C}$ be a triangulated category, and let $\mathcal{A}$ be an abelian category. A functor $F : \mathcal{C} \to \mathcal{A}$ is called a *cohomological functor* if for any distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \to \Sigma(X)$$

in $\mathcal{C}$, the sequence

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$$

is exact in $\mathcal{A}$.

Note that due to **T4**, we actually get a long exact sequence

$$\cdots \to F(Z[-1]) \to F(X) \to F(Y) \to F(Z) \to F(X[1]) \to \cdots$$

We now give a classic example of a cohomological functor, which leads to a rather useful theorem.

Example 3.5. Let $\mathcal{C}$ be a triangulated category, let $W \in \mathrm{Ob}(\mathcal{C})$, and consider the functor $\mathrm{Hom}_{\mathcal{C}}(W, -)$. As $\mathcal{C}$ is triangulated, it is also additive, so its Hom-sets are abelian groups. Thus $\mathrm{Hom}_{\mathcal{C}}(W, -)$ is a functor into the category of abelian groups, **Ab**, which is certainly an abelian category.

Let $X \xrightarrow{f} Y \xrightarrow{g} Z \to \Sigma(X)$ be a distinguished triangle in $\mathcal{C}$. We need to study the following sequence:

$$\mathrm{Hom}_{\mathcal{C}}(W, X) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(W,f)} \mathrm{Hom}_{\mathcal{C}}(W, Y) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(W,g)} \mathrm{Hom}_{\mathcal{C}}(W, Z)$$

As the square

$$\begin{array}{ccc}
X & \xrightarrow{\mathrm{id}_X} & X \\
{\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle f} \\
X & \xrightarrow{f} & Y
\end{array}$$

commutes, it can be embedded in a morphism of triangles as follows:

$$\begin{array}{ccccccc}
X & \xrightarrow{\mathrm{id}_X} & X & \longrightarrow & 0 & \longrightarrow & \Sigma(X) \\
{\scriptstyle \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\Sigma(X)}} \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & \Sigma(X)
\end{array}$$

Thus $gf = 0$, and also $\mathrm{Hom}_{\mathcal{C}}(W, g)\,\mathrm{Hom}_{\mathcal{C}}(W, f) = 0$, so

$$\mathrm{Im}(\mathrm{Hom}_{\mathcal{C}}(W, f)) \subseteq \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(W, g)).$$

Since we are working in **Ab**, we can prove the reverse inclusion just by looking at elements. Let $\phi \in \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(W, g)) \subseteq \mathrm{Hom}_{\mathcal{C}}(W, Y)$; then $g\phi = 0$. Since the following square commutes,

$$\begin{array}{ccc}
W & \longrightarrow & 0 \\
{\scriptstyle \phi}\downarrow & & \downarrow \\
Y & \xrightarrow{g} & X
\end{array}$$

it can be embedded in a morphism of triangles as follows:

$$
\begin{array}{ccccccc}
W & \xrightarrow{\mathrm{id}_W} & W & \longrightarrow & 0 & \longrightarrow & \Sigma(W) \\
\downarrow{\psi} & & \downarrow{\phi} & & \downarrow & & \downarrow{\Sigma(\psi)} \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & \Sigma(X)
\end{array}
$$

As $\phi = f\psi \in \mathrm{Im}(\mathrm{Hom}_{\mathcal{C}}(W, f))$, it follows that $\mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(W, g)) \subseteq \mathrm{Im}(\mathrm{Hom}_{\mathcal{C}}(W, f))$, and $\mathrm{Hom}_{\mathcal{C}}(W, -)$ is a cohomological functor.

It can be shown that $\mathrm{Hom}_{\mathcal{C}}(-, W)$ is a cohomological functor as well.

The following theorem turns our attention back to the internal workings of the triangulated category.

Theorem 3.6. *Let*

$$
\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma(X) \\
\downarrow{\phi} & & \downarrow{\psi} & & \downarrow{\theta} & & \downarrow{\Sigma(\phi)} \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma(X')
\end{array}
$$

be a morphism of triangles in a triangulated category $\mathcal{C}$. If ϕ and ψ are isomorphisms, then so is θ.

Proof. As we know that $\mathrm{Hom}(Z, -)$ is cohomological, we also know that the following commutative diagram has exact rows:

$$
\begin{array}{ccccccccc}
\mathrm{Hom}(Z', X) & \twoheadrightarrow & \mathrm{Hom}(Z', Y) & \twoheadrightarrow & \mathrm{Hom}(Z', Z) & \twoheadrightarrow & \mathrm{Hom}(Z', \Sigma(X)) & \twoheadrightarrow & \mathrm{Hom}(Z', \Sigma(Y)) \\
\downarrow{\mathrm{Hom}(Z',\phi)} & & \downarrow{\mathrm{Hom}(Z',\psi)} & & \downarrow{\mathrm{Hom}(Z',\theta)} & & \downarrow{\mathrm{Hom}(Z',\Sigma(\phi))} & & \downarrow{\mathrm{Hom}(Z',\Sigma(\psi))} \\
\mathrm{Hom}(Z', X') & \twoheadrightarrow & \mathrm{Hom}(Z', Y') & \twoheadrightarrow & \mathrm{Hom}(Z', Z') & \twoheadrightarrow & \mathrm{Hom}(Z', \Sigma(X')) & \twoheadrightarrow & \mathrm{Hom}(Z', \Sigma(Y))
\end{array}
$$

By the five lemma (lemma A.1), the fact that $\mathrm{Hom}(Z', \phi)$, $\mathrm{Hom}(Z', \psi)$, $\mathrm{Hom}(Z', \Sigma(\phi))$ and $\mathrm{Hom}(Z', \Sigma(\psi))$ are isomorphisms implies that $\mathrm{Hom}(Z', \theta)$ is an isomorphism. Particularly, is is surjective (and $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a set), so there exists a morphism $u \in \mathrm{Hom}(Z', Z)$ so that $\theta u = \mathrm{id}_{Z'} \in \mathrm{Hom}(Z', Z')$. Unfortunately, so far we only know that u is a right inverse of θ. However, if we redo the above proof with the following morphism of triangles

$$
\begin{array}{ccccccc}
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma(X') \\
\downarrow{\phi^{-1}} & & \downarrow{\psi^{-1}} & & \downarrow{u} & & \downarrow{\Sigma(\phi)^{-1}} \\
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma(X)
\end{array}
$$

we see that u has a right inverse v so that $uv = \mathrm{id}_Z$. Since θ is a left inverse of u it follows that u is an isomorphism, which means that $v = \theta$, and so θ is an isomorphism. $\square$

It may seem cumbersome to use cohomological functors to prove a property that is intrinsic to triangulated categories, but the only thing it is used for is to show a sequence of homomorphism groups to be exact. We could have shown the result without using the cohomological property but the proof would not have been as elegant.

3.2   Localization by a null system

So far, when we have studied localization it has been done by a class of morphisms (typically called S). However, we might prefer to consider a class of objects instead, and by the process of localization set the objects isomorphic to the zero object. The name for such a class is a *null system.*

Definition 3.7. Let $\mathcal{C}$ be a category triangulated with respect to the equivalence $\Sigma : \mathcal{C} \to \mathcal{C}$. A subclass N of $\mathrm{Ob}(\mathcal{C})$ is called a null system if:

N1 $0 \in N$.

N2 $X \in N$ if and only if $\Sigma(X) \in N$.

N3 If $X, Y \in N$ and $X \to Y \to Z \to \Sigma(X)$ is a distinguished triangle, then $Z \in N$.

Having a null system, we can define a multiplicative system.

Theorem 3.8. *Let N be a null system in the triangulated category $\mathcal{C}$. Define the class of morphisms $S(N)$ by*

$$S(N) = \{f : X \to Y \mid \exists \text{ a distinguished triangle } X \xrightarrow{f} Y \to Z \to \Sigma(X) \text{ with } Z \in S\}.$$

Then $S(N)$ is a multiplicative system.

Proof. We will prove the theorem by checking the conditions of definition 2.3 in order.

S1 For any $X \in \mathrm{Ob}\,\mathcal{C}$ we know that id_X can be embedded in a distinguished triangle $X \xrightarrow{\mathrm{id}_X} X \to 0 \to \Sigma(X)$. As $0 \in N$, it follows that $\mathrm{id}_X \in S(N)$

S2 Suppose $f : X \to Y$ and $g : Y \to Z$ are both elements in $S(N)$, in other words that there exists distinguished triangles

$$X \xrightarrow{f} Y \to Z' \to \Sigma(X) \text{ and } Y \xrightarrow{g} Z \to X' \to \Sigma(Y)$$

with $X', Z' \in N$. From axiom **T3** of triangulated categories we know that there exists a distinguished triangle $X \xrightarrow{gf} Z \to Y' \to \Sigma(X)$. Then by **T6** there must exist a distinguished triangle $Z' \to Y' \to X' \to \Sigma(Z')$. Applying **T4** twice we see that the triangle $X' \to \Sigma(Z') \to \Sigma(Y') \to \Sigma(X')$ is distinguished. Then we know that $\Sigma(Y') \in N$ and thus we have $Y' \in N$. It follows that $gf \in S(N)$.

S3 Suppose there exist morphisms $f : X \to Y$ and $x : Z \to Y$ with $f \in S(N)$. Then we know that there exists a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{k} Z' \to \Sigma(X)$, with $Z' \in N$. From **T3** and **T4** it follows that there exists a distinguished triangle $W \xrightarrow{g} Z \xrightarrow{kx} Z' \to \Sigma(W)$. As the following diagram is commutative

$$
\begin{array}{ccc}
Z & \xrightarrow{kx} & Z' \\
\downarrow{\scriptstyle x} & & \downarrow{\scriptstyle \mathrm{id}_{Z'}} \\
Y & \xrightarrow{k} & Z'
\end{array}
$$

it can be embedded in a morphism of distinguished triangles in the following manner:

$$
\begin{array}{ccccccc}
Z & \xrightarrow{kx} & Z' & \longrightarrow & \Sigma(W) & \xrightarrow{\Sigma(g)} & \Sigma(Z) \\
\downarrow{\scriptstyle x} & & \downarrow{\scriptstyle \mathrm{id}_{Z'}} & & \downarrow{\scriptstyle \Sigma(y)} & & \downarrow{\scriptstyle \Sigma(x)} \\
Y & \xrightarrow{k} & Z' & \longrightarrow & \Sigma(X) & \xrightarrow{\Sigma(f)} & \Sigma(Y)
\end{array}
$$

Using axiom **T4** to shift the diagram, we see that this ensures that the following is a morphism of distinguished triangles:

$$
\begin{array}{ccccccc}
W & \xrightarrow{g} & Z & \xrightarrow{kx} & Z' & \longrightarrow & \Sigma(W) \\
\downarrow{\scriptstyle y} & & \downarrow{\scriptstyle x} & & \downarrow & & \downarrow \\
X & \xrightarrow{f} & Y & \xrightarrow{k} & Z' & \longrightarrow & \Sigma(X)
\end{array}
$$

Since $Z' \in N$ we have that $g \in S(N)$, and the axiom is satisfied.

S4 First note that in an additive category this axiom is equivalent to saying that if $sf = 0$ with $s \in S$, then there exists $t \in S$ such that $ft = 0$ (and vice versa).

Therefore we suppose that for a morphism $f : Y \to Z$ there exists an $s \in S(N)$ such that $sf = 0$. Then, by the definition of $S(N)$ and **T4** there exists a distinguished triangle $Z \xrightarrow{g} Y \xrightarrow{s} Y' \to \Sigma(Z)$, where $Z \in N$. As the diagram

$$
\begin{array}{ccc}
X & \longrightarrow & 0 \\
\downarrow{\scriptstyle f} & & \downarrow \\
Y & \xrightarrow{s} & Y'
\end{array}
$$

commutes, it can be embedded in a morphism of distinguished triangles as follows:

$$
\begin{array}{ccccccc}
X & \xrightarrow{\mathrm{id}_X} & X & \longrightarrow & 0 & \longrightarrow & \Sigma(X) \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} & & \downarrow & & \downarrow \\
Z & \xrightarrow{g} & Y & \xrightarrow{s} & Y' & \longrightarrow & \Sigma(Z)
\end{array}
$$

By **T3** and **T4**, there exists a distinguished triangle $W \xrightarrow{t} X \xrightarrow{h} Z \to \Sigma(W)$. Note that since $Z \in N$ we must have $t \in S(N)$. Knowing that the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\;h\;} & Z \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\
Y & \xrightarrow{\;\mathrm{id}_Y\;} & Y
\end{array}
$$

commutes, we embed it in the following morphism of triangles:

$$
\begin{array}{ccccccc}
W & \xrightarrow{\;t\;} & X & \xrightarrow{\;h\;} & Z & \longrightarrow & \Sigma(U) \\
\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow \\
0 & \longrightarrow & Y & \xrightarrow{\;\mathrm{id}_Y\;} & Y & \longrightarrow & 0
\end{array}
$$

It follows that $ft = 0$. The reverse implication is proved in a highly similar manner.
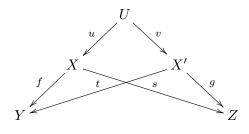
$\square$

To know that we can define a multiplicative system $S(N)$ from the null system N is not as interesting as to know that we have found the *right one*; in other words that localization by $S(N)$ preserves triangulation (since $\mathcal{C}$ is triangulated) and takes elements of N to zero. Perhaps most importantly, we want to check that the localization is universal.

Theorem 3.9. *Let $\mathcal{C}$ be a triangulated category and N a null system in $\mathcal{C}$. Then we know that*

(i) *The localization of $\mathcal{C}$ by $S(N)$, written $\mathcal{C}/N$, is triangulated. Moreover the localization functor $F : \mathcal{C} \to \mathcal{C}/N$ is a functor between triangulated categories.*

(ii) *$F(X) \cong 0$ for any $X \in N$.*

(iii) *The localization functor is universal among triangulated functors that have the above property.*

Proof.   (i) Suppose $\mathcal{C}$ is triangulated with respect to the equivalence functor Σ. By theorem 2.9, the category $\mathcal{C}/N$ must be additive. We define the functor $\Sigma_{\mathcal{C}/N} : \mathcal{C}/N \to \mathcal{C}/N$ by setting $\Sigma_{\mathcal{C}/N}(X) = \Sigma(X)$ for an object $X \in \mathrm{Ob}(\mathcal{C}/N) = \mathrm{Ob}(\mathcal{C})$, and setting $\Sigma_{\mathcal{C}/N}(X, s, f) = (\Sigma(X), \Sigma(s), \Sigma(f))$ for a morphism $(X, s, f) : Y \to Z$. To show that the latter is well-defined note first that since Σ is a functor we know that $\Sigma_{\mathcal{C}/N}(X, s, f)$ is a morphism $\Sigma_{\mathcal{C}/N}(Y) \to \Sigma_{\mathcal{C}/N}(Z)$. It remains to check that $\Sigma_{\mathcal{C}/N}$ will map two equivalent morphisms to the same equivalence class. Since Σ

is a functor, if this diagram commutes

$$
\begin{array}{c}
U \\
\swarrow u \quad v \searrow \\
X \quad\quad\quad X' \\
f \swarrow \quad t \quad s \quad \searrow g \\
Y \quad\quad\quad\quad\quad Z
\end{array}
$$

then so does this:

$$
\begin{array}{c}
\Sigma(U) \\
\Sigma(u) \swarrow \quad \searrow \Sigma(v) \\
\Sigma(X) \quad\quad\quad \Sigma(X') \\
\Sigma(f) \swarrow \; \Sigma(t) \quad \Sigma(s) \; \searrow \Sigma(g) \\
\Sigma(Y) \quad\quad\quad\quad\quad \Sigma(Z)
\end{array}
$$

The latter gives an equivalence between the image of two equivalent morphisms. Thus $\Sigma_{\mathcal{C}/N}$ is well-defined as a map. That it is a functor and furthermore an equivalence of categories follows directly from the that fact Σ is. Moreover we see that $F\Sigma = \Sigma_{\mathcal{C}/N}F$. We will denote $\Sigma_{\mathcal{C}/N}$ as simply Σ when there is no risk of confusion.

