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Geometric reduction and the three body problem

Thesis for the degree of Philosophiae Doctor

Trondheim, August 2012

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ISBN 978-82-471-3721-5 (printed ver.)
ISBN 978-82-471-3722-2 (electronic ver.)
ISSN 1503-8181

Doctoral theses at NTNU, 2012:211

Printed by NTNU-trykk

Abstract

This dissertation investigates a particular reduction of the three body problem, using a combination of Riemannian geometry and geometric invariant theory of three body motions in Euclidean space.

Our point of departure is the reduction that is described in [HS07]. Here, we present this reduction from a new point of view. This viewpoint emphasizes the flexibility in the choice of geometric invariants of three body motions, within one particular class of systems of invariants. Many of our important calculations are based on the singular value decomposition of matrices, and we show that the flexibility of the geometric invariants is strongly related to the flexibility of the singular value decomposition.

In addition, we go some steps further than [HS07]: In the context of the three dimensional three body problem, we calculate the reduced equations of motion in terms of our chosen system of invariants. The rotational part of this reduction is extended to the general case of many particle systems evolving in three dimensional space. We also include a large discussion on the conformal geometry of the shape invariants of the three body problem.

Preface

I submitted this dissertation to the Norwegian University of Science and Technology (NTNU) in partial fulfillment of the requirements for the degree ph.d. My funding has been provided by the Department of Mathematical Sciences, Faculty of Information Technology, Mathematics and Electrical Engineering, NTNU. I carried out the research at the Department of Mathematical Sciences, NTNU during the years 2007-2012, and at University of California, Berkeley in the spring semester 2011.

I would like to thank my adviser Eldar Straume for important feedback, inspiration and encouragement, as well as for teaching me the value of a good discussion. Likewise, I am in debt to Wu-Yi Hsiang for his hospitality and his willingness to share ideas and coffee on Brewed Awakening in Berkeley.

I would also like to thank my friend Mahdi K. Salehani for stimulating exchange of thoughts about celestial mechanics and Magnus B. Løberg, Haaken A. Moe, Hans J. Rivertz, Asgeir Steine and Marius Thaule for their mathematical friendship. Anne Kajander and the rest of administration staff at our department have always been very kind and helpful.

Finally, I want to thank my wife Vivianne and my children Anna and Sindre for their incomprehensible patience and confidence, and for making my life outside mathematics so rich.

Trondheim, June 2012
Lars Sydnes

Contents

1 Introduction

1.1 Background

The aim of this dissertation is to gain a better understanding of geometric reduction in classical mechanics. Rather than taking an abstract and coordinate free point of view, we try to understand reduction in general by consideration of particular examples, namely many particle systems with an emphasis on the three body problem. The three body problem provides a rich soil for such an investigation, due to the following:

(i) In the spatial three body problem, we have a minimal example of *non-Abelian reduction*. The rotation group SO(3), which is a symmetry group of the spatial three body problem, is without doubt the most important and also the simplest compact non-Abelian Lie group.

(ii) The group action of SO(3) on the configuration space of the three body problem has three different isotropy types. Hence, in the $SO(3)$-reduction of the three body problem we must handle the singularities of the group action and the associated quotient map. The number of such singularities is however not overwhelming.

(iii) In the three body problem, we can take significant advantage of the *democracy group*, which we will identify with the orthogonal group $O(2)$. This

> group represents the symmetries of the $SO(3)$-equivariant kinematic geometry of the configuration space. Hence, we can say that the three body problem is an example with a minimal non-trivial democracy group.

The three body problem can be taken as a "minimal example" exhibiting these features, i.e. a mechanical system with a non-Abelian group action with singularities and non-trivial democracy group. Rather than being directed towards astronomy, this study investigates these mathematical and geometrical structures, and aims at making them transparent.

We deliberately use the term *geometry* ambiguously: First, this term denotes the *kinematic geometry of the configuration space*, i.e. the Riemannian differential geometry of the configuration space. Our focus on Riemannian geometry is somehow in opposition with the more commonly used symplectic approach to classical mechanics. Secondly, the term *geometry* denotes the *Euclidean geometry* of three dimensional space. We aim at expressing the laws of motion in terms of Euclidean-geometric properties. In effect this amounts to expressing the laws of motion in terms of Euclidean-geometric invariants of three body configurations. Regarding the Euclidean geometry as a Klein geometry with symmetry group $E^+(3) = \mathbb{R}^3 \ltimes SO(3)$, we are led to the study of $SO(3)$-invariants of the configuration space.

The combination of the kinematic geometry and the Euclidean Klein geometry yields *the $SO(3)$-equivariant kinematic geometry of three body configurations*, which can be regarded as a synthesis of the kinematic geometry and the Euclidean geometry. This geometry will be taken as the background for our studies in dynamics.

This does not imply that the notions of equivariant geometry will always be in the foreground. The mathematical substance always boils down calculations with relations between various variables, and we try not to obscure this by using an advanced terminology. When doing mathematics, it may be tempting to prefer abstract terminology and seemingly advanced concepts. This may be a double-edged sword: Under an abstract point of view, we may see some important structures very clearly, but there is always a danger that we loose content. Facing this dilemma, we must always look for a healthy balance. In the present work, we tend towards using an elementary terminology. The more advanced concepts will rather serve as a guide through the resulting wilderness: Our chosen variables are meant to reveal the equivariant kinematic geometry of the three body problem, as well as the action of the democracy group. The resulting set of variables is however treated in the most elementary way.

The starting point of this investigation is the article [HS07] by Hsiang and Straume, which partially originates from [HS95]. One of the main objectives in the work of Hsiang and Straume is to find out to what extent the dynamics of three body motions is determined by the evolution of shape. In their work they parametrize three body shapes over a round sphere – the *shape sphere* – and in [HS08] they concluded that in the case of zero total angular momentum, three body motions are completely determined by geometric properties of the shape curves, i.e. the evolution of shape represented by a curve on the shape sphere. The generalization of this to general motions is still work in progress, and the present thesis can be regarded as a contribution in that direction. This contribution includes the following: (i) Interpretation and systematization of the variables of [HS07] by means of the singular value decomposition of matrices. (ii) Deduction of the reduced equations in the most general case, including a correction of the equations for the planar case in [HS07]. (iii) Generalization of the Euler equations in [HS07] to many particle systems and deformable bodies. (iv) Adaptation of Lemaitre's regularization [Lem64] of binary collisions.

1.1.1 Overview

This thesis consists of 5 chapters. The present chapter, which is Chapter 1, is the introduction chapter, and contains an overview over the dissertation, as well as a tiny discussion of the present results and future work. We also give a short note on various relations between geometry and many particle systems. Finally, we give an introduction to our approach to the three body problem.

Chapter 2 contains a discussion of many particle kinematics. In our terminology, *many* includes also infinitely many. This leads us to a Hilbert space formalism, which has two important advantages: (i) We see that our theory is a theory about inertial mass in motion in space that does not depend on finiteness of the number of particles, and (ii) we are moved in the direction of a coordinate-free approach. At the core of our application of Hilbert spaces lies the *constituent space*, which is a Hilbert space that is intended to represent the inertial mass of the system. In Section 2.3 we give a fairly comprehensive account on the notion of Jacobi vectors in many particle systems, with an emphasis on their flexibility, which we encode by the Jacobi groupoid. We need a thorough understanding of this topic in our study of the three body problem; several places we explicitly use the flexibility of the Jacobi groupoid to do important calculations. Our treatment of Jacobi vectors builds on [Str06], but in order to understand also transitions between Jacobi vectors associated with different mass distributions, we have to turn the topic upside down. In Section 2.5 we introduce the singular value decomposition of many particle configurations. We are particularly interested in some aspects of the perturbation theory of the singular value decomposition. Later, we will use this as the basis of our treatment of the three body problem, and also of our investigation of many particle motions with constant total angular momentum. Another important aspect of our study of the singular value decomposition is the introduction of the notion of multi-valued gauges. In Section 2.7, we generalize the Euler equations of [HS07] to arbitrary many-particle systems. The Euler equations arise in the study of many-particle motions using one particular rotating frame, namely the principal axes frame. To some extent, the Euler equations determine the rotation of the principal frame, and they can be regarded as a manifestation of the conservation of total angular momentum.

Chapter 3 contains the first part of our treatment of the three body problem, and focuses mainly on computations in terms of geometric invariants of three body motions. The first sections of this chapter concerns the specialization of some of the results in Chapter 2 to this particular case. In Section 3.7 we treat the reduction of Newton's equations of motion in the three body problem to a set of differential equations in a complete set of geometric invariants of three body motions. In Section 3.8 we give an alternative derivation of the reduced equations of motion, based on a method due to Poincaré [Poi01]. Finally, in Section 3.9, we treat a singular case which is not covered by the reduced equations of motion.

Chapter 4 investigates various shape spaces for the three body problem, i.e. spaces of three body configurations modulo scaling and rotation. In the first sections, we discuss various representations of three body shapes, and in particular, we investigate the flexibility in the choice of such a representation. This flexibility is closely related to the flexibility in choice of Jacobi vectors. In Section 4.3 we use this flexibility to discuss the hyperbolic geometry of three body shapes. In Section 4.4 we discuss the geometry of three body shapes that is induced by the kinematical structure of the three body problem itself. This leads to a study of three body shapes in terms of spherical geometry, a study that gives us a nice geometric representation of the reduced equations of motion and also a relation between the spherical area and the rotational orientation of three body motions with zero angular momentum. Finally, we demonstrate how the shape sphere may provide a fertile ground for the study of regularization of binary collisions.

In Chapter 5 we give some applications of the theory developed in the previous chapters. First, in Section 5.1 we apply the reduced equations of motion to classify the homographic motions of the three body problem. We do this for a 1-parameter family of potential functions, and present a slightly more detailed classification than the classification found in [Pyl41]. In Section 5.2 we sketch a path towards the solution of Cabral's constant inclination problem[Cab90].

1.1.2 Discussion

This thesis can be understood as a critical review of the approach of [HS07] and [Sal11]. With some exceptions most of the results stand the test. The most notable exceptions are the reduced equations of motion in the case of non-vanishing total angular momentum, and the proposed approach to the constant inclination problem in [Sal11]. Along with the revision of the reduced equations of motion, we also gain a better understanding of the singularities of the reduction procedure.

From personal correspondence with Eldar Straume, we know that the particular shape coordinates that are used in [HS07] grew out of the observation that the kinematic geometry of three body shapes is spherical, and that these coordinates were introduced in order to make this geometry transparent. With this point of departure, it was then desirable to find a physical interpretation of these variables. In the present work we started with a new definition of these variables based on the singular value decomposition of matrices, and in some sense, we turned everything upside down, and made a new point of departure for the investigation of the three body problem. It is not an unusual phenomenon in mathematics that the presentations of a theory move far away from the way the theory was conceived.

Above we mentioned the revisions suggested by the present work. Here, we will mention some points where we go beyond [HS07]:

Our treatment of the singularities of the reduction of the three body problem, and in particular the umbilic shape invariant motion in Section 3.9 is an important part of the understanding of the revised reduction procedure.

We recognize that the present reduction depends on a multi-valued choice of gauge, and that the resulting finite gauge group is a very nice book-keeping device.

In the application of the reduced equations of motion to the constant inclination problem, we are not able to proceed far. We only give a quite simple reformulation of the problem. An important aspect of this is however, that we are able to shed some light on the article [Sal11] by a straightforward application of the finite gauge group associated with the multi-valued choice of gauge.

The treatment of the regularization of binary collision also presents a new insight, namely that our approach to the reduction of the three body problem, and in particular the shape sphere is very well suited to understanding this topic. In particular, it reveals clearly the relation between the classical regularisation of the planar Kepler problem and the regularization of the three body problem.

The presence of hyperbolic geometry in the study of three body shapes is alluded to in [HS07] and mentioned in [Mon02]. In this thesis, we give a thorough presentation of this topic. In particular, we make the connection with the Jacobi vector flexibility very clear, and show also the limitations of this line of thought, namely that every hyperbolic shape invariant corresponds to mass distribution-invariant properties of three body configurations, but that the converse is not true.

Finally, we will claim that the present work provides a larger context for the study of the three body problem. In the generalization of the Euler equations of [HS07], we determine precisely the range of validity of these equations, in terms of which types of mechanical systems and which three body configurations they are valid for. Our investigation of the flexibility of the choice of Jacobi vectors provides a conceptual context for quite a few computational tricks that are used in our study of the three body problem. The multi-valued gauge and the associated finite gauge group yields a systematic way to distinguish between legal and illegal operations on the data involved in the reduction.

1.1.3 Main results

From the above discussion, we see that much of the material in this dissertation can be regarded as investigations and explorations of the three body problem as a rich mathematical landscape. Another important facet is the attempt to establish a simple and reliable computational framework, mainly by application of the singular value decomposition. There are however a few pertinent main results which we want to point out:

- Derivation of the reduced equations of motion og the three body problem (cf. Section 3.7 (Section 3.7)), as well as investigation of their range of validity.

- Derivation of the generalized Euler equations, and discussion of their range of validity (cf. Theorem 2.7.4 (Theorem 2.7.4)).

- The thorough investigation of Jacobi transformations (cf. Section 2.3), and clarification of their role in the conformal geometry of the shape space of the three body problem (cf. Section 4.3).

1.1.4 Future work

Finally, we mention some possible directions of future work.

- We want to find a more detailed characterization of the phenomena described in Definition 3.9.8 and Lemma 5.1.2, i.e. three body motions where the total angular momentum vector is always parallel to the plane spanned by the configuration.

- We want to get a better understanding of the possible application of conformal and hyperbolic geometry to the study of evolution of three body shapes. This seems to presuppose a better understanding of properties of the potential function that are independent of the mass distribution. One such property is reflected by the fact that the equilateral triangle is a relative equilibrium, for every mass distribution. Are there more such properties?

- We want to follow the path indicated in Section 5.2 to its end, and solve the constant inclination problem.

- We want to give a detailed account on the regularization of binary collisions.

- We want to study the limit of the present reduction when one or more of the two masses tend to 0, in order to connect the present approach with classical astronomical perturbation computations.

- We want to study periodic solutions of the three body problem, in continuation of [CM00] and [Sal12].

1.2 Geometry and many particle systems

In this section, we will formulate some guidelines of our research on many particle systems. In short, they are the following:

(i) Separation of *kinematic geometry* from *dynamics*. The kinematic geometry is regarded as the background on which the dynamics takes place.

(ii) Description of kinematic geometry and dynamics in terms of Euclidean *geometric invariants* of many particle configurations.

1.2.1 Kinematic geometry

The space of positions

Let us regard a system of n mass points $P_1, P_2, \ldots, P_n$ moving in Euclidean space $\mathbb{E}^3$. The *position of the system* is defined as the point $P = (P_1, P_2, \ldots, P_n) \in (\mathbb{E}^3)^n$.

We assume that there is a chosen segment AB in $\mathbb{E}^3$ which is defined to be of length 1. For a given orthonormal frame of reference centred at $O \in \mathbb{E}^3$, the position of the system is represented by n displacement vectors

$$\mathbb{a}_1, \mathbb{a}_2, \mathbb{a}_3, \ldots, \mathbb{a}_n \in \mathbb{R}^3, \quad \mathbb{a}_i = \overrightarrow{OP_i}$$

The tuple $X = (\mathbb{a}_1, \mathbb{a}_2, \ldots, \mathbb{a}_n)$ is called the *position vector* of the system with respect to the given frame of reference. Accordingly, *the space of position vectors* is identified with $(\mathbb{R}^3)^n = \mathbb{R}^{3\times n}$.

We note that for the given frame of reference, there is a canonical action of $\mathbb{R}^{3\times n}$ on $(\mathbb{E}^3)^n$, and that this action is free and transitive. Accordingly, there is an induced canonical trivialization of the tangent bundle

$$T(\mathbb{E}^3)^n \to (\mathbb{E}^3)^n \times \mathbb{R}^{3\times n}. \tag{1.1}$$

For a given mass distribution $m_1, m_2, \ldots, m_n$, we define an inner product

$$\langle X, Y \rangle = \sum_i m_i (\mathbb{a}_i \cdot \mathbb{a}_i) \tag{1.2}$$

on $\mathbb{R}^{3\times n}$, which can be regarded as a Riemannian metric on the tangent bundle (1.1) of $(\mathbb{E}^3)^n$. This metric is independent of the choice of orthonormal frame of reference, and depends only on the choice of length scale and the mass distribution.

Definition 1.2.1. The kinematic geometry of $(\mathbb{E}^3)^n$ associated with the mass distribution $m_1, \ldots, m_n$ is the Riemannian geometry associated with the quadratic form (1.2) on the tangent bundle (1.1).

For a given motion $P(t)$, the *velocity vector* $\dot{P}(t)$, which is a curve in $\mathbb{R}^{3\times n}$ is defined by

$$\dot{P}(t) = \lim_{\Delta t \to 0} \frac{\overrightarrow{P(t)P(t+\Delta t)}}{\Delta t}.$$

$\dot{P}$ is thus defined as a vector in $\mathbb{R}^{3\times n}$, and in virtue of (1.1) we can regard $(P, \dot{P})$ as a tangent vector to $(\mathbb{E}^3)^n$.

As a Riemannian geometry, the kinematic geometry provides us with a notion of covariant acceleration vectors. Because of the simple form of the kinematic metric, this assumes the usual form

$$\ddot{P}(t) = \left(\frac{d}{dt}\dot{\mathbb{a}}_1(t), \frac{d}{dt}\dot{\mathbb{a}}_2(t), \dots, \frac{d}{dt}\dot{\mathbb{a}}_n(t)\right),$$

for a motion $P(t)$ with $\dot{P}(t) = (\dot{\mathbb{a}}_i(t))_{i=1}^n$. Hence we note that for a position vector representation $X(t) = (\mathbb{a}_i(t))_{i=1}^n$ of the motion $P(t)$, $\dot{P} = \dot{X}$ and $\ddot{P} = \ddot{X}$.

Furthermore, we have the notion of the covariant gradient in the Riemannian geometry, and for a function $U(P)$, the gradient is the vector field satisfies

$$\nabla U(P) = \left(\frac{1}{m_1}\frac{\partial U}{\partial \mathbb{a}_1}, \dots, \frac{1}{m_n}\frac{\partial U}{\partial \mathbb{a}_n}\right) \in \mathbb{R}^{3\times n},$$

where $\frac{\partial U}{\partial \mathbb{a}_i}$ is the usual gradient given by

$$\frac{\partial U}{\partial \mathbb{a}_i} = \left[\frac{\partial U}{\partial x_i}, \frac{\partial U}{\partial y_i}, \frac{\partial U}{\partial z_i}\right]^t,$$

where x_i, y_i, z_i are the components of the position vector $\mathbb{a}_i$.

This construction can be descried in a quite different way. The kinematic metric yields an isomorphism $\mathbf{m}$ from the space of vector fields to the space of 1-forms, and we can define $\nabla U = \mathbf{m}^{-1}\mathrm{d}U$.

Now we can treat the equations of motion in the following way:

Proposition 1.2.2. *In the kinematic geometry of* $(\mathbb{E}^3)^n$ *associated with the mass distribution* $m_1, \dots, m_n$*, the Newtonian equations of motion for a conservative system with potential function U can be written as*

$$\ddot{P} = \nabla U, \quad \textit{or equivalently} \quad \mathrm{d}U = \mathbf{m}\ddot{P}. \tag{1.3}$$

Proof. This is a simple restatement of Newton's equations of motion

$$m_i \ddot{\mathbb{a}}_i = \frac{\partial U}{\partial \mathbb{a}_i}$$

associated with conservative systems. ∎

At the first glance, equations as (1.3) looks like a trivial restatement of Newton's equations, and surely, there is not very much more to it. There is however one big difference: Equation (1.3) is a statement within the language of Riemannian geometry, and as such coordinate free.

The usual notion of *kinetic* energy is closely related to the kinematic metric: For a motion $P(t)$ in $(\mathbb{E}^3)^n$, with velocity $\dot{P} = (\dot{\mathbb{a}}_i)$, the kinetic energy can be written as

$$T = \frac{1}{2}\sum_i m_i(\dot{\mathbb{a}}_i \cdot \dot{\mathbb{a}}_i) = \frac{1}{2}\langle \dot{P}, \dot{P}\rangle.$$

The notion of orthogonality in the kinematic geometry has a straightforward interpretation: Along a motion $P(t)$ of the system

$$\dot{T} = \langle \dot{P}, \ddot{P}\rangle.$$

Hence, the kinetic energy is preserved precisely when $\ddot{P} \perp \dot{P}$.

For this reason the notion of orthogonality also plays an important role in the study of constrained systems. If $M \subset (\mathbb{E}^3)^n$ is a sub-manifold, M inherits a Riemannian geometry, for which we have intrinsic notions of covariant acceleration and gradients which are defined by orthogonal projection onto the tangent spaces of M. Interpreted in the induced structure on M, the equations of motion on the form (1.3) correspond to the equations of motion of the system holonomically constrained to M, i.e. in the case where the constraint forces does not affect the total energy. This is the case precisely when the constraint forces are orthogonal to M.

After this discussion our notion of *Riemannian kinematic geometry* should be quite clear.

1.2.2 Euclidean geometry and symmetry

Regarded as Klein geometry [Kle72] the oriented Euclidean space $\mathbb{E}^3$ has the symmetry group $E^+(3)$ of translations and rotations. The space $(\mathbb{E}^3)^n$ inherits an induced $E^+(3)$-action, and physical processes are invariant under this action of $E^+(3)$. Accordingly, the actual points $P_1, P_2, \ldots, P_n$ are less significant than the geometric relations among the points. Classical physics is regarded to be invariant also under reflection, and hence admits a larger symmetry group, namely the group $E(3)$ generated by the reflections in Euclidean space. In this work, orientation plays an important role in the formalism, and hence, we find it convenient to restrict ourselves to the orientation preserving Euclidean transformations.

Geometric properties of point sets $P_1, \dots, P_n$ can usually be expressed by real functions

$$f \colon (\mathbb{E}^3)^n \to \mathbb{R}$$

that are invariant under the Euclidean group. Such invariants can be identified with functions on the congruence moduli space

$$\frac{(\mathbb{E}^3)^n}{E(3)},$$

i.e. the space of congruence classes of n-particle positions.

The oriented Euclidean group is the semi-direct product of the group $\mathbb{R}^3$ of translations and the group $SO(3)$ of rotations. Here, $\mathbb{R}^3$ is the normal subgroup, and we perform the reduction in two stages, first by consideration of the $\mathbb{R}^3$ symmetry on $(\mathbb{E}^3)^n$ and then subsequently by consideration of the symmetry action

$$\left(\frac{E^+(3)}{\mathbb{R}^3} \cong \mathrm{SO}(3), \frac{(\mathbb{E}^3)^n}{\mathbb{R}^3} \cong \mathbb{R}^{3(n-1)} \right).$$

In this thesis the first step is done by means of *Jacobi maps*

$$J \colon (\mathbb{E}^3)^n \to \mathbb{R}^{3\times(n-1)},$$

a special class of $SO(3)$-equivariant linear maps. A chosen Jacobi map gives a representation of n-particle positions by $3 \times (n-1)$-matrices. The columns of such matrices are called *Jacobi vectors.*

The second step in the reduction is taken care of by the standard (diagonal) action of $SO(3)$ on $\mathbb{R}^{3(n-1)}$, and its invariants, which can be regarded as functions on the congruence moduli space

$$\frac{\mathbb{R}^{3(n-1)}}{\mathrm{SO}(3)} \cong \frac{(\mathbb{E}^3)^n}{E^+(3)} \tag{1.4}$$

1.3 Reduction of the three body problem

In the study of the three body problem by means of geometric invariants, we follow [HS07] and describe three body motions in terms of a size variable, the *hyper-radius* ρ, together with two variables φ, θ that records the shape. Together the variables ρ, φ, θ determine the congruence class of three body configurations, and can, within some limitations, be regarded as coordinates of the congruence moduli space (1.4) of the three body problem. Hence, ρ, φ, θ yields a complete representation of static properties of three body configurations.

The fundamental Newtonian description of the three body problem has 9 degrees of freedom, i.e. an 18-dimensional phase space. When we fix the centre of mass at the origin, the number of degrees of freedom is reduced to 6, which yields a 12-dimensional phase space. Hence, on this level of reduction the three body problem is represented by a system of 12 ordinary differential equations. Using the 3-dimensional rotational symmetry, we are able to reduce this to a system of 9 ordinary differential equations. In other words, we have a 9-dimensional *reduced phase space.*

On the other hand, the quantities $\rho, \varphi, \theta, \dot{\rho}, \dot{\varphi}, \dot{\theta}$ can be regarded as coordinates in the tangent bundle of the congruence moduli space (1.4). This tangent bundle is a 6-dimensional subspace of the reduced phase space. Hence, we should not expect to be able to reduce the three body problem completely to the reduced configuration space. This seems obvious also from a physical point of view: Knowing the dynamics of shape and size of the three body configurations, we still miss information about the rotational motion.

To some extent, the rotational motion is determined by the total angular momentum vector Ω. This quantity is not $SO(3)$-invariant. We can however choose a rotating coordinate system in which the components g_1, g_2, g_3 of Ω yield $SO(3)$-invariant quantities. This can be done in several ways. In our approach, we use the rotating *principal axes frame*. A *gauge* for the $SO(3)$-symmetry of the three body problem is the same as a $SO(3)$-equivariant choice of a frame of reference for every three body configuration. Unfortunately, it is impossible to give a global choice of gauge in the three body problem. Using the principal frame, we construct a multi-valued choice of gauge which is defined almost everywhere on the configuration space. Fortunately, the principal frame yields analytic choices of gauge along analytic three body motions, and this analytical version of the principal axes frame extends throughout the configuration space.

Partially depending on the choice of principal axes gauge, we thus describe three body motions by means of the nine ambiguous variables

$$\rho, \varphi, \theta, \dot{\rho}, \dot{\varphi}, \dot{\theta}, g_1, g_2, g_3, \tag{1.5}$$

for which we will express the reduction of the Newtonian equations of motion. We rely on two different but equally effective and adequate methods:

The first method is very "old fashioned": Since the equations of motion are geometrically invariant, they can be expressed by means of our chosen set of basic geometric invariants (1.5). Practically, this is done by brute force: We do algebraic manipulations on the original Newtonian equations of motion, manipulations which are cumbersome, but completely feasible, in particular with some help of a computer algebra system.

The second method is due to Poincaré [Poi01]: We interpret the six variables

$$\dot{\rho}, \dot{\varphi}, \dot{\theta}, g_1, g_2, g_3$$

as a system of differential forms on the configuration space of the three body problem. After computation of the structure coefficients of this system, it is straightforward to write down the equations of motion.

None of these methods yield elegant treatments of the singularities of our description of the three body problem, and a large portion of the present work concerns the treatment of these singularities by various ad-hoc methods.

From the present investigations, we extract a particular view on geometric mechanics, namely that Poincaré's article [Poi01] from 1901 gives a highly adequate and flexible differential geometric framework for classical mechanics. Poincaré's method reveals the importance of the Lie bracket in Lagrangian mechanics, and combined with geometric and differential geometric ideas, we have a rich variety of tools, which can be organized into a toolbox which may be called *geometric mechanics*. We will discuss the relation between Poincaré's method and symplectic geometry in greater detail in Section 3.8.3.

2 Many particle systems

2.1 Introduction

This chapter concerns the symmetry reduction of many particle dynamics. By *many* we mean *finite*, *countably infinite*, or even *uncountably infinite*. Hence in our terminology, even a deformable body in the sense of continuum mechanics will be called a *many particle system*.

The aim of this chapter is (i) to investigate the notion of Jacobi vectors for finite many particle systems (ii) to introduce the singular value decomposition and the principal axes as computational tools adapted to the study of many particle systems (iii) to give a geometric expression of conservation of total angular momentum, in terms of the principal frame. This leads to the so-called *Euler equations*, which is a generalization of the classical Euler equations of rigid body dynamics.

As an important tool we introduce the *constituent space*. This is a Hilbert space that is intended to represent the physically significant information concerning the constituents of the system. The underlying vector space can be regarded as a subspace of the free vector space generated by the set of mass points, while the inner product represents the mass distribution.

The main motivation of the investigation of the Jacobi vectors is to be able to carry out a rigorous study of hyperbolic and conformal geometry in the three body problem (cf. Section 4.3.3).

The main motivation of the investigation of the Euler equations is to broaden the understanding of the Euler equations in [HS07]. It is not obvious that the generalization is valuable in itself. This generalization may however shed some light on the Euler equations of the three body problem.

Initially in the process that led to this dissertation, the introduction of the singular value decomposition was intended to give a foundation for the discussion of the Euler equations. Later it played an increasingly important role, and now it permeates completely our discussion of the three body problem in Chapter 3. It can be said to give a new point of departure for the study of the three body problem in the style of [HS07].

2.2 The position space

The spatial position of an n particle system is represented by n points

$$P_1, P_2, \ldots, P_n$$

in Euclidean space $\mathbb{E}^3$. We can think of the constituents of the system as a set $B = \{1, 2, \ldots, n\}$, and the position of the system as a mapping $B \to \mathbb{E}^3$.

Similarly, for an arbitrary set B, we can consider the set of mappings $B \to \mathbb{E}^3$, and thereby *positions* of systems with *constituent set* B. Hence, we define the *set of positions* by

$$C = \{F \colon B \to \mathbb{E}^3\}.$$

The inertial mass of the many particle system can be represented by a σ-algebra σ over B together with a positive measure m on (B, σ) satisfying $m(B) < \infty$. Without going into the details of measure theory, we simply assume the existence of a real Hilbert space

$$\mathscr{H} = \mathscr{L}^2(B, m)$$

whose elements can be represented by real functions on B that are square integrable with respect to the mass distribution. In the case where this construction is valid, we define $\mathscr{H}$ to be the *constituent space* of the system.

If we provide $\mathbb{E}^3$ with a length scale and an orthonormal frame of reference, i.e. an isometry $\mathbb{E}^3 \to \mathbb{R}^3$, we get a vector space structure on $\mathbb{E}^3$. Accordingly, for every position $B \to \mathbb{E}^3$, we get a unique linear map

$$\mathbb{R}\langle B\rangle \to \mathbb{R}^3$$

from the real vector space $\mathbb{R}\langle B\rangle$ freely generated by B. Hence, we can represent the *set of positions* by the vector space

$$C = \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}\langle B\rangle, \mathbb{R}^3)$$

In the light of the following considerations, we may find the space C too large in the case where B is infinite:

(i) Two different elements $f, g \in C$ may be physically indistinguishable, e.g. if they agree outside a mass-less subset of B.

(ii) An element $f \in C$ may yield a position of the system where the moment of inertia is infinite.

In order to cope with (i), we can identify elements $f, g \in C$ which agree outside a mass-less subset of B. (ii) can be dealt with by throwing away elements $f \in C$ such that

$$\int_B (f \cdot f)\, dm$$

is either undefined or infinite. Here, $f \cdot f$ denotes the point-wise inner product.

With this in mind, we find it reasonable to work with the following space:

Definition 2.2.1 (Position vector space)**.** For a system with constituent space $\mathscr{H}$, we define the *space of position vectors* to be the space

$$\mathscr{C} = \mathscr{B}(\mathscr{H}, \mathbb{R}^3)$$

of bounded linear transformations $\mathscr{H} \to \mathbb{R}^3$.

The elements of $\mathscr{C}$ can be regarded as equivalence classes of functions $F\colon B \to \mathbb{R}^3$ for which the x, y, and z-components are square-integrable functions on (B, m).

In the case of the n-body problem, there is a natural identification of $\mathscr{C}$ with the space $M_{3\times n}$ of real $3 \times n$-matrices where the i-th column of the matrix representative of a configuration is the position vector $\mathbb{a}_i$ of particle i. In terms of the standard basis e_i of $\mathbb{R}^n$, the following matrix represents a position vector X

$$\left[X(e_1)|X(e_2)|\cdots|X(e_n)\right] = \left[\mathbb{a}_1|\mathbb{a}_2|\cdots|\mathbb{a}_n\right],$$

where $\mathbb{a}_i$ is the position of particle i.

2.2.1 The constituent space

Above, the constituent space was defined as the real Hilbert space

$$\mathscr{H} = \mathscr{L}^2(B, m).$$

For a square-integrable function $f\colon B \to \mathbb{R}$, we let $[f]$ denote the corresponding element of $\mathscr{H}$, i.e. the $\mathscr{L}^2$-equivalence class.

We have the following important and familiar examples of constituent spaces:

(i) If B is a compact subset of $\mathbb{R}^3$ and m is given by integration of a smooth mass density function ρ, we can express the inner product in $\mathscr{H}$ as

$$\langle f, g\rangle = \int_B f\, g\, \rho\, dV$$

(ii) In the case of n bodies, we can represent the elements of $\mathscr{H}$ by row vectors $\mathbb{x} = [x_1, x_2, \dots x_n]$, $\mathbb{y} = [y_1, y_2, \dots, y_n]$. The inner product is then given by

$$\langle \mathbb{x}, \mathbb{y}\rangle = \sum_i m_i\, x_i\, y_i$$

For a subset $A \subset B$ we have the characteristic function χ_A, and since $m(A) \leq m(B) < \infty$, we can average elements of $\mathscr{H}$ and $\mathscr{C}$ over A: For an element $\mathbb{q} \in \mathscr{H}$, we have the mass weighted average value

$$\frac{\langle \mathbb{q}, \chi_A \rangle}{\|\chi_A\|},$$

and the centre of mass of the restriction of X to A is given by

$$\frac{X(\chi_A)}{\|\chi_A\|} \in \mathbb{R}^3.$$

As $m(B)$ is assumed to be finite, the constant function 1 is square integrable, and yields an important element of $\mathscr{H}$ that will be denoted by $\mathbb{1}$. This element satisfies

$$\langle \mathbb{1}, [f] \rangle = \int_B f\, dm,$$

where $[f] \in \mathscr{H}$ is the element representing $f\colon B \to \mathbb{R}$. The linear functional $\mathbb{1}/m(B)\colon \mathscr{H} \to \mathbb{R}$ is the same as the mass-weighted average over B. The orthogonal complement of $\mathbb{1}$ will be called *the barycentric constituent space* and denoted by $\mathscr{H}^0$. This yields the following important orthogonal decomposition of the constituent space

$$\mathscr{H} = \mathbb{R}\langle \mathbb{1} \rangle \oplus \mathscr{H}^0 \tag{2.1}$$

into the subspace $\mathbb{R}\langle \mathbb{1} \rangle$ spanned by $\mathbb{1}$ and the barycentric constituent space $\mathscr{H}^0$.

The symmetry groups $O(\mathscr{H})$ and $O(\mathscr{H}^0)$ are called *democracy groups*. Later we will have much use $O(\mathscr{H}^0)$, which can be thought of as the subgroup of $O(\mathscr{H})$ for which $\mathbb{1}$ is fixed. In the case of the n-body problem with equal masses, both of these groups contains the group of permutations of n the indices. Hence, in this case, the democracy symmetry enforces that particles with equal mass are equal.

The notion of the constituent space allows for the following abstraction: The kinematic geometry is completely determined by the constituent space $\mathscr{H}$, and the only reminiscent of the constituent set B that we possibly will need, is the characteristic function $\chi_B = \mathbb{1}$. Hence, from a kinematic point of view, we only need to consider the Hilbert space $\mathscr{H}$, and when discussing mean values, we may have to take into consideration the distinguished vector $\mathbb{1} \in \mathscr{H}$, or at least the linear subspace spanned by $\mathbb{1}$. In the study of dynamics, B will however usually be indispensable.

2.2.2 The Hilbert space $\mathscr{C}$ of positions

The Euclidean geometry of $\mathbb{R}^3$ is represented by the scalar product

$$\mathbb{a}\cdot\mathbb{b} = xx' + yy' + zz', \quad \text{where } \mathbb{a} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \mathbb{b} = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}.$$

Since $\mathscr{H}$ is self dual, we can describe the position space as a product

$$\mathscr{C} = \mathscr{B}(\mathscr{H}, \mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}) = \mathscr{B}(\mathscr{H},\mathbb{R}) \oplus \mathscr{B}(\mathscr{H},\mathbb{R}) \oplus \mathscr{B}(\mathscr{H},\mathbb{R}) = \mathscr{H}\oplus\mathscr{H}\oplus\mathscr{H}, \tag{2.2}$$

and hence, a configuration is represented by a tuple

$$X = \begin{bmatrix} \mathbb{x} \\ \mathbb{y} \\ \mathbb{z} \end{bmatrix} \in \mathscr{H}\oplus\mathscr{H}\oplus\mathscr{H}. \tag{2.3}$$

The corresponding linear operator $\mathscr{H} \to \mathbb{R}^3$ is given by

$$X(\mathbb{q}) = \begin{bmatrix} \langle \mathbb{x}, \mathbb{q}\rangle \\ \langle \mathbb{y}, \mathbb{q}\rangle \\ \langle \mathbb{z}, \mathbb{q}\rangle \end{bmatrix}.$$

Accordingly, for an n-body position represented by a matrix

$$X = \left[\mathbb{a}_1 | \mathbb{a}_2 | \cdots | \mathbb{a}_n\right] = \begin{bmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \\ z_1 & \cdots & z_n \end{bmatrix}, \tag{2.4}$$

the corresponding elements $\mathbb{x}, \mathbb{y}, \mathbb{z} \in \mathscr{H} = \mathbb{R}^n$ are given by

$$\begin{aligned} \mathbb{x} &= \begin{bmatrix} \frac{x_1}{m_1} & \frac{x_2}{m_2} & \cdots & \frac{x_n}{m_n} \end{bmatrix} \\ \mathbb{y} &= \begin{bmatrix} \frac{y_1}{m_1} & \frac{y_2}{m_2} & \cdots & \frac{y_n}{m_n} \end{bmatrix} \\ \mathbb{z} &= \begin{bmatrix} \frac{z_1}{m_1} & \frac{z_2}{m_2} & \cdots & \frac{z_n}{m_n} \end{bmatrix}. \end{aligned}$$

We can give $\mathscr{C}$ a Hilbert space structure as the orthogonal sum of three copies of $\mathscr{H}$. This yields the following inner product:

$$\langle X, X' \rangle = \langle \mathbb{x}, \mathbb{x}' \rangle + \langle \mathbb{y}, \mathbb{y}' \rangle + \langle \mathbb{z}, \mathbb{z}' \rangle.$$

Using the transformation interpretation of the position vectors, we can write this as

$$\langle X', X \rangle = \operatorname{tr}(X' X^t),$$

where $X^t \colon \mathbb{R}^3 \to \mathscr{H}$ is the Hilbert-space transpose of $X \colon \mathscr{H} \to \mathbb{R}^3$ in the following sense: If $\mathbb{v} \in \mathbb{R}^3$ and $\mathbb{u} \in \mathscr{H}$, then $\langle X^t \mathbb{v}, \mathbb{u} \rangle_{\mathscr{H}} = \langle \mathbb{v}, X\mathbb{u} \rangle_{\mathbb{R}^3}$.

Expressed by integrals of functions on B, this inner product satisfies

$$\langle [F], [G] \rangle = \int_B (F \cdot G)\, dm,$$

where $F, G \colon B \to \mathbb{R}^3$ and $F \cdot G$ is the point-wise inner product. In the case of n point masses, this specializes to

$$\langle X, X' \rangle = \sum_{i=1}^{n} m_i (x_i x_i' + y_i y_i' + z_i z_i') = \sum_{i=1}^{n} m_i (\mathbb{a}_i \cdot \mathbb{a}_i),$$

i.e. a sum of Euclidean inner products $(\mathbb{a}_i \cdot \mathbb{a}_i)$, weighted by the respective masses of the particles.

2.2.3 Kinematics

For a motion $X(t)$ of the system, i.e. a curve in the position space $\mathscr{C}$, we have the time derivative

$$\dot{X}(t) = \lim_{\Delta t \to 0} \frac{X(t + \Delta t) - X(t)}{\Delta t},$$

where we take the limit in the Hilbert space $\mathscr{C}$. In this way differentiable curves $X(t)$ yield velocity curves $\dot{X}(t)$ in $\mathscr{C}$. Under the usual identification of the tangent bundle $T\mathscr{C}$ with $\mathscr{C} \times \mathscr{C}$, $(X(t), \dot{X}(t))$ is the tangent lift of $X(t)$.

In the infinite-dimensional case, a motion $X(t)$ does not specify particular mappings $F_t \colon B \to \mathbb{R}^3$, and hence it does not make sense to talk about a velocity field $\dot{F}_t \colon B \to \mathbb{R}^3$ associated with a motion $X(t)$. On the other hand, if χ_A is the characteristic function of a subset $A \subset \mathscr{B}$ with mass $m(A) > 0$, $\langle X(t), [\chi_A]\rangle / m(A)$ yields the motion of the centre of mass of A. As long as $\dot{X}(t)$ exists, $\langle \dot{X}(t), [\chi_A]\rangle / m(A)$ yields the velocity of the centre of mass of A. This follows from differentiability of the inner product $\langle -, - \rangle$. Hence, $\dot{X}(t)$ seems to give a good way to keep track of velocities of physically significant parts of the system.

In the case of n mass points, the position space is a finite dimensional vector space. Consequently there is only one notion of convergence, and for a motion given in the matrix representation (2.4) the velocity is

$$\dot{X}(t) = \begin{bmatrix} \dot{x}_1(t) & \cdots & \dot{x}_n(t) \\ \dot{y}_1(t) & \cdots & \dot{y}_n(t) \\ \dot{z}_1(t) & \cdots & \dot{z}_n(t) \end{bmatrix}.$$

If B is a compact subset of $\mathbb{R}^3$, the mass distribution is given by a smooth density function $\rho\colon B \to \mathbb{R}$, and $X(t)$ is represented by a smooth function $F\colon B \times [a,b] \to \mathbb{R}^3$, then the velocity field $\frac{\partial F}{\partial t}\colon B \times [a,b] \to \mathbb{R}^3$ yields a curve $\left[\frac{\partial F}{\partial t}\right]$ in $\mathscr{C}$. Now $\dot{X}$ is represented by $\frac{\partial F}{\partial t}$, i.e.

$$(*) \qquad \dot{X} = \left[\frac{\partial F}{\partial t}\right] \in \mathscr{C}.$$

This is proved as follows: If $A \subset B$, then the velocity of the centre of mass $\overline{A}(t)$ of A is given by

$$\frac{d}{dt}\overline{A}(t) = \frac{d}{dt}\int_A F(b,t)\rho\, dV = \int_A \frac{\partial F}{\partial t}\rho\, dV = \frac{\left\langle \left[\frac{\partial F}{\partial t}\right], \chi_A \right\rangle}{m(A)}.$$

$(*)$ is now seen to hold since the characteristic functions χ_A generate a dense subspace of $\mathscr{H}$.

In the case of the n-body problem, where $B = \{1,2,\ldots,n\}$, the application of this formalism to the one-point set $\{i\} \subset B$ yields

$$\langle X(t), \chi_{\{i\}}\rangle = \mathbb{a}_i(t) \quad \text{and} \quad \langle \dot{X}(t), \chi_{\{i\}}\rangle = \dot{\mathbb{a}}_i(t),$$

where $\mathbb{a}_i(t)$ is the position of particle i, and $\dot{\mathbb{a}}_i(t)$ its velocity.

Finally, we present the kinetic energy, which is essentially equivalent to the inner product on $\mathscr{C}$. In the case where B is a compact subset of $\mathbb{R}^3$, the total kinetic energy T of the motion $F\colon B \times [a,b]$ is usually defined by

$$T = \int_B \left(\frac{1}{2}\rho \frac{\partial F}{\partial t}(b,t)\right) dV = \frac{1}{2}\left\langle \left[\frac{\partial F}{\partial t}\right], \left[\frac{\partial F}{\partial t}\right] \right\rangle = \frac{1}{2}\langle \dot{X}(t), \dot{X}(t)\rangle.$$

Similarly, for n-body motions,

$$T = \sum_i \left(\frac{1}{2} m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)\right) = \frac{1}{2}\langle \dot{X}, \dot{X}\rangle.$$

Conforming to these examples will employ the following general definition of the kinetic energy of a motion $X(t)$ in $\mathscr{C}$:

$$T = \frac{1}{2}\langle \dot{X}, \dot{X}\rangle$$

2.2.4 The configuration space

Within the Galilean theory of relativity [Gal32], absolute positions are physically insignificant. This phenomenon can be discussed by means of transformations of the Galilean space-time. We will not go into that topic here, but simply state that we will consider only systems for which the relative positions yield an adequate description, and hence go directly to formulations in terms of relative positions.

We will define a *configuration space* $\mathscr{M}$ whose points represent relative positions of the system. This contrasts the position space $\mathscr{C}$ that records absolute positions. The relation between $\mathscr{M}$ and $\mathscr{C}$ can be described as follows:

The physical translation symmetry is represented by an affine action of $\mathbb{R}^3$ on $\mathscr{C}$. In terms of functions $F\colon B \to \mathbb{R}^3$, this action is represented by point-wise translation, i.e.

$$\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} + F\right)(b) = F(b) + \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

In terms of the matrix representation of n-body systems (2.4) this yields

$$\mathbb{v} + \left[\cdots | \mathbb{a}_i | \cdots\right] = \left[\cdots | \mathbb{a}_i + \mathbb{v} | \cdots\right],$$

where $\mathbb{v}, \mathbb{a}_i \in \mathbb{R}^3$. In terms of the representation (2.3), this action can be described as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} + X = \begin{bmatrix} \mathbb{x} + x\mathbb{1} \\ \mathbb{y} + y\mathbb{1} \\ \mathbb{z} + z\mathbb{1} \end{bmatrix},$$

and hence we see that the action of $\mathbb{R}^3$ on $\mathscr{C}$ is properly represented by the subspace

$$\mathbb{R}^3\langle\mathbb{1}\rangle = \mathbb{R}\langle\mathbb{1}\rangle \oplus \mathbb{R}\langle\mathbb{1}\rangle \oplus \mathbb{R}\langle\mathbb{1}\rangle \subset \mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H} \cong \mathscr{C}, \tag{2.5}$$

in the following sense: Two position vectors X, X' represent the same configuration if and only if $(X - X') \in \mathbb{R}^3\langle\mathbb{1}\rangle$. The configuration space is defined as follows:

Definition 2.2.2 (Configuration space)**.** The *configuration space* associated with the position space $\mathscr{C}$ is the quotient

$$\mathscr{M} = \frac{\mathscr{C}}{\mathbb{R}^3\langle\mathbb{1}\rangle},$$

i.e. the space of many particle position vectors modulo translation symmetry.

We will regard $\mathscr{M}$ as a Hilbert space where the metric is induced by the projection $\mathscr{C} \to \mathscr{M}$.

The configuration space $\mathscr{M}$ can be identified with a subspace $\mathscr{M}^0 \subset \mathscr{C}$, namely the orthogonal complement of $\mathbb{R}^3\langle\mathbb{1}\rangle \subset \mathscr{C}$:

Definition 2.2.3 (Barycentric configuration space)**.** The orthogonal complement $\mathscr{M}^0 = \mathbb{R}^3\langle\mathbb{1}\rangle^\perp \subset \mathscr{C}$ is called *the barycentric configuration space.* As a subspace of $\mathscr{C}$, $\mathscr{M}^0$ inherits a Hilbert space structure.

Clearly, for any given mass distribution, there is a canonical isometry

$$\mathscr{M} \cong \mathscr{M}^0.$$

$\mathscr{M}$ is universally well defined as a quotient of $\mathscr{C}$. This contrasts $\mathscr{M}^0$, which depends on the choice of mass distribution. When we consider a fixed mass distribution, we will however not distinguish between the configuration space and the barycentric configuration space. On the other hand, when we are interested in different mass distributions, we will sometimes retain the distinction between $\mathscr{M}$ and $\mathscr{M}^0$.

The barycentric configuration space can be described as the space of positions $X = (\mathbb{x}, \mathbb{y}, \mathbb{z})$ such that

$$\langle \mathbb{x}, \mathbb{1} \rangle = \langle \mathbb{y}, \mathbb{1} \rangle = \langle \mathbb{z}, \mathbb{1} \rangle = 0, \quad \text{i.e.} \quad X(\mathbb{1}) = 0.$$

Hence $\mathscr{M}^0$ is the same as the space of positions with the centre of mass at the origin. In the case of the n-body problem, the above conditions read

$$\sum_i m_i x_i = \sum_i m_i y_i = \sum_i m_i z_i = 0.$$

Using the decompositions (2.2) and (2.1), we have the following identifications

$$\mathscr{M} = \frac{\mathscr{H}}{\mathbb{R}\langle \mathbb{1} \rangle} \oplus \frac{\mathscr{H}}{\mathbb{R}\langle \mathbb{1} \rangle} \oplus \frac{\mathscr{H}}{\mathbb{R}\langle \mathbb{1} \rangle} \quad \text{and} \quad \mathscr{M}^0 = \mathscr{H}^0 \oplus \mathscr{H}^0 \oplus \mathscr{H}^0,$$

and the identification of $\mathscr{M}$ with $\mathscr{M}^0$ is identical to the direct sum of three copies of the identification $\mathscr{H}/\mathbb{R}\langle \mathbb{1} \rangle \cong \mathscr{H}^0$.

2.2.5 Rotational symmetries

By the standard representation of $SO(3)$ on $\mathbb{R}^3$, the position space $\mathscr{C}$ inherits a natural $SO(3)$-action that can be described as follows:

$$(Q, X) \mapsto Q \circ X, \quad \text{where} \quad Q \in SO(3),\ X \in \mathscr{C} \cong \mathscr{B}(\mathscr{H}, \mathbb{R}^3)$$

This action of $SO(3)$ on $\mathscr{C}$ is isometric and reflects the rotational symmetry of Euclidean geometry.

The subspace $\mathbb{R}^3\langle \mathbb{1} \rangle \subset \mathscr{C}$ defined in (2.5) is clearly $SO(3)$-invariant, and accordingly, we have an induced representation of $SO(3)$ on the configuration space $\mathscr{M} = \mathscr{C}/\mathbb{R}^3\langle \mathbb{1} \rangle$. For the same reason, $\mathscr{M}^0 = \mathbb{R}^3\langle \mathbb{1} \rangle^\perp \subset \mathbb{C}^*$ is $SO(3)$-invariant. Accordingly, the barycentric configuration space $\mathscr{M}^0$ also inherits an action of $SO(3)$.

The natural maps $\mathscr{M}^0 \to \mathscr{C} \to \mathscr{M}$ are $SO(3)$ equivariant, and in the following, we will take the described $SO(3)$-actions as the default actions of $SO(3)$ on $\mathscr{M}$, $\mathscr{M}^0$ and $\mathscr{C}$.

2.3 Jacobi vectors in the n-body problem

In this section we will consider the case where $\mathscr{H}$ is finite dimensional, i.e. the case of n mass points. It is possible to extend parts of the following discussion to the case where $\mathscr{H}$ is infinite dimensional and separable, using infinite orthonormal sequences. We will however not go into that discussion here.

In many cases, the position space $\mathscr{C}$ has a fairly natural coordinatization, which although depends on a choice of frame of reference in Euclidean space: The set of coordinates of all the particles constitutes such a coordinatization. On the other hand, the configuration space $\mathscr{M}$ has in general no natural coordinatization, and in this section we will consider a particular class of such coordinatizations, namely the coordinatization given by a set of *Jacobi vectors*. The special feature of Jacobi vectors is that they yield a good representation if the $SO(3)$-equivariant kinematic geometry of the n-body problem, and thus gives optimally simple expressions of the kinetic energy and the total angular momentum.

Classically, the Jacobi vectors dates back at least to Jacobi[Jac43]. In the literature of astronomy, quantum mechanics and molecular dynamics it is common to work with one fixed choice of Jacobi vectors. We are however interested in understanding the flexibility in choice of Jacobi vectors, and in the present work, we extend the discussion of Jacobi vectors that is found in [Str06], and investigate transitions between different choices of Jacobi vectors from a slightly different perspective.

In our terminology, a *Jacobi map* represents a choice of Jacobi vectors, while a transition between two choices of Jacobi vectors is represented by a *Jacobi transformation*. We can take the Jacobi maps and the Jacobi transformations respectively as objects and arrows in a groupoid, which will be called the *Jacobi groupoid*. Hence, our main goal can be expressed as understanding the so-called Jacobi groupoid.

Among the particular applications of our understanding of the Jacobi groupoid, we can mention the elegant analysis of the potential function found in Section 3.4 and the introduction of hyperbolic geometry in the study of the three body problem in Section 4.3.

Finally, we mention that from the point of view of invariant theory, a choice of Jacobi vectors yields a complete system of $SO(3)$-equivariant translation invariant functions $\mathscr{C} \to \mathbb{R}^3$.

2.3.1 A non-standard definition of Jacobi vectors

For a given mass distribution and corresponding mass dependent inner product, we present the position space as an orthogonal direct sum

$$\mathscr{C} = \mathscr{M}^0 \oplus \mathbb{R}^3\langle \mathbb{1} \rangle \qquad \left(\mathbb{R}^3\langle \mathbb{1} \rangle = \mathbb{R}\mathbb{1} \oplus \mathbb{R}\mathbb{1} \oplus \mathbb{R}\mathbb{1} \subset \mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H} = \mathscr{C}\right),$$

with the corresponding orthogonal decomposition of the constituent space

$$\mathscr{H} = \mathscr{H}^0 \oplus \mathbb{R}\langle \mathbb{1} \rangle,$$

with respect to the mass dependent inner product on $\mathscr{H}$. Accordingly we have the following description of the barycentric configuration space:

$$\mathscr{M}^0 = \mathscr{H}^0 \oplus \mathscr{H}^0 \oplus \mathscr{H}^0.$$

Now we define the notion of Jacobi vectors, in the following non-standard way: For a given orthonormal basis $\mathscr{B} = (\mathbb{q}_1, \mathbb{q}_2, \ldots, \mathbb{q}_{n-1})$ of $\mathscr{H}^0$ and a position vector $X = (\mathbb{x}, \mathbb{y}, \mathbb{z}) \in \mathscr{C} = \mathscr{H} \oplus \mathscr{H} \oplus \mathscr{H}$, we define the vectors

$$\mathbb{x}_i = X(\mathbb{q}_i) = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} = \begin{bmatrix} \langle \mathbb{x}, \mathbb{q}_i \rangle \\ \langle \mathbb{y}, \mathbb{q}_i \rangle \\ \langle \mathbb{z}, \mathbb{q}_i \rangle \end{bmatrix} \in \mathbb{R}^3,$$

to be the *Jacobi vectors associated with the position vector X and the basis $\mathscr{B}$ of $\mathscr{H}^0$*. Note that the notion of orthonormality of $\mathscr{B}$ depends on the mass distribution. Hence, different mass distributions yield different classes of allowed choices of Jacobi vectors.

$O(\mathscr{H}^0)$ acts freely and transitively on the set of bases of $\mathscr{H}^0$, and accordingly, the set of Jacobi vectors allowed by a given mass distributions can be put in bijection with $O(\mathscr{H}^0)$. Equivalently, we have the free and transitive action of $O(n-1)$ on the set of orthonormal bases of $\mathscr{H}^0$, which implies that we can put the set of allowed Jacobi vectors in bijective correspondence with $O(n-1)$. The actions of $O(\mathscr{H})$ and $O(n-1)$ are of course essentially the same.

From the point of view of Jacobi vectors, we can say that the democracy groups $O(\mathscr{H}^0)$ and $O(n-1)$ enforces that different choices of Jacobi vectors are born equal.

2.3.2 Jacobi maps

The Jacobi vectors $\mathbb{x}_1, \ldots, \mathbb{x}_{n-1}$ can be regarded as the columns of a $3 \times (n-1)$-matrix $J(X) \in M_{3\times(n-1)}$ with real coefficients. For the given basis $\mathscr{B}$ of $\mathscr{H}^0$, this defines a linear transformation $J\colon \mathscr{C} \to M_{3\times(n-1)}$, which has the following properties:

(J1) J annihilates $\mathbb{R}^3\langle\mathbb{1}\rangle \subset \mathscr{C}$.
This is a manifestation of translation invariance (c.f. (2.5)).

(J2) J restricts to an isometry

$$\mathscr{M}^0 \to M_{3\times(n-1)},$$

where $M_{3\times(n-1)}$ is equipped with the Frobenius inner product

$$\langle X, Y\rangle = \operatorname{tr}(XY^t)$$

and $\mathscr{M}^0$ is equipped with the mass dependent inner product inherited from $\mathscr{C}$.
This means that J respects the kinematic geometry.

(J3) For every $Q \in SO(3)$ and every position $X \in \mathscr{C}$, $J(QX) = QJ(X)$.
This means that J respects the rotational symmetries of $\mathscr{C}$ and $M_{3\times(n-1)}$.

Following [Str06] we characterize the choice of Jacobi vectors by these properties. We will however use a somehow different terminology:

Definition 2.3.1 (Jacobi map)**.** A linear map $J\colon \mathscr{C} \to M_{3\times(n-1)}$ satisfying (J1), (J2) and (J3) above with respect to a mass distribution m is called *a Jacobi map admitted by m*.

Above we observed that orthonormal bases $\mathscr{B}$ of $\mathscr{H}^0$ yield Jacobi maps. The converse is also true:

Lemma 2.3.2. *Suppose that $J \colon \mathscr{C} \to M_{3\times(n-1)}$ is a Jacobi map. Then there exists an orthonormal basis $\mathbb{q}_1, \dots, \mathbb{q}_{(n-1)}$ of $\mathscr{H}^0$ such that*

$$J(X) = \begin{bmatrix} \cdots \langle \mathbb{x}, \mathbb{q}_i \rangle \cdots \\ \cdots \langle \mathbb{y}, \mathbb{q}_i \rangle \cdots \\ \cdots \langle \mathbb{z}, \mathbb{q}_i \rangle \cdots \end{bmatrix} \tag{2.6}$$

Proof. Let $\mathscr{C}_x \subset \mathscr{C}$ be the space of configurations where all the particles lie on the x-axis, and let $M_x \subset M_{3\times(n-1)}$ be the space of matrices for which the second and third rows are 0.

