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Homology Theory from the Geometric Viewpoint

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Idar, en stor takk til deg for våre ukentlige møter, dine fantastiske forelesninger, og for ditt smittende engasjement for faget.

Tone, du er den som betyr aller mest. Takk for at du alltid passer på meg.

Abstract. Given a multiplicative cohomology theory, h^*, represented by a spectrum, E, we define its associated *geometric homology theory*, h_*^{geo}, by means of bordism. Restricted to CW pairs, we show how h_*^{geo} is naturally equivalent to h_*, the homology theory associated to E. This was done by M. Jakob in the paper [Jak00], and we give an exposition following his approach. We also consider a naturally occurring cap product.

Introduction. In 1982, P. Baum and R. G. Douglas presented the paper *K Homology and Index Theory*, [BD82], there introducing a geometric description of *K-homology*—the spectrally defined homology theory associated to (complex) K-theory. Their version of K-homology resembles K-theory as well as oriented bordism and is in some sense a fusion of these. Given a space X, one considers the set of triples of the form (M, E, f), where M is a closed *$Spin^c$*-manifold, E is a complex vector bundle over M, and where $f\colon M \to X$ is a continuous map. A suitable equivalence relation is defined on such triples, extending the traditional bordism relation on *singular manifolds* (M, f). There is the obvious disjoint union of triples. This operation passes on to the set of equivalence classes, which by that becomes an abelian group.

This construction would turn out to be applicable to more than K-theory. In the triple (M, E, f), the vector bundle E may be regarded as an element of $K^*(M)$. In his paper *A Bordism-Type Description of Homology*, [Jak98], M. Jakob shows how the description of Baum and Douglas generalizes to a great range of multiplicative cohomology theories. Suppose given a multiplicative cohomology theory h^* represented by a spectrum. For a pair of spaces, (X, A), Jakob considers triples (M, x, f), for which M is a compact, h-oriented manifold, x is an element of $h^*(M)$, and where $f\colon (M, \partial M) \to (X, A)$ is a continuous map. Under a suitable equivalence relation, generalizing that of Baum and Douglas, this becomes an abelian group as above. This evolves to *geometric homology*, h_*^{geo}, a homology theory defined on topological pairs. Restricted to CW pairs, h_*^{geo} is naturally equivalent to the spectrally defined homology theory, h_*. Jakob published a second version, [Jak00], which is our approach to the subject.

Terminology. Certain categories have been given names.

- Ab_*
  Graded abelian groups with graded group homomorphisms.

- Top
  Topological spaces with continuous maps. Synonymous names for objects are **space**, **topological space**.

- Top^2
  Pairs of topological spaces and subspaces with continuous maps of pairs. Synonymous names for objects are **pair**, **topological pair**, **pair of spaces**.

- CW
  CW complexes with continuous maps. Synonymous names for objects are **CW space**, **CW complex**.

- CW^2
  Pairs of CW complexes and subcomplexes with continuous maps of pairs. Synonymous names for objects are **CW pair**, **pair of CW spaces**, **pair of CW complexes**.

We write pt for the generic one-point space. Disjoint union of spaces X and Y is denoted $X \sqcup Y$. For a space X, we define $X_+ \coloneqq X \sqcup \mathrm{pt}$.

Homology and **cohomology theories** are always assumed to be *additive*. Moreover, we do not include the axiom of *dimension*.

When we speak of a **manifold** M, we shall assume the following.

- M is real and smooth (C^∞-differentiable).
- M has a finite number of connected components, but we will allow the components' dimensions to vary.
- M has a boundary ∂M (possibly empty), and we have $\partial M \subseteq M$ as a smooth submanifold.
- $M = \varnothing$ is regarded to be a manifold of any dimension.

By **submanifold** we mean *imbedded* submanifold, i.e. the image of some smooth imbedding. All imbeddings are assumed to be smooth. Being locally Euclidean, the connected and path-connected components of a manifold coincide, and **components** will therefore be used. By an n**-manifold** M, we mean a manifold whose components all have the the same dimension n. We may then write M^n. When necessary, **single-** and **multi-dimensional** will be used to distinguish the two cases. Throughout, **vector bundles** are assumed to be *real*. As with manifolds, we also allow a vector bundle to have varying rank on its different components.

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1 PRELIMINARIES

Throughout this text, we shall let h^* be a fixed, multiplicative cohomology theory on CW^2 represented by an Ω-spectrum, E. To E, there is the associated homology theory, h_*. We shall postpone the definitions of these terms to Chapter 3, and we use h_* for now without further discussion.

In this chapter, we give the notion of *h-orientations* on vector bundles and manifolds, before defining the *Gysin homomorphism* in cohomology and demonstrating some of its basic properties.

1.1 Defining orientations on vector bundles and manifolds

DEFINITION. n-dimensional **Euclidean half-space** is

$$\mathbb{H}^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0\} \subseteq \mathbb{R}^n. \qquad \square$$

As a submanifold, we identify $\mathbb{R}^{n-1} = \partial\mathbb{H}^n \subseteq \mathbb{H}^n$ and have the following imbedding theorem:

THEOREM (IMBEDDING). *Let M be a compact manifold. Then for some n, there is an imbedding*

$$(M, \partial M) \hookrightarrow (\mathbb{H}^n, \mathbb{R}^{n-1}). \qquad \square$$

More frequently, we shall be using imbeddings $M \hookrightarrow \mathbb{R}^n$ obtained by composing with the standard inclusion $\mathbb{H}^n \hookrightarrow \mathbb{R}^n$.

THEOREM (COLLARING). *Let M be a manifold. Then there is an imbedding*

$$f : \partial M \times [0, 1) \hookrightarrow M$$

such that $f(x, 0) = x$ for every $x \in \partial M$. $\square$

Such an imbedding, as well as its image, is referred to as a **collar** on M. When identifying diffeomorphic manifolds, we may assume f is an inclusion map, i.e. $\partial M \times [0, 1) \subseteq M$ as an open neighborhood of ∂M.

DEFINITION. Suppose M and N are manifolds. We write τ_M and τ_N for their respective tangent bundles. Let $f : M \hookrightarrow N$ be an imbedding. Identifying τ_M as a subbundle of $f^*\tau_N$, we define the **normal bundle** of f to be the quotient bundle

$$\nu_f \coloneqq f^*\tau_N \,/\, \tau_M. \qquad \square$$

REMARK. This makes $\nu_f \downarrow M$ a smooth vector bundle. If M and N are of constant codimension, then $\operatorname{rank} \nu_f = \dim N - \dim M$. The sequence

$$0 \to \tau_M \to f^*\tau_N \to \nu_f \to 0$$

is exact. Moreover, manifolds being paracompact, the sequence splits. When N is a Euclidean space, we thus get that $\tau_M \oplus \nu_f$ is trivial. $\square$

The following theorem can be found in [Hir76].

1.3 THEOREM. *Let M^n be a compact manifold. Then $(M, \partial M)$ has the homotopy type of a CW pair.* $\square$

Suppose given a continuous map of compact manifolds. Then a choice of CW representatives and homotopy equivalences defines a homotopy class of maps between the representatives. Passing to the homotopy category of CW pairs, this becomes independent of the choice of representatives and a functorial assignment. In homology and cohomology, we may therefore treat compact manifolds as if they were CW pairs. This we shall use freely. For the following, we refer to [Hus75].

DEFINITION. Suppose ξ is a vector bundle with a *metric*, ξ having total space E and base space X. Then the associated **disk** and **sphere bundle** of ξ are the fiber bundles with respective total spaces

$$DE \coloneqq \big\{x \in E \;\big|\; ||x|| \leq 1\big\}, \quad SE \coloneqq \big\{x \in E \;\big|\; ||x|| = 1\big\},$$

and base space X, denoted $D\xi$ and $S\xi$. The **Thom space** of ξ is the pointed quotient space

$$\operatorname{Th}\xi \coloneqq DE/SE. \qquad \square$$

Most often, we shall use the same notation π for the projections of SE, DE and E onto X.

Given two metrics on ξ, the fiber bundle equivalence $D\xi \xrightarrow{\cong} D'\xi$, on the total spaces locally given by

$$(x, v) \mapsto (x, \frac{||v||}{||v||'} v), \quad v \in \pi^{-1}(x),$$

allows us to identify $D\xi$ and $D'\xi$ in a canonical way. This identifies the associated sphere bundles as well. We shall only consider vector bundles ξ over manifolds and CW spaces. These base spaces are paracompact and hence admit partitions of unity and thus metrics on ξ. Also, we note that $\operatorname{Th}\xi$ is independent of this metric. The passing from vector bundles to Thom spaces can be made covariantly functorial. Given a bundle map $\xi \to \eta$, the Thom functor induces the pointed map $\operatorname{Th}\xi \to \operatorname{Th}\eta$. This we shall be using freely.

We note that for a vector bundle over a CW space, the associated Thom space is a CW space as well.