Let a triangle in $\mathcal{C}/N$ be distinguished if and only if it is isomorphic to the image of a distinguished triangle in $\mathcal{C}$. We check that the triangle axioms are satisfied:

T1 Follows from the definition of distinguished triangles in $\mathcal{C}/N$.

T2 Suppose $X \in \mathrm{Ob}(\mathcal{C}/N) = \mathrm{Ob}(\mathcal{C})$. The triangle $X \xrightarrow{(X,\mathrm{id}_X,\mathrm{id}_X)} X \to 0 \to \Sigma(X)$ is isomorphic (by the identity isomorphism) to $F(X) \xrightarrow{F(\mathrm{id}_X)} F(X) \to 0 \to \Sigma(X)$ and is thus distinguished.

T3 Suppose $(U,s,f) : X \to Y$ is a morphism in $\mathcal{C}/N$. We know that (U,s,id_U) is an isomorphism (because (U,id_U,s) is its inverse). Consider the following morphism of triangles:

$$
\begin{array}{ccccccc}
X & \xrightarrow{(U,s,f)} & Y & \longrightarrow & Z & \longrightarrow & \Sigma(X) \\
{\scriptstyle (U,s,\mathrm{id}_U)} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_Y} & & \downarrow {\scriptstyle \mathrm{id}_Z} & & \downarrow {\scriptstyle \Sigma(U,s,\mathrm{id}_U)} \\
U & \xrightarrow[(U,\mathrm{id}_U,f)]{} & Y & \longrightarrow & Z & \longrightarrow & \Sigma(U)
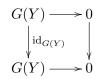\end{array}
$$

As the diagram shows an isomorphism of triangles, and the lower triangle is the image of a distinguished triangle (the embedding of f in a triangle), the upper triangle is distinguished as well.

T4 Suppose $X \xrightarrow{\phi} Y \xrightarrow{\psi} Z \xrightarrow{\theta} \Sigma(X)$ is a distinguished triangle in $\mathcal{C}/N$. This is true if and only if it is isomorphic to the image of a distinguished triangle $X' \xrightarrow{\phi'} Y' \xrightarrow{\psi'} Z' \xrightarrow{\theta'} \Sigma(X')$ in $\mathcal{C}$. This triangle is distinguished if and only if $Y' \xrightarrow{\psi'} Z' \xrightarrow{\theta'} \Sigma(X') \xrightarrow{\Sigma(\phi')} \Sigma(Y')$ is distinguished. However, the image of this triangle is isomorphic to $Y \xrightarrow{\psi} Z \xrightarrow{\theta} \Sigma(X) \xrightarrow{\Sigma(\phi)} \Sigma(Y)$, which must be distinguished.

T5 and **T6** follows from image chasing proofs similar to the one for **T4**.

We have already seen that $F\Sigma = \Sigma_{\mathcal{C}/N}F$, and the image of a distinguished triangle in $\mathcal{C}$ is obviously distinguished, so F is a functor between triangulated categories.

(ii) Consider $X \in N$. The triangle $X \xrightarrow{\mathrm{id}_X} X \to 0 \to \Sigma(X)$ is distinguished in $\mathcal{C}$. So must $X \to 0 \to \Sigma(X) \to \Sigma(X)$ be. Thus the morphism $X \to 0$ is in $S(N)$. Its image will be an isomorphism in $\mathcal{C}/N$ and thus $F(X) \cong 0$.

(iii) Suppose $G : \mathcal{C} \to \mathcal{C}'$ is a functor between triangulated categories so that $G(X) \cong 0$ for all $X \in N$. Suppose $f \in S(N)$, then there exists a distinguished triangle $X \xrightarrow{f} Y \to Z \to \Sigma(X)$ in $\mathcal{C}$ where $Z \in N$. The functor G maps this to the distinguished triangle $G(X) \xrightarrow{G(f)} G(Y) \to 0 \to \Sigma(G(X))$ (we have used the fact that $G(Z) \cong 0$ and $G\Sigma = \Sigma'G$).

It is trivial to see that the following square commutes:

$$
\begin{array}{ccc}
G(Y) & \longrightarrow & 0 \\
\Big\downarrow{\scriptstyle \mathrm{id}_{G(Y)}} & & \Big\downarrow \\
G(Y) & \longrightarrow & 0
\end{array}
$$

Furthermore we can embed it in a morphism of distinguished triangles in $\mathcal{C}'$ as follows:

$$
\begin{array}{ccccccc}
G(Y) & \xrightarrow{\mathrm{id}_{G(Y)}} & G(Y) & \longrightarrow & 0 & \longrightarrow & \Sigma'G(Y) \\
{\scriptstyle f^*}\Big\downarrow & & {\scriptstyle \mathrm{id}_{G(Y)}}\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
G(X) & \xrightarrow{G(f)} & G(Y) & \longrightarrow & 0 & \longrightarrow & \Sigma'G(X)
\end{array}
$$

$G(f)f^* = \mathrm{id}_Y$, so f^* is a left-sided inverse of $G(f)$. Symmetrically we can find the following morphism of distinguished triangles:

$$
\begin{array}{ccccccc}
G(X) & \xrightarrow{G(f)} & G(Y) & \longrightarrow & 0 & \longrightarrow & \Sigma'G(X) \\
{\scriptstyle \mathrm{id}_{G(X)}}\Big\downarrow & & {\scriptstyle f^{**}}\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
G(X) & \xrightarrow{\mathrm{id}_{G(X)}} & G(X) & \longrightarrow & 0 & \longrightarrow & \Sigma'G(Y)
\end{array}
$$

which gives f^{**}, a right sided inverse of $G(f)$. Thus $G(f)$ is an isomorphism.

Since we have shown that G takes elements of $S(N)$ to isomorphisms we know, by the universality of F, that there exists a functor $H : \mathcal{C}/N \to \mathcal{C}'$ so that $G = HF$.

$\square$

4 Derived categories

In [8], we started out studying abelian categories, and moved on to categories of complexes and the homotopy category. This was done in order to define the derived category, which we will do in this section. The plan for this construction can be summarized as follows:

$$\mathcal{C} \to \mathrm{C}(\mathcal{C}) \to \mathrm{K}(\mathcal{C}) \to \mathrm{D}(\mathcal{C})$$

We start in an abelian category $\mathcal{C}$, and form the category of complexes, $\mathrm{C}(\mathcal{C})$, where the objects are long sequences of the form

$$X : \cdots \to X^{n-1} \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1} \to \cdots$$

with $d_X^n d_X^{n-1} = 0$ for all n. We constructed the homotopy category $\mathrm{K}(\mathcal{C})$ by creating an equivalence relation on the Hom-sets of $\mathrm{C}(\mathcal{C})$ where $f \sim g$ if and only if f was homotopic to g. Finally, we studied the homotopy functor $H^n : \mathrm{K}(\mathcal{C}) \to \mathcal{C}$, defined by setting $H^n(X) = \mathrm{Ker}(d_X^n)/\mathrm{Im}(d_X^n)$.

Now we will pick up this thread and use H^n to define a multiplicative system in $\mathrm{K}(\mathcal{C})$. Localizing by this system will give us $D(\mathcal{C})$. However, we would like to use a null system for this purpose, so we have to start by showing the homotopy category to be triangulated.

4.1 Triangulation of $\mathrm{K}(\mathcal{C})$

Let $\mathcal{C}$ be an abelian category. We will now show that the homotopy category $\mathrm{K}(\mathcal{C})$ (which is additive) is triangulated.

Let $\Sigma : \mathrm{K}(\mathcal{C}) \to \mathrm{K}(\mathcal{C})$ be the map taking the complex $X \in \mathrm{K}(\mathcal{C})$ to the complex $\Sigma(X)$ where $\Sigma(X)^n = X^{n-1}$ and $d_{\Sigma(X)}^n = -d_X^{n-1}$. The map works on morphisms of complexes by shifting their respective elements in the same manner; if $\phi \in \mathrm{Hom}_{\mathrm{K}(\mathcal{C})}(X, Y)$ then we set

$$\{\Sigma(X)^n \xrightarrow{\Sigma(\phi)^n} \Sigma(Y)^n\} = \{X^{n+1} \xrightarrow{\Sigma(\phi)^n} Y^{n+1}\} = \{X^{n+1} \xrightarrow{\phi^{n+1}} Y^{n+1}\}.$$

To check that this is well-defined in the homotopy category, it is enough to check that if a map $\phi : X \to Y$ is homotopic to zero, say by the collection $(p^n)_{n \in \mathbb{Z}}$, then $\Sigma(\phi)$ is homotopic to zero as well. This is shown by the fact that

$$\Sigma(\phi)^n = \phi^{n+1} = d_Y^n p^{n+1} + p^{n+2} d_X^{n+1} = d_{\Sigma(Y)}^{n-1} q^n + q^{n+1} d_{\Sigma(X)}^n,$$

where $q^n = p^{n+1}$. It is easy to see that Σ is a functor and an equivalence of categories. We call this functor the *shift functor*, and typically write $\Sigma(X)$ as $X[1]$. Extending the notation, we set $X[n] = \Sigma^n[X]$ for a natural number n.

This gives us the required suspension functor on $K(\mathcal{C})$, but to proceed with defining its triangles we also need the notion of a *mapping cone*. Let $\phi : X \to Y$ be a morphism of complexes; its mapping cone is the complex $M(X)$ where $M(X)^n = X^{n+1} \oplus Y^n$ and $d_{M(X)}^n = \begin{pmatrix} -d_X^{n+1} & 0 \\ \phi^{n+1} & d_Y^n \end{pmatrix}$. As

$$d_Z^{n+1} d_Z^n = \begin{pmatrix} -d_X^{n+2} & 0 \\ \phi^{n+2} & d_Y^{n+1} \end{pmatrix} \cdot \begin{pmatrix} -d_X^{n+1} & 0 \\ \phi^{n+1} & d_Y^n \end{pmatrix}$$
$$= \begin{pmatrix} d_X^{n+2} d_X^{n+1} & 0 \\ -\phi^{n+2} d_X^{n+1} + d_Y^{n+1} \phi^{n+1} & d_Y^{n+1} d_Y^n \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

the mapping cone $M(X)$ is a complex. We define maps $Y \xrightarrow{\alpha_f} M(f)$ and $M(f) \xrightarrow{\beta_f} X[1]$ by setting $\alpha_f^n = \begin{pmatrix} 0 \\ \mathrm{id}_{Y^n} \end{pmatrix}$ and $\beta_f^n = \begin{pmatrix} \mathrm{id}_{X^{n+1}} & 0 \end{pmatrix}$. These maps are well-defined in $K(\mathcal{C})$. If g is homotopic to f, it has the same domain and range as f, so $M(g) = M(f)$, $\alpha_g = \alpha_f$ and $\beta_g = \beta_f$.

Theorem 4.1. *[9, 1.4.4] Let the distinguished triangles of $K(\mathcal{C})$ be the triangles isomorphic to a triangle*

$$X \xrightarrow{f} Y \xrightarrow{\alpha_f} M(X) \xrightarrow{\beta_f} X[1]$$

for some $X, Y \in \mathrm{Ob}(K(\mathcal{C}))$ and some $f \in \mathrm{Hom}_{K(\mathcal{C})}(X, Y)$. The category $K(\mathcal{C})$ is triangulated with respect to this family of distinguished triangles.

Proof. We consider the conditions of definition 3.1, but out of order.

T1 Follows from the definition of the distinguished triangles in $K(\mathcal{C})$.

T3 Follows from the definition of the distinguished triangles in $K(\mathcal{C})$.

T4 We will only show one direction. Suppose that the triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is distinguished. We want to show that the triangle $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ is distinguished as well. We can assume that $Z^n = M^n(f)$ as the triangle is isomorphic to a triangle where this is true.

We know that there exists a distinguished triangle $Y \xrightarrow{g} Z \xrightarrow{\alpha_g} M(g) \xrightarrow{\beta_g} Y[1]$, and that by definition $M(g)^n = Y^{n+1} \oplus Z^n = Y^{n+1} \oplus X^{n+1} \oplus Y^n$. We define the map $\phi : X[1] \to M(g)$ by $\phi^n = \begin{pmatrix} -f^{n+1} \\ \mathrm{id}_X^{n+1} \\ 0 \end{pmatrix}$, and the map $\psi : M(g) \to X[1]$ by $\psi^n = \begin{pmatrix} 0 & \mathrm{id}_X^{n+1} & 0 \end{pmatrix}$. It can be checked that ϕ and ψ are morphisms of complexes. We see that

$$\psi^n \phi^n = \begin{pmatrix} 0 & \mathrm{id}_X^{n+1} & 0 \end{pmatrix} \begin{pmatrix} -f^{n+1} \\ \mathrm{id}_X^{n+1} \\ 0 \end{pmatrix} = \mathrm{id}_X^{n+1},$$

so the morphisms ψ and ϕ are at the very least one-sided inverses of each other, but

$$\phi^n \psi^n = \begin{pmatrix} -f^{n+1} \\ \mathrm{id}_X^{n+1} \\ 0 \end{pmatrix} \begin{pmatrix} 0 & \mathrm{id}_X^{n+1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -f^{n+1} & 0 \\ 0 & \mathrm{id}_X^{n+1} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

tells us that we have got to do work to in order to show that they are two-sided inverses. We know that

$$\mathrm{id}^n_{M(g)} = \begin{pmatrix} \mathrm{id}^{n+1}_Y & 0 & 0 \\ 0 & \mathrm{id}^{n+1}_X & 0 \\ 0 & 0 & \mathrm{id}^n_Y \end{pmatrix}.$$

Setting $s^n = \begin{pmatrix} 0 & 0 & \mathrm{id}^n_Y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, we get

$$s^{n+1}d^n_{M(g)} + d^{n-1}_{M(g)}s^n =$$

$$= \begin{pmatrix} 0 & 0 & \mathrm{id}^{n+1}_Y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -d^{n+1}_Y & 0 & 0 \\ 0 & -d^{n+1}_X & 0 \\ \mathrm{id}^{n+1}_Y & f^{n+1} & d^n_Y \end{pmatrix} + \begin{pmatrix} -d^n_Y & 0 & 0 \\ 0 & -d^n_X & 0 \\ \mathrm{id}^n_Y & f^n & d^{n-1}_Y \end{pmatrix} \begin{pmatrix} 0 & 0 & \mathrm{id}^n_Y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{id}^{n+1}_Y & f^{n+1} & d^n_Y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -d^n_Y \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{id}^n_Y \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{id}^{n+1}_Y & f^{n+1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{id}^n_Y \end{pmatrix}$$

$$= \mathrm{id}^n_{M(g)} - \phi^n\psi^n,$$

which means that $\phi\psi$ is homotopic to $\mathrm{id}_{M(g)}$, and so $\phi\psi = \mathrm{id}_{M(g)}$ in K($\mathcal{C}$). This means that ϕ is an isomorphism (and so is ψ). Consider the following diagram:

$$\begin{array}{ccccccc}
Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] & \xrightarrow{-f[1]} & Y[1] \\
\downarrow{\scriptstyle \mathrm{id}_Y} & & \downarrow{\scriptstyle \mathrm{id}_Z} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathrm{id}_{Y[1]}} \\
Y & \xrightarrow{g} & Z & \xrightarrow{\alpha_g} & M(g) & \xrightarrow{\beta_g} & Y[1]
\end{array}$$

It can be checked that the diagram commutes. As each of the downward maps is an isomorphism we see that $(\mathrm{id}_Y, \mathrm{id}_Z, \phi)$ is an isomorphism of triangles, and so $Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$ is distinguished.

T2 For $X \in \mathrm{K}(\mathcal{C})$, there exists a map $0 \xrightarrow{0} X$. Given that $\mathrm{M}(0) = X$, the triangle $0 \to X \xrightarrow{\mathrm{id}_X} X \to 0$ is distinguished, and then by **T4** the triangle $X \xrightarrow{\mathrm{id}_X} X \to 0 \to X[1]$ is distinguished.