By (J3), J is $SO(3)$-equivariant and thus, J must map $\mathscr{C}_x$ into M_x; they are both fix-point sets of the group of rotations around the x-axis. Hence, J induces a linear map $J_x \colon \mathscr{C}_x \to M_x$.

$\mathscr{C}_x$ is naturally identified with $\mathscr{H}$, while M_x is naturally identified with $\mathbb{R}^{n-1}$. Hence, J_x is represented by $(n-1)$ linear functionals $J_x^i \colon \mathscr{H} \to \mathbb{R}$, and since $\mathscr{H}$ is a Hilbert space, this gives us $(n-1)$ elements $\mathbb{q}_i \subset \mathscr{H}$.

By (J1) $J(\mathbb{R}^3\langle \mathbb{1} \rangle) = 0$, i.e.

$$\langle \mathbb{q}_i, \mathbb{1} \rangle = J_x^i(\mathbb{1}) = 0,$$

and hence, $\mathbb{q}_1, \dots, \mathbb{q}_{n-1} \in \mathscr{H}^0$.

We can carry out similar constructions with the y-axis and the z-axis, and thus provide similar functionals $J_y^i, J_z^i \colon \mathscr{H} \to \mathbb{R}$.

The Jacobi map is now given by

$$J(X) = \begin{bmatrix} \cdots J_x^i(\mathbb{x}) \cdots \\ \cdots J_y^i(\mathbb{y}) \cdots \\ \cdots J_z^i(\mathbb{z}) \cdots \end{bmatrix}$$

Following (J3), this expression is invariant under permutation of the axes. It follows that $J_x^i = J_y^i = J_z^i$, and hence that (2.6) above holds, since $J_x^i(\mathbb{v}) = \mathbb{q}_i(\mathbb{v})$ for all $\mathbb{v} \in \mathscr{H}$. It remains to show that the frame $(\mathbb{q}_1, \dots, \mathbb{q}_{n-1})$ is orthonormal.

Define the elements of $\mathscr{M}^0$ on the form

$$E_x^i = \begin{bmatrix} \mathbb{q}_i \\ 0 \\ 0 \end{bmatrix} \qquad i = 1, 2, \dots, n-1.$$

Now,

$$J(E_x^i) = \begin{bmatrix} \langle \mathbb{q}_1, \mathbb{q}_i \rangle & \cdots & \langle \mathbb{q}_{n-1}, \mathbb{q}_i \rangle \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{bmatrix},$$

and by the isometry condition (J2) we deduce that

$$\langle \mathbb{q}_i, \mathbb{q}_j \rangle = \langle E_x^i, E_x^j \rangle = \operatorname{tr}\left(J\left(E_x^i\right) J\left(E_x^j\right)^t \right) = \sum_{k=1}^{n-1} \langle \mathbb{q}_i, \mathbb{q}_k \rangle \langle \mathbb{q}_k, \mathbb{q}_j \rangle.$$

Hence the frame $\mathscr{B} = (\mathbb{q}_1, \dots, \mathbb{q}_{n-1})$ is orthonormal. ■

We sum up this section and the previous one with the following proposition:

Proposition 2.3.3. *Consider a many particle system with a given mass distribution m. Then the formula*

$$J(X) = \left[X(\mathbb{q}_1) | \cdots | X(\mathbb{q}_{n-1}) \right]$$

defines a bijective correspondence between Jacobi maps $J \colon \mathscr{C} \to M_{3\times(n-1)}$ and orthonormal frames $(\mathbb{q}_1, \mathbb{q}_2, \dots, \mathbb{q}_{n-1})$ in $\mathscr{H}^0$.

2.3.3 Jacobi vectors and the kinetic energy

Since Jacobi maps $J \colon \mathscr{C} \to M_{3\times(n-1)}$ restrict to isometries $\mathscr{M}^0 \to M_{3\times(n-1)}$, we have the following proposition:

Proposition 2.3.4. *For an n-body motion $X(t)$ with the centre of mass at the origin and a given Jacobi map J with corresponding evolution of Jacobi vectors $\mathbb{x}_1(t), \dots, \mathbb{x}_{n-1}(t)$, the kinetic energy can be written as*

$$T = \frac{1}{2} \operatorname{tr}(J(\dot{X}) J(\dot{X})^t) = \frac{1}{2} \langle J(\dot{X}), J(\dot{X}) \rangle = \frac{1}{2} \sum_i (\dot{\mathbb{x}}_i \cdot \dot{\mathbb{x}}_i).$$

Later, in Proposition 2.4.6, we will see that the total angular momentum obeys a similar simplification when we work with Jacobi vectors.

2.3.4 Jacobi transformations and the Jacobi groupoid

Here, we will investigate transitions between different Jacobi maps.

Definition 2.3.5 (Jacobi transformation)**.** Let J, J' be two Jacobi maps, possibly associated with different mass distributions. A linear automorphism A of $M_{3\times(n-1)}$ is called a *Jacobi transformation* from J^1 to J^2 if $J^2 = A \circ J^1$. Symbolically, we write this as

$$A\colon J^1 \to J^2.$$

This suggests that the Jacobi transformations fit nicely into a groupoid, since they are by definition invertible. We isolate this groupoid as a separate object:

Definition 2.3.6 (Jacobi groupoid)**.** The *Jacobi groupoid* $\mathscr{J}$ is defined in the following way:

Objects: Jacobi maps $J^1, J^2, \ldots$

Arrows: Jacobi transformations $A\colon J^1 \to J^2$

Composition: Composition of automorphisms of $M_{3\times(n-1)}$.

The rest of this section can be regarded as an investigation of the Jacobi groupoid. First we show that each Jacobi transformation can be represented by an invertible $(n-1)\times(n-1)$-matrix:

For any given matrix $A \in \mathrm{GL}_{n-1}(\mathbb{R})$ we get a linear automorphism of $M_{3\times(n-1)}$ given by matrix multiplication from the right. This gives an anti-representation $GL_{n-1}(\mathbb{R}) \to \mathrm{Aut}\big(M_{3\times(n-1)}\big)$, which we denote by $A \mapsto A^*$, where $A^*(B) = BA$ for every $B \in M_{3\times(n-1)}$.

Lemma 2.3.7. *For every pair of Jacobi maps $J^1, J^2 \in \mathscr{J}$, there exists a unique arrow*

$$J^1 \to J^2.$$

This arrow is of the form $A^*\colon J^1 \to J^2$ where $A \in \mathrm{GL}_{n-1}(\mathbb{R})$. This yields an anti-homomorphism of groupoids,

$$\mathscr{J} \to \mathrm{GL}_{n-1}(\mathbb{R}).$$

Proof. Assume that we are given two possibly different mass distributions m^1, m^2 for the n-body problem. Denote by $\langle -,- \rangle^1, \langle -,- \rangle^2$ the two associated inner products on $\mathscr{H}$ and by $\| - \|_1, \| - \|_2$ the associated norms.

Assume that we are given Jacobi maps J^1, J^2 associated with the two mass distributions, and let $(\mathbb{q}_1^1, \ldots, \mathbb{q}_{n-1}^1)$, $(\mathbb{q}_1^2, \ldots, \mathbb{q}_{n-1}^2)$ be the corresponding orthonormal frames (cf. Proposition 2.3.3). This yields the following two bases for $\mathscr{H}$:

$$\mathscr{B}_1 = \left(\mathbb{q}_0^1, \mathbb{q}_1^1, \ldots, \mathbb{q}_{n-1}^1\right), \quad \mathscr{B}_2 = \left(\mathbb{q}_0^2, \ldots, \mathbb{q}_1^2, \ldots, \mathbb{q}_{n-1}^2\right), \tag{2.7}$$

where we have defined

$$\mathbb{q}_0^1 = \frac{\mathbb{1}}{\|\mathbb{1}\|_1} \quad \mathbb{q}_0^2 = \frac{\mathbb{1}}{\|\mathbb{1}\|_2}.$$

Both of these bases are orthonormal in the respective inner products, and are related by a matrix $\tilde{A} = [a_i^j]$ such that

$$\mathbb{q}_i^2 = \sum_j a_i^j \mathbb{q}_j^1, \quad i.e. \quad a_i^j = \langle \mathbb{q}_i^2, \mathbb{q}_j^1 \rangle^1$$

An important observation is that $a_0^j = 0$. Hence, for a position vector X, the associated Jacobi vectors $\mathbb{x}_i^k = X(\mathbb{q}_i^k)$, $k = 1, 2$ are related by

$$\mathbb{x}_i^2 = \sum_{j=1}^{n-1} a_i^j \mathbb{x}_j^1, \quad i.e. \quad J^2 = A^* \circ J^1,$$

where

$$A = \begin{bmatrix} a_1^1 & \cdots & a_{n-1}^1 \\ \vdots & & \vdots \\ a_1^{n-1} & \cdots & a_{n-1}^{n-1} \end{bmatrix}.$$

The matrix A induces a Jacobi transformation $A^* \colon J^1 \to J^2$, which is by construction the only Jacobi transformation $J^1 \to J^2$. ∎

The following commutative diagram describes the situation:

$$
\begin{tikzcd}
 & & \mathscr{M}_1^0 \arrow[dll, hook', "\supset"'] \arrow[r, "\cong"] \arrow[dd, bend left, "\pi_1^2"] & M_{3\times(n-1)} \arrow[dd, bend left, "A_1^2"] \\
\mathscr{C} \arrow[r, "\pi"] \arrow[urrr, bend left=50, "J^1"] \arrow[drrr, bend right=50, "J^2"'] & \mathscr{M} \arrow[ur, dash, "\cong"] \arrow[dr, dash, "\cong"'] & & \\
 & & \mathscr{M}_2^0 \arrow[ull, hook, "\supset"] \arrow[r, "\cong"'] \arrow[uu, bend left, "\pi_2^1"] & M_{3\times(n-1)} \arrow[uu, bend left, "A_2^1"]
\end{tikzcd}
$$

Here, $\mathscr{M}_1^0, \mathscr{M}_2^0$ denotes the orthogonal complements of the subspace

$$\mathbb{R}^3\langle\mathbb{1}\rangle \subset \mathscr{C} \qquad \text{cf.(2.5)}.$$

with respect to the inner products associated with the different mass distributions m^1, m^2. Both of them are naturally identified with the configuration space $\mathscr{M} = \mathscr{C}/\mathbb{R}^3\langle\mathbb{1}\rangle$, and therefore also with each other via the projection maps $\pi^i : \mathscr{C} \to \mathscr{M}^i$. J^1, J^2 are the given Jacobi maps, and the transition functions A^i_j are uniquely defined by commutativity of the diagram. We see that J^i, A^i_j corresponds to π^i, π^i_j by the pair of isomorphisms $M^i \to M_{3\times(n-1)}$.

The fact that A^j_i is given by right matrix multiplication can not be read directly out of the above diagram, unless we take into consideration that this is a diagram in the category of representations of $SO(3)$.

2.3.5 The Jacobi groupoid associated with a fixed mass distribution

Definition 2.3.8. For a fixed mass distribution m, we let $\mathscr{J}_m \subset \mathscr{J}$ denote the full sub-groupoid of Jacobi transformations between Jacobi maps admitted by the mass distribution m. $\mathscr{J}_m$ will be called the *Jacobi groupoid associated with the mass distribution m*.

By Proposition 2.3.3 and Lemma 2.3.7, we see that there is a canonical action of $O(n-1)$ on the set of Jacobi maps $J \in \mathscr{J}_m$, and that this action is free and transitive. From this we conclude the following:

Corollary 2.3.9. *A choice of reference object $J \in \mathscr{J}_m$ defines a bijection from the set of objects of $\mathscr{J}_m$ to $O(n-1)$. In other words: If J is given, every $J' \in \mathscr{J}_m$ is uniquely determined by the unique element $Q \in O(n-1)$ with*

$$Q^* : J \to J'.$$

2.3.6 Mass distributions with common Jacobi maps

In this section we will prove that the mass distributions m are classified by the associated Jacobi groupoids $\mathscr{J}_m$. Since each $\mathscr{J}_m$ is $O(n-1)$-homogenous, it is necessary and sufficient to investigate the following claim: *If two mass distributions m, m' admit a common Jacobi map J, then $m = m'$.*

The two body problem

It will be useful for us to consider an example where this breaks down, namely the two body problem: For two mass points P_1, P_2 with position vectors $\mathbb{a}_1, \mathbb{a}_2$ and mass distribution (m_1, m_2), the orthogonal complement of $\mathbb{1} = [1,1]^t$ is the line spanned by $\pm[-m_2, m_1]^t$, and hence the possible choices of Jacobi vectors are given by unit vectors of the form

$$\mathbb{q}_1 = \pm \begin{bmatrix} -\sqrt{\frac{m_2}{m_1(m_1+m_2)}} \\ \sqrt{\frac{m_1}{m_2(m_1+m_2)}} \end{bmatrix}.$$

The corresponding Jacobi vectors are

$$\mathbb{x}_1 = \pm\sqrt{\frac{m_1 m_2}{m_1+m_2}}(\mathbb{a}_1 - \mathbb{a}_2).$$

Hence, for different mass distributions $(m_i), (m'_i)$, the associated Jacobi vectors are related by an element of $O(1)$ if and only if the reduced masses

$$\frac{m_1 m_2}{m_1+m_2}, \quad \frac{m'_1 m'_2}{m'_1+m'_2}$$

are equal. Hence, there will exist one-parameter families of mass distributions with common Jacobi maps.

The n body problem

With $n > 2$ in mind, let us study the general case. We look at two mass distributions $(m_i), (m_i')$, with a common Jacobi map

$$J : \mathscr{C} \to M_{3\times(n-1)}.$$

Along with the two mass distributions, we have inner products $\langle,\rangle_m, \langle,\rangle_{m'}$ on $\mathscr{C}$, and also possibly different notions of *centre of mass*, which is reflected by possibly different orthogonal complements $\mathscr{H}^0, \mathscr{H}^{0\prime}$ of $\mathbb{1}$ in $\mathscr{H}$.

For each pair of labels i, j, we can find a transformation $Q_{ij} \in O(n-1)$ which gives a Jacobi map

$$J_{ij} = Q_{ij}^* \circ J$$

such that the corresponding $\langle,\rangle_m$-orthonormal basis $(\mathbb{q}_1, \ldots, \mathbb{q}_{n-1})$ of $\mathscr{H}^0$ satisfies

$$\mathbb{q}_1 \in \operatorname{span}(e_i, e_j), \tag{2.8}$$

where $(e_1, e_2, \ldots, e_n)$ is the standard basis for $\mathbb{R}^n$. By Corollary 2.3.9 the existence of Q_{ij} is clear.

Similarly as in the case of the two body problem, we must have

$$\mathbb{q}_1 = \pm\left(-\sqrt{\frac{m_i}{m_j(m_i+m_j)}}\,e_j + \sqrt{\frac{m_j}{m_i(m_i+m_j)}}\,e_i\right),$$

and the corresponding first Jacobi vector is

$$\mathbb{x}_1 = J_{ij}(X) = \pm\sqrt{\frac{m_i m_j}{m_i+m_j}}\,(\mathbb{a}_i - \mathbb{a}_j), \tag{2.9}$$

i.e. the Jacobi vector of the two body system P_i, P_j with mass distribution m_i, m_j.

Modulo sign, the first element of the representation of J_{ij} by an $\langle,\rangle_{m'}$-orthonormal basis $(\mathbb{q}_1', \dots, \mathbb{q}_{n-1}')$ for $\mathscr{H}^{0'}$ will satisfy $\mathbb{q}_1' \in \operatorname{span}(e_i, e_j)$, and as above,

$$\mathbb{q}_1' = \pm\left(-\sqrt{\frac{m_i'}{m_j'(m_i' + m_j')}}\,e_j + \sqrt{\frac{m_j'}{m_i'(m_i' + m_j)'}}\,e_i\right).$$

The corresponding first Jacobi vector is

$$\mathbb{x}_1' = J_{ij}(X) = \pm\sqrt{\frac{m_i' m_j'}{m_i' + m_j'}}\,(\mathbb{a}_i - \mathbb{a}_j),$$

i.e. the Jacobi vector of the two body system P_i, P_j with mass distribution (m_i', m_j'). Since $\mathbb{x}_1 = J_{ij}(X) = \mathbb{x}_1'$, the reduced masses

$$\mu_{ij} = \frac{m_i m_j}{m_i + m_j} \quad \text{and} \quad \mu_{ij}' = \frac{m_i' m_j'}{m_i' + m_j'}$$

must be equal, and we conclude the following:

Lemma 2.3.10. *The mass distributions m, m' have a Jacobi map in common if and only if all the reduced masses μ_{ij} and μ_{ij}' $(i \neq j)$ are pairwise equal.*

In the case of $n = 2$, equality of the reduced masses is clearly not sufficient to pin down the mass distribution. However, in the case of $n \geq 3$, the mass distribution is completely determined by the distribution of the reduced masses. This follows from the following simple algebraic observation:

Let a_i, A_i range over $(0, \infty)$. Then the equations

$$A_k = \frac{a_i a_j}{a_j + a_k} \qquad \{i, j, k\} = \{1, 2, 3\}$$

have the unique solution

$$a_i = \frac{2 A_i A_j A_k}{A_i A_j - A_j A_k + A_k A_i} \qquad \{i, j, k\} = \{1, 2, 3\}.$$

When $n > 2$, we can choose triples $1 \le \iota_1 < \iota_2 < \iota_3 \le n$ and apply the algebraic observation to the masses $m_{\iota_k} = a_k$ and the reduced masses $\mu_{\iota_j \iota_k} = A_i$. In this way we see that the mass distribution is completely determined by the distribution of reduced masses:

Lemma 2.3.11. *For $n > 2$, the mass distribution $m = (m_1, \dots, m_n)$ is uniquely determined by the corresponding distribution of pairwise reduced masses*

$$\mu_{ij} = \frac{m_i m_j}{m_i + m_j}, \quad i, j = 1, 2, \dots, n$$

Together, the two above results yields:

Theorem 2.3.12. *Consider an n body system with $2 < n < \infty$, and suppose that the two mass distributions m, m' admit a common Jacobi map $J \colon \mathscr{C} \to M_{3\times(n-1)}$. Then $m = m'$.*

Proof. Lemma 2.3.10 tells us that the distribution of reduced masses is determined by the set of admitted Jacobi maps, while Lemma 2.3.11 tells us that the mass distribution is uniquely determined by the distribution of reduced masses when $n > 2$. ∎

2.3.7 Similar mass distributions

Two mass distributions $m = (m_1, \dots, m_n)$, $m' = (m'_1, \dots, m'_n)$ are called *similar* if there exists a real number λ such that

$$m' = \lambda m = (\lambda m_1, \dots, \lambda m_n).$$

Similarly, for two similar mass distributions, the associated kinematic inner products $\langle , \rangle_m$, $\langle , \rangle_{m'}$ on the position space $\mathscr{C}$ are related by the scaling factor $\sqrt{\lambda}$: If $\mathbb{v}, \mathbb{w} \in \mathscr{C}$, then

$$\langle \mathbb{v}, \mathbb{w} \rangle_{m'} = \lambda \langle \mathbb{v}, \mathbb{w} \rangle_m = \left\langle \sqrt{\lambda}\mathbb{v}, \sqrt{\lambda}\mathbb{w} \right\rangle_m .$$

Accordingly, the decompositions $\mathscr{H} = \mathbb{R}\mathbb{1} \oplus \mathscr{H}^0$ are identical for the two inner products, and if the basis $(\mathbb{q}_1, \dots, \mathbb{q}_{n-1})$ of $\mathscr{H}^0$ yields a Jacobi map $J' \in \mathscr{J}_{m'}$, then $(\sqrt{\lambda}\mathbb{q}_1, \ldots, \sqrt{\lambda}\mathbb{q}_{n-1})$ yields a Jacobi map $J \in \mathscr{J}_m$. The corresponding Jacobi transformation $\phi\colon J' \to J$ is then given by multiplication with $\sqrt{\lambda}$.

Now, if $\psi\colon J' \to J''$ is another Jacobi transformation where $J'' \in \mathscr{J}_m$, then $\psi \circ \phi^{-1}\colon J \to J''$ is a Jacobi transformation within $\mathscr{J}_m$, and is accordingly represented by an orthogonal matrix $Q \in O(n-1)$. Hence, ψ is clearly represented by the similarity matrix $\sqrt{\lambda}Q$. This yields the following result:

Corollary 2.3.13. *Two mass distributions m, m' are similar if and only if every Jacobi transformation from any $J \in \mathscr{J}_m$ to any $J' \in \mathscr{J}_{m'}$ is represented by a similarity matrix.*

2.3.8 How large is the Jacobi groupoid?

We ask the following question:

Question 2.3.14. Which matrices $A \in \mathrm{GL}_{n-1}\mathbb{R}$ can represent Jacobi transformations $A^*\colon J \to J'$ for the n-body problem?

In other words: What is the image of the groupoid mapping

$$\mathscr{J} \to \mathrm{GL}_{n-1}(\mathbb{R})$$

(cf. Lemma 2.3.7)?

This question is important for our understanding of the hyperbolic geometry of the three body problem. Proposition 4.3.11 yields a solution in the case of the three body problem. In the following, we present some small steps towards a general answer to Question 2.3.14.

Reformulation in terms of diagonal matrices

Let us assume that $A \in \mathrm{GL}_{n-1}\mathbb{R}$ yields a Jacobi transformation $A^*\colon J^1 \to J^2$ between Jacobi maps associated with two mass distributions m^1, m^2. Since each of the $\mathscr{J}_{m^i}$ are in bijection with $O(n-1)$, we can find Jacobi maps J^1_o, J^2_o and $P, Q \in O(n-1)$ such that PAQ is diagonal and gives an arrow

$$(PAQ)^*\colon J^1_o \to J^2_o.$$

PAQ is the matrix of singular values of A. In this way we can restrict the problem to the class of diagonal matrices. Equivalently, we can aim at formulating an answer to Question 2.3.14 in terms of relations between the singular values of A.

This problem can be now phrased as follows: For a given set $(a_1, \ldots, a_{n-1})$ of real numbers, we want to find bases $\mathbb{q}_0, \ldots, \mathbb{q}_{n-1}, \mathbb{q}'_0, \ldots, \mathbb{q}'_{n-1}$ of $\mathscr{H} = \mathbb{R}^n$ such that $\mathbb{q}_0, \mathbb{q}'_0 \in \operatorname{span}(\mathbb{1})$, and a set $(b_1, \ldots, b_{n-1})$ of real numbers such that

$$\mathbb{q}_i = a_i \mathbb{q}'_i + b_i \mathbb{q}'_0 \quad i = 1, \ldots, n-1,$$

and such that the bases $(\mathbb{q}_i)$, $(\mathbb{q}'_i)$ are orthonormal in a mass dependent inner product on $\mathscr{H} = \mathbb{R}^n$, i.e. in an inner product in which the standard basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$ is orthogonal.

Diagonality of inner products in $\mathbb{R}^n$ yields $\frac{1}{2}n(n-1)$ constraints, since the off-diagonal elements are all zero. On the other hand the bases $(\mathbb{q}_i), (\mathbb{q}'_i)$ are determined by the basis $(\mathbb{q}_i)$ and the $(n-1)$ real numbers $b_1, \ldots, b_{n-1}$ as well as the proportion $b_0 = \mathbb{q}_0 : \mathbb{q}'_0$. When we choose the basis $(\mathbb{q}_i)$, we must obey the constraint that $\mathbb{q}_0 \| \mathbb{1}$. This gives $n-1$ real constraints, and the basis $\mathbb{q}_i$ can hence be chosen in a space of dimension $n^2 - (n-1)$. Hence we are allowed to adjust $n^2 - (n-1) + n = n(n-1) + 1$ parameters in order to satisfy $\frac{1}{2}n(n-1)$ constraints. From this argument, it is tempting to believe that every element of $\mathrm{GL}_{n-1}\mathbb{R}$ can represent at Jacobi transformation.

Anyhow, we leave this as an open problem:

Question 2.3.15. Find relations among the singular values $a_1, \ldots, a_{n-1}$ of matrices A which let us determine whether or not A represents a Jacobi transformation.

2.3.9 A note on translation invariant functions

Here we return to the general case of many particle systems with possibly infinitely many particles.

For a vector $\mathbb{v}$ in the constituent space $\mathscr{H} = \mathscr{L}^2(B, m)$ we get an $SO(3)$-equivariant linear function

$$T_{\mathbb{v}} : \mathscr{C} \to \mathbb{R}^3 \quad T_{\mathbb{v}}(X) = X(\mathbb{v}) = \begin{bmatrix} \langle \mathbb{x}, \mathbb{v} \rangle \\ \langle \mathbb{y}, \mathbb{v} \rangle \\ \langle \mathbb{z}, \mathbb{v} \rangle \end{bmatrix},$$

where $\mathbb{x}, \mathbb{y}, \mathbb{z} \in \mathscr{H}$ are the coordinate elements associated with the position $X \in \mathscr{C}$. By the method of the proof of Lemma 2.3.2 we can demonstrate that every $SO(3)$-equivariant linear function $\mathscr{C} \to \mathbb{R}^3$ is of the form $T_{\mathbb{v}}$ for a vector $\mathbb{v} \in \mathscr{H}$.

On the other hand, $T_{\mathbb{v}}$ is translation-invariant if and only if $\mathbb{v} \perp \mathbb{1}$, i.e. if $\mathbb{v} \in \mathscr{H}^0$ with respect to the given mass distribution.

We conclude that every translation invariant $SO(3)$-equivariant linear function $T : \mathscr{C} \to \mathbb{R}^3$ is of the form $T = T_{\mathbb{v}}$ where $\mathbb{v} \in \mathscr{H}^0$.

This observation yields a nice approach to the following example, which is taken from [Str06]:

Example 2.3.16. Consider the n-body problem. For a pair (i, j) of indices, the mapping

$$\rho_{ij} : X = [\mathbb{a}_1, \mathbb{a}_2, \ldots, \mathbb{a}_n] \mapsto \mathbb{a}_i - \mathbb{a}_j$$

is $SO(3)$-equivariant and translation invariant. Accordingly, there exists a vector $\mathbb{r}_{ij} \in \mathscr{H}^0$ such that

$$\mathbb{a}_i - \mathbb{a}_j = X(\mathbb{r}_{ij}).$$

This vector satisfies

$$\mathbb{r}_{ij} = \frac{1}{m_i} e_i - \frac{1}{m_j} e_j$$

where e_i, e_j are standard basis vectors of $\mathbb{R}^n$. [Str06] investigates the structure of the subset $\{\mathbb{r}_{ij}\} \subset \mathscr{H}^0$, defining the notion of *weighted root system of n body systems*, proposing to use it as a book-keeping device for the relative distances.

With this device, the inter-particle distances are computed by the formula

$$r_{ij} = \|X(\mathbb{r}_{ij})\|$$

Another example is given by a choice of Jacobi vectors $\mathbb{x}_1, \ldots, \mathbb{x}_{n-1}$. Since each individual Jacobi vector $\mathbb{x}_i$ can be regarded as a $SO(3)$-equivariant translation invariant function $\mathscr{C} \to \mathbb{R}^3$, we know that there have to exist vectors $\mathbb{q}_1, \ldots, \mathbb{q}_{n-1} \in \mathscr{H}^0$ such that the Jacobi vectors of a configuration X are given by $\mathbb{x}_i = X(\mathbb{q}_i)$.

2.4 Rotational symmetries and momentum maps

The introduction of Jacobi vectors gives us a method of elimination of the translational symmetry of many particle systems. Here, we turn to the rotational symmetry. In this section the dimension of $\mathscr{H}$ is allowed to be infinite.

The material in this section can be applied without elimination of the translational degrees of freedom. Hence, it will be convenient to abandon the distinction between the position space $\mathscr{C}$ and the configuration space $\mathscr{M}$ as well as the distinction between $\mathscr{H}$ and $\mathscr{H}^0$. In the following we will talk about the *configuration space*

$$\mathscr{M} = \mathscr{B}(\mathscr{H}, \mathbb{R}^3),$$

having in mind that $\mathscr{H}$ can stand for either $\mathscr{H}$ or $\mathscr{H}^0$ (using the previous notation), and that $\mathscr{M}$ can stand for either $\mathscr{C}$ or $\mathscr{M}$ (using the previous notation).

However, the interpretation of the results depends on the interpretation of the symbol $\mathscr{M}$: When $\mathscr{M}$ denotes the position vector space, the quantities are defined with respect to a *fixed spatial origin*. On the other hand, when $\mathscr{M}$ denotes the space of configurations, all the quantities are defined relative to the *centre of mass*.

Using the identification $\mathscr{M} = B(\mathscr{H}, \mathbb{R}^3)$ of configurations with linear operators, we have actions φ and ψ of $O(3)$ and $O(\mathscr{H})$ respectively on $\mathscr{M}$ given by composition of linear operators:

$$\varphi(U, X) = U \circ X, \quad \psi(Q, X) = X \circ Q^t,$$

where $U \in O(3)$ and $Q \in O(\mathscr{H})$. These actions are isometric, since

$$\mathrm{tr}(U \circ X \circ Q^t \circ Q \circ Y^t \circ U^t) = \mathrm{tr}(X \circ (Q^t \circ Q) \circ Y^t \circ (U^t \circ U)) = \mathrm{tr}(XY^t),$$

and they obviously commute. Following [Str06], these groups form a maximal pair of commuting subgroups of $O(\mathscr{M})$. Hence, $O(\mathscr{H})$ can be regarded as the symmetry group of the $O(3)$-equivariant geometry of the configuration space.

2.4.1 Momentum maps of simple mechanical systems

Consider a manifold M with a given Lagrange function

$$L\colon TM \to \mathbb{R},$$

such that L is of the form

$$L = K + U,$$

where K is a Riemannian metric on M and U is a smooth function. Such a Lagrange system is called a *simple mechanical system*.

Let $G \times M \to M$ be a smooth action of a Lie group G with Lie algebra $\mathfrak{g}$. On the infinitesimal leve, the action is characterized by a vector bundle map

$$\xi\colon \mathfrak{g} \times M \to TM.$$

The Riemannian metric K yields a canonical self duality $TM \cong T^*M$, and an associated dual map $\xi^*: TM \to \mathfrak{g}^* \times M$, which gives a linear map

$$\Omega_G\colon TM \to \mathfrak{g}^*.$$

Definition 2.4.1 (Momentum map)**.** Ω_G is called the *momentum map* associated with the action of G on the Lagrange system (M, L).

The momentum map is characterized in the following way: If $\varepsilon \in \mathfrak{g}$ and $\mathbb{v} \in T_pM$, then

$$K\left(\xi((\varepsilon, p)), \mathbb{v}\right) = \langle \varepsilon, \Omega_G(\mathbb{v})\rangle,$$

where $\langle -, - \rangle$ denotes the dual pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$.

An inner product space V can be regarded as a simple mechanical system, where the kinetic energy is given by the inner product and the potential function is constant. Hence, the notion of *momentum maps* extends also to G-actions on inner product spaces.

In the case where G acts by symmetries of the Lagrange function L, Ω_G is conserved:

Proposition 2.4.2 (Noether's theorem)**.** *For a simple mechanical system (M, L) with a Lie symmetry group G, Ω_G is conserved along the orbits of the Lagrange system.*

In the case where the bundle map $\xi\colon \mathfrak{g} \times M \to TM$ is injective, i.e. when the G-action is free, there is an induced inner product $\langle -,-\rangle_\xi$ in the bundle $\mathfrak{g} \times M$, and accordingly a smooth family $(I_p)_{p\in M}$ of isomorphisms $I_p\colon \mathfrak{g} \to \mathfrak{g}^*$. This observation forms the basis for the following definition:

Definition 2.4.3 (G-velocity)**.** When the action of G on the simple mechanical system (M, L) is free, we define the G-velocity to be the mapping $\omega_G\colon TM \to \mathfrak{g}$ given by

$$\omega_G(\mathbb{v}) = I_p^{-1}\Omega_G(\mathbb{v}), \quad \text{for} \quad \mathbb{v} \in T_m M.$$

The G-velocity ω_G can be described in the following way: By orthogonal projection onto the image of $\xi\colon \mathfrak{g} \times M \to TM$, we can define a bundle map $TM \to \mathfrak{g} \times M$, which gives a fibre-wise linear map $\omega_G\colon TM \to \mathfrak{g}$. It is straightforward to check that ω_G is equivariant with respect to the adjoint representation of G on $\mathfrak{g}$, and hence an Ehresmann connection in the principal bundle $M \to M/G$. In the literature this connection is called *the mechanical connection* (cf. [CMR01]).

2.4.2 Momentum maps for many particle systems

The Lie algebra of the orthogonal group $O(n)$ can be identified with the Lie algebra $\mathfrak{so}(n)$ of skew-symmetric $n \times n$-matrices, which can be given a bi-invariant inner product defined by the formula

$$\langle A, B\rangle = \frac{1}{2}\operatorname{tr}(AB^t).$$

This inner product will be regarded as the *standard inner product* on $\mathfrak{so}(n)$, and yields a natural identification of the dual vector space $\mathfrak{so}(n)^*$ with $\mathfrak{so}(n)$ itself.

For the case $n = 3$, we note that the usual identification of $\mathfrak{so}(3)$ with $\mathbb{R}^3$,

$$\begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{2.10}$$

is isometric with respect to the standard inner products on $\mathfrak{so}(3)$ and $\mathbb{R}^3$. Notationally, we represent this important identification as follows:

Definition 2.4.4. Consider the correspondence (2.10).

For a vector $\mathbb{v} \in \mathbb{R}^3$, we let $[\mathbb{v}]$ denote the corresponding 3×3-matrix, while for a matrix $A \in \mathfrak{so}(3)$, we let $\overrightarrow{A}$ denote the corresponding vector in $\mathbb{R}^3$.

With this identification, the actions of elements $U \in \mathrm{SO}(3)$ are related as follows:

$$U\overrightarrow{A} = \overrightarrow{UAU^t} \quad \text{and} \quad [U\mathbb{v}] = U[\mathbb{v}]U^t. \tag{2.11}$$

A useful exterior product

Let us regard a Hilbert space $\mathscr{H}$ together with the space $V = \mathscr{B}(\mathscr{H}, \mathbb{R}^n)$ of linear transformations from $\mathscr{H}$ to $\mathbb{R}^n$. We define an exterior product

$$\times\colon\ V \otimes V \to \mathfrak{so}(n)$$

by

$$X \times Y = YX^t - XY^t,$$

where $X^t\colon \mathbb{R}^n \to \mathscr{H}$ is the Hilbert space dual of $X\colon \mathscr{H} \to \mathbb{R}^n$.

Restricted to $\mathfrak{so}(n) \subset \mathscr{B}(\mathbb{R}^n, \mathbb{R}^n)$ this product yields the common Lie bracket:

$$A \times B = BA^t - AB^t = AB - BA = [A, B] \quad \text{for} \quad A, B \in \mathfrak{so}(n),$$

and in the case $n = 3$ the composite

$$(\mathbb{R}^3)^{\otimes 2} \xrightarrow{\cong} \mathscr{B}(\mathbb{R}^1, \mathbb{R}^3)^{\otimes 2} \xrightarrow{\times} \mathfrak{so}(3) \xrightarrow{\cong} \mathbb{R}^3$$

is identical to the standard $\times$-product on $\mathbb{R}^3$.

Our most important application of this product is to give a neat representation of the momentum map associated with the action of the orthogonal group $O(n)$ that is given by matrix multiplication:

Lemma 2.4.5 (Momentum map)**.** *Let $\mathscr{H}$ be a real Hilbert space. The momentum map*

$$\Omega\colon\ TV \to \mathfrak{so}(n)$$

associated with the left action of the orthogonal group $O(n)$ *on the Hilbert space* $V = \mathscr{B}(\mathscr{H}, \mathbb{R}^n)$ *of linear transformations* $\mathscr{H} \to \mathbb{R}^n$ *is given by*

$$\Omega(X, \dot{X}) = X \times \dot{X}$$

where we have identified TV *with* $V \times V$ *in the way that is usual for vector spaces.*

Proof. The group action corresponds to the a Lie algebra action

$$(A, X) \mapsto AX \quad \text{for} \quad A \in \mathfrak{so}(n),\ X \in V,$$

and following Section 2.4.1, the momentum map is characterized by the formula

$$\langle \Omega(X, \dot{X}), A \rangle_{\mathfrak{so}(n)} = \langle AX, \dot{X} \rangle_V.$$

The calculation

$$\begin{aligned} \langle \Omega(X, \dot{X}), A \rangle_{\mathfrak{so}(n)} &= \frac{1}{2} \operatorname{tr}\big((X \times \dot{X}) A^t\big) \\ &= \frac{1}{2} \operatorname{tr}(\dot{X} X^t A^t) + \frac{1}{2} \operatorname{tr}(A X \dot{X}^t) \\ &= \langle AX, \dot{X} \rangle_V \end{aligned}$$

proves that the characterizing formula holds when Ω is defined as in the Lemma. ∎

Applied to a motion $X(t)$ in a configuration space $\mathscr{M} = \mathscr{B}(\mathscr{H}, \mathbb{R}^3)$, Ω is the total angular momentum with respect to the centre of mass. Using the identification of $\mathbb{R}^3$ with $\mathfrak{so}(3)$ (cf. Definition 2.4.4), we arrive

$$\overrightarrow{\Omega(X, \dot{X})} = \overrightarrow{X \times \dot{X}} = \begin{bmatrix} \langle \mathbb{y}, \dot{\mathbb{z}} \rangle_{\mathscr{H}} - \langle \mathbb{z}, \dot{\mathbb{y}} \rangle_{\mathscr{H}} \\ \langle \mathbb{z}, \dot{\mathbb{x}} \rangle_{\mathscr{H}} - \langle \mathbb{x}, \dot{\mathbb{z}} \rangle_{\mathscr{H}} \\ \langle \mathbb{x}, \dot{\mathbb{y}} \rangle_{\mathscr{H}} - \langle \mathbb{y}, \dot{\mathbb{x}} \rangle_{\mathscr{H}} \end{bmatrix}. \tag{2.12}$$

In the case of n mass points with masses $m_1, \ldots, m_n$, we can translate this back to the language of position vectors $\mathbb{a}_1, \ldots, \mathbb{a}_n$ and velocity vectors $\dot{\mathbb{a}}_1, \ldots, \dot{\mathbb{a}}_n$ of configurations with the centre of mass at the origin. This yields the expression

$$\overrightarrow{\Omega(X, \dot{X})} = \begin{bmatrix} \Sigma_i m_i (y_i \dot{z}_i - \dot{y}_i z_i) \\ \Sigma_i m_i (z_i \dot{x}_i - \dot{z}_i x_i) \\ \Sigma_i m_i (y_i \dot{y}_i - \dot{x}_i y_i) \end{bmatrix} = \sum_{i=1}^n m_i (\mathbb{a}_i \times \dot{\mathbb{a}}_i).$$

Hence, we recover the standard notion of total angular momentum with respect to the centre of mass.

In the case where $\dim \mathscr{H} = n$, we can introduce Jacobi vectors. This yields a particularly simple form of the total angular momentum:

Proposition 2.4.6. *In terms of Jacobi vectors $[\mathbb{x}_1, \ldots, \mathbb{x}_{n-1}] = J(X)$, the total angular momentum of a motion $X(t)$ with the centre of mass at the origin satisfies*

$$\Omega(X, \dot{X}) = J(X) \times \left(\frac{d}{dt} J(X) \right) = \left[\sum_{i=1}^{n-1} (\mathbb{x}_i \times \dot{\mathbb{x}}_i) \right] \in \mathfrak{so}(3),$$

Hence, the Jacobi vectors allow us to express the total angular momentum independently of the choice of mass distribution. The situation is similar to that of Proposition 2.3.4, and we can in fact use the properties given by these two propositions to characterize Jacobi vectors.

We prove Proposition 2.4.6 in the following way: Let $(\mathbb{q}_1, \ldots, \mathbb{q}_{n-1})$ be the basis of $\mathscr{H}^0$ associated with the Jacobi map J. Then formula (2.12) reads

$$\vec{\Omega} = \sum_i \begin{bmatrix} \langle \mathbb{y}, \mathbb{q}_i \rangle \langle \dot{\mathbb{z}}, \mathbb{q}_i \rangle - \langle \mathbb{z}, \mathbb{q}_i \rangle \langle \dot{\mathbb{y}}, \mathbb{q}_i \rangle \\ \langle \mathbb{z}, \mathbb{q}_i \rangle \langle \dot{\mathbb{x}}, \mathbb{q}_i \rangle - \langle \mathbb{x}, \mathbb{q}_i \rangle \langle \dot{\mathbb{z}}, \mathbb{q}_i \rangle \\ \langle \mathbb{x}, \mathbb{q}_i \rangle \langle \dot{\mathbb{y}}, \mathbb{q}_i \rangle - \langle \mathbb{y}, \mathbb{q}_i \rangle \langle \dot{\mathbb{x}}, \mathbb{q}_i \rangle \end{bmatrix} = \sum_i \mathbb{x}_i \times \dot{\mathbb{x}}_i$$

Equivariance of the momentum map

The adjoint representation Ad of $O(n)$ on $\mathfrak{so}(n)$ and the co-adjoint representation Ad^* of $O(n)$ on $\mathfrak{so}(n)^*$, are given by

$$Ad_U A = UAU^t, \qquad Ad_U^* B = U^t BU, \quad \text{for } B \in \mathfrak{so}(n) \cong \mathfrak{so}(n)^*,$$

where we use the identifications given by the standard inner product. For a given element $U \in O(n)$ and $X, Y \in \mathscr{B}(\mathscr{H}, \mathbb{R}^n)$, we have

$$(UX) \times UY = UYX^tU^t - UXY^tU^t = U(X \times Y)U^t.$$

This shows that the momentum map can be regarded as an equivariant map from $\mathscr{M}$ to $(\mathfrak{so}(n)^*, Ad^*)$.

In the case $n = 3$, where we identify $\mathfrak{so}(3)$ and $\mathfrak{so}(3)^*$ with $\mathbb{R}^3$, the adjoint representation and the co-adjoint representation is represented by the standard action of $SO(3)$ on $\mathbb{R}^n$, and when we interpret $X \times Y$ as a vector in $\mathbb{R}^3$, we have

$$(UX) \times (UY) = U(X \times Y).$$

This formula fails for elements $U \in O(3)$ with negative determinant.

2.5 Invariant theory and the singular value decomposition

In [Hsi99], the orbit structures of the actions of $SO(3)$ and $O(\mathscr{H})$ on the configuration space $\mathscr{M} = \mathscr{B}(\mathscr{H}, \mathbb{R}^3)$ are investigated closely by means of classical invariant theory, in the case of a finite number of mass points. Here, we partially reproduce this description, and extend it to the infinite dimensional context. For the most, we will relate this invariant theory to the singular value decomposition of linear transformations. The singular value decomposition will so to speak be our our interface to the $SO(3)$,$O(\mathscr{H})$-invariant theory.

When it comes to the definition of the configuration space $\mathscr{M}$ and the constituent space $\mathscr{H}$, we still follow the intentionally ambiguous convention of Section 2.4. This implies that we the use of the singular value decomposition is independent of the use of Jacobi vectors.

2.5.1 Invariant theory

The matrix coefficients of $X \mapsto XX^t$ give a complete set of $O(\mathscr{H})$-invariants on $\mathscr{M}$, and since symmetric matrices are orthogonally diagonalizable, the eigenvalues give a complete set $O(3)$-invariants of XX^t. Hence, the set

$$\mu_1, \mu_2, \mu_3$$

of eigenvalues of XX^t give a complete set of simultaneous $O(\mathscr{H})$- and $O(3)$-invariants. In this way [Hsi99] identifies the orbit space

$$\frac{\mathscr{M}}{O(3) \times O(\mathscr{H})} \tag{2.13}$$

with the cone $\{\mu_1 \geq \mu_2 \geq \mu_3 \geq 0\} \subseteq \mathbb{R}^3$.

Now we will describe a fundamental domain by embedding this cone in $\mathscr{M}$: Choose an element $A \in \mathscr{M}$ in the following way: If $\dim(\mathscr{H}) \geq 3$, take any orthonormal set $\mathbb{q}_1, \mathbb{q}_2, \mathbb{q}_3$ in $\mathscr{H}$, and let

$$A = \begin{bmatrix} \mathbb{q}_1 \\ \mathbb{q}_2 \\ \mathbb{q}_3 \end{bmatrix} \tag{2.14}$$

In the case where $\dim \mathscr{H} = 2$, we let

$$A = \begin{bmatrix} \mathbb{q}_1 \\ \mathbb{q}_2 \\ 0, \end{bmatrix} \tag{2.15}$$

where $(\mathbb{q}_1, \mathbb{q}_2)$ is an orthonormal basis of $\mathscr{H}$.

We introduce the *gyration radii* r_i satisfying $r_i^2 = \mu_i$. It follows from the identification of the orbit space (2.13) with the cone $\{\mu_1 \geq \mu_2 \geq \mu_2 \geq 0\}$ that

$$\Delta = \left\{ \begin{bmatrix} r_1 & & \\ & r_2 & \\ & & r_3 \end{bmatrix} \cdot A \mid r_1 \geq r_2 \geq r_3 \geq 0 \right\} \subset \mathscr{M}$$

is a fundamental domain for the action of $SO(3) \times O(\mathscr{H})$ on $\mathscr{M}$. This means that each $SO(3) \times O(\mathscr{H})$-orbit intersects Δ in exactly one point. This implies that for every configuration X, there exists a real diagonal matrix R together with elements $U \in SO(3)$ and $P \in O(\mathscr{H})$ such that

$$X = URAP.$$

The statement that Δ is a fundamental domain is hence equivalent to the existence of singular value decompositions and the uniqueness of the partially ordered set of positive singular values. Hence we will regard r_1, r_2, r_3 as fundamental invariants of the $O(3)$-equivariant kinematic geometry of many particle systems, and in the following we will have great use of both the r_i and the singular value decomposition.

2.5.2 The singular value decomposition

Definition 2.5.1. *The space S of singular value decompositions associated with* the configuration space $\mathscr{M} = \mathscr{B}(\mathscr{H}, \mathbb{R}^3)$ is defined to be the product

$$S = SO(3) \times D_{3,\mathscr{H}} \times V_{3,\mathscr{H}}, \tag{2.16}$$

where $SO(3)$ is the rotation group, while $D_{3,\mathscr{H}}$, $V_{3,\mathscr{H}}$ are defined as follows:

$V_{3,\mathscr{H}} =$ The orbit $A \cdot O(\mathscr{H}) \subset \mathscr{M}$. (cf. (2.14) and (2.15))

$$D_{3,\mathscr{H}} = \begin{cases} \{\text{Diagonal } 3\times 3\text{-matrices, third diag. elem. } = 0\} & \text{if } \dim \mathscr{H} = 2 \\ \{\text{Diagonal } 3\times 3\text{-matrices}\} & \text{if } \dim \mathscr{H} > 2 \end{cases}$$

The composition map $S \to \mathscr{M}$ will be denoted by Φ:

$$\Phi(U, R, Q) = URQ \quad \text{for} \quad (U, R, Q) \in S.$$

Note that $V_{3,\mathscr{H}}$ has the following description: For $\dim \mathscr{H} > 2$, the elements of this space are of the form

$$\begin{bmatrix} \mathbb{q}_1 \\ \mathbb{q}_2 \\ \mathbb{q}_3 \end{bmatrix}$$

where $\mathbb{q}_1, \mathbb{q}_2, \mathbb{q}_3 \in \mathscr{H}$ are orthonormal. Hence in this case, $V_{3,\mathscr{H}}$ is simply the Stiefel manifold of 3-frames in $\mathscr{H}$. In the case where $\dim \mathscr{H} = 2$, $V_{3,\mathscr{H}}$ consists of elements

$$\begin{bmatrix} \mathbb{q}_1 \\ \mathbb{q}_2 \\ 0 \end{bmatrix}$$

where $(\mathbb{q}_1, \mathbb{q}_2)$ forms an orthonormal basis for $\mathscr{H} \cong \mathbb{R}^2$. Hence in this case $V_{3,\mathscr{H}}$ can be put in bijection with $O(2)$.

Definition 2.5.2. For an element $(U, R, Q) \in S$, we use the following terminology:

(i) The columns $\mathbb{u}_1, \mathbb{u}_2, \mathbb{u}_3$ of U are called *principal axes vectors*, and constitutes the *principal axes frame* $(\mathbb{u}_1, \mathbb{u}_2, \mathbb{u}_3)$. The matrix U will be called the *principal axes matrix*.

(ii) The diagonal elements r_1, r_2, r_3 of R are called *gyration radii*.

(iii) The vectors $\mathbb{q}_1, \mathbb{q}_2, \mathbb{q}_3 \in \mathscr{H}$ forming Q are called *inner configuration vectors*.

This terminology comes from the following sources: The term *inner configuration vector* is used only in this thesis. The term *gyration radius* is found in the literature of molecule dynamics, see e.g.[YKMK07]. For a discussion of the *principal axes*, see Section 2.6.3.

In the case where we study the barycentric configuration space $\mathscr{M}^0$, $\mathscr{H}$ represents the barycentric constituent space $\mathscr{H}^0$. Hence, in this case, the inner configuration vectors $\mathbb{q}_i \in \mathscr{H}^0$. When we study the n-body problem we can work with the coordinate vectors of $\mathbb{q}_1, \mathbb{q}_2, \mathbb{q}_3$ with respect to an orthonormal basis $(\mathbb{q}_1, \ldots, \mathbb{q}_{n-1})$ of $\mathscr{H}^0$. Such a basis can be given by a Jacobi map (cf. Lemma 2.3.2). As we see, the singular value decomposition works equally well with or without the Jacobi vectors.

A very important property of the singular value decomposition is its universal existence, which we will express in the following way:

Lemma 2.5.3. *The multiplication map* $\Phi\colon S \to \mathscr{M}$ *is surjective.*

Proof. This is simply a restatement of the fact that every element in $\mathscr{B}(\mathscr{H}, \mathbb{R}^3)$ admits a real singular value decomposition. As noted above, this is also a direct consequence of the discussion of $O(3) \times O(\mathscr{H})$-invariants, which is also found in [Hsi99]. Furthermore, we will also implicitly prove this result later, with an emphasis on the differentiability properties of the singular value decomposition. ■

The actions of $O(3)$ and $O(\mathscr{H})$ on $\mathscr{M}$ lifts to the obvious actions on S:

$$A(U, R, Q) = (AU, R, Q) \quad B(U, R, Q) = (U, R, QB^t),$$

where $A \in O(3)$ and $B \in O(\mathcal{H})$. Clearly

$$\Phi(A(U,R,Q)) = A(URQ) = A\Phi(U,R,Q)$$
$$\Phi(B(U,R,Q)) = URQB^t = B\Phi(U,R,Q).$$

We learn from this the following good reasons to consider the singular value decomposition:

(i) Using a singular value decomposition, the basic kinematic invariants r_1, r_2, r_3 are readily available.

(ii) The singular value decomposition gives the actions of $O(3)$ and $O(\mathcal{H})$ on $\mathcal{M}$ optimally transparent representations.

2.5.3 Analytic perturbation theory of the singular value decomposition

In this section we turn to the analytic perturbation theory of the singular value decomposition. It is surprisingly difficult to find references to this in the literature. As a consequence of this, it seems necessary to include it here. We will build on the analytic perturbation theory of diagonalization, which is sufficiently well developed, and in this respect we rely on [Kat66].

Since the configuration space $\mathcal{M} = \mathcal{B}(\mathcal{H}, \mathbb{R}^3)$ is regarded as a Hilbert space, the notion of analytic curves is completely unproblematic. Since absolute convergence implies convergence, the theory is for most practical purposes identical to the theory of real analytic functions in one variable.

As a subspace of $\mathcal{M}$, $V_{3,\mathcal{H}}$ inherits a notion of analyticity, while $SO(3)$ and $D_{3,\mathcal{H}}$ possess standard analytic structures, regarded matrix spaces. This defines an analytic structure on the space $S = SO(3) \times D_{3,\mathcal{H}} \times V_{3,\mathcal{H}}$ of singular value decompositions.

Now we set out to prove that any analytic curve $X(t)$ in $\mathcal{M}$ admits an analytic lifting to S. Because of the complications in the case $\dim \mathcal{H} < 3$, we state and prove this result in the following general form:

Lemma 2.5.4 (Analyticity of the singular value decomposition)**.** *Let* E, E' *be Hilbert spaces with* $\dim E \geq \dim E'$ *and* $\dim E' = k < \infty$*. An analytic curve* $X(t)$

in $\mathcal{M} = \mathcal{B}(E, E')$ admits a singular value decomposition $(U(t), R(t), Q(t))$ such that $U(t)$, $R(t)$ and $Q(t)$ depends analytically on t.

Proof. If $X(t)$ is an analytic curve in $\mathcal{M} = \mathcal{B}(E, E')$, then $X(t)X(t)^t$ is an analytic family of self-adjoint operators on E'. According to [Kat66], such a family admits analytic diagonalization. Hence there exists a positively oriented analytic orthonormal family $\mathbb{u}_1, \mathbb{u}_2, \ldots, \mathbb{u}_k$ of vectors in E' diagonalizing XX^t and corresponding analytic families $\mu_1, \mu_2, \ldots, \mu_k$ of eigenvalues.

Since XX^t is positive semi-definite, the eigenvalues $\mu_i \geq 0$, and the power series expansions of the μ_i must be of the form

$$\mu_i(t) = (t-a)^{2n}(a_0 + a_1(t-a) + \cdots) \quad \text{where } n \geq 0.$$

We can now choose gyration-radii $r_i = \pm\sqrt{\mu_i}$ in such a way that they depend analytically on t, by using one of the following expansions at $t = a$:

$$r_i(t) = (t-a)^n(\sqrt{a_0} + \frac{a_1}{2\sqrt{a_0}}(t-a) + \cdots)$$
$$r_i(t) = -(t-a)^n(\sqrt{a_0} + \frac{a_1}{2\sqrt{a_0}}(t-a) + \cdots)$$

Hence, from the diagonalization of XX^t, we get analytic families $\mathbb{u}_1, \mathbb{u}_2, \cdots, \mathbb{u}_k$ of principal axes vectors, as well as analytic families of gyration-radii $r_1, r_2, \ldots, r_k$, and in order to complete the singular value decomposition, we have to specify the inner configuration vectors $\mathbb{q}_1, \mathbb{q}_2, \ldots, \mathbb{q}_k$.

Without loss of generality, we can assume that the singular values r_i are ordered in such a way that there exist a natural number $K \leq k$ such that $r_i(t) = 0$ for all t if and only if $i > K$.

First, let us define $\mathbb{q}_1, \ldots, \mathbb{q}_K$: For $i \leq K$ and for all t such that $r_i(t) \neq 0$, we define

$$\mathbb{q}_i(t) = \frac{1}{r_i(t)} X^t(t)\mathbb{u}_i(t).$$

This formula determines $\mathbb{q}_i(t)$ for all t either directly or by means of analytic continuation: For all t such that $r_i(t) \neq 0$, we can easily check that $\|\mathbb{q}_i(t)\| = 1$. Hence, if r_i has a zero at $t = t_0$, $X^t\mathbb{u}_i$ must have a zero of exactly the same order. Hence, $\mathbb{q}_i(t)$ can be defined by analytic continuation through the zeros of $r_i(t)$.

After the construction of $\mathbb{q}_1, \dots, \mathbb{q}_K$ we choose the vectors $\mathbb{q}_{K+1}, \dots, \mathbb{q}_k$ by analytic extension of the frame $\mathbb{q}_1, \dots, \mathbb{q}_K$ to an orthonormal frame in E'. The existence and analyticity of such an extension follows directly from the properties Gram-Schmidt procedure, which can be expressed entirely by rational functions. ■

This result has the following specialization:

Lemma 2.5.5. *If $X(t)$ is an analytic curve in the configuration space $\mathscr{M} = \mathscr{B}(\mathscr{H}, \mathbb{R}^3)$, then there exists an analytic curve $(U(t), R(t), Q(t))$ in the space S of singular value decompositions such that*

$$\Phi(U(t), R(t), Q(t)) = U(t)R(t)Q(t) = X(t)$$

Proof. In the case where $\dim \mathscr{H} \geq 3$, this is covered by direct application of Lemma 2.5.4. In the case where $\dim \mathscr{H} < 3$, we apply the same lemma to the transpose $X(t)^t$, i.e. the case where $E = \mathscr{H}$ and $E' = \mathbb{R}^3$. ■

2.5.4 Smooth perturbation theory of the singular value decomposition

We can also lift continuous curves from $\mathscr{M}$ to S provided that they stay inside a specific domain, namely the domain of *regular configurations*:

Definition 2.5.6. For $\dim \mathscr{H} > 3$, we define the space $\mathscr{M}_r$ of *regular configurations* to consist of configurations $X \in \mathscr{M}$ such that XX^t has three distinct non-zero eigenvalues. The space $S_r \subset S$ of *regular singular value decompositions* consists of singular value decompositions such that the squares r_i^2 of the gyration-radii are non-zero and distinct.

For $\dim \mathscr{H} \leq 3$, we define the spaces $\mathscr{M}_r, S_r$ are defined to consist of singular value decompositions where the squares r_i^2 of the gyration-radii are distinct.

$\mathscr{M}_r$ is clearly an open and dense subset of $\mathscr{M}$.