DEFINITION. Let X be a compact manifold or a CW space and let $\xi = E \downarrow X$ be a vector bundle of rank n. Then a **Thom class** of ξ is an element $u \in \widetilde{h}^n(\mathrm{Th}\,\xi)$ which on each fiber of ξ is pulled back to an $\widetilde{h}^0(\mathbb{S}^0)$-module generator

$$i_x^*(u) \in \widetilde{h}^n(DE_x/SE_x) \cong \widetilde{h}^n(\mathbb{D}^n/\mathbb{S}^{n-1}) \cong \widetilde{h}^n(\mathbb{S}^n) \cong \widetilde{h}^0(\mathbb{S}^0).$$

Now, let $\xi = E \downarrow X$ be a vector bundle, not necessarily of constant rank on different components. Let $\sqcup E_k \downarrow \sqcup X_k$ be the component decomposition of ξ, $\xi_k = E_k \downarrow X_k$, $r_k := \operatorname{rank} \xi_k$. Then an h**-orientation** of ξ is a class $u \in \widetilde{h}^*(\mathrm{Th}\,\xi)$ such that

- u is non-zero only in dimensions r_k,
- for each k, the restriction $u_k \in \widetilde{h}^{r_k}(\mathrm{Th}\,\xi_k)$ is a Thom class of ξ_k.

Such a class will also be referred to as a Thom class. If a Thom class exists, ξ is said to be h**-orientable**. ξ is h**-oriented** when a choice of Thom class is made. Most often, we will suppress the prefix h when speaking of orientations. □

REMARK. [Dye69] is our main reference for orientations of vector bundles and manifolds. However, note how our definition of a Thom class differs from the one found there. We *do not* require the restriction of a Thom class to correspond to $1 \in \widetilde{h}^0(\mathbb{S}^0)$, but only to *some* $\widetilde{h}^0(\mathbb{S}^0)$-module generator. This is essential to us as we shall be needing that $-u$ is a Thom class whenever u is. □

1.4 LEMMA. *Let M be a compact manifold and let $f_i : M \hookrightarrow \mathbb{R}^{k_i}$ be two imbeddings of M, $i = 1, 2$. Then there are integers a_i yielding a bundle isomorphism $\nu_{f_1} \oplus a_1 \cong \nu_{f_2} \oplus a_2$.*

PROOF. For any imbedding f in Euclidean k-space, we have the canonical bundle isomorphism $\nu_f \oplus \tau_M \cong k$: By definition, at each fiber, ν_f is the linear quotient of Euclidean (tangent) space and a tangent space of M. Thus—ν_f being the orthogonal complement at each fiber—this isomorphism is evident. We get

$$\begin{aligned} &\nu_{f_i} \oplus \tau_M \cong k_i \\ \Longrightarrow\ &\nu_{f_1} \oplus k_2 \cong \nu_{f_1} \oplus \nu_{f_2} \oplus \tau_M \cong \nu_{f_2} \oplus k_1. \end{aligned}$$ □

Having fixed a manifold M as base space, two vector bundles are called **stably isomorphic** when they become isomorphic by adding suitable trivial bundles. This defines an equivalence relation on bundles over M. The lemma above shows that any two normal bundles are stably isomorphic. Hence all normal bundles belong to the same equivalence class. We call this class the **stable normal bundle** of M. We also have the notion of a **stable orientation class** giving an orientation to the stable normal bundle:

1.5 PROPOSITION. *Let M be a compact n-manifold and let $u_1 \in \widetilde{h}^{k_1}(\mathrm{Th}\,\nu_1)$ be an orientation of the normal bundle ν_1 of some imbedding f_1. Then if ν_2 is the normal bundle of some other imbedding f_2, u_1 determines in a unique way an orientation $u_2 \in \widetilde{h}^{k_2}(\mathrm{Th}\,\nu_2)$ of ν_2.*

PROOF. The two imbeddings may be composed with standard imbeddings of Euclidean spaces to give two new imbeddings, $i_1 \circ f_1$ and $i_2 \circ f_2$, into the same $\mathbb{R}^{n+l}$. When l is chosen sufficiently high, these imbeddings are isotopic, i.e. homotopic through imbeddings. The new normal bundles, ν_1' and ν_2', are thus isomorphic. Each standard imbedding of Euclidean spaces yields the addition of a trivial bundle to the normal bundle:

$$\nu_1' \cong \nu_1 \oplus (l - k_1), \quad \nu_2' \cong \nu_2 \oplus (l - k_2).$$

Hence we get $\nu_1 \oplus (l - k_1) \cong \nu_2 \oplus (l - k_2)$. Passing to Thom spaces, we get

$$\mathrm{Th}(\nu_1 \oplus (l - k_1)) \approx \mathrm{Th}(\nu_2 \oplus (l - k_2)).$$

Even more, this homeomorphism can be chosen so that it preserves disks, and as such it is unique up to isotopy: cf. [Dye69] section D.1. Hence the induced isomorphism in cohomology is uniquely determined.

When $\xi \downarrow X$ and $\eta \downarrow Y$ are vector bundles, so is $(\xi \times \eta) \downarrow (X \times Y)$, and one has the canonical homeomorphism $\mathrm{Th}(\xi \times \eta) \approx \mathrm{Th}\,\xi \wedge \mathrm{Th}\,\eta$. Letting $Y = \mathrm{pt}$, $\eta = 1$ and identifying $\xi \oplus 1 \downarrow X$ with $(\xi \times 1) \downarrow (X \times \mathrm{pt})$, the homeomorphism becomes $\mathrm{Th}(\xi \oplus 1) \approx \Sigma\,\mathrm{Th}\,\xi$. We get

$$\begin{aligned}
\widetilde{h}^{k_1}(\mathrm{Th}\,\nu_1) &\cong \widetilde{h}^{l}(\Sigma^{l-k_1}\,\mathrm{Th}\,\nu_1) \\
&\cong \widetilde{h}^{l}(\mathrm{Th}(\nu_1 \oplus (l - k_1))) \\
&\cong \widetilde{h}^{l}(\mathrm{Th}(\nu_2 \oplus (l - k_2))) \\
&\cong \widetilde{h}^{l}(\Sigma^{l-k_2}\,\mathrm{Th}\,\nu_2) \\
&\cong \widetilde{h}^{k_2}(\mathrm{Th}\,\nu_2),
\end{aligned}$$

and thus we have defined an element $u_2 \in \widetilde{h}^{k_2}(\mathrm{Th}\,\nu_2)$ corresponding to u_1 via the given isomorphisms. u_2 is now seen to be an orientation of ν_2, as the Thom map in the middle is disk preserving and the suspension isomorphism commutes with induced maps. □

If M instead is multi-dimensional, the exact same argument applies by isomorphisms of graded cohomology groups. Taking M to be an n-manifold as we have done, carrying out the proof is more illustrative exposing the cohomology dimensions in question.

From the proposition above, we conclude that it makes sense speaking of an orientation of the stable normal bundle of the compact manifold M: If u is an orientation of one normal bundle, then to any other normal bundle, there is the corresponding orientation. Hence we say that u is an **orientation** of the stable normal bundle if it is an orientation of the normal bundle of *some imbedding* $M \hookrightarrow \mathbb{R}^k$. On the stable normal bundle, we identify corresponding orientations of normal bundles coming from different imbeddings. This justifies the following definition.

Definition. An **h-manifold** is a pair (M, u), where M is a *compact* manifold and u is an orientation of its stable normal bundle. We write M for (M, u), and we say that M is **oriented** by u. If M is an h-manifold oriented by u, we write M^- for the h-manifold M oriented by $-u$. □

We now have a conflict regarding notation. However, with no risk of confusion, we shall proceed speaking of a manifold of dimension n as an *n-manifold*.

We give a trivial but important example of an h-manifold: Every imbedding of the empty set into Euclidean space yields the same normal bundle, namely the empty set. The Thom space of this normal bundle reads $\varnothing/\varnothing = \mathrm{pt}$. It is vacuously verified that $0 \in \widetilde{h}^*(\mathrm{pt}) = 0$ is a Thom class of this empty bundle, $\varnothing$, and clearly there can be no other. Thus there is precisely one orientation making $\varnothing$ an h-manifold, and we will assume this structure on $\varnothing$ hereafter.

1.2 Induced orientations

If M is an h-manifold, it imposes a canonical orientation on its boundary, ∂M, as well as on any codimension zero submanifold $B \subseteq M$:

First, let M^n be an h-manifold. Choosing an imbedding $f : (M, \partial M) \hookrightarrow (\mathbb{H}^{n+k}, \mathbb{R}^{n+k-1})$, we get normal bundles $\nu_f \downarrow M$ and $\nu_{f|_{\partial M}} \downarrow \partial M$, both of rank k. Composing f with the inclusion $\mathbb{H}^{n+k} \hookrightarrow \mathbb{R}^{n+k}$ also yield normal bundles, naturally identified with ν_f and $\nu_{f|_{\partial M}}$. Thus the orientation of M determines a Thom class $u \in \widetilde{h}^*(M^{\nu_f})$ orienting ν_f. Restricting the total space of ν_f to ∂M yields the bundle $\nu_f|_{\partial M} \downarrow \partial M$. By the collaring theorem, we can assume $\nu_f|_{\partial M} = \nu_{f|_{\partial M}} \downarrow \partial M$. Hence $\nu_{f|_{\partial M}} \to \nu_f$ is a bundle inclusion map:

$$\begin{array}{ccc} \nu_{f|_{\partial M}} & \hookrightarrow & \nu_f \\ \downarrow & & \downarrow \\ \partial M & \hookrightarrow & M \end{array}$$

Passing to Thom spaces, we get the disk preserving map $\mathrm{Th}\,\nu_{f|_{\partial M}} \to \mathrm{Th}\,\nu_f$. It induces $h^k(\mathrm{Th}\,\nu_f) \to h^k(\mathrm{Th}\,\nu_{f|_{\partial M}})$, evidently taking Thom classes to Thom classes. The image of u in $h^k(\mathrm{Th}\,\nu_{f|_{\partial M}})$ is thus an orientation of $\nu_{f|\partial M}$, making ∂M an h-manifold. Again, this does not depend on the choice of imbedding.