T5 As all distinguished triangles are isomorphic to one of the form $X \xrightarrow{f} Y \xrightarrow{\alpha_f} M(f) \xrightarrow{\beta_f} X[1]$, we will assume that we are dealing with two such triangles, namely $X \xrightarrow{f} Y \xrightarrow{\alpha_f} M(f) \xrightarrow{\beta_f} X[1]$ and $\bar{X} \xrightarrow{\bar{f}} \bar{Y} \xrightarrow{\alpha_{\bar{f}}} M(\bar{f}) \xrightarrow{\beta_{\bar{f}}} \bar{X}[1]$. We also assume that

there exists morphisms $u : X \to \bar{X}$ and $v : Y \to \bar{Y}$ so that the following diagram commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} \\
\bar{X} & \xrightarrow{\ \bar{f}\ } & \bar{Y}
\end{array}
$$

As we are working in $K(\mathcal{C})$, this means that there exist maps $s^n : X^n \to \bar{Y}^{n-1}$ so that $v^n f^n - \bar{f}^n u^n = s^{n+1} d_X^n + d_{\bar{Y}}^{n-1} s^n$. Define $w : M(f) \to M(\bar{f})$ by letting $w^n = \begin{pmatrix} u^{n+1} & 0 \\ s^{n+1} & v^n \end{pmatrix}$ (remember that $M(f) = X^{n+1} \oplus Y^n$). We see that

$$
\begin{aligned}
d_{M(\bar{f})}^n w^n &= \begin{pmatrix} -d_{\bar{X}}^{n+1} & 0 \\ \bar{f}^{n+1} & d_{\bar{Y}}^n \end{pmatrix} \begin{pmatrix} u^{n+1} & 0 \\ s^{n+1} & v^n \end{pmatrix} \\
&= \begin{pmatrix} -d_{\bar{X}}^{n+1} u^{n+1} & 0 \\ \bar{f}^{n+1} u^{n+1} + d_{\bar{Y}}^n s^{n+1} & d_{\bar{Y}}^n v^n \end{pmatrix} = \begin{pmatrix} -u^{n+2} d_X^{n+1} & 0 \\ -s^{n+2} d_X^{n+1} + v^{n+1} f^{n+1} & v^{n+1} d_Y^n \end{pmatrix} \\
&= \begin{pmatrix} u^{n+2} & 0 \\ s^{n+2} & v^{n+1} \end{pmatrix} \begin{pmatrix} -d_X^{n+1} & 0 \\ f^{n+1} & d_Y^n \end{pmatrix} = w^{n+1} d_{M(f)}^n,
\end{aligned}
$$

so w is a morphism of complexes. Moreover,

$$
w^n \alpha_f^n = \begin{pmatrix} u^{n+1} & 0 \\ s^{n+1} & v^n \end{pmatrix} \begin{pmatrix} 0 \\ \mathrm{id}_Y^n \end{pmatrix} = \begin{pmatrix} 0 \\ v^n \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{id}_{\bar{Y}}^n \end{pmatrix} v^n = \alpha_{\bar{f}}^n v^n
$$

and

$$
\beta_{\bar{f}}^n w^n = \begin{pmatrix} \mathrm{id}_{\bar{X}}^{n+1} & 0 \end{pmatrix} \begin{pmatrix} u^{n+1} & 0 \\ s^{n+1} & v^n \end{pmatrix} = \begin{pmatrix} u^{n+1} & 0 \end{pmatrix} = u^{n+1} \begin{pmatrix} \mathrm{id}_X^{n+1} & 0 \end{pmatrix} = u^n \beta_f^n.
$$

Thus the following diagram commutes

$$
\begin{array}{ccccccc}
X & \xrightarrow{\ f\ } & Y & \xrightarrow{\ \alpha_f\ } & M(f) & \xrightarrow{\ \beta_f\ } & X[1] \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow \\
\bar{X} & \xrightarrow{\ \bar{f}\ } & \bar{Y} & \xrightarrow{\ \alpha_{\bar{f}}\ } & M(\bar{f}) & \xrightarrow{\ \beta_{\bar{f}}\ } & X[1]
\end{array}
$$

which is what the definition requires.

T6 We assume that there exist distinguished triangles

$$
X \xrightarrow{\ f\ } Y \xrightarrow{\ \alpha_f\ } \bar{Z} \xrightarrow{\ \beta_f\ } X[1]
$$

$$
Y \xrightarrow{\ g\ } Z \xrightarrow{\ \alpha_g\ } \bar{X} \xrightarrow{\ \beta_g\ } Y[1]
$$

$$
X \xrightarrow{\ gf\ } Z \xrightarrow{\ \alpha_{gf}\ } \bar{Y} \xrightarrow{\ \beta_{gf}\ } X[1]
$$

and that these triangles are so that $\bar{Z} = M(f) = X[1] \oplus Y$, $\bar{X} = M(g) = Y[1] \oplus Z$ and $\bar{Y} = M(gf) = X[1] \oplus Z$ [5].

Let $u : \bar{Z} \to \bar{Y}$ be defined by $u^n = \begin{pmatrix} \mathrm{id}_X^{n+1} & 0 \\ 0 & g^n \end{pmatrix}$, let $v : \bar{X} \to \bar{Y}$ be defined by $v^n = \begin{pmatrix} f^{n+1} & 0 \\ 0 & \mathrm{id}_n^Z \end{pmatrix}$ and let $w = \alpha_f[1]\beta_g = \begin{pmatrix} 0 & 0 \\ \mathrm{id}_Y^{n+1} & 0 \end{pmatrix}$. Then the following diagram commutes:

$$
\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{\alpha_f} & \bar{Z} & \xrightarrow{\beta_f} & X[1] \\
\downarrow{\scriptstyle \mathrm{id}_X} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \mathrm{id}_{X[1]}} \\
X & \xrightarrow{gf} & Z & \xrightarrow{\alpha_{gf}} & \bar{Y} & \xrightarrow{\beta_{gf}} & X[1] \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}_Z} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle f[1]} \\
Y & \xrightarrow{g} & Z & \xrightarrow{\alpha_g} & \bar{X} & \xrightarrow{\beta_g} & Y[1] \\
\downarrow{\scriptstyle \alpha_f} & & \downarrow{\scriptstyle \alpha_{gf}} & & \downarrow{\scriptstyle \mathrm{id}_{\bar{X}}} & & \downarrow{\scriptstyle \alpha_f[1]} \\
\bar{Z} & \xrightarrow{u} & \bar{Y} & \xrightarrow{v} & \bar{X} & \xrightarrow{w} & \bar{Z}[1]
\end{array}
$$

Thus we only need to show that $\bar{Z} \xrightarrow{u} \bar{Y} \xrightarrow{v} \bar{X} \xrightarrow{w} \bar{Z}[1]$ is distinguished. We will do this by finding an isomorphism $\phi : M(u) \to \bar{X}$, because then the following diagram will commute,

$$
\begin{array}{ccccccc}
\bar{Z} & \xrightarrow{u} & \bar{Y} & \xrightarrow{\alpha_u} & M(u) & \xrightarrow{\beta_u} & \bar{Z}[1] \\
\downarrow{\scriptstyle \mathrm{id}_{\bar{Z}}} & & \downarrow{\scriptstyle \mathrm{id}_{\bar{Y}}} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathrm{id}_{\bar{Z}[1]}} \\
\bar{Z} & \xrightarrow{u} & \bar{Y} & \xrightarrow{v} & \bar{X} & \xrightarrow{w} & \bar{Z}[1]
\end{array}
$$

$(\mathrm{id}_{\bar{Z}}, \mathrm{id}_{\bar{Y}}, \phi)$ will be an isomorphism of triangles and $\bar{Z} \xrightarrow{u} \bar{Y} \xrightarrow{v} \bar{X} \xrightarrow{w} \bar{Z}[1]$ is isomorphic to a distinguished triangle.

We start by noticing that that $\bar{X}^n = Y^{n+1} \oplus Z^n$ and $M(u)^n = \bar{Z}^{n+1} \oplus \bar{Y}^n = X^{n+2} \oplus Y^{n+1} \oplus X^{n+1} \oplus Z^n$, and define ϕ by setting

$$
\phi^n = \begin{pmatrix} 0 & \mathrm{id}_Y^{n+1} & f^{n+1} & 0 \\ 0 & 0 & 0 & \mathrm{id}_Z^n \end{pmatrix}
$$

We check that ϕ is a morphism of complexes:

---

[5] This can once again be assumed without loss of generality because any distinguished triangle is isomorphic to a triangle of the mapping cone sort.

$$
\phi^{n+1} d^n_{M(u)} = \begin{pmatrix} 0 & \mathrm{id}^{n+2}_Y & f^{n+2} & 0 \\ 0 & 0 & 0 & \mathrm{id}^{n+1}_Z \end{pmatrix} \begin{pmatrix} d^{n+2}_X & 0 & 0 & 0 \\ -f^{n+2} & -d^{n+1}_Y & 0 & 0 \\ \mathrm{id}^{n+2}_X & 0 & -d^{n+1}_X & 0 \\ 0 & g^{n+1} & (fg)^{n+1} & d^n_Z \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 & -d^{n+1}_Y & -f^{n+2} d^{n+1}_X & 0 \\ 0 & g^{n+1} & (fg)^{n+1} & d^n_Z \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 & -d^{n+1}_Y & -d^{n+1}_Y f^{n+1} & 0 \\ 0 & g^{n+1} & (fg)^{n+1} & d^n_Z \end{pmatrix}
$$

$$
= \begin{pmatrix} -d^{n+1}_Y & 0 \\ g^{n+1} & d^n_Z \end{pmatrix} \begin{pmatrix} 0 & \mathrm{id}^{n+1}_Y & f^{n+1} & 0 \\ 0 & 0 & 0 & \mathrm{id}^n_Z \end{pmatrix} = d^n_{\tilde{X}} \phi^n.
$$

To show that it is an isomorphism as well, we define the map ψ by

$$
\psi^n = \begin{pmatrix} 0 & 0 \\ \mathrm{id}^{n+1}_Y & 0 \\ 0 & 0 \\ 0 & \mathrm{id}^n_Z \end{pmatrix}.
$$

That ψ is a morphism of complexes is verified by

$$
\psi^{n+1} d^n_{\tilde{X}} = \begin{pmatrix} 0 & 0 \\ \mathrm{id}^{n+2}_Y & 0 \\ 0 & 0 \\ 0 & \mathrm{id}^{n+1}_Z \end{pmatrix} \begin{pmatrix} -d^{n+1}_Y & 0 \\ g^{n+1} & d^n_Z \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -d^{n+1}_Y & 0 \\ 0 & 0 \\ g^{n+1} & d^n_Z \end{pmatrix}
$$

$$
= \begin{pmatrix} d^{n+2}_X & 0 & 0 & 0 \\ -f^{n+2} & -d^{n+1}_Y & 0 & 0 \\ \mathrm{id}^{n+2}_X & 0 & -d^{n+1}_X & 0 \\ 0 & g^{n+1} & (fg)^{n+1} & d^n_Z \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \mathrm{id}^{n+1}_Y & 0 \\ 0 & 0 \\ 0 & \mathrm{id}^n_Z \end{pmatrix} = d^n_{M(u)} \psi^n.
$$

We observe that

$$
\phi^n \psi^n = \begin{pmatrix} 0 & \mathrm{id}^{n+1}_Y & f^{n+1} & 0 \\ 0 & 0 & 0 & \mathrm{id}^n_Z \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \mathrm{id}^{n+1}_Y & 0 \\ 0 & 0 \\ 0 & \mathrm{id}^n_Z \end{pmatrix} = \begin{pmatrix} \mathrm{id}^{n+1}_Y & 0 \\ 0 & \mathrm{id}^n_Z \end{pmatrix} = \mathrm{id}^n_{\tilde{X}}
$$

so ψ is a right-handed inverse of ϕ. Next, we check that

$$
\psi^n \phi^n = \begin{pmatrix} 0 & 0 \\ \mathrm{id}^{n+1}_Y & 0 \\ 0 & 0 \\ 0 & \mathrm{id}^n_Z \end{pmatrix} \begin{pmatrix} 0 & \mathrm{id}^{n+1}_Y & f^{n+1} & 0 \\ 0 & 0 & 0 & \mathrm{id}^n_Z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathrm{id}^{n+1}_Y & f^{n+1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{id}^n_Z \end{pmatrix}
$$

which is not obviously equal to $\mathrm{id}^n_{M(u)} = \begin{pmatrix} \mathrm{id}^{n+2}_X & 0 & 0 & 0 \\ 0 & \mathrm{id}^{n+1}_Y & 0 & 0 \\ 0 & 0 & \mathrm{id}^{n+1}_X & 0 \\ 0 & 0 & 0 & \mathrm{id}^n_Z \end{pmatrix}$. However, setting

$s^n = \begin{pmatrix} 0 & 0 & \mathrm{id}^{n+1}_X & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, we get that

$$s^{n+1} d^n_{M(u)} + d^{n-1}_{M(u)} s^n = \begin{pmatrix} 0 & 0 & \mathrm{id}^{n+2}_X & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} d^{n+2}_X & 0 & 0 & 0 \\ -f^{n+2} & -d^{n+1}_Y & 0 & 0 \\ \mathrm{id}^{n+2}_X & 0 & -d^{n+1}_X & 0 \\ 0 & g^{n+1} & (fg)^{n+1} & d^n_Z \end{pmatrix}$$

$$+ \begin{pmatrix} d^{n+1}_X & 0 & 0 & 0 \\ -f^{n+1} & -d^n_Y & 0 & 0 \\ \mathrm{id}^{n+1}_X & 0 & -d^n_X & 0 \\ 0 & g^n & (fg)^n & d^{n-1}_Z \end{pmatrix} \begin{pmatrix} 0 & 0 & \mathrm{id}^{n+1}_X & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{id}^{n+2}_X & 0 & 0 & 0 \\ 0 & 0 & -f^{n+1} & 0 \\ 0 & 0 & \mathrm{id}^{n+1}_X & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \mathrm{id}^n_{M(u)} - \psi^n \phi^n$$

Thus $\psi^n \phi^n$ is homotopic to $\mathrm{id}^n_{M(u)}$ and they are equal in the category $\mathrm{K}(\mathcal{C})$, which means that ϕ is an isomorphism from $M(u)$ to $\bar{X}$, as we set out to prove. Finally, we need to show that the diagram gives an isomorphism of triangles, so we calculate:

$$\phi^n \alpha^n_u = \begin{pmatrix} 0 & \mathrm{id}^{n+1}_Y & f^{n+1} & 0 \\ 0 & 0 & 0 & \mathrm{id}^n_Z \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \mathrm{id}^{n+1}_X & 0 \\ 0 & \mathrm{id}^n_Z \end{pmatrix} = \begin{pmatrix} f^{n+1} & 0 \\ 0 & \mathrm{id}^n_Z \end{pmatrix} = v^n$$

and

$$\beta^n_u \psi^n = \begin{pmatrix} \mathrm{id}^{n+1}_X & 0 & 0 & 0 \\ 0 & \mathrm{id}^{n+1}_Y & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \mathrm{id}^{n+1}_Y & 0 \\ 0 & 0 \\ 0 & \mathrm{id}^n_Z \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \mathrm{id}^{n+1}_Y & 0 \end{pmatrix} = w^n$$

Thus the two triangles are isomorphic, and $\bar{Z} \xrightarrow{u} \bar{Y} \xrightarrow{v} \bar{X} \xrightarrow{w} \bar{Z}[1]$ is distinguished.

$\square$

4.2   Derived categories

Since the homotopy category is triangulated, localization by a null system can be applied to it. The goal of this section is to use that to construct the derived category $\mathrm{D}(\mathcal{C})$ where $\mathcal{C}$ is an abelian category. To get there we need the *homology functor*, which will be used

to define our null system. Note that throughout this section, $\mathcal{C}$ is assumed to be an abelian category. The material presented in this section is well covered in [9], which has been our main source for the presentation.

Definition 4.2. The homology functor $H^n : \mathrm{K}(\mathcal{C}) \to \mathcal{C}$ is defined by

$$H^n(X) = \mathrm{Ker}(d_X^n) / \mathrm{Im}(d_X^{n-1})$$

for an object $X \in \mathrm{Ob}(\mathrm{K}(\mathcal{C}))$ and $n \in \mathbb{Z}$.