In the context of Banach manifolds (cf. [Lan99]), we have the following result:

Lemma 2.5.7. *The restriction*

$$\Phi_r\colon S_r \to \mathcal{M}_r$$

of $\Phi\colon S \to \mathcal{M}$ *is a local diffeomorphism of class* C^ω.

Proof. We investigate the invertibility of the derivative

$$\Phi_*\colon TS \to T\mathcal{M}$$

of the multiplication map $\Phi\colon S \to \mathcal{M}$, and use the inverse function theorem to draw our conclusion. This is done under the assumption that $\dim \mathcal{H} \geq 3$. The case $\dim \mathcal{H} = 2$ differs only in insignificant details, and we leave that out here.

Thus we let $\mathcal{H}$ be a Hilbert space of dimension ≥ 3 and $\mathcal{M}$ the corresponding configuration space. Furthermore, let $M_{3\times 3}$ be the vector space of real 3×3-matrices, $D_3 \subset M_{3\times 3}$ the subspace of diagonal matrices, and $\mathcal{V}$ the Stiefel manifold of orthonormal 3-frames in $\mathcal{H}$. We regard the space of singular value decomposition as a submanifold

$$S = O(3) \times D_3 \times \mathcal{V} \subset M_{3\times 3} \times D_3 \times \mathcal{M}.$$

The tangent bundle $T\mathcal{M}$ of $\mathcal{M}$ is identified with $\mathcal{M} \times \mathcal{M}$, and the tangent bundle TS of S can be regarded as a sub-bundle of

$$T(M_{3\times 3} \times D_3 \times M) = (M_{3\times 3} \times D_3 \times \mathcal{M}) \times (M_{3\times 3} \times D_3 \times \mathcal{M}).$$

Under these identifications,

$$\Phi_*(U, R, Q, \dot{U}, \dot{R}, \dot{Q}) = (URQ, \dot{U}RQ + U\dot{R}Q + UR\dot{Q}).$$

Our aim is to show that Φ_* is a fibre-wise isomorphism.

For a given point $(U, R, Q) \in S$, we want to understand the derivative

$$\Phi_*\colon T_{(U,R,Q)}S_r \to T_{URQ}\mathcal{M}_r.$$

Since Φ is $SO(3)$-equivariant, we loose no generality in assuming that U is the identity matrix, and hence that $\dot{U}$ is skew symmetric.

Since R is diagonal, $\dot{R}$ is also diagonal.

Let $\mathbb{q}_1, \mathbb{q}_2, \mathbb{q}_3 \in \mathcal{H}$ be the triple representing Q, let V denote the subspace $span\{\mathbb{q}_1, \mathbb{q}_2, \mathbb{q}_3\} \subset \mathcal{H}$ and let $W = V^\perp$. When regarding Q and $\dot{Q}$ as a linear operators $\mathcal{H} \to \mathbb{R}^3$, we observe the following facts:

(i) In the basis $(\mathbb{q}_1, \mathbb{q}_2, \mathbb{q}_3)$ of V, $Q|_V$ is represented by the identity matrix.

(ii) In the basis $(\mathbb{q}_1, \mathbb{q}_2, \mathbb{q}_3)$ of V, $\dot{Q}|_V$ is represented by a skew symmetric matrix that we will denote by A.

(iii) $Q|_W = 0$.

(iv) $\dot{Q}|_W$ can be *any* linear transformation $W \to \mathbb{R}^3$.

In the basis $(\mathbb{q}_1, \mathbb{q}_2, \mathbb{q}_3)$ of V we thus get

$$\begin{aligned} \Phi_*(U, R, Q, \dot{U}, \dot{R}, \dot{Q})|_V &= \dot{U}R + R + RA \\ \Phi_*(U, R, Q, \dot{U}, \dot{R}, \dot{Q})|_W &= R\dot{Q}|_W \end{aligned} \tag{2.17}$$

Now, for any given $B \in \mathscr{M} = T_{URQ}\mathscr{M}$, we ask if there exist a unique solution of the equation

$$\Phi_*(U, R, Q, \dot{U}, \dot{R}, \dot{Q}) = B$$

among the tangent vectors at $(U, R, Q) \in S$.

Considering the restrictions to the subspace $W \subset \mathscr{H}$, we see that

$$\Phi_*(U, R, Q, \dot{U}, \dot{R}, \dot{Q})|_W = B|_W$$

always gives a unique value of $\dot{Q}|_W$, provided that the diagonal elements of R are all non-zero. This is the origin of the *non-zero condition* in Definition 2.5.6. In the case where $\dim \mathscr{H} \leq 3$, $\dim W = 0$, and hence for $\dim \mathscr{H} \leq 3$ we do not need the *non-zero* condition in Definition 2.5.6.

Under the restriction to V, we get the equations

$$\begin{aligned} \dot{r}_i &= \langle \mathbb{b}_i, \mathbb{q}_i \rangle \\ r_j u_i + r_k a_i &= \langle \mathbb{b}_k, \mathbb{q}_j \rangle, \\ r_k u_i + r_j a_i &= \langle \mathbb{b}_j, \mathbb{q}_k \rangle \end{aligned} \tag{2.18}$$

where $\mathbb{b}_1, \mathbb{b}_2, \mathbb{b}_3$ is the triple in $\mathscr{H}$ representing $B \in \mathscr{M}$, and $\{i, j, k\} = \{1, 2, 3\}$,

$$\dot{U} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix}, \quad \text{and} \quad A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$

Clearly, the system (2.18) has a unique solution $(\dot{r}_i, u_i, a_i)$ if and only if the determinant $r_j^2 - r_i^2 \neq 0$. This is the origin of the *distinct eigenvalue condition* in Definition 2.5.6.

From these considerations it is clear that Φ_* is a fibre-wise isomorphism precisely when the squares r_i^2 of the gyration-radii are non-zero and distinct, i.e. over the space $\mathscr{M}_r$ of regular configurations.

By the inverse function theorem for smooth mappings between open sets of Banach spaces (cf. [Lan99]), it follows that Φ induces a local diffeomorphism $S_r \to \mathscr{M}_r$. Since Φ is analytic, the local inverses will also be analytic. This proves Lemma 2.5.7.

■

As an immediate consequence, we have the following curve lifting lemma:

Lemma 2.5.8. *Let $X(t)$ be a curve in $\mathscr{M}_r$ of smoothness class C^n, where $n \in \{\infty, \omega, 0, 1, 2, \dots\}$. If $\Sigma_0 \in S_r$ is a singular value decomposition of $X(0)$, then there exists a unique curve $\Sigma(t)$ in S_r such that $\Phi(\Sigma(t)) = X(t)$ and $\Sigma(0) = \Sigma_0$. The curve $\Sigma(t)$ is of the same smoothness class as $X(t)$.*

At the first sight, it may seem probable that we can extend Lemma 2.5.8 outside $\mathscr{M}_r$. Following Lemma 2.5.5, analyticity of $X(t)$ is clearly a sufficient condition for extension to all of $\mathscr{M}$. Extension to the space of matrices X with $\operatorname{rank} X > 1$, clearly requires more than C^∞-smoothness: A C^∞-motion $X(t)$ with $\operatorname{rank} X > 1$, may have discontinuous singular value decomposition data. An obvious example of this is given by the following family of 2×2-matrices:

$$X(t) = \begin{cases} \begin{bmatrix} 1 + e^{-\frac{1}{t}} & 0 \\ 0 & 1 - e^{-\frac{1}{t}} \end{bmatrix} & t > 0 \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & t = 0 \\ \begin{bmatrix} 1 & e^{\frac{1}{t}} \\ e^{\frac{1}{t}} & 1 \end{bmatrix} & t < 0 \end{cases}$$

For $t > 0$, $X(t)$ is diagonalized by the standard basis of $\mathbb{R}^2$ while for $t < 0$, $X(t)$ is diagonalized by

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Accordingly, the diagonalizing basis is not even continuous.

2.5.5 The principal axes frame and the inertia operator

For a motion $X(t)$ with singular value decomposition $(U(t), R(t), Q(t))$ the column vectors $\mathbb{u}_1, \mathbb{u}_2, \mathbb{u}_3$ of U constitutes the *principal axes frame*. Classically, the principal frame is characterized as a diagonalizing frame for the inertia operator, which is defined as follows:

We consider a purely rotational motion through the configuration X_0, given by the angular velocity $\xi \in \mathbb{R}^3$. The motion is then described as $X(t) = exp([\xi]t)X_0$, and the corresponding total angular momentum vector $\Omega \in \mathbb{R}^3$ is given by

$$[\Omega] = X_0 \times \dot{X} = X_0 \times ([\xi]X_0)$$

(cf. Definition 2.4.4). This yields a linear operator

$$I_{X_0} : \mathbb{R}^3 \to \mathbb{R}^3, \quad \xi \mapsto \Omega,$$

which is called the *inertia operator*. This operator is identical to the operator I_p that appeared in Definition 2.4.3.

The coordinate vectors of Ω and ξ in the principal frame are respectively $U^t\Omega$ and $U^t\xi$, and they are related by

$$\begin{aligned} [U^t\Omega] &= U^t[\Omega]U \\ &= (U^tX) \times (U^t[\xi]UU^tX \\ &= (RQ) \times (U^t[\xi]URQ) \\ &= [U^t\xi]R^2 + R^2[U^t\xi] \end{aligned}$$

Hence if we write

$$U^t\Omega = \mathbf{g} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} \quad \text{and} \quad U^t\xi = \mathbf{h} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} r_2^2 + r_3^2 & & \\ & r_3^2 + r_1^2 & \\ & & r_1^2 + r_2^2 \end{bmatrix}$$

we can conclude, after some calculations, that

$$\mathbf{g} = \Lambda\mathbf{h} \quad \text{i.e.} \quad I_{X_0} = U\Lambda U^t. \tag{2.19}$$

Hence, the principal frame diagonalizes the inertia operator. The diagonal elements

$$\lambda_1 = r_2^2 + r_3^2, \quad \lambda_2 = r_3^2 + r_1^2, \quad \lambda_3 = r_1^2 + r_2^2 \tag{2.20}$$

of Λ are the eigenvalues of the inertia operator I_X, and are called the *principal moments of inertia* of the configuration X with respect to the principal frame $(\mathbb{u}_1, \mathbb{u}_2, \mathbb{u}_3)$.

For later reference, we note that

$$(RQ) \times ([\mathbb{v}]RQ) = [\Lambda\mathbb{v}], \tag{2.21}$$

where R, Q belongs to a singular value decomposition $(U, R, Q) \in S$ of $X \in \mathscr{M}$.

2.6 The principal axes gauge and many particle systems

2.6.1 General discussion in terms of principal bundles

In this section we take a step back, in order to consider simple mechanical systems for which the configuration space is a principal bundle $\pi\colon P \to B$ with compact structure group G. Let $\mathfrak{g}$ denote the Lie algebra of G. Hence, the following is of relevance only for the principal stratum P of the action of $SO(3)$ on the configuration space $\mathscr{M}$. We assume that G acts by isometries of the kinematic geometry of P.

The kernel of the derivative π_* of the projection $\pi\colon P \to B$ yields a distribution $VP \subset TP$, which is called *vertical distribution*. The orthogonal complement $HP = VP^\perp \subset TP$ is called the *horizontal distribution*. The vertical distribution is integrable, and can be regarded as an attribute of the projection π itself; the leaves of the corresponding foliation are the G-orbits in P. On the other hand, the horizontal distribution depends on the kinematic geometry, and can be non-integrable, both locally and globally.

The derivative of the group action induces a vector bundle isomorphism $\mathfrak{g} \times P \to VP$. On the other hand, the orthogonal projection $TP \to VP$ induces a 1-form

$$\omega\colon TP \to \mathfrak{g},$$

the *mechanical connection form*, which is an Ehresmann connection. In the sense of Section 2.4.1, ω is the same as the G-velocity. The curvature

$$d\omega - \frac{1}{2}[\omega, \omega]$$

of ω measures to what extent the horizontal distribution HP is locally integrable.

In [LR97] a *local choice of gauge over an open set $U \subset B$* is defined to be a local section

$$\sigma\colon U \to P.$$

Since G acts freely, such a section gives an isomorphism

$$\pi^{-1}(U) \to U \times G.$$

Equivalently, we may represent the local gauge by a mapping

$$\tilde{\sigma}\colon \pi^{-1}(U) \to G,$$

where $\tilde{\sigma}(gp) = g\tilde{\sigma}(p)$ for $g \in G$, and $p \in P$. We can reconstruct σ from $\tilde{\sigma}$, provided that $\tilde{\sigma}$ is transversal to the projection π. Hence, a choice of gauge may be interpreted as a local trivialization of the principal bundle. The Darboux derivative associated with $\tilde{\sigma}$ is defined by means of the left trivialization $TG \to \mathfrak{g}$ as the composite

$$\omega_\sigma\colon T(\pi^{-1}(U)) \to TG \to \mathfrak{g},$$

and gives a complete characterization of $\tilde{\sigma}$ modulo G-translation. By the fundamental theorem of the Darboux derivative [Sha97] the 1-form ω_σ contains all the interesting local information about the choice of gauge.

The two $\mathfrak{g}$-valued 1-forms ω, ω_σ have different benefits:

- ω is a natural part of the kinematic geometry, but may have non-trivial curvature.

- ω_σ has no curvature, but is not naturally given.

The curvature of ω is the local obstruction to finding a gauge σ for which $\omega = \omega_\sigma$. From a geometric point of view, it would be favourable to find such a gauge: Locally, we would be able to find a trivialization $\pi^{-1}(U) \cong U \times G$, where the G-component is everywhere orthogonal to U. For a G-invariant simple mechanical system on P, we would be able to fully reduce the system (locally) to the base space B, in the sense that we would acquire a simple mechanical system on B, which would determine the moduli curves $\overline{\gamma}(t)$ in B directly. From the moduli curves we would be able to find curves $g(t)$ in G such that the motions of the original system were of the form $g(t)\sigma(\gamma(t))$.

When ω has non-trivial curvature, it is impossible to reduce the system to the level of B in such a simple way.

Our main example of a simple mechanical system in the form of a principal bundle is given by the action of $SO(3)$ on the regular part $\mathscr{M}_r$ of configuration space $\mathscr{M}$.

2.6.2 Global considerations: Multi-valued gauges

The local considerations stop here. When it comes to the question of existence of a global gauge $\sigma\colon B \to P$ we also have to take into consideration global topological obstructions. It is well known that such obstructions exist for many body systems [Eck35]; they give rise to inevitable *body frame singularities*. In [LMAC98] it is suggested to consider multi-valued choices of gauge, and in the following, we will develop that idea.

We can formulate the core of the idea of multi-valued gauges in the following way: When we lack global sections in $P \to B$, we may consider the following options:

(i) Consider gauges defined by restriction of P to subsets $U \subset B$, i.e. local gauges.

(ii) Consider gauges defined on the pullback of P through mappings $\tilde{B} \to B$. The multi-valued gauges that we will define below falls into this category.

Regarding our terminology, we note the following: Many authors reserve the term *covering space* to path connected spaces. For us it is favourable to admit non-connected covering spaces: The essential properties are (i) surjectivity, (ii) path lifting property, and (iii) local diffeomorphism. We can always restrict the discussion to one path component of the covering space, but when we do this change in terminology, we gain the liberty not having to count path components.

Definition 2.6.1. A multi-valued gauge in the principal bundle $\pi\colon P \to B$ consists of a covering space

$$\pi\colon \tilde{B} \to B$$

and a mapping $\sigma\colon \tilde{B} \to P$ which gives a commutative diagram

$$\begin{tikzcd} \tilde{B} \arrow[r, "\sigma"] \arrow[dr, "\pi"'] & P \arrow[d, "\pi_P"] \\ & B \end{tikzcd}$$

An equivalent characterization is the following: A multi-valued gauge consists of covering space $\tilde{B} \to B$ together with a global section in the pullback bundle $\tilde{P} \to \tilde{B}$. Using the set-theoretical definition

$$\tilde{P} = P \times_B \tilde{B} = \{(p, b) \in P \times \tilde{B}\colon \pi(p) = \pi(b)\}$$

of this pullback, the connection to Definition 2.6.1 is revealed: There is a bijection between the set of global sections $\tilde{\sigma}\colon \tilde{B} \to \tilde{P}$ and mappings $\sigma\colon \tilde{B} \to P$ such that $\pi_P \circ \sigma = \pi_{\tilde{B}}$.

Every covering space $\tilde{B} \to B$ comes with its own symmetry group Σ that acts by permutations in the fibres. The global picture can be captured by the fundamental group $\pi_1(B)$ together with the group $\Gamma(\tilde{B})$ of permutations of the path components of $\tilde{B}$. Their product $\pi_1(B) \times \Gamma$ then acts transitively on every fibre, and the kernel of ineffectiveness is $\pi_1(\tilde{B}) \times \Gamma(B)$, uniformly. Hence, there is a group

$$\Sigma = \frac{\pi_1(B) \times \Gamma(\tilde{B})}{\pi(\tilde{B}) \times \Gamma(B)}$$

which makes $\tilde{B} \to B$ a principal bundle with structure group Σ.

In other words: When we use a multi-valued gauge, we replace the original principal bundle $P \to B$ with a principal bundle $\tilde{B} \to B$ with discrete structure group. Hence, when the multi-valued gauge $\sigma\colon \tilde{B} \to P$ is given, we can specify a local gauge in $P \to B$ by giving a local gauge in $\tilde{B} \to B$. In our study of the three body problem, we will meet the question of giving local gauges in a bundle of the type $\tilde{B} \to B$ disguised as questions about conventions for singular value decomposition data, and also carry out explicit computations with the structure group Σ associated with a multi-valued gauge.

2.6.3 The principal axes frame as a multi-valued gauge

For a given regular configuration $X \in \mathcal{M}_r$, there exists a finite number of principal frames

$$(\mathbb{u}_1, \mathbb{u}_2, \mathbb{u}_3),$$

which are related by permutations and changes in sign. If we decide to work in the gauge of positively oriented principal axes, the freedom in choice of principal frame is encoded by the finite subgroup $SO(3,\mathbb{Z}) \subset SO(3)$ consisting of matrices with integer coefficients.

We want to describe the principal axes gauge as a multi-valued gauge in the principal $SO(3)$-bundle

$$\mathcal{M}_r \to \overline{\mathcal{M}}_r = \frac{\mathcal{M}_r}{SO(3)}.$$

In our description, we will use the space

$$S_r = SO(3) \times \overline{S}_r$$

of singular value decompositions where

$$\overline{S}_r = D_{3,\mathcal{H}} \times V_{3\times H}$$

(cf. (2.16)). S_r can be regarded as a trivial principal bundle

$$S_r \to \overline{S}_r$$

with structure group $SO(3)$.

We observe that the map $S_r \to \mathscr{M}_r$ is a principal bundle with a certain finite structure group Σ, which can be described as follows: The group $SO(3,\mathbb{Z})$ acts on S_r in the following way: For $A \in SO(3,\mathbb{Z})$ and $(U,R,Q) \in S_r$, we let

$$A(U,R,Q) = (UA^t, ARA^t, AQ)$$

Additionally, the subgroup $O_{diag}(3) \subset O(3)$ of diagonal matrices in acts on S_r in the following way: For $B \in O_{diag}(3)$, we let

$$B(U,R,Q) = (U, RB, BQ).$$

Let Σ be the group of transformations of S_r generated by $SO(3,\mathbb{Z})$ together with $O_{diag}(3)$.

The multiplication map $\Phi_r\colon S_r \to \mathscr{M}_r$ is clearly Σ-invariant. On the other hand, if (U,R,Q) and (U',R',Q') are two singular value decompositions of the same element $X \in \mathscr{M}_r$, then they are related by a unique transformation in Σ. This transformation is given by application of $A = U^tU' \in SO(3,\mathbb{Z})$ before application of $B = (A^tR^{-1}A)R' \in O_{diag(3)}$. Hence, Σ acts freely and transitively on the fibres of Φ_r.

Since $\Phi_r\colon S_r \to \mathscr{M}_r$ is $SO(3)$-equivariant and $S_r/SO(3) \cong \overline{S}_r$, we get an induced mapping $\overline{\Phi}_r\colon \overline{S}_r \to \overline{\mathscr{M}_r}$. The action of Σ commutes with the action of $SO(3)$ on S_r, and we have the following diagram where the rows and columns are principal bundles:

$$\begin{tikzcd}
& SO(3) \arrow[d, phantom, "\circlearrowright"] & SO(3) \arrow[d, phantom, "\circlearrowright"] \\
\Sigma \circlearrowright & S_r \arrow[r, "\Phi_r"] \arrow[d, "\pi"] & \mathscr{M}_r \arrow[d, "\pi"] \\
\Sigma \circlearrowright & \overline{S}_r \arrow[r, "\overline{\Phi}_r"] & \overline{\mathscr{M}_r}
\end{tikzcd},$$

Referring to this diagram, we note the following:

(i) S_r can be regarded as the pullback of $\mathscr{M}_r \to \overline{\mathscr{M}}_r$ along $\overline{S}_r \to \overline{\mathscr{M}}_r$: The pullback $\overline{\Phi}_r^* \mathscr{M}_r$ can be identified with the space

$$\left\{((R,Q),X) \in \overline{S}_r \times \mathscr{M}_r \colon RQ = X \mod SO(3)\right\}$$

Since $\mathscr{M}_r$ is a principal bundle, there exists precisely one $U \in SO(3)$ such that $URQ = X$, if $((R,Q),X) \in \overline{\Phi}_r^* \mathscr{M}_r$. Hence, the tuple $((R,Q),X) \in \overline{\Phi}_r^* \mathscr{M}_r$ contains precisely the same information as the tuple $(U,R,Q) \in S_r$, and $S_r \to \overline{S}_r$ is thus isomorphic to the pullback bundle.

(ii) There is a global section in $S_r \to \overline{S}_r$, given by $(R,Q) \mapsto (I,R,Q)$. The corresponding map $\overline{S}_r \to \mathscr{M}_r$ is given by $(R,Q) \mapsto RQ$.

(iii) Σ acts freely and transitively on the fibres of $\overline{S}_r \to \overline{\mathscr{M}}_r$.

The connection

As we noted above, action of $SO(3)$ on $\mathscr{M}_r$ lifts to the obvious $SO(3)$-action on S_r. We will use this to compute the momentum map of $(\mathscr{M}_r, SO(3))$, lifted to the language of singular value decompositions.

Let us consider a motion $(U(t),R(t),Q(t))$ in S_r with velocity $(\dot{U}(t),\dot{R}(t),\dot{Q}(t))$. On the level of many particle configurations, the corresponding velocity is

$$\dot{X} = \dot{U}RQ + U\dot{R}Q + UR\dot{Q}.$$

The principal axes gauge is now represented by the mapping $S_r \to \mathrm{SO}(3)$ given by

$$\tilde{\sigma}(U,R,Q) \mapsto U,$$

and the corresponding flat connection is represented by

$$\omega_\sigma \colon (U,R,Q,\dot{U},\dot{R},\dot{Q}) \mapsto \dot{U}U^t$$

This mapping is Σ-invariant, and accordingly ω_σ descends to a well defined differential geometric object on $\mathscr{M}_r$. In terms of ω_σ, we can interpret the gauge map $\overline{S}_r \to \mathscr{M}_r$ as an integral manifold of the distribution $\ker \omega_\sigma$ on $\mathscr{M}_r$. This yields an interesging approach to construction of multi-valued gauges from flat connections.

We compute the mechanical connection as follows: By the proof of Proposition 2.7.3 later in this thesis, we will see that

$$\begin{aligned}[\Omega] &= (URQ) \times (\dot{U}RQ + P\dot{R}Q + UR\dot{Q}) \\ &= U[(\Lambda\mu - \widehat{R}\nu)]U^t,\end{aligned}$$

where $\mu, \nu \in \mathbb{R}^3$ respectively represents the skew symmetric matrices $U^t\dot{U}$ and $Q \times \dot{Q}$, and where

$$\widehat{R} = \begin{bmatrix} r_2 r_3 & & \\ & r_3 r_1 & \\ & & r_1 r_2 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} r_2^2 + r_3^2 & & \\ & r_3^2 + r_1^2 & \\ & & r_1^2 + r_2^2 \end{bmatrix}.$$

From this we can conclude that the mechanical connection ω satisfies

$$\omega = U\Lambda U^T \Omega = U(\mu - \Lambda^{-1}\hat{R}\nu),$$

(cf. (2.19)).

2.7 Characterization of motions with constant total angular momentum

In this section, the configuration space $\mathcal{M}$ is still ambiguously defined, as in Section 2.4. Hence $\mathcal{M}$ can denote (i) the position vector space or (ii) the configuration space. In case (i) we work with the total angular momentum with respect to a fixed spatial origin, while in case (ii) the total angular momentum is defined with respect to the centre of mass.

Motivated by the lifting results of Lemma 2.5.5 and 2.5.8, we will now study motions $X(t)$ in $\mathcal{M}$ together with their lifting $(U(t), R(t), Q(t))$ to the space S of singular value decompositions.

First, we will establish some formulae that are valid in any rotating frame. Then, we will study our favourite rotating frame, namely the principal axes frame. Finally, we use this to deduce the Euler equations for many body motions with constant angular momentum.

2.7.1 Rotating frames

A rotating frame is a time dependent positively oriented orthonormal basis $(\mathbb{u}_1, \mathbb{u}_2, \mathbb{u}_3)$ of $\mathbb{R}^3$. Such a frame can be represented by a curve $U(t)$ in $SO(3)$, and we consider here only the case where $U(t)$ is a differentiable function of t. Define $\mu(t)$ to be the coordinate vector of the *angular velocity of the rotating frame* within the rotating frame. In other words, $\mu = \overrightarrow{U^t\dot{U}} \in \mathbb{R}^3$ (cf. Definition 2.4.4.)

Proposition 2.7.1. *Consider a rotating frame in $\mathbb{R}^3$ with differentiable matrix representation $U(t) = [\mathbb{u}_1(t)|\mathbb{u}_2(t)|\mathbb{u}_3(t)]$ and angular velocity $\mu(t)$. If $\Omega(t)$ is a differentiable time-dependent vector in $\mathbb{R}^3$ with coordinate vector $\mathbf{g} = U^t\Omega$, then*

$$\dot{\mathbf{g}} = -[\mu]\mathbf{g} + U^t\dot{\Omega}$$

(cf. Definition 2.4.4).

Proof. The proof is a straightforward application of the product rule for differentiation of matrix products. ∎

We are particularly interested in the following immediate application of Proposition 2.7.1:

Corollary 2.7.2. *Suppose that X is a continuously differentiable many body motion with a continuously differentiable rotating frame represented by a curve $U(t)$ in* SO(3) *with continuous angular velocity vector $\mu(t)$ in $\mathbb{R}^3$.*

In this case the total angular momentum $\Omega = X \times \dot{X}$ is conserved if and only if

$$\dot{\mathbf{g}} = -[\mu]\mathbf{g},$$

where $\mathbf{g} = U^t\Omega$ is the coordinate vector of the total angular momentum.

Proof. Let us suppose that $X(t)$ is a continuously differentiable many body motion, and that $U(t)$ is a continuously differentiable rotating frame.

Assume that $X(t)$ has conserved total angular momentum Ω. Then by the product rule, $\mathbf{g}$ is differentiable, and satisfies

$$\dot{\mathbf{g}} = \frac{d}{dt}(U^t\Omega) = \dot{U}^t U\mathbf{g} + U^t\dot{\Omega} = -[\mu]\mathbf{g}.$$

Suppose on the other hand that $\dot{\mathbf{g}} = -[\mu]\mathbf{g}$. Then we can judge the rate of change of $\Omega = U\mathbf{g}$ by the product rule, which gives

$$\dot{\Omega} = \frac{d}{dt}(U\mathbf{g}) = \dot{U}\mathbf{g} + U\dot{\mathbf{g}} = (\dot{U} - U(U^t\dot{U}))\mathbf{g} = 0.$$

■

We note that this theorem applies to any rotating frame, not only the principal frame.

2.7.2 Euler equations

Here we will give a characterization of many particle motions with conserved total angular momentum. This characterization can be regarded as a generalization of the Euler equations for the rigid body (cf. [Arn89])

For a twice differentiable motion $X(t)$ in $\mathscr{M}$ with a twice differentiable singular value decomposition $(U(t), R(t), Q(t))$, we define the following dependent objects:

We have the matrices R, $\widehat{R}$

$$R = \begin{bmatrix} r_1 & & \\ & r_2 & \\ & & r_3 \end{bmatrix} \quad \widehat{R} = \begin{bmatrix} r_2 r_3 & & \\ & r_3 r_1 & \\ & & r_2 r_3 \end{bmatrix}$$

the matrix of *principal moments of inertia*,

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix} \quad \text{where} \quad \lambda_i = r_j^2 + r_k^2 \quad (\{i, j, k\} = \{1, 2, 3\}),$$

and the associated matrix

$$\widehat{\Lambda} = \begin{bmatrix} \lambda_2\lambda_3 & & \\ & \lambda_3\lambda_1 & \\ & & \lambda_1\lambda_2 \end{bmatrix}.$$

These matrices satisfy $(\Lambda X)\times(\Lambda Y) = \widehat{\Lambda}(X\times Y)$ and $(RX)\times(RY) = \widehat{R}(X\times Y)$ for configurations $X, Y \in \mathscr{M}$. We also recall the coordinate vector

$$\mathbf{g} = U^t\Omega$$

of the total angular momentum in the principal frame, the angular velocity of the principal frame $\mu = \overrightarrow{U^t\dot{U}}$, and the inner angular velocity

$$\nu = \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix} = \overrightarrow{Q\times\dot{Q}}$$

The data

$$R, \widehat{R}, \Lambda, \widehat{\Lambda}, \mathbf{g}, \mu, \nu \tag{2.22}$$

depends on the choice of principal axes. We shall see that there exist relations among these data that determines the motion of the principal frame. Hence, the data (2.22) can be regarded as an adequate description of the rotational motion of many particle systems.

First we see how the angular velocity μ of the principal frame is determined by the other quantities.

Proposition 2.7.3. *Under reference to the above definitions, the angular velocity $\mu(t)$ of the principal axes frame satisfies*

$$\Lambda\mu = \mathbf{g} - \widehat{R}\nu.$$

Proof. We can express $\mathbf{g}$ in terms of the singular value decomposition data in the following way:

$$\begin{aligned}[]
[\mathbf{g}] &= [U^t\Omega] = U^t(X\times\dot{X})U^t \\
&= \left[(RQ)\times\left([\mu]RQ\right)\right] + \left[(RQ)\times(\dot{R}Q)\right] + \left[(RQ)\times(R\dot{Q})\right].
\end{aligned} \tag{2.23}$$

We prove the proposition by investigation of the terms of this expression:

The first term: Equation (2.21) yields the formula

$$(RQ)\times([\mu]RQ) = [\Lambda\mu] \tag{2.24}$$

for the first term in (2.23).

The second term: By the definition of the product $\times$, we have

$$(RQ) \times (\dot{R}Q) = \dot{R}QQ^t R^t - RQQ^t \dot{R}^t = \dot{R}R^t - R\dot{R}^t = 0, \tag{2.25}$$

since Q is orthogonal and $R, \dot{R}$ are diagonal.

The third term: The matrix $\hat{R}$ was defined in such a way that

$$(RQ) \times (R\dot{Q}) = \widehat{R}\,(Q \times \dot{Q}) = \widehat{R}\,\nu, \tag{2.26}$$

where $\nu = Q \times \dot{Q}$.

The proposition now follows directly from (2.23), (2.24), (2.25) and (2.26). ∎

The reconstruction of the principal frame

For non-collinear motions, Λ is invertible, and

$$\mu = \Lambda^{-1}(\mathbf{g} - \widehat{R}\nu).$$

Hence, in this case U is determined by its initial and a differential equation on the form

$$U^t \dot{U} = F(r_i, g_i, \nu_i). \tag{2.27}$$

For given $r_i(t), g_i(t), \nu_i(t)$, this can be regarded as a quadrature in $SO(3)$. Later, we will see that these required data are given in an intrinsic way, and hence, we can determine $U(t)$ by first determining the data r_i, g_i, ν_i intrinsically, and then integrate (2.27). In general, we are however not interested in the extrinsic rotation $U(t)$. A small exception is given in the proof of Lemma 2.8.1.

For collinear motions, Λ is singular and $\mu(t)$ will be well defined modulo a summand along the collinearity. Hence, in this case $U(t)$ will be well defined modulo rotations about the axis of collinearity. Since such rotations are physically insignificant, we see that the variables r_i, g_i, ν_i yields a satisfying geometric description of the motion even in this case.

We remark that the above quadrature (2.27) in SO(3) yields an interesting indication of the difficulties of non-Abelian reduction, and similar problems can in principle involve various types of groups. In our case, the calculation of the quadrature is complicated by the fact that SO(3) is a simple group.

On the other hand, in reduction with a solvable symmetry group G, the corresponding quadrature could be solved by a sequence of 1-dimensional quadratures. Correspondingly, we could work with a chain $G_0 \subset G_1 \subset \cdots \subset G_n = G$ of normal subgroups where G_i/G_{i-1} is Abelian, and do a stepwise Abelian reduction.

The case where G is Abelian, the situation is particularly simple, since we can diagonalize (2.27), and thus reduce the problem to a finite number of one-dimensional quadratures.

According to the level of difficulty of solving (2.27), we should distinguish between the following types of reduction:

(i) Non-Abelian vs Abelian reduction.

(ii) Solvable vs non-solvable reduction.

(iii) Reduction with (semi)simple group vs other groups.

The Euler equation

The main result in this section is the following:

Theorem 2.7.4 (Euler equation)**.** *Suppose that $X(t)$ is a curve in $\mathcal{M}$ with a differentiable singular value decomposition* $(U(t), R(t), Q(t))$.

If the total angular momentum Ω is constant, then

$$\widehat{\Lambda}\dot{\mathbf{g}} = -\mathbf{g} \times (\Lambda \mathbf{g}) + (\widehat{R}v) \times (\Lambda \mathbf{g}) \tag{2.28}$$

(For the notation cf. the discussion before (2.22)*).*

Conversely, for motions such that Λ is invertible, this equation implies conservation of total angular momentum.

Proof. By Corollary 2.7.2 and the definition of the objects in (2.22), we see that conservation of angular momentum Ω implies

$$\widehat{\Lambda}\dot{\mathbf{g}} = -(\Lambda\mu) \times (\Lambda \mathbf{g}).$$

Together with (2.7.3), this gives (2.28). Thus it is clear that the validity of the Euler equations follows from the conservation of angular momentum.

The matrices $\widehat{\Lambda}, \Lambda$ are both at the same time either non-singular or non-invertible. If Λ is invertible, then the Euler equation (2.28) implies that $\dot{\mathbf{g}} = -\mu \times \mathbf{g}$. This implies that the total angular momentum Ω is conserved (cf. Corollary 2.7.2). ■

On component form, the Euler equations reads

$$\begin{aligned}
\lambda_2\lambda_3\dot{g}_1 &= g_2 g_3 (\lambda_2 - \lambda_3) + r_3 r_1 v_2 \lambda_3 g_3 - r_1 r_2 v_3 \lambda_2 g_2 \\
\lambda_3\lambda_1\dot{g}_2 &= g_3 g_1 (\lambda_3 - \lambda_1) + r_1 r_2 v_3 \lambda_1 g_1 - r_2 r_3 v_1 \lambda_3 g_3 \\
\lambda_1\lambda_2\dot{g}_3 &= g_1 g_2 (\lambda_1 - \lambda_2) + r_2 r_3 v_1 \lambda_2 g_2 - r_3 r_1 v_2 \lambda_1 g_1.
\end{aligned}$$

Analytic motions

For a motion $X(t)$ with an analytic singular value decomposition $(U(t), R(t), Q(t))$, we may add the following refinement: Since the zeros of non-zero real analytic functions in one real variable are isolated, the motion $X(t)$ is collinear at isolated instances of time, unless the motion is purely collinear. Hence we have the following slight extension of Theorem 2.7.4, which follows from continuity.

Theorem 2.7.5. *For analytic many body motions that are not purely collinear, the Euler equation* (2.28)*, referring to an analytic singular value decomposition is equivalent to conservation of total angular momentum.*

Motions that are never collinear

For differentiable motions which are never collinear, we have the following result:

Theorem 2.7.6. *If $X(t)$ is a differentiable many particle motion which is never collinear, and $(U(t), R(t), Q(t))$ is a differentiable singular value decomposition, then $X(t)$ has constant angular momentum if and only if*

$$\dot{\mathbf{g}} = -(\Lambda^{-1}\mathbf{g}) \times \mathbf{g} + (\Lambda^{-1}\widehat{R}v) \times \mathbf{g} \tag{2.29}$$

Proof. Since Λ is invertible when the configuration is not collinear, Proposition 2.7.3 yields

$$\mu = \Lambda^{-1}\mathbf{g} - \Lambda^{-1}\widehat{R}\nu.$$

Using this formula, the theorem follows directly from Corollary 2.7.2. ∎

In this case, i.e. when the motions are never collinear, the Euler equation (2.29) can be written as

$$\begin{aligned}
\dot{g}_1 &= g_2 g_3\left(\frac{1}{\lambda_3} - \frac{1}{\lambda_2}\right) + \frac{r_3 r_1}{\lambda_2}\nu_2 g_3 - \frac{r_1 r_2}{\lambda_3}\nu_3 g_2 \\
\dot{g}_2 &= g_3 g_1\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_3}\right) + \frac{r_1 r_2}{\lambda_3}\nu_3 g_1 - \frac{r_2 r_3}{\lambda_1}\nu_1 g_3. \\
\dot{g}_3 &= g_1 g_2\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) + \frac{r_2 r_3}{\lambda_1}\nu_1 g_2 - \frac{r_3 r_1}{\lambda_2}\nu_2 g_1
\end{aligned}$$

2.7.3 Collinear motions and the Euler equations

As we saw above, the case of collinear motions needs some special treatment. For the sake of completeness, we include the following brief discussion:

Definition 2.7.7. A many particle *configuration* $X \in \mathcal{M}$ is *collinear* if $X\colon \mathcal{H} \to \mathbb{R}^3$ is of rank 1. A many body *motion* $X(t)$ is *collinear* if the configurations $X(t)$ are always collinear.

Using the singular value decomposition data, we could rather say that the motion is collinear if two of the gyration-radii $r_i = 0$, or equivalently, if one of the moments of inertia $\lambda_i = 0$.

Without loss of generality, we can hence assume that $\lambda_1 = r_2^2 + r_3^2 = 0$. Accordingly, $\lambda_2 = \lambda_3 = r_1^2$. In this case the particles are always contained in the line spanned by the first principal axes vector $\mathbb{u}_1$, and the Euler equations are

$$\lambda_2\lambda_3\dot{g}_1 = 0 \quad 0 = g_3 g_1 r_1^2 \quad 0 = -g_1 g_2 r_1^2$$

On the other hand by Proposition 2.7.3,

$$g_1 = \lambda_1\mu_1 + r_2 r_3 \nu_1 = 0.$$

This proves the following:

Proposition 2.7.8. *Every collinear many body motion satisfies the Euler equation* (2.28).

Hence, in this case the Euler equations are useless, and we need another characterization of collinear motions with conserved total angular momentum:

Proposition 2.7.9. *For a purely collinear continuously differentiable many body motion with non-zero total angular momentum, the total angular momentum is conserved if and only if the following conditions are satisfied:*

(i) The motion takes place in a fixed plane.

(ii) The length $\|\Omega\|$ of the angular momentum is constant.

Proof. A collinear motion $X(t)$ can be written as

$$X(t) = \begin{bmatrix} x\mathbb{q} \\ y\mathbb{q} \\ z\mathbb{q} \end{bmatrix},$$

where $x(t), y(t), z(t) \in \mathbb{R}$ and $\mathbb{q}(t) \in \mathscr{H}$. With $\mathbb{x}(t) = [x(t), y(t), z(t)]^t$ and the usual identification of $\mathfrak{so}(3)$ with $\mathbb{R}^3$, we get

$$\Omega = X \times \dot{X} = \|\mathbb{q}\|^2(\mathbb{x} \times \dot{\mathbb{x}}) \tag{2.30}$$

Hence, $\mathbb{x}$ is always perpendicular to Ω, and if Ω is constant and non-zero, $\mathbb{x}$ is confined to the fixed plane perpendicular to Ω.

On the other hand, if the motion $X(t)$ takes place in a fixed plane Π, then Ω is perpendicular to Π, and clearly, Ω is constant and non-zero if and only if the length $\|\Omega\|$ is constant and non-zero. ∎

In order to characterize collinear motion with zero angular momentum, we have to extend our terminology a little bit: By a *rectilinear* motion, we mean motion $X(t)$ in $\mathscr{M}$ such that all the particles lie on a *fixed* line, and a *total collapse* occurs when they all collide. With this terminology, we have the following result:

Proposition 2.7.10. *A collinear many body motion with zero angular momentum is rectilinear on any time interval that is free of total collapses.*

Proof. Using the terminology from the preceding proof, as well as equation (2.30), the condition $\Omega = 0$ can be rewritten as

$$\left(\sum (s_i^2)\right) \mathbb{u} \times \dot{\mathbb{u}} = 0.$$

Now, on a time interval $a < t < b$ without total collapses, $\sum (s_i^2) > 0$, and hence $\mathbb{u} \times \dot{\mathbb{u}} = 0$. This implies that $\mathbb{u} \parallel \dot{\mathbb{u}}$. But since $\|\mathbb{u}\| = 1$, this implies that $\dot{\mathbb{u}} = 0$, i.e., that $\mathbb{u}$ is constant. Thus, for $a < t < b$, the columns of X are all contained in the linear span of the constant vector $\mathbb{u}$, and hence, the motion is rectilinear on this time interval. ∎

2.8 Applications

There are several applications of Theorem 2.7.4, Theorem 2.7.5 and Theorem 2.7.6. They seem to fall in two categories:

First, we have systems where the natural description of the system is strongly linked to the principal axes frame. An important example of this is the rigid body, where the principal axes frame actually rotates along with the body. The simplest extension of this is the class of "rigid" bodies with inner angular momentum, say for instance a satellite with momentum wheel. Another example is the spinning top (cf. [Arn89]), which however needs a treatment of the external force. Using Proposition 2.7.1, this is straightforward.

There exist a slightly more general class of systems that admits strong links between the principal axes frame and the natural description, namely the class of systems that are rectangularly deformable along the principal axes with respect to the centre of mass. An example of this is the system of three point masses m_1, m_2, m_3, where $m_1 = m_3$ and m_1, m_3 are connected with m_2 by rigid links of equal length. For such a system, $\dot{Q} = 0$, and hence the classical Euler equations of the rigid body are still valid.

The *second* category of applications utilizes weaker couplings between the principal axes frame and the configurations. A striking example of this occurs for coplanar many body configurations: In this case, one of the principal axes will be orthogonal to the plane spanned by the bodies. In particular, a configuration of three bodies will always lie in a plane. Hence, we can expect that the Euler equations may be more useful in the study of the three body problem than in the n body problem for $n > 3$. For the application of the Euler equations in the three body problem, see [HS95, HS07] and also Chapter 3 in the present dissertation.

2.8.1 Planar and coplanar motions with conserved angular momentum

By *planar motion*, we understand a motion $X(t)$ in the configuration space $\mathscr{M}$ such that the image of $X\colon \mathscr{H} \to \mathbb{R}^3$ is contained in a fixed plane Π_0 through the origin of $\mathbb{R}^3$. On the other hand, under *coplanar motion*, the image $\operatorname{im} X$ is contained in a moving plane $\Pi(t)$ through the origin of $\mathbb{R}^3$.

From our ambiguous definition of the configuration space $\mathscr{M}$ (cf. the beginning of Section 2.4), the notion of *planarity* is also quite flexible, and using a coordinate system with the centre of mass fixed in the origin, the notions of *planarity* and *coplanarity* has the following meanings: A motion is planar (resp. coplanar) if the bodies at all instances of time lie in a fixed (resp. possibly varying) affine plane in $\mathbb{R}^3$.

In [Saa88], Theorem 2.8.2 below is proved under the dynamics imposed by a potential function depending only on the relative distances of the bodies. Here we prove this result as a consequence of *analyticity*. This can be regarded as a solution in a conjecture proposed by Straume in [Str01].

In order to simplify our argument, we use the following characterization of planar motion:

Lemma 2.8.1. *A many particle motion $X(t)$ is planar if and only if it has a singular value decomposition such that*

$$r_3(t) = 0 \quad \textit{and} \quad \mathbf{g}(t) = (0, 0, \|\Omega\|), \tag{2.31}$$

where r_3 is the third gyration-radius and $\mathbf{g}$ is the coordinate vector in the principal axes frame of the total angular momentum vector Ω.

Proof. Clearly, for a planar motion, the total angular momentum vector Ω is perpendicular to the fixed plane Π_0, and we can choose a principal axes frame such that (2.31) holds for all t.

Let us assume that (2.31) is satisfied in a principal axes frame for an analytic many body motion $X(t)$. By Proposition 2.7.3 and equation (2.31), the angular velocity μ of the principal axes frame satisfies

$$\Lambda\mu = \mathbf{g} - \widehat{R}\nu = \begin{bmatrix} g_1 - r_2 r_3 \nu_1 \\ g_2 - r_3 r_1 \nu_2 \\ g_3 - r_1 r_2 \nu_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \|\Omega\| - r_1 r_2 \nu_3 \end{bmatrix}.$$

The principal axes frame matrix $U(t)$ is hence determined by the differential equation

$$U^t \dot{U} = \frac{1}{r_1^2 + r_2^2} \begin{bmatrix} 0 & -(\|\Omega\| - r_1 r_2 \nu_3) & 0 \\ (\|\Omega\| - r_1 r_2 \nu_3) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and consequently $U(t)$ can be written on the form

$$U(t) = U_0 \begin{bmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & 1 \end{bmatrix},$$

where $U_0 \in \mathrm{SO}(3)$ and

$$\theta'(t) = \frac{\|\Omega\| - r_1 r_2 \nu_3}{r_1^2 + r_2^2}.$$

It follows that

$$X(t) = U_0 \begin{bmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & 1 \end{bmatrix} \begin{bmatrix} r_1(t) & & \\ & r_2(t) & \\ & & 0 \end{bmatrix} Q,$$

and hence the motion $X(t)$ is confined to the fixed plane Π_0 spanned by the first two columns of U_0.

Thus, equation (2.31) gives necessary and sufficient conditions for planar motion. ∎

Using this characterization of planar motion, we can easily prove the following result:

Theorem 2.8.2. *Consider an analytic coplanar many body motion with conserved angular momentum. Assume that the bodies span a plane perpendicular to the total angular momentum at one instance of time. Then the motion is confined to this plane.*

Proof. Let $X(t)$ be a coplanar motion. Without loss of generality we can assume that $r_3(t) = 0$ for all t. Hence the first part of (2.31) is automatically satisfied.

Now, we assume that $X(t)$ spans a plane perpendicular to Ω at $t = 0$. Under this assumption, we will show that also the second part of (2.31) is satisfied, and hence that the motion is planar.

The proof goes as follows: By induction, we prove that all the derivatives $\mathbf{g}^{(n)}$ of $\mathbf{g}$ are 0 at $t = 0$. Then the result follows from the assumption of analyticity.

Write the Euler equations on the form

$$\widehat{\Lambda}\dot{\mathbf{g}} = -\mathbf{g} \times (\Lambda \mathbf{g}) + \mathbb{w} \times (\Lambda \mathbf{g}), \tag{2.32}$$

where $\mathbb{w} = (r_2 r_3 v_1 \,, r_3 r_1 v_2 \,, r_1 r_2 v_3)$. By $r_3 = 0$, we see that $\mathbb{w} = (0, 0, r_1 r_2 v_3)$. At $t = 0$, all the vectors on the right side of (2.32) are parallel, since Ω is perpendicular to the plane spanned by the configuration at $t = 0$. Since $\widehat{\Lambda}$ is invertible when the configuration spans a plane, we conclude that $\dot{\mathbf{g}} = 0$ at $t = 0$.

As induction hypothesis, we assume that $\mathbf{g}^{(i)}(0) = 0$ for $1 \le i \le K$. By repeated derivation of (2.32) and application of the induction hypothesis, we arrive

$$\widehat{\Lambda}(0)\left(\mathbf{g}^{(K+1)}(0)\right) = \mathbb{w}^{(K)}(0) \times (\Lambda(0)\mathbf{g}(0))$$

But $\mathbb{w}^{(K)} = (0, 0, *)$ is parallel to $\Lambda(0)\mathbf{g}(0) = (0, 0, \lambda_3 \|\Omega\|)$, and hence

$$\widehat{\Lambda}(0)\mathbf{g}^{(K+1)}(0) = 0. \tag{2.33}$$

Since we assumed that the columns of $X(0)$ span a plane, $\widehat{\Lambda}(0)$ must be invertible. Thus, by equation (2.33), the derivative $\mathbf{g}^{(K+1)}(0) = 0$. By induction, we conclude that $\mathbf{g}^{(n)} = 0$ whenever $n \ge 1$. The analyticity assumption implies that $\mathbf{g}$ is constant.

This proves that $\mathbf{g}(t) = \mathbf{g}(0) = (0, 0, \|\Omega\|)$ for all t. Since $r_3(t) = 0$ for all t, Lemma 2.8.1 implies that the motion is planar, and thus confined to the plane spanned by the bodies at $t = 0$. ■

In the case that $\Omega = 0$, this implies the following result:

Corollary 2.8.3. *A coplanar analytic many body motion with zero total angular momentum is planar.*

Proof. When the total angular momentum is zero, every plane is perpendicular to the angular momentum. Hence if the configuration spans a plane at one instance of time, the rest of the motion will be confined to this plane.

If the case where the configuration never spans a plane, we rely on Proposition 2.7.9, which shows that the motion is planar also in this case. ■

2.8.2 Existence of the angular velocity

The angular velocity vector $\omega \in \mathbb{R}^3$ of a motion $X(t)$ is defined by

$$[\Omega] = X \times ([\omega] X) \tag{2.34}$$

where Ω is the total angular momentum. ω is the same as the $SO(3)$-velocity defined in Section 2.4 and the mechanical connection defined in Section 2.6.

When X is not collinear, the map $\omega \mapsto \Omega$ is invertible, and hence for such motion, we have a well defined notion of angular velocity. Here we will point out the observation that in the case of analytic motions $X(t)$, ω can be defined by means of analytic continuation.

Since purely collinear motions are planary, it is possible to define ω to be the unique solution of (2.34) where $\omega \| \Omega$.

Accordingly, the interesting part is to study analytic motions that are not purely collinear.

By (2.19) we have the relation

$$U^t \Omega = \Lambda U^t \omega,$$

and hence

$$\omega = U \Lambda U^t \Omega = U \Lambda \mathbf{g} = U(\mu + \Lambda^{-1} \hat{R} v).$$

By analyticity of the singular value decomposition, the data $U, \mu, \nu, \hat{R}$ are analytic in time. Trivially, Λ^{-1} is also analytic when all the $\lambda_i > 0$. It remains to show that $\Lambda^{-1}\hat{R}$ is analytic in the case where some of the $\lambda_i(t) = 0$ at $t = 0$. The non-zero elements of this matrix are all on the diagonal, and of the form

$$\frac{x(t)y(t)}{x(t)^2 + y(t)^2} \tag{2.35}$$

where $x(t), y(t)$ are analytic real valued functions which may satisfy $x(0) = y(0) = 0$. Now, if 0 this is a zero of $x(t)$ (resp. $y(t)$) of order p (resp. q), then $x(t)y(t)$ has a zero of order $P = pq$, while $x(t)^2 + y(t)^2$ has a zero of order $Q = \min\{p^2, q^2\}$. In any case, $P \geq Q$, and hence the expression (2.35) admits analytic continuation through 0.

This shows that $\Lambda^{-1}\hat{R}$ admits analytic continuation even through collinear configurations and accordingly

$$\omega = U(\mu + \Lambda^{-1}\hat{R}\nu)$$

admits analytic continuation through the collinearities. We summarize this as follows:

Proposition 2.8.4. *Any analytic many body motion $X(t)$ admits a choice of angular velocity $\omega(t)$ that is analytic in time.*

The angular velocity ω can be regarded as the angular velocity of the rotating *Tisserand frame*, which seems to be useful in astronomy and geophysics. In the study of the planar three body problem, some authors give this frame the name *Fujiwara coordinates* [DFPCS08]. By integration of the analytic angular velocity, we get the following result:

Corollary 2.8.5. *The Tisserand frame of an analytic many particle motion $X(t)$ can be chosen in such a way that it depends analytically on time, even at passages through collinearities.*

2.8.3 Rigid body with internal rotors.

As a final simple case study, let us consider a rigid body with one or more internal rotors. We can think of this a solid hull with some attached momentum wheels, i.e. axially symmetric rotors, so that the direction of the principal axes is not affected by the internal rotations. Furthermore, we assume that the motion of the momentum wheels relative to the hull is prescribed in advance, even though this assumption is quite unphysical.

The prescribed motion of the momentum wheels can be represented by the internal angular momentum

$$\Omega_I = [w_1, w_2, w_3]^t,$$

which is defined to be the total angular momentum of the motion with the prescribed internal rotations in the case where the principal axes frame coincides with the standard basis of $\mathbb{R}^3$. We will assume that Ω_I is fixed. For such a motion Proposition 2.7.3 yields

$$\Omega_I = \mathbf{g} = \widehat{R}v = [r_2 r_3 v_1, r_3 r_1 v_2, r_1 r_2 v_3]^t.$$

The identities $w_i = r_j r_k v_i$ holds even for a motion where the principal frame is not fixed, and the Euler equations now read

$$\begin{aligned}
\dot{g}_1 &= g_2 g_3 \left(\frac{1}{\lambda_3} - \frac{1}{\lambda_2}\right) + \frac{w_2}{\lambda_2} g_3 - \frac{w_3}{\lambda_3} g_2 \\
\dot{g}_2 &= g_3 g_1 \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_3}\right) + \frac{w_3}{\lambda_3} g_1 - \frac{w_1}{\lambda_1} g_3 \\
\dot{g}_3 &= g_1 g_2 \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) + \frac{w_1}{\lambda_1} g_2 - \frac{w_2}{\lambda_2} g_1
\end{aligned}$$

By conservation of the length of the total angular momentum, this system has the following first integral

$$(I_1) \quad g_1^2 + g_2^2 + g_3^2 = C_1.$$

We also have the following integral,

$$(I_2) \quad \frac{1}{2}\left(\frac{(g_1 - w_1)^2}{\lambda_1} + \frac{(g_2 - w_2)^2}{\lambda_2} + \frac{(g_3 - w_3)^2}{\lambda_3}\right) = C_2,$$

which is the kinetic energy of the motion modified in such a way that the internal rotation i eliminated. Note that the total kinetic energy is not conserved: In order to force the internal angular momentum to be constant, we may have to increase or decrease the total energy.

In a similar way as for the rigid body, we can describe the motion by the curves of intersection between the family (I_1) of spheres and the family (I_2) of ellipsoids, but in contrast to the case of the rigid body, the ellipsoids are now centred at (w_1, w_2, w_3).

This example indicates how the Euler equations may yield very simple calculations in cases where the motion of the principal axes frame is tightly linked to the motion of the system itself.

3 The three body problem

3.1 Introduction

The three body problem concerns the gravitational interaction between three mass points and the resulting dynamics. By P_1, P_2, P_3, we will denote the positions of three bodies in Euclidean space, and for a given origin O, the position vectors $\overrightarrow{OP_i}$ are denoted by $\mathbb{a}_i$. We let m_1, m_2, m_3 respectively denote the masses associated with mass points located at P_1, P_2, P_3. The inter-particle distances $|P_i P_j|$ are denoted by r_{ij}.

The aggregate

$$(m_1, m_2, m_3, P_1, P_2, P_3),$$

or equivalently

$$(m_1, m_2, m_3, \mathbb{a}_1, \mathbb{a}_2, \mathbb{a}_3)$$

will be called an *m-triangle*. This is a spatial triangle with positive weights associated with the vertices.

In the most important case, the particles interact according to Newton's law of gravitation:

$$m_i \ddot{\mathbb{a}}_i = \sum_{j \neq i} \frac{m_i m j}{r_{ij}^3} (\mathbb{a}_j - \mathbb{a}_i), \quad i, j = 1, 2, 3,$$

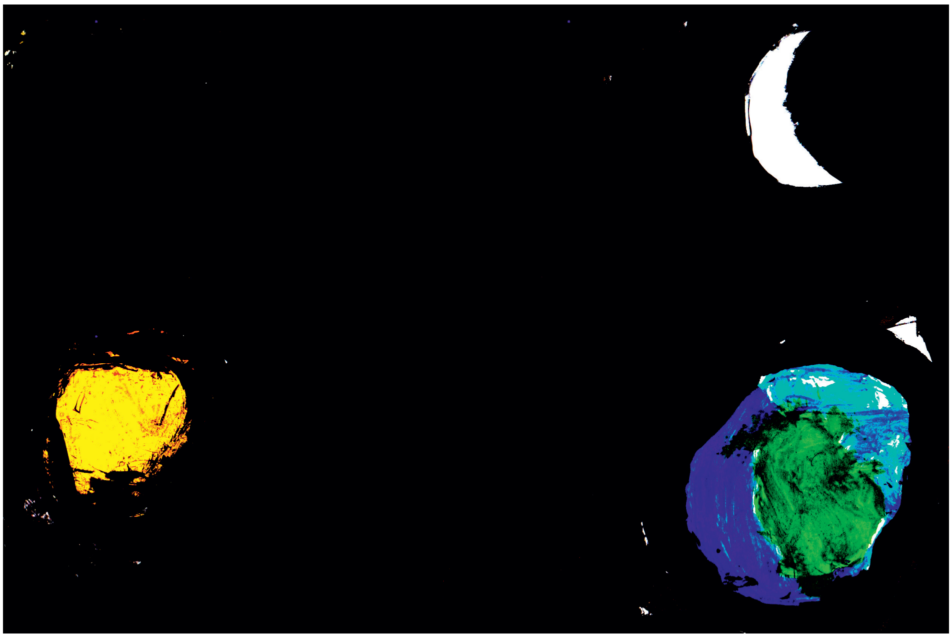

Figure 3.1: The most studied instance of the three body problem. Artist: Sindre Sydnes.

but we will also consider the more general case of a potential of power e, which gives

$$m_i \ddot{\mathbb{x}}_i = e \sum_{j \neq i} \frac{m_i m j}{r_{ij}^{e+2}} (\mathbb{x}_j - \mathbb{x}_i), \quad i, j = 1, 2, 3, \tag{3.1}$$

where the Newtonian gravitation corresponds to the case $e = 1$. The data are scaled in such a way that Newton's constant of gravitation $G = 1$.