Now let $B^n \subseteq M^n$ be a submanifold. The imbeddings $B^n \hookrightarrow M^n \hookrightarrow \mathbb{R}^{n+k}$ yield rank k normal bundles, with $\nu_B \to \nu_M$ a bundle inclusion map:

$$\begin{array}{ccc} \nu_B & \hookrightarrow & \nu_M \\ \downarrow & & \downarrow \\ B & \hookrightarrow & M \end{array}$$

The fiber over x in ν_B is the fiber over x in ν_M, and hence a Thom class orienting ν_M restricts to a Thom class orienting ν_B.

The two cases easily extend to the general setting: When M is a multi-dimensional h-manifold, an orientation is imposed on its boundary—resp. on its codimension zero submanifold—componentwise, in the obvious fashion.

DEFINITION. If $i\colon M \hookrightarrow M'$ is a codimension zero imbedding of h-manifolds, a choice of imbedding $M' \hookrightarrow \mathbb{R}^k$ gives rise to a bundle inclusion map $\nu_{i(M)} \hookrightarrow \nu_{M'}$ of normal bundles. $\nu_{i(M)}$ also being a normal bundle of M via i, we say that the imbedding i is **orientation preserving** if the orientation of M agrees with the orientation $\nu_{i(M)}$ inherits from $\nu_{M'}$. $\square$

REMARK. We shall from this point always assume that imbeddings and submanifolds of codimension zero h-manifolds preserve orientations. $\square$

Now let (M_1, u_1) and (M_2, u_2) be h-manifolds. We want to assign an orientation to their disjoint union:

First we let M_1 and M_2 have the common dimension n. We can assume $f_i : M_i \hookrightarrow \mathbb{R}^{n+k}$ are disjoint imbeddings of M_1 and M_2 such that $u_i \in \widetilde{h}^k(\operatorname{Th}\nu_{f_i})$. We get the imbedding $f_1 \sqcup f_2 : M_1 \sqcup M_2 \hookrightarrow \mathbb{R}^{n+k}$. Identifying $\nu_{f_1 \sqcup f_2} = \nu_{f_1} \sqcup \nu_{f_2} \downarrow M_1 \sqcup M_2$, we have the canonical isomorphisms

$$\widetilde{h}^k(\operatorname{Th}\nu_{f_1 \sqcup f_2}) \cong \widetilde{h}^k(\operatorname{Th}\nu_{f_1} \vee \operatorname{Th}\nu_{f_2}) \cong \widetilde{h}^k(\operatorname{Th}\nu_{f_1}) \oplus \widetilde{h}^k(\operatorname{Th}\nu_{f_2}).$$

These give the element $u_1 \sqcup u_2 \in \widetilde{h}^k(\operatorname{Th}\nu_{f_1 \sqcup f_2})$ corresponding to

$$(u_1, u_2) \in \widetilde{h}^k(\operatorname{Th}\nu_{f_1}) \oplus \widetilde{h}^k(\operatorname{Th}\nu_{f_2}).$$

The first isomorphism is the induced map of a disk preserving homeomorphism. The second isomorphism is on each summand the induced map of the inclusion in the wedge sum, again preserving disks. Hence $u_1 \sqcup u_2$ is a Thom class.

If M_1 and M_2 instead have dimensions n_1 and n_2, the isomorphisms

$$\widetilde{h}^*(\operatorname{Th}\nu_{f_1 \sqcup f_2}) \cong \widetilde{h}^*(\operatorname{Th}\nu_{f_1} \vee \operatorname{Th}\nu_{f_2}) \cong \widetilde{h}^*(\operatorname{Th}\nu_{f_1}) \oplus \widetilde{h}^*(\operatorname{Th}\nu_{f_2})$$

again give the orientation $u_1 \sqcup u_2$ of $M_1 \sqcup M_2$. Now we extend componentwise when M_1 and M_2 are general h-manifolds. Thus we have a canonical h-orientation $u_1 \sqcup u_2$ on $M_1 \sqcup M_2$, making $(M_1 \sqcup M_2, u_1 \sqcup u_2)$ an h-manifold.

REMARK. Let $\xi \downarrow X$ and $\eta \downarrow Y$ be vector bundles and $\xi \xrightarrow{f} \eta$ a bundle map. The fibers over $x \in X$, $f(x) \in Y$, may be regarded as vector bundles $\xi_x \downarrow \{x\}$, $\eta_{f(x)} \downarrow \{f(x)\}$. $\xi \to \eta$ restricts to the linear map $\xi_x \to \eta_{f(x)}$, which we consider to be a bundle map covering $\{x\} \to \{f(y)\}$. This setting makes the Thom functor more applicable. $\square$

When ξ and η are oriented vector bundles over the same base space, there is a canonical orientation on their sum $\xi \oplus \eta$:

Let $u_\xi \in \widetilde{h}^*(\mathrm{Th}\,\xi)$ and $u_\eta \in \widetilde{h}^*(\mathrm{Th}\,\eta)$ be respective Thom classes for the vector bundles $\xi \downarrow X$ and $\eta \downarrow X$. $\xi \oplus \eta$ is the pullback bundle of $\xi \times \eta$ via the diagonal map $X \to X \times X$. Hence $\xi \oplus \eta \to \xi \times \eta$ is a bundle map, inducing $\mathrm{Th}\,\xi \oplus \eta \to \mathrm{Th}\,\xi \wedge \mathrm{Th}\,\eta$ on Thom spaces when identifying $\mathrm{Th}\,\xi \times \eta$ with $\mathrm{Th}\,\xi \wedge \mathrm{Th}\,\eta$. We have the diagram

$$\begin{array}{ccccc}
\widetilde{h}^*(\mathrm{Th}\,\xi) \otimes \widetilde{h}^*(\mathrm{Th}\,\eta) & \xrightarrow{\wedge} & \widetilde{h}^*(\mathrm{Th}\,\xi \wedge \mathrm{Th}\,\eta) & \longrightarrow & \widetilde{h}^*(\mathrm{Th}\,\xi \oplus \eta) \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{h}^*(\mathrm{Th}\,\xi_x) \otimes \widetilde{h}^*(\mathrm{Th}\,\eta_x) & \xrightarrow{\wedge} & \widetilde{h}^*(\mathrm{Th}\,\xi_x \wedge \mathrm{Th}\,\eta_x) & \longrightarrow & \widetilde{h}^*(\mathrm{Th}(\xi \oplus \eta)_x)\ .
\end{array}$$

The left-hand square commutes by naturality of the reduced external product, while the right-hand square is induced by a commutative square of Thom spaces. As the diagram commutes, $u_\xi \otimes u_\eta$ gives a Thom class in $\widetilde{h}^*(\mathrm{Th}\,\xi \oplus \eta)$, i.e. an orientation of $\xi \oplus \eta$.

Even more, if ξ and η are vector bundles over X such that ξ and $\xi \oplus \eta$ are oriented, then η has a unique orientation such that the orientations of ξ and η induce that of $\xi \oplus \eta$ (cf. [Dye69]).