For a proof that this is well-defined, see [8]. For a morphism $f : X \to Y$ we set $H^n(f)$ to be the morphism from $H^n(X) \to H^n(Y)$ induced by f. A morphism f is called a *quasi-isomorphism* if $H^n(f)$ is an isomorphism for all $n \in \mathbb{Z}$.

We are going to need some important properties of the homology functor.

Theorem 4.3. *Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence in $\mathrm{C}(\mathcal{C})$.[6] Then there exists a long exact sequence in $\mathcal{C}$*

$$\cdots \to H^n(X) \to H^n(Y) \to H^n(Z) \to H^{n+1}(X) \to H^{n+1}(Y) \to H^{n+1}(Z) \to \cdots$$

Proof. We have a commutative diagram with exact rows in $\mathcal{C}$:

$$
\begin{array}{ccccccc}
\mathrm{Im}(d_X^{n-1}) & \longrightarrow & X^n & \longrightarrow & \mathrm{Cok}(d^{n-1}) & \longrightarrow & 0 \\
\downarrow{\scriptstyle u} & & \| & & \downarrow{\scriptstyle v} & & \\
0 \longrightarrow \mathrm{Ker}(d_X^n) & \longrightarrow & X^n & \longrightarrow & \mathrm{Im}(d_X^n) & &
\end{array}
$$

By the snake lemma (lemma A.2), the following sequence exists and is exact: $0 \to \mathrm{Ker}(v) \overset{\phi}{\to} \mathrm{Cok}(u) \to 0$, so ϕ must be an isomorphism. However $H^n(X) = \mathrm{Cok}(u)$, so there exists a monomorphism $w : H^n(X) \to \mathrm{Cok}(d_X^{n-1})$ defined by $w = i_v \phi^{-1}$, where i_v is the kernel monomorphism of v. As v is an epimorphism (because $X^n \to \mathrm{Cok}(d^{n-1}) \overset{v}{\to} \mathrm{Im}(d^n) = X^n \to \mathrm{Im}(d^n)$, and the latter is an epimorphism), we know the following sequence to be exact:

$$0 \to H^n(X) \to \mathrm{Cok}(d_X^{n-1}) \to \mathrm{Im}(d_X^n) \to 0$$

We glue this together with the exact sequence

$$0 \to \mathrm{Im}(d_X^n) \to \mathrm{Ker}(d_X^{n+1}) \to H^{n+1}(X) \to 0$$

to form the exact sequence

$$0 \to H^n(X) \to \mathrm{Cok}(d_X^{n-1}) \to \mathrm{Ker}(d_X^{n+1}) \to H^{n+1}(X) \to 0 \tag{4.1}$$

---

[6]A short exact sequence in $\mathrm{C}(\mathcal{C})$ is defined to be a sequence of complexes $0 \to X \to Y \to Z \to 0$ so that for each $n \in \mathbb{Z}$, the sequence $0 \to X^n \to Y^n \to Z^n \to 0$ is short exact.

Studying the following diagram with exact middle rows

$$
\begin{array}{ccccc}
\mathrm{Ker}(d_X^n) & & \mathrm{Ker}(d_Y^n) & & \mathrm{Ker}(d_Z^n) \\
\downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow X^{n-1} \longrightarrow Y^{n-1} \longrightarrow Z^{n-1} \longrightarrow 0 \\
\quad\downarrow d_X^n \qquad\quad \downarrow d_Y^n \qquad\quad \downarrow d_Z^n \\
0 \longrightarrow X^n \longrightarrow Y^n \longrightarrow Z^n \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Cok}(d_X^n) & & \mathrm{Cok}(d_Y^n) & & \mathrm{Cok}(d_Z^n)
\end{array}
$$

we see that by the snake lemma there must exist exact sequences $0 \to \mathrm{Ker}(d_X^n) \to \mathrm{Ker}(d_X^n) \to \mathrm{Ker}(d_X^n)$ and $\mathrm{Cok}(d_X^n) \to \mathrm{Cok}(d_Y^n) \to \mathrm{Cok}(d_Z^n) \to 0$.

Using the exact sequence (4.1), we see that the following diagram is commutative with exact middle rows and exact columns:

$$
\begin{array}{ccccc}
H^n(X) & & H^n(Y) & & H^n(Z) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Cok}(d_X^{n-1}) \longrightarrow \mathrm{Cok}(d_Y^{n-1}) \longrightarrow \mathrm{Cok}(d_Z^{n-1}) \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow \mathrm{Ker}(d_X^n) \longrightarrow \mathrm{Ker}(d_Y^n) \longrightarrow \mathrm{Ker}(d_Z^n) \\
\downarrow & & \downarrow & & \downarrow \\
H^{n+1}(X) & & H^{n+1}(Y) & & H^{n+1}(Z)
\end{array}
$$

Using the snake lemma one final time we get the long exact sequence

$$\cdots \to H^n(X) \to H^n(Y) \to H^n(Z) \to H^{n+1}(X) \to H^{n+1}(Y) \to H^{n+1}(Z) \to \cdots$$

$\square$

The long exact sequence of the above theorem is important in homological algebra. We use it to prove the following theorem.

Theorem 4.4. $H^0 : \mathrm{K}(\mathcal{C}) \to \mathcal{C}$ *is a cohomological functor.*

Proof. $\mathrm{K}(\mathcal{C})$ is triangulated, $\mathcal{C}$ is abelian and H^0 is an additive functor, so it remains to show that H^0 maps distinguished triangles to exact sequences. Consider the distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1].$$

We know that also

$$Z[-1] \xrightarrow{h[-1]} X \xrightarrow{f} Y \xrightarrow{g} Z$$

is distinguished. The square

$$
\begin{array}{ccc}
Z[-1] & \xrightarrow{h[-1]} & X \\
\downarrow{\scriptstyle \mathrm{id}_{Z[1]}} & & \downarrow{\scriptstyle \mathrm{id}_X} \\
Z[-1] & \xrightarrow{h[-1]} & X
\end{array}
$$

can be embedded in a morphism of triangles:

$$
\begin{array}{ccccccc}
Z[-1] & \xrightarrow{h[-1]} & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\
{\scriptstyle \mathrm{id}_{Z[-1]}}\downarrow & & {\scriptstyle \mathrm{id}_X}\downarrow & & {\scriptstyle \phi}\downarrow & & {\scriptstyle \mathrm{id}_Z}\downarrow \\
Z[-1] & \xrightarrow{h[-1]} & X & \xrightarrow{\binom{0}{\mathrm{id}_X}} & M(h[-1]) & \xrightarrow{(\mathrm{id}_Z\ 0)} & Z
\end{array}
$$

Since $\mathrm{id}_{Z[-1]}$ and id_X are isomorphisms, so is ϕ by Theorem 3.6. Thus the following is an isomorphism of triangles:

$$
\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & X[1] \\
{\scriptstyle \mathrm{id}_X}\downarrow & & {\scriptstyle \phi}\downarrow & & {\scriptstyle \mathrm{id}_Z}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{X[1]}} \\
X & \xrightarrow{\binom{0}{\mathrm{id}_X}} & M(h[-1]) & \xrightarrow{(\mathrm{id}_Z\ 0)} & Z & \xrightarrow{h} & X[1]
\end{array}
$$

Since functors preserve isomorphisms, the sequences $H^0(X) \to H^0(Y) \to H^0(Z)$ and $H^0(X) \to H^0(M(h[-1])) \to H^0(Z)$ are isomorphic. The sequence

$$
0 \to X \xrightarrow{\binom{0}{\mathrm{id}_X}} M(h[-1]) \xrightarrow{(\mathrm{id}_Z\ 0)} Z \to 0
$$

is split exact and thus short exact, so $H^0(X) \to H^0(M(h[-1])) \to H^0(Z)$ is exact, and so is $H^0(X) \to H^0(Y) \to H^0(Z)$. $\qquad\square$

The following corollary is more useful.

Corollary 4.5. H^n *is cohomological for any* $n \in \mathbb{Z}$.

Proof. This follows from the fact that $H^n(X) = H^0(X[n])$. $\qquad\square$

Since we now know that H^n is a rather well-behaved functor, we are ready to define a null system in $\mathrm{K}(\mathcal{C})$:

Theorem 4.6. *The set of morphisms*

$$
N = \{X \in \mathrm{Ob}(\mathrm{K}(\mathcal{C})) | H^n(X) = 0 \ \forall \, n \in \mathbb{Z}\}
$$

is a null system.

Proof. We check the conditions of definition 3.7 in order.

N1 As $H^n(0) = 0$, we have $0 \in N$.

N2 As $H^n(X) = H^{n-1}(X[1])$, it follows that $X \in N$ if and only if $X[1] \in N$.

N3 Consider a triangle $X \to Y \to Z \to X[1]$, where $X, Y \in N$. We know that $Y \to Z \to X[1] \to Y[1]$ is distinguished as well. As H^n is a cohomological functor, we then know that $H^n(Y) \to H^n(Z) \to H^n(X[1])$ is exact. However $H^n(Y) = 0 = H^{n+1}(X) = H^n(X[1])$, so the sequence $0 \to H^n(Z) \to 0$ is exact and $H^n(Z) = 0$, so $Z \in N$.

$\square$

The corresponding multiplicative system, as defined in theorem 3.8, is interesting:

Lemma 4.7. *Let N be defined as in the above theorem. Then $S(N)$ is the collection of quasi-isomorphisms in* $\mathrm{K}(\mathcal{C})$.

Proof. By definition, the elements of $S(N)$ are the morphisms $f : X \to Y$ that can be embedded as the first morphism of a distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$ where $Z \in N$ (which in this case means that $H^n(Z) = 0$ for all n).

Suppose $f : X \to Y$ is a quasi-isomorphism. The triangle $X \xrightarrow{f} Y \to M(f) \to X[1]$ is distinguished. As H^n is a cohomological functor, we know that

$$H^n(X) \xrightarrow{H^n(f)} H^n(Y) \to H^n(M(f)) \to H^n(X[1]) \xrightarrow{H^n(f[1])} H^n(Y[1])$$

is exact. As $H^n(f)$ is an isomorphism, it follows that $H^n(M(f)) = 0$, and $f \in S(N)$.

Conversely, suppose $f \in S(N)$, and let $X \xrightarrow{f} Y \to Z \to X[1]$ be the embedding of f in a distinguished triangle with $H^n(Z) = 0$. We know that there exists an isomorphism of triangles

$$
\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
\downarrow{\mathrm{id}_X} & & \downarrow{\mathrm{id}_Y} & & \downarrow{\phi} & & \downarrow{\mathrm{id}_{X[1]}} \\
X & \xrightarrow{f} & Y & \longrightarrow & M(f) & \longrightarrow & X[1]
\end{array}
$$

It follows that $H^n(M(f)) = 0$. As the triangles $X \xrightarrow{f} Y \to M(f) \to X[1]$ and $M(f)[-1] \to X \xrightarrow{f} Y \to M(f)$ are distinguished, and H^n is cohomological, the sequences $H^n(X) \xrightarrow{H^n(f)} H^n(Y) \to H^n(M(f))$ and $H^n(M(f)[-1]) \to H^n(X) \xrightarrow{f} H^n(Y)$ are exact. Gluing these two sequences together we get an exact sequence $0 \to H^n(X) \xrightarrow{f} H^n(Y) \to 0$, and thus $H^n(f)$ is an isomorphism and f is a quasi-isomorphism. $\square$

We can now define the derived category, as promised.

Definition 4.8. The *derived category* is $\mathrm{D}(\mathcal{C}) = \mathrm{K}(\mathcal{C})/N$.

From theorem 3.9 we know that $\mathrm{D}(\mathcal{C})$ is triangulated. For a module category $\mathrm{mod}(\Lambda)$ we will write the derived category $\mathrm{D}(\Lambda)$.

4.3   Localization of subcategories

We call a complex *bounded below* if there exists an integer N so that $X^n = 0$ for $n < N$. Similarly it is called *bounded above* if $X^n = 0$ for $n > N$. If a complex is bounded above and below, it is called a *bounded complex*.

   We see that the class of bounded below complexes form a full subcategory of $C(\mathcal{C})$; we call this subcategory $C^-(\mathcal{C})$. Similarly, we define $C^+(\mathcal{C})$ as the subcategory of complexes bounded above and $C^b(\mathcal{C})$ as the subcategory of bounded complexes.

   If we restrict the canonical functor $F : C(\mathcal{C}) \to K(\mathcal{C})$ to $C^-(\mathcal{C})$, we denote the image $K^-(\mathcal{C})$. Similar notation is used for category of complexes bounded above and the category of bounded complexes.

   To see that we can go just as easily from $K^*(\mathcal{C})$ (where $*$ stands for $-$, $+$ or b) to $D^*(\mathcal{C})$, we use the following theorem, which is a combination of [9, 1.6.5] and [9, 1.7.7]:

Theorem 4.9. *Let $\mathcal{C}'$ be a full subcategory of the category $\mathcal{C}$. Then the following holds:*

(i) *Let S be a multiplicative system in $\mathcal{C}$ and let S' be the morphisms in S that also are in $\mathcal{C}'$. Suppose the following, or its dual, holds: for any $f : X \to Y$ with $f \in S$ and $Y \in \mathrm{Ob}(\mathcal{C})$, there exists $g : Z \to X$, with $Z \in \mathrm{Ob}(\mathcal{C}')$, so that $fg \in S$ (and thus $fg \in S'$). If S' is a multiplicative system then $\mathcal{C}'_{S'}$ is a full subcategory of $\mathcal{C}_S$.*

(ii) *Suppose $\mathcal{C}$ and $\mathcal{C}'$ moreover are triangulated so that any distinguished triangle $X \to Y \to Z \to \Sigma X$ in $\mathcal{C}$ with $X, Y \in \mathrm{Ob}(\mathcal{C}')$ is distinguished in $\mathcal{C}'$.*

   Let N be a null system in $\mathcal{C}$ and let $N' = N \cap \mathcal{C}'$. Suppose any morphism $f : X \to Y$ with $X \in \mathrm{Ob}(\mathcal{C}')$ and $Y \in N$ factorizes through an object of N'. Then $\mathcal{C}'/N'$ is a full subcategory of $\mathcal{C}/N$.

Proof.   (i) We see that $\mathrm{Ob}\,\mathcal{C}'_{S'} = \mathrm{Ob}\,\mathcal{C}' \subseteq \mathrm{Ob}\,\mathcal{C} = \mathrm{Ob}\,\mathcal{C}_S$, so $\mathcal{C}'_{S'}$ is a subcategory of $\mathcal{C}_S$. It remains to show that it is full.
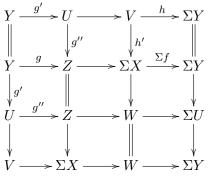
   Suppose the property mentioned in the theorem is fulfilled. Let $X, Y \in \mathrm{Ob}\,\mathcal{C}'$ and let $(U, s, f) \in \mathrm{Hom}_{\mathcal{C}_S}(X, Y)$. If $U \notin \mathrm{Ob}\,\mathcal{C}'_{S'}$, then there exists some morphism $V \xrightarrow{t} U$ so that $V \in \mathrm{Ob}\,\mathcal{C}'_{S'}$ and $st \in S'$. Moreover, $(V, st, ft) \sim (U, s, f)$; thus $\mathrm{Hom}_{\mathcal{C}_S}(X, Y) = \mathrm{Hom}_{\mathcal{C}'_{S'}}(X, Y)$.

   To see that the theorem also holds if the dual property is fulfilled, we use the fact that while we have considered the morphisms in $\mathcal{C}_S$ as being left fractions fs^{-1} we may just as well write them as right fractions $s^{-1}g$. The proof then follows dually.

(ii) That N' is a null system is easy to see by checking the requirements of null systems. Hence we can form the multiplicative system $S(N')$, and if this satisfies the condition from part (i), we are done.