Our aim is to provide a background for the geometric study of three body motions satisfying (3.1). Such motions are analytic in t as long as all the $r_{ij} \neq 0$. When $r_{ij} \to 0$ as $t \to t_0$, the motion suffers a collision at t_0. The maximal time-interval of existence for a motion $X(t)$ will be an open interval $(a, b) \subset \mathbb{R}$ where a, b are either infinite or instances of collision. As long as the initial configuration is not a collision configuration, the maximal interval of existence will be non-empty. Except of our discussion of the regularization of binary collision in Section 4.5, we will not mention the maximal interval of existence. Hence our results should be interpreted to be valid *on the maximal interval of existence*.

We can describe the three body problem as a Lagrange system in $(\mathbb{R}^3)^3$ with Lagrange function

$$L = \frac{1}{2}\sum_{i=1}^{3} m_i(\dot{\mathbb{a}}_i \cdot \dot{\mathbb{a}}_i) + U,$$

where $\cdot$ denotes the usual scalar product in $\mathbb{R}^3$ and the potential function U is given by

$$U = \frac{m_1 m_2}{r_{12}^e} + \frac{m_2 m_3}{r_{23}^e} + \frac{m_3 m_1}{r_{31}^e}.$$

We see that one of the main features of the potential function is that it is a homogeneous function of the position vectors of degree $-e$.

The three body problem satisfies in general the three following classical conservation laws: The integral of linear momentum takes the form

$$\sum_i m_i \dot{\mathbb{a}}_i = \text{ Constant },$$

the integral of angular momentum takes the form

$$\Omega = \sum_i \mathbb{a}_i \times \dot{\mathbb{a}}_i = \text{ Constant}, \tag{3.2}$$

and the energy integral takes the form

$$h = \frac{1}{2}\sum_i m_i \dot{\mathbb{a}}_i \cdot \dot{\mathbb{a}}_i - U(\mathbb{a}_1, \mathbb{a}_2, \mathbb{a}_3) = \text{ Constant}.$$

The study of the three body problem aims at understanding the dynamics defined by (3.1). This problem dates back to Newton's *Principia*, where Newton failed to give a good account on the lunar motion (cf. Figure 3.1). The subsequent development of the three body problem is too rich to be discussed here.

3.2 Jacobi vectors: Coordinates on the configuration space

The topic of Jacobi vectors is thoroughly investigated in Section 2.3. We will present the application of this theory on the three body problem, i.e. the case where $n = 3$. In order to make the chapter about the three body problem relatively self contained, we will repeat some of the material of Section 2.3.

We will employ coordinates with respect to an inertial system with the origin at the centre of mass of the m-triangle, i.e. *barycentric coordinates*. Equivalently, the coordinate vectors $\mathbb{a}_i$ and their velocities $\dot{\mathbb{a}}_i$ satisfy

$$\sum_i m_i \mathbb{a}_i = 0, \quad \sum_i m_i \dot{\mathbb{a}}_i = 0,$$

Hence the *configuration space* of the three body problem with mass distribution m_1, m_2, m_3 is given by

$$M = \{(\mathbb{a}_1, \mathbb{a}_2, \mathbb{a}_3) : \Sigma_i m_i \mathbb{a}_i = 0\}.$$

This implies that we use the terms *configuration space* and *barycentric configuration space* synonymously. Thus, we regard the configuration space as a subspace of the position space. The inner product on M is

$$\langle (\mathbb{a}_1, \mathbb{a}_2, \mathbb{a}_3), (\mathbb{b}_1, \mathbb{b}_2, \mathbb{b}_3) \rangle = \sum_i m_i (\mathbb{a}_i \cdot \mathbb{b}_i),$$

and the natural $SO(3)$-action is given by

$$Q(\mathbb{a}_1, \mathbb{a}_2, \mathbb{a}_3) = (Q\mathbb{a}_1, Q\mathbb{a}_2, Q\mathbb{a}_3), \quad Q \in O(3)$$

In the following we will also consider the inner product space $M_{3\times 2}$ of real 3×2-matrices with the inner product

$$\langle X, Y \rangle = \operatorname{tr}(XY^t),$$

and $SO(3)$-action given by left matrix multiplication, $(Q, X) \mapsto QX$.

3.2.1 Jacobi maps

A choice of Jacobi vectors is represented by a Jacobi map

$$J\colon (\mathbb{a}_1, \mathbb{a}_2, \mathbb{a}_3) \mapsto X = [\mathbb{x}_1 | \mathbb{x}_2] \in M_{3\times 2}$$

that restricts to an $SO(3)$-equivariant isometry $M \to M_{3\times 2}$. Under a choice of Jacobi vectors, we have the formulae

$$T = \frac{1}{2}\sum_i m_i(\dot{\mathbb{a}}_i \cdot \dot{\mathbb{a}}_i) = \frac{1}{2}(\dot{\mathbb{x}}_1 \cdot \dot{\mathbb{x}}_1 + \dot{\mathbb{x}}_2 \cdot \dot{\mathbb{x}}_2) = \frac{1}{2}\operatorname{tr}(\dot{X}\dot{X}^t)$$

and

$$\Omega = \sum_i m_i(\mathbb{a}_i \times \dot{\mathbb{a}}_i) = \mathbb{x}_1 \times \dot{\mathbb{x}}_1 + \mathbb{x}_2 \times \dot{\mathbb{x}}_2 = \overrightarrow{X \times \dot{X}},$$

where we use the identification between vectors in $\mathbb{R}^3$ and skew symmetric 3×3-matrices given in Definition 2.4.4.

According to Corollary 2.3.9, if J, J' are two different choices of Jacobi vectors, then there exists an element $Q \in O(2)$ such that

$$J'(\mathbb{a}_1, \mathbb{a}_2, \mathbb{a}_3) = J(\mathbb{a}_1, \mathbb{a}_2, \mathbb{a}_3)Q^t. \tag{3.3}$$

Conversely, if J is a valid choice of Jacobi vectors and $Q \in O(2)$, then (3.3) defines an equally valid choice of Jacobi vectors.

3.2.2 Jacobi vectors and dynamics

When we study isolated systems within the Galilean theory of relativity, the barycentric configuration space M is invariant under the dynamics, and the dynamics on M can be studied by means of the Lagrange function

$$L = T + U,$$

interpreted as a function on the barycentric configuration space.

A Jacobi map $J\colon M \to M_{3\times 2}$ is simply a reparametrization of M, and the dynamics will be determined by the corresponding Lagrange function

$$L = T(\dot{X}) + U(X),$$

where $X(t)$ denotes the Jacobi vector representation of three body motions. As noted above, the kinetic energy term is on the simple mass independent form $T = \frac{1}{2}\operatorname{tr}(\dot{X}\dot{X}^t)$. On the other hand, the form of the potential function U is rather complicated. Later we will show how we can use the flexibility that is encoded by the Jacobi groupoid to give U a more transparent form.

3.2.3 A specific choice of Jacobi vectors

The following formulae yield a family of Jacobi maps:

$$\begin{aligned} \mathbb{x}_1 &= \sqrt{\frac{m_j m_k}{m_j + m_k}}(\mathbb{a}_j - \mathbb{a}_k) \\ \mathbb{x}_2 &= \sqrt{\frac{m_i(m_j + m_k)}{m_i + m_j + m_k}}\left(\mathbb{a}_i - \frac{m_j\mathbb{a}_j + m_k\mathbb{a}_k}{m_j + m_k}\right) \end{aligned} \tag{3.4}$$

where $\{i, j, k\} = \{1, 2, 3\}$.

3.3 The singular value decomposition

From now on we will identify the configuration space of the three body problem with the space $M = M_{3\times 2}$ of real 3×2-matrices by means of a Jacobi map $J\colon M \cong M_{3\times 2}$.

Unless otherwise stated, the choice of Jacobi map is regarded as fixed. Hence, we will apply the singular value decomposition *after* the Jacobi map. Since the singular value decomposition trivializes the democracy action, it will provide a transparent representation of Jacobi transformations between different Jacobi maps.

This section will serve as a brief recapitulation of Section 2.5. We will however make some adjustments. Instead of working with the full space $SO(3) \times D_{3,\mathscr{H}} \times V_{3,\mathscr{H}}$ of singular value decompositions, we will regard singular value decompositions on the form

$$X = P \cdot R \cdot Q = \Phi(P, R, Q), \tag{3.5}$$

where $P \in SO(3)$, R is a diagonal matrix where the third diagonal element is 0, and Q is a matrix where the two first rows forms a positively oriented frame in $\mathbb{R}^2$. Hence, we focus on one path component of the previously defined space S of singular value decompositions, and we redefine S accordingly.

From the singular value decomposition, we get a positively oriented orthonormal frame $\mathbb{u}_1, \mathbb{u}_2, \mathbb{u}_3$ in $\mathbb{R}^3$ such that

$$P = [\mathbb{u}_1|\mathbb{u}_2|\mathbb{u}_3],$$

and we define variables ρ, φ, θ such that

$$R = \frac{\rho}{\sqrt{2}} \begin{bmatrix} \cos\frac{\varphi}{2} + \sin\frac{\varphi}{2} & 0 & 0 \\ 0 & \cos\frac{\varphi}{2} - \sin\frac{\varphi}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \\ 0 & 0 \end{bmatrix}$$

For notational convenience, we will occasionally use the short-hands r_1, r_2, r_3 for the diagonal elements of R. Clearly

$$\begin{aligned} r_1 &= \frac{\rho}{\sqrt{2}}\left(\cos\frac{\varphi}{2} + \sin\frac{\varphi}{2}\right) \\ r_2 &= \frac{\rho}{\sqrt{2}}\left(\cos\frac{\varphi}{2} - \sin\frac{\varphi}{2}\right) \\ r_3 &= 0 \end{aligned} \tag{3.6}$$

Consequently $\rho^2 = r_1^2 + r_2^2 + r_3^2$. The normalization of φ, θ may seem unmotivated, but we will have great advantage of this later. Note that the values of φ, θ are significant modulo 4π. Hence, we can interpret (φ, θ) as point in the torus

$$\mathbb{T}^2 = \frac{\mathbb{R}^2}{4\pi\mathbb{Z}^2}.$$

Using the singular value decomposition and the variables defined above, we will consider the following representation of the matrix of Jacobi vectors:

$$X = P \cdot R \cdot Q, \quad \text{for} \quad (P, R, Q) \in S,$$

i.e.

$$X = \frac{\rho}{\sqrt{2}} \left[\mathbb{u}_1 | \mathbb{u}_2 | \mathbb{u}_2 \right] \cdot \begin{bmatrix} \cos\frac{\varphi}{2} + \sin\frac{\varphi}{2} & 0 & 0 \\ 0 & \cos\frac{\varphi}{2} - \sin\frac{\varphi}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \\ 0 & 0 \end{bmatrix} \tag{3.7}$$

Occasionally, we will also parametrize P by Euler angles α, β, γ.

We recall the notion of regular and singular configuration defined in Chapter 2:

Definition 3.3.1 (Regular and singular configurations)**.** A configuration $X \in M$ is called *regular* if the matrix XX^t has three distinct eigenvalues, otherwise, it is called *singular*.

We let M_r (M_s) denote the space of regular (singular) configurations, $S_r = \Phi^{-1}(M_r)$ ($S_s = \Phi^{-1}(M_s)$) the space of singular value decompositions of regular (singular) configurations, and $\Phi_r \colon S_r \to M_r$ ($\Phi_s \colon S_s \to M_s$) the restriction of Φ.

Using Section 2.5, we easily deduce the following facts:

Lemma 3.3.2. *(i)* $\Phi \colon S \to M$ *is surjective.*

(ii) $\Phi_r \colon S_r \to M_r$ *is a local diffeomorphism of class* C^ω.

(iii) The fibres of $\Phi_s \colon S_s \to M_s$ *are diffeomorphic to disjoint unions of circles, with one exception, namely* $\Phi^{-1}(0)$*, which is diffeomorphic to* $SO(3) \times \mathrm{SO}(2)$.

(iv) An analytic curve $X(t)$ *in* M *can be lifted to an analytic curve* $(U(t), R(t), Q(t))$ *in* S*. If* $X(0)$ *is regular, this lifting is uniquely determined by* $U(0), R(0), Q(0)$.

Proof. (i) is proved in Lemma 2.5.4, while (ii) is proved in Lemma 2.5.7. In order to prove (iii), we simply divide S_s into three regions characterized by (1) exactly one of r_1, r_2 is 0, (2) $r_1^2 = r_2^2$ and (3) $r_1 = r_2 = r_3 = 0$. The result is then verified individually for these three cases. (iv) follows from Lemma 2.5.4. ■

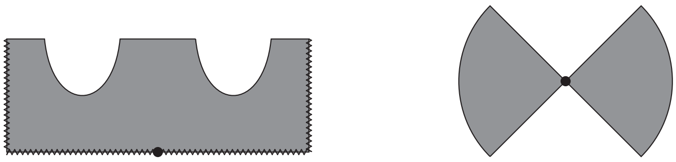

Figure 3.2: The effect of change between polar and rectangular coordinates for rectangular variations. The singularity is marked by a black dot. To the left: Splitting of variations after transfer from rectangular to polar coordinates. To the right: Pinching of variation after transfer from polar coordinates to rectangular coordinates.

3.3.1 Lifting of the dynamics from M to S

Since the laws of motion of Lagrangian systems can be expressed by means of variational principles [Arn89], it is important for us to understand to what extent it is possible to transfer variations of paths through the multiplication map $\Phi\colon S \to M$ defined in (3.5), i.e. to describe the three body dynamics on the level of the space S of singular value decompositions. The general result is disappointing: Because of the singularities of Φ, we can not globally lift the variational principle from M to S: For a motion passing through a singular value of the map Φ, it is in general impossible to lift variations through Φ; the singularities will split them up.

In order to see this, we can consider a toy model: Instead of Φ, we consider the polar coordinate mapping $\mathbb{R}^2 \to \mathbb{R}^2$ given by

$$(r,\theta) \mapsto (r\cos\theta, r\sin\theta)$$

An attempt to lift the rectangular variation $\gamma_\varepsilon(t) = (1-t,\varepsilon)$ through this map yields the situation at the left side of Figure 3.2.

Conversely, we can not push variational principles down from S to M: Variations containing singularities of Φ are pinched at the singularities after mapping through Φ. See Figure 3.2.

We conclude that a global lifting of the dynamics from M to S through Φ is not valid, since the variational principles are distorted by the singularities.

On the other hand, lifting of dynamics from M_r to S_r through Φ_r is still unproblematic: Since Φ_r is a local diffeomorphism, it has nice path lifting properties. Also lifting of variations of paths from M_r to S_r through Φ_r is unproblematic. We can say that Φ_r *admits lifting of variations of paths*: For a given variation γ_ε in M_r of the path $X(t) = \gamma_0(t)$ in M_r, and a prescribed point $s \in S_r$ with $\Phi_r(s) = X(0)$, there exist a unique variation $\tilde{\gamma}_\varepsilon$ such that $\tilde{\gamma}_0(0) = s$ and $\gamma_\varepsilon = \Phi_r \circ \tilde{\gamma}_\varepsilon$. Hence, we can lift variational calculus from M_r to S_r, and consequently also the Lagrangian dynamics.

Hence, on the regular part $S_r \to M_r$, we have an unproblematic lifting of the dynamics. In order to manage the global picture, we use the following strategy, following Lemma 3.3.2: For a three body motion $X(t)$ which is analytic in t, we can choose a singular value decomposition representation $U(t), R(t), Q(t)$ which is also analytic in t. With this device we can usually handle the singularities by analytic continuation:

For a given analytic singular value decomposition of $X(t)$, r_1, r_2 becomes analytic functions of t and since M_s is characterized by algebraic relations among r_1, r_2, we have three possible relations between $X(t)$ and M_s: (1) $X(t)$ never visits M_s, (2) $X(t)$ visits M_s at isolated instances of time, (3) $X(t)$ stays in M_s forever. The situations (1) and (2) will be handled by an understanding of situation (1), which allows us to treat situation (2) by analytic continuation. The situation (3) calls for a separate treatment.

3.3.2 Various quantities related to the singular value decomposition

Consider a three body motion given in term of its Jacobi vector matrix $X(t) = [\mathbb{x}_1(t)|\mathbb{x}_2(t)]$, and an associated singular value decomposition $(P(t), R(t), Q(t))$. As noted above, the columns $\mathbb{u}_1, \mathbb{u}_2, \mathbb{u}_3$ of P yields a diagonalization of XX^t, and hence, these vectors give an instantaneous *principal frame* for the configuration X.

In terms of multi-valued choices of gauge, we can lay this out as follows, at least over the set of regular configurations:

$$M_r \to \overline{M_r} = \frac{M_r}{\mathrm{SO}(3)} \quad \text{and} \quad S_r \to \overline{S_r} = \mathbb{R}^2 \times SO(2)$$

are principal bundles with $\Phi\colon S_r \to M_r$ as an intertwining operator. Since $\overline{S}_r \to \overline{M}_r$ is a covering map, the map

$$(R, Q) \in \overline{S}_r \mapsto RQ \in M_r$$

can be understood as a multi-valued choice of gauge.

Let r_1, r_2, r_3 denote the diagonal elements of R. The instantaneous *principal moments of inertia* of X relative to the principal frame are

$$\begin{aligned}
\lambda_1 &= r_2^2 + r_3^2 = r_2^2 = \frac{\rho^2}{2}(1 - \sin\varphi) \\
\lambda_2 &= r_3^2 + r_1^2 = r_1^2 = \frac{\rho^2}{2}(1 + \sin\varphi) \\
\lambda_3 &= r_1^2 + r_2^2 = \rho^2
\end{aligned} \tag{3.8}$$

(cf. (3.6) and (2.20))

Furthermore, we define the *total angular momentum in the principal frame*, $\mathbf{g} = P^t\Omega$, where Ω is the total angular momentum vector $\overrightarrow{X \times \dot{X}}$. The three components

$$g_i = \mathbb{u}_i \cdot \Omega, \quad i = 1, 2, 3$$

can be regarded as three functions on the tangent bundle of the configuration space. Note that g_1, g_2, g_3 depend on our choice of singular value decomposition of the motion. Modulo this choice, g_1, g_2, g_3 are $SO(3)$-invariant functions on the tangent bundle of the configuration space.

Our phase space invariants have now entered the stage: ρ represents the size, while φ, θ represents the shape of three body configurations. Finally, g_1, g_2, g_3 represent the relation between the total angular momentum vector and the spatial motion of m-triangles. Together they yield our preferred system

$$\rho, \varphi, \theta, \dot{\rho}, \dot{\varphi}, \dot{\theta}, g_1, g_2, g_3 \tag{3.9}$$

of multi-valued $SO(3)$-invariants on the phase space, for which we will deduce the equations of motion in the following sections. The fact that these variables yield a complete geometric description of three body motions follows from the reduction and reconstruction results below, i.e. from the fact that they simply *do the job.* For us, this serves as a sufficient confirmation of the completeness of this set of variables.

In the following section, we give our variables another interpretation, which yields another indication on the completeness of (3.9).

3.3.3 Vector-bundle interpretation of the variables

ρ, φ, θ can be regarded as local coordinates on the reduced configuration space $\overline{M}_r = M_r / \mathrm{SO}(3)$. Together with $\dot{\rho}, \dot{\varphi}, \dot{\theta}$ they yield a set of local bundle coordinates on the tangent bundle $T\overline{M}_r$.

Together, all the variables (3.9) yields local bundle coordinates on the reduced tangent bundle $(TM_r) / \mathrm{SO}(3)$. This bundle has the natural structure of a Lie algebroid [Wei96], and in Section 3.8 we will see that these variables are well adapted to application of Poincaré's equations, which can be regarded as an application of the Lie algebroid structure of $(TM_r)/SO(3)$.

Our variables are also adapted to the Lagrangian reduction that laid out in [CMR01]:

From the action of $SO(3)$ on M_r, we get the vertical distribution VM_r and the horizontal distribution HM_r (cf. Section 2.6). By the natural isomorphisms of $(HM_r)/\mathrm{SO}(3)$ with $T(\overline{M}_r)$, we can regard $\rho, \varphi, \theta, \dot{\rho}, \dot{\varphi}, \dot{\theta}$ as bundle coordinates on $(HM_r)/\mathrm{SO}(3)$. On the other hand, $\rho, \varphi, \theta, g_1, g_2, g_3$ can be regarded as bundle coordinates on $(VM_r)/SO(3)$, since g_1, g_2, g_3 are SO(3)-invariant and annihilate HM_r. The vector bundle $(VM_r)/\mathrm{SO}(3)$ can be identified with the adjoint bundle

$$\left(\mathfrak{so}(3), Ad_{\mathrm{SO}(3)}\right) \times_{\mathrm{SO}(3)} M_r \to \overline{M}_r,$$

an identification which is natural with respect to a chosen connection on the principal bundle $M_r \to \overline{M}_r$. Our approach implies application of the mechanical connection (cf. Section 2.6), as in [CMR01], and our choice of coordinates is well adapted to that approach, which to a large extent is formulated on the bundle

$$\left(\mathfrak{so}(3), Ad_{\mathrm{SO}(3)}\right) \times_{\mathrm{SO}(3)} M_r \oplus T\overline{M}_r,$$

which is – with respect to the chosen connection – naturally isomorphic to $(TM_r)/\mathrm{SO}(3)$.

3.3.4 Historical remark

The variables (3.9) are extensively in use in [HS07], and they were present already in [HS95]. Within that context, the emergence of the variables φ, θ seems to start with the observation that the natural kinematic geometry of the space of m-triangle shapes is isometric to a hemisphere of radius $1/2$. Accordingly, the spherical coordinates φ, θ is a choice of coordinates on the shape space which fits the kinematic geometry very well. Hence, the introduction of these variables seems to be based on the kinematic geometry.

We also find these spherical variables φ, θ in the earlier work [Lem64], where they are introduced as spherical coordinates on a shape sphere, which is used the global regularization of binary collisions (cf. Section 4.5).

3.3.5 The finite gauge group associated with the singular value decomposition

As indicated in Section 2.6.3, the choice of a singular value decomposition (P, R, Q) of a given configuration X depends on a finite set of choices; the application of a singular value decomposition implies an arbitrary choice of gauge. An important aspect of theories that depend on a choice of gauge is the understanding of their gauge symmetries. This is treated in general in Section 2.6. Here we give a detailed description of the gauge group in the case of the three body problem.

First, let us count the number of different singular value decompositions in the case of regular three body configurations. Since the principal frame matrix P is assumed to have determinant 1, and the columns of P are eigenvectors of XX^t where the third eigenvector belongs to the eigenvalue 0, we count 8 different choices of P.

When P is fixed we have the following situation: The squares r_1^2, r_2^2 of the gyration-radii are determined by the order of $\mathbb{u}_1, \mathbb{u}_2, \mathbb{u}_3$, and thus given once and for all. We can however freely choose the sign of r_1. When r_1 is given, the first row of Q is

$$\mathbb{q}_1 = \frac{1}{r_1} X^t \mathbb{u}_1.$$

Finally, the sign of r_2 is determined by the corresponding formula for the second row $\mathbb{q}_2$ of Q together with the requirement that the frame $(\mathbb{q}_1, \mathbb{q}_2)$ is positively oriented. Hence, when P is given in advance, the remaining gauge-freedom can be represented by freedom in choice of sign of r_1. This proves the following lemma:

Lemma 3.3.3. *A regular three body configuration has* $8 \times 2 = 16$ *different singular value decompositions.*

In other words: The multiplication map $\Phi_r : S_r \to M_r$ *is* $16-1$.

Now, we will represent the freedom in choice of singular value decomposition by a finite group Σ which acts on the space S of singular value decompositions. We will need several different expressions for this group action: We want to know how the group acts on the variables $\varphi, \theta, g_1, g_2, g_3$, and how it acts on the columns of P. In order to achieve this, we start by giving generators in terms of their action on the $(\mathbb{u}_1, \mathbb{u}_2, \mathbb{u}_3, r_1, r_2)$-data. Since Q is determined by X, P and R, expressions in terms of P, R-data are sufficient.

The following transformations generate the group of gauge symmetries for the singular value decomposition:

$$\begin{aligned}
\sigma_1 &: (\mathbb{u}_1, \mathbb{u}_2, \mathbb{u}_3, r_1, r_2) \mapsto (-\mathbb{u}_1, \mathbb{u}_2, -\mathbb{u}_3, -r_1, r_2) \\
\sigma_2 &: (\mathbb{u}_1, \mathbb{u}_2, \mathbb{u}_3, r_1, r_2) \mapsto (-\mathbb{u}_1, -\mathbb{u}_2, \mathbb{u}_3, -r_1, -r_2) \\
\sigma_3 &: (\mathbb{u}_1, \mathbb{u}_2, \mathbb{u}_3, r_1, r_2) \mapsto (\mathbb{u}_2, \mathbb{u}_1, -\mathbb{u}_3, r_2, -r_1) \\
\sigma_4 &: (\mathbb{u}_1, \mathbb{u}_2, \mathbb{u}_3, r_1, r_2) \mapsto (\mathbb{u}_1, \mathbb{u}_2, \mathbb{u}_3, -r_1, -r_2)
\end{aligned} \tag{3.10}$$

Strictly speaking, this set of generators is redundant, since $\{\sigma_1, \sigma_3\}$ generates the whole group. Hence, we could obviously have given a more elegant set of generators.

We translate this group action to a $(\mathbb{u}_i, \rho, \varphi, \theta)$-formulation: Since the effect on $\mathbb{u}_i$ is described above, and $\rho = (r_1^2 + r_2^2)^{1/2}$ is clearly unaffected, it is sufficient to describe this in terms of φ and θ. Using the above formulae, we arrive at the following transformation rules for φ, θ:

$$\begin{aligned}
\sigma_1(\varphi,\theta) &= (3\pi - \varphi, \theta) (\mod 4\pi) \\
\sigma_2(\varphi,\theta) &= (2\pi + \varphi, \theta) (\mod 4\pi) \\
\sigma_3(\varphi,\theta) &= (\pi + \varphi, \pi + \theta) (\mod 4\pi) \\
\sigma_4(\varphi,\theta) &= (2\pi + \varphi, 2\pi + \theta) (\mod 4\pi).
\end{aligned} \tag{3.10'}$$

For the components g_1, g_2, g_3 of the total angular momentum in the principal frame, the gauge transformations act in the following way:

$$\begin{matrix}
\sigma_1(g_1,g_2,g_3) = (-g_1,g_2,-g_3) & \sigma_2(g_1,g_2,g_3) = (-g_1,-g_2,g_3) \\
\sigma_3(g_1,g_2,g_3) = (g_2,g_1,-g_3) & \sigma_4(g_1,g_2,g_3) = (g_1,g_2,g_3)
\end{matrix}. \tag{3.11}$$

When it comes to the λ_i, the only non-trivial transformation is σ_3, which satisfies

$$\sigma_3(\lambda_1,\lambda_2) = (\lambda_2,\lambda_1). \tag{3.12}$$

The fundamental result concerning the group Σ is the following:

Lemma 3.3.4. *For a regular three body configuration X, the group Σ generated by $\sigma_1,\sigma_2,\sigma_3,\sigma_4$ acts freely and transitively on the set of admitted singular value decompositions.*

This is proved rigorously by exhibition of the 16 elements of the group generated by the σ_i, for instance by using a representation by 5×5-matrices suggested by the definitions of the σ_i in terms of $(\mathbb{u}_1,\mathbb{u}_2,\mathbb{u}_3,r_1,r_2)$. This way we see (i) that the action is free and (ii) that the number of elements of the group matches the number of possible gauges.

Finally, we make the following remarks:

(i) Over the set of singular configurations, the action of Σ is neither free nor transitive.

(ii) Σ is isomorphic to $D_4\times\mathbb{Z}_2$, where D_4 is the symmetry group of the square.

3.4 The potential function

The Euclidean geometric invariants of m-triangles can be expressed by means of the basic invariants ρ, φ, θ. Another set of basic invariants is formed by the relative distances r_{ij}. The potential function

$$U = \sum_{i<j} \frac{m_i m_j}{r_{ij}^e},$$

is yet another geometrically invariant quantity.

U is most easily expressed as above, namely in terms of the relative distances r_{ij}. It seems to be difficult to find a transparent formula for U in terms of the Jacobi vectors $\mathbb{x}_1, \mathbb{x}_2$. On the other hand, we can deduce a quite simple formula for U in terms of ρ, φ, θ. As a first step, we find formulae for the relative distances r_{ij} in terms of ρ, φ, θ.

Since U is $SO(3)$-invariant, we may consider a configuration X for which the principal frame matrix $P = I_{3\times 3}$. Hence, we can regard a configuration with Jacobi vector representation of the form

$$X = [\mathbb{x}_1 | \mathbb{x}_2] = \frac{\rho}{\sqrt{2}} \begin{bmatrix} \cos\frac{\varphi}{2} + \sin\frac{\varphi}{2} & 0 & 0 \\ 0 & \cos\frac{\varphi}{2} - \sin\frac{\varphi}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \\ 0 & 0 \end{bmatrix}.$$

Accordingly

$$\mathbb{x}_1 = \frac{\rho}{\sqrt{2}} \begin{bmatrix} (\cos\frac{\varphi}{2} + \sin\frac{\varphi}{2})\cos\frac{\theta}{2} \\ (\sin\frac{\varphi}{2} - \cos\frac{\varphi}{2})\sin\frac{\theta}{2} \\ 0 \end{bmatrix}$$

and hence

$$||\mathbb{x}_1||^2 = \frac{\rho^2}{2}(1 + \sin\varphi\cos\theta). \tag{3.13}$$

Jacobi transformations $J^1 \to J^2$ between different Jacobi maps for the three body problem with the fixed mass distributions corresponds to right matrix multiplications

$$X \mapsto XQ$$

for $Q \in O(2)$. Since the action of $SO(2)$ gives a rotation of the Jacobi vectors $\mathbb{x}_1, \mathbb{x}_2$ in the plane, and the action of the Jacobi transformation given by the matrix

$$\begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$

amounts to the transformation $\theta \mapsto \theta + 2\phi$ of the shape variable θ, there exist constants θ_i such that

$$\mathbb{x}_1(\theta - \theta_i) = \sqrt{\frac{m_j m_k}{m_j + m_k}}(\mathbb{a}_j - \mathbb{a}_k), \tag{3.14}$$

where $\mathbb{a}_j, \mathbb{a}_k$ are position vectors (cf. (2.9)). Together with (3.13), this observation yields the following lemma:

Lemma 3.4.1. *For a given choice of Jacobi vectors and singular value decomposition for the three body problem, there exist constants $\theta_1, \theta_2, \theta_3$ such that*

$$r_{jk} = \rho\sqrt{\frac{m_j + m_k}{2m_j m_k}}\sqrt{1 + \sin(\varphi)\cos(\theta - \theta_i)}, \quad \textit{for} \quad \{i, j, k\} = \{1, 2, 3\}$$

This proves the following proposition about the potential function:

Proposition 3.4.2. *For a given choice of Jacobi vectors and singular value decomposition for the three body problem, there exist constants $\theta_1, \theta_2, \theta_3$ such that the potential function U has the following expression:*

$$U = \frac{1}{\rho^e}\sum_{i=1}^{3}\frac{\mu_i}{\left(1 + \sin(\varphi)\cos(\theta - \theta_i)\right)^{\frac{e}{2}}}$$

where

$$\mu_i = m_j m_k \left(\frac{2m_j m_k}{m_j + m_k}\right)^{\frac{e}{2}} \quad \textit{for} \quad \{i, j, k\} = \{1, 2, 3\}.$$

We find similar expressions of the potential function both in [HS07] and [Lem64].

Definition 3.4.3. We define the *shape potential* U^* by

$$U^* = \sum_{i=1}^{3} \frac{\mu_i}{\left(1 + \sin(\varphi)\cos(\theta - \theta_i)\right)^{\frac{e}{2}}}$$

In terms of the shape potential U^*, we will write the potential function as $U = U^* / \rho^e$.

The constants $\theta_1, \theta_2, \theta_3$ depend on the choice of Jacobi vectors, but the set of relative angles $|\theta_1 - \theta_2|, |\theta_2 - \theta_3|, |\theta_3 - \theta_1|$ is uniquely determined by the mass distribution m; a transition between different Jacobi maps $J, J' \in \mathscr{J}_m$ will translate the values θ_i with a common phase shift. Later, we will see that the converse is almost true: Proposition 4.3.7 indirectly shows that the similarity class of the mass distribution is uniquely determined by $\theta_1, \theta_2, \theta_3$.

Lemma 3.4.1 tells us that the binary collision of particle j with particle k occurs for

$$\varphi = 2n\pi + \frac{\pi}{2} \quad \text{and} \quad \theta = 2n\pi + (\theta_i + \pi) \tag{3.15}$$

and

$$\varphi = 2n\pi - \frac{\pi}{2} \quad \text{and} \quad \theta = 2n\pi + \theta_i, \tag{3.16}$$

where $n \in \mathbb{Z}$ and $\{i, j, k\} = \{1, 2, 3\}$. In this way θ_1, θ_2 and θ_3 determine the φ, θ-coordinates that correspond to binary collision configurations.

Finally, we will make a remark on the differentiability of the shape potential U^*. From the formula in Definition 3.4.3 we see that U^* is a smooth function of the variables φ, θ *except* at the points where $\sin\varphi\cos(\theta - \theta_i) = -1$, i.e. precisely at the collision points. Later, we will interpret φ, θ as spherical polar coordinates. Under this interpretation, we can regard U^* as a smooth function on the sphere:

3.4.1 Spherical representation of the shape potential

Let us regard φ, θ as spherical polar coordinates on the unit sphere $S^2 \subset \mathbb{R}^3$. This interpretation will be further justified in Section 4.1. We do this in order to treat the vector-algebraic presentation of the shape potential found in [HS07].

In such a representation, the binary collisions (3.15) and (3.16) are mapped to precisely three points $B_1, B_2, B_3 \in S^2 \subset \mathbb{R}^3$. For a given point $P = P(\varphi, \theta)$ on the sphere S^2, we apply the spherical cosine law to the spherical distance γ_i to the binary collision B_i. This yields

$$\cos\gamma_i = \cos\left(\frac{\pi}{2}-\varphi\right)\cos(\theta-(\pi+\theta_i)) = -\sin(\varphi)\cos(\theta-\theta_i)$$

Accordingly, the Euclidean distance $|\overline{PB_i}|$ between P and B_i satisfies

$$|\overline{PB_i}|^2 = \|P\|^2 + \|B_i\|^2 - 2P\cdot B_i = 2-2\cos\gamma_i = 2(1+\sin\varphi\cos(\theta-\theta_i)),$$

since $P\cdot B_i = \cos\gamma$. Following Lemma 3.4.1, we thus get the following relation between the relative distances r_{jk} and the Euclidean distances $|\overline{PB_i}|$ between points on the sphere S^2:

$$r_{jk} = \frac{1}{2}\rho\sqrt{\frac{m_j+m_k}{m_jm_k}}|\overline{PB_i}| \tag{3.17}$$

Accordingly, the shape potential can be written as

$$U^* = 2^{e/2}\sum_{i=1}^{3}\frac{\mu_i}{|\overline{PB_i}|^e}, \tag{3.18}$$

where the μ_i are given in Proposition 3.4.2.

This formula shows that when we regard φ,θ as spherical polar coordinates on the sphere $S^2\subset\mathbb{R}^3$, U can be regarded as a function on S^2 which is analytic everywhere except at the binary collision points B_1, B_2, B_3.

An asymptotic formula

Recall that we defined γ_i to be the spherical distance between a point P on the sphere and the binary collision point B_i. By an analysis of the behaviour of the shape potential near the binary collision points B_i, we have

$$U^* = \frac{1}{\gamma_i^e}F^i(\gamma_i) + G^i(\gamma_j,\gamma_k), \tag{3.19}$$

where F, G are strictly positive functions which are analytic near B_i. Hence, asymptotically, at the binary collision points,

$$U^* \sim \frac{F_0^i}{\gamma_i^e}, \tag{3.20}$$

for a constant F_0^i. Near the binary collision point B_i the shape potential thus resembles the potential of the planar Kepler problem. This observation forms the basis for our regularization of binary collisions in the three body problem in Section 4.5. The idea is that we can regularize each binary collision in the same way as we would do with a planar Kepler problem, and then patch together these regularizations to a global regularization.

3.4.2 The critical points of the shape potential

The potential function can be regarded as a function $U(\rho, \varphi, \theta)$, i.e. a function on the ρ, φ, θ-space. The shape potential U^* can be regarded as the restriction of the potential function U to the surface $\rho = 1$.

In the following analysis of the shape potential, we will use the relative distance formulation of the potential function. In terms of the relative distances r_{23}, r_{31}, r_{12} and the masses m_1, m_2, m_3, the potential function is

$$U(r_{23}, r_{31}, r_{12}) = \frac{m_1 m_2}{r_{12}^e} + \frac{m_2 m_3}{r_{23}^e} \frac{m_3 m_1}{r_{31}^e},$$

and the polar moment of inertia with respect to the centre of mass $I = \rho^2$ satisfies Lagrange's formula [Lag72]

$$(m_1 + m_2 + m_3) I = m_1 m_2 r_{12}^2 + m_2 m_3 r_{23}^2 + m_3 m_1 r_{31}^2. \tag{3.21}$$

We have the following geometric restrictions on the relative distances:

$$r_{ij} + r_{jk} \geq r_{ki} \geq 0 \quad \text{for} \quad \{i, j, k\} = \{1, 2, 3\} \tag{3.22}$$

Together with the restriction $I = \rho^2 = 1$, equation (3.22) singles out a triangle shaped ellipsoidal region Δ with vertices given by

$$r_{ij} = 0,$$

and edges given by

$$r_{ij} + r_{jk} = r_{ki}.$$

The restriction of U to this region is clearly equivalent to the shape potential.

Lagrange's multiplicator method on the interior of Δ gives the equation

$$\nabla U = \lambda(m_1 + m_2 + m_3)\nabla I \quad i.e. \quad \frac{-e}{r_{jk}^{e+1}} = \lambda r_{jk},$$

and accordingly, the only critical point of U in the interior of Δ is the equilateral triangle, which is given by

$$r_{23} = r_{31} = r_{12}.$$

If we turn to the interior of the edges of Δ, which are given by $r_{ij} = r_{jk} + r_{ki}$, Lagrange's multiplicator method leads to a rather complicated discussion. Here, we find it useful to apply the shape potential $U^*(\varphi, \theta)$. Since each point of the triangle Δ represents one triangle shape, there is a mapping $\mathbb{R}^2 \to \Delta$ which sends each (φ, θ) to the corresponding point in Δ, and since U^* and $U|_\Delta$ both are defined given by restriction to $\rho = 1$, we can regard U^* as the pullback of $U|_\Delta$ along this mapping. Finally, since the edges of Δ corresponds to $\varphi = \frac{\pi}{2} \mod \pi$, we can parametrize the edges by the variable θ. Hence, it is useful to investigate the following function of θ:

$$U^* = \sum_i \frac{\mu_i}{(1 \pm \cos(\theta - \theta_i))^{\frac{e}{2}}},$$

The interior of the edge $r_{23} = r_{31} + r_{12}$ is parametrized over the interval (θ_2, θ_3), if we assume that $0 \leq \theta_1 < \theta_2 < \theta_3 < 2\pi$. The terms

$$U_i^*(\theta) = \frac{\mu_i}{(1 \pm \cos(\theta - \theta_i))^{\frac{e}{2}}}$$

are strictly convex functions of θ on the intervals $(\theta_i + 2k\pi, \theta_i + 2(k+1)\pi)$. Accordingly $U^* = U_1^* + U_2^* + U_3^*$ is a strictly convex function on the interval (θ_2, θ_3), and since $U^* \to \infty$ at the endpoint of this interval, we conclude that U^* has six critical points on the boundary of Δ, namely 3 poles at the binary collision points B_1, B_2, B_3 and 3 relative minima E_1, E_2, E_3. The relative minima are saddle points of U.

U is clearly singular at the vertices of Δ, which are given by $r_{ij} = 0$, i.e. at the points that correspond to binary collisions.

We summarize this discussion as follows:

Lemma 3.4.4. *For $\varphi \neq 0 \mod \pi$, $U*$ is critical at (φ, θ)-values corresponding to the points $L, E_1, E_2, E_3, B_1, B_2, B_3 \in \Delta$ above. L corresponds to the equilateral configuration. B_i corresponds to binary collision configurations. E_i corresponds to certain collinear configurations.*

Note that the each of the points L, E_i, B_i are represented by several values of (φ, θ).

3.5 Geometric invariants of triangles

Along the lines of the previous section, we will here find φ,θ-expressions of some geometric invariants of triangles. Later, we will need some information about the triangle shape given by $\varphi = 0 \mod \pi$. Therefore we give explicit formulae for this case here. In particular, we need to know under what circumstances $\varphi = 0 \mod \pi$ corresponds to the equilateral shape.

Following Lemma 3.4.1, the relative distances r_{kj} satisfies

$$r_{jk} = \frac{\rho}{\sqrt{2}} \sqrt{\frac{m_j + m_k}{m_j m_k}} \sqrt{1 + \sin\varphi \cos(\theta - \theta_i)}$$

and for $\varphi = 0 \mod \pi$ we conclude as follows:

Lemma 3.5.1. *Consider the three body problem with mass distribution m_1, m_2, m_3, and shape variables φ, θ as defined above.*

In a configuration with $\varphi = 0 \mod \pi$, the relative distances satisfies

$$r_{jk} = \frac{\rho}{\sqrt{2}} \sqrt{\frac{m_j + m_k}{m_j m_k}}$$

Hence $\varphi = 0 \mod \pi$ represents the equilateral triangle if and only if $m_1 = m_2 = m_3$.

Along these lines, we can also compute the lengths of the position vectors $\mathbb{a}_i$ relative to the centre of mass. Following (3.4), the other Jacobi vector corresponding to (3.14) is

$$\mathbb{x}_2(\theta-\theta_i) = \sqrt{\frac{m_i(m_j+m_k)}{m_i+m_j+m_k}}\left(\mathbb{a}_i - \frac{m_j\mathbb{a}_j + m_k\mathbb{a}_k}{m_j+m_k}\right).$$

Since $\Sigma_i \mathbf{m}_i \mathbb{a}_i = 0$, we can write this as

$$\begin{aligned}\mathbb{x}_2(\theta-\theta_i) &= \sqrt{\frac{m_i(m_j+m_k)}{m_i+m_j+m_k}}\left(\frac{m_i+m_j+m_k}{m_j+m_k}\mathbb{a}_i - \frac{m_i\mathbb{a}_i + m_j\mathbb{a}_j + m_k\mathbb{a}_k}{m_i+m_j+m_k}\right)\\ &= \sqrt{\frac{m_i(m_i+m_j+m_k)}{(m_j+m_k)}}\mathbb{a}_i.\end{aligned}$$

In analogy with (3.13), we have

$$\|\mathbb{x}_2\|^2 = \frac{\rho^2}{2}(1-\sin\varphi\cos(\theta-\theta_i)),$$

and hence if we denote by r_i the length $\|\mathbb{a}_i\|$ of $\mathbb{a}_i$, then we have

$$r_i = \frac{\rho}{\sqrt{2}}\sqrt{\frac{m_j+m_k}{m_i(m_i+m_j+m_k)}}\sqrt{1-\sin\varphi\cos(\theta-\theta_i)}.$$

By specializing to the case $\varphi = 0 \mod \pi$, we have some nice observations:

The orthocentre of the triangle ABC is defined as follows: Let l_A be the line through A which is perpendicular to the line BC. The three lines l_A, l_B, l_C intersects at a point which is called the *orthocentre*. It is straightforward to check that $\mathbb{x}_1 \perp \mathbb{x}_2$ if and only if $\sin(\theta-\theta_i)\sin\varphi = 0$. Hence $(\mathbb{a}_j - \mathbb{a}_k) \perp \mathbb{a}_i$ for all $\{i,j,k\} = \{1,2,3\}$ if and only if $\varphi = 0 \mod \pi$. Hence the configurations where $\varphi = 0 \mod \pi$ have the characterizing property that the orthocentre and the centre of mass are coincident. This observation is present in [Lem64], based on a very different geometric argument.

Applied to the case where $\varphi = 0 \mod \pi$, we get:

Lemma 3.5.2. *For a three body configuration with $\varphi = 0 \mod \pi$, the distance r_i between particle i and the centre of mass satisfies*

$$r_i = \frac{\rho}{\sqrt{2}} \sqrt{\frac{m_j + m_k}{m_i(m_i + m_j + m_k)}}. \tag{3.23}$$

We have similar formulae for the central angles in the m-triangle: Let β_i be the angle between $\mathbb{a}_j$ and $\mathbb{a}_k$. From the cosine law we get

$$\cos\beta_i = -\frac{r_{jk}^2 - r_j^2 - r_k^2}{2r_j r_k} = -\sqrt{\frac{m_j m_k}{(m_i + m_j)(m_i + m_k)}} \tag{3.24}$$

Hence, β_i lies in the open interval $(\pi/2, \pi)$. Similarly, for the angles δ_i between $\mathbb{a}_k - \mathbb{a}_i$ and $\mathbb{a}_j - \mathbb{a}_i$, we get

$$\cos\delta_i = \frac{r_{ij}^2 + r_{ik}^2 - r_{jk}^2}{2r_{ij} r_{ik}} = \sqrt{\frac{m_j m_k}{(m_i + m_j)(m_i + m_k)}}. \tag{3.25}$$

This last formula can also be verified in the following way: Since the orthocentre and the centre of mass are coincident when $\varphi = 0 \mod \pi$, we can easily check that $\beta_i = \pi - \delta_i$, and hence $\cos\delta_i = -\cos\beta_i$.

3.6 Regular and singular configurations

Above, we defined a configuration X to be regular if the three eigenvalues r_1^2, r_2^2, r_3^2 of XX^t are distinct. In terms of the variables ρ, ϕ, θ defined in Section 3.3, a configuration X is regular if and only if $\varphi \neq 0 \mod \frac{\pi}{2}$. This follows from the relation (3.6) between r_1, r_2, r_3 and ρ, φ.

We denoted by M_r the set of regular configurations. Define M_s to be the set of singular configurations. $X \in M$ can be singular in three different ways

Umbilic shape: This is the case where $r_1^2 = r_2^2$, i.e. $\varphi = 0 \mod \pi$.

Collinear shape: This is the case where either $r_1 = 0$ or $r_2 = 0$, i.e. $\varphi = \frac{\pi}{2} \mod \pi$.

Triple collision shape: This is the case where $r_1 = r_2 = 0$.

We will not investigate triple collisions in this thesis, and hence this shape will for the most be left out of the discussion. From our point of view, three body motions cease to exist as they approach the triple collision.

The different types of singularity can be characterized by their isotropy types in the SO(3) × SO(2)-action on the configuration space. The principal isotropy type is represented by the group of elements of the type

$$\left(\begin{bmatrix} \pm 1 & & \\ & \pm 1 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} \pm 1 & \\ & \pm 1 \end{bmatrix}\right) \in SO(3) \times SO(2)$$

The triple collision is the unique point where the action of SO(3) × SO(2) on the configuration space has isotropy group SO(3) × SO(2).

Collinear configurations can be characterized as the points of isotropy type $O(2)$, where $O(2)$ is identified with the group of elements of the type

$$\left(\begin{bmatrix} \pm 1 & & \\ & \pm\cos\vartheta & \mp\sin\vartheta \\ & \sin\vartheta & \cos\vartheta \end{bmatrix}, \begin{bmatrix} \pm 1 & \\ & \pm 1 \end{bmatrix}\right) \in SO(3) \times SO(2)$$

The isotropy groups associated with the umbilic shape are conjugate to the subgroup of elements of the form

$$\left(\begin{bmatrix} \pm\cos\vartheta & \mp\sin\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}, \begin{bmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{bmatrix}\right) \in SO(3) \times SO(2)$$

In any case, the different types of singularity corresponds to different isotropy types in the SO(3) × SO(2)-action.

We observe that except of the umbilic shape, the singularities are mainly related to the rotational action of $SO(3)$. Hence, the umbilic shape singularity can be regarded as an artefact of our formalism. On the other hand, the umbilic shape singularity is related to the democracy representation, and hence from this point of view a natural part of the $SO(3)$-equivariant kinematic geometry.

3.7 The reduced dynamical equations

In this section we will express the equations of motion of the three body problem in terms of the variables $\rho, \varphi, \theta, g_1, g_2, g_3$ and their derivatives.

Our method is very straightforward: Since $\rho, \varphi, \theta, \dot{\rho}, \dot{\varphi}, \dot{\theta}, g_1, g_2, g_3$ can be taken as basic $SO(3)$-invariants on the tangent bundle TM_r of the space M_r of regular three body configurations, the equations of motion for the three body problem should be expressible in terms of these variables and their derivatives. In this section this will be provided by means of elementary algebraic manipulations. Later, in Section 3.8, we will demonstrate a more differential-geometric approach, using the method that was described by Poincaré in [Poi01].

Following the discussion of regular and singular configurations, we are forced to distinguish between the following cases:

Regular motion: Motions confined to the space M_r of regular configurations, i.e. motions where the squares r_1^2, r_2^2 of the gyration-radii are non-zero and distinct.

Collinear motion: These are motions where $r_1 r_2 = 0$, i.e. $\varphi = \frac{\pi}{2} \mod \pi$. Following Section 2.7.3, we know that collinear motions with conserved total angular momentum are planar – for purely kinematic reasons. This allows us to embed the investigation of this case into an investigation of the general case of planar motion.

Umbilic shape invariant motion: These are motions where $r_1^2 = r_2^2$, i.e. $\varphi = 0 \mod \pi$. This very special case is treated in Section 3.9. For $e \neq 2$ such motions occur only in the case of three equal masses, in which case the umbilic shape is equilateral, and the motion is Lagrange's equilateral motion [Lag72]. In the case $e = 2$, Section 3.9 opens for the existence of another class of umbilic shape invariant motions.

In the following, we will first discuss the planar case, and then the general case. The techniques applied to the two cases are essentially equivalent, and the planar case can be deduced directly from the general case by imposing some restrictions on the variables. We find it however useful to exhibit both of the calculations. In this way we can show the essential features of the calculations in the simple case of planar motion, and hide away the details in the more complicated case of general motion, which will anyway be treated in detail in Section 3.8. Again, we emphasise that our treatment of the planar case is essential to our understanding of collinear configurations.

In Section 3.9, we will investigate the case of umbilic shape invariant motions, while in Chapter 4 we will see that the differential geometry of three body shapes allows for an elegant and complete presentation of the reduction of the planar three body problem to the evolution of shape and size. Hence, our conclusive account on our geometric reduction is postponed to Section 4.4.8.

3.7.1 The planar case

Without loss of generality, we assume that the motion takes place in the xy-plane. By neglecting the z-component, we get the following singular value decomposition of three body configurations:

$$X = \frac{\rho}{\sqrt{2}} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\frac{\varphi}{2} + \sin\frac{\varphi}{2} & 0 \\ 0 & \cos\frac{\varphi}{2} - \sin\frac{\varphi}{2} \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \tag{3.26}$$

Let us denote by M_2 the space of xy-planar configurations and

$$S_2 = \mathrm{SO}(2) \times \mathbb{R}^2 \times \mathrm{SO}(2)$$

the corresponding "z-neglected" space of singular value decompositions. This restriction of the set of singular value decompositions is determined by the choice of the normal vector $\mathbb{u}_3$, which is taken to be the unit vector in positive z-direction.

The multiplication map $S_2 \to M_2$ is singular only for $\varphi = 0 \mod \pi$. Hence, we can take $\alpha, \rho, \varphi, \theta$ as proper coordinates of the planar three body problem, valid for $\varphi \neq 0 \mod \pi$. Accordingly, the umbilic shape singularity of the general problem is the only singularity that survives to the planar case.

In this coordinate system, the kinetic energy satisfies

$$T = \frac{1}{2}\mathrm{tr}(\dot{X}\dot{X}^t) = \frac{1}{2}\dot{\rho}^2 + \frac{\rho^2}{8}\left(\dot{\varphi}^2 + \dot{\theta}^2\right) + \frac{\rho^2}{2}\dot{\alpha}^2 - \frac{\rho^2}{2}\cos\varphi\dot{\alpha}\dot{\theta},$$

and total angular momentum is represented by the scalar α-momentum (in the Lagrangian sense),

$$\Omega_s = \frac{\partial T}{\partial \dot{\alpha}} = \rho^2(\dot{\alpha} - \frac{1}{2}\cos\varphi\dot{\theta}).$$

The total angular momentum is the vector $\Omega = \Omega_s \mathbb{k}$, which can also be calculated by direct application of the formula $[\Omega] = X \times \dot{X}$ after an embedding of the x, y-plane into three dimensional space.

Using the formula for Ω_s, we can express $\dot{\alpha}$ by the other variables as

$$\dot{\alpha} = \frac{\Omega_S}{\rho^2} + \frac{1}{2}\cos\varphi\dot{\theta}. \tag{3.27}$$

The equations of motion of the three body problem are equivalent to the Euler-Lagrange equations associated with the Lagrange function

$$L = T + \frac{U^*(\varphi,\theta)}{\rho^e}.$$

Direct computation yields

$$\begin{aligned}
\dot{\Omega}_s &= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\alpha}}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\alpha}}\right) = \frac{\partial L}{\partial \alpha} = 0,\\
\ddot{\rho} &= \rho\left(\frac{\dot{\varphi}^2 + \dot{\theta}^2}{4} + \dot{\alpha}^2 - \cos\varphi\dot{\theta}\dot{\alpha}\right) - \frac{eU^*}{\rho^{e+1}},\\
\frac{\rho^2\ddot{\varphi}}{4} &= -\frac{\rho\dot{\rho}}{2}\dot{\varphi} + \frac{\rho^2}{2}\sin\varphi\dot{\theta}\dot{\varphi} + \frac{U^*_\varphi}{\rho^e},\\
\frac{\rho^2\ddot{\theta}}{4} &= \frac{\rho^2}{2}\cos\varphi\ddot{\alpha} + \rho\dot{\rho}\cos\varphi\dot{\alpha} - \frac{\rho^2}{2}\sin\varphi\dot{\alpha}\dot{\varphi} - \frac{\rho\dot{\rho}}{2}\dot{\theta} + \frac{U^*_\theta}{\rho^e}.
\end{aligned}$$

After elimination of $\dot{\alpha}$ and $\ddot{\alpha}$ by (3.27), the Euler-Lagrange equations assumes the form

$$
\begin{aligned}
\dot{\Omega}_s &= 0 \\
\ddot{\rho} &= \frac{\Omega_s^2}{\rho^3} + \frac{\rho}{4}\left(\dot{\varphi}^2 + \sin^2\varphi\dot{\theta}\right) - \frac{eU^*}{\rho^{e+1}} \\
\ddot{\varphi} &= \cos\varphi\sin\varphi\dot{\theta}^2 - 2\frac{\dot{\rho}}{\rho}\dot{\varphi} + 2\frac{\Omega_s\sin\varphi}{\rho^2}\dot{\theta} + 4\frac{U_\varphi^*}{\rho^{2+e}} \\
\ddot{\theta} &= -2\cot\varphi\dot{\theta}\dot{\varphi} - 2\frac{\dot{\rho}}{\rho}\dot{\theta} - 2\frac{\Omega_s}{\rho^2\sin\varphi}\dot{\varphi} + 4\frac{U_\theta^*}{\rho^{2+e}\sin^2\varphi}
\end{aligned}
\tag{3.28}
$$

These equations determines the evolution of ρ, φ, θ for three body motions avoiding the umbilic shape. Ω_s can be regarded as a constant parameter in these equations. In Section 4.4.7, we show that these equations have a nice differential geometric presentation which removes the umbilic singularity.

If we know $\Omega_s, \rho(t), \varphi(t), \theta(t)$, we can determine $\alpha(t)$ by quadrature from (3.27), and the evolution of the Jacobi vector matrix $X(t)$ is then determined by equation (3.26) together with an initial value for α.

3.7.2 Regular motions

Here we will investigate motions in the space M_r of regular three body configurations. It is straightforward to study the dynamics on the level of singular value decompositions, since $\Phi_r: S_r \to M_r$ is a local diffeomorphism.

We will make use of a full coordinatization of S_r, and for that reason we will use Euler angles in $SO(3)$ as auxiliary variables. This implies that we introduce some new singularities, namely the gimbal lock singularities of the Euler angles. Since our problem is $SO(3)$-invariant, these singularities can be moved freely around and away from the motion under consideration. Accordingly, our particular choice of Euler angle gauge has no effect on the final result.