1.6 LEMMA. *Suppose $f\colon X \to Y$ is a map and $\xi \downarrow Y$ an h-vector bundle. Then the pull-back bundle $f^*\xi \downarrow X$ of ξ along f becomes an h-vector bundle in a canonical way.*

PROOF. We have the bundle map

$$\begin{array}{ccc}
f^*\xi & \longrightarrow & \xi \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}\ .$$

For each $x \in X$, this restricts to a (linear) bundle map $(f^*\xi)_x \to \xi_{f(x)}$, as remarked above. These give commutative squares of bundle maps. Applying the functors Th and $\widetilde{h}^*$ successively yield commutative diagrams of Thom spaces and their corresponding cohomology groups

$$\begin{array}{ccc}
\mathrm{Th}(f^*\xi)_x & \longrightarrow & \mathrm{Th}\,\xi_{f(x)} \\
\downarrow & & \downarrow \\
\mathrm{Th}\,f^*\xi & \longrightarrow & \mathrm{Th}\,\xi
\end{array}\ ,\qquad
\begin{array}{ccc}
\widetilde{h}^*\big(\mathrm{Th}\,\xi\big) & \longrightarrow & \widetilde{h}^*\big(\mathrm{Th}\,f^*\xi\big) \\
\downarrow & & \downarrow \\
\widetilde{h}^*\big(\mathrm{Th}\,\xi_{f(x)}\big) & \xrightarrow{\cong} & \widetilde{h}^*\big(\mathrm{Th}(f^*\xi)_x\big)
\end{array}\ ,$$

the vertical maps in the left-hand diagram being the inclusions. The isomorphism to the right follows from the identification $(f^*E)_x = \{x\} \times E_{f(x)}$ in the total space of $f^*\xi$. The following diagram commutes by the naturality of the reduced external product.

$$\begin{array}{ccc} \widetilde{h}^0(\mathbb{S}^0) \otimes \widetilde{h}^*(\operatorname{Th}\xi_{f(x)}) & \xrightarrow{\wedge} & \widetilde{h}^*(\operatorname{Th}\xi_{f(x)}) \\ \downarrow & & \downarrow \cong \\ \widetilde{h}^0(\mathbb{S}^0) \otimes \widetilde{h}^*(\operatorname{Th}(f^*\xi)_x) & \xrightarrow{\wedge} & \widetilde{h}^*(\operatorname{Th}(f^*\xi)_x) \end{array}$$

Let $u \in \widetilde{h}^*(\operatorname{Th}\xi)$ be the Thom class of ξ. Now the corresponding class in $\widetilde{h}^*(\operatorname{Th} f^*\xi)$ is seen to be a Thom class of $f^*\xi$, since its restriction to any $\widetilde{h}^*(\operatorname{Th}(f^*\xi)_x)$ is an $\widetilde{h}^0(\mathbb{S}^0)$-module generator by the last diagram. □

Now we let M be an h-manifold and ξ a *smooth* h-vector bundle over M. We show how the total spaces of the disk and sphere bundles $D\xi$ and $S\xi$ become h-manifolds in a natural way:

Let $\pi\colon E \to M$ be the (smooth) projection of ξ. First of all, there are smooth metrics on ξ. This makes DE and SE (smooth) submanifolds of E. Again, this is independent of the choice of metric, now up to canonical diffeomorphism. And since M is compact, so are DE and SE.

Since M is an h-manifold and $\tau_M \oplus \nu_M$ is trivial for some imbedding $M \hookrightarrow \mathbb{R}^k$, τ_M has a canonical orientation as an h-vector bundle. Thus $\xi \oplus \tau_M \downarrow M$ gets an orientation, and by the lemma above, so does $\pi^*(\xi \oplus \tau_M) \downarrow E$. From the canonical isomorphism $\pi^*(\xi \oplus \tau_M) \cong \pi^*\xi \oplus \pi^*\tau_M$, we obtain a short exact sequence of vector bundles over E (cf. [Lan02]),

$$0 \to \pi^*\xi \to \tau_E \to \pi^*\tau_M \to 0.$$

We restrict the bundles in the sequence to DE, preserving exactness. The zero-section $M \to DE$ in $D\xi$ gives the injective bundle map $\tau_M \hookrightarrow \tau_{DE}$, which again gives a splitting $\pi^*\tau_M|_{DE} \to \tau_{DE}$ of the new short exact sequence. This determines an isomorphism

$$\tau_{DE} \cong \pi^*(\xi \oplus \tau_M)|_{DE},$$

making DE an h-manifold.

SE sits inside the boundary of DE as a codimension zero submanifold, by that inheriting an orientation.

We shall always assume this orientation on the associated disk bundle—resp. sphere bundle—whenever we speak of a smooth h-vector bundle having an h-manifold as base space.

1.3 The Gysin homomorphism

REMARK. Given a compact manifold M^n, there is a bijective correspondence between its orientations and its *fundamental classes* $[M, \partial M] \in h_n(M, \partial M)$. When M is given an orientation, we thus speak of *the* fundamental class of M. This readily generalizes to multi-dimensional h-manifolds. When $[M, \partial M]$ is the fundamental class of M, $-[M, \partial M] = [M^-, \partial M^-]$ is the fundamental class of M^-. □

For the remark and the next three results, we refer to [Swi75].

1.7 PROPOSITION. *Suppose M is an h-manifold and that $[M, \partial M] \in h_*(M, \partial M)$ is its fundamental class, with $[\partial M] \in h_*(\partial M)$ the associated fundamental class of ∂M. Then*

$$\partial[M, \partial M] = [\partial M],$$

where ∂ is the boundary homomorphism in the long exact sequence associated to the pair $(M, \partial M)$. □

1.8 PROPOSITION. *Suppose C is a closed h-manifold and that $M \subseteq C$ is a codimension zero submanifold. We write $[C]$ for the fundamental class of C, and $[M, \partial M]$ for the fundamental class of M corresponding to the orientation M inherits from C. Then we have*

$$i_*[C] = j_*[M, \partial M],$$

where i_*, j_*, are the maps induced by the inclusions

$$(C, \varnothing) \overset{i}{\hookrightarrow} (C, C - \operatorname{int} M) \overset{j}{\hookleftarrow} (M, \partial M).$$ □

THEOREM (POINCARÉ DUALITY). *Let M^n be an h-manifold and $[M, \partial M] \in h_n(M, \partial M)$ the fundamental class of M. Then the maps*

$$D_M \colon h^k(M) \xrightarrow{\cong} h_{n-k}(M, \partial M),$$
$$\widehat{D}_M \colon h^k(M, \partial M) \xrightarrow{\cong} h_{n-k}(M),$$

each given by $x \mapsto x \cap [M, \partial M]$, are isomorphisms. □

Next, we define the *Gysin homomorphism* associated to a continuous map of oriented manifolds. This map and maps similar to it go by various names and descriptions, e.g. *Umkehr maps* and *transfers*. In [Dye69], there is a construction of an Umkehr map in terms of a Thom collapsing map and Thom isomorphisms. For the Gysin map we are about to define, we shall be deducing several properties. The Gysin map seems to share these properties with the Umkehr map, and there is much speaking in favor of the two maps being equal. According to Jakob, [Wür71] proves that this is the case. We will not rely on this fact, but bear it in mind for the following discussion: The construction of the Umkehr map has the advantage of avoiding homology groups entirely, being a strict composition of homomorphisms in *co*homology. Our purpose of involving the Gysin map in the first place is to be able to describe the dual homology theory h_* associated to h^* in another way. The use of the Gysin map may therefore seem a bit odd; indeed, by definition, it factors through homology groups of the homology theory we are trying to describe. Using instead the Umkehr map would eliminate this oddity, which is the approach of [Jak00]. Nevertheless, we shall stick to the Gysin map. This is partly because it is cleaner, in some sense, and more comprehensible with respect to computations, and partly to do things in a way different than that of [Jak00].

DEFINITION. Let $f : (M^m, \partial M) \to (N^n, \partial N)$ be a continuous map of h-manifolds. Then we define the map $f_!$ as the composition making the following diagram commute:

$$\begin{array}{ccc} h^k(M) & \overset{f_!}{\dashrightarrow} & h^{k+n-m}(N) \\ {\scriptstyle D_M}\downarrow & & \uparrow{\scriptstyle D_N^{-1}} \\ h_{m-k}(M, \partial M) & \xrightarrow{f_*} & h_{m-k}(N, \partial N) \end{array}$$

We call $f_!$ the **Gysin homomorphism** induced by f. □

More generally, let $f : (M, \partial M) \to (N, \partial N)$ be a continuous map between two multi-dimensional h-manifolds. Of course, each component of M is mapped to a component of N. Hence we extend the Gysin homomorphism componentwise to the map $f_! \colon h^*(M) \to h^*(N)$ in the obvious manner: We write $M = \sqcup M_k$ and $N = \sqcup N_k$ for the component decompositions, with $f = \sqcup f_k$ such that f restricts to $f_k \colon M_k \to N_k$. With $x_k \in h^*(M_k)$, we have $f_{k!}(x_k) \in h^r(N_k)$ for some r. Then with the corresponding $x = \sqcup x_k \in h^*(M)$, we define

$$f_!(x) = \big(\sqcup f_k\big)_!\big(\sqcup x_k\big) := \sqcup f_{k!}(x_k) \in h^*(N).$$

Immediate from this definition, we have that the Gysin construction is (covariantly) functorial in the sense $f_! g_! = (fg)_!$. We also note the homotopy invariance.