   Suppose $f : X \to Y$ is such that $f \in S(N)$, $Y \in \mathrm{Ob}(\mathcal{C}')$; this means that there exists a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \to X[1]$ with $Z \in N$. By the conditions of the theorem, we know that $Y \xrightarrow{g} Z = Y \xrightarrow{g'} U \xrightarrow{g''} Z$ with $U \in N'$. Using axiom

TR6, we find the following commutative diagram where the rows are distinguished triangles:

$$
\begin{array}{ccccccc}
Y & \xrightarrow{g'} & U & \longrightarrow & V & \xrightarrow{h} & \Sigma Y \\
\| & & \downarrow{g''} & & \downarrow{h'} & & \| \\
Y & \xrightarrow{g} & Z & \longrightarrow & \Sigma X & \xrightarrow{\Sigma f} & \Sigma Y \\
\downarrow{g'} & & \| & & \downarrow & & \downarrow \\
U & \xrightarrow{g''} & Z & \longrightarrow & W & \longrightarrow & \Sigma U \\
\downarrow & & \downarrow & & \| & & \downarrow \\
V & \longrightarrow & \Sigma X & \longrightarrow & W & \longrightarrow & \Sigma Y
\end{array}
$$

We see that $h = (\Sigma f)h'$, and so $\Sigma^{-1}h = f(\Sigma^{-1}h')$. Moreover, $\Sigma^{-1}h$ can be embedded in a distinguished triangle $\Sigma^{-1}V \xrightarrow{\Sigma^{-1}h} Y \to U \to V$; and since $U \in N'$, it follows that $\Sigma^{-1}h \in S(N')$. Thus $\mathcal{C}'/N' = \mathcal{C}'_{S(N')}$ is a full subcategory of $\mathcal{C}/N$. $\qquad\square$

We would like to use this theorem for $\mathrm{K}^-(\mathcal{C})$. Consider a distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$ in $\mathrm{K}(\mathcal{C})$; since $Z \cong M(f)$ we can assume $Z \in \mathrm{K}^-(\mathcal{C})$, and thus the triangle is distinguished in $\mathrm{K}^-(\mathcal{C})$ as well. Since $\mathrm{K}^-(\mathcal{C})/N'$ (where N' are the objects of the null system N that are bounded below) is the full subcategory of $\mathrm{K}(\mathcal{C})/N = \mathrm{D}(\mathcal{C})$ containing all objects that are bounded above, it follows that $\mathrm{K}^-(\mathcal{C})/N' = \mathrm{D}^-(\mathcal{C})$. The same holds for $\mathrm{K}^+(\mathcal{C})$ and $\mathrm{K}^b(\mathcal{C})$.

4.4 Projective resolutions

In this section, we will see that if the category $\mathcal{C}$ fulfills certain requirements, the category $\mathrm{D}^-(\mathcal{C})$ has a simpler description; namely we get an equivalence of categories $\mathrm{D}^-(\mathcal{C}) \cong \mathrm{K}^-(\mathcal{P})$, where $\mathcal{P}$ is the subcategory of $\mathcal{C}$ consisting of projective objects. The material presented here appears in [9] and [10] as well, but these books consider injective objects, the dual of projective objects.

Obviously, we need to start by defining a projective object.

Definition 4.10. An object P in a category $\mathcal{C}$ is projective if the functor $\mathrm{Hom}_{\mathcal{C}}(P, -)$ is exact.

We need the following equivalent definitions of a projective object (here given without proof):

- An object P is projective if and only if for any morphism $P \to Y$ and any epimorphism $X \to Y$ there exists a morphism $P \to Y$ so that the following diagram commutes:

$$
\begin{array}{ccc}
& & P \\
& \swarrow & \downarrow \\
X & \longrightarrow & Y
\end{array}
$$

- P is projective if and only if any epimorphism into P splits.

It can be shown that a summand of a projective object and a direct sum of projective objects are projective as well.

We will also need the following lemma:

Lemma 4.11. *Let $0 \to X \to Y \to P \to 0$ be an exact sequence and P a projective object. Then X is projective if and only if Y is projective.*

Proof. Since $Y \to P$ is an epimorphism, it splits, and so the sequence splits and consequently $Y \cong X \oplus P$. If X is projective then Y is the sum of two projective objects and projective itself. If Y is projective then X is a summand of a projective object and is thus projective. $\qquad\square$

We say that a category has *enough projectives* if for any object X there exists a projective object P and an epimorphism $P \to X$.

We are now ready to give our big theorem:

Theorem 4.12. *Let $\mathcal{C}$ be an abelian category with enough projectives. Then $\mathrm{D}^-(\mathcal{C}) \cong \mathrm{K}^-(\mathcal{P})$*

Proof. We first note that $\mathcal{P}$ forms a full, additive subcategory of $\mathcal{C}$, and also that $\mathrm{K}^-(\mathcal{P})$ is a full subcategory of $\mathrm{K}^-(\mathcal{C})$. Let N be the null system in $\mathrm{K}^-(\mathcal{C})$ consisting of objects mapped to zero by the homology functor. Consider $N' = N \cap \mathrm{Ob}(\mathrm{K}^-(\mathcal{P}))$, the class of complexes P of projectives with a lower bound so that $H^n(P) = 0$ for all n. This means that for any n we have that $\mathrm{Cok}(d_P^n) \cong \mathrm{Ker}(d_P^{n+1})$. Accordingly, the sequence

$$0 \to \mathrm{Ker}(d_P^n) \xrightarrow{i^n} P^n \xrightarrow{p^n} \mathrm{Ker}(d_P^{n+1}) \to 0$$

is exact. For some integer M we must have $P^n = 0$ for all $n > M$ and so also $\mathrm{Ker}(d_P^{n+1}) = 0$. By induction using lemma 4.11, we see that this holds for any n. The sequence splits, so there must exist morphisms j^n and q^n as below

$$0 \longrightarrow \mathrm{Ker}(d_P^n) \underset{j^n}{\overset{i^n}{\rightleftarrows}} P^n \underset{q^n}{\overset{p^n}{\rightleftarrows}} \mathrm{Ker}(d_P^{n+1}) \longrightarrow 0$$

so that:

$$j^n i^n = \mathrm{id}_{\mathrm{Ker}(d_P^n)} \qquad\qquad p^n q^n = \mathrm{id}_{\mathrm{Ker}(d_P^{n+1})}$$
$$j^n q^n = 0 \qquad\qquad\qquad i^n j^n + q^n p^n = \mathrm{id}_{P^n}$$

Note also that $d_P^n = i^{n+1} p^n$.

Let $s^n = q^{n-1}j^n$. By considering the following diagram:

we find that

$$
\begin{aligned}
d_P^{n-1}s^n + s^{n+1}d_P^n &= i^n p^{n-1} q^{n-1} j^n + q^n j^{n+1} i^{n+1} p^n \\
&= i^n j^n + q^n p^n \\
&= \mathrm{id}_P .
\end{aligned}
$$

Thus id_P is homotopic to 0 and $P \cong 0$ in $\mathrm{K}^-(\mathcal{C})$. It follows that $N' = \{0\}$. From this and theorem 4.9 we know that $\mathrm{D}^-(\mathcal{P}) = \mathrm{K}^-(\mathcal{P})/N' \cong \mathrm{K}^-(\mathcal{P})$.

We now need to show that $\mathrm{D}^-(\mathcal{C}) \cong \mathrm{D}^-(\mathcal{P})$. The embedding functor $\mathrm{D}^-(\mathcal{P}) \to \mathrm{D}^-(\mathcal{C})$ is full and faithful, so we only need it to be dense.

Suppose $X \in \mathrm{Ob}(K^-(\mathcal{C}))$; we will construct a complex $P \in \mathrm{Ob}(K^-(\mathcal{P}))$ with a quasi-isomorphism $P \to X$. Assume, without loss of generality, that $X^n = 0$ for $n > 0$; in other words

$$
X = \cdots \to X^{-2} \to X^{-1} \to X^0 \to 0 \to \cdots
$$

Start constructing the complex P by setting $P^n = 0$ for $n > 0$ and choose $P^0 \in \mathcal{P}$ so that there exists an epimorphism $f^0 : P^0 \to X^0$.

Suppose $n < 0$ and that we have chosen P^i for $i > n$ (and that these P^i are so that $d_P^i d_P^{i-1} = 0$), in other words that we have the following situation:

$$
\begin{array}{ccc}
P^{n+1} & \xrightarrow{d_P^{n+1}} & \cdots \\
\downarrow{\scriptstyle f^{n+1}} & & \\
X^n \xrightarrow{d_X^n} X^{n+1} & \xrightarrow{d_X^{n+1}} & \cdots
\end{array}
$$

Let Q^n be the pullback completion of $X^n \xrightarrow{d_X^n} X^{n+1} \to \mathrm{Ker}(d_X^{n+1})$ and the morphism $\mathrm{Ker}(d_P^{n+1}) \to P^{n+1} \to X^{n+1} \to \mathrm{Ker}(d_X^{n+1})$, and let P^n be a projective object with an epimorphism into Q^n. Define f^n as the composition $P^n \to Q^n \to X^n$ and d_P^n as the composition $P^n \to Q^n \to X^n$. Then $d_P^{n+1}d_P^n = 0$, as d_P^n factors through the kernel of d_P^{n+1}.

Note that this construction agrees with what we did in degree 0, as since $P^1 = 0$, we had $Q^0 = X^0$. We have now constructed a complex $P \in \mathrm{Ob}(K^-(\mathcal{P}))$; we need to show that it is quasi-isomorphic to X.

For $i > 1$, we have $X^i = 0 = P^i$ and thus $H^i(f)$ is an isomorphism. Suppose f^i is a quasi-isomorphism for all $i > n$, and consider f^n; specifically the following commutative diagram with exact rows:

$$
\begin{array}{ccccccc}
P^{n-1} & \longrightarrow & \mathrm{Ker}(d_P^{n-1}) & \longrightarrow & H^n(P) & \longrightarrow & 0 \\
\downarrow {\scriptstyle f^{n-1}} & & \downarrow & & \downarrow {\scriptstyle H^n(f)} & & \\
X^{n-1} & \longrightarrow & \mathrm{Ker}(d_X^{n-1}) & \longrightarrow & H^n(X) & \longrightarrow & 0
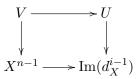\end{array}
$$

Since there exists an epimorphism from P^{n-1} into Q^{n-1}, it follows by lemma A.4 that $H^n(f)$ is an epimorphism. To see that it is a monomorphism, consider the following commutative diagram:

$$
\begin{array}{ccccc}
P^{n-1} & & & & \mathrm{Ker}(H^n(f)) \\
\downarrow & & & & \downarrow \\
Q^{n-1} & \longrightarrow & \mathrm{Ker}(d_P^n) & \longrightarrow & H^n(P) \\
\downarrow & & \downarrow & & \downarrow {\scriptstyle H^n(f)} \\
X^{n-1} & \longrightarrow & \mathrm{Ker}(d_P^n) & \longrightarrow & H^n(X)
\end{array}
$$

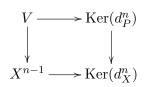Let U be the pullback completion of $\mathrm{Ker}(H^n(f)) \to H^n(P)$ and $\mathrm{Ker}(d_P^n) \to H^n(P)$. Since

$$ U \to \mathrm{Ker}(d_P^n) \to \mathrm{Ker}(d_X^n) \to H^n(X) = U \to \mathrm{Ker}(H^n(f)) \to H^n(P) \to H^n(X) = 0, $$
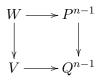
there exists a morphism $U \to \mathrm{Im}(d_X i - 1)$.
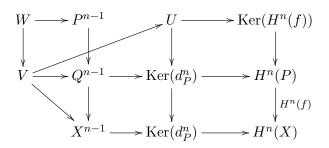
Let V be the pullback completion

$$
\begin{array}{ccc}
V & \longrightarrow & U \\
\downarrow & & \downarrow \\
X^{n-1} & \longrightarrow & \mathrm{Im}(d_X^{i-1})
\end{array}
$$

There exists a morphism $V \to Q^{n-1}$ because the following square commutes.

$$
\begin{array}{ccc}
V & \longrightarrow & \mathrm{Ker}(d_P^n) \\
\downarrow & & \downarrow \\
X^{n-1} & \longrightarrow & \mathrm{Ker}(d_X^n)
\end{array}
$$

Finally, let W be the pullback completion

$$
\begin{array}{ccc}
W & \longrightarrow & P^{n-1} \\
\downarrow & & \downarrow \\
V & \longrightarrow & Q^{n-1}
\end{array}
$$

We sum up the situation in the following diagram:

$$
\begin{array}{ccccc}
W & \longrightarrow & P^{n-1} & & U & \longrightarrow & \mathrm{Ker}(H^n(f)) \\
\downarrow & & \downarrow & \nearrow & \downarrow & & \downarrow \\
V & \longrightarrow & Q^{n-1} & \longrightarrow & \mathrm{Ker}(d_P^n) & \longrightarrow & H^n(P) \\
& \searrow & \downarrow & & \downarrow & & \downarrow {\scriptstyle H^n(f)} \\
& & X^{n-1} & \longrightarrow & \mathrm{Ker}(d_P^n) & \longrightarrow & H^n(X)
\end{array}
$$

We note that $U \to \mathrm{Ker}(H^n(F))$, $V \to U$ and $W \to V$ must be epimorphisms. This means that since

$$W \to V \to U \to \mathrm{Ker}(H^n(f)) \to H^n(P) =$$
$$= W \to P^{n-1} \to \mathrm{Im}(d_P^{i-1}) \to \mathrm{Ker}(d_P^n) \to H^n(P) = 0,$$

we must have the kernel monomorphism $\mathrm{Ker}(H^n(f)) \to H^n(P) = 0$ and thus $H^n(f)$ is also a monomorphism and f is a quasi-isomorphism.

Thus the inclusion functor $D^-(\mathcal{P}) \to D^-(\mathcal{C})$ is dense and an equivalence of categories. □

A consequence of the above is that in the derived bounded category an object (complex) which is non-zero in all but one term (i. e. of the form $\cdots \to 0 \to X \to 0 \to \cdots$) will be isomorphic to the complex containing the projective resolution of its non-zero term.

4.5 Example: Derived module category of a hereditary algebra

In this section we will consider one specific abelian category, and describe its derived category. We will start with a module category, namely the finitely generated modules over the quiver algebra $k\Gamma$, where Γ is the quiver $1 \to 2 \to 3$.

As Γ is a quiver without oriented cycles, $k\Gamma$ is a hereditary algebra [1], so the global dimension of $\mathrm{mod}(k\Gamma)$ is 1 (in other words the maximal length of a minimal projective resolution of a $k\Gamma$-module is 1). Moreover, in a category that has global dimension 1, any subobject of a projective object is projective, which gives the objects of its derived category an unusually nice structure.

Lemma 4.13. *Let $\mathcal{C}$ be an abelian category with enough projectives and global dimension 1. Then any indecomposable object in $\mathrm{D}^b(\mathcal{C})$ is of the form $\cdots \to 0 \to X \to 0 \to \cdots$ where X is an indecomposable object.*

Proof. Given an indecomposable object in $\mathrm{D}^b(\mathcal{C})$, we already know that it is isomorphic to one in $\mathrm{D}^b(\mathcal{P})$, and if one is indecomposable then so is the other. Thus we consider an object $P \in \mathrm{D}^b(\mathcal{P})$ and assume it to be indecomposable. We know that there can be no two non-zero terms in P separated by a zero term, since in that case the complex

would be decomposable. Suppose the length (number of non-zero terms) of P is larger than two, and assume without loss of generality that the highest non-zero term in P is situated in the zeroth position. The first few terms of P then become:

$$\cdots \to P^{-2} \xrightarrow{d_P^{-2}} P^{-1} \xrightarrow{d_P^{-1}} P^0 \to 0 \to \cdots$$

We assume P^0 to be non-zero.