In the present calculations, we use Euler angles α, β, γ in the z-x-z-gauge. Accordingly, the principal axes matrix $P = [\mathbb{u}_1|\mathbb{u}_2|\mathbb{u}_3]$ equals

$$
\begin{bmatrix}
\cos\alpha\cos\gamma - \cos\beta\sin\alpha\sin\gamma & -\cos\gamma\sin\alpha - \cos\alpha\cos\beta\sin\gamma & \sin\beta\sin\gamma \\
\cos\alpha\sin\gamma + \cos\beta\cos\gamma\sin\alpha & -\sin\alpha\sin\gamma + \cos\alpha\cos\beta\cos\gamma & -\cos\gamma\sin\beta \\
\sin\alpha\sin\beta & \cos\alpha\sin\beta & \cos\beta
\end{bmatrix}
$$

Combined with the variables ρ, φ, θ, this gives a set $(\alpha, \beta, \gamma, \rho, \varphi, \theta)$ of coordinates on the set M_r of regular three body configurations.

By direct computation, we find that the kinetic energy T is given by

$$\begin{aligned}
T = &\frac{1}{2}\dot{\rho}^2 + \frac{\rho^2}{8}(\dot{\varphi}^2 + \dot{\theta}^2) \\
&+ \frac{\rho^2}{2}\dot{\alpha}^2 + \frac{\rho^2}{4}(1 - \cos 2\alpha \sin\varphi)\dot{\beta}^2 \\
&+ \frac{\rho^2}{2}\left(1 - \frac{1}{4}(1 - \cos 2\beta)(1 - \cos 2\alpha \sin\varphi)\right)\dot{\gamma}^2 \\
&- \frac{\rho^2}{2}\cos\varphi\,\dot{\theta}\dot{\alpha} - \frac{\rho^2}{2}\cos\beta\cos\varphi\,\dot{\theta}\dot{\gamma} \\
&+ \rho^2\cos\beta\,\dot{\alpha}\dot{\gamma} - \frac{\rho^2}{2}\sin\beta\sin 2\alpha\sin\varphi\,\dot{\beta}\dot{\gamma}
\end{aligned} \tag{3.29}$$

and that

$$\begin{aligned}
g_1 &= (\dot{\beta}\cos\alpha + \dot{\gamma}\sin\alpha\sin\beta)\lambda_1 \\
g_2 &= (-\dot{\beta}\sin\alpha + \dot{\gamma}\cos\alpha\sin\beta)\lambda_2 \\
g_3 &= (\dot{\alpha} + \dot{\gamma}\cos\beta - \frac{1}{2}\dot{\theta}\cos\varphi)\lambda_3,
\end{aligned} \tag{3.30}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the principal moments of inertia

$$\begin{aligned}
\lambda_1 &= \frac{1}{2}\rho^2(1 - \sin\varphi) \\
\lambda_2 &= \frac{1}{2}\rho^2(1 + \sin\varphi) \\
\lambda_2 &= \rho^2
\end{aligned}$$

Except of the Euler angle singularities $\beta = 0 \mod \pi$, the relation (3.30) is invertible, and we get

$$\begin{aligned}
\dot{\alpha} &= \frac{1}{2}\dot{\theta}\cos\varphi - \frac{g_1\cos\beta\sin\alpha}{\lambda_1\sin\beta} - \frac{g_2\cos\alpha\cos\beta}{\lambda_2\sin\beta} + \frac{g_3}{\rho^2} \\
\dot{\beta} &= \frac{g_1\cos\alpha}{\lambda_1} - \frac{g_2\sin\alpha}{\lambda_2} \\
\dot{\gamma} &= \frac{g_1\sin\alpha}{\lambda_1\sin\beta} + \frac{g_2\cos\alpha}{\lambda_2\sin\beta}
\end{aligned} \quad . \tag{3.31}$$

Substituted into the expression (3.29), we get the following expression for the kinetic energy:

$$T = T^{\rho} + T^{\sigma} + T^{\omega} = \frac{1}{2}\dot{\rho}^2 + \frac{\rho^2}{8}(\dot{\varphi}^2 + \sin\varphi\dot{\theta}^2) + \frac{1}{2}\left(\frac{g_1^2}{\lambda_1} + \frac{g_2^2}{\lambda_2} + \frac{g_3^2}{\lambda_3}\right) \qquad (3.29')$$

This is in complete accordance with [HS07], where it is pointed out that this gives a decomposition of the kinetic energy T into kinetic energy due to change of size

$$T^{\rho} = \frac{1}{2}\dot{\rho}^2,$$

change of shape

$$T^{\sigma} = \frac{\rho^2}{8}(\dot{\varphi}^2 + \sin^2\varphi\dot{\theta}^2)$$

and rotation

$$T^{\omega} = \frac{1}{2}\left(\frac{g_1^2}{\lambda_1} + \frac{g_2^2}{\lambda_2} + \frac{g_3^2}{\lambda_3}\right)$$

In the literature, the decomposition $T = T^{\rho} + T^{\sigma} + T^{\omega}$ is often referred to as Saari's decomposition.

The equations of motion of three body motions within the set of regular configurations are equivalent to the Euler-Lagrange equations associated with the Lagrange function

$$L = T + \frac{U^*}{\rho^e}.$$

We can thus express the equations of motion in terms of $\alpha, \beta, \gamma, \rho, \varphi, \theta$ and their derivatives up to order 2. By relation (3.31), we are able to eliminate $\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \ddot{\alpha}, \ddot{\beta}, \ddot{\gamma}$ from the equations of motion, in favour of $g_1, g_2, g_3, \dot{g}_1, \dot{g}_2, \dot{g}_3$. Practically, this requires a huge amount of algebraic operations, but the result is quite transparent:

For the variables g_1, g_2, g_3 we get the following equations

$$
\begin{aligned}
\dot{g}_1 &= g_2 \left[g_3 \left(\frac{1}{\lambda_3} - \frac{1}{\lambda_2} \right) + \frac{1}{2}\dot{\theta}\cos\varphi \right] \\
\dot{g}_2 &= g_1 \left[g_3 \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_3} \right) - \frac{1}{2}\dot{\theta}\cos\varphi \right] \\
\dot{g}_3 &= g_1 g_2 \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1} \right).
\end{aligned}
\tag{3.32}
$$

We have the equivalent form

$$
\begin{aligned}
\dot{g}_1 &= -\left(\frac{1-\sin\varphi}{1+\sin\varphi} \right) \frac{g_2 g_3}{\rho^2} + \frac{1}{2} g_2 \dot{\theta}\cos\varphi \\
\dot{g}_2 &= \left(\frac{1+\sin\varphi}{1-\sin\varphi} \right) \frac{g_3 g_1}{\rho^2} - \frac{1}{2} g_1 \dot{\theta}\cos\varphi \\
\dot{g}_3 &= -\left(\frac{\sin\varphi}{\cos^2\varphi} \right) \frac{4 g_1 g_2}{\rho^2}.
\end{aligned}
\tag{3.32'}
$$

These equations are identical to the *Euler equations* (2.29); by direct computation we easily verify that

$$
v_1 = v_2 = 0 \quad \text{and} \quad v_3 = \dot{\theta}, \quad \text{and} \quad \frac{r_1 r_2}{\lambda_3} = -\frac{1}{2}\cos\varphi
$$

in the present representation of the three body problem. As we saw in Chapter 2, the Euler equations are valid for all non-collinear configurations, and in the domain of validity, they are equivalent to conservation of total angular momentum. The Euler equations (3.32) of the three body problem were deduced in [HS07].

For the variables ρ, φ, θ, we get the following equations:

$$
\begin{aligned}
\ddot{\rho} &= \frac{g_1^2}{\rho\lambda_1} + \frac{g_2^2}{\rho\lambda_2} + \frac{g_3^2}{\rho^3} + \frac{1}{4}\rho\dot{\varphi}^2 + \frac{1}{4}\rho\dot{\theta}^2\sin^2\varphi - e\frac{U^*}{\rho^{1+e}} \\
\ddot{\varphi} &= \dot{\theta}^2\cos\varphi\sin\varphi + \frac{g_2^2\cos\varphi}{\lambda_2^2} - \frac{g_1^2\cos\varphi}{\lambda_1^2} + 2\frac{\dot{\theta} g_3 \sin\varphi}{\rho^2} - 2\frac{\dot{\rho}\dot{\varphi}}{\rho} + 4\frac{U^*_\varphi}{\rho^{2+e}} \\
\ddot{\theta} &= -2\frac{\dot{\theta}\dot{\varphi}\cos\varphi}{\sin\varphi} - 2\frac{g_1 g_2\cos\varphi}{\lambda_1\lambda_2\sin\varphi} - 2\frac{\dot{\rho}\dot{\theta}}{\rho} - 2\frac{\dot{\varphi} g_3}{\rho^2\sin\varphi} + 4\frac{U^*_\theta}{\rho^{2+e}\sin^2\varphi},
\end{aligned}
\tag{3.33}
$$

where U^*_φ, U^*_θ denotes the partial derivatives with respect to φ, θ.

Following (3.29'), the total energy h satisfies

$$h = \frac{1}{2}\dot{\rho}^2 + \frac{\rho^2}{8}(\dot{\varphi}^2 + \sin\varphi\dot{\theta}^2) + \frac{1}{2}\left(\frac{g_1^2}{\lambda_1} + \frac{g_2^2}{\lambda_2} + \frac{g_3^2}{\lambda_3}\right) - \frac{U^*(\varphi,\theta)}{\rho^e},$$

and this allows us to rewrite the equations of motion as

$$\begin{aligned}
\ddot{\rho} &= -\frac{\dot{\rho}^2}{\rho} + \frac{1}{\rho}\left(\frac{2-e}{\rho^e}U^* + 2h\right) \\
\ddot{\varphi} &= -2\frac{\dot{\rho}\dot{\varphi}}{\rho} + \frac{1}{2}\sin 2\varphi\dot{\theta}^2 + 2\frac{\dot{\theta}g_3\sin\varphi}{\rho^2} + \frac{4U^*_\varphi}{\rho^{2+e}} \\
&\quad - \frac{4\cos\varphi}{\rho^4}\left(\frac{g_1^2}{(1-\sin\varphi)^2} - \frac{g_2^2}{(1+\sin\varphi)^2}\right) \\
\ddot{\theta} &= -2\frac{\dot{\rho}\dot{\theta}}{\rho} - 2\cot\varphi\,\dot{\theta}\dot{\varphi} - 2\frac{\dot{\varphi}g_3}{\rho^2\sin\varphi} + \frac{4U^*_\theta}{\rho^{2+e}\sin^2\varphi} - 16\frac{g_1g_2}{\rho^4\sin 2\varphi},
\end{aligned} \tag{3.33'}$$

The equations (3.33') and (3.32) are clearly equivalent to Newton's equations of motion for the three body problem within their domain of validity, which is the set M_r of regular configurations.

3.7.3 A regular form of the reduced equations of motion

In order to allow for an interpretation of the reduced equations of motion outside the set M_r of regular configurations, we can rewrite the Euler equations as

$$\begin{aligned}
\cos^2(\varphi)\,\dot{g}_1 &= -\frac{1}{\rho^2}(1-\sin\varphi)^2 g_2 g_3 + \frac{1}{2}\cos^3(\varphi)\dot{\theta}g_2 \\
\cos^2(\varphi)\,\dot{g}_2 &= \frac{1}{\rho^2}(1+\sin\varphi)^2 g_3 g_1 - \frac{1}{2}\cos^3(\varphi)\dot{\theta}g_1 \\
\cos^2(\varphi)\,\dot{g}_3 &= -\frac{4}{\rho^2}\sin\varphi g_1 g_2,
\end{aligned} \tag{3.34}$$

and the reduced equations as

$$\begin{aligned}
0 &= \ddot{\rho} + \frac{\dot{\rho}^2}{\rho} - \frac{1}{\rho}\left(\frac{2-e}{\rho^e}U^* + 2h\right) \\
0 &= \cos^3\varphi\,\ddot{\varphi} + 2\cos^3\varphi\,\frac{\dot{\rho}\dot{\varphi}}{\rho} - \sin\varphi\cos^4\varphi\,\dot{\theta}^2 - 2\cos^3\varphi\,\frac{\dot{\theta}g_3\sin\varphi}{\rho^2} - \cos^3\varphi\frac{4U^*_{\varphi}}{\rho^{2+e}} \\
&\quad + \frac{4}{\rho^4}\left(g_1^2(1+\sin\varphi)^2 - g_2^2(1-\sin\varphi)^2\right) \\
0 &= \rho^2\sin^2(\varphi)\ddot{\theta} + 2\sin^2(\varphi)\rho\dot{\rho}\dot{\theta} + 2\rho^2\sin\varphi\cos\varphi\,\dot{\theta}\dot{\varphi} \\
&\quad + 2\sin\varphi\dot{\varphi}g_3 - \frac{4U^*_{\theta}}{\rho^e} - 2\cos\varphi\dot{g}_3.
\end{aligned} \tag{3.35}$$

These equations are equivalent to (3.32),(3.33) on M_r. Additionally, they also give meaning when $\varphi = 0 \mod \frac{\pi}{2}$. The physical meaning is not obvious at this point, but will be clarified in Section 3.7.6.

3.7.4 The Lagrange-Jacobi equation

The Lagrange-Jacobi equation (cf. [Lag72][Jac43]) is a general relation which holds for all many particle systems with motion determined by a homogeneous potential function. Here we point out that the $\ddot{\rho}$-equation in (3.35) is simply a restatement of the Lagrange-Jacobi equation, and hence universally valid.

If the potential function $U(\mathbb{a}_1,\dots,\mathbb{a}_n)$ is homogeneous of degree e, then the expression $U(\mathbb{x}_1,\dots,\mathbb{x}_n)$ of the potential function U in terms of the Jacobi vectors $\mathbb{x}_1,\dots,\mathbb{x}_n$ is still homogeneous of degree e. The polar moment of inertia with respect to the centre of mass is

$$I = \langle X, X\rangle = \rho^2,$$

Using that U is homogeneous of degree e, we arrive

$$\ddot{I} = 2\langle \dot{X}, \dot{X}\rangle + 2\langle X, \ddot{X}\rangle = 4T + 2\langle X, \nabla U\rangle = 4T - 2eU.$$

This can also be written as

$$\ddot{I} = 2(e-2)U + 4h.$$

Here $h = T - U$ is the total energy. Using $I = \rho^2$, we arrive

$$\ddot{\rho} = -\frac{\dot{\rho}^2}{\rho} + \frac{1}{\rho}\left((e-2)U + 2h\right), \tag{3.36}$$

an equation which is clearly equivalent to the $\ddot{\rho}$-equation in (3.35). This proves the following:

Proposition 3.7.1. *Equation* (3.36) *is valid for all many particle systems with dynamics given by a homogeneous potential function.*

3.7.5 Relation between the planar case and the general case

With our conventions for the three body problem, the discussion of Section 2.8.1 implies that a three body motion is planar if and only if $g_1 = g_2 = 0$. Setting $g_1 = g_2 = 0$, $g_3 = \Omega_s$, we see that the reduced equations (3.33) in the general case are identical to the reduced equations in the planar case (3.28).

The non-planar reduction does not apply to the case of collinear motions. Since collinear motions are planar (cf. Section 2.7.3), we can apply the general equations (3.33) to the collinear case if we adopt the same convention as the preferred convention for the planar case, namely

$$g_1 = g_2 = 0, g_3 = \Omega_s. \tag{3.37}$$

Since this convention is compatible with the modified Euler equations (3.34), we see that the general system (3.34),(3.35) holds also for collinear three body motions following this convention.

3.7.6 Extension of the domain of validity by means of analyticity

By the Cauchy-Kowalevski theorem, we know that collision free three body motions are analytic in time, since Newton's equations of motion are analytic away from the collision points. By the analyticity of the singular value decomposition, we infer that $\rho, \varphi, \theta, g_1, g_2, g_3$ can be taken to be analytic functions of time.

When φ is analytic in t, we know that $\varphi = \frac{\pi}{2} \mod \pi$ either for all t or for isolated instances of t. Hence in order to study collinearity, it is sufficient to study either purely collinear motions or passages through collinear configurations.

At the singularities at $\varphi = 0 \mod \pi$, we have the same situation: Because of analyticity we have either an umbilic configuration for all t, or passages through umbilic configurations at isolated instances of time.

The modified equations (3.34),(3.35) are given by analytic expressions, and will hence remain valid at passages through the singular set. By analyticity, they determine the three body motions passing through the singularities.

Since collinear motions are planar, we see that our reduced equations cover all three body motions except of the umbilic shape invariant motions. This class of motions is discussed in Section 3.9.

3.7.7 The reconstruction problem

We have seen how three body motions determine $\rho, \varphi, \theta, g_1, g_2, g_3$, whose evolution is governed by the reduced equations of motion (3.32) and (3.33). In this situation, the following question arise: To what extent is the spatial three body motion determined by the evolution of $\rho, \varphi, \theta, g_1, g_2, g_3$?

Since $\rho, \varphi, \theta, g_1, g_2, g_3$ is intended to be a complete set of geometrical invariants of three body motions, the answer should be obvious: They determine three body motions modulo choice of inertial system. We demonstrate this by indication of the steps of the reconstruction procedure:

Let us assume that we are given an analytic solution

$$\rho(t), \varphi(t), \theta(t), g_1(t), g_2(t), g_3(t)$$

of (3.34) and (3.35). The Euler angles $\alpha(t), \beta(t), \gamma(t)$ are determined by equation (3.31) together initial values, which can be represented by any element $P_0 \in SO(3)$. If we avoid the singularities $\beta = 0$, $\lambda_1 = 0$, $\lambda_2 = 0$ this works fine. Hence, if we restrict the discussion to motions in M_r, we can get along by patching up different Euler angle gauges. In this way we reconstruct a principal frame matrix $P(t)$ as an analytic curve in $SO(3)$. The evolution of $\rho(t), \varphi(t), \theta(t), P(t))$, specifies a singular value decomposition $(P(t), R(t), Q(t))$ of the three body motion X, from which we can determine the Jacobi vectors by

$$\left[\mathbb{x}_1(t) | \mathbb{x}_2(t)\right] = X(t) = P(t) \cdot R(t) \cdot Q(t).$$

In the case of purely collinear motions, we get along in a similar way, using the similar reconstruction procedure for the planar problem, based on (3.27).

Thus we see that we are able to recover the evolution of the Jacobi vectors $\mathbb{x}_1(t), \mathbb{x}_2(t)$. For a prescribed linear motion $\overline{\mathbb{a}}(t) = \mathbb{b} \cdot t + \mathbb{c}$ of the centre of mass, we can invert (3.4) in order to find the actual motions $\mathbb{a}_1(t), \mathbb{a}_2(t), \mathbb{a}_3(t)$ of the particles in space.

The reconstruction process depends on our choice of the linear motion of the centre of mass and the constant rotation matrix P_0. Hence, the reconstructed motion $\mathbb{a}_1(t), \mathbb{a}_2(t), \mathbb{a}_3(t)$ is – modulo choice of inertial system – uniquely determined by

$$\rho(t), \varphi(t), \theta(t), g_1(t), g_2(t), g_3(t).$$

3.8 Poincaré's principle

Here we give a more conceptual and differential geometric deduction of the reduced equations (3.34),(3.35) in the case of regular three body configurations.

3.8.1 General considerations

Consider a smooth manifold M with a given Lagrange function $L\colon TM \to \mathbb{R}$. As is well known, the motions of such a system are stationary points for the action integral

$$\int L \mathrm{d}t$$

when we consider variations keeping endpoints fixed. As long as L is smooth, it is safe to consider smooth variations in the class of smooth curves; the motions are at least as smooth as the Lagrange function.

Following [Poi01], we will derive the equations of motion for a Lagrange system in an anholonomic frame, the so-called *Poincaré* equations.

Anholonomic frames

An anholonomic frame is a system

$$\omega^1, \omega^2, \ldots, \omega^n$$

of point-wise linearly independent 1-forms spanning the cotangent bundle T^*M. Equivalently, we can consider the dual frame

$$X_1, X_2, \ldots, X_n$$

of vector fields on M, satisfying $\omega^i(X_j) = \delta^i_j$, where δ^i_j is the Kronecker delta. Locally, an anholonomic frame is characterized by the structure coefficients c^k_{ij}, which can be defined in any one of the following equivalent ways:

$$\mathrm{d}\omega^k = -c^k_{ij}\omega^i \wedge \omega^j, \quad d\omega^k(X_i, X_j) = -c^k_{ij}, \quad [X_i, X_j] = c^k_{ij} X_k.$$

In these formulae as well as in the following computations, we use Einstein's summation convention.

The frame $\omega^1, \omega^2, \ldots, \omega^n$ yields a trivialization $\psi\colon TM \to M \times \mathbb{R}^n$ of the tangent bundle:

$$\psi\colon v \mapsto [\pi(v), (\omega^1(v), \ldots, \omega^n(v))],$$

where π is the projection $TM \to M$. Hence, using a local coordinate system $(x^1, \ldots, x^n)$ and the frame $\omega^1, \ldots, \omega^n$, any function F on the tangent bundle is locally represented by a function $F\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$:

$$F = F(x^1, \ldots, x^n, \omega^1, \ldots, \omega^n),$$

where we implicitly interpret the ω^i as coordinate functions on TM. With this notation, the function $\frac{\partial F}{\partial \omega^i}(x^1, \ldots, \omega^n)$ can be interpreted as a function

$$\frac{\partial F}{\partial \omega^i}\colon TM \to \mathbb{R}$$

on the tangent bundle. For a vector field X on M which is locally represented by $f^i \frac{\partial}{\partial x^i}$, we define as usual the derivative

$$XF = f^i \frac{\partial F}{\partial x^i}\colon TM \to \mathbb{R}.$$

With these definitions, XF and $\frac{\partial F}{\partial \omega^i}$ depend on the choice of frame $\omega^1, \ldots, \omega^n$, but not on the choice of coordinates $x^1, \ldots, x^n$, since $\frac{\partial F}{\partial \omega^i}$ depends only on the restrictions of F to the fibres T_pM.

Calculus of variations

Let us consider a smooth curve $\gamma(t)$ $t \in [t_0, t_1]$ together with a smooth variation $\gamma_u(t) = \alpha(t,u)$, where $(t,u) \in [t_0,t_1] \times (-\varepsilon, \varepsilon)$ and $\alpha(t,0) = \gamma(t)$.

From the variation α, we get the variational vector field

$$\alpha_u = \frac{\partial \alpha}{\partial u} = \alpha_* \left(\frac{\partial}{\partial u}\right) = \alpha_u^i X_i,$$

i.e. $\alpha_u^i = \omega^i(\alpha_u) = \alpha^* \omega^i(\frac{\partial}{\partial u})$. Recall that X_i denotes the vector field dual to ω^i. Similarly, the velocity

$$\alpha_t = \frac{\partial \alpha}{\partial t} = \alpha_* \left(\frac{\partial}{\partial t}\right) = \alpha_t^i X_i,$$

i.e. $\alpha_t^i = \omega^i(\alpha_t) = \alpha^* \omega^i(\frac{\partial}{\partial t})$.

α_t^i, α_u^i are now regarded as functions of (t,u), and since $[\frac{\partial}{\partial t}, \frac{\partial}{\partial u}] = 0$ on $[t_0,t_1] \times (-\varepsilon, \varepsilon)$, we get

$$\begin{aligned}
\frac{\partial \alpha_t^k}{\partial u} &= \frac{\partial}{\partial u}\left(\alpha^* \omega^k \left(\frac{\partial}{\partial t}\right)\right) \\
&= L_{\frac{\partial}{\partial u}}(\alpha^* \omega^k)\left(\frac{\partial}{\partial t}\right) + \alpha^* \omega^k \left(\left[\frac{\partial}{\partial u}, \frac{\partial}{\partial t}\right]\right) \\
&= d(\alpha^* \omega^k)\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial t}\right) + d\left(\alpha^* \omega^k \left(\frac{\partial}{\partial u}\right)\right)\left(\frac{\partial}{\partial t}\right) + 0 \\
&= d\omega^k \left(\alpha_* \frac{\partial}{\partial u}, \alpha_* \frac{\partial}{\partial t}\right) + \frac{\partial \alpha_u^k}{\partial t} \\
&= d\omega^k(\alpha_u^i X_i, \alpha_t^j X_j) + \frac{\partial \alpha_u^k}{\partial t} \\
&= -c_{ij}^k \alpha_u^i \alpha_t^j + \frac{\partial \alpha_u^k}{\partial t}.
\end{aligned} \tag{3.38}$$

In local coordinates, we represent respectively $\alpha(u,t)$ and $\gamma(t)$ by $x^i(u,t)$

and $x^i(t)$, and thus get

$$\begin{aligned}\frac{d}{du}\Gamma[\gamma_u] &= \int_{t_0}^{t_1} \frac{\partial}{\partial u} L(\dots, x^i(u,t), \dots, \alpha_t^i, \dots)\mathrm{d}t \\ &= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x_i}\frac{\partial x_i}{\partial u} + \frac{\partial L}{\partial \omega^k}\frac{\partial \alpha_t^k}{\partial u} \right)\mathrm{d}t \\ &= \int_{t_0}^{t_1} \left(\alpha_u^i X_i L + \frac{\partial L}{\partial \omega^k} c_{ij}^k \alpha_u^i \alpha_t^j + \frac{\partial L}{\partial \omega^k}\frac{\partial \alpha_u}{\partial t} \right) dt\end{aligned}$$

and finally, by integration of parts, keeping the end-points fixed

$$\frac{d}{du}\Gamma[\gamma_u] = \int_{t_0}^{t_1} \left(X_i L - \frac{\partial L}{\partial \omega^k} c_{ij}^k \alpha_t^j - \frac{d}{dt}\left(\frac{\partial L}{\partial \omega^k}\right)\right) \alpha_u^i dt$$

Hence, we see that stationarity of the action integral at γ is equivalent to the equations

$$\frac{d}{dt}\frac{\partial L}{\partial \omega^i} = -\frac{\partial L}{\partial \omega^k} c_{ij}^k \omega^j + X_i L, \quad i = 1, 2, \dots, n, \tag{3.39}$$

which characterizes the motions of the Lagrange system. These equations can be called the *Poincaré equations associated with the Lagrange function L and the anholonomic frame* $\omega^1, \dots, \omega^n$.

Let us make the correct interpretation of this equation precise: Along a curve $\gamma(t)$ in M, we can interpret $\frac{\partial L}{\partial \omega^i}$ and $-\frac{\partial L}{\partial \omega^k} c_{ij}^k \omega^j + X_i L$ as functions of t, and equation (3.39) shall be interpreted as an identity of functions of t.

3.8.2 Application to the three body problem

Using the coordinates $(\alpha, \beta, \gamma, \rho, \varphi, \theta)$ defined in Section 3.7, we will now apply Poincaré's equations to the three body problem. This implicitly amounts to a lifting from M to S, and we must have in mind that this lifting is straightforward only over the regular part $M_r \subset M$.

The anholonomic frame

Let $\mathbf{g} = (g_1, g_2, g_3)$ denote the total angular momentum in the principal frame, i.e. $\mathbf{g} = P^t\Omega = P^t(X \times \dot{X})$, where $P(t)$ is the principal frame matrix and $X(t)$ is the three body motion under consideration. We can regard g_1, g_2, g_3 as 1-forms on the configuration space. We will rather use the 1-forms $\omega^i = \lambda_i^{-1} g_i$ for our computations. By direct computation, we have

$$\begin{aligned}
\omega^1 &= \cos\alpha \mathrm{d}\beta + \sin\alpha \sin\beta \mathrm{d}\gamma \\
\omega^2 &= -\sin\alpha \mathrm{d}\beta + \cos\alpha \sin\beta \mathrm{d}\gamma \\
\omega^3 &= \mathrm{d}\alpha + \cos\beta \mathrm{d}\gamma - \frac{1}{2}\cos\varphi \mathrm{d}\theta.
\end{aligned} \tag{3.40}$$

We will now apply Poincaré's principle to the anholonomic frame

$$\mathrm{d}\rho, \mathrm{d}\varphi, \mathrm{d}\theta, \omega^1, \omega^2, \omega^3. \tag{3.41}$$

In the following we will use

$$\{\rho, \varphi, \theta, 1, 2, 3\}$$

as index set, and in order to gain conformity with the above notation, we can define

$$\omega^\rho = \mathrm{d}\rho, \quad \omega^\varphi = \mathrm{d}\varphi \quad \text{and} \quad \omega^\theta = \mathrm{d}\theta.$$

The structure coefficients c_{ij}^k of the frame (3.41) are determined by

$$\begin{aligned}
\mathrm{d}^2\rho &= \mathrm{d}^2\varphi = \mathrm{d}^2\theta = 0 \\
\mathrm{d}\omega^1 &= -\omega^2 \wedge \omega^3 + \frac{1}{2}\cos\varphi \, \mathrm{d}\theta \wedge \omega^2 \\
\mathrm{d}\omega^2 &= -\omega^3 \wedge \omega^1 - \frac{1}{2}\cos\varphi \, \mathrm{d}\theta \wedge \omega^1 \\
\mathrm{d}\omega^3 &= -\omega^1 \wedge \omega^2 + \frac{1}{2}\sin\varphi \, \mathrm{d}\varphi \wedge \mathrm{d}\theta,
\end{aligned}$$

and the dual frame is given by

$$\begin{aligned}
X_\rho &= \frac{\partial}{\partial\rho}, \quad X_\varphi = \frac{\partial}{\partial\varphi}, \quad X_\theta = \frac{\partial}{\partial\theta} + \frac{1}{2}\cos\varphi \frac{\partial}{\partial\alpha} \\
X_1 &= -\cot\beta \sin\alpha \frac{\partial}{\partial\alpha} + \cos\alpha \frac{\partial}{\partial\beta} + \frac{\sin\alpha}{\sin\beta}\frac{\partial}{\partial\gamma} \\
X_2 &= -\cot\beta \cos\alpha \frac{\partial}{\partial\alpha} - \sin\alpha \frac{\partial}{\partial\beta} + \frac{\cos\alpha}{\sin\beta}\frac{\partial}{\partial\gamma} \\
X_3 &= \frac{\partial}{\partial\alpha}.
\end{aligned}$$

Note that these formulae shows that X_1, X_2, X_3 spans the tangent spaces of the SO(3) orbits.

From the equations (3.40) above, we see that the 1-forms $\omega^1, \omega^2, \omega^3$ are globally well defined, but that ω^1, ω^2 seem to be linearly dependent for $\sin\beta = 0$. On the other hand, $\omega^1, \omega^2, \omega^3$ has a physical interpretation which is independent of the choice of Euler angle gauge. Hence, the $\sin\beta = 0$-singularity is just associated with singularity of the Euler angles α, β, γ. Hence, we can regard $\omega^1, \omega^2, \omega^3$ are globally linearly independent and well defined on S. Dually, we seemingly experience a blow-up of the vector fields X_1, X_2, X_3 for $\sin\beta = 0$. The degree of well-definedness of X_1, X_2, X_3 does not depend on the choice of Euler angles, and hence, this singularity should also be blamed on the Euler angle singularity. Accordingly, we will regard X_1, X_2, X_3 as globally well defined on S.

Hence when we apply the one-forms $\omega^1, \omega^2, \omega^3$, we overcome the problem of Euler-angle-singularities. The singularities of $\Phi\colon S \to M$ for $\varphi = 0 \mod \frac{\pi}{2}$, however, still remain.

The Lagrange function

In our system of 1-forms, the Lagrange function can be expressed as

$$L = \frac{1}{2}\mathrm{d}\rho^2 + \frac{\rho^2}{8}\left(\mathrm{d}\varphi^2 + \sin^2\varphi\,\mathrm{d}\theta^2\right) + \frac{1}{2}\left(\lambda_1(\omega^1)^2 + \lambda_2(\omega^2)^2 + \lambda_3(\omega^3)^2\right) + U(\rho, \phi, \theta). \tag{3.42}$$

As announced above, the generalized momenta associated with $\omega^1, \omega^2, \omega^3$ are indeed

$$\frac{\partial L}{\partial \omega^i} = g_i,$$

where g_i are the components of the total angular momentum, as defined above. The other momenta are

$$\frac{\partial L}{\partial(\mathrm{d}\rho)} = \mathrm{d}\rho, \quad \frac{\partial L}{\partial(\mathrm{d}\varphi)} = \frac{\rho^2}{4}\mathrm{d}\varphi, \quad \frac{\partial L}{\partial(\mathrm{d}\theta)} = \frac{\rho^2\sin^2\varphi}{4}\mathrm{d}\theta$$

This notation may look odd. However, when we regard $\mathrm{d}\rho, \mathrm{d}\varphi, \mathrm{d}\theta$ as coordinate functions on the tangent bundle TM, it makes sense.

The equations of motion

Now we are ready to present the Poincaré equations for three body motions within the set of regular configurations:

The $\mathrm{d}\rho$-equation: We have

$$X_\rho L = \frac{\rho}{4}\left(\mathrm{d}\varphi^2 + \sin^2\varphi \mathrm{d}\theta^2\right) + \frac{1}{\rho}\left(\lambda_1(\omega^1)^2 + \lambda_2(\omega^2)^2 + \lambda_3(\omega^3)^2\right) + \frac{\partial}{\partial\rho}U,$$

since $\frac{\partial}{\partial\rho}\lambda_i = \frac{2}{\rho}\lambda_i$. Since all the structure coefficients $c^i_{\rho j} = 0$, we get the following equation:

$$\ddot{\rho} = \frac{\rho}{4}\left(\dot{\varphi}^2 + \sin^2\varphi\dot{\theta}^2\right) + \frac{1}{\rho}\left(\lambda_1(\omega^1)^2 + \lambda_2(\omega^2)^2 + \lambda_3(\omega^3)^2\right) + \frac{\partial}{\partial\rho}U. \tag{3.43}$$

The $\mathrm{d}\varphi$-equation: We have

$$X_\varphi L = \frac{\rho^2}{4}\sin\varphi\cos\varphi \mathrm{d}\theta^2 + \frac{\rho^2\cos\varphi}{4}\left((\omega^2)^2 - (\omega^1)^2\right) + \frac{\partial}{\partial\varphi}U.$$

The only relevant structure coefficient is $c^3_{\varphi\theta} = -\frac{1}{2}\sin\varphi$, which yields the following equation:

$$\begin{aligned}\frac{\rho^2}{4}\ddot{\varphi} + \frac{\rho\dot{\rho}\dot{\varphi}}{2} &= \frac{1}{2}\sin\varphi g_3\dot{\theta} + \frac{\rho^2}{4}\sin\varphi\cos\varphi\dot{\theta}^2 \\ &\quad + \frac{\rho^2\cos\varphi}{4}\left((\omega^2)^2 - (\omega^1)^2\right) + \frac{\partial}{\partial\varphi}U.\end{aligned} \tag{3.44}$$

The $\mathrm{d}\theta$-equation: We have

$$X_\theta L = \frac{\partial}{\partial\theta}U,$$

and the relevant structure coefficients are

$$c^3_{\theta\varphi} = \frac{1}{2}\sin\varphi, \quad c^2_{\theta 1} = -c^1_{\theta 2} = \frac{1}{2}\cos\varphi.$$

Accordingly, the corresponding Poincaré equation is

$$\begin{aligned}\frac{\rho^2\sin^2\varphi}{4}\ddot{\theta}+\frac{\rho\sin^2\varphi}{2}\dot{\rho}\dot{\theta}+\frac{\rho^2\sin\varphi\cos\varphi}{2}\dot{\varphi}\dot{\theta}&\\ =-\frac{1}{2}\sin\varphi g_3\dot{\varphi}+\frac{1}{2}\cos\varphi\left(g_1\omega^2-g_2\omega^1\right)+\frac{\partial}{\partial\theta}U&\end{aligned} \tag{3.45}$$

The ω^i-equations We note that $X_1L = X_2L = X_3L = 0$, $c^1_{2\theta} = -c^2_{1\theta} = \frac{1}{2}\cos\varphi$ and $c^k_{ij} = 1$ when (i,j,k) is an even permutation of $(1,2,3)$. Hence the three remaining equations are

$$\begin{aligned}\dot{g}_1 &= -c^3_{12}g_3\omega^2 - c^2_{13}g_2\omega^3 - c^2_{1\theta}g_2\dot{\theta} = g_2g_3\left(\frac{1}{\lambda_3}-\frac{1}{\lambda_2}\right)+\frac{1}{2}\cos\varphi\, g_2\dot{\theta}\\ \dot{g}_2 &= -c^3_{21}g_3\omega^1 - c^1_{23}g_1\omega^3 - c^1_{2\theta}g_1\dot{\theta} = g_3g_1\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_3}\right)-\frac{1}{2}\cos\varphi\, g_1\dot{\theta}\\ \dot{g}_3 &= -c^2_{31}g_2\omega^1 - c^1_{32}g_2\omega^2 = g_1g_2\left(\frac{1}{\lambda_2}-\frac{1}{\lambda_1}\right)\end{aligned} \tag{3.46}$$

By inspection, we see that the equations (3.43)–(3.46) are identical to the reduced equations of motion given in Section 3.7.

3.8.3 A remark on Lagrangian and Hamiltonian formalism

The intention of this section is to demonstrate that Poincaré's equations reveal important aspects of the relation between Lagrangian and Hamiltonian formalism in classical mechanics. The symplectic structure of cotangent bundles, and the Poisson structure on their function algebras plays a preeminent role in the Hamiltonian formalism. On the Lagrangian side, the Lie bracket plays an almost equivalent role, and the connection is revealed by the following observation:

Vector fields $X, Y, \ldots$ on M can be regarded as smooth functions on the cotangent bundle T^*M, and the Lie bracket $[-,-]$ and the Poisson bracket $\{-,-\}$ are related by

$$[X,Y] = \{Y,X\}.$$

This justifies the claim that *the Poisson bracket and the Lie bracket are essentially the same thing.* There are several extensions to this correspondence. An example of this is the extension to symmetric contravariant tensors given by the Schouten bracket.

As an extension of the Schouten bracket, we can consider the Lie derivative $L_X F$ along the vector field X of a function $F\colon TM \to \mathbb{R}$. This is given by

$$L_X F(\mathbb{v}) = \frac{d}{ds} F(\psi_*^s \mathbb{v}), \quad \mathbb{v} \in TM,$$

where ψ^s is the flow on M generated by X, and ψ_*^s the corresponding flow on the tangent bundle.

Let us consider a simple mechanical system on a manifold M, i.e. a Lagrange system where the Lagrange function is on the form

$$L = \frac{1}{2} K + U \tag{3.47}$$

where K is a Riemannian metric on M and U is a smooth function on M.

K yields a correspondence between TM and $T^* M$, and hence a correspondence between covariant and contravariant tensors. Thereby we can define a correspondence between vector fields and 1-forms: For a given vector field X, we let Ω_X denote the corresponding 1-form, which is defined to satisfy $\Omega_X(\mathbb{v}) = K(X, \mathbb{v})$ for every $\mathbb{v} \in TM$. We will call Ω_X the canonical momentum associated with the vector field X. Similarly, for every 1-form Ω, we will define a vector field X_Ω by $\Omega_{X_\Omega} = \Omega$. We can thus define a bracket $\{-, -\}$ acting on 1-forms on M and smooth function on the tangent bundle TM by

$$\{\Omega, F\} = L_{X_\Omega} F$$

Notationally, we intentionally allude to the Poisson-bracket defined for functions on the tangent bundle TM.

With our notation, we are able to reproduce the Poisson equations in a Lagrangian setting:

Proposition 3.8.1. *$\gamma\colon \mathbb{R} \to M$ is a motion of the Lagrange system on M with Lagrange function* (3.47) *if and only if*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Omega = \{\Omega, L\} \tag{3.48}$$

is satisfied along γ for every 1-form Ω on M.

Proof. Here we will use the fact that motions of Lagrange systems in local coordinate systems are characterized by the usual Euler-Lagrange equations.

First we note that if Ω belongs to a coordinate vector field $\frac{\partial}{\partial x^k}$ in a coordinate system $(x^1, \ldots, x^n)$, i.e. $\Omega = \Omega_{\frac{\partial}{\partial x^i}}$, then

$$\Omega(\dot{\gamma}(t)) = g_{kj}\, dx^j(\dot{\gamma}(t)) = g_{kj}\dot{x}^j = \frac{\partial L}{\partial \dot{x}^k},$$

where g_{ij} are the coefficients of the Riemannian metric. On the other hand,

$$\begin{aligned}
\{\Omega, L\} &= L_X(L)(\dot{\gamma}(t)) \\
&= L_{\frac{\partial}{\partial x^k}}\left(\frac{1}{2} g_{ij}\, dx^i dx^j + U\right)(\dot{\gamma}(t)) \\
&= \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k} dx^i dx^j(\dot{\gamma}(t)) + \frac{\partial U}{\partial x^k}(\gamma(t)). \\
&= \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\dot{x}^i \dot{x}^j + \frac{\partial U}{\partial x^k}(x^i) \\
&= \frac{\partial L}{\partial x^k}.
\end{aligned}$$

Hence, if γ satisfies (3.48) for all Ω, then this holds for all $\Omega_{\frac{\partial}{\partial x^i}}$. Accordingly,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right) = \frac{\partial L}{\partial x^k} \quad k = 1, 2, \ldots, n$$

along γ. This proves the first implication in the theorem above, and we turn our attention to the converse implication:

Without loss of generality, we can look at the situation in a small neighborhood of $t = 0$. We now assume that γ satisfies the Euler-Lagrange equations, and thus that the Euler-Lagrange equations are satisfied in any coordinate system. Let Ω be any given 1-form on M. If $\Omega|_{\gamma(0)} = 0$, the equations are trivially satisfied,and if $\Omega|_{\gamma(0)} \neq 0$, it is possible to choose a small coordinate domain $(x^1, \ldots, x^n)$ around $\gamma(0)$ such that $\Omega = \Omega_{\frac{\partial}{\partial x^1}}$. By the above formulae for Ω and $\{\Omega, L\}$ together with the assumption that the Euler-Lagrange equations hold for γ, we see that (3.48) holds for every motion $\gamma(t)$ of the Lagrange system. ■

In this way, we see that there is a formal similarity between Poincaré's equations for Lagrangian systems and Poisson's equations for Hamiltonian system, and that these equations are equivalent when the following conditions are satisfied:

- The Lagrange function is of the form $L = \frac{1}{2}K + U$, where K is a Riemannian metric and U is a smooth function on M.
- We restrict the application of Poisson's equations to functions which are linear in the generalized momenta.

Under these conditions, we see that the structure coefficients in Poincaré's equations can be regarded as a manifestation of the symplectic structure on the cotangent bundle.

3.9 Umbilic shape invariant motion

3.9.1 Introduction

Here we will study the following particular type of shape invariant three body motions:

Definition 3.9.1. The *umbilic shape* is the shape of a configuration satisfying one of the following equivalent conditions:

(i) The gyration-radii r_1, r_2 are of the same magnitude.

(ii) The moments of inertia λ_1, λ_2 are equal, i.e.

$$\lambda_1 = \lambda_2 = \frac{\rho^2}{2}, \quad \lambda_3 = \rho^2$$

(iii) $\mathbb{u}_1, \mathbb{u}_2$ can be chosen freely among the orthonormal bases of the linear space spanned by the configurations.

(iv) $\varphi = 0 \mod \pi$.

Note that umbilic configurations can not be collinear, since collinearity is characterized by $\varphi = \frac{\pi}{2} \mod \pi$. Hence, if a three body configuration is not collinear and not regular, then it is umbilic.

Our main reason to study this class of motions, is that they can not be taken care of with our reduced equations (3.35), because of the coordinate singularity at $\varphi = 0 \mod \pi$.

The conclusion of this investigation is the following: The class of umbilic shape-invariant three body motions consists of motions of the following type:

(i) Planar motions of three equal masses forming the shape of an equilateral triangle, namely Lagrange's solution [Lag72].

(ii) Motions where the power of the potential function $e = 2$ and the total angular momentum Ω is contained in the plane spanned by the three body configuration.

Remark: It is not completely clear whether or not the class of three body motions of type (ii) is empty, and this is formulated as an open problem.

3.9.2 Preliminary investigations

The umbilic shape

We recall the following facts from Section 3.4: Let r_{jk} denote the distance between particle j and particle k, and let r_i denote the length of the position vector $\mathbb{a}_i$ relative to the centre of mass, and let β_i denote the angle between $\mathbb{a}_j$ and $\mathbb{a}_k$.

Lemma 3.5.1 tells us that

$$r_{jk}^2 = \frac{\rho^2}{2} \frac{m_j + m_k}{m_j m_k}, \tag{3.49}$$

while (3.23) yields

$$r_i^2 = \frac{\rho^2}{2}\frac{m_j + m_k}{m_i M}, \tag{3.50}$$

where $M = m_1 + m_2 + m_3$. Finally (3.24) yields

$$\cos\beta_i = -\sqrt{\frac{m_j m_k}{(M - m_j)(M - m_k)}}. \tag{3.51}$$

With these formulae, we have a very good understanding of how the umbilic shape depends on the mass distribution.

The Lagrange-Jacobi-equation

According to Proposition 3.7.1, the $\ddot{\rho}$-equation in (3.35) is valid also for the umbilic shape. Here, we will write this equation as

$$\ddot{\rho} = -\frac{\dot{\rho}^2}{\rho} + \frac{1}{\rho}\left(\frac{2-e}{\rho^e}u^* + 2h\right), \tag{3.52}$$

where h is total energy and u^* is the value of the potential function U on the umbilic configuration with $\rho = 1$.

Since $\varphi = 0 \mod \pi$, the energy integral reads

$$h = \frac{1}{2}\dot{\rho}^2 + \frac{1}{2}\left(\frac{g_1^2}{\lambda_1} + \frac{g_2^2}{\lambda_2} + \frac{g_3^2}{\rho^2}\right) - \frac{u^*}{\rho^e} = \frac{1}{2}\dot{\rho}^2 + \frac{1}{2}\left(\frac{2g_1^2}{\rho^2} + \frac{2g_2^2}{\rho^2} + \frac{g_3^2}{\rho^2}\right) - \frac{u^*}{\rho^e}.$$

We can use this to eliminate $\dot{\rho}$ from (3.52). This yields the following useful form of the Lagrange-Jacobi equation:

$$\ddot{\rho} = \frac{2g_1^2}{\rho^3} + \frac{2g_2^2}{\rho^3} + \frac{g_3^2}{\rho^3} - \frac{e\,u^*}{\rho^{e+1}} \tag{3.53}$$

The Euler equations and the non-planar case

The deduction of the Euler equations (cf. Theorem 2.7.4) is valid for every non-collinear three body motion and hence, we conclude that the Euler equations are valid also for the umbilic shape. For umbilic motions the Euler equations assumes the following form:

$$\dot{g}_1 = -g_2\left(\frac{g_3}{\rho^2} - \frac{\dot{\theta}}{2}\right), \quad \dot{g}_2 = g_1\left(\frac{g_3}{\rho^2} - \frac{\dot{\theta}}{2}\right), \quad \dot{g}_3 = 0. \tag{3.54}$$

Note that the application of the Euler equations presupposes a choice of singular value decomposition of the motion, and that the term $\dot{\theta}$ depends on such a choice.

As stated in Definition 3.9.1 (iii) above, the vectors $\mathbb{u}_1, \mathbb{u}_2$ can be rotated freely in the variable plane $\Pi(t)$ spanned by the three body configuration. Hence, we can adopt the following convention:

Convention 3.9.2. For umbilic motions, we choose the first principal axes vector $\mathbb{u}_1$ to be perpendicular to the total angular momentum Ω, i.e. $g_1 = 0$

Since $g_1^2 + g_2^2 + g_3^2 = \|\Omega\|$, we conclude that

$$g_2^2 = \|\Omega\|^2 - g_3^2$$

Accordingly g_1, g_2, g_3 are constant under this convention, and $\dot{\theta}$ is thus determined by the Euler equations. This yields the following lemma:

Lemma 3.9.3. *For a non-planar umbilic shape-invariant three body motion,*

$$\dot{\theta} = \frac{2g_3}{\rho^2}.$$

Without loss of generality, we may align the total angular momentum Ω with the positive z-axis, and hence write $\Omega = \|\Omega\|\mathbb{k}$, where $\mathbb{k}$ is the unit vector in z-direction. Using this convention, $\mathbb{u}_1$ is in the x, y-plane, and hence we can parametrize the motion of $\mathbb{u}_1$ by an angle $\gamma(t)$ such that

$$\mathbb{u}_1 = \begin{pmatrix} \cos\gamma \\ \sin\gamma \\ 0 \end{pmatrix}$$

γ is called *the precession angle*. The other principal axes vectors can be written as

$$\mathbb{u}_2 = \begin{pmatrix} -\cos\beta\sin\gamma \\ \cos\beta\cos\gamma \\ \sin\beta \end{pmatrix} \qquad \mathbb{u}_3 = \begin{pmatrix} \sin\beta\sin\gamma \\ -\sin\beta\cos\gamma \\ \cos\beta \end{pmatrix}.$$

Here we recognize the Euler angles (α, β, γ) given in Section 3.7.2, with $\alpha = 0$. The constraint $\alpha = 0$ is a result of our convention $g_1 = 0$.

The angle β, which is called the *inclination angle*, is the angle between the normal vector $\mathbb{u}_3$ and the total angular momentum vector Ω. It follows from $\dot{g}_3 = 0$ that β is constant.

The rate of change of the precession angle γ is found by (3.31), which yields

$$\dot{\gamma} = \frac{2g_2}{\rho^2 \sin\beta} = \frac{2\|\Omega\| \sin\beta}{\rho^2 \sin\beta} = \frac{2\|\Omega\|}{\rho^2}. \tag{3.55}$$

Adaption to the planar case

The above equations are valid also in the planar case. In this case, we take

$$\mathbb{u}_1 = \begin{bmatrix} \cos\gamma \\ \sin\gamma \\ 0 \end{bmatrix}, \quad \mathbb{u}_2 = \begin{bmatrix} -\sin\gamma \\ \cos\gamma \\ 0 \end{bmatrix}, \quad \mathbb{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We may and shall assume that $\dot{\gamma}$ satisfies (3.55). The purpose of this seemingly odd convention is to unify the treatment of the planar and the non-planar case. With this convention, Lemma 3.9.3 is still valid. This can be seen from (3.27), if we note that in the planar case, the precession angle γ is identical to the variable α in our reduction of the three body problem in the planar case (cf. (3.26)).

3.9.3 Particle kinematics

We will use the singular value decomposition to investigate the motion of the Jacobi vectors $\mathbb{x}_1, \mathbb{x}_2$. Let $P = [\mathbb{u}_1|\mathbb{u}_2|\mathbb{u}_3]$ and $X = [\mathbb{x}_1, \mathbb{x}_2]$. The gyration-radii are $(\pm\frac{1}{2}\sqrt{2})\rho$, and hence, the Jacobi vectors are given by

$$X = \left[\mathbb{x}_1|\mathbb{x}_2\right] = \rho\left[\mathbb{u}_1|\mathbb{u}_2|\mathbb{u}_3\right] \begin{bmatrix} \pm\frac{1}{2}\sqrt{2} & & \\ & \pm\frac{1}{2}\sqrt{2} & \\ & & 0 \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \\ 0 & 0 \end{bmatrix},$$

i.e.

$$\begin{aligned} \mathbb{x}_1 &= \pm\frac{1}{2}\sqrt{2}\rho(\mathbb{u}_1 \cos\frac{\theta}{2} - \mathbb{u}_2 \sin\frac{\theta}{2}) \\ \mathbb{x}_2 &= \pm\frac{1}{2}\sqrt{2}\rho(\mathbb{u}_1 \sin\frac{\theta}{2} + \mathbb{u}_2 \cos\frac{\theta}{2}) \end{aligned}$$

This implies that $\sqrt{2}\mathbb{x}_i/\rho$ are vectors of unit length rotating in the plane spanned by $\mathbb{u}_1, \mathbb{u}_2$ with angular velocity $-\frac{\dot{\theta}}{2}$ relative to $(\mathbb{u}_1, \mathbb{u}_2)$.

On the other hand, the position vectors $\mathbb{a}_1, \mathbb{a}_2, \mathbb{a}_3$ relative to the centre of mass are, as usual, given by fixed linear combinations of the $\mathbb{x}_i$. Hence, we conclude the following:

Lemma 3.9.4. *The normalized position vectors $\mathbb{a}_i(t)/\rho$ rotate with angular velocity $-\frac{\dot{\theta}}{2}$ relative to the frame $(\mathbb{u}_1, \mathbb{u}_2)$.*

Since the motion is shape invariant and takes place in the plane $\Pi(t) = \mathrm{span}(\mathbb{u}_1, \mathbb{u}_2)$, there exist real numbers r_i, α_i^0 such that the position vector of particle i is given by

$$\mathbb{a}_i = \rho r_i \left[\cos(\alpha_i)\mathbb{u}_1 + \sin(\alpha_i)\mathbb{u}_2\right], \quad \alpha_i = \alpha + \alpha_i^0, \tag{3.56}$$

where $\alpha = \alpha(t)$ depends on time t. Note that all the $r_i \neq 0$, since the configuration is not collinear. Lemma 3.9.3 and 3.9.4 give the rate of change of the rotation angle α:

Lemma 3.9.5. *For an umbilic shape invariant three body motion with constant angular momentum $\Omega \neq 0$,*

$$\dot{\alpha} = -\frac{g_3}{\rho^2}, \tag{3.57}$$

where α is defined in terms of the position vectors and the principal frame by (3.56)*, and the principal frame follows the above conventions, i.e. that $g_1 = 0$ in the non-planar case, and that $\dot{\gamma}$ follows* (3.55) *in the planar case.*

The angles α_i of rotation

Using (3.57), we have the following integral formula for the rotational angle:

$$\alpha_i(s) = \alpha_i(0) - g_3 \int_0^s \frac{dt}{\rho^2}. \tag{3.58}$$

Hence, for a three body motion with an infinite interval of existence, we have the following: If $g_3 \neq 0$ and there exist a real number M such that $\rho(t) \in (0, M)$ for all $t \in \mathbb{R}$, then α_i will be unbounded. Regarded as an angle, α_i will hence experience an infinite number of revolutions.

The kinematic acceleration

Differentiation of (3.56) and application of Lemma 3.9.5 and equations (3.53) and (3.55) yield

$$\begin{aligned}\ddot{\mathbb{a}}_i = &-\frac{2g_2^2}{\rho^4}\left(\rho r_i(\cos\alpha_i \mathbb{u}_1 - \sin\alpha_i \mathbb{u}_2)\right)\\ &-\frac{eu^*}{\rho^{e+2}}(\rho r_i(\cos\alpha_i \mathbb{u}_1 + \sin\alpha_i \mathbb{u}_2))\end{aligned} \tag{3.59}$$

We will write this as

$$\ddot{\mathbb{a}}_i = -\frac{eu^*}{\rho^{e+2}}\mathbb{a}_i - \frac{2g_2^2}{\rho^4}\mathbb{a}_i^*$$

where

$$\mathbb{a}_i^* = \rho r_i\left[\cos\alpha_i \mathbb{u}_1 - \sin\alpha_i \mathbb{u}_2\right],$$

i.e. the reflection of $\mathbb{a}_i$ in the $\mathbb{u}_1$-axis.

The dynamical acceleration

Here we will analyse the gravitational forces acting on each body, in order to determine the acceleration due to the dynamics. Note that we assume that the centre of mass is at the origin, i.e. $\Sigma_i m_i \mathbb{a}_i = 0$.

Let $\{i, j, k\} = \{1, 2, 3\}$, and let F_i denote the force on particle i coming from the interaction with particle j and particle k. Then the dynamic acceleration is

$$\frac{F_i}{m_i} = \frac{em_j}{r_{ij}^{e+2}}(\mathbb{a}_j - \mathbb{a}_i) + \frac{em_k}{r_{ik}^{e+2}}(\mathbb{a}_k - \mathbb{a}_i),$$

where e is the power of the potential, and r_{ik}, r_{ij} are inter-particle distances. Using $m_k\mathbb{a}_k = -m_i\mathbb{a}_i - m_j\mathbb{a}_j$, we see that

$$\frac{\rho^{e+2}F_i}{m_i} = A_{ij}\mathbb{a}_i + B_{ij}\mathbb{a}_j \tag{3.60}$$

where

$$A_{ij} = -e\left(\frac{m_j}{d_{ij}^{e+2}} + \frac{m_j + m_k}{d_{ik}^{e+2}}\right) \quad B_{ij} = em_j\left(\frac{1}{d_{ij}^{e+2}} - \frac{1}{d_{ik}^{e+2}}\right)$$

Here, we have introduced normalized particle distances $d_{ij} = \rho^{-1} r_{ij}$. Following (3.49),

$$d_{ij}^2 = \frac{m_i + m_j}{2m_i m_j},$$

and accordingly,

$$B_{ij} = 0 \iff m_j = m_k. \tag{3.61}$$

3.9.4 The force balance

The balance between the dynamical acceleration F_i/m_i and the kinematic acceleration $\ddot{\mathbb{a}}_i$, i.e.

$$\frac{\rho^{e+2} F_i}{m_i} = \rho^{e+2} \ddot{\mathbb{a}}_i,$$

yields

$$A_{ij}\mathbb{b}_i + B_{ij}\mathbb{b}_j = -eu^*\mathbb{b}_i - 2\rho^{e-2} g_2^2 \mathbb{b}_i^*, \tag{3.62}$$

where

$$\mathbb{b}_i = \frac{\mathbb{a}_i}{\rho} \quad \text{and} \quad \mathbb{b}_i^* = \frac{\mathbb{a}_i^*}{\rho}, \quad i = 1,2,3.$$

$A_{ij}, B_{ij}, e, u^*, g_2^2$ are constant coefficients, and the $\mathbb{b}_i$, $\mathbb{b}_i^*$ are of fixed length. Since umbilic configurations can not be collinear, $\mathbb{b}_j$ and $\mathbb{b}_i$ must be linearly independent.