REMARK. When a square of groups and homomorphisms commutes up to sign, we shall say that it **commutes with sign** s if multiplying some map in the square by the sign s makes it commutative. In the following proposition, we determine the sign with which the square commutes. This sign we shall be needing at a later point. □

1.10 PROPOSITION. *Let M^m and N^n be h-manifolds and $f : (M^m, \partial M) \to (N^n, \partial N)$ a continuous map. We write $\partial f : \partial M \to \partial N$ for the restriction. The following diagram commutes with sign $(-1)^{n-m}$.*

$$\begin{array}{ccc} h^k(M) & \xrightarrow{i_M^*} & h^k(\partial M) \\ {\scriptstyle f_!}\downarrow & & \downarrow{\scriptstyle (\partial f)_!} \\ h^{k+n-m}(N) & \xrightarrow{i_N^*} & h^{k+n-m}(\partial N) \end{array}$$

Proof. There is the diagram

$$\begin{array}{ccccc}
 & h_{m-k}(M,\partial M) & \xrightarrow{\ \partial\ } & h_{m-k-1}(\partial M) & \\
 & {\scriptstyle -\cap[M,\partial M]}\Big\uparrow{\scriptstyle \cong} & & {\scriptstyle \cong}\Big\uparrow{\scriptstyle -\cap[\partial M]} & \\
 & h^k(M) & \xrightarrow{\ i_M^*\ } & h^k(\partial M) & \\
f_* \Big\downarrow & {\scriptstyle f_!}\Big\downarrow & & \Big\downarrow{\scriptstyle (\partial f)_!} & \Big\downarrow (\partial f)_* \\
 & h^{k+n-m}(N) & \xrightarrow{\ i_N^*\ } & h^{k+n-m}(\partial N) & \\
 & {\scriptstyle -\cap[N,\partial N]}\Big\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\Big\downarrow{\scriptstyle -\cap[\partial N]} & \\
 & h_{m-k}(N,\partial N) & \xrightarrow{\ \partial\ } & h_{m-k-1}(\partial N) &
\end{array}.$$

By the definition of the Gysin map, the left- and right-hand parts both commute, and naturality of ∂ gives $\partial \circ f_* = (\partial f)_* \circ \partial$. We now consider the top square of the diagram. We have $\partial[M, \partial M] = [\partial M] \in h_{m-1}(\partial M)$. Then by XIII §7($d$) of [Mas91], this square commutes with sign $(-1)^k$. Likewise, the bottom square commutes with sign $(-1)^{k+n-m}$. The four isomorphisms now imply that the middle square commutes with sign $(-1)^{n-m}$. □

Definition. For pairs (X, A) and (X, B), we write $(X; A, B)$ and refer to it as a **triad**. A **map of triads** $f : (X'; A', B') \to (X; A, B)$ is a map $f \colon X' \to X$ for which $f(A') \subseteq A$ and $f(B') \subseteq B$. A **full triad** is a triad for which $X = A \cup B$. A full triad is called **excisive** with respect to h_* if the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ induces an isomorphism

$$h_*(A, A \cap B) \xrightarrow{\cong} h_*(X, B). \qquad □$$

Remark. Let $M \subseteq C$ be a compact, codimension zero submanifold, where C is a closed h-manifold. Then $C - \operatorname{int} M \subseteq C$ is also a compact, codimension zero submanifold. Moreover, $\partial(C - \operatorname{int} M) = \partial M = M \cap (C - \operatorname{int} M)$. □

The following lemma now follows from the more general case of full CW triads always being excisive (cf. [Swi75]). Such an isomorphism will be referred to as **excision**.

1.11 Lemma. *Let $M^n \subseteq C^n$ be a compact codimension zero submanifold of the* closed *h-manifold C. Then the map*

$$h_*(M, \partial M) \xrightarrow{\cong} h_*(C, C - \operatorname{int} M),$$

induced by the inclusion, is an isomorphism. □

For a triad $(X; A, B)$, the cap product

$$\cap\colon h^*(X,A) \otimes h_*(X, A \cup B) \to h_*(X,B)$$

is defined whenever the full triad $(X \times B \cup A \times X; X \times B, A \times X)$ is excisive. We note that this is so for the respective cases $A = \varnothing$, $B = \varnothing$, as $(X \times B; X \times B, \varnothing)$, $(A \times X; \varnothing, A \times X)$ respectively yield isomorphisms

$$1\colon h_*(X \times B, \varnothing) \xrightarrow{\cong} h_*(X \times B, \varnothing), \quad 0\colon h_*(\varnothing, \varnothing) \xrightarrow{\cong} h_*(A \times X, A \times X).$$

For all this and the next proposition, we refer to [Swi75].

1.12 PROPOSITION. *Suppose $f\colon (X'; A', B') \to (X; A, B)$ is a map of triads such that $(X \times B \cup A \times X; X \times B, A \times X)$ is excisive. Then for $x' \in h_*(X', A' \cup B')$, the following diagram commutes*

$$\begin{array}{ccc} h^*(X', A') & \xleftarrow{f^*} & h^*(X, A) \\ \big\downarrow{\scriptstyle - \cap x'} & & \big\downarrow{\scriptstyle - \cap f_*(x')} \\ h_*(X', B') & \xrightarrow{f_*} & h_*(X, B) \end{array},$$

i.e. for any $x \in h^*(X, A)$, we have the equality

$$f_*(f^*(x) \cap x') = x \cap f_*(x'). \qquad \square$$

1.13 LEMMA. *For any pair (X, B) and elements $x \in h^*(X)$, $y \in h_*(X)$, the inclusion $j\colon (X, \varnothing) \hookrightarrow (X, B)$ gives*

$$j_*(x \cap y) = x \cap j_*(y),$$

i.e. we have the following commutative diagram

$$\begin{array}{ccc} h^*(X) & & \\ {\scriptstyle - \cap y}\big\downarrow & \searrow^{- \cap j_*(y)} & \\ h_*(X) & \xrightarrow{j_*} & h_*(X, B) \end{array}.$$

PROOF. As noted above, the previous proposition applies to the triads

$$(X'; A', B') := (X, \varnothing, \varnothing), \quad (X; A, B) := (X; \varnothing, B).$$

The identity map on X restricts to $j\colon (X, \varnothing) \to (X, B)$. This gives the commutative diagram

$$\begin{array}{ccc} h^*(X, \varnothing) & \xleftarrow{1} & h^*(X, \varnothing) \\ \big\downarrow{\scriptstyle - \cap y} & & \big\downarrow{\scriptstyle - \cap j_*(y)} \\ h_*(X, \varnothing) & \xrightarrow{j_*} & h_*(X, B) \end{array},$$

and the result follows. $\square$

1.14 LEMMA. *Let $M^n \subseteq C^n$ be a compact codimension zero submanifold of the closed h-manifold C. Then the following diagram commutes.*

$$\begin{array}{ccc}
h^k(C) & \xrightarrow{i^* \oplus j^*} & h^k(M) \oplus h^k(C - \operatorname{int} M) \\
 & & \cong \big\downarrow D_M \oplus D_{C - \operatorname{int} M} \\
\cong \big\downarrow D_C & & h_{n-k}(M, \partial M) \oplus h_{n-k}(C - \operatorname{int} M, \partial M) \\
 & & \cong \big\downarrow i'_* + j'_* \\
k_{n-k}(C) & \xrightarrow{l_*} & h_{n-k}(C, \partial M)
\end{array}$$

PROOF. The maps induced by the inclusions I', I, J' and J give the composition maps

$$I_*{}^{-1} I'_* \colon h_n(C) \xrightarrow{I'_*} h_n(C, C - \operatorname{int} M) \xrightarrow{I_*{}^{-1}} h_n(M, \partial M),$$
$$J_*{}^{-1} J'_* \colon h_n(C) \xrightarrow{J'_*} h_n(C, M) \xrightarrow{J_*{}^{-1}} h_n(C - \operatorname{int} M, \partial M).$$

Here, I_* and J_* are excisions. In terms of the fundamental class $[C]$ of C, the fundamental classes of M and $C - \operatorname{int} M$ are by Proposition 1.8 given by

$$[M, \partial M] = I_*{}^{-1} I'_*[C], \qquad [C - \operatorname{int} M, \partial M] = J_*{}^{-1} J'_*[C].$$

We see that the inclusions I' and J' each factor through the pair $(C, \partial M)$, and we define new inclusions,

$$\begin{array}{ccccc}
 & \swarrow^{I'} & (C, \varnothing) & \searrow^{J'} & \\
 & & \big\downarrow l & & \\
(C, C - \operatorname{int} M) & \xleftarrow{I''} & (C, \partial M) & \xrightarrow{J''} & (C, M)\ .
\end{array}$$

The following diagram clearly commutes, the maps $J''_* i'_*$ and $I''_* j'_*$ respectively factoring through $h_n(M, M)$ and $h_n(C - \operatorname{int} M, C - \operatorname{int} M)$, thus being zero.

$$\begin{array}{ccc}
 & & h_n(M, \partial M) \oplus h_n(C - \operatorname{int} M, \partial M) \\
 & \swarrow^{i'_* + j'_*}_{\cong} & \cong \big\downarrow I_* \oplus J_* \\
h_n(C, \partial M) & \xrightarrow{I''_* \oplus J''_*} & h_n(C, C - \operatorname{int} M) \oplus h_n(C, M)
\end{array}$$