We can factorize d_P^{-1} into $P^{-1} \xrightarrow{\pi} \mathrm{Im}(d_P^{-1}) \xrightarrow{i} P^0$, where π is projective and i is injective. Since $\mathrm{Im}(d_P^{-1})$ is a subobject of P^0 it is projective, and thus π splits, and $P^{-1} \cong Q \oplus \mathrm{Im}(d_P^{-1})$ with Q some object of $\mathcal{C}$, necessarily projective. Thus we can write P as

$$\cdots \to P^{-2} \xrightarrow{\binom{d_P^{-2}}{0}} Q \oplus \mathrm{Im}(d_P^{-1}) \xrightarrow{(0\ \iota)} P^0 \to 0 \to \cdots$$

Thus P decomposes if its length is larger than two. However, if it does have length two (or less), it is the complex of the projective resolution of $X = P^0/P^1$ and is thus isomorphic to the complex having X in its zeroth degree and zero elsewhere. Since P is indecomposable, X must be indecomposable.   □

For our example category $\mathrm{mod}(k\Gamma)$, we know the indecomposable objects, here given by their corresponding quiver representations, to be as follows:

$$S_1 : k \to 0 \to 0 \qquad\qquad P_1 : \ k \xrightarrow{1} k \xrightarrow{1} k$$

$$S_2 : 0 \to k \to 0 \qquad\qquad P_2 : \ k \xrightarrow{1} k \to 0$$

$$S_3 = P_3 : 0 \to 0 \to k \qquad\qquad M : \ 0 \to k \xrightarrow{1} k$$

An Auslander-Reiten quiver (or AR-quiver) is a diagrammatical way of showing the structure of a category. Each node in the quiver represents an isomorphism class of indecomposable objects. We call a morphism f irreducible if it is not an isomorphism, and for any factorization $f = gh$ through an indecomposable object g or h (or both) will be an isomorphism. Each arrow in the graph represents an indecomposable morphism (up to isomorphism).

When the category is of finite representation type (the AR-quiver has finitely many vertices), all morphisms are linear combinations of compositions of irreducible maps and isomorphisms [10], and thus the AR-quiver contains all information about the category.

A morphism $f : X \to Y$ is called *right almost split* if it is not a split epimorphism, and any morphism $g : Z \to Y$ that is not a split epimorphism factors through g. The morphism f is called *right minimal* if for any g such that $fg = f$ we must have that g is an isomorphism. If f is both right almost split and right minimal, it is called *minimal right almost split*. The definition of a left almost split, left minimal and a minimal left almost split morphism is dual.

An *almost split sequence* is an exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ so that either of the following equivalent conditions hold[1]:

- f is left almost split and g is right almost split.

- f is minimal left almost split.

- g is minimal right almost split.

- Z is indecomposable and f is left almost split.

- X is indecomposable and g is right almost split.

It is known that if X is an indecomposable object then $f : X \to Y$ is an irreducible morphism if and only if there exists a morphism $f' : X \to Y'$ so that $\left(\begin{smallmatrix} f \\ f' \end{smallmatrix} \right) : X \to Y \oplus Y'$ is a minimal right almost split morphism. But then the sequence

$$0 \to A \xrightarrow{\left(\begin{smallmatrix} f \\ f' \end{smallmatrix} \right)} Y \oplus Y' \to \mathrm{Cok} \left(\begin{smallmatrix} f \\ f' \end{smallmatrix} \right) \to 0$$
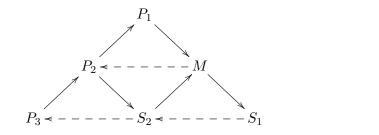
is an almost split sequence. Thus we can find all the irreducible morphisms from an indecomposable object by looking at almost split sequences; and dually we can find all irreducible morphisms to an indecomposable object the same way[1].

There exists a functor $\tau : \mathcal{C} \to \mathcal{C}$ which for each indecomposable non-projective object X associates an indecomposable non-injective object $\tau(X)$; moreover there exists an almost split sequence $0 \to \tau(X) \to Y \to X \to 0$. There also exists a dual functor τ^{-} which will take indecomposable non-injective objects to indecomposable non-projective objects; for X indecomposable and non-projective $\tau^{-}\tau(X) = X$; see [7]. The functor τ is also known as the Auslander-Reiten translation (or AR-translation). If the sequence $0 \to X \to Y \to Z \to 0$ is almost split, then $X \cong \tau(Z)$ and $Z \cong \tau^{-}(X)$; see [1], and we often draw the AR-translation as well when drawing the AR-quiver.

In the case of $\mathrm{mod}(k\Gamma)$ the almost split sequences are

$$0 \to P_3 \to P_2 \to S_2 \to 0$$
$$0 \to P_2 \to P_1 \oplus S_2 \to M \to 0$$
$$0 \to S_2 \to M \to S_3 \to 0$$

and the AR-quiver is

$$(4.2)$$

The dashed arrows show the AR-translation τ.

When we have drawn the AR-translation on the quiver, we can read the almost split sequences directly from the AR-quiver. An almost split sequence will occur when there is an AR-translation between two objects. The end terms will be the two objects, and the middle term will be the sum of all objects that are the middle term of a path of length two between the objects.
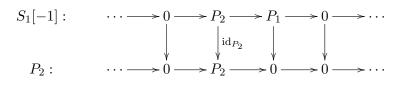
An analogue concept of the almost split sequence in an abelian category is the *almost split triangle* in a triangulated category. The distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \to \Sigma(X)$ is called almost split if f and g satisfy the same conditions as listed in the definition of an exact triangle. It can be shown that for a finite dimensional k-algebra Λ, the bounded derived category of the module category $\mathrm{D}^b(\Lambda)$ has almost split triangles if and only if Λ has finite global dimension[6]. In this case there also exists an AR-quiver. Note that in the derived category $\mathrm{D}(\mathcal{C})$, the AR-translation τ is defined for all indecomposable objects in $\mathrm{D}(\mathcal{C})$, included the non-projective.

In particular, $k\Gamma$ has finite global dimension, and we study $\mathrm{D}(k\Gamma)$. We will again draw dashed arrows equivalent to the ones for τ in the previous quiver, but this time they will help us find the almost split triangles.
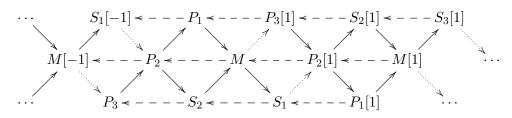
We know the structure of the indecomposable objects from lemma 4.13, so for each $n \in \mathbb{Z}$ there exists a subquiver with the same structure as (4.2) with the nodes being the chain complexes with an indecomposable object from $\mathrm{mod}(k\Gamma)$ in degree n.

Abusing notation, we let X denote the complex with the object X in degree 0 and zero elsewhere. As before $X[n]$ is the complex X shifted n times to to the right, so it has the object X in position $-n$, and so forth.

Are there any non-zero morphisms from (for example) $S_1[-1]$ to P_2? At first glance, it seems unlikely, but remembering that we have identified complexes concentrated in one degree with the projective resolution of their non-zero object, we find the following morphism:

$$S_1[-1]: \qquad \cdots \longrightarrow 0 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow 0 \longrightarrow \cdots$$
$$\qquad\qquad\qquad\qquad\qquad \Big\downarrow \quad\ \Big\downarrow{\scriptstyle \mathrm{id}_{P_2}} \quad\ \Big\downarrow \quad\ \Big\downarrow$$
$$P_2: \qquad \cdots \longrightarrow 0 \longrightarrow P_2 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

We find that the AR-quiver of $\mathrm{D}(k\Gamma)$ will look like this:

The dotted arrows represent the morphisms between different copies of the original AR-quiver. The dashed arrows help us read almost split triangles from the graph. For example $S_1[-1] \to P_2 \to P_1 \to S_1$ is an almost split triangle, and so is $M \to S_1 \oplus P_3[1] \to P_2[1] \to M[1]$.

5 Localization to a module category

Let $\mathcal{C}$ be an additive category, $T \in \mathrm{Ob}(\mathcal{C})$, and consider the functor $H = \mathrm{Hom}_{\mathcal{C}}(T, -)$. Its domain is clearly the category $\mathcal{C}$, and so far we have used the category of abelian groups as its range. Now consider the ring $\Lambda = \mathrm{End}_{\mathcal{C}}(T) = \mathrm{Hom}_{\mathcal{C}}(T, T)$. Its elements are the endomorphisms on T, and clearly these can act on any group $\mathrm{Hom}_{\mathcal{C}}(T, X)$. We can even check that this makes $\mathrm{Hom}_{\mathcal{C}}(T, X)$ a right Λ-module, or equivalently a left Λ^{op}-module. Thus we may take the range of H to be $\mathrm{mod}(\mathrm{End}_{\mathcal{C}}(T)^{op})$.

In this section we will study the localization described by Buan and Marsh in [3], where we find that for a restricted class of categories, there exists a localization equivalent to $\mathrm{mod}(\mathrm{End}_{\mathcal{C}}(T)^{op})$. Specifically, we will take a rigid object T in a Hom-finite Krull-Schmidt triangulated category $\mathcal{C}$ over a field k and specify a suitable class of morphisms by which $\mathcal{C}$ localizes to a category equivalent to $\mathrm{mod}\,\mathrm{End}_{\mathcal{C}}(T)^{op}$.

Obviously, we need to start by giving some definitions.

Definition 5.1. A category $\mathcal{C}$ is *Hom-finite* over a field k if for all $X, Y \in \mathrm{Ob}(\mathcal{C})$ the set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ are finite-dimensional vector spaces over k.

Definition 5.2. A *Krull-Schmidt* category $\mathcal{C}$ is a category where the Krull-Schmidt theorem holds; that is to say that any $X \in \mathrm{Ob}(\mathcal{C})$ can be written as a finite, direct sum of indecomposable objects, unique up to permutation of factors.

Definition 5.3. An object T in a triangulated category $\mathcal{C}$ is rigid if $\mathrm{Ext}_{\mathcal{C}}(T, T) = 0$.

Note that when $\mathcal{C}$ is triangulated, we define $\mathrm{Ext}_{\mathcal{C}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, \Sigma Y)$, where Σ is the suspension functor of $\mathcal{C}$. We note that the requirements for the category and object in question are rather strong, but the requirements for the category will hold in the case of $D^b(\Lambda)$ where Λ is a finite dimensional algebra of finite global dimension.

Having set the stage, we will define the class of morphisms shortly. Let $\mathcal{X}$ be a full subcategory of $\mathcal{C}$; then we set $\mathcal{X}^{\perp} = \{C \in \mathcal{C} \,|\, \mathrm{Ext}_{\mathcal{C}}(X, C) = 0 \,\forall\, X \in \mathcal{X}\}$. For an object X we set $X^{\perp} = \mathrm{Add}(X)^{\perp}$ (here, $\mathrm{Add}(X)$ is the smallest additive subcategory of $\mathcal{C}$ containing X). Now, fix a rigid object T in $\mathcal{C}$ and let

$$\tilde{S} = \{f : X \to Y \,|\, \text{for } \Sigma^{-1}Z \xrightarrow{h} X \xrightarrow{f} Y \xrightarrow{g} Z \text{ distinguished},$$
$$g \text{ and } h \text{ factorize through } \Sigma T^{\perp}\}$$
$$S = \{f : X \to Y \,|\, \text{for } \Sigma^{-1}Z \xrightarrow{h} X \xrightarrow{f} Y \xrightarrow{g} Z \text{ distinguished},$$
$$g \text{ factorizes through } \Sigma T^{\perp} \text{ and } \Sigma^{-1}Z \in \Sigma T^{\perp}\}$$

We see that for these classes, which can be shown to be well-defined[7], it holds that $S \subseteq \tilde{S}$. Denoting the localization of $\mathcal{C}$ by a class of morphisms M as $\mathcal{C}_M$, with the localization functor denoted L_M, we see that since $S \subseteq \tilde{S}$, and $L_{\tilde{S}}$ makes every element

---

[7]the proof boils down to showing that there is an isomorphism between any two triangles where f is the middle morphism
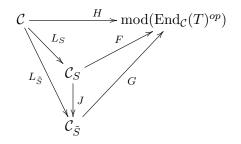
of $\tilde{S}$ invertible, then by universality, there must exist a functor $J : \mathcal{C}_S \to \mathcal{C}_{\tilde{S}}$, such that $L_{\tilde{S}} = JL_S$.

It turns out that there is a different description of $\tilde{S}$. Let $f \in \tilde{S}$, and choose a distinguished triangle $\Sigma^{-1}Z \xrightarrow{h} X \xrightarrow{f} Y \xrightarrow{g} Z$. Applying the functor $\mathrm{Hom}_{\mathcal{C}}(T, -)$ and using the fact that it is homological, we see that the following sequence is exact:

$$\mathrm{Hom}(T, \Sigma^{-1}Z) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(T,h)} \mathrm{Hom}_{\mathcal{C}}(T, X) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(T,f)} \mathrm{Hom}_{\mathcal{C}}(T, Y) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(T,g)} \mathrm{Hom}_{\mathcal{C}}(T, Z)$$

However, since g and h factorize through $\Sigma T^{\perp}$, if follows that $\mathrm{Hom}_{\mathcal{C}}(T, g) = 0 = \mathrm{Hom}_{\mathcal{C}}(T, h)$. Since the sequence is exact it follows that $\mathrm{Hom}_{\mathcal{C}}(T, f)$ is an isomorphism. Conversely, if a morphism $f : X \to Y$ is such that $\mathrm{Hom}_{\mathcal{C}}(T, f)$ is an isomorphism, then we have $\mathrm{Hom}_{\mathcal{C}}(T, Z) \cong 0$, thus $Z \in \Sigma T^{\perp}$ and $f \in \tilde{S}$. This means that $\tilde{S}$ is exactly the collection of morphisms f for which $\mathrm{Hom}_{\mathcal{C}}(T, f)$ is an isomorphism.

We now know that the functor $H = \mathrm{Hom}_{\mathcal{C}}(T, -) : \mathcal{C} \to \mathrm{mod}\,\mathrm{End}(T)^{op}$ turns elements of $\tilde{S}$ into isomorphisms; then by universality there must exist a functor $G : \mathcal{C}_{\tilde{S}} \to \mathrm{mod}\,\mathrm{End}(T)^{op}$, such that $H = GL_{\tilde{S}}$. Since $S \subseteq \tilde{S}$, there must also exist $F : \mathcal{C}_S \to \mathrm{mod}\,\mathrm{End}(T)^{op}$ with $H = FL_S$. The situation is summed up in the following commutative diagram:

We will show that F is an equivalence of categories, and that J is an isomorphism of categories. Thus it will follow that $\mathcal{C}_{\tilde{S}} \cong \mathrm{mod}(\mathrm{End}_{\mathcal{C}}(T)^{op})$.

Lemma 5.4. $F : \mathcal{C}_S \to \mathrm{mod}(\mathrm{End}_{\mathcal{C}}(T)^{op})$ *is an equivalence of categories*

Proof. By definition, F is an equivalence of categories if and only if it is full, faithful and dense.

We will show these in turn:

Dense First, we state that the following lemma can be proven by translating the proof given for proposition II.2.1 in [1] to category theory:

Lemma 5.5. *Let $\mathcal{C}$ be an additive Krull-Schmidt category with enough projectives, let $X \in \mathrm{Ob}(\mathcal{C})$ and let $e_X = \mathrm{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathrm{mod}(\mathrm{End}_{\mathcal{C}}(X)^{op})$. Then it holds that*

(i) e_X acts as an isomorphism on the Hom-sets of $\mathcal{C}$.

(ii) If $Y \in \mathrm{Add}(X)$ then $e_X(Y)$ is a projective module.

(iii) e_X is an equivalence of categories between $\mathrm{Add}(X)$ and $\mathcal{P}(\mathrm{End}_{\mathcal{C}}(X)^{op})$, the projective modules over $\mathrm{End}_{\mathcal{C}}(X)^{op}$.

This implies that H is dense; consider any object $M \in \mathrm{mod}(\mathrm{End}_{\mathcal{C}}(T)^{op})$, let $P_1 \xrightarrow{f} P_0 \to M$ be its minimal projective presentation (which exists since $\mathrm{End}_{\mathcal{C}}(T)$ is finite dimensional as a k-vector space and thus a finite dimensional k-algebra). By the above lemma, H is an equivalence between $\mathcal{P}(\mathrm{End}_{\mathcal{C}}(T)^{op})$ and $\mathrm{Add}(T)$, so there must exist a morphism $T_1 \xrightarrow{g} T_0$ so that $H(T_1) = P_1$, $H(T_2) = P_2$ and $H(g) = f$. We can complete g to a triangle $T_1 \to T_0 \to X \to \Sigma T_1$. If we the use the fact that H is a cohomological functor, we get the following commutative diagram:

$$
\begin{array}{ccccccc}
\mathrm{Hom}(T, T_1) & \longrightarrow & \mathrm{Hom}(T, T_0) & \longrightarrow & \mathrm{Hom}(T, X) & \longrightarrow & \mathrm{Hom}(T, \Sigma T_1) \\
\| & & \| & & \downarrow & & \| \\
P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}
$$

Using the five lemma (lemma A.1), we get that $\mathrm{Hom}(T, X) \to M$ is an isomorphism; thus H is dense. Since $H = FL_S$, this means that F is dense as well.