Application to the planar case

Following Lemma 2.8.1 and the assumption that $g_1 = 0$, we see that the motion is planar if and only if $g_2 = 0$. From the linear independence of $\mathbb{b}_i, \mathbb{b}_j$ and (3.61) we conclude that $m_j = m_k$ for all $j, k = 1,2,3$. Hence, planar motion occurs only in the case of three equal masses, in which case the umbilic shape is identical to the equilateral shape.

For later reference we summarize this as follows:

Lemma 3.9.6. *Let $X(t)$ be a planar umbilic shape invariant three body motion for the three body problem with mass distribution m_1, m_2, m_3. Then* *(i)* $m_1 = m_2 = m_3$

(ii) *The triangle formed by the configuration* $X(t)$ *is equilateral for all* t.

Hence, in this case we encounter Lagrange's solution, and the motion is determined by the following:

(i) The Lagrange-Jacobi equation (3.53), which now assumes the form

$$\ddot{\rho} = \frac{\Omega_s^2}{\rho^3} - \frac{2u^*}{\rho^{e+1}},$$

where $\Omega_s = g_3$ is the scalar angular momentum.

(ii) Conservation of total angular momentum.

The non-planar case

Recall that under the present conventions, planarity is equivalent to the condition $g_2 = 0$. Hence, throughout the following study of the planar case, we will assume that $g_2 \neq 0$.

If $\rho(t)$ is bounded and non-zero for all real t and $g_3 \neq 0$, then α_i will be unbounded, and hence we can find an instance of time where $\mathbb{b}_i = \mathbb{b}_i^*$. In the following, we will see that this leads to contradiction: Since $\mathbb{b}_i, \mathbb{b}_j$ are always linearly independent, $\mathbb{b}_i = \mathbb{b}_i^*$ implies that $B_{ij} = 0$, i.e. that $m_j = m_k$. Application of this argument to $\mathbb{a}_j, \mathbb{a}_k$ implies that $m_1 = m_2 = m_3$. Hence, we encounter the case of three identical masses, and consequently the equilateral triangle. In this case all $B_{ij} = 0$, and hence the force balance assumes the form

$$\frac{\rho^{e+2} F_i}{m_i} = A_{ij}\mathbb{a}_i, \quad i, j = 1, 2, 3.$$

This implies that $\mathbb{a}_i = \pm\mathbb{a}_i^*$ for $i = 1, 2, 3$ and for all t. We conclude that each of the tree particles lies on the lines spanned by $\mathbb{u}_1$ and $\mathbb{u}_2$. Since the centre of mass coincides with the origin, this can not be the case unless the configuration is collinear. Since collinear configurations are not umbilic, the assumed behaviour of ρ can not occur. Accordingly, we can assume either that $\rho \to 0$ in finite time or that $\sup_t \rho(t) = \infty$.

If ρ^{e-2} is not bounded away from 0, we see that we can make the $\mathbb{b}_i^*$-term in the force balance as small as we want by consideration of instances of time sufficiently close to the boundary of the maximal interval of existence of the motion, and since $\mathbb{b}_i, \mathbb{b}_j$ are linearly independent, we again conclude that $B_{ij} = 0$ for $i, j = 1, 2, 3$, and hence that $m_1 = m_2 = m_3$. Again, this contradicts the assumption of umbilic shape.

If ρ^{e-2} is not bounded away from ∞, we conclude that $g_2^2 = 0$. This implies planar motion.

Following this discussion, the remaining possibilities for umbilic shape invariant non-planar motions are the following:

(i) The case $g_3 = 0$

(ii) The case where $e = 2$ and $g_3 \neq 0$.

The case $e = 2$, $g_3 \neq 0$.

In the case where $e = 2$, the force balance reads

$$A_{ij}\mathbb{b}_i + B_{ij}\mathbb{b}_j = -eu^*\mathbb{b}_i - 2g_2\mathbb{b}_i^*,$$

and by taking the inner product of this equation with $\mathbb{k}$, and dividing by $\sin\beta$, which is nonzero since $g_3 \neq 0$, we get

$$A_{ij}r_i \sin\alpha_i + B_{ij}r_j \sin\alpha_j = -eu^* r_i \sin\alpha_i + 2g_2 r_i \sin\alpha_i.$$

Since we assumed $g_3 \neq 0$, $\dot{\alpha} \neq 0$, and as a consequence of this, $\sin(\alpha_i), \sin(\alpha_j)$ are linearly independent as functions of t. Accordingly, the constant coefficient $B_{ij} = 0$. Again this contradicts the umbilicity assumption. Hence, this case can not occur.

The case $g_3 = 0$, $e \neq 2$

Here, we can assume that $\Omega \neq 0$, since the opposite assumption implies planarity (cf. Corollary 2.8.3). From the convention $g_1 = 0$ we see that this case is characterized by

$$\Omega = g_2 \mathbb{u}_2.$$

In other words, we can take the angle of inclination β to be $\frac{\pi}{2}$, and the principal frame now reads:

$$\mathbb{u}_1 = \begin{bmatrix} \cos\gamma \\ \sin\gamma \\ 0 \end{bmatrix}, \quad \mathbb{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbb{u}_3 = \begin{bmatrix} \sin\gamma \\ -\cos\gamma \\ 0 \end{bmatrix}.$$

From $\dot{\alpha} = -g_2/\rho^2 = 0$ we conclude that the normalized position vectors

$$\mathbb{b}_i = \mathbb{a}_i/\rho, \quad \mathbb{b}_i^* = \mathbb{a}_i/\rho \tag{3.63}$$

are constant, and the force balance (3.62) can be written as

$$\rho^{e-2}\mathbb{b}_i^* = \frac{1}{2g_2^2}\left(eu^*\mathbb{b}_i - A_{ij}\mathbb{b}_i - B_{ij}\mathbb{b}_j\right).$$

Hence $\rho^{(e-2)}$ must be constant, and under the assumption $e \neq 2$, we can conclude that ρ is constant.

When ρ is constant, the Lagrange-Jacobi equation (3.53) yields

$$\frac{2g_2^2}{\rho^2} = \frac{eu^*}{\rho^e},$$

and following (3.59) we see that

$$\ddot{\mathbb{a}}_i = -\frac{4g_2^2}{\rho^3} r_i \cos\alpha_i \mathbb{u}_1,$$

i.e. that the kinematic accelerations are horizontal. Such horizontal accelerations can be balanced by the dynamical accelerations if and only if the three particles all lie on the line spanned by $\mathbb{u}_1$, i.e. in the case of collinear motion. This contradicts the umbilicity assumption, and hence this case can not occur.

The case $g_3 = 0$, $e = 2$

In this case, the variability of ρ can be eliminated from the force balance. Using (3.63), we write the force balance of particle i as

$$(B_i): \quad A_{ij}\mathbb{b}_i + B_{ij}\mathbb{b}_j = -2u^*\mathbb{b}_i - 2g_2^2\mathbb{b}_i^*,$$

and the question is now if all the equations B_1, B_2, B_3 can be satisfied at the same time. Since the centre of mass coincides with the origin, the three equations B_1, B_2, B_3 are dependent: The linear combination $m_1B_1 + m_2B_2 + m_3B_3$ yields the trivial equation $0 = 0$. Hence it is sufficient to investigate whether or not two of the equations can be satisfied.

Now we ask the following question: Which parameters can we adjust in order to satisfy B_1, B_2, B_3?

(i) g_2 can in principle take any real value.

(ii) The lengths and relative angles of the normalized position vectors $\mathbb{b}_i$ are determined respectively by (3.50) and (3.51). The particles can however freely undergo a collective rotation in the $\mathbb{u}_1, \mathbb{u}_2$-plane. Hence, we can regard the equations B_1, B_2, B_3 as a system of rank 4 over $\mathbb{R}$ with two unknowns, namely g_2 and the angular orientation of the normalized position vectors.

Let us define the following basis for the plane spanned by $\mathbb{u}_1, \mathbb{u}_2$:

$$\mathscr{B}_i = (\mathbb{v}_i, \mathbb{v}_2) = (\cos\alpha_i\mathbb{u}_1 + \sin\alpha_i\mathbb{u}_2, -\sin\alpha_i\mathbb{u}_1 + \cos\alpha_i\mathbb{u}_2).$$

In this basis, $\mathbb{b}_j$ has the coordinate vector

$$[\mathbb{b}_j]_{\mathscr{B}_i} = \begin{bmatrix} r_j\cos\beta_k \\ r_j\sin\beta_k \end{bmatrix}$$

where $\beta_k = \alpha_j - \alpha_i$, when (i, j, k) is a cyclic permutation of $(1,2,3)$. Note that the angles β_k are given by (3.51). The coordinate vectors of $\mathbb{b}_i, \mathbb{b}_i^*$ are

$$[\mathbb{b}_i]_{\mathscr{B}_i} = \begin{bmatrix} r_i \\ 0 \end{bmatrix} \qquad [\mathbb{b}_i^*]_{\mathscr{B}_i} = \begin{bmatrix} r_i\cos 2\alpha_i \\ -r_i\sin 2\alpha_i \end{bmatrix}$$

In the basis $\mathscr{B}_i$, the components of the force balance (B_i) are

$$\begin{aligned} A_{ij}r_i + B_{ij}r_j\cos\beta_k + 2u^*r_i &= -2g_2^2 r_i\cos 2\alpha_i \\ B_{ij}r_j\sin\beta_k &= 2g_2^2 r_i\sin 2\alpha_i \end{aligned} \tag{3.64}$$

In the case $m_1 = m_2 = m_3$, every $B_{ij} = 0$, and since $g_2, r_i \neq 0$, we conclude that $\sin 2\alpha_i = 0$, i.e. that each $\alpha_i = 0 \mod \frac{\pi}{2}$. Since the centre of mass is at the origin, this must imply collinearity, and hence this contradicts umbilicity. We conclude that in the case of three equal masses, the problem does not have a solution.

In the case where the mass distribution is uneven, we can proceed as follows:

Since this problem is invariant under change of the sign of g_2, we can assume that $\alpha_1 \in (-\pi/2, \pi/2)$. Hence, we can work with the variable $x = \sin\alpha_1$ instead of α_1. Since $\alpha_2 = \alpha_1 + \beta_3$ and $\alpha_3 = \alpha_1 - \beta_2$, we have

$$\sin\alpha_2 = x\cos\beta_3 + \sqrt{1-x^2}\sin\beta_3 \quad \text{and} \quad \sin\alpha_3 = x\cos\beta_2 - \sqrt{1-x^2}\sin\beta_2$$

In this way, we can eliminate the angles α_i from the force balance (3.64), and acquire a complicated set of non-linear equations in the unknowns $x = \sin\alpha_1$ and $y = g_2^2$, depending on the mass distribution m_1, m_2, m_3 through the relations (3.49), (3.50), (3.51). This leads to the following open problem:

Question 3.9.7. For which mass distributions m_1, m_2, m_3 and which

$$(x, y) \in (-1, 1) \times (0, \infty)$$

are the equations (3.64) satisfied?

In this formulation, there are 5 parameters. There is however a scaling symmetry among the parameters: If we scale the mass distribution with the factor λ, (3.64) is affected in the following way: A_{ij}, B_{ij}, u^* are scaled with the factor λ^3, the r_i are scaled with the factor $1/\sqrt{\lambda}$, while $y = g_2^2$ is scaled with a factor λ^3. Hence, this problem is invariant under rescaling of the mass distribution. Accordingly, we can reduce the problem to the four variables

$$\frac{m_2}{m_1}, \frac{m_3}{m_1}, x, y.$$

As noted above, we can eliminate one of the force balance equations by the relation $m_1 B_1 + m_2 B_2 + m_3 B_3 = 0$. Hence our problem can be formulated as four non-linear equations in four unknowns. A more detailed investigation of this question could not be included here because of time constraints.

In order to simplify references to this problem, we add the following definition:

Definition 3.9.8 (Exceptional three body motions)**.** An umbilic shape invariant three body motion $X(t)$ with non-zero angular momentum Ω such that Ω and is always contained in the variable plane spanned by the three particles is called an *exceptional three body motion.*

According to the above discussion, we do not know whether or not any exceptional three body motions exist. The only thing that we know is the following:

Proposition 3.9.9. *(i) Exceptional three body motions can occur only in the case $e = 2$.*

(ii) Exceptional three body motions do not occur in the case of three equal masses

3.9.5 Conclusions

After this investigation, this is our conclusion:

Theorem 3.9.10 (Umbilic shape-invariant motion)**.** *Assume that*

$$X(t) = (\mathbb{a}_1, \mathbb{a}_2, \mathbb{a}_3)$$

is an umbilic shape-invariant motion of the three body problem with potential function

$$U = \sum \frac{m_i m_j}{r_{ij}^e} \quad (e > 0).$$

If the motion is not exceptional in the sense of Definition 3.9.8 we have the following:

(i) The mass distribution is even, i.e. $m_1 = m_2 = m_3$

(ii) *The configuration is equilateral.*

(iii) *The motion is planar.*

(iv) *The angular velocity*

$$\dot{\alpha} = \pm \frac{\|\Omega\|}{\rho^2}.$$

(v) *The hyper-radius*

$$\rho = \sqrt{\sum_i m_i \mathbb{a}_i^2}$$

is determined by

$$\ddot{\rho} = -\frac{\dot{\rho}^2}{\rho} - \frac{1}{\rho}\left(\frac{2-e}{\rho^e} u^* + 2h\right),$$

where h is the total energy and u^* is the constant $\rho^e U$

Since the motion is shape-invariant, the motion is determined by the angular velocity and the hyper-radius ρ. Hence, to determine such a motion, we first find $\rho(t)$ by (v), and then $\alpha(t)$ by quadrature of (iv). Hence, the only non-exceptional umbilic shape invariant three body motion is Lagrange's equilateral solution [Lag72] in the case of three equal masses.

4 Shape spaces

4.1 Introduction

In this chapter we will investigate various spaces of three body shapes. The shape spaces vary both in size and structure, depending on which aspects of the three body problem we want to emphasise.

First we will discuss various representations of three body shapes, as well as the transitions between different representations. As we shall see, this is intimately related to the Jacobi groupoid (cf. Definition 2.3.6). We will use this flexibility to point out how the geometry of the hyperbolic plane can be given a natural place in the study of the three body problem. Later, we introduce the kinematic geometry of three body shapes, and demonstrate how this can be used to give a differential geometric description of the reduced equations of motion of the three body problem. In the planar case, this allows us to eliminate the umbilic shape singularity of the reduction. Finally, we show how the space of three body shapes yields a good background for discussing regularization of binary collisions in the three body problem.

4.2 Representations of m-triangle shapes

4.2.1 The obvious representation of triangular shapes

Congruence classes of m-triangles can be parametrized by the relative distances r_{23}, r_{31}, r_{12} subject to the conditions

$$0 \le r_{jk} \le r_{ki} + r_{ij} \quad \{i, j, k\} = \{1,2,3\}. \tag{4.1}$$

When we exclude the trivial triangle $r_{23} = r_{31} = r_{12} = 0$, each similarity class can be represented by an m-triangle satisfying

$$\frac{m_2 m_3 r_{23}^2 + m_3 m_1 r_{31}^2 + m_1 m_2 r_{12}^2}{m_1 + m_2 + m_3} = 1 \tag{4.2}$$

(cf. (3.21)). Together, (4.1) and (4.2) single out an ellipsoidal triangle $\Delta \subset \mathbb{R}^3$ with non-trivial angles at each vertex. The edges of Δ correspond to configurations where $r_{jk} = r_{ki} + r_{ij}$, i.e. collinear configurations, while the vertices corresponds to configurations where $r_{jk} = 0$, i.e. binary collisions. We summarize this as follows:

Proposition 4.2.1. *The set of triangle shapes can be identified with an ellipsoidal triangle $\Delta \subset \mathbb{R}^3$ together with an isolated point representing the triple collision.*

As a subspace of $\mathbb{R}^3$, the triangle Δ inherits notions of analytic curves and smooth curves, notions which in general do not coincide with the notion of smooth and analytic motions of the three body problem. We note however that the analytic structure on the boundary of Δ fits the analytic structure of the three body problem quite well: An analytic curve on the boundary of the triangle must be contained in one edge, say $r_{23} = r_{31} + r_{12}$ for all t. Hence, the binary collision $r_1 = 0$ is excluded for a collinear motion where P_1 initially lies between P_2, P_3. This fits very well with Sundman's regularization of the three body problem (in the case $e = 1$), where the binary collisions are regularized by means of elastic collisions. This implies that the mass points will not pass through each other, and accordingly, purely collinear motions have precisely one unreachable binary collision, namely the collision between the two mass points that are separated by the third mass. Hence, the induced analytic structure of $\Delta \subset \mathbb{R}^3$ fits very well with Sundman's regularization.

4.2.2 The shape sphere

The space M^o of *oriented three body configurations* which is introduced in [HS07] contains tuples $(\mathbb{x}_1, \mathbb{x}_2, \mathbb{n})$, where $\mathbb{x}_1, \mathbb{x}_2$ are Jacobi vectors and $\mathbb{n}$ is a unit vector perpendicular to $\operatorname{span}(\mathbb{x}_1, \mathbb{x}_2)$. We can hence say that an oriented m-triangle has positive or negative orientation according to the sign of

$$\det(\mathbb{x}_1, \mathbb{x}_2, \mathbb{n}).$$

This determinant is 0 for collinear configurations and for the trivial configuration, i.e. the triple collision.

We will also consider oriented three body positions, namely tuples

$$(\mathbb{a}_1, \mathbb{a}_2, \mathbb{a}_3, \mathbb{n}) \quad \text{where} \quad \mathbb{n} \perp \{\mathbb{a}_3 - \mathbb{a}_2, \mathbb{a}_3 - \mathbb{a}_1\},$$

and we will extend Jacobi maps to act on oriented positions: If we consider a Jacobi map J and an oriented three body position $(\mathbb{a}_1, \mathbb{a}_2, \mathbb{a}_3, \mathbb{n})$, and $J(\mathbb{a}_1, \mathbb{a}_2, \mathbb{a}_3) = [\mathbb{x}_1 | \mathbb{x}_2]$, then we define

$$J(\mathbb{a}_1, \mathbb{a}_2, \mathbb{a}_3, \mathbb{n}) = (\mathbb{x}_1, \mathbb{x}_2, \mathbb{n}).$$

Since $\{\mathbb{a}_3 - \mathbb{a}_2, \mathbb{a}_3 - \mathbb{a}_1\}$ and $\{\mathbb{x}_1, \mathbb{x}_2\}$ span precisely the same subspace of $\mathbb{R}^3$, this extension is unproblematic.

Note however that for different Jacobi maps J, J' two oriented three body configurations $(\mathbb{x}_1, \mathbb{x}_2, \mathbb{n})$ and $(\mathbb{x}_1', \mathbb{x}_2', \mathbb{n})$ associated with the same oriented position $(\mathbb{a}_1, \mathbb{a}_2, \mathbb{a}_3, \mathbb{n})$ can have opposite orientation, but if the Jacobi transformation $J \to J'$ is represented by a matrix of positive determinant, the orientations will be the same.

Choosing $\mathbb{n}$ to be the third principal axes vector $\mathbb{u}_3$ associated with a singular value decomposition yields a factorization

$$S \to M^o \to M$$

of the multiplication map $\Phi \colon S \to M$ associated with the singular value decomposition. In this respect, our application of the singular value decomposition can be regarded as an extension of the space of oriented configurations in [HS07].

In order to gain flexibility, we will temporarily use some more redundant information: We will represent a three body configuration by a matrix $[\mathbb{v}_1, \mathbb{v}_2, \mathbb{v}_3] \in$ SO(3) together with two complex numbers ξ_1, ξ_2 such that the corresponding Jacobi vectors are

$$\mathbb{x}_i = \mathrm{re}(\xi_i)\mathbb{v}_1 + \mathrm{im}(\xi_i)\mathbb{v}_2. \tag{4.3}$$

In this way, we represent three body configurations by elements of $\mathrm{SO}(3) \times \mathbb{C}^2$. We will call ξ_1, ξ_2 *complex Jacobi vectors* with respect to the frame $\mathbb{v}_1, \mathbb{v}_2, \mathbb{v}_3$.

The choice $\mathbb{v}_i = \mathbb{u}_i$, where $\mathbb{u}_i$ belongs to a principal axes frame gives a mapping $S \to \mathrm{SO}(3) \times \mathbb{C}^2$ and a commutative diagram

$$\begin{tikzcd}
& \mathrm{SO}(3) \times \mathbb{C}^2 \arrow[dr, "\text{onto}"] & \\
S \arrow[ur, "\text{1-1}"] \arrow[r] & M^o \arrow[r] & M
\end{tikzcd}.$$

In [HS07] it is noted that the projection $(\mathbb{x}_1, \mathbb{x}_2, \mathbb{n}) \mapsto \mathbb{n}$,

$$\pi_3 \colon M^0 \to S^2 \tag{4.4}$$

is a vector bundle of rank 4. Under the restriction to the case where $\mathbb{v}_i = \mathbb{u}_i$ (the principal axes vectors), the present introduction of the complex numbers ξ_1, ξ_2 yields local trivialisations of that bundle. This trivialization depends on $\mathbb{u}_1, \mathbb{u}_2$, and hence it can not be interpreted as a global trivializaton of (4.4).

The assignment $(\mathbb{x}_i) \mapsto (\xi_i)$ can be lifted up to the level of three body positions in space, where an oriented three body position $(\mathbb{v}_1, \mathbb{v}_2, \mathbb{v}_3, \mathbb{a}_1, \mathbb{a}_2, \mathbb{a}_3)$ with the centre of mass at the origin is represented by a complex triple η_1, η_2, η_3 in such a way that

$$\mathbb{a}_i = \mathrm{re}(\eta_i)\mathbb{v}_1 + \mathrm{im}(\eta_i)\mathbb{v}_2. \tag{4.5}$$

When $(\mathbb{v}_1, \mathbb{v}_2, \mathbb{v}_3)$ are given, this gives a bijection between $(\mathbb{a}_i)$-data and (η_i)-data.

The complex numbers ξ_1, ξ_2 clearly represents the shape and size of m-triangles properly. This justifies that we leave the vectors $\mathbb{v}_1, \mathbb{v}_2, \mathbb{v}_3$ out of the discussion for a while.

In the complex plane, the group of proper similarity transformations is represented by

$$\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$$

and the space of non-trivial m-triangles modulo oriented similarity transformations is

$$\frac{\mathbb{C}^2 \setminus \{0\}}{\mathbb{C}^\times} = \mathbb{C}P^1 = \mathbb{C}^*,$$

where $\mathbb{C}P^1$ is the complex projective line and $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere. Accordingly, we have the following proposition:

Proposition 4.2.2. *The space of non-trivial oriented three body configurations modulo similarity transformations is diffeomorphic to the sphere S^2.*

This should be compared with Proposition 4.2.1, which tells us that the space of unoriented shapes is an ellipsoidal triangle.

When considering homogeneous coordinates on $\mathbb{C}P^1$, the shape of an oriented m-triangle with complex Jacobi vectors ξ_1, ξ_2 is represented by

$$[\xi_1 : \xi_2] \in \mathbb{C}P^1.$$

When considering the Riemann sphere, the shape of an oriented m-triangle with complex Jacobi vectors ξ_1, ξ_2 is represented by the extended complex number

$$\zeta = \frac{\xi_2}{\xi_1} \in \mathbb{C}^* = \mathbb{C} \cup \infty.$$

In the following we will consider this representation of three body shapes, i.e. regard oriented three body shapes as points on the Riemann sphere.

The space of unoriented shapes

We discussed the space of unoriented shapes in Section 4.2.1 by means of the relative distances r_{23}, r_{31}, r_{12}. But in this section we study unoriented shapes by means of oriented three body configurations $(\mathbb{x}_1, \mathbb{x}_2, \mathbb{n})$, together with the involution "change of orientation"

$$(\mathbb{x}_1, \mathbb{x}_2, \mathbb{n}) \mapsto (\mathbb{x}_1, \mathbb{x}_2, -\mathbb{n})$$

On the level of $(\mathbb{v}_1, \mathbb{v}_2, \mathbb{v}_3, \mathbb{x}_1, \mathbb{x}_2)$-data (cf. (4.3)), we can represent this involution in several ways, but from Lemma 4.2.6 below, we see that the effect on the $\mathbb{v}_1, \mathbb{v}_2$-data are inessential. Hence we can represent the change of orientation by the involution

$$\tau\colon (\mathbb{v}_1, \mathbb{v}_2, \mathbb{v}_3) \mapsto (\mathbb{v}_1', \mathbb{v}_2', \mathbb{v}_3') = (-\mathbb{v}_1, \mathbb{v}_2, -\mathbb{v}_3)$$

On the level of complex Jacobi vectors and the shape-sphere $\mathbb{C}^*$, this yields

$$\begin{aligned} \tau\colon & (\xi_1, \xi_2) \mapsto (\overline{\xi}_1, \overline{\xi}_2) \\ \tau\colon & \zeta \mapsto \overline{\zeta} \end{aligned}$$

Hence, *change of orientation* induces a reflection of the shape-sphere $\mathbb{C}^*$ in the extended real line $\mathbb{R}^* \subset \mathbb{C}^*$, and three body configurations with $\mathrm{im}(\zeta) \neq 0$ are seen to have precisely two shape-sphere representations corresponding to opposite orientations, on opposite hemispheres. From this point of view, the space of unoriented triangular shapes, i.e. oriented triangles modulo change of orientation, is a closed disk:

Proposition 4.2.3. *The space M^* of unoriented triangle shapes is homeomorphic to a closed disk.*

In Section 4.2.1, we identified this space with an ellipsoidal triangle. In the triangle representation and the disc representation, the smooth structures differ on the boundary. This turns out to be a fruitful contradiction, which we will discuss in Section 4.5 in connection with the regularization of binary collisions in the three body problem.

4.2.3 Interpretation of φ and θ as spherical coordinates on the shape sphere.

In our coordinatization of the three body problem, we can regard φ, θ as shape parameters. This raises the following question: For a three body configuration with given values of φ, θ, what is the corresponding point on the shape-sphere $\mathbb{C}^*$?

In the case where $(\mathbb{v}_1, \mathbb{v}_2, \mathbb{v}_3)$ is taken to be the principal frame, such a configuration is represented by the complex Jacobi vectors

$$\xi_1 = (\cos\frac{\varphi}{2} + \sin\frac{\varphi}{2})\cos\frac{\theta}{2} - i(\cos\frac{\varphi}{2} - \sin\frac{\varphi}{2})\sin\frac{\theta}{2}$$
$$\xi_2 = (\cos\frac{\varphi}{2} + \sin\frac{\varphi}{2})\sin\frac{\theta}{2} + i(\cos\frac{\varphi}{2} - \sin\frac{\varphi}{2})\cos\frac{\theta}{2}$$

(cf. Section 3.3 and (4.3)), and the corresponding shape representation is

$$\zeta = \frac{\sin\theta\sin\varphi + i\cos\varphi}{1 + \sin\varphi\cos\theta} \in \mathbb{C}^* \tag{4.6}$$

Consider the unit sphere $S^2 \subset \mathbb{R}^3$, and let us identify the complex plane $\mathbb{C}$ with the yz-plane in such a way that the imaginary axis coincides with the z-axis. Let us consider the stereographic projection of the sphere from the point $(-1,0,0)$ onto the complex plane. From the three-point homogeneity of the conformal sphere [Jah11] we see that this stereographic projection is the unique conformal map of S^2 onto $\mathbb{C}^*$ satisfying the following constraints:

$$\begin{aligned}(-1,0,0) &\mapsto \infty\\ (1,0,0) &\mapsto 0\\ (0,0,1) &\mapsto i.\end{aligned} \tag{4.7}$$

Under the inverse projection $\mathbb{C}^* \to \mathbb{S}^2 \subset \mathbb{R}^3$, the point ζ in (4.6) is mapped to the point

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \sin\varphi\cos\theta \\ \sin\varphi\sin\theta \\ \cos\varphi \end{bmatrix} \in \mathbb{R}^3$$

Accordingly, we have the following interpretation of the shape variables φ, θ:

Proposition 4.2.4. *Under the identification of the shape sphere $\mathbb{C}^*$ with the unit sphere $S^2 \subset \mathbb{R}^3$ given by* (4.7)*, we can regard the shape variables φ, θ as spherical polar coordinates centred at $x = y = 0$, $z = 1$.*

In section Section 3.4, we saw that the shape potential $U^*(\varphi,\theta)$ can be interpreted as a function on the sphere $S^2 \subset \mathbb{R}^3$. The present discussion gives a firm confirmation of this point of view. We will develop this theme further in Section 4.4, and in particular note that the unit-sphere geometry of S^2 can be given a central role in the study of the dynamics of three body problem.

4.2.4 The groupoid of shape transformations

Here we will consider the space of *framed three body positions*,

$$C^F = \{(\mathbb{v}_1, \mathbb{v}_2, \mathbb{v}_3, \mathbb{a}_1, \mathbb{a}_2, \mathbb{a}_3) \colon \mathbb{v}_3 \perp (\mathbb{a}_2 - \mathbb{a}_1), (\mathbb{a}_3 - \mathbb{a}_1)\}$$

as well as the space of *oriented three body positions*,

$$C^O = \{(\mathbb{n}, \mathbb{a}_1, \mathbb{a}_2, \mathbb{a}_3) \colon \mathbb{n} \perp (\mathbb{a}_2 - \mathbb{a}_1), (\mathbb{a}_3 - \mathbb{a}_1)\}.$$

Obviously, the vectors $\mathbb{v}_1, \mathbb{v}_2$ are physically insignificant, and we will see that the shape-sphere representations of three body configurations – as we should expect – factors through the obvious projection $C^F \to C^O$.

Definition 4.2.5. The mapping $C^F \to \mathbb{C}^*$

$$(\mathbb{v}_1, \mathbb{v}_2, \mathbb{v}_3, \mathbb{a}_1, \mathbb{a}_2, \mathbb{a}_3) \mapsto (\mathbb{v}_1, \mathbb{v}_2, \mathbb{v}_3, \mathbb{x}_1, \mathbb{x}_2) \mapsto (\xi_1, \xi_2) \mapsto \zeta = \frac{\xi_2}{\xi_1}$$

associated with a Jacobi map $J\colon (\mathbb{a}_1, \mathbb{a}_2, \mathbb{a}_3) \mapsto (\mathbb{x}_1, \mathbb{x}_2)$ will be called a *shape map* for the three body problem.

We observe that shape maps σ, σ' are related by Jacobi transformations together with transformations of the orthonormal frames $(\mathbb{v}_1, \mathbb{v}_2, \mathbb{v}_3)$. Such transformations between shape maps yields the *shape groupoid*, where the objects are shape maps, and the arrows are shape transformations.

How do transformations of $\mathbb{v}_1, \mathbb{v}_2, \mathbb{v}_3$ affect shape maps?

Heuristically, we *know* that shape maps factor through $C^F \to C^O$. Here, we will verify that this is indeed the case.

When we consider oriented m-triangles, the vector $\mathbb{v}_3 = \mathbb{u}_3$ should be regarded as given, and $\mathbb{v}_1, \mathbb{v}_2$ are allowed to vary in such a way that $(\mathbb{v}_1, \mathbb{v}_2, \mathbb{v}_3)$ is always a positively oriented orthonormal frame. This allows for transformations

$$\tau_\vartheta\colon (\mathbb{v}_1, \mathbb{v}_2) \mapsto (\mathbb{v}_1', \mathbb{v}_2'),$$

where

$$[\mathbb{v}_1'|\mathbb{v}_2'] = [\mathbb{v}_1|\mathbb{v}_2] \cdot \begin{bmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{bmatrix}.$$

τ_ϑ induces the following transformation of the variables ξ_i, ζ:

$$\xi_i \mapsto \xi_i' = e^{-i\vartheta}\xi_i, \quad \text{i.e.} \quad \zeta \mapsto \zeta' = \zeta,$$

and accordingly, we can state the following result:

Lemma 4.2.6. *In the study of shapes of oriented m-triangles, we have the following:*

(i) Shape transformations given by transformations of the frame $\mathbb{v}_1, \mathbb{v}_2, \mathbb{v}_3$ are trivial.

(ii) Shape maps can be regarded as maps $C^O \to \mathbb{C}^*$

$$\sigma\colon (\mathbb{n}, \mathbb{a}_1, \mathbb{a}_2, \mathbb{a}_3) \mapsto \zeta.$$

How do Jacobi transformations affect shape maps?

A given Jacobi map J can be represented by real matrix $[J_i^j]$ such that

$$\mathbb{x}_i = \sum_j J_i^j \mathbb{a}_j.$$

For a given frame $(\mathbb{v}_1, \mathbb{v}_2, \mathbb{v}_3)$ we have the same relation between the associated complex numbers:

$$\xi_i = \sum_j J_i^j \eta_j,$$

where ξ_i are complex Jacobi vectors (4.3) and η_i are complex position vectors (4.5).

In this situation, application of Lemma 2.3.7, Corollary 2.3.9 and Corollary 2.3.13 yields the following facts:

Fact I: If $(\xi_i), (\xi'_i)$ are complex Jacobi vectors associated with possibly different Jacobi maps J, J', then there exist an invertible real matrix

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$$

such that

$$\xi_i = \alpha_{1i}\xi'_1 + \alpha_{2i}\xi'_2.$$

Fact II: A is orthogonal *if and only if* the Jacobi maps J, J' are admitted by the same mass distribution, i.e. if there exist a mass distribution m such that $J, J' \in \mathscr{J}_m$.

Fact III: A is a similarity matrix if and only if J, J' are admitted by similar mass distributions.

For different Jacobi maps J, J', we get different shape maps

$$\sigma : (\mathbb{v}_1, \mathbb{v}_2, \mathbb{v}_3, \mathbb{a}_1, \mathbb{a}_2, \mathbb{a}_3) \mapsto (\xi_1, \xi_2) \mapsto \zeta \in \mathbb{C}^*,$$
$$\sigma' : (\mathbb{v}_1, \mathbb{v}_2, \mathbb{v}_3, \mathbb{a}_1, \mathbb{a}_2, \mathbb{a}_3) \mapsto (\xi'_1, \xi'_2) \mapsto \zeta' \in \mathbb{C}^*,$$

and following Fact I above, for two such shape maps, there exist a real matrix $[\alpha_i^j]$ such that

$$\zeta' = \frac{\xi'_2}{\xi'_1} = \frac{\alpha_2^1\xi_1 + \alpha_2^2\xi_2}{\alpha_1^1\xi_1 + \alpha_1^2\xi_2} = \frac{\alpha_2^1\zeta + \alpha_2^2}{\alpha_1^1\zeta + \alpha_1^2}.$$

Using this formula and Fact I,II,III above we arrive at the following theorem:

Theorem 4.2.7. *Let J, J' be two Jacobi maps for the three body problem, possibly associated with different mass distributions, and let*

$$(\mathbb{a}_1, \mathbb{a}_2, \mathbb{a}_3, \mathbb{m}) \mapsto \begin{cases} \zeta \\ \zeta' \end{cases}$$

respectively be the corresponding shape maps represented on the Riemann sphere $\mathbb{C}^*$. Then we have the following:

(i) *Associated with* (J, J') *there exist an invertible real* 2×2*-matrix* $[\alpha_i^j]$ *such that*

$$\zeta' = \frac{\alpha_2^1 \zeta + \alpha_2^2}{\alpha_1^1 \zeta + \alpha_1^2}. \tag{4.8}$$

Hence the Jacobi transformation induce a fractional linear transformation of the Riemann sphere $\mathbb{C}^*$.

(ii) J, J' *belong to similar mass distributions if and only if* $[\alpha_i^j]$ *can be chosen among the orthogonal matrices.*

(iii) J *and* J' *yield the same shape map if and only if the Jacobi transformation* $J \to J'$ *is given by scalar multiplication.*

In other words, for every pair σ, σ' of shape maps, there exist a unique real fractional linear transformation $\phi\colon \mathbb{C}^* \to \mathbb{C}^*$ such that $\sigma' = \phi \circ \sigma$. In this situation, we will regard ϕ as an arrow $\phi\colon \sigma \to \sigma'$ in the shape groupoid.

The democracy representation of $O(2)$ on the shape sphere.

The democracy group $O(2)$ acts on the configuration space M by matrix multiplication from the right, and represents the freedom in choice of Jacobi vectors for a given mass distribution. Now we will investigate the shape-sphere manifestation of the democracy action. Following Theorem 4.2.7, we deduce that the democracy representation yields fractional linear transformation of the Riemann sphere $\mathbb{C}^*$.

First we consider democracy transformations of positive determinant. The corresponding transformations are fractional linear transformations of the type

$$\zeta \mapsto \frac{\cos\vartheta\,\zeta - \sin\vartheta}{\sin\vartheta\,\zeta + \cos\vartheta}.$$

The only fixed points of such transformations are

$$\zeta = \pm i,$$

and the extended real line $\mathbb{R}^* \subset \mathbb{C}^*$ is an invariant subset. Under this transformation

$$0 \mapsto -\tan\vartheta.$$

According to [Jah11], such a fractional linear transformation is uniquely determined by the constraints

$$\begin{aligned} i &\mapsto i \\ -i &\mapsto -i \\ 0 &\mapsto -\tan\vartheta. \end{aligned} ,$$

Under the identification of $\mathbb{C}^*$ with $S^2 \subset \mathbb{R}^3$ given by Proposition 4.2.4, this corresponds to a linear transformation of $\mathbb{R}^3$ with matrix

$$\begin{bmatrix} \cos 2\vartheta & -\sin 2\vartheta & \\ \sin 2\vartheta & \cos 2\vartheta & \\ & & 1 \end{bmatrix}.$$

Hence, on the level of φ, θ-variables, the corresponding transformation is

$$(\varphi, \theta) \mapsto (\varphi, \theta + 2\vartheta).$$

Democracy transformations of negative determinant correspond to fractional linear transformations of the type

$$\zeta \mapsto \frac{-\cos\vartheta\,\zeta + \sin\vartheta}{\sin\vartheta\,\zeta + \cos\vartheta}.$$

The fixed points of a transformation of this type are of the form

$$\zeta^{\pm} = -\frac{\cos\vartheta \pm 1}{\sin\vartheta},$$

and clearly lies on the extended real line, and since such a transformation maps the imaginary unit i to $-i$, this transformation is the unique fractional linear transformation of the sphere satisfying

$$\begin{aligned} i &\mapsto -i \\ \zeta^{\pm} &\mapsto \zeta^{\pm}. \end{aligned}$$

The corresponding transformation of $\mathbb{R}^3$ is of the form

$$\begin{bmatrix} -\cos 2\vartheta & \sin 2\vartheta & \\ \sin 2\vartheta & \cos 2\vartheta & \\ & & -1 \end{bmatrix},$$

and the corresponding transformation of the shape variables is

$$(\varphi, \theta) \mapsto (\pi - \varphi, 2\vartheta - \theta).$$

4.3 Hyperbolic geometry of triangular shapes

From Theorem 4.2.7 we see that the freedom in choice of shape map is related to a representation of $GL_2\mathbb{R}$ by fractional linear transformations on the shape-sphere $\mathbb{C}^*$. In this way, we encounter the group Möb $\mathbb{R}$ of real Möbius transformations. Regarded as a transformation group of the sphere, Möb $\mathbb{R}$ is a subgroup of the group of oriented conformal transformations. The group of oriented conformal transformations of the round sphere is equivalent to the group Möb $\mathbb{C}$ of complex fractional linear transformations

$$z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathrm{GL}_2\,\mathbb{C}$$

of the Riemann sphere.

Mass-invariance and Möb $\mathbb{R}$-invariance

We propose that it may be fruitful to study invariants of the group action (Möb $\mathbb{R}, \mathbb{C}^*$) in connection with the three body problem. This is based on the following specialization of Theorem 4.2.7:

Theorem 4.3.1. *Properties of oriented three body configurations which can be expressed as invariants and covariants of the action of Möb($\mathbb{R}$) on the shape-sphere $\mathbb{C}^*$ are independent of the choice of mass distribution and Jacobi map.*

At this point of the exposition, it is not completely clear whether or not the converse theorem is also true, namely whether or not properties of three body configurations which are independent of choice of mass distributions and Jacobi maps yields Möb$\mathbb{R}$-invariants. This question is related to whether or not the set of fractional linear transformations coming from shape transformations generates the group Möb$\mathbb{R}$. In Section 4.3.4, we will give a partial solution to this problem.

Decomposition of the shape sphere

The extended real line $\mathbb{R}^* \subset \mathbb{C}^*$ is invariant under Möb$\mathbb{R}$, and will be called the *equator* of the shape sphere. The equator divides the shape-sphere into the *northern hemisphere* which is defined to contain $i \in \mathbb{C}^*$, and the *southern hemisphere*, which contains $-i$. Correspondingly, we will call i *the north pole* and $-i$ *the south pole*.

The connected component Möb$^+\mathbb{R}$ of the identity in Möb$\mathbb{R}$ consists of fractional linear transformations with positive determinant. This subgroup leaves the decomposition

$$\mathbb{C}^* = \text{Southern hemisphere} \cup \text{Northern hemisphere} \cup \text{Equator}$$

invariant. The elements of negative determinant in Möb$\mathbb{R}$ interchanges the hemispheres and leaves the equator invariant (cf. the above discussion of the shape-sphere representation of the democracy group.).

Regarded as Klein geometries [Kle72], the actions of Möb$^+\mathbb{R}$ on each of the hemispheres are equivalent to hyperbolic geometry [Jah11], while the action of Möb$^+\mathbb{R}$ on the extended real line $\mathbb{R}^*$ is equivalent to 1-dimensional real projective geometry. We will use the following sections to discuss these geometries.

4.3.1 Excursions into the Möb$\mathbb{R}$-geometry of the shape sphere

The equator and collinear configurations

We can give a very simple example of Theorem 4.3.1: Since the equator $\mathbb{R}^* \subset \mathbb{C}^*$ is Möb$\mathbb{R}$-invariant, the property

$$(*) \quad \zeta \in \text{ equator} = \mathbb{R}^*$$

must reflect a property of three body configurations which is independent of the mass distribution. We will see that this is indeed the case: $(*)$ is equivalent to

$$\xi_1 = r\xi_2 \quad \text{for} \quad r \in \mathbb{R}.$$

In terms of Jacobi vectors and position vectors, this yields

$$\mathbb{x}_1 = r\mathbb{x}_2, \quad i.e. \quad \mathbb{a}_i = r_{jk}\mathbb{x}_1, \quad \text{for} \quad r_{jk} \in \mathbb{R}.$$

Accordingly, $(\mathbb{a}_1, \mathbb{a}_2, \mathbb{a}_3)$ is collinear. Collinearity is a property which is clearly independent of choice of mass distribution and Jacobi map.

Proposition 4.3.2. *The equator of the shape-sphere corresponds to the set of non-trivial collinear three body configurations.*

We will continue along these lines in Section 4.3.2.

Circles and linear motions

When the three mass points move with uniform velocity along straight lines in an inertial system, we call the three body motion *linear*. The statement "this three body motion is linear" is independent of the mass distribution, and hence we can hope to characterize such motions in a Möb$\mathbb{R}$-invariant way on the shape sphere.

In the case of planar motions, we can accomplish this in the following very simple way: For a planar linear motion, the complex Jacobi vectors are of the form

$$\xi_i(t) = (\xi_i^1 t + \xi_i^0), \quad \xi_i^j \in \mathbb{C},$$

within a fixed frame $(\mathbb{v}_1, \mathbb{v}_2, \mathbb{v}_3)$ of reference. The corresponding shape curve is

$$\zeta(t) = \frac{\xi_2^1 t + \xi_2^0}{\xi_1^1 t + \xi_1^0}, \quad t \in \mathbb{R} \subset \mathbb{R}^*$$

Hence, the shape curve $\zeta(t)$ lies on the image of the extended real line under a Möbius transformation $\mathbb{C}^* \to \mathbb{C}^*$. Since Möbius transformations maps circles to circles, we have the following:

Proposition 4.3.3. *For a linear three body motion in a fixed plane, the corresponding shape curve lies on a circle in the shape sphere.*

In connection to this, we note the following:

(i) The notion of circles is the same in the Riemann sphere $\mathbb{C}^*$ and the Euclidean sphere S^2. This means that circularity is a geometric notion within the Möb$\mathbb{R}$-geometry.

(ii) We can realize every circle on the shape-sphere as the shape curve of a linear planar motion.

(iii) In the non-planar case, the story is more complicated since linear motions generically give non-linear complex Jacobi vectors $\xi_i(t)$.

The shape curves of planar linear motions have an equivalent characterization in terms of the Schwartz-derivative: $\zeta(t)$ is the shape curve of a linear planar motion if and only if the Schwartz derivative

$$\frac{\zeta'''}{\zeta'} - \frac{3}{2}\left(\frac{\zeta''}{\zeta'}\right)^2 = 0.$$

(cf. [Neh52]) The linear three body motions are the geodesics of the kinematic geometry of the configuration space, and the dynamics of three body motions can be described as the deviation from geodesic motion. In the planar case, such geodesic motions correspond to circular motions on the shape sphere. Hence, we can characterize the shape curves of planar linear motions in terms of the Schwartz-derivative. This opens the following question:

Question 4.3.4. To what extent can we characterize the dynamics of three body motions in terms of the Schwartz derivative or other conformal invariants of curves on the shape sphere?

4.3.2 Shape sphere geography

As noted above, the shape-sphere $\mathbb{C}^* \cong S^2 \subset \mathbb{R}^3$ of oriented m-triangles is spherical, and we considered the following decomposition:

$$\begin{aligned} \text{Equator: } \mathbb{R}^* &= \{z \in \mathbb{C}^* : \operatorname{im}(z) = 0\} \\ \text{Northern hemisphere: } \mathbb{C}^*_+ &= \{z \in \mathbb{C}^* : \operatorname{im}(z) > 0\} \\ \text{Southern hemisphere: } \mathbb{C}^*_- &= \{z \in \mathbb{C}^* : \operatorname{im}(z) < 0\} \end{aligned} \tag{4.9}$$

The flexibility in choice of Jacobi vectors is represented by a group action of $\text{Möb}\,\mathbb{R}$ on $\mathbb{C}^*$, and the partition $\mathbb{C}^* = \mathbb{C}^*_+ \cup \mathbb{R}^* \cup \mathbb{C}^*_-$ is preserved by the identity component $\text{Möb}^+\,\mathbb{R} \subset \text{Möb}\,\mathbb{R}$. This splits the geometry $(\text{Möb}^+\,\mathbb{R}, \mathbb{C}^*)$ into one copy of projective geometry on $\mathbb{R}^*$ and two copies of the hyperbolic geometry on $\mathbb{C}^*_+$ and $\mathbb{C}^*_-$.

Hence, when we want to describe features of the shape-sphere geography in a mass-distribution invariant way, we should be able to apply the languages of hyperbolic geometry and projective geometry.

Since the hyperbolic description is independent of the choice of Jacobi vectors and mass distribution, this can be worked out in one specific setting. We choose the case of equal masses $m_1 = m_2 = m_3 = 1$ and the particular choice of Jacobi vectors given by (3.4). In the language of complex configuration vectors and complex Jacobi vectors, this is the following Jacobi transformation:

$$\begin{aligned} \xi_1 &= \sqrt{\frac{1}{2}}(\eta_2 - \eta_1) \\ \xi_2 &= \sqrt{\frac{2}{3}}\left(\eta_3 - \frac{\eta_1 + \eta_2}{2}\right) = \sqrt{\frac{2}{3}}\left(\frac{\eta_3 - \eta_1}{2} + \frac{\eta_3 - \eta_2}{2}\right) \end{aligned} \tag{4.10}$$

(cf. (4.5) and (3.4)).

Isosceles and equilateral triangles

An isosceles triangle (P_1, P_2, P_3) of type i is a triangle where the distances $|P_i P_j| = |P_i P_k|$. For an isosceles triangle of type 1, there must exist a complex number

$$z = re^{i\vartheta} = r(\cos\vartheta + i\sin\vartheta)$$

such that the complex position vectors η_1, η_2, η_3 satisfy

$$\eta_1 - \eta_2 = z(\eta_2 - \eta_3), \quad \eta_1 - \eta_3 = -\overline{z}(\eta_2 - \eta_3).$$

Here ϑ represents the base angle in the triangle.

Using the particular mass distribution $m_1 = m_2 = m_3 = 1$ and the particular Jacobi map given by (4.10), the corresponding points in shape space satisfy

$$\zeta = \frac{z - \overline{z}}{\sqrt{3}} = \frac{2i}{3}\sqrt{3}\sin\vartheta \tag{4.11}$$

Hence, the isosceles triangles of type 1 are represented by the extended imaginary line $i\mathbb{R}^* \subset \mathbb{C}^*$. The intersections $i\mathbb{R}^* \cap \mathbb{C}^*_+, i\mathbb{R}^* \cap \mathbb{C}^*_-$ are thus hyperbolic lines respectively in $\mathbb{C}^*_+$ and $\mathbb{C}^*_-$, since the hyperbolic lines in $\mathbb{C}^*_\pm$ are precisely the circle segments which meets the equator orthogonally. Hence, the set of isosceles triangles of type 1 is represented by a hyperbolic line in $\mathbb{C}^*_\pm$.

Using the relAbeling symmetry, we can conclude the following: The set of isosceles triangles is represented by three hyperbolic lines in $\mathbb{C}^*_\pm$.Since $\pm i$ are the only fixed point of non-trivial democracy transformations, we conclude that the three lines intersect precisely at $\pm i$. Because of the relAbeling symmetry, the intersection angles of the hyperbolic lines are $\frac{\pi}{3}$.

The fact that $\pm i$ represents equilateral triangles can also be seen directly from (4.11).

Returning to arbitrary mass distributions and Jacobi maps, we get the following general result in the language of hyperbolic geometry:

Proposition 4.3.5. *The equilateral configurations are represented by three circles S_1, S_2, S_3 in $\mathbb{C}^*$ such that $S_i \cap \mathbb{C}^*_\pm$ are hyperbolic lines with triple intersections with intersection angles $\frac{\pi}{3}$ (cf. Figure 4.1).*

4.3.3 The monotonicity theorem

In [Mon02] Montgomery shows that bounded three body motions with zero angular momentum and no triple collisions suffers infinitely many collinearities. In the language of the shape-sphere $\mathbb{C}^*$, we would rather say that the shape

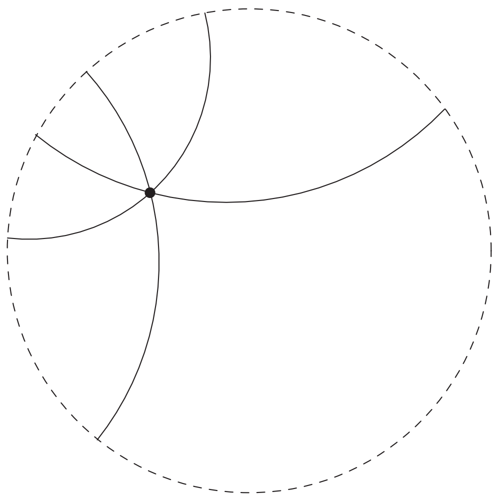

Figure 4.1: Stereographic projection of the hemisphere $\mathbb{C}^*_+$ with the hyperbolic lines of Proposition 4.3.5 indicated. This particular figure represents an arbitrary mass distribution.

curve crosses the equator infinitely many times. The article [Mon02] applies the conformal geometry of the shape sphere. In [HS08], an auxiliary result – which is called the *monotonicity theorem* – is proved somewhat differently than in [Mon02]. As an indication on how the hyperbolic geometry of the three body shapes can be applied, we will reformulate an auxiliary result in the language of hyperbolic geometry.

We will consider an oriented three body motion $X(t)$ with zero angular momentum, and consider a time interval (t_1, t_2) where the shape curve $\gamma(t)$ is on the northern hemisphere $\mathbb{C}^*_+ \subset \mathbb{C}^*$. Let $L \in \mathbb{C}^*$ denote the equilateral shape. For a given choice of length unit on $\mathbb{C}^*_+$, we let $d(t)$ denote the hyperbolic distance from L to $\gamma(t)$.

Note that $d(t) \to \infty$ when $\gamma(t) \to \mathbb{R}^* \subset \mathbb{C}^*$; the hyperbolic distance from L to the equator is infinite. Hence, if $d(t) \to \infty$ as $t \to t_1$, then the three body motion approaches a collinearity as $t = t_1$.

Under the above assumptions, we can reformulate Montgomery's result as follows: Let (t_1, t_2) be a maximal time interval for which the shape curve $\gamma(t)$ associated with the motion $X(t)$ stays at the northern hemisphere. Then we have the following:

(i) t_1, t_2 are finite.

(ii) There exists an instance of time $t_c \in (t_1, t_2)$ such that $d(t)$ is monotonically decreasing on (t_1, t_2) and increasing at (t_c, t_2).

From this result, we can see that there will be finite time intervals between each collinearity, and hence the motions under considerations will suffer infinitely many collinearities. This result – which is the main result in [Mon02] – seems to have very little in common with the hyperbolic geometry of the shape sphere. Now we have seen how we can formulate an important lemma to this by means of hyperbolic geometry. This indicates that the hyperbolic geometry can play an important role in the study of three body motions. It remains however to show that this approach gives technical or conceptual advantages.

4.3.4 Binary collisions and projective geometry

Binary collision configurations are clearly collinear, and hence they are represented by three points B_1, B_2, B_3 on the equator $\mathbb{R}^*$, which we will label by the index of the non-colliding body.

The projective line is three point homogeneous, and hence we see that every distinct triple $\zeta_1, \zeta_2, \zeta_3 \in \mathbb{R}^*$ can be mapped to the binary collision points of the three body problem. Hence, in the language of the projective line, the only thing that we can say is the following: *The binary collisions are represented by three distinct points on the equator* $\mathbb{R}^*$.

Note that this does not imply that every distinct triple in $\mathbb{R}^*$ can represent the binary collision points of the three body problem under some shape map. This indicates that the converse of Theorem 4.3.1 is not true.

If (B_1, B_2, B_3), $(\zeta_1, \zeta_2, \zeta_3)$ are two distinct triples in $\mathbb{R}^*$ there is a unique transformation

$$\zeta' = \frac{a\zeta + b}{c\zeta + d}, \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2\,\mathbb{R}$$

relating them. In this way we see that the position of the binary collision points determines the shape map completely, since two shape maps with coincident binary collision points must be identical. By Theorem 4.2.7 item (iii) we conclude that the position of the binary collision points determines the Jacobi maps modulo scalar multiplication, and following Corollary 2.3.13 we conclude the following:

Proposition 4.3.6. *The similarity class of mass distributions in the three body problem is uniquely determined by the relative position of the images of the binary collision configurations under an associated shape map $C^O \to \mathbb{C}^*$.*

We can go even further, and give simple formulae relating the positions of the binary collision points on the shape-sphere with the normalized mass distribution.

For a general mass distribution m_1, m_2, m_3 and the Jacobi vectors (3.4), we get

$$B_1 = \sqrt{\frac{m_1 m_3}{m_2(m_1+m_2+m_3)}}, \quad B_2 = -\sqrt{\frac{m_2 m_3}{m_1(m_1+m_2+m_3)}}, \quad B_3 = \infty, \tag{4.12}$$

and in terms of the normalized masses $\overline{m}_i = m_i/(m_1+m_2+m_3)$ we thus have the formulae

$$B_1 = \sqrt{\frac{\overline{m}_1\overline{m}_3}{\overline{m}_2}}, \quad B_2 = -\sqrt{\frac{\overline{m}_2\overline{m}_3}{\overline{m}_1}}, \quad B_3 = \infty,$$

and together with $\Sigma_i \overline{m}_i = 1$, and an interpretation of B_1, B_2 as real numbers, we get

$$\overline{m}_1 = \frac{1+B_1B_2}{1-\frac{B_2}{B_1}}, \quad \overline{m}_2 = \frac{1+B_1B_2}{1-\frac{B_1}{B_2}}, \quad \overline{m}_3 = -B_1B_2. \tag{4.13}$$

This gives an explicit way to reconstruct the mass distribution from the position of the binary collision points B_1, B_2, B_3, provided that $B_3 = \infty$. First, we will see how this can be related to the spherical distances between the collision points.

Recall that every Jacobi map J is related to (3.4) by a unique democracy transformation $Q \in O(2)$. Hence, if $B_1', B_2', B_3' \in \mathbb{C}^*$ are the binary collision points given by a specific shape map σ' induced by a Jacobi map J', there is a unique democracy transformation $Q \in O(2)$ that transforms the shape map and the Jacobi map in such a way that the binary collision points are mapped to points B_1, B_2, B_3 satisfying

$$B_1 > 0 \quad B_2 < 0 \quad B_3 = \infty, \tag{4.14}$$

Following the uniqueness of Q, this set of collision point mus coincide with the collision points (4.12) belonging to (3.4), and accordingly, the normalized mass distribution is given by (4.13).