The sum of the induced inclusions $i'_* + j'_*$ is part of the Mayer-Vietoris sequence of the full excisive triad $(C; M, C - \operatorname{int} M)$ (cf. [Swi75]). As $M \cap (C - \operatorname{int} M) = \partial M$, this is an isomorphism. By the triangle above, the composition

$$h_n(C,\partial M) \xrightarrow{I''_*\oplus J''_*} h_n(C, C-\operatorname{int} M)\oplus h_n(C,M)$$
$$\Big\downarrow I_*{}^{-1}\oplus J_*{}^{-1}$$
$$h_n(M,\partial M)\oplus h_n(C-\operatorname{int} M,\partial M) \xrightarrow{i'_*+j'_*} h_n(C,\partial M)$$

is therefore the identity map on $h_n(C,\partial M)$. We get

$$\begin{aligned} i'_*[M,\partial M]+j'_*[C-\operatorname{int} M,\partial M] &= i'_*I_*{}^{-1}I'_*[C]+j'_*J_*{}^{-1}J'_*[C] \\ &= i'_*I_*{}^{-1}I''_*l_*[C]+j'_*J_*{}^{-1}J''_*l_*[C] \\ &= (i'_*+j'_*)(I_*{}^{-1}\oplus J_*{}^{-1})(I''_*\oplus J''_*)l_*[C]=l_*[C]. \end{aligned}$$

From Proposition 1.12—resp. Lemma 1.13—we have the equalities (the first equality also for j', j)

$$i'_*(i^*(x)\cap[M,\partial M]) = x\cap i'_*[M,\partial M], \qquad x\cap l_*[C]=l_*(x\cap[C]).$$

We can now see that the two ways around the initial diagram are the same,

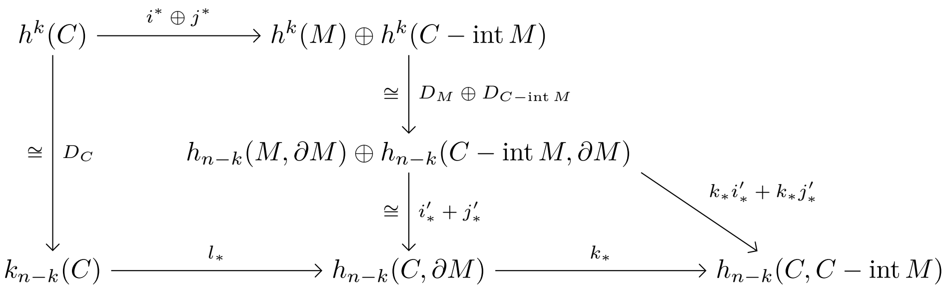
$$\begin{aligned} &(i'_*+j'_*)(D_M\oplus D_{C-\operatorname{int} M})(i^*\oplus j^*)(x) \\ =&i'_*(i^*(x)\cap[M,\partial M])+j'_*(j^*(x)\cap[C-\operatorname{int} M,\partial M]) \\ =&x\cap i'_*[M,\partial M]+x\cap j'_*[C-\operatorname{int} M,\partial M] \\ =&x\cap l_*[C]=l_*(x\cap[C])=l_*D_C(x). \end{aligned}$$

□

1.15 COROLLARY. *Let $M^n\subseteq C^n$ be a compact codimension zero submanifold of the closed h-manifold C. Then the following diagram commutes.*

$$\begin{array}{ccc} h^k(C) & \longrightarrow & h^k(M) \\ & & \cong\Big\downarrow D_M \\ \cong\Big\downarrow D_C & & h_{n-k}(M,\partial M) \\ & & \cong\Big\downarrow \\ k_{n-k}(C) & \longrightarrow & h_{n-k}(C,C-\operatorname{int} M) \end{array}$$

PROOF. Beginning with the commutative diagram of the previous lemma, we attach the group $h_{n-k}(C,C-\operatorname{int} M)$.

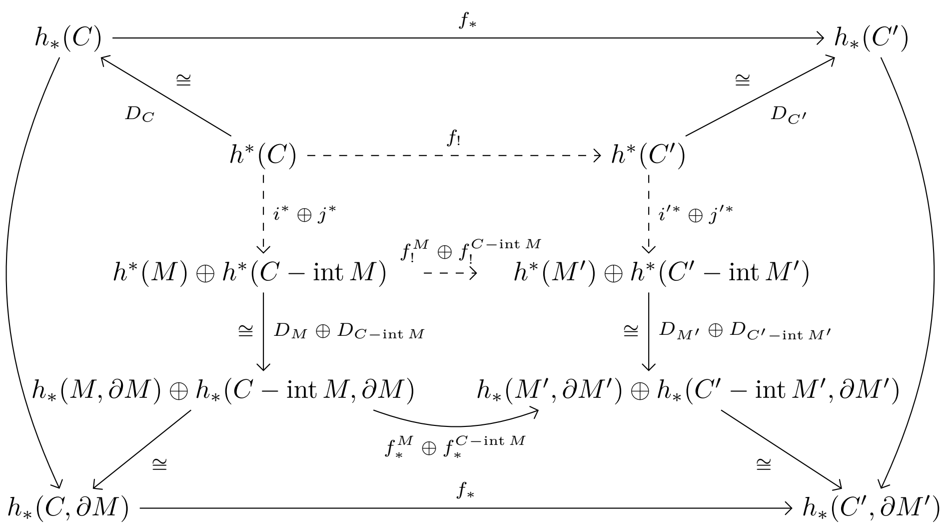


The map $k_* j'_*$ factors through $h_{n-k}(C - \operatorname{int} M, C - \operatorname{int} M) = 0$, thus being zero. The result follows. $\square$

1.16 Proposition. *Let $f : C \to C'$ be a map of closed h-manifolds and let $M \subseteq C$ and $M' \subseteq C'$ be compact, codimension zero submanifolds. If f restricts to $f^M : (M, \partial M) \to (M', \partial M')$ on M and to $f^{C-\operatorname{int} M} : (C - \operatorname{int} M, \partial M) \to (C' - \operatorname{int} M', \partial M')$ on $C - \operatorname{int} M$, then the following diagram commutes.*

$$\begin{array}{ccc} h^*(C) & \xrightarrow{f_!} & h^*(C') \\ \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i'^*} \\ h^*(M) & \xrightarrow{f_!^M} & h^*(M') \end{array}$$

Proof. The result follows from proving commutativity of the dashed square in the following diagram.



By definition of the Gysin homomorphisms, the upper square and the lower middle square commute. The squares on the left- and right-hand sides commute by Lemma 1.14, where the isomorphisms on the lower corners are addition of the maps induced by inclusions. Finally, the square at the bottom commutes by inspection. The different isomorphisms in the diagram now impose commutativity on the dashed square. □

2 GEOMETRIC HOMOLOGY

Given a pair of spaces, (X, A), we define the set $h_*^{geo}(X, A)$—*geometric homology* of (X, A) associated to h^*. We get the covariant functor $h_*^{geo}\colon \mathsf{Top}^2 \to \mathsf{Ab}_*$ by endowing each $h_*^{geo}(X, A)$ with an addition, before verifying that h_*^{geo} satisfies the axioms of a homology theory.

2.1 Triples and equivalence relations

NOTATION. For an inclusion map $i\colon X \hookrightarrow Y$, we often write

$$y|_X \coloneqq i^*(y) \in h^*(X)$$

for the restriction of y to $h^*(X)$. □

REMARK. We shall be using the phrase **straightening the angle** for the procedure described in [Con79] to give a smooth structure to certain topological manifolds. □

For the following definition, we stress that h-manifolds in general are required to be compact.

DEFINITION. For $(X, A) \in \mathsf{Top}^2$, we define $\Lambda(X, A)$ to be the set of all (M, x, f) such that

- M is an h-manifold,
- $x \in h^*(M)$ is a cohomology element,
- $f\colon (M, \partial M) \to (X, A)$ is a continuous map,
- for each component $M_k \subseteq M$, the inclusion pulls x back to a homogeneous element $x_k \coloneqq i_k^*(x) \in h^*(M_k)$,
- for all components M_k of M satisfying $x_k \neq 0$, the number $\dim M_k - \dim x_k$ is the same.

An element $(M, x, f) \in \Lambda(X, A)$ is called a **triple** in (X, A). When the pair (X, A) is understood, we simply call it a triple.

If (M, x, f) is a triple with $x \neq 0$, we define

$$\dim(M, x, f) \coloneqq \dim M_k - \dim x_k,$$

where M_k is any component of M where we have $x_k \neq 0$. This we call the **dimension** of the triple. We have to treat triples with $x = 0$ as a special case: Analogous to the dimension of the empty set, we let a triple on the form $(M, 0, f)$ have any finite dimension. □

REMARK. Concerning manifolds, we alert the reader that the notation M_k will be used in two different ways. Sometimes M_k will denote some component of a manifold M, and sometimes the subscript k will be used to index a collection of (not necessarily connected) manifolds. The latter is the case in the next definition. The analogous ambiguity applies to cohomology elements denoted x_k. However, this will in each case be made clear or will follow from context. □

DEFINITION. Two triples (M_1, x_1, f_1) and (M_2, x_2, f_2) in (X, A) are **bordant** if there exists a triple (W, y, F) in (X, X) such that

- $M_1 \sqcup M_2^- \subseteq \partial W$ as a codimension zero submanifold,
- $y|_{M_1} = x_1$ and $y|_{M_2} = x_2$,
- $F|_{M_1} = f_1$ and $F|_{M_2} = f_2$,
- $F\big(\partial W - (M_1 \sqcup M_2)\big) \subseteq A$.