Full Let $\mathcal{C}(T)$ be the class of $X \in \mathrm{Ob}(\mathcal{C})$ so that X can be embedded in a distinguished triangle $T_1 \to T_0 \to X \to \Sigma T_1$ with $T_1, T_0 \in \mathrm{Add}(T)$.

It is shown in [3] that H induces an equivalence between the categories $\mathcal{C}(T)/\Sigma T^{\perp}$ and $\mathrm{mod}(\mathrm{End}_{\mathcal{C}}(T)^{op})$. In the same article it is also shown that for any $Y \in \mathrm{Ob}\,\mathcal{C}$ there exists a morphism $f : X \to Y$ where $X \in \mathcal{C}(T)$ and $f \in S$.

Suppose $\alpha : F(X) \to F(Y)$ is a morphism in $\mathrm{mod}(\mathrm{End}_{\mathcal{C}}(T)^{op})$ (we can make these assumptions on its domain and range without loss of generality because F is dense). There must exist morphisms $X' \xrightarrow{u} X$ and $Y' \xrightarrow{v} Y$ so that $X', Y' \in \mathcal{C}(T)$ and $u, v \in S$. We define $\alpha' = H(v)^{-1}\alpha H(u)$. Thus the following diagram is commutative and the vertical morphisms are invertible.

$$
\begin{array}{ccc}
F(X') & \xrightarrow{\alpha'} & F(Y') \\
\downarrow{\scriptstyle H(u)} & & \downarrow{\scriptstyle H(v)} \\
F(X) & \xrightarrow{\alpha} & F(Y)
\end{array}
$$

As α' is a morphism of images of objects in $\mathcal{C}(T)$ and H is a full functor when restricted to $\mathcal{C}(T)$, there must exist $f' : X' \to Y'$ so that $H(f') = \alpha'$. But then $F(L_S(v)L_S(f')L_S(u)^{-1}) = H(v)H(f')H(u)^{-1} = \alpha$. Thus F is full.

Faithful Let $X, Y \in \mathrm{Ob}(\mathcal{C})$ and suppose $f, g : L_S(X) \to L_S(Y)$ are such that $F(f) = F(g)$. Let X', Y', u and v be as above; it can then be shown that there exist morphisms $f', g' : X' \to Y'$ (in $\mathcal{C}$, as opposed to f and g, which are in $\mathcal{C}_S$) so that

$$L_S(f') = L_S(v)^{-1} f L_S(u) \text{ and } L_S(g') = L_S(v)^{-1} g L_S(u). \text{ Then}$$

$$H(f') = F(L_S(f')) = F(L_S(v)^{-1} f L_S(u)) = H(v)^{-1} F(f) H(u)$$
$$= H(v)^{-1} F(g) H(u) = H(g')$$

Consequently $H(g' - f') = 0$. Since H is an equivalence between $\mathcal{C}(T)/\Sigma T^{\perp}$ and $\mathrm{mod}(\mathrm{End}_{\mathcal{C}}(T)^{op})$, we must have that $g' - f'$ factorize through $\Sigma T^{\perp}$. Thus $L_S(f') = L_S(f') + L_S(g') - L_S(f') = L_S(g')$, and so $f = L_S(v) L_S(f') L_S(u)^{-1} = L_S(v) L_S(g') L_S(u)^{-1} = g$. Consequently, F is faithful.

Since F is full, faithful and dense, it is an equivalence of categories. $\square$

We now consider J:

Lemma 5.6. $J : \mathcal{C}_S \to \mathcal{C}_{\tilde{S}}$ *is an isomorphism of categories*

Proof. We know that $H = F L_S$, so since F is an equivalence of categories and H make morphisms in $\tilde{S}$ invertible, the functor L_S must do so as well. Thus there must exist a functor $I : \mathcal{C}_{\tilde{S}} \to \mathcal{C}_S$ with $I L_{\tilde{S}} = L_S$. Since $IJL_S = IL_{\tilde{S}} = L_S$ we have by universality of L_S that $IJ = \mathrm{id}_{\mathcal{C}_S}$. Similarly, since $JIL_{\tilde{S}} = L_{\tilde{S}}$, we have $JI = \mathrm{id}_{\mathcal{C}_{\tilde{S}}}$. Thus J is an isomorphism of categories. $\square$

We have proven:

Theorem 5.7.
$$\mathcal{C}_{\tilde{S}} \cong \mathrm{mod}(\mathrm{End}_{\mathcal{C}}(T)^{op})$$

Example 5.8. Once again we study the quiver $\Gamma : 1 \to 2 \to 3$ and the category $\mathrm{D}^b(k\Gamma)$. Let $T = P_3 \oplus P_1 \oplus S_1$. We redraw the AR-quiver of $\mathrm{D}^b(k\Gamma)$ and mark the summands of T for reference

We see that the only composition of irreducible morphisms from the indecomposable summands of T to the indecomposable summands of $T[1] \cong P_3[1] \oplus P_1[1] \oplus S_1[1]$ factor through almost split (and thus distinguished) triangles, and so must be zero. Consequently, $\mathrm{Ext}_{\mathrm{D}^b(k\Gamma)}(T, T[1]) = 0$ and T is rigid. We thus know that if we perform the localization described above with respect to the object T we will end up in the category $\mathrm{mod}(\mathrm{End}_{\mathrm{D}^b(k\Gamma)}(T)^{op})$.

What does $\mathrm{End}^{op}_{\mathrm{D}^b(k\Gamma)}(T)$ look like? Since T is a complex concentrated in (i. e is non-zero only in) one term, we only need to look at the endomorphisms on the object in that one term, namely the endomorphisms of the *module* $P_3 \oplus P_1 \oplus S_1$. These are

limited to the linear combination of the (non-zero) morphisms between the components; and as we see from the AR-quiver 4.2, these are, up to isomorphism, the monomorphism $P_3 \xrightarrow{\iota} P_1$, the epimorphism $P_1 \xrightarrow{\pi} S_1$ and the identity morphisms. Also, $\pi\iota = 0$. We thus see that $\mathrm{End}_{\mathrm{D}^b(k\Gamma)}(T)^{op} \cong k\Gamma_T / \langle \rho_T \rangle$ where (Γ_T, ρ_T) is the quiver with relations

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

The corresponding AR-quiver is

6   Summary

We have described two main ways of constructing a localization. Gabriel-Zisman localization works for any skeletally small category, but give us very little structure to work with in the localized category. Localization by a multiplicative system gives us a very simple structure thanks to the Ore condition, but the restrictions on the class to be localized are rather strong.

We have also studied two concrete examples where localization is used. The derived category was constructed by using localization of a multiplicative system, while the localization to a module category used Gabriel-Zisman localization. The former turned an abelian category into a triangulated category, while the latter turned a triangulated category into an abelian category.

References

[1] Maurice Auslander, Idun Reiten, and Sverre O. Smalø. *Representation Theory of Artin Algebras*. Cambridge University Press, 1997.

[2] P. B. Bhattacharya, S. K. Jain, and S. R. Nagpaul. *Basic abstract algebra*. Cambridge University Press, second edition, 1994.

[3] Aslak Bakke Buan and Robert J. Marsh. From triangulated categories to module categories via localisation. *Transactions of the American Mathematical Society*, 2011.

[4] Peter Freyd. *Abelian categories : an introduction to the theory of functors*. New York : Harper & Row, 1966.

[5] Sergei I. Gelfand and Yuri I. Manin. *Methods of homological algebra.* Berlin : Springer, second edition, 2003.

[6] Dieter Happel. Auslander-reiten triangles in derived categories of finite-dimensional algebras. *Proceedings of the American Mathematical Society*, 112(3), July 1991.

[7] Lidia Angeleri Hügel. An introduction to Auslander-Reiten theory. Lecture notes retrieved 21.1.2012, `http://www.math.jussieu.fr/~keller/ictp2006/lecturenotes/angeleri.pdf`.

[8] Karin Marie Jacobsen. Abelian categories, 2011. Student project in mathematics.

[9] Masaki Kashiwara and Pierre Schapira. *Sheaves on manifolds.* Berlin : Springer, 1990.

[10] Steffen König and Alexander Zimmermann. *Derived equivalences for group rings.* Berlin : Springer, 1998.

A Diagram lemmas for abelian categories

The following are technical lemmas needed in the rest of the thesis. For the whole of this appendix we assume that we are working in an abelian category. For other technical lemmas used, see [4] or [8].

A.1 The five lemma

Lemma A.1. *Suppose the following diagram in an abelian category is commutative with exact rows:*

$$
\begin{array}{ccccccccc}
X_0 & \xrightarrow{a} & X_1 & \xrightarrow{b} & X_2 & \xrightarrow{c} & X_3 & \xrightarrow{d} & X_4 \\
\downarrow{f_0} & & \downarrow{f_1} & & \downarrow{f_2} & & \downarrow{f_3} & & \downarrow{f_4} \\
Y_0 & \xrightarrow{a'} & Y_1 & \xrightarrow{b'} & Y_2 & \xrightarrow{c'} & Y_3 & \xrightarrow{d'} & Y_4
\end{array}
\tag{A.1}
$$

(i) *if f_0 is an epimorphism and f_1 and f_3 are monomorphisms, then f_2 is a monomorphism.*

(ii) *if f_4 is a monomorphism and f_1 and f_3 are epimorphisms, then f_2 is an epimorphism.*

(iii) *if f_0 is an epimorphism, f_4 is a monomorphism and f_1 and f_3 are isomorphisms, then f_2 is an isomorphism.*

Proof. As statement *(ii)* is the dual of statement *(i)* and statement *(iii)* follows from statement *(i)* and *(ii)*, we will only prove statement *(i)*.

Suppose that f_0 is an epimorphism and f_1 and f_2 are monomorphisms. Let $i_{f_2} :$ $\mathrm{Ker}(f_2) \to X_2$ be the kernel monomorphism[8]. We see that

$$f_3 c i_{f_2} = c' f_2 i_{f_2} = 0$$

As f_3 is a monomorphism, we must have $c i_{f_2} = 0$. Thus there exists a unique morphism $i : \mathrm{Ker}(f_2) \to \mathrm{Ker}(c)$ such that $i_c i = i_{f_2}$. Moreover, i is a monomorphism. Using the fact that $\mathrm{Ker}(c) = \mathrm{Im}(b) = \mathrm{Coim}(b) = \mathrm{Cok}(a)$ (by exactness of the upper sequence of diagram A.1), we study the following pullback completion:

$$
\begin{array}{ccc}
P & \xrightarrow{\ u\ } & \mathrm{Ker}(f_2) \\
\downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle i} \\
X_1 & \xrightarrow{\ p_a\ } & \mathrm{Im}\, b
\end{array}
$$

As p_a is an epimorphism, so is u and as i is a monomorphism, so is u'. We see that

$$b' f_1 u' = f_2 b u' = f_2 i_c p_a u' = f_2 i_c i u = f_2 i_{f_2} u = 0.$$

Thus there exists a unique morphism $j : P \to \mathrm{Ker}(b')$ so that $i_{b'} j = f_1 u'$. As $f_1 u'$ is a composition of monomorphisms, j is a monomorphism. We perform another pullback completion (where we use that $\mathrm{Ker}(b') = \mathrm{Im}(a)$).

$$
\begin{array}{ccc}
Q & \xrightarrow{\ v\ } & P \\
\downarrow{\scriptstyle v'} & & \downarrow{\scriptstyle j} \\
X_0 & \xrightarrow{\ p f_0\ } & \mathrm{Im}\, a
\end{array}
$$

Again we observe that as $p f_0$ is an epimorphism and j is a monomorphism, v is an epimorphism and v' is a monomorphism. We see that $f_1 a v' = a' f_0 v = f_1 u' v$. As f_1 is a monomorphism, it follows that $a v' = u' v$. But then we have that $i_{f_2} u v = b u' v = b a v' = 0$. As i_{f_2} is a monomorphism, we must have $u v = 0$. However $u v$ is a composition of epimorphisms and is thus itself an epimorphism onto $\mathrm{Ker}(f_2)$, so it follows that $\mathrm{Ker}(f_2) = 0$, and thus f_2 is a monomorphism. □

A.2   The snake lemma

The Snake Lemma is a classical lemma for abelian categories. There are two main variants of the lemma; we will first show the weaker version, and then give the stronger version as a corollary.

---

[8]for the remainder of this appendix we will, given a morphism f, denote the kernel monomorphism i_f and the cokernel epimorphism p_f

Lemma A.2. *Suppose that in an abelian category $\mathcal{C}$ the following is a commutative diagram with exact rows:*

$$
\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 \longrightarrow & X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' &
\end{array}
$$

Then there exists a unique exact sequence

$$\mathrm{Ker}(\alpha) \to \mathrm{Ker}(\beta) \to \mathrm{Ker}(\gamma) \xrightarrow{\phi} \mathrm{Cok}(\alpha) \to \mathrm{Cok}(\beta) \to \mathrm{Cok}(\gamma).$$

Proof. We will first show that the sequence $\mathrm{Ker}(\alpha) \to \mathrm{Ker}(\beta) \to \mathrm{Ker}(\gamma)$ exists and is exact. As $\beta f i_\alpha = f' \alpha i_\alpha = 0$, there must exist a unique morphism $\bar{f} : \mathrm{Ker}(\alpha) \to \mathrm{Ker}(\beta)$ so that $i_\beta \bar{f} = f i_\alpha$. Similarly there must exist a unique morphism $\bar{g} : \mathrm{Ker}(\beta) \to \mathrm{Ker}(\gamma)$ such that $i_\gamma \bar{g} = g i_\beta$.

We now need to show that $\mathrm{Ker}(\bar{g}) = \mathrm{Im}(\bar{f})$. Note first that $i_\gamma \bar{g} \bar{f} = g f i_\alpha = 0$. Since i_γ is a monomorphism, this means that $\bar{g}\bar{f} = 0$. Thus there exists a morphism $u : \mathrm{Ker}(\alpha) \to \mathrm{Ker}(\bar{g})$, so that $i_{\bar{g}} u = \bar{f}$.

We see that $g i_\beta i_{\bar{g}} = i_\gamma \bar{g} i_{\bar{g}} = 0$, and that there therefore must exist a unique morphism $i : \mathrm{Ker}(\bar{g}) \to \mathrm{Ker}(g)$, which is necessarily a monomorphism, so that $i_g i = i_\beta i_{\bar{g}}$. Letting $p : X \to \mathrm{Im}(f)$ be the image mapping for f and using that $\mathrm{Im}(f) = \mathrm{Ker}(g)$, we consider the following diagram:

$$
\begin{array}{ccc}
\mathrm{Ker}(\alpha) & \xrightarrow{u} & \mathrm{Ker}(\bar{g}) \\
\downarrow{\scriptstyle i_\alpha} & & \downarrow{\scriptstyle i} \\
X & \xrightarrow{p} & \mathrm{Im}(f)
\end{array}
$$

We see that $i_g p i_\alpha = f i_\alpha = i_\beta \bar{f} = i_\beta i_{\bar{g}} u = i_g i u$. As i_g is a monomorphism, it follows that $i u = p i_\alpha$, and the above diagram commutes. Suppose Q is an object with morphisms v and v' so the following diagram commutes:

$$
\begin{array}{ccc}
Q & \xrightarrow{v} & \mathrm{Ker}(\bar{g}) \\
\downarrow{\scriptstyle v'} & & \downarrow{\scriptstyle i} \\
X & \xrightarrow{p} & \mathrm{Im}(f)
\end{array}
$$

As $f' \alpha v' = \beta f v' = \beta i_\beta i_{\bar{g}} v = 0$ and f' is a monomorphism, it follows that $v' \alpha = 0$. Thus there must exist a unique morphism $w : Q \to \mathrm{Ker}(\alpha)$, such that $v' = i_\alpha w$. Furthermore, $i u w = p i_\alpha w = p v' = i v'$. Given that i is a monomorphism, we get that $u w = v'$. Thus the first square is actually a pullback square, and it follows that u is an epimorphism. As the factorization $\bar{f} = i_{\bar{g}} u$, where u is an epimorphism and $i_{\bar{g}}$ is a monomorphism exists, it follows that $\mathrm{Ker}(\bar{g}) = \mathrm{Im}(\bar{f})$, and the kernel sequence $\mathrm{Ker}(\alpha) \xrightarrow{\bar{f}} \mathrm{Ker}(\beta) \xrightarrow{\bar{g}} \mathrm{Ker}(\gamma)$ is exact. It follows dually that the cokernel sequence $\mathrm{Cok}(\alpha) \xrightarrow{\bar{\bar{f}}} \mathrm{Cok}(\beta) \xrightarrow{\bar{\bar{g}}} \mathrm{Cok}(\gamma)$ is exact.