Using the inverse stereographic projection $\mathbb{C}^* \to S^2$, democracy transformations acts by isometries of the spherical geometry, and hence preserves spherical distances. Furthermore, we find a relation between the spherical distances $|B'_i B'_j|$ between B'_i and B'_j regarded as points on the sphere $S^2 \subset \mathbb{R}^3$ and the binary collision positions $B_i \in \mathbb{R}^*$ satisfying (4.14). This relation is as follows:

$$B_1 = \tan\left(\frac{\pi - |B'_1 B'_3|}{2}\right), \quad B_2 = -\tan\left(\frac{\pi - |B'_2 B'_3|}{2}\right)$$

From this and (4.13) we conclude the following:

Proposition 4.3.7. *The similarity class of mass distributions in the three body problem are uniquely determined by the spherical distances*

$$|B'_1 B'_2|, |B'_2 B'_3|, |B'_3 B'_1|$$

between the points $B'_1, B'_2, B'_3 \in \mathbb{C}^* \cong S^2 \subset \mathbb{R}^3$ *representing binary collisions.*

Central triangles and the Jacobi groupoid

Equation (4.14), which can be made valid for every mass distribution by the correct choice of Jacobi map is quite powerful. It tells us that B_i, B_j always lie on opposite sides of the spherical antipodal point B_k^* of B_k, measured along the equator $\mathbb{R}^*$.

Regarding this in the hyperbolic plane $\mathbb{C}_+^*$, we see the three hyperbolic lines of the form $B_i B_i^*$ intersect precisely at the imaginary unit i. This implies that the hyperbolic asymptotic triangle $B_1 B_2 B_3$ contains i in the interior. This is a very important property, and translating the terminology of [HS07] to the language of hyperbolic geometry, we make the following definition:

Definition 4.3.8. An asymptotic triangle in the hyperbolic geometry of the hemisphere $\mathbb{C}^*_+$ is called *central* if it contains i in the interior.

An equivalent condition is that the binary collision points B_i and their antipodal points B_i^* can be ordered the following way along the equator circle $\mathbb{R}^* \subset \mathbb{C}^*$:

$$B_1, B_3^*, B_2, B_1^*, B_3, B_2^*, B_1$$

From the above discussion, we see that the binary collision points $B_1, B_2, B_3 \in \mathbb{C}^*$ will always form a central triangle, since B_3^* always is situated between B_1 and B_2 along the equator. By the following proposition, every central triangle can be taken as the triangle of collision points for the three body problem:

Proposition 4.3.9. *Let B_1, B_2, B_3 be three distinct points on the equator $\mathbb{R}^* \subset \mathbb{C}^*$ and δ the democracy transformation of the sphere $\mathbb{C}^*$ satisfying $\delta(B_3) = \infty$ and $\delta(B_1) > 0$.*

Then the B_1, B_2, B_3 is a central triangle if and only if the images $w_i = \delta(B_i)$ satisfies

$$\overline{m}_1 = \frac{1 + w_1 w_2}{1 - w_2/w_1} > 0, \quad \overline{m}_2 = \frac{1 + w_1 w_2}{1 - w_1/w_2} > 0 \quad \textit{and} \quad \overline{m}_3 = -w_1 w_2 > 0. \quad (4.15)$$

Proof. Let us regard B_1, B_2, B_3 as points on the circle S^1 and regard δ as the stereographic projection from S^1 to $\mathbb{R}^*$ defined by $\delta(B_3) = \infty$, $\delta\left(\frac{B_1}{|B_1|}\right) = 1$ and $\delta(B_3) = 0$. Hence, without loss of generality, we may assume that

$$w_3 = \delta(B_3) = \infty, \quad \delta(B_3^*) = 0, \quad w_1 = \delta(B_1) > 0.$$

Under these assumptions, the third inequality of (4.15) is satisfied if and only $w_2 < 0$, i.e. if and only if B_3^* is situated between B_1 and B_2 along the equator circle $\mathbb{R}^* \subset \mathbb{C}^*$.

Now, we will look at the relation between B_3^*, B_1 and B_2^*. From Figure 4.2 we see that B_1 is situated between B_3^* and B_2^* if and only if $0 < w_1 < w_2^*$.

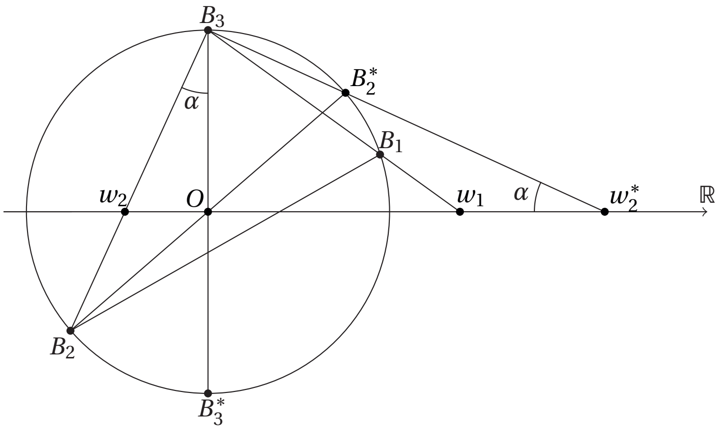

Figure 4.2: Geometric construction showing that every central triangle can represent the binary collision points for some mass distribution m_1, m_2, m_3.

The angle $\angle B_2 B_3 B_2^*$ is orthogonal, since $B_2 B_2^*$ is a diameter. This implies that the angles $\angle w_2 B_3 O$ and $\angle B_3 w_2^* O$ are equal. Accordingly, the triangles $\Delta w_2 B_3 O$ and $\Delta B_3 w_2^* O$ are congruent and since $|OB_3| = 1$ we conclude that

$$w_2^* = -\frac{1}{w_2}$$

when we regard w_2, w_2^* as points on the real line.

From this we conclude that the ordering

$$B_2, B_3^*, B_1, B_2^*$$

along the half circle from B_2 to B_2^* is satisfied if and only if

$$-1 = w_2^* w_2 < w_1 w_2 < 0,$$

i.e. if and only if

$$w_1 w_2 < 0 \quad \text{and} \quad 1 + w_1 w_2 > 0.$$

The latter condition is clearly equivalent to (4.15). Since the ordering of the B_i, B_i^* is uniquely determined by the restriction to the circular segment $B_2 B_2^*$, and because of the characterization of central triangles by means of the ordering of the collision points and their antipodes, we conclude that the proposition is valid. ■

Using the interpretation suggested by (4.13), this proposition establishes a $1-1$-correspondence between normalized mass distributions $\overline{m}_1, \overline{m}_2, \overline{m}_3$ of the three body problem and central triangles $\{B_1, B_2, B_3\} \subset \mathbb{R}^* \subset \mathbb{C}^*$ modulo rotation.

In the following, we apply this observation to our study of the Jacobi groupoid:

Definition 4.3.10. We define *the central triangle groupoid* as follows:

Objects: Central triangles (B_1, B_2, B_3) on the hemisphere $\mathbb{C}^*_+$

Arrows: 2×2-matrices with positive determinant: $A \in \mathrm{GL}_2^+(\mathbb{R})$ yields an arrow $(B_1, B_2, B_3) \to (B_1', B_2', B_3')$ if and only if $B_i \mapsto B_i'$ under the induced fractional linear transformation of the shape-sphere $\mathbb{C}^*$ (cf. (4.8)).

Composition: Matrix multiplication.

Following the correspondence between mass distributions and central triangles we see the following:

(i) Jacobi transformations with positive determinant induce arrows in the central triangle groupoid.

(ii) If $A\colon (B_1, B_2, B_3) \to (B_1', B_2', B_3')$ is an arrow in the central triangle groupoid, then A will yield a Jacobi transformation $A^*: J \to J'$, where J (J') is a Jacobi map associated with a mass distribution m_1, m_2, m_3 (m_1', m_2', m_3') associated with the central triangles: Let $(\overline{m}_i), (\overline{m}_i')$ be the normalized mass distributions associated with $(B_i), (B_i')$, and let J_0, J_0' be Jacobi maps respectively associated with the two mass distributions. After application of appropriate democracy transformations, we can assume that $(B_i), (B_i')$ are the positions of the binary collision points under the shape maps associated with J_0, J_0'.

Now, there exist a unique Jacobi transformation $A_0^*: J_0 \to J_0^*$, which must map the central triangle (B_i) to the central triangle (B_i'). Accordingly, A_0

and A induce the same fractional linear transformation of $\mathbb{C}^*$, and we conclude that there exist a real number λ such that

$$A = \lambda A_0$$

Accordingly,we can regard A as a Jacobi transformation from the Jacobi map $J = J_0$ to the Jacobi map $J' = \lambda J_0'$, which is associated with the mass distribution $(\overline{m}_i'/|\lambda|)$.

Following this discussion we have the following characterization of the Jacobi transformations:

Proposition 4.3.11. *An invertible* 2×2*-matrix* A *is the matrix of a Jacobi transformation if and only if either*

$$A \quad or \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A$$

represents an arrow in the central triangle groupoid.

The reason for introducing the second matrix in the above equation is that the discussion of the central triangle groupoid concerns only matrices with positive determinants, while in the discussion of the Jacobi groupoid, we consider also matrices with negative determinant.

This raises the following question: For which matrices

$$A = \begin{bmatrix} \alpha_1^1 & \alpha_1^2 \\ \alpha_2^1 & \alpha_2^2 \end{bmatrix} \in GL_2^+(\mathbb{R})$$

do there exist two central triangles $(B_i), (B_i')$ in $\mathbb{C}_+^*$ that are mapped onto each other by the Möbius transformation

$$\zeta \mapsto \frac{\alpha_2^1 \zeta + \alpha_2^2}{\alpha_1^1 \zeta + \alpha_1^2}.$$

We can give a simple answer to this question: Suppose that the matrix A is given. We may diagonalize A using the singular value decomposition and democracy transformations of the shape sphere. Hence, we can restrict our discussion to the fractional linear transformations of the form

$$\zeta \mapsto \alpha\zeta \quad \text{where} \quad \alpha > 0.$$

Since A belongs to the central triangle groupoid if and only if the inverse A^{-1} belongs to the same groupoid, we can assume that $\alpha \leq 1$. The question now goes as follows: Given $\alpha \in (0,1]$, do there exist a central triangle B_1, B_2, B_3 such that $\alpha B_1, \alpha B_2, \alpha B_3$ is central?

The answer is clearly *yes*: Let $\Delta = (B_1, B_2, B_3)$ be any triangle with $B_3 = \infty$ and $B_2 = -B_1$. Then Δ is central if and only if $|B_1| < 1$. This property is clearly invariant under multiplication with any $\alpha \in (0,1]$. Accordingly, there exist a central triangle which is mapped to a central triangle under multiplication with α.

This discussion leads to the following conclusion:

Theorem 4.3.12. *For every element $A \in GL_2\mathbb{R}$, there exist two Jacobi maps J, J' for the three body problem such that A induces an arrow*

$$A^* : J \to J'.$$

This solves Question 2.3.14 in the case of $n = 3$, but do not give us the converse of Theorem 4.3.1. The central triangles themselves yield a universal counter-example: The class of central triangles is *not* invariant under the group of Möbius transformations.

Hence, in order to refine Theorem 4.3.1 we should replace the geometry given by the group of Möbius transformations by a "geometry" given by the central triangle groupoid. On the other hand, we can regard Theorem 4.3.1 as a sufficient basis for our application of hyperbolic geometry in the study of the three body problem, when we are aware of the directions of the implications.

4.4 Kinematic geometry of the shape spaces

As above will mostly neglect the triple collision configuration, i.e. the zero configuration. This will be reflected by our use of the notation $V^\times$ for the set $V \setminus \{0\}$ of non-trivial elements of a vector space V. Since M denotes the vector space of three body configurations, $M^\times$ will denote the space of non-zero three body configurations.

In order to study the kinematic geometry of the shape spaces, we will use Figure 4.3 as a major tool. Our main goal is to understand the central column

$$M^\times \to \frac{M^\times}{\mathrm{SO}(3)} \to M^*,$$

where the points in the moduli space $M^\times / \mathrm{SO}(3)$ represent non-trivial three body configurations modulo congruence and the points of M^* represent non-trivial three body configurations modulo similarity, i.e. three body shapes. This sequence is naturally given by the three body problem. We want to study this sequence by means of the two other parallel sequences in Figure 4.3. The sequence on the left hand side is associated with the singular value decomposition of three body configurations. In this sequence, we have a simple description of the Riemannian geometry. The sequence on the right hand side is related to the shape-sphere construction (cf. Proposition 4.2.2), and here we have a good understanding of the quotient construction in the case of oriented m-triangles.

4.4.1 Topological preliminaries

The involution $\tau : \mathbb{C}^* \to \mathbb{C}^*$ given by complex conjugation represents change in orientation of oriented m-triangles. This yields the above identification

$$M^* \cong \frac{\mathbb{C}^*}{\tau} \cong \mathbb{C}^*_+ \cup \mathbb{R}^* \cong \mathbb{D}^2$$

of the shape space with the closed 2-disc.

On the other side we have the torus $\mathbb{T}^2$ which is parametrized by the variables φ, θ over the interval $[-2\pi, 2\pi]$. The finite gauge group Σ defined in Section 3.3.5 acts on $\mathbb{T}^2$ according to (3.10'). The mapping $\mathbb{T}^2 \to M^*$ is Σ-invariant.

$$\begin{tikzcd}
& S \arrow[rr] & & SO(3)\times\mathbb{C}^2 & \\
\mathrm{SO}(3)\times(\mathbb{R}^2)^\times\times\mathrm{SO}(2) \arrow[ur, "\subset"] \arrow[rrrr] \arrow[d, "\cong"'] \arrow[drr, "\Phi"] & & & & \mathrm{SO}(3)\times(\mathbb{C}^2)^\times \arrow[ul, "\supset"'] \arrow[dll] \arrow[dd] \\
\mathrm{SO}(3)\times(0,\infty)\times\mathbb{T}^2 \arrow[d] & & M^\times \arrow[d] & & \\
(0,\infty)\times\mathbb{T}^2 \arrow[rr] \arrow[d] & & \frac{M^\times}{\mathrm{SO}(3)} \arrow[d] & & (\mathbb{C}^2)^\times/U(1) \arrow[ll] \arrow[d] \\
\mathbb{T}^2 \arrow[rr] & & M^* & & \mathbb{C}^* \arrow[ll]
\end{tikzcd}$$

Figure 4.3: Commutative diagram which shows the relation between the shape-sphere construction and the singular value decomposition.
For a vector space V we use $V^\times$ to denote the space of non-trivial elements.

Hence, there exist an induced map

$$\frac{\mathbb{T}^2}{\Sigma} \to M^*$$

which is a diffeomorphism on the open complement of $\{(\varphi,\theta)\colon \varphi = 0 \mod \frac{\pi}{2}\} \subset \mathbb{T}^2$. By inspection of the generators of Σ, we can infer that

$$\frac{\mathbb{T}^2}{\Sigma} \cong \frac{[0,\frac{\pi}{2}]\times S^1}{\sim},$$

where $\sim$ is the equivalence relation given by

$$(0,\theta) \sim (0,\theta+\pi)$$

and the quotient map

$$\mathbb{T}^2 \to \frac{[0,\frac{\pi}{2}] \times S^1}{\sim}$$

is given by

$$(\varphi \mod 4\pi, \theta \mod 4\pi) \mapsto \left[(\varphi', \theta \mod 2\pi) \mod \sim\right],$$

where $\varphi' = min_{k\in\mathbb{Z}}(|\varphi + k\pi|, |k\pi - \varphi|)$.

The quotient space $([0,\frac{\pi}{2}] \times S^1)/\sim$ has the following topological description: It is a cylinder A where one of the boundary circles $\partial_\pm A$ is glued to itself along a $2-1$ mapping $S^1 \to \partial_\pm A$. Topologically, $\frac{\mathbb{T}^2}{\Sigma}$ is a Möbius band.

All the points $(0,\theta)$ represent the same shape. Hence, the set of points of this form is collapsed to one point by the projection

$$\frac{\mathbb{T}^2}{\Sigma} \to M^*.$$

On the other hand, the set of point $(\varphi,\theta) \in (0,\frac{\pi}{2}) \times S^1$ is mapped injectively. This gives an interpretation of φ, θ as polar coordinates on the disc $\mathbb{D}^2$, where we use the above identification of the shape space with a closed disk.

By commutativity of diagrams, we see that this coordinatization coincides with the coordinatization of $\mathbb{C}^*$ given in Section 4.2.3, modulo the projection $\mathbb{C}^* \to \mathbb{C}^*/\tau = \mathbb{D}^2$.

In the study of oriented m-triangles by means of the left column of figure 4.3, we implicitly work with the subgroup $\Sigma' \subset \Sigma$ given by

$$\sigma \in \Sigma' \iff \sigma \mathbb{u}_3 = \mathbb{u}_3,$$

i.e. the group of gauge transformations of the singular value decompositions keeping $\mathbb{u}_3$ fixed. The quotient space

$$\frac{\mathbb{T}^2}{\Sigma'} = \frac{[0,\pi] \times S^1}{\sim'}$$

where the equivalence relation $\sim'$ is given by

$$(0,\theta) \sim' (0,\theta+\pi) \quad \text{and} \quad (\pi,\theta) \sim' (\pi,\theta+\pi).$$

The resulting quotient space is homeomorphic to the Klein bottle.

There is a surjection

$$\frac{[0,\pi]\times S^1}{\sim'} \to \mathbb{C}^*$$

which collapses the circles $\varphi = 0, \varphi = \pi$ to two distinct points, and this a local diffeomorphism for $\varphi \neq 0, \pi$. This yields another reason for interpreting φ, θ as spherical polar coordinates on the shape-sphere $\mathbb{C}^*$, over the ranges $0 \le \varphi \le \pi$, $0 \le \theta \le 2\pi$.

4.4.2 The analytic structures

Kinematic analyticity of shape curves

Since we are mainly interested in studying analytic three body motions $X(t)$, the analytic (C^ω) structures of $M^\times, M^\times/\mathrm{SO}(3), M^*$ are of interest, and our main lies in the following question: Which curves in M^* come from analytic virtual three body motions.

An analytic singular value decomposition $U(t), R(t), Q(t)$ representation of $X(t)$ yields an analytic curve $(U(t), \xi_1(t), \xi_2(t))$ in $SO(3)\times\mathbb{C}^2$. The corresponding curve $\zeta(t) = \xi_1(t)/\xi_2(t)$ on the Riemann sphere $\mathbb{C}^*$ is analytic with respect to the usual analytic structure on the Riemann sphere, which is the same as the analytic structure of the round sphere regarded as analytic submanifold of $\mathbb{R}^3$. The opposite is obviously also true: Any smooth curve in $\mathbb{C}^*$ lifts to an analytic virtual three body motion. We summarize this as follows:

Proposition 4.4.1. *The curves in the shape-sphere $\mathbb{C}^*$ which are analytic in the usual sense corresponds precisely to the shape curves of analytic virtual three body motions.*

Now, we turn our attention to the shape space M^*. In this case the boundary needs some attention.

If we regard the shape space $M^* \cong \mathbb{D}^2$ as the quotient of $\mathbb{C}^*$ by the conjugation map, we get an analytic structure on M^*. Over the interior region $(int)(M^*) = M^* \setminus \partial M^*$, the projection $\mathbb{C}^*_+ \cup \mathbb{C}^*_- \to \mathrm{int}(M^*)$ is an analytic local diffeomorphism, and hence the analytic structure of $\mathrm{int}(M^*)$ is the usual analytic structure of the open disk in $\mathbb{R}^2$.

When it comes to the boundary ∂M^*, the situation is quite different. If we identify M^* with the closure of the northern hemisphere, i.e. $M^* = \overline{\mathbb{C}^*_+}$, the quotient map is represented by

$$\zeta = a + ib \mapsto \zeta' = a + i|b|$$

From this we see that a C^1-smooth curve $\gamma(t)$ in $\mathbb{C}^*$ with $\gamma(0) = \in \mathbb{R}^* \subset \mathbb{C}^*$ projects to a curve $\overline{\gamma}(t)$ in M^* where the two one-sided derivatives satisfies

$$\mathrm{im}(\overline{\gamma}'(0^+)) = -\mathrm{im}(\overline{\gamma}'(0^-)).$$

Since the stereographic projection is angle-preserving, we have the following proposition:

Proposition 4.4.2. *The shape space M^* of the three body problem admits a homeomorphism $M^* \cong \mathbb{D}^2$. In the natural analytic structure induced by the analytic structure of the three body problem, a curve $\overline{\gamma}(t)$ in $\mathbb{D}^2$ is C^1 smooth if and only if*

- *The part of $\overline{\gamma}$ that lies in the interior $\mathrm{int}(\mathbb{D}^2)$ is C^1-smooth in the usual sense.*
- *At points where $\overline{\gamma}(t) \in \partial\mathbb{D}^2$, the oriented angles $\vartheta^\pm$ between the one sided velocity vectors $\overline{\gamma}'(t^\pm)$ and the inward pointing normal N of $\partial\mathbb{D}^2$ satisfy $\vartheta^- = -\vartheta^+$.*

Hence, smooth curves with non-vanishing velocity through points at the boundary of $M^* \cong \mathbb{D}^2$ follow the usual reflection law, where the incoming angle is equal to the outgoing angle. See figure 4.4

Now we turn to the study of analyticity. An analytic curve through 0 in $\mathbb{C}^*$

$$\gamma(t) = (a_1 + ib_1)t + (a_2 + ib_2)t^2 + \cdots$$

is transformed to a curve

$$\overline{\gamma}(t) = a_1 t + a_2 t^2 + \cdots + i|b_1 t + b_2 t^2 + \cdots|$$

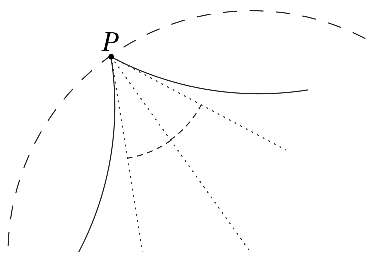

Figure 4.4: Reflection of C^1-curves in a point P at the boundary of $\mathbb{D}^2$.

in $M^* \cong \mathbb{C}_+^* \cup \mathbb{R}^*$.

As above, we are interested in the change of sign of the imaginary part of $\overline{\gamma}(t)$, which measures the part of the motion which is transversal to the equator of the shape sphere. For small t, the first non-zero term $b_k t^k$ dominates the change in sign of $\sum_i b_i t^i$ near 0. If k is even, there is no change in sign near 0. If k is odd, there is a local change in sign, and we conclude that $\overline{\gamma}(t)$ is analytic at 0 if and only if it is analytic on a pointed neighbourhood of 0, and the first non-zero one sided derivatives satisfies

$$\mathrm{im}(\overline{\gamma}^{(k)}(0^-)) = -\mathrm{im}(\overline{\gamma}^{(k)}(0^+)) \quad \text{if} \quad k \text{ is odd.} \tag{4.16}$$

It may also be of some interest to describe the analytic functions on the shape space M^*. Since we can identify analytic functions on M^*, with conjugation invariant functions on $\mathbb{C}^*$, we can conclude that the analytic functions on the shape space M^* can be identified with analytic functions f on the disc $\mathbb{D}^2 \subset \mathbb{R}^2$ for which the normal derivative at the boundary is 0. Similarly, we see that the normal derivatives of any odd order must vanish.

Proposition 4.4.3. *The smooth structure of the shape space M^* of the three body problem can be described in the following way:*

Topology: *M^* is homeomorphic to the closed unit disc $\mathbb{D}^2$ via the mapping*

$$M^* \xrightarrow{f_1} \mathbb{C}^*_+ \cup \mathbb{R}^* \xrightarrow{f_2} \mathbb{D}^2$$

where f_1 is a section of $\mathbb{C}^* \to M^*$ and f_2 is given by stereographic projection in $-i \in \mathbb{C}^*$.

Smooth structure in terms of analytic functions: *A function $f\colon M^* \cong \mathbb{D}^2 \to \mathbb{R}$ is analytic if and only if it is an analytic function on the subset $\mathbb{D}^2 \subset \mathbb{R}^2$, and that the non-vanishing normal derivatives at the boundary are all of even order.*

Smooth structure in terms of analytic curves: *A curve in $M^* \cong \mathbb{D}^2$ is analytic if the induced curve in $\mathbb{D}^2 \subset \mathbb{R}^2$ is analytic in the interior of $\mathbb{D}^2$ and that the non-zero one-sided derivatives of the lowest order at the boundary satisfies* (4.16).

The practical consequences of this proposition are the following: (i) In the study of shape curves, we have a notion of continuation at the boundary of M^*, which is given by the reflection law. (ii) If we want to study the quantum mechanics of the three body problem in terms of smooth functions on the shape space, we can work with smooth functions on the unit disc where the normal derivatives at the boundary of odd orders vanish.

4.4.3 The Riemannian geometry

For a compact group G with a continuous isometric action on a metric space (M, d), the orbit space $B = M/G$ has a natural metric, namely the *orbital distance metric*, which is defined as follows: For two moduli classes $[m], [m'] \in B$, we define the orbital distance by

$$d([m], [m']) = \min_{g \in G}\{d(gm, m')\}$$

as long as G is compact and everything is continuous, this minimum value exists. Hence B inherits an orbital distance metric.

Needless to say, the topology induced by the orbital distance metric is identical to the quotient topology.

When M is a Riemannian manifold, d is the induced distance function and G is a compact Lie group, we can give a differential geometric description of the orbital distance metric. The fact that the orbit space B in general is not a manifold complicates the discussion. The smooth structure is however as described above, and it is possible give a stratification $B_0 \subset B_1 \subset \ldots \subset B_k = B$ where the smooth structure of $B_i \setminus B_{i-1}$ is that of a smooth manifold (cf. [BdCH09])

If we restrict the discussion to a subset $M_{(K)} \subset M$ of fixed isotropy type (K), the quotient map yields a smooth map between smooth manifolds

$$\pi\colon M_{(K)} \to B_{(K)}.$$

The subsets $M(K) \subset M$ of fixed isotropy type yields a partition of M into smooth submanifolds, and the induced distance function on $B_{(K)}$ is associated with a Riemannian metric which is defined as follows:

The tangent bundle $TM_{(K)}$ splits orthogonally as $TM_{(K)} = \ker(\pi_*) \oplus HM_{(K)}$. There is precisely one Riemannian metric on $B_{(K)}$ such that $HM_{(K)} \to TB_{(K)}$ is a fibre-wise isometry. Hence, each stratum $B_{(K)}$ has a natural Riemannian metric. The space B is called a stratified Riemannian manifold. The essential features of B are the following: (i) The orbital distance function $d\colon B \times B \to (0,\infty)$. (ii) The Riemannian manifold structures on the strata $B_{(K)} \subset B$. The structure of the projection map $M \to B$ is referred to as a *stratified Riemannian submersion* (cf. [BdCH09]).

Let us turn our attention to the three body problem. On the open subset $M_r \subset M$ consisting of regular configurations, SO(3) acts freely. Hence, the corresponding quotient map

$$\pi\colon M_r \to \overline{M}_r \subset \overline{M} = M/\,\mathrm{SO}(3)$$

is a smooth map of smooth manifolds.

Using the singular value decomposition we can take U, ρ, φ, θ as coordinates on M_r over the ranges

$$U \in \mathrm{SO}(3), \quad \rho \in (0,\infty), \quad \varphi \in (0, \frac{\pi}{2}), \quad \theta \in (a, 2\pi + a)$$

where a is an arbitrary real number. In these coordinates, we can express the quotient map as follows:

$$(U,\rho,\varphi,\theta) \mapsto (\rho,\varphi,\theta).$$

We will study the tangent bundle TM_r using the coframe of Section 3.8, namely

$$\omega^1,\omega^2,\omega^3,\mathrm{d}\rho,\mathrm{d}\varphi,\mathrm{d}\theta.$$

In this frame, the kinematic metric reads

$$K = \mathrm{d}\rho^2 + \rho^2\frac{1}{4}\left(\mathrm{d}\varphi^2 + \sin^2\varphi\mathrm{d}\theta^2\right) + \frac{1}{2}\left(\lambda_1(\omega^1)^2 + \lambda_2(\omega^2)^2 + \lambda_3(\omega^3)^2\right)$$

(cf. (3.42)), and we observe the following:

(i) $v \in \ker(\pi_*)$ if and only if $\mathrm{d}\rho(v) = \mathrm{d}\varphi(v) = \mathrm{d}\theta(v) = 0$, since ρ,φ,θ can be taken as basic invariants of the SO(3)-action.

(ii) $v \perp \ker(\pi_*)$ if and only if $\omega^i(v) = 0$ for $i = 1,2,3$, since the three body motions perpendicular to $\ker(\pi_*)$ are precisely the motions with zero total angular momentum.

Accordingly, the Riemannian orbital distance metric in $\overline{M}_r$ is

$$\overline{\mathrm{d}s}^2 = \mathrm{d}\rho^2 + \rho^2\frac{1}{4}\left(\mathrm{d}\varphi^2 + \sin^2\varphi\mathrm{d}\theta^2\right), \tag{4.17}$$

and we recognize $\overline{M}_r$ as the Riemannian cone over a subset of a sphere of radius $\frac{1}{2}$.

The orbit type stratification of M with respect to the $SO(3)$-action is as follows:

$$\begin{aligned} M_{(SO(3))} &= \{\text{Triple collision configuration}\}\\ M_{(SO(2))} &= \{\text{Collinear configurations}\}\\ M_{(Id)} &= \{\text{Configurations spanning a plane}\}\end{aligned}$$

$M_{(SO(3))}$ is only a point, and the restriction of the quotient map is simply the mapping of a one point set onto one point.

The stratum $M_{(Id)}$ contains M_r as a dense subset, and $M_{(Id)} \setminus M_r$ is precisely the set of configurations of umbilic shape. From the theory of stratified Riemannian submersions, we know that

$$M_{(Id)} \to \frac{M_{(Id)}}{SO(3)}$$

is a Riemannian submersion. Hence, the singularity in the expression (4.17) at the umbilic shape is only a coordinate singularity. Since φ ranges over $[0, \frac{\pi}{2})$ in $M_{(Id)}$, we conclude that $M_{(Id)}/SO(3)$ is the Riemannian cone over an open hemisphere of radius $\frac{1}{2}$.

The stratum $M_{(SO(2))}$ can be parametrized by U, ρ, φ, θ, with $\varphi = \frac{\pi}{2}$. $M_{(SO(s))}/SO(3)$ is thus a Riemannian cone over a circle of radius $\frac{1}{2}$.

We see here that the metric structure of the stratified Riemannian manifold $M/SO(3)$ is determined by continuity together with the restriction to the open and dense subset M_r of the principal stratum $M_{SO(3)}$.

When we want to define a relevant metric on the shape space $M_r^* = \overline{M}_r/(\text{scaling})$, we can not rely on the orbital distance metric, since that metric is trivial. Instead we will rather regard M^* as a subset of $\overline{M}$:

$$M^* = \{(\rho, \varphi, \theta) : \rho = 1\}.$$

Clearly, the induced metric on M_r^* is the following:

$$\mathrm{d}s_*^2 = \frac{1}{4}\left(\mathrm{d}\varphi^2 + \sin^2\varphi\, \mathrm{d}\theta^2\right),$$

which is locally isometric to the metric of the round sphere of radius $\frac{1}{2}$. Hence, M_r^* is isometric to a subset of the sphere of radius $\frac{1}{2}$. By continuity, we conclude that the metric space M^* is isometric to a closed hemisphere of radius $\frac{1}{2}$, and that φ, θ yields a spherical coordinate system covering this hemisphere over the parameter domain $[0, \frac{\pi}{2}] \times [0, 2\pi]$. We summarize this discussion with the following definition

Definition 4.4.4. The kinematic geometry of the shape space M^* is the stratified Riemannian geometry of the closed hemisphere of radius $\frac{1}{2}$.

The pullback of the kinematic metric through $\mathbb{C}^* \to M^*$, endows $\mathbb{C}^*$ with the geometry of the round sphere of radius $\frac{1}{2}$. This suggests the following definition:

Definition 4.4.5. The kinematic geometry of the shape-sphere $\mathbb{C}^*$ is the Riemannian geometry of the sphere of radius $\frac{1}{2}$.

4.4.4 The complex structure of the shape sphere

Through the identification of the shape-sphere with the Riemann sphere $\mathbb{C}^*$, we get a canonical complex structure on the shape sphere. In the variables φ, θ, this complex structure can be represented by the $1,1$-tensor

$$J = -\sin\varphi \frac{\partial}{\partial\varphi} \otimes \mathrm{d}\theta + \frac{1}{\sin\varphi}\frac{\partial}{\partial\theta} \otimes \mathrm{d}\varphi. \tag{4.18}$$

This object is an intrinsic part of the spherical geometry of the shape sphere, but also as a manifestation of the conformal structure.

Since the projection $\mathbb{C}^* \to M^*$ is not everywhere orientation preserving, the complex structure can not be regarded as an object which is intrinsic to M^*.

4.4.5 Shape sphere area and geometric phase

We choose the orientation of the shape-sphere $\mathbb{C}^*$ in such a way that the area form – given by the Riemannian geometry and the choice of orientation – is represented by the differential form

$$\mathrm{d}A = \frac{1}{4}\sin\varphi \, \mathrm{d}\varphi \wedge \mathrm{d}\theta, \tag{4.19}$$

which refers to the (φ, θ)-coordinatization given in Section 4.2.3. This choice of orientation is compatible with the chosen complex structure, in the sense that

$$\mathrm{d}A(\mathbb{v}, J\mathbb{v}) = \|\mathbb{v}\|^2$$

for every tangent vector $\mathbb{v} \in T\mathbb{C}^*$

Here, we will investigate the curious fact that the area form is intimately related to the overall rotation of three body motions with total angular momentum $\Omega = 0$ – the so-called *geometric phase*. This topic is investigated in [HS07]. Here we give a treatment of this topic which is adapted to the terminology of this thesis. There is a discrepancy in sign between my approach and that of [HS07], the source of this discrepancy is presumably $2n+1$ sign conventions which differs between the present work and [HS07].

Rotation of Jacobi vectors

Since we are interested in motions with $\Omega = 0$, we can assume that the motion takes place in the xy-plane, i.e. that $\mathbb{u}_3$ is the positive unit vector on the z-axis. We consider the case of oriented m-triangles, i.e. three body motions represented by curves on the shape-sphere $\mathbb{C}^*$.

For a three body motion $X(t)$ with Jacobi vectors $\mathbb{x}_1(t), \mathbb{x}_2(t)$ we define the 1-forms

$$\Phi_i = \frac{\mathbb{x}_i \times \mathrm{d}\mathbb{x}_i}{\|\mathbb{x}_i\|^2} = \frac{\mathbb{x}_i \times \dot{\mathbb{x}}_i}{\|\mathbb{x}_i\|^2}\mathrm{d}t \quad i = 1, 2. \tag{4.20}$$

These forms represent the angular velocity of the individual Jacobi vectors. In the notation above, the Φ_i are 1-forms with values in $\mathbb{R}^3$, but since the motion is assumed to take place in the x, y-plane, we will regard Φ_i as $\mathbb{R}$-valued 1-forms, given by projection to the z-axis. We note that the Φ_i are singular at $\mathbb{x}_i = 0$.

We need a principal axes system which we will represent as

$$\mathbb{u}_1 = \begin{bmatrix} \cos\vartheta(t) \\ \sin\vartheta(t) \end{bmatrix} \quad \text{and} \quad \mathbb{u}_2 = \begin{bmatrix} -\sin\vartheta(t) \\ \cos\vartheta(t) \end{bmatrix}$$

in the x, y-plane. Using the singular value decomposition (3.7), we write the Jacobi vectors as

$$\mathbb{x}_1 = \frac{\rho}{\sqrt{2}}(r_1 b_2 \mathbb{u}_1 - r_2 b_2 \mathbb{u}_2) \quad \text{and} \quad \mathbb{x}_2 = \frac{\rho}{\sqrt{2}}(r_1 b_1 \mathbb{u}_1 + r_2 b_2 \mathbb{u}_2),$$

where

$$r_1 = \cos\frac{\varphi}{2} + \sin\frac{\varphi}{2}, \quad r_2 = \cos\frac{\varphi}{2} - \sin\frac{\varphi}{2}$$

and

$$b_1 = \sin\frac{\theta}{2}, \quad b_2 = \cos\frac{\theta}{2}.$$

The squared norms are

$$\|\mathbb{x}_1\|^2 = \frac{\rho^2}{2}(r_1^2 b_2^1 + r_2^1 b_1^2) = \frac{\rho^2}{2}(1 + \sin\varphi\cos\theta)$$

and

$$\|\mathbb{x}_2\|^2 = \frac{\rho^2}{2}(r_1^2 b_1^2 + r_2^2 b_2^2) = \frac{\rho^2}{2}(1 - \sin\varphi\cos\theta).$$

If we represent the cross products by their z-components, we get

$$\mathbb{x}_1 \times \mathrm{d}\mathbb{x}_1 = \|\mathbb{x}_1\|^2 \mathrm{d}\vartheta + \frac{\rho^2}{4}\sin\theta\mathrm{d}\varphi - \frac{\rho^2}{4}\cos\varphi\mathrm{d}\theta$$

and

$$\mathbb{x}_2 \times \mathrm{d}\mathbb{x}_2 = \|\mathbb{x}_2\|^2 \mathrm{d}\vartheta - \frac{\rho^2}{4}\sin\theta\mathrm{d}\varphi - \frac{\rho^2}{4}\cos\varphi\mathrm{d}\theta.$$

Under the assumption that $\Omega = 0$, i.e. $\mathbb{x}_1 \times \mathrm{d}\mathbb{x}_1 + \mathbb{x}_2 \times \mathrm{d}\mathbb{x}_2 = 0$, we have

$$\mathrm{d}\vartheta = \frac{1}{2}\cos\varphi\mathrm{d}\theta.$$

When we put all this together, we arrive at

$$\begin{aligned}
\Phi_1 &= \mathrm{d}\vartheta + \left(\frac{\sin\theta}{2(1+\sin\varphi\cos\theta)}\right)\mathrm{d}\varphi - \left(\frac{\cos\varphi}{2(1+\sin\varphi\cos\theta)}\right)\mathrm{d}\theta \\
&= \left(\frac{\cos\varphi\sin\varphi\cos\theta}{2(1+\sin\varphi\cos\theta)}\right)\mathrm{d}\theta + \left(\frac{\sin\theta}{2(1+\sin\varphi\cos\theta)}\right)\mathrm{d}\varphi \\
\Phi_2 &= \mathrm{d}\vartheta - \left(\frac{\sin\theta}{2(1-\sin\varphi\cos\theta)}\right)\mathrm{d}\varphi - \left(\frac{\cos\varphi}{2(1-\sin\varphi\cos\theta)}\right)\mathrm{d}\theta \\
&= -\left(\frac{\cos\varphi\sin\varphi\cos\theta}{2(1-\sin\varphi\cos\theta)}\right)\mathrm{d}\theta - \left(\frac{\sin\theta}{2(1-\sin\varphi\cos\theta)}\right)\mathrm{d}\varphi,
\end{aligned} \tag{4.21}$$

wherever this makes sense, i.e. away from the singularities $\mathbb{x}_i = 0$.

This suggests that we can regard Φ_1, Φ_2 as 1-forms on the shape sphere, a point of view which is confirmed by the fact that these expressions are invariant under the finite gauge group Σ' consisting of transformations of singular value decomposition data leaving $\mathbb{u}_3$ fixed. (cf. (3.10)).

By a straightforward calculation we see that

$$\mathrm{d}\Phi_1 = \mathrm{d}\Phi_2 = -\frac{1}{2}\sin\varphi \mathrm{d}\varphi \wedge \mathrm{d}\theta = -2\mathrm{d}A.$$

i.e. that the exterior derivatives of Φ_1, Φ_2 are both identical to -2 times the area form $\mathrm{d}A$ associated with the kinematic geometry of the shape sphere.

As we have already pointed out, the 1-forms Φ_i are singular at $\mathbb{x}_i = 0$, i.e. at some points P_i on the equator $\mathbb{R}^*$ of the shape-sphere $\mathbb{C}^*$. More mysteriously, when regarded as 1-forms on the shape sphere, Φ_i appears to be singular at the poles $N, S = \pm i \in \mathbb{C}^*$, i.e. for $\varphi = 0 \mod \pi$. To the lowest order in φ, Φ_1 is of the form

$$\varphi\cos\theta\mathrm{d}\theta + \sin\theta\mathrm{d}\varphi = \sin 2\theta dx - \cos 2\theta dy, \tag{4.22}$$

where x, y are rectangular coordinates at the pole, defined by $x = \varphi\cos\theta$, $y = \varphi\sin\theta$. There is a similar formula valid for Φ_2. These formulae are not compatible with smoothness of Φ_i at the poles. Hence, we should always be aware that (4.20) and (4.21) are in a strict sense *not* equivalent. However, if we stay away from the singularities, they are equivalent.

In the following, we will refer to the definition of Φ_1, Φ_2 given in (4.20), and insist that the integral

$$\int_\gamma \Phi_i$$

is well defined even at passages through the singularities. This implies that we regard Φ_i as a generalized differential form, i.e. something similar to a Schwartz distribution. Such an interpretation is well-founded: The direction of $\mathbb{x}_i$ can be regarded as a piecewise smooth function which may be discontinuous at instances where $\mathbb{x}_i = 0$. Regarded as the derivative of the direction of $\mathbb{x}_i$, it obviously make sense to integrate Φ_i along three body motions.

In the following, we will discuss the significance of these singularities for integration of closed curves on the shape sphere, and we will consider curves γ in $\mathbb{C}^*$ and mappings $\Gamma : \mathbb{D}^2 \to \mathbb{C}^*$ of the disk $\mathbb{D}^2$ into the shape-sphere which are piecewise smooth and continuous. Our goal is to use Stoke's theorem to relate the geometric phase to the spherical area.

First, let us look at the singularity at the north pole N: From (4.22), we see that the Φ_i are bounded at N. Hence, path integrals of Φ_i along small curve segments close to the pole can be ignored. Hence, for a closed curve in the upper hemisphere, $\gamma\colon S^1 \to \mathbb{C}_+^*$, with an extension $\Gamma\colon \mathbb{D}^2 \to \mathbb{C}_+^*$, Stoke's formula

$$\int_{S^1} \gamma^* \Phi_i = \int_{\mathbb{D}^2} \Gamma^* (\mathrm{d}\Phi_i) = -2 \int_{\mathbb{D}^2} \Gamma^* (\mathrm{d}A) \tag{4.23}$$

is valid, in spite of the apparent singularity. Furthermore, if γ is a curve passing through the pole, we can approximate the integral

$$\int_{S^1} \gamma^* \Phi_i$$

as accurately as we want by replacing the passage through the singularity by a small circular arc. Accordingly, (4.23) is valid even for a curve passing through the north pole. These results are equally valid for the south pole S. This discussion is summarized by formula (4.23), and the observation that polar singularities are inessential to the integration of the forms Φ_i along shape curves.

The next step is to consider the singularities at P_i. Let r, ϑ be a spherical polar coordinate system centred at P_i. For small r , we have

$$\Phi_i \approx \mathrm{d}\vartheta. \tag{4.24}$$

Accordingly, we can make

$$\int_\gamma \Phi_i - 2\pi$$

as small as we want choosing a loop γ with winding number 1 with respect to P_i in a sufficiently small neighbourhood of P_i, and hence, if we work modulo 2π, we can neglect path integrals of small loops around this singularity. In other words: if $\gamma\colon S^1 \to \mathbb{C}^*$ is a closed loop which do not pass through P_i and $\Gamma\colon \mathbb{D}^2 \to \mathbb{C}^*$ is an extension, then

$$\begin{aligned} \int_{S^1} \gamma^* \Phi_i &= \int_{\mathbb{D}^2} \Gamma^* (\mathrm{d}\Phi_i) \mod 2\pi \\ &= -2 \int_{\mathbb{D}^2} \Gamma^* (\mathrm{d}A) \mod 2\pi \end{aligned}$$

Since we are for the most interested in measuring angles modulo 2π, this formula is clearly sufficient for our needs.

Let us consider passages through the singularity P_1 with well defined incoming and outgoing directions. Suppose that

$$\gamma\colon S^1 = \frac{\mathbb{R}}{2\pi\mathbb{Z}} \to \mathbb{C}^*$$

is a closed loop with one passage through P_1, at $t = 0$, belonging to a three body motion $X(t)$ with $\Omega = 0$. For $\varepsilon > 0$, we let γ_ε be defined as follows: Let (a, b) be the maximal interval containing 0 such that $d(\gamma(t), P_1) < \varepsilon$ when $a < t < b$. Define γ_ε to be the restriction of γ to the interval $[a, b]$.

Let $\overline{\gamma}_\varepsilon$ be the circular arc going from $\gamma(a)$ to $\gamma(b)$ in the positive direction, and $\overline{\gamma}$ be the modification of γ where we have replaced γ_ε with $\overline{\gamma}_\varepsilon$. Then $\overline{\gamma}$ and $\delta = \overline{\gamma}_\varepsilon^{-1} \circ \gamma_\varepsilon$ are closed loops. If we lift δ to a three body motion $Y(t)$ with total angular momentum $\Omega = 0$, we see that we can compute the total rotational motion of $Y(t)$ by integration of Φ_2 along δ. If ε is sufficiently small, then Φ_2 will be non-singular along δ. The integral of Φ_2 along δ tends to 0 as $\varepsilon \to 0$, since $\mathrm{d}\Phi_2 = -2\mathrm{d}A$. Accordingly, the rotation of $Y(t)$ around the loop δ can be made as small as we want by choosing ε sufficiently small. Hence the error in computation of the rotation angle when we replace the original motion $X(t)$ with the lift $\overline{X}(t)$ of $\overline{\gamma}$ vanishes when $\varepsilon \to 0$. Accordingly, the integration of the rotation of $\mathbb{x}_1$ yields

$$\int_{X(t)} \Phi_1 = -2\int_{\mathbb{D}^2} \Gamma^*(\mathrm{d}A) \mod 2\pi, \tag{4.25}$$

where the integral of Φ_1 is taken along the given three body motion $X(t)$ defined over an interval $[a, b]$, which induces a closed loop $\gamma\colon S^1 \to \mathbb{C}^*$ with extension $\Gamma\colon \mathbb{D}^* \to \mathbb{C}^*$. By a subdivision argument, we see that this formula holds for any number of passages through the singularities.

Following this discussion, we arrive at the following result:

Theorem 4.4.6. *Let $X(t)$ be a piecewise smooth three body motion with vanishing total angular momentum, defined for $a \le t \le b$. Assume that $X(a)$ and $X(b)$ are of the same oriented shape. Let*

$$\gamma\colon S^1 = \to \mathbb{C}^*$$

be a reparametrization of the associated shape curve, and let

$$\Gamma\colon \mathbb{D}^2 \to \mathbb{C}^*$$

be an extension of γ*, i.e. a mapping such that the restriction to the boundary* $S^1 = \partial\mathbb{D}^2 \subset \mathbb{D}^2$ *equals* γ*. Then we conclude the following:*

The angle of rotation ψ_a^b *relating* $X(a)$ *to* $X(b)$ *satisfies*

$$\psi_a^b = -2\int_{\mathbb{D}^2} \Gamma^*(\mathrm{d}A) \mod 2\pi, \tag{4.26}$$

i.e. it is (-2) *times the oriented area of the image of* Γ *– modulo* 2π*.*

Proof. This follows from the fact that we can calculate the angular difference between the configurations $X(a)$ and $X(b)$ by means of the angular difference between $\mathbb{x}_1(a)$ and $\mathbb{x}_1(b)$, i.e. by integration of Φ_1 along the motion. ∎

We can give a simple but illustrative example of this result. In the case where $m_2 = m_3$, we consider the following three body motion: $X(0)$ is an isosceles triangle with particle 1 between particle 2 and 3. For $0 \le t \le 1$, we let $X(t)$ belong to the positively oriented isosceles triangles based at the line between particle 2 and particle 3. At $t = 1$, we let the particles 2 and 3 collide. For $1 \le t \le 2$, we let $X(t)$ be collinear. For some $t \in (1,2)$, we pass through the binary collision between particle 1 and particle 2 and finally, at $t = 2$ we arrive at a configuration $X(2)$ with the same shape as $X(0)$, i.e. an isosceles triangle with particle 1 between the other particles.

At the level of the shape sphere, this motion yields a closed loop γ which first traverses a meridian through the north pole from equator to equator, and then returns along the equator to the initial position, in the positive direction. Accordingly, for any extension Γ of γ to the disc $\mathbb{D}^2$, we have

$$\int_{\mathbb{D}^2} \Gamma^*(\mathrm{d}A) = \frac{\pi}{4}, \tag{4.27}$$

since the area of the sphere of radius $\frac{1}{2}$ is π.

Let us do a direct calculation of the angular difference: On the interval $[0,1]$, we can assume that the line spanned by particles $2,3$ is parallel to the x-axis, and that particle 1 lies above that line, and on the y-axis. Since the triangles are assumed to be positively oriented, we know that particle 2 must be to the left of particle 3. On the interval $[1,2]$, the particles will be collinear along the y-axis. Since we pass through a collision between the particles 1,2, we see that we end up with a collinear isosceles triangle aligned with the y-axis, and that particle 2 has the largest y-value. Accordingly, the angle of rotation between $t = 0$ and $t = 1$ is

$$\psi_0^2 = -\frac{\pi}{2}. \tag{4.28}$$

Clearly (4.27) and (4.28) are in complete accordance with Theorem 4.4.6.

If we assume that $\mathbb{x}_1$ is the Jacobi vector which is always parallel to the line between particle 2 and particle 3 (cf. (3.4)), $\gamma^*\Phi_1 = 0$ except at the singularity P_i, where this differential form is strictly speaking *not* well defined. We can in other words regard $\gamma^*\Phi_1$ as a delta-function with a spike at $t = 1$. On the other hand, when we calculate the angle of rotation by integration of the area form, we avoid this difficulty, and can work with well defined differential-geometric objects. Hence, there are significant advantages in computing the geometric phase with the aid of the spherical area.

4.4.6 The equations of motion in spherical geometry

In the following section, it is essential to work with the shape-sphere $\mathbb{C}^*$ rather than the shape space M^*, since the complex structure of the shape space M^* is not well defined. This affects the g_3-terms in the reduced equations; if we change the orientation of the configurations, the sign of g_3 will also have to change. We avoid these problems by working on the shape sphere, where the two hemispheres represent the different orientations.

For an oriented three body motion $(\mathbb{m}(t), X(t))$, we let $\gamma(t)$ denote the corresponding shape curve in $\mathbb{C}^*$. In the spherical shape coordinates, $\gamma(t)$ is represented by $\varphi(t), \theta(t)$, and the covariant acceleration satisfies

$$\ddot{\gamma}(t) = \left(\ddot{\varphi} - \sin\varphi\cos\varphi\dot{\theta}^2\right)\frac{\partial}{\partial\varphi} + \left(\ddot{\theta} + 2\cot\varphi\dot{\varphi}\dot{\theta}\right)\frac{\partial}{\partial\vartheta},$$

wherever this makes sense, i.e. for $\varphi \neq 0 \mod \pi$.

In the following we will express the reduced equations (3.35) in terms of the geometry of the shape space. Because of the coupling to the Euler equations (3.34), we have to take into account the different choices of principal axes gauge.

The equations of motion: For a shape curve $\gamma(t)$ represented in spherical polar coordinates as $(\varphi(t), \theta(t))$, we express the φ, θ-components of the reduced equations of motion (3.35) as

$$\ddot{\gamma} = -2\frac{\dot{\rho}}{\rho}\dot{\gamma} + \frac{1}{\rho^{2+e}}\nabla U^* - \omega_3 J\dot{\gamma} - G, \tag{4.29}$$

where J is the complex structure and

$$G = \left(\omega_1^2 - \omega_2^2\right)\cos\varphi\frac{\partial}{\partial\varphi} + \omega_1\omega_2\cot\varphi\frac{\partial}{\partial\theta} \quad \text{for} \quad \omega_i = \frac{g_i}{\lambda_i}$$

Note that the geometric gradient of the shape potential is

$$\nabla U^* = 4U^*_\varphi\frac{\partial}{\partial\varphi} + \frac{4U^*_\theta}{\sin^2\varphi}\frac{\partial}{\partial\theta}.$$

Hence, the dynamics of thee body motions in the set of regular configurations can be represented by the spherical geometric equation (4.29) together with the Lagrange-Jacobi equation (3.36) and the Euler equations (3.34). If we want to apply analytic continuation of this system through the poles, we must be aware that G is singular at the poles, and also Σ-gauge-dependent. Hence, we have to keep track of our choice of gauge along the motion.

In the planar case, $G = 0$. As we will see, we can benefit a lot from this fact:

4.4.7 Planar motion

In the planar case, $\omega_1 = \omega_2 = 0$ and $\omega_3 = \rho^{-2}\Omega_s$. The reduced equations (4.29) assume the form

$$\ddot{\gamma} = -2\frac{\dot{\rho}}{\rho}\dot{\gamma} + \frac{1}{\rho^{2+e}}\nabla U^* - \frac{\Omega_s}{\rho^2}J\dot{\gamma}. \tag{4.30}$$

In contrast to (4.29), this can be regarded as an equality between smooth differential geometric objects which are globally defined at the shape sphere. The only external data are the scalar angular momentum Ω_s and the logarithmic derivative $\dot{\rho}/\rho$.

The Lagrange-Jacobi equation can now be written as

$$\ddot{\rho} = \frac{\Omega_s^2}{\rho^3} + \rho v^2 - \frac{eU^*}{\rho^{e+1}}, \tag{4.31}$$

where v^2 is length of the shape velocity vector $\dot{\gamma}$ in the kinematic geometry of the shape sphere. Together (4.30) and (4.31) are equivalent to the reduced equations (3.28), wherever this makes sense, i.e. away from the poles $\pm i$ of the shape sphere.

In the planar case, we can take some significant advantage of the differential geometric description: From Section 3.7.1, we know that except of the umbilic shape invariant motions, planar motions $X(t)$ satisfy the Newtonian equations of motion for the three body problem if and only if the associated set $\rho, \varphi, \theta, \Omega_s$ satisfies the reduced equations (4.30). With the differential geometric description, we can include the umbilic shape invariant motions:

Suppose that $X(t)$ is an umbilic shape invariant planar motion of the three body problem. Following Theorem 3.9.10, we know that $m_1 = m_2 = m_3$, that the configuration is always equilateral, and that the motion is determined by conservation of total angular momentum and the Lagrange-Jacobi equation.

On the other hand, if $\rho(t), \varphi(t), \theta(t), \Omega$ is a solution of (4.31) and (4.30) with $\varphi(t) = 0 \mod \pi$. Then the associated shape curve $\gamma(t)$ satisfies $\dot{\gamma} = 0$, and (4.30) degenerates to

$$\nabla U^* = 0.$$

Following Lemma 3.5.1 and Lemma 3.4.4, this occurs if and only if the mass distribution satisfies $m_1 = m_2 = m_3$, i.e. if and only if $\varphi = 0 \mod \pi$ corresponds to the equilateral triangle. Hence, the differential geometric equation (4.30) singles out precisely Lagrange's equilateral solution in the case of three equal masses, which is the only existing umbilic shape invariant three body motion.

Accordingly, we can conclude as follows:

Theorem 4.4.7. *Let $X(t)$ be a virtual planar three body motion, let $\gamma(t)$ be the associated shape curve given by a shape map $M \to \mathbb{C}^* = S^2$, let $\rho(t)$ be the size variable given by $\rho^2 = I = \sum_i m_u \|\mathfrak{a}_i\|^2$, and let Ω_s be the scalar total angular momentum of the motion.*

Then $X(t)$ is a motion of the three body problem if and only if

(i) $\dot{\Omega}_s = 0$

(ii) ρ, γ *and* Ω_s *satisfies the reduced equations* (4.30), (4.31).

Proof. Three body motions exist and are analytic in t precisely on open intervals which are free from binary and triple collisions. Independently of this, we can see that solutions $\rho(t), \gamma(t), \Omega_s$ (4.30) and (4.31) exists and are analytic in t on time intervals where the singularities of (4.30) and (4.31) are avoided, i.e. at points corresponding to binary and triple collisions.

Using the analyticity of shape maps $M \cong \mathbb{C}^2 \to \mathbb{C}^*$, the singular value decomposition, as well as the lifting condition (3.27), we conclude that virtual motions $X(t)$ are analytic in t if and only if the associated data $\rho(t), \gamma(t)$ are analytic in t.

Since the reduced equations of motion are equivalent to the equations of motion for the three body problem both for motions with $\gamma(t) = \pm i$ and motions such that $\gamma(t)$ avoids $\pm i$, and all the involved data are analytic in t, we conclude that the theorem holds by continuity. ■

Now we are in position to give our final account on the reduction of the three body problem:

4.4.8 Overview over the reduction of the three body problem

Theorem 4.4.7 shows that in the planar case, the formulation of the three body problem by means of shape $\gamma(t)$ and size $\rho(t)$ has a global smooth description in terms of the kinematic geometry of the shape sphere. This reduction is complete in the following sense: The reduced equations (4.31), (4.30) yields a complete characterization of three body motions. In this reduction, the kinematic geometry of the shape-sphere is indispensable. We note that there is no trace left of the singular value decomposition in this formulation of the planar three body problem; the shape curve $\gamma(t)$ is determined by a shape map $M \cong \mathbb{C}^2 \to \mathbb{C}^*$, while the size $\rho(t)$ is determined by $\rho^2 = I = \sum_i m_i \|ba_i\|^2$.

The non-planar reduction, which is given in Section 3.7.2 is valid for analytic motions such that $\varphi = 0 \mod \frac{\pi}{2}$ only at isolated instances of time (cf. Section 3.7.6). It should be possible to prove directly that these reduced equations yields analytic solutions even at passages through $\varphi = 0 \mod \frac{\pi}{2}$, provided that these passages are not binary collisions. For the moment, we can give an indirect justification using analyticity of motions $X(t)$ of the three body problem together with the analyticity of the singular value decomposition. In this reduction, there are some traces of the singular value decomposition; this is for instance manifested by the fact that g_1, g_2, g_3 depends on the choice of principal axes gauge.