This defines the relation **bordism** on $\Lambda(X, A)$. We shall denote it by the symbol $\mathcal{B}$. We also refer to the triple (W, y, F) itself as a bordism, and in the case above, (W, y, F) is said to **give** a bordism between (M_1, x_1, f_1) and (M_2, x_2, f_2). □

REMARK. In (X, A), we stress that a bordism (W, y, F) is an element of $\Lambda(X, X)$, not of $\Lambda(X, A)$. Of course, if it were to lie in $\Lambda(X, A)$, the fourth condition above would be vacuous.

We have written $M_1 \sqcup M_2^- \subseteq \partial W$, while we actually mean $i(M_1 \sqcup M_2^-) \subseteq \partial W$; the *image* of some orientation preserving imbedding, i. Suppressing the imbedding from our notation is a mild sin, which we will allow. We also stress that imbeddings and submanifolds of codimension zero h-manifolds are assumed to preserve orientations.

To ease notation, we shall throughout identify triples (M, x, f) and (M', x', f') when there exists an orientation preserving diffeomorphism $\varphi\colon (M, \partial M) \to (M', \partial M')$ such that $\varphi^*(x') = x$ and $f'\varphi = f$. Such triples are especially identified under the bordism relation. □

2.1 PROPOSITION. *The bordism relation, $\mathcal{B}$, is an equivalence relation.*

PROOF. We rely on the methods found in [Con79].

(REFLEXIVITY.) Two copies of (M, x, f) are bordant via $(M \times I, \pi^*(x), \widehat{f\pi})$, as

$$M \sqcup M^- \subseteq (M \times \{0\}) \cup (M^- \times \{1\}) \cup (\partial M \times I) = \partial(M \times I).$$

Here, $\widehat{f\pi}\colon (M \times I, \partial(M \times I)) \to (X, X)$ is the map obtained from

$$f\pi\colon (M \times I, \partial M \times I) \xrightarrow{\pi} (M, \partial M) \xrightarrow{f} (X, A).$$

Also, $M \times I$ has been given a smooth structure by straightening the angle, and it is an h-manifold in a canonical way.

(SYMMETRY.) If (W, y, F) is a bordism between (M_1, x_1, f_1) and (M_2, x_2, f_2), then (W^-, y, F) is a bordism between (M_2, x_2, f_2) and (M_1, x_1, f_1).

(TRANSITIVITY.) Let (W_{12}, y_{12}, F_{12}) and (W_{23}, y_{23}, F_{23}) be consecutive bordisms of (M_1, x_1, f_1), (M_2, x_2, f_2) and (M_3, x_3, f_3). We choose collars on W_{12} and W_{23}^- and obtain a smooth structure on

$$W := W_{12} \cup_{M_2} W_{23}^-$$

by straightening the angle. By requiring that $W_{12} \hookrightarrow W$ and $W_{23}^- \hookrightarrow W$ are orientation preserving imbeddings, a unique orientation is imposed on W. Since $\partial W_{12} - \operatorname{int} M_2$ and $\partial W_{23}^- - \operatorname{int} M_2$ both imbed in ∂W, we have

$$M_1 \sqcup M_3^- \subseteq \partial W.$$

The maps F_{12} and F_{23} agree on M_2, and so they piece together to give the continuous map $F\colon W \to X$. Now, $f_2(\partial M_2) \subseteq A$, and hence $F\big(\partial W - (M_1 \sqcup M_3)\big) \subseteq A$.

To the full excisive triad $(W; W_{12}, W_{23})$, there is the associated Mayer-Vietoris sequence (cf. [May99])

$$\ldots \to h^*(W) \to h^*(W_{12}) \oplus h^*(W_{23}) \to h^*(M_2) \to \ldots .$$

The homomorphism to the right maps the element $(y_{12}, y_{23}) \in h^*(W_{12}) \oplus h^*(W_{23})$ to $y_{12}|_{M_2} - y_{23}|_{M_2} = x_2 - x_2 = 0$. Exactness thus yields an element $y \in h^*(W)$ such that $y|_{W_{12}} = y_{12}$ and $y|_{W_{23}} = y_{23}$. This implies $y|_{M_1} = x_1$ and $y|_{M_3} = x_3$.

Hence (W, y, F) is a bordism between (M_1, x_1, f_1) and (M_3, x_3, f_3). □

By taking h^* to be singular cohomology with integral coefficients, an h-oriented vector bundle of rank k is just an ordinary $SO(k)$-vector bundle. A single-dimensional h-manifold M is thus a compact manifold which is *oriented* in the traditional sense. Let (X, A) be a pair and $f\colon (M, \partial M) \to (X, A)$ a continuous map. As defined in [Con79], (M, f) is then a *singular manifold* in (X, A). There is a natural way of regarding (M, f) as an element of $\Lambda(X, A)$, namely via the correspondence $(M, f) \mapsto (M, 1, f)$. We note how bordant singular manifolds (in the sense of [Con79]) are taken to bordant triples. That is, our notion of *bordism* may be seen as a generalization of the classical one.

REMARK. From this point forward, our language will be less formal regarding vector bundles. Specifically, we will most often use the projection or the total space and speak of these as the vector bundle itself. □

We shall now define another relation on $\Lambda(X, A)$. Let (M, x, f) be a triple in (X, A). If $\pi\colon E \to M$ is a smooth h-vector bundle, then the restrictions $\pi\colon D(E \oplus 1) \to M$ and $\pi\colon S(E \oplus 1) \to M$ are smooth fiber bundles. Now let $s\colon M \to S(E \oplus 1)$ be a smooth section. Then

$$(S(E \oplus 1), s_!(x), f\pi)$$

is also a triple in (X, A). This is clear from the definition of the Gysin homomorphism: The restriction $s_!(x)_k$ of $s_!(x)$ to any component $S(E \oplus 1)_k \subseteq S(E \oplus 1)$ yields the same difference $\dim S(E \oplus 1)_k - \dim s_!(x)_k$, namely $\dim(M, x, f)$. (We inspect this more closely in Proposition 2.2 below.) The following is therefore well-defined.

DEFINITION. Let (M, x, f) be a triple in (X, A) and $\pi\colon E \to M$ a smooth h-vector bundle. We let $s\colon M \to S(E \oplus 1)$ be the section given by $x \mapsto (0_{E_x}, 1_{\mathbb{R}})$. Now

$$(S(E \oplus 1), s_!(x), f\pi)$$

is said to be a **sphere triple** of (M, x, f). If one triple is a sphere triple of the other, they are **$\mathcal{S}$-related**. This will be referred to as **sphere bundle modification**. □

As opposed to bordism, $\mathcal{S}$ is by no means an equivalence relation: There is no vector bundle E such that $S(E \oplus 1) = \mathrm{pt}$. Hence the triple $(\mathrm{pt}, 0, c)$ in $(\mathrm{pt}, \varnothing)$ is not $\mathcal{S}$-related to itself, and so the $\mathcal{S}$-relation is not even reflexive.

NOTATION. We use the symbols $\sim_{\mathcal{B}}$ and $\sim_{\mathcal{S}}$ between triples which are respectively $\mathcal{B}$- and $\mathcal{S}$-related. □

DEFINITION. We let $\mathcal{BS}$ be the equivalence relation generated by $\mathcal{B}$ and $\mathcal{S}$, i.e. the smallest such containing both of them. To be precise, (M, x, f) and (M', x', f') are $\mathcal{BS}$-related if and only if there for some n exist triples (M_i, x_i, f_i), $1 \leq i \leq n$, such that

$$\begin{aligned}
&(M, x, f) = (M_1, x_1, f_1),\\
&(M_i, x_i, f_i) \sim_{\mathcal{R}_i} (M_{i+1}, x_{i+1}, f_{i+1}),\\
&(M_n, x_n, f_n) = (M', x', f'),
\end{aligned}$$

each $\mathcal{R}_i$ being either $\mathcal{B}$ or $\mathcal{S}$. We then write $(M, x, f) \sim_{\mathcal{BS}} (M', x', f')$. By construction, this is an equivalence relation, and we let $[M, x, f]$ denote the class of triples $\mathcal{BS}$-related to (M, x, f). □

REMARK. As we will be using $\sim_{\mathcal{B}}$, $\sim_{\mathcal{S}}$ and $\sim_{\mathcal{BS}}$, we stress that $\mathcal{B}$ and $\mathcal{BS}$ are equivalence relations, whereas $\mathcal{S}$ only is a symmetric relation. □

One might ask whether the relation $\mathcal{BS}$ is strictly larger than $\mathcal{B}$, i.e. if there exist non-bordant triples that *are* $\mathcal{BS}$-related. In any case, an affirmative answer will be given at a later point.

2.2 Defining h_*^{geo}—geometric homology

Definition. The **dimension** of $[M, x, f] \in \Lambda(X, A)/\sim_{\mathcal{BS}}$ is

$$\dim[M, x, f] := \dim(M, x, f). \qquad \square$$

The dimension of the equivalence class of a triple is well-defined, as the following proposition shows. For the proof, we note that functoriality of the Gysin construction implies that the induced map $s_!$ of a section s is a *monomorphism*, as we have

$$1 = 1_! = (\pi s)_! = \pi_! s_! \colon h^*(M) \xrightarrow{s_!} h^*(S(E \oplus 1)) \xrightarrow{\pi_!} h^*(M).$$

The statement that $\dim(M, x, f)$ is equal to $\dim(M', x', f')$ shall be interpreted as *true* if either x or x' is zero.