The next step is to construct the morphism $\phi : \mathrm{Ker}(\gamma) \to \mathrm{Cok}(\alpha)$. Consider the following commutative diagram with exact rows (where the dashed arrows will be shown to exist):

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}(u) & \xrightarrow{\;i_u\;} & \mathrm{Ker}(\gamma g) & \xrightarrow{\;u\;} & \mathrm{Ker}(\gamma) & \longrightarrow & 0 \\
& & & & \downarrow{\scriptstyle i_{\gamma g}} & & \downarrow{\scriptstyle i_\gamma} & & \\
& & X & \xrightarrow{\;f\;} & Y & \xrightarrow{\;g\;} & Z & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \alpha} & {\scriptstyle v} & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & X' & \xrightarrow{\;f'\;} & Y' & \xrightarrow{\;g'\;} & Z' & & \\
& & \downarrow{\scriptstyle p_\alpha} & & \downarrow{\scriptstyle p_{f'\alpha}} & {\scriptstyle v'} & & & \\
0 & \longrightarrow & \mathrm{Cok}(\alpha) & \xrightarrow{\;u'\;} & \mathrm{Cok}(f'\alpha) & \xrightarrow{\;p_{u'}\;} & \mathrm{Cok}(u') & \longrightarrow & 0
\end{array}
$$

As $\gamma g i_\gamma = 0$, there must exist a morphism $u : \mathrm{Ker}(\gamma g) \to \mathrm{Ker}(\gamma)$, so that $i_\gamma u = g i_{\gamma g}$. Consider any object U with morphisms so that the following diagram commutes:

$$
\begin{array}{ccc}
U & \xrightarrow{\;x\;} & \mathrm{Ker}(\gamma) \\
\downarrow{\scriptstyle y} & & \downarrow{\scriptstyle i_\gamma} \\
Y & \xrightarrow{\;g\;} & Z
\end{array}
$$

As $\gamma g y = \gamma i_\gamma x = 0$, there exists a map $z : U \to \mathrm{Ker}(\gamma g)$ so that $i_{\gamma g} z = gy$. Furthermore, $i_\gamma u z = g i_{\gamma g} z = gy = i_\gamma x$. As i_γ is a monomorphism, this means $uz = x$. Thus $\mathrm{Ker}(\gamma g)$ is the pullback completion of the square. As g is an epimorphism, this means that u is an epimorphism. Dually the morphism $u' : \mathrm{Cok}(\alpha) \to \mathrm{Cok}(f'\alpha)$ exists and is a monomorphism.

We see that $g'\beta i_{\gamma g} = \gamma g i_{\gamma g} = 0$. As $X' = \mathrm{Ker}(g')$, with f' being the kernel monomorphism, this means that there exists a morphism $v : \mathrm{Ker}(\gamma g) \to X'$ so that $f'v = \gamma i_{\gamma g}$. Dually there exists a morphism $v' : Z \to \mathrm{Cok}(f'\alpha)$ so that $p_{f'\alpha}\beta = v'g$.

Since $p_{u'}v'i_\gamma u = pu'p_\alpha v = 0$, and u is an epimorphism, $p_{u'}v'i_\gamma = 0$. However, by exactness, u' is the kernel of $p_{u'}$. Thus there must exist a map $\phi : \mathrm{Ker}(\gamma) \to \mathrm{Cok}(\alpha)$, such that $u'\phi = v'i_\gamma$. We have shown the existence of the sequence

$$
\mathrm{Ker}(\alpha) \xrightarrow{\bar{f}} \mathrm{Ker}(\beta) \xrightarrow{\bar{g}} \mathrm{Ker}(\gamma) \xrightarrow{\phi} \mathrm{Cok}(\alpha) \xrightarrow{\bar{\bar{f}}} \mathrm{Cok}(\beta) \xrightarrow{\bar{\bar{g}}} \mathrm{Cok}(\gamma).
$$

It remains to show exactness. Consider the pullback diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\;a\;} & \mathrm{Ker}(\phi) \\
\downarrow{\scriptstyle a'} & & \downarrow{\scriptstyle i_\gamma i_\phi} \\
Y & \xrightarrow{\;g\;} & Z
\end{array}
$$

As $g'\beta a' = \gamma g a' = \gamma i_\gamma i_\phi = 0$ and X is the kernel of g', there exists a morphism $b : A \to X'$ such that $\beta a' = f'b$. In addition, there exists a morphism $c : A \to \mathrm{Ker}(\gamma g)$ such that

$i_{\gamma g}c = a'$. Furthermore, $i_\gamma uc = gi_{\gamma g}c = ga' = i_\gamma i_\phi a$. As i_γ is a monomorphism, it follows that $uc = i_\phi a$.
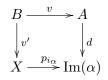
By definition $u'\phi u = p_{f'\alpha}\beta i_{\gamma g}$, and we have that

$$0 = u'\phi i_\phi a = u'\phi uc = p_{f'\alpha}\beta i_{\gamma g}c = p_{f'\alpha}\beta a' = p_{f'\alpha}f'b = u'p_\alpha b.$$

As u' is a monomorphism, this means that $p_\alpha b = 0$. As $\text{Im}(\alpha) = \text{Ker}(p_\alpha)$, this means that there exists a unique morphism $d : A \to \text{Im}(\alpha)$ so that $i_{p_\alpha}d = b$. We have that $p_\beta f'i_{p_\alpha} = \bar{\bar{f}}p_\alpha i_{p_\alpha} = 0$, so there exists a unique morphism $w : \text{Im}(\alpha) \to \text{Im}(\beta)$ so that $i_{p_\beta}w = f'i_{p_\alpha}$. We have

$$i_{p_\beta}p_{i_\beta}a' = \beta a' = f'b = f'i_\alpha d = i_{p_\beta}wd,$$

and since i_{p_β} is a monomorphism, this means that $p_{i_\beta}a' = wd$. We use this fact to see that $f'i_{p_\alpha}d = i_{p_\beta}wd = i_{p_\beta}p_{i_\beta}a' = \beta a'$. Let B be the following pullback completion:

$$
\begin{array}{ccc}
B & \xrightarrow{\;v\;} & A \\
\downarrow{\scriptstyle v'} & & \downarrow{\scriptstyle d} \\
X & \xrightarrow{\;p_{i_\alpha}\;} & \text{Im}(\alpha)
\end{array}
$$

As p_{i_α} is an epimorphism, so is v. We summarize the work we have done so far in the following diagram:

We get that

$$\beta fv' = f'\alpha v' = f'i_{p_\alpha}p_{i_\alpha}v' = f'i_{p_\alpha}dv = f'bv = \beta av,$$

and thus $\beta(fv' - a'v) = 0$, so there must exist a morphism $e : B \to \text{Ker}(\beta)$ so that $i_\beta e = fv' - a'v$. As

$$i_\gamma \bar{g}e = gi_\beta e = ga'v = i_\gamma i_\phi av$$

and i_γ is a monomorphism, we have $\bar{g}e = i_\phi av$. Since $\bar{g}\phi = 0$, there exists a morphism $x : \mathrm{Ker}(\beta) \to \mathrm{Ker}(\phi)$ so that $i_\phi x = \bar{g}$. However, this means that $i_\phi xe = \bar{g}e = i_\phi av$ and since i_ϕ is injective, that means $xd = av$. Both a and v are epimorphisms, so x must be an epimorphism. This means that we have constructed a factorization $\bar{g} = i_\phi x$ through $\mathrm{Ker}(\phi)$ where x is an epimorphism and i_ϕ is a monomorphism. Thus $\mathrm{Ker}(\phi) = \mathrm{Im}(\bar{g})$, and the sequence is exact in $\mathrm{Ker}(\gamma)$. Dually it is exact in $\mathrm{Cok}(\alpha)$. $\qquad\square$

Corollary A.3. *Suppose the following is a commutative diagram with exact rows:*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \overset{f}{\longrightarrow} & Y & \overset{g}{\longrightarrow} & Z & \longrightarrow & 0 \\
& & \downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} & & \\
0 & \longrightarrow & X' & \overset{f'}{\longrightarrow} & Y' & \overset{g'}{\longrightarrow} & Z' & \longrightarrow & 0
\end{array}
$$

then there exists a unique long exact sequence

$$0 \to \mathrm{Ker}(\alpha) \to \mathrm{Ker}(\beta) \to \mathrm{Ker}(\gamma) \overset{\phi}{\to} \mathrm{Cok}(\alpha) \to \mathrm{Cok}(\beta) \to \mathrm{Cok}(\gamma) \to 0.$$

Proof. Using the terminology from the above proof, what we need to prove is that if f is a monomorphism, so is $\bar{f}$ (that $\bar{g}$ is an epimorphism if g is, follows dually). However, as $i_\beta \bar{f} = f i_\alpha$ and $f i_\alpha$ is a monomorphism, $\bar{f}$ is a monomorphism. $\qquad\square$

A.3 Technical lemma for theorem 4.12

Finally, we give a rather technical lemma required in the proof of theorem 4.12. The lemma is given as an exercise in [9].

Lemma A.4. *Suppose the following square commutes:*

$$
\begin{array}{ccc}
X' & \overset{f'}{\longrightarrow} & Y' \\
\downarrow{g'} & & \downarrow{g} \\
X & \overset{f}{\longrightarrow} & Y
\end{array}
$$

The following are equivalent:

  (i) If P is the pullback completion of f and g, then $X' \to P$ is an epimorphism.

  (ii) If Q is the pushout completion of f' and g', then $Y \to Q$ is a monomorphism.

(iii) The following diagram is commutative with exact rows and columns:

$$(A.2)$$

$$
\begin{array}{ccccccc}
& 0 & & 0 & & 0 & \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow \operatorname{Ker}(\phi) \longrightarrow \operatorname{Ker}(g') \longrightarrow \operatorname{Ker}(g) \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow \operatorname{Ker}(f') \longrightarrow X' \xrightarrow{f'} X \longrightarrow \operatorname{Cok}(f') \longrightarrow 0 \\
& \downarrow{\phi} & & \downarrow{g'} & & \downarrow{g} & & \downarrow{\psi} \\
0 \longrightarrow \operatorname{Ker}(f) \longrightarrow Y' \xrightarrow{f} Y \longrightarrow \operatorname{Cok}(f) \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow \operatorname{Cok}(g') \longrightarrow \operatorname{Cok}(g) \longrightarrow \operatorname{Cok}(\psi) \longrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& 0 & & 0 & & 0 &
\end{array}
$$

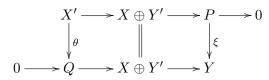*Proof. (i)⇔(ii)* Let P and Q be defined as the respective pullback and pushout completion. We then have a commutative diagram with exact rows:

$$
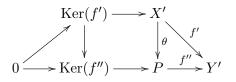\begin{array}{ccccccc}
X' & \longrightarrow & X \oplus Y' & \longrightarrow & P & \longrightarrow & 0 \\
\downarrow{\theta} & & \| & & \downarrow{\xi} & & \\
0 \longrightarrow Q & \longrightarrow & X \oplus Y' & \longrightarrow & Y & &
\end{array}
$$

By the snake lemma there exists an exact sequence $0 \to \operatorname{Ker}(\xi) \to \operatorname{Cok}(\theta) \to 0$; thus $\operatorname{Ker}(\xi) \cong \operatorname{Cok}(\theta)$ and θ is an epimorphism if and only if ξ is a monomorphism.

(i)⇒(iii) We start by looking at the pullback square

$$
\begin{array}{ccc}
P & \xrightarrow{f''} & Y' \\
\downarrow{g''} & & \downarrow{g} \\
X & \xrightarrow{f} & Y
\end{array}
$$
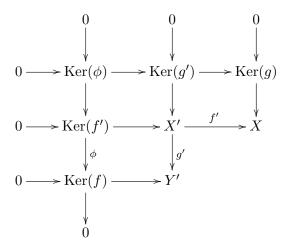
We let θ be the epimorphism $X' \to P$ and consider the following commutative diagram, where the bottom row is exact:

$$
\begin{array}{ccc}
\operatorname{Ker}(f') & \longrightarrow & X' \\
\nearrow & \downarrow & \searrow{f'} \\
0 \longrightarrow \operatorname{Ker}(f'') & \longrightarrow P & \xrightarrow{f''} Y'
\end{array}
$$

By a theorem proved in [8] we know that the square is pullback because $0 \to \mathrm{Ker}(f') \to X' \to Y'$ is exact. Hence $\mathrm{Ker}(f') \to \mathrm{Ker}(f'')$ is an epimorphism. However, we have $\mathrm{Ker}(f'') \cong \mathrm{Ker}(f)$ (because P is pullback), and so $\mathrm{Ker}(f') \to \mathrm{Ker}(f)$ is an epimorphism.

By using the "half version" of the nine lemma proved in [8], we find that the following diagram is commutative and has exact rows and columns:

$$
\begin{array}{ccccc}
 & 0 & & 0 & & 0 \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & \mathrm{Ker}(\phi) & \longrightarrow & \mathrm{Ker}(g') & \longrightarrow & \mathrm{Ker}(g) \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & \mathrm{Ker}(f') & \longrightarrow & X' & \xrightarrow{\;f'\;} & X \\
 & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle g'} & & \\
0 \longrightarrow & \mathrm{Ker}(f) & \longrightarrow & Y' & & \\
 & \downarrow & & & & \\
 & 0 & & & &
\end{array}
$$

Using the fact that *(i)⇔(ii)*, the rest of diagram A.2 follows dually.

(iii)⇒(i) Suppose P is the pullback completion in the diagram

$$
\begin{array}{ccc}
P & \xrightarrow{\;f''\;} & Y' \\
\downarrow{\scriptstyle g''} & & \downarrow{\scriptstyle g} \\
X & \xrightarrow{\;f\;} & Y
\end{array}
$$

The morphism $X' \to P$ exists by the universality of the pullback completion. We consider the following commutative diagram with exact rows:

$$
\begin{array}{ccccccc}
\mathrm{Ker}(f') & \longrightarrow & X' & \xrightarrow{\;f'\;} & Y' & \longrightarrow & \mathrm{Cok}(f') \\
\downarrow & & \downarrow & & \| & & \downarrow \\
\mathrm{Ker}(f'') & \longrightarrow & P & \xrightarrow{\;f''\;} & Y' & \longrightarrow & \mathrm{Cok}(f'') \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Ker}(f) & \longrightarrow & X & \xrightarrow{\;f\;} & Y & \longrightarrow & \mathrm{Cok}(f)
\end{array}
$$

Since *(iii)* holds, we have that $\phi : \mathrm{Ker}(f') \to \mathrm{Ker}(f)$ is an epimorphism and $\psi : \mathrm{Cok}(f') \to \mathrm{Cok}(f)$ is a monomorphism. Furthermore $\mathrm{Ker}(f'') \cong \mathrm{Ker}(f)$, and hence $\mathrm{Ker}(f') \to \mathrm{Ker}(f'')$ is an epimorphism and since $\mathrm{Cok}(f') \to \mathrm{Cok}(f'') \to \mathrm{Cok}(f) = \mathrm{Cok}(f') \to \mathrm{Cok}(f)$, we must have that $\mathrm{Cok}(f') \to \mathrm{Cok}(f'')$ is a monomorphism. Thus we can use the five lemma on the two upper rows to show that $X' \to P$ is an epimorphism. $\qquad\square$