Since purely collinear three body motions are planar, this case is covered by the planar reduction.

Finally, we have the exceptional motion given in Definition 3.9.8, which is the only candidate for a three body motion which is not covered by the non-planar reduction of Section 3.7.2 and the planar reduction of Theorem 4.4.7.

We summarize this as follows:

Theorem 4.4.8 (Geometric reduction of the three body problem)**.** *According to the type of relevant reduction, the set of three body motions can be divided into the following three groups:*

(i) Exceptional motions (cf. Definition 3.9.8), which may or may not exist for $e = 2$ (cf. Proposition 3.9.9).

(ii) Planar motions: The reduction is given by Theorem 4.4.7.

(iii) Motions which are neither purely planar nor purely exceptional: The reduction is given by Section 3.7.2 and analytic continuation (cf. Section 3.7.6).

4.5 Regularization of binary collisions in the three body problem.

The equations of motion of the three body problem are analytic in all the involved parameters except at the collision points. Sundman [Sun12] proved the existence of a new independent parameter τ, in which three body motions admit analytic continuation through binary collisions. Sundman also proved that for motions with non-zero total angular momentum this continuation method defines the three body motion for all $t \in \mathbb{R}$. In the case of vanishing total angular momentum, Sundman proves that if the maximal time interval of the form (a, b) where $a < \infty$ (resp. $b > -\infty$), then the motion tends to a triple collision (resp. triple explosion) as $t \to a$ (resp. $t \to b$).

In this section, we present aspects of Lemaitre's regularization of the three body problem [Lem64] in the case $e = 1$. We do not intend to give a detailed exposition, but rather to show that the shape-sphere can be given a natural role in the regularization of the three body problem, and that this can be accomplished by application of the classical regularization of the planar Kepler problem to the binary collision points $B_1, B_2, B_3 \in \mathbb{C}^*$. This builds on the observation that the dynamics of the shape curve at the approach of binary collision points closely resembles the dynamics of the planar Kepler problem.

4.5.1 Jacobi's dynamical metric

When we study a Lagrangian system on a smooth manifold M where the Lagrange function can be expressed as the combination

$$L = \frac{1}{2} ds^2 + U,$$

of a Riemanninan metric ds^2 and a smooth function U, the dynamics of motions on a fixed energy level

$$\frac{1}{2}\left(\frac{ds}{dt}\right)^2 - U = h$$

can be studied by means of the conformally modified metric

$$ds_h^2 = (U + h) ds^2,$$

which yields a Riemannian metric on the subspace

$$M_h = \{p \in M : U(p) + h \geq 0\} \subset M.$$

This metric will be called *Jacobi's dynamical metric* for the energy level h.

Modulo reparametrization, the geodesics in the Riemannian manifold

$$(M_h, ds_h^2)$$

correspond precisely to the motions of the Lagrange system (M, L) with total energy h. Hence, in order go understand the unparametrized motions of the system (M, L), it is sufficient to understand the geodesics of the Riemannian manifolds (M_h, ds_h^2); the time parametrization can be reconstructed using the equation

$$\frac{ds}{dt} = \sqrt{U + h}.$$

4.5.2 Regularization of the Kepler problem

The planar Kepler problem can be defined as the Lagrange system given by the Lagrange function

$$L = \frac{1}{2}(dx^2 + dy^2) + \frac{G}{\sqrt{x^2 + y^2}},$$

defined on the punctured x, y-plane $M = \mathbb{R}^2 \setminus \{0\}$. In polar coordinates r, θ, this reads

$$L = \frac{1}{2}(dr^2 + r^2 d\theta^2) + \frac{G}{r}.$$

Jacobi's dynamical metric for the energy level h is

$$ds_h^2 = \left(h + \frac{G}{r}\right)\left(dr^2 + r^2 d\theta^2\right).$$

The Gaussian curvature of this metric is

$$\kappa = -\frac{Gh}{2(hr + G)^3},$$

and we find it interesting to note that $\kappa = 0$ precisely when $h = 0$. For $h = 0$, the Kepler orbits are parabolic. In the light of Jacobi's dynamical metric we see that these parabolas have an interpretation as geodesics on the flat Riemannian manifold (M_h, ds_h^2).

κ is negative for positive energy levels h, and positive for negative energy levels h. Combined with Jacobi-Morse theory, this yields an interesting approach to explain the Poincaré-stability of solutions of the Kepler problem. For positive energy levels, the Kepler orbits tend to diverge from each other. This tendency becomes more intense at passages through regions where κ is large and negative, i.e. when r is close to 0. On the other hand, when h is negative, κ is positive and the orbits will always remain close to each other. Note that for two orbits passing on opposite sides of the singularity in $r = 0$, there may – and will – occur dramatic effects which lie outside the scope of Jacobi-Morse theory.

For an interesting investigation of the relation between the curvature of Jacobi's dynamical metric and phase transitions in statistical mechanics, see [Pet07].

Regularization of unparametrized curves

We take the flatness of ds_0^2 as an encouragement to study the Kepler problem using the geometry defined by Jacobi's dynamical metric for $h = 0$,

$$ds_0^2 = \frac{G}{r} dr^2 + Gr\, d\theta^2$$

as a background geometry.

Fortunately, in this geometry, the distance to the singularity $r = 0$ is always finite. Let ρ denote this distance. In this way define a new set of coordinates, (ρ, θ) in which the dynamical metric is of the form

$$ds_0^2 = d\rho^2 + \frac{\rho^2}{4} d\theta^2.$$

By integration, we find that $\rho = 2\sqrt{Gr}$.

By inspection of the new form of the metric, we see that the space (M_h, ds_0^2) can be regarded as a Riemannian cone over the circle of radius $\frac{1}{2}$, and that the singularity at $r=0$ is equivalent to a cone with vertex angle $\frac{\pi}{3}$. Such a cone can be straightened out by application of the double covering $S^1(1) \to S^1(\frac{1}{2})$ of the circle of radius 1 onto the circle of radius $\frac{1}{2}$. We can extend this covering to a double covering $\mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ given by

$$(\rho,\vartheta) \mapsto (\rho,\theta), \quad \text{where} \quad \theta = 2\vartheta,$$

and in the new variables ρ, ϑ, Jacobi's dynamical metric for the energy level $h=0$ in the Kepler problem reads

$$ds_0^2 = d\rho^2 + \rho^2 d\vartheta^2.$$

These two steps are captured by the mapping

$$(\rho,\vartheta) \mapsto (r,\theta) = \left(\frac{\rho^2}{4G}, 2\vartheta\right).$$

Using complex notation $z = \rho e^{\vartheta i}$, $w = re^{\theta i}$, we can write this mapping as

$$z \mapsto w = \frac{z^2}{4G}.$$

In the new coordinates ρ, φ, Jacobi's dynamical metric for the energy level h in the Kepler problem satisfies

$$\begin{aligned} ds_h^2 &= \left(\frac{G}{r} + h\right)\left(dr^2 + r^2 d\theta^2\right) \\ &= \left(\frac{4G^2}{\rho^2} + h\right)\left(\frac{\rho^2}{4G^2}\mathrm{d}\rho^2 + \frac{\rho^4}{4G}d\vartheta^2\right) \\ &= \left(1 + \frac{h}{4G^2}\rho^2\right)\left(d\rho^2 + \rho^2 d\vartheta^2\right). \end{aligned}$$

Hence, under the mapping $(\rho,\vartheta) \mapsto (r,\theta)$, Jacobi's dynamical metric for the energy level h in the Kepler problem transforms to Jacobi's dynamical metric for the energy level 1 for the Lagrangian system with Lagrange function

$$L = \frac{1}{2}(dx^2 + dy^2) + \frac{1}{2}\left(\frac{h}{2G^2}\right)(x^2 + y^2),$$

i.e. the harmonic oscillator with spring constant $-\frac{h}{2G^2}$. Note that for $h < 0$, the spring constant is positive and for $h > 0$, the spring constant is negative. The two cases correspond respectively to bounded elliptic motions and unbounded hyperbolic motions of the harmonic oscillator.

This shows that there is a strong correspondence between the orbits of the harmonic oscillator and the orbits of the Kepler problem, at least from a geometric point of view.

Kepler motions cease to exist at the collision point $r = 0$. Now we will see how the correspondence with the harmonic oscillator can help us to regularize motions which pass through this point. As above, we let t denote the natural time in the Kepler problem. Let τ denote the natural time in the harmonic oscillator, i.e. the curve parameter given by the energy level.

When we study collision orbits, we can assume –without loss of generality – that the motion takes place along the x-axis. Modulo reparametrization of time and space, the collision orbits of the harmonic oscillator are of one of the forms

$$x(\tau) = \sin\tau, \quad x(\tau) = \tau, \quad x(\tau) = \sinh\tau,$$

depending on the sign of the spring constant. Modulo reparametrization, this yields Kepler orbits of the form

$$x(\tau) = \frac{1}{4G}\sin^2\tau, \quad \frac{1}{4G}\tau^2, \quad x(\tau) = \frac{1}{4G}\sinh^2\tau. \tag{4.32}$$

If parametrize the motion with the natural time parameter t of the Kepler problem, $x(t)$ will satisfy the equations of motion of the Kepler problem, except at the point $x = 0$. Hence, we can regard $x(\tau)$ as an extension of the solution of the Kepler problem.

From the formulae (4.32) we see that the regularized Kepler-collisions – modulo reparametrization – resemble elastic collisions, as if the direction of time is reversed at the time of collision.

Hence, if $\gamma(t)$ is a Kepler motion with maximal interval of existence $(0,1)$, then $\gamma(t)$ will experience collisions at $0,1$. With the above technique, we extend the motion $\gamma(t)$ to a motion $\tilde{\gamma}(t)$ defined for all t by letting

$$\tilde{\gamma}(t) = \gamma(t \mod 1),$$

where $t \mod 1$ is interpreted as a real number in the interval $[0,1)$. This motion satisfies the equations of motion for the Kepler problem when $t \notin \mathbb{Z}$. Note that $\tilde{\gamma}(t)$ is not analytic for $t \in \mathbb{Z}$. On the other hand, if we parametrize $\tilde{\gamma}$ by the natural time parameter τ for the corresponding harmonic oscillator, we get a curve $\tilde{\gamma}(\tau)$ which is analytic in τ for all $\tau \in \mathbb{R}$.

Time regularization

As above, we let t denote the time scale of the Kepler problem, and τ the time scale of the harmonic oscillator. Since τ can be used as a regularizing parameter for the Kepler problem, we want to investigate the relation between t and τ. Because of the rotational symmetry of the problem, it should be sufficient to study purely radial motions.

For a motion $r(t)$ of the Kepler problem we have

$$\frac{dr}{dt} = \pm\sqrt{\frac{G}{r} + h}.$$

For a motion $\rho(\tau)$ of the harmonic oscillator, we similarly get

$$\frac{d\rho}{d\tau} = \pm\sqrt{1 + \frac{h}{4G^2}\rho^2} = \pm\sqrt{1 + \frac{h}{G}r},$$

and since

$$dr = \frac{\rho}{2G}d\rho = \sqrt{\frac{r}{G}}d\rho,$$

we have

$$dt = \frac{dr}{\sqrt{\frac{G}{r} + h}} = \frac{\sqrt{\frac{r}{G}}d\rho}{\sqrt{\frac{G}{r} + h}} = \sqrt{\frac{\frac{hr}{G}\left(\frac{1}{h} + \frac{r}{G}\right)}{\frac{G}{r} + h}}d\tau = \frac{r}{G}d\tau,$$

i.e.

$$d\tau = \frac{G}{r}dt = U dt. \tag{4.33}$$

4.5.3 Regularization of the Kepler problem for $e \neq 1$

We now ask the following question: Can we do a similar reduction in the case where the potential function is of the form

$$U(x,y) = \frac{1}{r^e}, \quad \text{where } r^2 = x^2 + y^2$$

and $e \neq 1$.

Jacobi's dynamical metric for the energy level $h = 0$ reads

$$ds_0^2 = \frac{1}{r^e}\left(dr^2 + r^2 d\theta\right).$$

We introduce a variable ρ, which satisfies

$$d\rho = \frac{dr}{r^{\frac{e}{2}}}$$

In the case $e \neq 2$, we can take

$$\rho = \frac{2}{2-e} r^{\frac{2-e}{2}}, \tag{4.34}$$

and the dynamical metric can be written as

$$ds_0^2 = d\rho^2 + \left(\frac{2-e}{2}\right)^2 \rho^2 d\theta^2. \tag{4.35}$$

For every $e \neq 0$ this is the metric of a cone, where ρ measures the distance to the vertex.

For $e > 2$, the vertex of this cone corresponds to $r = \infty$. Hence, for $e > 2$, a regularization by means of a flattening of the cone will yield a regularization of infinity. From the following considerations, this seems reasonable: For large r, we have $d\tau \approx \frac{1}{2}\sqrt{2} r^{-e/2} dr$. Hence in the case $e > 2$, we can reach infinity for finite values of τ. On the other hand, using the parameter τ, we will never be able to reach $r = 0$. We see that if we insist on using the independent variable τ in the case $e > 2$, it makes sense to regularize infinity rather than the collision point. After these considerations, we will place $e > 2$ outside the scope of our discussion.

For $e < 2$ the vertex of the cone corresponds to $r = 0$. Hence, a flattening of the cone will yield a regularization of the collision point.

For $e = 2$, we can take

$$\rho = \ln r, \tag{4.36}$$

and the dynamical metric is now

$$ds_0^2 = d\rho^2 + d\theta^2,$$

i.e. the metric of an infinite cylinder $\mathbb{R} \times S^1$. In the case $e = 2$ it is obviously impossible to find a regularising map $\mathbb{R}^2 \to \mathbb{R}^2$ for the metric ds_0^2.

In the case $e < 2$ we will consider the possibility of regularizing maps $\mathbb{R}^2 \to \mathbb{R}^2$ of the form $z \mapsto Az^n$. Such a regularization restricts to a map $S^1 \to S^1$ of winding number n. By measuring the ratio between the radius and circumference of circles around the vertex of the cone, using the metric (4.35), we see that such a regularizing map can exist only if

$$\frac{2}{2-e} = n, \quad \text{i.e.} \quad e = 2\frac{n-1}{n}.$$

Since we are interested in $0 \le e < 2$, n is allowed to range over all positive integers. The corresponding regularization maps are

$$z \mapsto \frac{z^n}{n^2}.$$

In the variables ρ, ϑ, Jacobi's dynamical metric assumes the form

$$ds_h^2 = \left(\frac{h}{n^2}\rho^{2(n-1)} + n^{\frac{2n-4}{n}}\right)\left(d\rho^2 + \rho^2 d\vartheta^2\right).$$

This can be regarded as Jacobi's dynamical metric associated with the Lagrange function

$$L = \frac{1}{2}(dx^2 + dy^2) + \frac{h}{n^2}(x^2 + y^2)^{(n-1)}$$

and the energy level $n^{\frac{2n-4}{n}}$. Since $n > 0$, these Lagrange systems yield analytic motions.

Accordingly, we see that the regularization of the Kepler problem can be generalized only to a discrete set of values of e, and in particular not to the case $e = 2$. We give the following examples of n, e-values and regularisation maps:

$$\begin{aligned} n = 1, \quad & e = 0, \quad z \mapsto z \\ n = 2, \quad & e = 1, \quad z \mapsto \frac{z^2}{4} \\ n = 3, \quad & e = \frac{4}{3}, \quad z \mapsto \frac{z^3}{9} \end{aligned}$$

Note that when n is odd, such a regularization yields passages *through* $\rho = 0$, while for n even, it corresponds to *elastic collisions* at $\rho = 0$.

4.5.4 Regularization of the three body problem

A binary collision of the three body problem is – from a heuristic point of view – mainly a question of the dynamics of shape. The overall rotational motion, as well as the dynamics of the variable ρ is mainly determined by the motion of the non-colliding body relative to the centre of mass of the colliding bodies. This gives a reason for us to believe that the interesting features of a binary collision are properly represented by the shape curve $\gamma(t)$ in $\mathbb{C}^*$. Authors like Sundman [Sun12] and Levi-Civita [LC20] can be interpreted as giving rigorous confirmations of this point of view.

Following our heuristics, we should neglect data which are not directly related to the shape curve $\gamma(t)$. This leads to the following simplification of the differential geometric representation (4.29) of the shape dynamics:

$$\ddot{\gamma} \sim \frac{1}{\rho^3} \nabla U^*, \tag{4.37}$$

where we should regard ρ^3 as a constant. The equivalence sign $\sim$ indicates equality in the lowest order in the expansion in t at binary collisions points.

Formula (3.20) expresses the asymptotic behaviour of the shape potential U^* near the binary collision points B_i. Following our restriction to the case

$e = 1$, we have the asymptotic formula

$$U^* \sim F_0^i / r_i \tag{4.38}$$

where $r_i$denotes the spherical distance to B_i.

Following the heuristic formulae (4.37),(4.38), we expect that the dynamics of shape curves approaching the binary collision points B_i resembles the collision dynamics of the planar Kepler problem. This observation can be taken the heuristic background of the following treatment of the binary collisions of the three body problem.

Note that we do not claim that the dynamics of the shape curve near the binary collision point in general resembles the dynamics of the planar Kepler problem. As an example of this, we can take the following "counter-example", which is similar to Sitnikov's solution [Cab90]: Let $\varepsilon > 0$ given. In the case of two equal masses, say $m_2 = m_3$, there exist three body motion where the shape curve moves along a meridian passing through the binary collision point B_1, such that the minimal distance between $\gamma(t)$ and B_1 is less than ε, but where the motion can be extended to all $t \in \mathbb{R}$ without suffering any binary collisions. This contrasts the Kepler problem, where a motion confined to a straight line through the origin must experience a collision either in the future or in the past. Hence, every neighbourhood of B_1 meets shape curves which do not resemble Kepler orbits.

Let ds^2 denote the kinematic metric of the shape sphere, and let

$$ds_0^2 = U^* ds^2.$$

This is *not* a dynamical metric in the sense of Jacobi, but is chosen in order to resemble Jacobi's dynamical metric for the energy level $h = 0$ in the Kepler problem. In the following we will study the binary collisions by means of ds_0^2:

Let us investigate the asymptotic behaviour of this metric near binary collision point B_i on the shape-sphere $\mathbb{C}^*$. If r_i, θ_i is a spherical coordinate system centred at the binary collision $B_i \in \mathbb{R}^* \subset \mathbb{C}^*$, the kinematic metric reads

$$ds^2 = \frac{1}{4}(dr_i^2 + \sin^2(r_i) d\theta_i^2) = \frac{1}{4}(dr_i^2 + (r_i^2 + \cdots) d\theta_i^2).$$

The conformally modified metric is

$$ds_0^2 = F_0^i \left(\frac{1}{r_i}(1 + a_1 r_i + \cdots) dr_i^2 + r_i(1 + b_1^i r_i + \cdots +) d\theta_i^2 \right).$$

If we identify (r_i, θ_i) with the complex number $w = r_i e^{i\theta_i}$, we can define a mapping $\mathbb{C}^* \to \mathbb{C}^*$ by

$$z \mapsto w = \frac{z^2}{4F_0^i}.$$

Define auxiliary variables ρ_i, ϑ_i satisfying

$$r_i = \frac{\rho_i^2}{4F_0^i} \quad \theta_i = 2\vartheta_i.$$

In this new set of variables, we have

$$ds_0^2 = (1 + \alpha_1 \rho_i^2 + \cdots) d\rho_i^2 + (\rho_i^2 + \beta_1 \rho_i^4 + \cdots) d\vartheta_i^2 \sim d\rho_i^2 + \rho_i^2 d\vartheta_i^2.$$

Hence, the metric ds_0^2 is regularized at B_i by pullback to the variables ρ, ϑ_i.

Now, we will extend this and describe a map $\mathbb{C}^* \to \mathbb{C}^*$ which yields a simultaneous regularization of the three regularization points. We modify Lemaitre's regularization map [Lem64] slightly, and consider mappings $\mathbb{C}^* \to \mathbb{C}^*$ which are of the form

$$z \mapsto \zeta(z) = \frac{-4 w_2 w_3 z^2}{(w_2 - w_3) z^4 - 2(w_2 + w_3) z^2 + 1(w_2 - w_3)}, \tag{4.39}$$

where we assume that w_2, w_3 are distinct and non-zero real numbers. Figure 4.5 explains some features of this map.

As $\mathbb{C}^*$-valued function, $\zeta(z)$ is singular at $S = \{0, 1, i, -1, -i, \infty\}$, and the singularities are mapped in the following way:

$$\zeta(0) = \zeta(\infty) = 0, \quad \zeta(1) = \zeta(-1) = w_2, \quad \zeta(i) = \zeta(-i) = w_3. \tag{4.40}$$

If we exclude the singularities, we get a holomorphic local diffeomorphism

$$\zeta \colon \mathbb{C}^* \setminus S \to \mathbb{C}^* \setminus \{0, w_2, w_3\}.$$

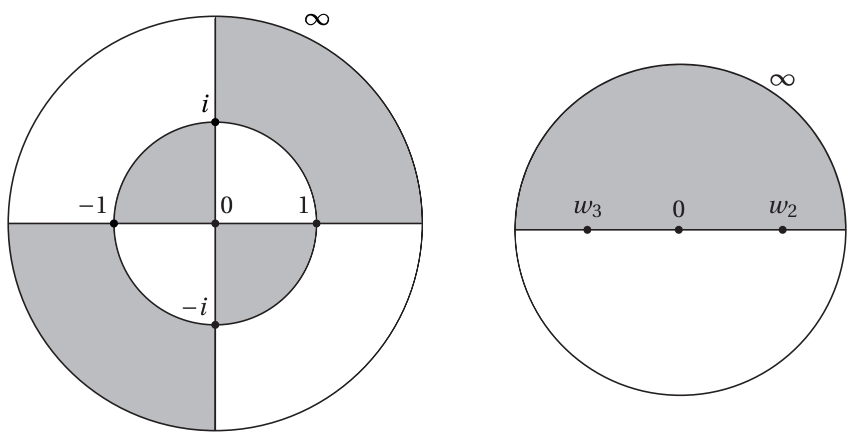

Figure 4.5: Diagram showing main features of the regularization map $z \mapsto \zeta(z)$ given by (4.39). Note that the point in infinity, $\infty \in \mathbb{C}^*$ is represented by the boundary circles. The gray (resp. white) triangles in the left figure are mapped to the gray (resp. white) hemisphere in the right figure. The mapping is singular at the dots and the boundary circle in the left figure. Singular values are marked by dots in the right figure. The correspondence between the dots in the figures is given by (4.40).

At the singularities, this mapping is locally analytically equivalent to

$$z \mapsto z^2.$$

Hence, if we choose a democracy transformation such that the third binary collision point $B_3 = 0 \in \mathbb{R}^*$, then we can apply the mapping $z \mapsto \zeta(z)$ in the case $w_2 = B_1, w_3 = B_2$. Since the triangle B_1, B_2, B_3 is always central, w_2, w_3 will always be distinct, finite and non-zero (cf. Definition 4.3.8).

The pullback of

$$ds_0^2 = U^* ds^2$$

through the mapping $z \mapsto \zeta(z)$ will be a smooth Riemannian metric on the Riemann sphere $\mathbb{C}^*$: This follows from the fact that $z \mapsto \zeta(z)$ is a local diffeomorphism away from the singularities, and locally equivalent to $z \mapsto z^2$ at the singularities.

Let us define a new independent parameter τ satisfying $d\tau = U^* dt$. Since U^* is always positive, and also analytic away from the binary collision points B_1, B_2, B_3, the shape curves of collision free three body motions will be analytic in τ. Let us now consider at shape curve $\gamma(t)$ which approach the binary collision point B_i as $t \to 0$, and apply the above regularization of the planar Kepler problem to this situation. Let $\tilde{\gamma}$ be the lifting of γ through ζ. Then the motion $\tilde{\gamma}(\tau)$ will asymptotically – at the binary collision – look like the motion of a harmonic oscillator. Hence, $\tilde{\gamma}(\tau)$ admits analytic continuation through the collision point. Now we can extend γ through $t = 0$ by projection of $\tilde{\gamma}$ through $z \mapsto \zeta(z)$.

First, we conclude from this that if we parametrize the shape curve γ by the variable τ, then γ admits analytic continuation though the binary collision points. Secondly, we conclude that this continuation implies that the shape curve undergoes a collision-ejection motion at the binary collision, i.e. a motion similar to an elastic collision with a fixed point. Hence, under this regularization, the shape curve will describe cusps at the binary collision points.

Note that this discussion of the regularization of the three body problem does little justice to Sundman [Sun12]. Some of the most non-trivial parts of his work concerns the behaviour of triple collisions, and in particular the fact that three body motions with non-vanishing total angular momentum avoids triple collisions. For a treatment of related questions in the flavour of this thesis, see [HS07].

4.5.5 The shape-space revisited

Let us for a moment return to the discussion of the shape of the shape space M^*. Following Proposition 4.2.1, we should regard the shape space as a triangular sub-region of an ellipsoid. Proposition 4.2.3 contradicts this, and claims that the shape-sphere should be regarded as a closed disk $\mathbb{D}^2 \subset \mathbb{R}^2$.

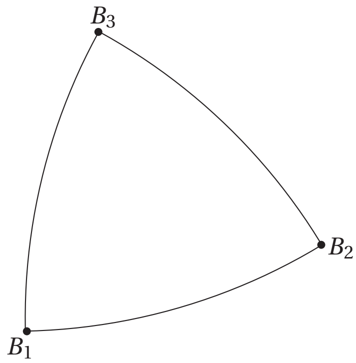

Figure 4.6: The shape space M^* represented as a spherical triangle with orthogonal corners. This can be regarded as the stereographic projection of one of the gray regions of Figure 4.5, and hence this planar figure yields an appropriate representation of the conformal structure.

In our discussion of the regularization of the three body problem, we have a found third representation of the shape sphere: After pullback through the map $\zeta\colon \mathbb{C}^* \to \mathbb{C}^*$ of (4.39), the upper hemisphere of the shape-sphere can be identified with the spherical triangle with corners $0, i, -1$, i.e. a spherical triangle with three orthogonal corners (cf. Figure 4.5). After stereographic projection in the antipodal point of the pre-image of $i \subset \mathbb{C}^*$, we get a conformal equivalence with a convex planar figure consisting of three congruent circular arcs joined at orthogonal vertices, as indicated in Figure 4.6.

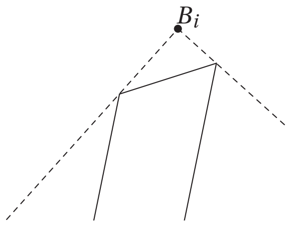

Figure 4.7: Near-collision shape curve, in the representation of the space of three body shapes as a spherical triangle with orthogonal corners. The outgoing ray is parallel to the outgoing ray since the corner is orthogonal.

This representation of the shape space is very well adapted to the study of analytic m-triangle motions: In Section 4.4.2, we defined a non-standard notion of analytic shape curves. Following Proposition 4.4.2 we will regard a shape curve as analytic if it is analytic on the interior of the shape space M^* and is reflected at encounters with the boundary (cf. Figure 4.4). According to our shape-sphere-description of Sundman's regularization of the three body problem, we propose to call a curve in M^* which pass through a collision point B_i analytic if it is analytic on the interior of M^*, suffers reflections at the boundary, and behaves like a collision-ejection motions at B_i.

Following our representation of the shape space as a spherical triangle with orthogonal corners at the binary collision points B_i, we have the situation of Figure 4.7, which indicates the parallelism between the incoming and outgoing direction of shape curves with near-collision trajectories. Hence near-collision trajectories will remain close to the collision-ejection trajectories associated with binary collisions. This observation can be regarded as an independent justification of the naturality of the collision-ejection regularization; it is the natural choice given by continuity and the analytic structure of the shape sphere.

5 Applications to the three body problem

5.1 Homographic solutions of the three body problem

A homographic solution of the three body problem is a motion where the triangle formed by the three bodies is always of the same shape. If we choose a representative $(\overline{\mathbb{a}}_1, \overline{\mathbb{a}}_2, \overline{\mathbb{a}}_3)$ for the similarity class of such a motion, the motion can always be represented as

$$\mathbb{a}_i(t) = \rho(t)R(t)\overline{\mathbb{a}}_i,$$

where $\rho(t)$ is a smooth real valued function in one variable and $R(t)$ is a curve in SO(3). It follows from the description of three body motions by means of the singular value decomposition (cf. (3.7)) and the definition of the variables φ, θ that homographic motions satisfy either

(A) $\sin\varphi = 0$,

or

(B) $\dot{\varphi} = \dot{\theta} = 0$.

Case (A) clearly corresponds to the umbilic shape invariant motion which was treated in Section 3.9, and in that case we concluded that umbilic shape invariant motions may occur (i) in the case of equal masses, in the form of Lagrange's equilateral solution and (ii) in the case of exceptional motions in the sense of Definition 3.9.8. In case (ii) it follows from Proposition 3.9.9 that the power e of the potential is necessarily equal to 2.

Leaving the exceptional three body motions to the future, we concentrate on the case where $\sin\varphi \neq 0$. I.e. case (B) above, which is characterized by

$$\dot{\varphi} = \dot{\theta} = 0.$$

5.1.1 The planar case

In the planar case, it follows from Theorem 4.4.7 that shape invariant motions occur only for

$$\nabla U^* = 0$$

where U^* is the shape potential interpreted as a function on the shape sphere. From Lemma 3.4.4 we know that this occurs only for the equilateral shape L and three collinear shapes E_1, E_2, E_3. The corresponding solutions are precisely Lagrange's equilateral solutions [Lag72] and Euler's collinear solutions [Eul67].

5.1.2 The non-planar case

Since collinearity implies planarity (cf. Proposition 2.7.9), we can in this case assume that $\cos\varphi \neq 0$, i.e. $\sin\varphi \neq \pm 1$. Additionally, case (A) above included $\sin\varphi = 0$. Hence, in the following we assume that

$$\sin\varphi, \cos\varphi \neq -1, 0, 1.$$

In this situation we write the reduced equations (3.33') as

$$\begin{aligned}
&(R1) \quad \ddot{\rho} = \frac{-\dot{\rho}^2}{\rho} + \frac{1}{\rho}\left(\frac{(2-e)u^*}{\rho^e} + 2h\right) \\
&(R2) \quad g_1^2(1+\sin\varphi)^2 - g_2^2(1-\sin\varphi)^2 = A\rho^{2-e} \\
&(R3) \quad g_1 g_2 = B\rho^{2-e},
\end{aligned}$$

where $A = U^*_\varphi(\varphi,\theta)\cos^3\varphi$ and $B = \frac{1}{2}\cot\varphi U^*_\theta$. Following the assumption of non-planarity together with Lemma 2.8.1 we conclude that either g_1 or g_2 must be non-zero, i.e. that

$$g_1^2 + g_2^2 > 0.$$

The Euler equations (3.32') now read

$$\begin{aligned}
(E1) \quad \dot g_1 &= -\left(\frac{1-\sin\varphi}{1+\sin\varphi}\right)\frac{g_2 g_3}{\rho^2}\\
(E2) \quad \dot g_2 &= \left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)\frac{g_3 g_1}{\rho^2}\\
(E3) \quad \dot g_3 &= -\left(\frac{4\sin\varphi}{\cos^2\varphi}\right)\frac{g_1 g_2}{\rho^2},
\end{aligned}$$

since $\dot\theta = 0$. The combination $(1+\sin\varphi)^2 g_1(E1) + (1-\sin\varphi)^2 g_2(E2)$ yields

$$(1+\sin\varphi)^2 g_1\dot g_1 + (1-\sin\varphi)^2 g_2 \dot g_2 = 0.$$

Since $\dot\varphi = 0$, this implies the existence of a constant C such that

$$(1+\sin\varphi)^2 g_1^2 + (1-\sin\varphi)^2 g_2^2 = C. \tag{5.1}$$

Since $g_1^2 + g_2^2 > 0$, we conclude that $C > 0$.

Combined with (R2), (5.1) yields

$$g_1^2 = \frac{C + A\rho^{2-e}}{2(1+\sin\varphi)^2}, \quad g_2^2 = \frac{C - A\rho^{2-e}}{2(1-\sin\varphi)^2}, \quad i.e. \quad g_1^2 g_2^2 = \frac{C^2 - A^2\rho^{2(2-e)}}{4\cos^4\varphi} \tag{5.2}$$

On the other hand, (R3) yields $g_1^2 g_2^2 = B^2\rho^{2(2-e)}$, and accordingly we conclude that for non-umbilic non-planar shape invariant three body motions,

$$4\cos^4\varphi B^2 + A^2 = C^2\rho^{2(e-2)}. \tag{5.3}$$

Since $C > 0$, this implies that ρ^{e-2} is constant, and from (5.2), we see that g_1^2, g_2^2 and $g_3^2 = \Omega^2 - g_2^2 - g_2^2$ are constant. By continuity, we conclude that g_1, g_2, g_3 must be constant.

From (E3) it follows that $g_1 g_2 = 0$. Both of them can not be zero, since this implies planar motion. Accordingly, by an appropriate choice of gauge, we can assume that $g_1 \neq 0$ and $g_2 = 0$. From (E2) we then conclude that $g_3 = 0$ i.e. that

$$\Omega = g_1 \mathbb{u}_1 \tag{5.4}$$

The case $e \neq 2$

In the following we show that (5.4) is absurd in the case where $e \neq 2$. By (5.3) we note that ρ is constant in this case.

The plane spanned by the three body configuration $\Pi(t) = \mathrm{span}(\mathbb{u}_1, \mathbb{u}_2)$ contains the total angular momentum vector $\Omega = g_1 \mathbb{u}_1$ and rotates around this vector. On the other hand, the configuration moves within the variable plane $\Pi(t)$ in such a way that there exists functions $x_i(t), y_i(t)$ with

$$\mathbb{a}_i(t) = x_i(t)\mathbb{u}_1 + y_i(t)\mathbb{u}_2$$

Since $\sin\varphi \neq 0$, the principal axes are uniquely determined by the configuration, modulo a finite number of choices. This implies that $x_i(t), y_i(t)$ are constant, since $\dot{\rho} = 0$.

Now, we shall show that $x_1 = x_2 = x_3$, and hence the motion is collinear along $\mathbb{u}_2$: Suppose that $x_1 > x_2 \geq x_3$. Then, by the attraction of the other particles on particle 1, particle 1 should experience a negative acceleration $\ddot{x}_1 < 0$. This acceleration is not balanced by the rotation of the configuration, since the rotation axis lies along $\mathbb{u}_1$. This contradicts the constancy of the x_i. Accordingly, $x_1 = x_2 = x_3$, and the motion is collinear. Since collinear motions always take place in the invariable plane, this contradicts the assumption of non-planarity.

From this we conclude the following:

Lemma 5.1.1. *There do not exist any non-planar non-umbilic shape-invariant motions for the three body problem with* $e \neq 2$.

The case $e = 2$

In the choice of gauge giving (5.4), i.e. $g_2 = g_3 = 0$, $g_1 \neq 0$, (R2) and (R3) yields

$$U^*_\theta = 0, \quad U^*_\varphi = \frac{g_1^2 \cos\varphi}{(1 - \sin\varphi)^2}. \tag{5.5}$$

Hence, these are necessary conditions the shape (φ, θ) has to satisfy for shape invariant (i.e. homographic) non-planar non-umbilic three body motions with $e = 2$.

On the other hand, for a given constant value of g_1, shape variables φ, θ satisfying (5.5) and $\sin\varphi \neq -1, 0, 1$ we can, in fact, construct a homographic three body motion solution with $e = 2$: Let $g_2 = g_3 = 0$ and let $\rho(t)$ be a solution of (R1), which in this case reads

$$\ddot{\rho} + \frac{\dot{\rho}^2}{\rho} - \frac{2h}{\rho} = 0. \tag{5.6}$$

This yields a solution

$$\rho(t), \varphi, \theta, g_1, g_2, g_3$$

of the reduced equations. Consequently, by straightforward reconstruction we obtain a three body motion $X(t) = (\mathbb{a}_1(t), \mathbb{a}_2(t), \mathbb{a}_3(t))$ with the centre of mass fixed at the origin which satisfies:

(i) Shape-invariance

(ii) $\Omega \in \operatorname{span}(\mathbb{a}_1, \mathbb{a}_2, \mathbb{a}_3)$

We summarize this as follows:

Lemma 5.1.2. *In the case* $e = 2$, (5.5), $g_2 = g_3 = 0$ *and* (5.6) *yield necessary and sufficient conditions for non-planar non-umbilic homographic three body motions.*

For every pair φ, θ such that

$$\varphi \neq 0 \mod \frac{\pi}{2}, \qquad U^*_\theta(\varphi, \theta) = 0 \quad \text{and} \quad \frac{U^*_\varphi}{\cos\varphi} > 0, \tag{5.7}$$

we can find a value of g_1 such that (5.5) is satisfied. Thus, the crucial question is wether there exist values of φ, θ satisfying (5.7). We will not discuss this question in full generality here, but rather discuss some special cases:

Let us consider the case of two equal masses $m_2 = m_3$. We choose the Jacobi vectors such that $\varphi = \frac{\pi}{2}, \theta = 0$ corresponds to the binary collision B_1. Since $m_2 = m_3$, there is a symmetry of the system that is given by relabeling of the equal masses, and which yields a democracy transformation that permutes the binary collisions B_2, B_3. On the level of the shape shere $\mathbb{C}^*$ this symmetry is represented by reflection (i.e. inversion) in the circle through $0, i, \infty \in \mathbb{C}^*$. From this, one can see that $U^*_\theta = 0$ when

$$(*) \quad (\varphi, \theta) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \{0\},$$

and that $U^*(\varphi, 0)$ is a monotonically increasing function of φ sufficiently close to $\frac{\pi}{2}$. For such values of (φ, θ), $U^*_\varphi / \cos\varphi > 0$, since $\cos\varphi \geq 0$ when $\varphi \in (-\pi/2, \pi/2)$. We conclude that there will exist solutions of (5.5) in the case of two equal masses.

Banachiewitz [Ban06] and Wintner's book [Win47] have given examples of three body motions of the above kind, namely with isosceles triangle shape and with two equal masses.

5.1.3 Conclusions

Following the discussion in this section, we can give the following classification of homographic three body motions:

Theorem 5.1.3. *The homographic motions of the three body problem with potential function*

$$U = \sum_{i<j} \frac{m_i m_j}{r_{ij}^e}$$

belongs to the following classes:

(i) Planar motions with one of the shapes E_1, E_2, E_3, L given in Lemma 3.4.4.

(ii) Motions with $\Omega \neq 0$ and non-umbilic shape given by Lemma 5.1.2.
Such motions exist only for $e = 2$.

(iii) Exceptional motions (cf. Definition 3.9.8).
Such motions exist only for $e = 2$.

Pylarinos presents a similar result in [Pyl41]: He identifies the planar motions (i) above as the only homographic solutions in the case $e \neq 2$. In the case $e = 2$ Pylarinos states that there exist motions that, in the terminology of this dissertation, satisifies $g_1 \neq 0, g_2 = g_3 = 0$. We go beyond Pylarinos' work when we give the characterizations (ii) and (iii) of this class of motion.

As a step towards a more precise classification of homographic motions [Pyl49] gives a partial characterisation of general three body motions satisfying $g_1 \neq 2, g_2 = g_3 = 0$; they have to satisfy an equation of the form

$$\frac{A_1}{r_{23}^e} + \frac{A_2}{r_{31}^e} + \frac{A_3}{r_{12}^e} = 0,$$

where A_1, A_2, A_3 are some constants, which depends on the motion, and r_{ij} are the relative distances.

Within the formalism of the present dissertation, we easily arrive the following conditions:

$$\begin{aligned}
\dot{\theta} &= 0 \\
U_\theta^* &= 0 \\
\ddot{\varphi} &= -\frac{2\dot{\rho}\dot{\varphi}}{\rho} + \frac{4U_\varphi^*}{\rho^{2+e}} \\
\ddot{\rho} &= \frac{2g_1^2}{\rho^3(1-\sin\varphi)} + \frac{1}{4}\dot{\varphi}^2 - \frac{eU^*}{\rho^{2+e}}
\end{aligned}$$

It seems plausible that Pylarinos' partial characterization is equivalent to the second equation of our characterization, namely $U_\theta^* = 0$.

5.2 The constant inclination problem

Along a three body motion $X(t)$, the configurations span a variable plane $\Pi(t)$ with normal vector $\mathbf{n}(t)$. The *inclination angle* $\beta(t)$ is defined to be the angle between the normal vector $\mathbf{n}(t)$ and the total angular momentum vector Ω. The inclination angle $\beta(t)$ can equally well be characterized as the angle between the variable plane $\Pi(t)$ and the invariable plane $\Pi_0 = \Omega^{\perp}$.

In order to eliminate time in the three body problem, [Bir27] suggests that it is possible to use the inclination angle as independent parameter. From this perspective, it is of some interest to understand three body motions where the inclination angle is constant. For planar three body motions, the angle of inclination is evidently 0. In the case $e = 1$, [Cab90] gives a complete description of the non-empty class of three body motions with constant inclination angle $\frac{\pi}{2}$. He also puts forth the conjecture that the only possible constant angles of inclination are 0 and $\frac{\pi}{2}$. In this section, we will indicate a possible path towards a proof of this conjecture.

Since we can take the third principal axes vector $\mathbb{u}_3(t)$ as the normal vector of the variable plane, we see that the inclination β is constant if and only if $g_3 = \Omega \cdot \mathbb{u}_3$ is constant. Hence, $\beta = 0$ if and only if $\Omega = g_3\mathbb{u}_3$, while $\beta = \frac{\pi}{2}$ if and only if $\Omega = g_1\mathbb{u}_1 + g2\mathbb{u}_2$. In light of this, we can pose Cabral's question in the following way:

Question. Does there exist a motion for the three body problem with such that

$$\dot{g}_3 = 0, \quad 0 < g_3^2, \quad \text{and} \quad 0 < g_1^2 + g_2^2$$

for all instances of time?

Is the answer dependent on the power e of the potential?

Since the three body motions under consideration are all analytic, it is sufficient to investigate this question over arbitrarily small non-empty time intervals.

5.2.1 Reduction of the problem

Lemma 5.2.1. *For a three body motion with $\varphi = 0 \mod \frac{\pi}{2}$, the inclination angle $\beta = 0 \mod \frac{\pi}{2}$.*

Proof. Since collinear motions are planar (cf. Proposition 2.7.9), the assertion holds for $\varphi = \frac{2k+1}{2}\pi$, i.e. the φ-values associated with collinear configurations.

Since umbilic shape invariant motions are either planar or exceptional (cf. Theorem 3.9.10), the assertion also holds for $\varphi = k\pi$, i.e. the φ-values associated with umbilic configurations. ∎

Accordingly, in our quest for three body motions with constant angle of inclination $\beta \in (0, \frac{\pi}{2})$, we can assume that $\varphi \neq \frac{k\pi}{2}$ for all t in the time interval under consideration.

5.2.2 Restriction of the gauge group

From the third Euler equation (3.34), we get

$$0 = \cos^2\varphi\, \dot{g}_3 = -\frac{4}{\rho^2}\sin\varphi\, g_1 g_2$$

Hence, we conclude that either g_1 or g_2 is zero. The finite gauge group Σ (cf. (3.10)) contains a transformation that interchanges g_1 and g_2 at the same time as $(-\varphi)$ is substituted for φ. Hence, by considering both positive and negative values for φ, we can, without loss of generality, assume that $g_1 = 0$. This implies that $g_2^2 = \|\Omega\|^2 - g_3^2$ and hence g_1, g_2, g_3 are constant along the motion.

In assuming $g_1 = 0$, we have made a choice which restricts our remaining freedom in choice of gauge: $\mathbb{u}_1$ must be one of the two unit vectors perpendicular to both Ω and $\mathbb{u}_3$. Hence, $\mathbb{u}_1, \mathbb{u}_2, \mathbb{u}_3$ are uniquely determined modulo their individual sign. This limits the freedom in choice of gauge. For instance, the generator σ_3 in (3.10) violates our gauge assumptions. By direct inspection of the group elements, one finds that the gauge freedom is restricted to a subgroup $\Sigma' \subset \Sigma$ with 8 elements, and that this group is generated by $\sigma_1, \sigma_2, \sigma_4$.

Following (3.10') and (3.11), we can take the following set of generators for Σ':

$$\begin{aligned}\sigma_2\colon (g_1,g_2,g_3,\varphi,\theta) &\mapsto (-g_1,-g_2,g_3,2\pi+\varphi,\theta)\\ \sigma_2\circ\sigma_1\colon (g_1,g_2,g_3,\varphi,\theta) &\mapsto (g_1,-g_2,-g_3,\pi-\varphi,\theta)\\ \sigma_2\circ\sigma_4\colon (g_1,g_2,g_3,\varphi,\theta) &\mapsto (-g_1,-g_2,g_3,\varphi,2\pi+\theta),\end{aligned} \tag{5.8}$$

where we should have in mind that φ,θ are interpreted modulo 4π (cf. (3.7))

5.2.3 Investigation of $\dot\theta$

According to our choice of gauge, and the assumed non-planarity, we know that $g_2^2 > 0$. Hence we deduce from the first Euler equation (3.34) that

$$\dot\theta = \eta(\varphi)\frac{2g_3}{\rho^2}, \tag{5.9}$$

where

$$\eta(\varphi) = \frac{1-\sin\varphi}{(1+\sin\varphi)\cos\varphi} = \frac{(1-\sin\varphi)^2}{\cos^3\varphi} = \frac{\cos\varphi}{(1+\sin\varphi)^2}$$

We notice that

$$\eta'(\varphi) = -\frac{2-\sin\varphi}{(1+\sin\varphi)^2} \le -\frac{1}{4}$$

Accordingly, $\eta(\varphi)$ is well defined, continuous and monotonically decreasing on intervals where $\sin\varphi \neq -1$. Knowing the limits

$$\eta(\varphi) \to \begin{cases} +\infty & \varphi \to -\frac{\pi}{2} \text{ from above} \\ 1 & \varphi \to 0 \\ 0 & \varphi \to \frac{\pi}{2} \\ -1 & \varphi \to \pi \\ -\infty & \varphi \to \frac{3\pi}{2} \text{ from below} \end{cases},$$

and the periodicity of $\eta(\varphi)$, it seems like we have a good understanding of this function, and this gives us the following result:

Lemma 5.2.2. *For three body motions with constant inclination, $\dot\theta$ is non-zero and hence of one sign on time intervals such that $\varphi \neq \frac{k\pi}{2}$, for $k = 0, \pm1, \pm2, \ldots$.*

5.2.4 On the effect of gauge transformations

In [Sal11] we find a proposed solution of the constant inclination problem based on surprising effects of gauge transformations. Here we refute this approach by means of our careful investigation of the finite group of gauge symmetries.

Let us consider a given three body motion $X(t)$ with constant inclination $\beta \neq 0 \mod \frac{\pi}{2}$, and $\varphi(t) \neq 0 \mod \frac{\pi}{2}$ on an open time interval. On this interval, we can choose several descriptions of the motion in terms of $g_i, \rho, \varphi, \theta$. The assumption $\mathbb{u}_1 \perp \Omega$ above restricts the gauge freedom, which is now parametrized by the subgroup $\Sigma' \subset \Sigma$ generated by $\sigma_1, \sigma_2, \sigma_4$. [Sal11] proposes that we can acquire new information from (5.9) by regarding this equation in various different gauges. We see however that this is not the case:

From (5.8) we see that the quantity $\dot{\theta}$ is obviously invariant under the group Σ'. g_3 and $\eta(\varphi)$ are clearly invariant under σ_2 and $\sigma_2 \circ \sigma_4$. Hence, it remains to investigate the action of $\sigma_2 \circ \sigma_1$: Under this transformation, g_3 is mapped to $-g_3$, and φ is mapped to $\pi - \varphi$. After noting that $\eta(\pi - \varphi) = -\eta(\varphi)$, we see that the transformed equation is

$$\dot{\theta} = (-\eta(\varphi))\frac{2(-g_3)}{\rho^2} = \eta(\varphi)\frac{2g_3}{\rho^2}.$$

As we should expect, this equation is Σ'-invariant, and hence, the gauge-invariance can not help us to come closer to a solution of the constant inclination problem.

5.2.5 Investigation of $\dot{\varphi}$

After differentiation of (5.9), we get

$$\frac{d}{dt}(\rho^2\dot{\theta}) = 2g_3\eta'(\varphi) \quad \text{i.e. } \ddot{\theta} = \eta'(\varphi)\frac{2g_3}{\rho^2} - \frac{2\dot{\theta}\dot{\rho}}{\rho}. \tag{5.9'}$$

Comparison with the third equation in (3.33) now yields

$$\dot{\varphi} = \frac{2}{g_3}\frac{U_\theta^*}{\rho^e}H(\varphi), \tag{5.10}$$

where

$$H(\varphi) = \frac{1}{3}\frac{\cot\varphi}{\eta(\varphi)} = \frac{\cos^2\varphi\,(1+\sin\varphi)}{3\sin\varphi\,(1-\sin\varphi)} = \frac{(1+\sin\varphi)^2}{3\sin\varphi}$$

This can be expressed by the simple formula

$$\rho^2\dot\theta\dot\varphi = \frac{4}{3}\cot\varphi\frac{U_\theta^*}{\rho^e}$$

Following (5.10), we have the following result:

Lemma 5.2.3. *For three body motions with constant inclination, $\dot\varphi$ does not change sign on time intervals where $U_\theta^* \neq 0$, $\sin\varphi \neq -1, 0$.*

5.2.6 Investigation of $\dot\rho$

Similarly as above, we differentiate (5.10), and compare with the second reduced equation in (3.33), which yields

$$\begin{aligned}\dot\rho = \frac{2g_3\rho^e}{(2-e)U_\theta^*}\Bigg(&\frac{g_3^2\eta(\eta\cos\varphi+1)\sin\varphi}{\rho^3 H} - \frac{K\cos\varphi}{\rho^3 H}\\ &+\frac{U_\varphi^* - U_{\theta\theta}^* H\eta}{\rho^{e+1}H} - \frac{U_\theta}{\rho^e}\frac{U_{\theta\varphi}^* H + U_\theta^* H'}{g_3^2\rho^{e-1}}\Bigg),\end{aligned} \tag{5.11}$$

where

$$K(\varphi) = -\left(\frac{g_2}{1+\sin\varphi}\right)^2.$$

We can also write this as

$$\begin{aligned}\dot\rho = &\frac{2g_3\rho^{e-1}}{(2-e)U_\theta^*}\frac{3\tan\varphi(1-\sin\varphi)}{(1+\sin\varphi)^4}\left(2\sin\varphi\frac{g_3^2}{\rho^2} + (1+\sin\varphi)\frac{g_2^2}{\rho^2}\right)\\ &+\frac{2g_3\rho^{e-1}}{(2-e)U_\theta^*}\frac{1}{(1+\sin\varphi)^2}\left(3\sin\varphi\frac{U_\varphi^*}{\rho^e} - \cos\varphi\frac{U_{\theta\theta}^*}{\rho^e}\right)\\ &-\frac{2\rho}{(2-e)g_3}\frac{1+\sin\varphi}{3\sin^2\varphi}\left(\sin\varphi(1+\sin\varphi)\frac{U_{\theta\varphi}^*}{\rho^e} - \cos\varphi(1-\sin\varphi)\frac{U_\theta^*}{\rho^e}\right).\end{aligned} \tag{5.11'}$$

If we multiply this equation by $(e-2)$, we see that in the case $e=2$, we have a relation
$$F(\rho,\varphi,\theta)=0 \tag{5.12}$$
between the variables ρ,φ,θ, and hence, we get a much simpler situation.

5.2.7 Reformulation of the problem

The case $e \neq 2$:

From the assumption $\dot{g}_3 = 0$ and the gauge-restriction $g_1 = 0$ we have deduced a system of first order ordinary differential equations for ρ,φ,θ. We regard this as a vector field
$$\begin{bmatrix} \dot{\rho} \\ \dot{\varphi} \\ \dot{\theta} \end{bmatrix} = X(\rho,\varphi,\theta),$$
defined on the ρ,φ,θ-space. Note that X depends on the constant parameters g_2, g_3, as well as the power e in the potential function U.

We have not yet invoked the first reduced equation, (3.33), which is equivalent to conservation of the total energy h. Since $\dot{\rho},\dot{\varphi},\dot{\theta}$ are expressed as functions of ρ,φ,θ, we can express the energy h as a function of ρ,φ,θ,
$$h = f(\rho,\varphi,\theta).$$
Observe that both the vector field X and the function f depend on the constant parameters $g2, g3$ and e.

Clearly, an integral curve for X such that f is constant along the curve satisfies the reduced equations (3.33). Hence for our purpose, we can find both necessary and sufficient conditions by studying the relation between X and f.

The case $e = 2$

In this case, the $\dot{\rho}$-equation is degenerated. In this case, we will determine $\dot{\rho}$ from the conservation of energy. In this way we acquire an energy-dependent family of vector fields

$$X_h = \begin{bmatrix} \dot{\theta} \\ \dot{\varphi} \\ \dot{\rho} \end{bmatrix},$$

and the question is now whether or not (5.12) can be satisfied along integral curves of any of the vector fields X_h.

U_θ^*-singularity

We note that our data are singular for $U_\theta^* = 0$. By analyticity, we can localize the constant inclination problem, and hence consider the two separate cases:

(i) $U_\theta^* = 0$ for all t.

(ii) $U_\theta^* \neq 0$ for all t.

However, it may be of considerable interest also to study how curves approach $U_\theta^* = 0$. Note that case (i) implies that $\dot{\varphi} = 0$.

General conclusions

For $e \neq 2$ and modulo some exceptional cases, we can reduce the constant inclination problem to the following question:

Question 5.2.4. Does there exist integral curves of $X(\rho, \varphi, \theta; g_2, g_3, e)$ along which $h = f(\rho, \varphi, \theta; g_2, g_3, e)$ is constant?

During the work with this thesis, there has been some attempts to prove the non-existence of such paths in the case $e = 1$ and three equal masses. After numerical computations of $\dot{h}, \ddot{h}, \dddot{h}$ along the vector field X, we have found some non-decisive evidence that such paths are quite exceptional. In the present implementation, these computations demand lots of computational power. As a consequence of this the evidence is for the moment too sparse for making a reliable judgement. Hence, our present knowledge is contained in Lemma 5.2.2, Lemma 5.2.3 and Question 5.2.4, and we can hope that this is a fruitful starting point for future investigations.

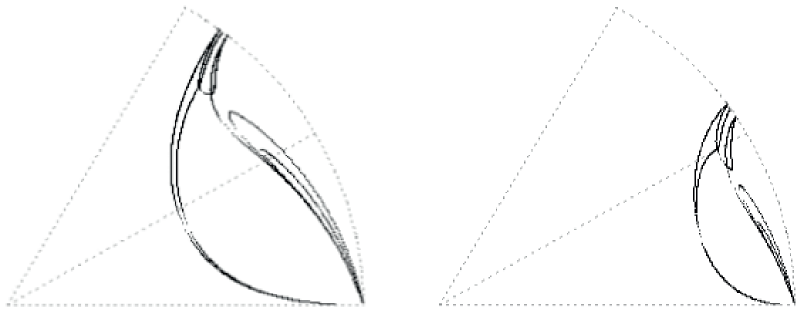

Figure 5.1: Examples of plots of the curves $\Gamma_1, \Gamma_2, \Gamma_3$ for $g_2 = 0.9, g_3 = 0.215$ (left) and $g_2 = 0.9, g_3 = 0.9$ (right) in the case of three equal masses. The dotted lines represent the grid, while the black curves represent $\Gamma_1, \Gamma_2, \Gamma_3$. The lower right corner represents a binary collision.

These preliminary investigations have been carried out in the following way: Using the scaling symmetry, we see that we can limit our investigation to the case $\rho = 1$, if we allow g_2 and g_3 to vary freely. Hence, we can investigate the three curves $\Gamma_1, \Gamma_2, \Gamma_3$ in the φ, θ-plane that are defined respectively by

$$\dot{h} = 0, \quad \ddot{h} = 0 \quad \text{and} \quad \dddot{h} = 0,$$

and search for values of g_2 and g_3 such that the curves $\Gamma_1, \Gamma_2, \Gamma_3$ experiences a triple crossing. In the absence of such triple crossings for given values of g_2 and g_3, we conclude that h can not be constant along any of the integral curves of the vector field X.

Illustrations of examples of numerically computed curves $\Gamma_1, \Gamma_2, \Gamma_3$ are given in Figure 5.1.

Using numerical computations, it is in principle difficult to confirm the existence of such triple crossings of $\Gamma_1, \Gamma_2, \Gamma_3$. On the other hand, such crossings can be detected quite rigorously by studying the relative topology of Γ_1 and the set of common points of Γ_2 and Γ_3. If we can detect a difference in this relative topology, we can be quite sure about the existence of a triple intersection, and thus conclude that there exist values of g_2 and g_3 and a point φ, θ where

$$\dot{h} = \ddot{h} = \dddot{h} = 0.$$

This raises however the question about the higher derivatives of h, a question which must be answered in order to confirm that h is actually constant along the integral curve. This can however be investigated indirectly by finding appropriate initial conditions for the possible constant inclination motion with the initial shape identical to the shape represented by the triple intersection of Γ_1, Γ_3 and Γ_3. After integration from these initial conditions, we can compute the angle of inclination in order to support or discredit the hypothesis that the inclination angle is constant.

We can use this numerical approach to survey the constant inclination problem, in order to find possible routes for rigorous proofs, which must however rely on entirely different principles. Hopefully, the analytical computations presented in this section can be useful in such an attempt.

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