2.2 Proposition. *The dimension of a triple is a $\mathcal{BS}$-invariant, i.e.*

$$(M, x, f) \sim_{\mathcal{BS}} (M', x', f') \Rightarrow \dim(M, x, f) = \dim(M', x', f').$$

Proof. It suffices to show this when $(M, x, f) \sim_{\mathcal{R}} (M', x', f')$, where $\mathcal{R}$ is either sphere bundle modification or bordism. As commented above, we may assume $x, x' \neq 0$:

- Let $(M', x', f') = (S(E \oplus 1), s_!(x), f\pi)$ be a sphere triple of (M, x, f). Denote by M_k a component of M where the restriction of x is non-zero, and let $S(E \oplus 1)_k$ be the corresponding component of $S(E \oplus 1)$. We write i for both inclusions. By assumption, the restriction $x_k := i^*(x) \in h^*(M_k)$ of x is homogeneous, hence so is $s_{k!}(x_k) \in h^*(S(E \oplus 1)_k)$. Not being concerned about the potential commutativity of

$$\begin{array}{ccc} h^*(M) & \xrightarrow{\quad i^* \quad} & h^*(M_k) \\ \Big\downarrow{\scriptstyle s_!} & & \Big\downarrow{\scriptstyle s_{k!}} \\ h^*(S(E \oplus 1)) & \xrightarrow{\quad i^* \quad} & h^*(S(E \oplus 1)_k)\,, \end{array}$$

  this diagram shows that $s_{k!}(x_k)$ and $s_!(x)_k := i^* s_!(x)$ are of the same dimension: They are both non-zero, and by definition, $s_!$ shifts the dimension of x componentwise agreeing with $s_{k!}$ on M_k.

  Thus we get

$$\begin{aligned} \dim(M', x', f') &= \dim S(E \oplus 1)_k - \dim s_!(x)_k \\ &= \dim E|_{M_k} - \dim s_{k!}(x_k) \\ &= \dim E|_{M_k} - \big(\dim x_k + \dim E|_{M_k} - \dim M_k\big) \\ &= \dim M_k - \dim x_k \\ &= \dim(M, x, f). \end{aligned}$$

- Let (W, y, F) be a bordism between (M, x, f) and (M', x', f'). Pick components $M_k \subseteq M$ and $M'_{k'} \subseteq M'$ on which x and x' are non-zero. We write $W_k \subseteq W$ and $W_{k'} \subseteq W$ for the components corresponding to $M_k \subseteq \partial(W_k)$ and $M_{k'} \subseteq \partial(W_{k'})$, and we write $y_k \in h^*(W_k)$ and $y_{k'} \in h^*(W_{k'})$ for the restrictions of $y \in h^*(W)$. This gives

$$\begin{aligned}\dim(M, x, f) &= \dim M_k - \dim x_k \\ &= \dim \partial(W_k) - \dim y_k \\ &= \dim(W, y, F) - 1 \\ &= \dim \partial(W_{k'}) - \dim y_{k'} \\ &= \dim M'_{k'} - \dim x'_{k'} \\ &= \dim(M', x', f').\end{aligned}$$

□

NOTATION. For any pair (X, A), we write

$$\mathrm{ef}\colon (\varnothing, \varnothing) \to (X, A)$$

for the **empty-function**. □

REMARK. Let $i\colon \mathbb{S}^1 \hookrightarrow \mathbb{D}^2$ be the inclusion. Now $1 \in H^0(\mathbb{D}^2)$ restricts to $1 \in H^0(\mathbb{S}^1)$, where H^* is singular cohomology with integral coefficients. This makes $(\mathbb{D}^2, 1, 1)$ a bordism between $(\mathbb{S}^1, 1, i)$ and $(\varnothing, 0, \mathrm{ef})$ in $(\mathbb{D}^2, \varnothing)$. We have $\dim(\mathbb{S}^1, 1, i) = 1$, while the triple $(\varnothing, 0, \mathrm{ef})$ is of any dimension. Our remark is that this does not conflict with the previous definition or proposition: Since $[\mathbb{S}^1, 1, i] = [\varnothing, 0, \mathrm{ef}]$, we have that $[\mathbb{S}^1, 1, i]$ is of any dimension, and *especially* of dimension $\dim[\mathbb{S}^1, 1, i] = \dim(\mathbb{S}^1, 1, i) = 1$. □

In the following definition, the proper name for h_*^{geo} would be *geometric* h-*homology*. However, the cohomology theory h^* being implicit, the h is left out.

DEFINITION. For each $(X, A) \in \mathsf{Top}^2$, we define the sets

$$\begin{aligned} h_d^{geo}(X, A) &:= \big\{[M, x, f] \in \Lambda(X, A)/{\sim_{\mathcal{BS}}} \;\big|\; d = \dim[M, x, f]\big\}, \\ h_*^{geo}(X, A) &:= \coprod_{d \in \mathbb{Z}} h_d^{geo}(X, A). \end{aligned}$$

The assignment $(X, A) \mapsto h_*^{geo}(X, A)$ we denote by h_*^{geo} and refer to is as **geometric homology**. □

For an element $[M, x, f] \in h_d^{geo}(X, A)$, we will also write $[M, x, f] \in h_*^{geo}(X, A)$ for its image via the natural inclusion $h_d^{geo}(X, A) \hookrightarrow h_*^{geo}(X, A)$.

2.3 Group structure on $h_*^{geo}(X, A)$

In time, h_*^{geo} will become a full-fledged homology theory. The first step is to give $h_d^{geo}(X, A)$ the structure of an abelian group by endowing it with an addition, reading

$$[M, x, f] + [N, y, g] = [M \sqcup N, x \sqcup y, f \sqcup g].$$

Here, $x \sqcup y \in h^*(M \sqcup N)$ is the element corresponding to $(x, y) \in h^*(M) \oplus h^*(N)$ via the canonical isomorphism. $M \sqcup N$ has been given an orientation from M and N as described earlier. The symbol of disjoint union suggests a functorial nature, on maps $f\colon (M, \partial M) \to (X, A)$ and $g\colon (N, \partial N) \to (X, A)$ yielding $f \sqcup g\colon (M \sqcup N, \partial(M \sqcup N)) \to (X \sqcup X, A \sqcup A)$. Nevertheless, to ease notation we shall instead use a slight deviation: We compose the suggested map with the obvious projection $(X \sqcup X, A \sqcup A) \to (X, A)$. Hence, disjoint union on maps shall mean

$$f \sqcup g\colon (M \sqcup N, \partial(M \sqcup N)) \to (X, A).$$

In order to show addition being well-defined, we first need some technical results:

2.3 Lemma. *Let $(N, y, g) \sim_{\mathcal{S}} (M, x, f)$ in (X, A). Then there are normal bundles ν_N and ν_M and sphere triples*

$$(S(\nu_N \oplus 1), s_{N!}(y), g\pi_N) \sim_{\mathcal{S}} (N, y, g),$$
$$(S(\nu_M \oplus 1), s_{M!}(x), f\pi_M) \sim_{\mathcal{S}} (M, x, f),$$

such that there is a bordism

$$(S(\nu_N \oplus 1), s_{N!}(y), g\pi_N) \sim_{\mathcal{B}} (S(\nu_M \oplus 1), s_{M!}(x), f\pi_M).$$

Proof. Let $(N, y, g) = (S(E \oplus 1), s_!(x), f\pi)$ be a sphere triple of (M, x, f). We choose an imbedding $i_N\colon N \hookrightarrow \mathbb{H}^{n-1} \hookrightarrow \mathbb{H}^n$ and get another imbedding, $i_M\colon M \overset{s}{\hookrightarrow} N \overset{i_N}{\hookrightarrow} \mathbb{H}^n$. ν_N and ν_M denoting the normal bundles of $N, M \hookrightarrow \mathbb{H}^{n-1}$, the normal bundles of i_N and i_M thus are $\nu_N \oplus 1$ and $\nu_M \oplus 1$. We take the disk bundles $D(\nu_N \oplus 1)$ and $D(\nu_M \oplus 1)$ to be closed tubular neighborhoods in $\mathbb{H}^n$ and take the radius of $D(\nu_M \oplus 1)$ small enough to get the subset

$$D(\nu_M \oplus 1) \subseteq D(\nu_N \oplus 1) - S(\nu_N \oplus 1).$$

We have the following diagram:

$$\begin{array}{ccc} D(\nu_M \oplus 1) & \subseteq & D(\nu_N \oplus 1) \\ \cup\!\!\mid & \overset{i}{\nearrow} & \cup\!\!\mid \\ S(\nu_M \oplus 1) & & S(\nu_N \oplus 1) \\ s_M \uparrow\downarrow \pi_M & & \pi_N \downarrow\uparrow s_N \\ M & \underset{s}{\overset{\pi}{\leftrightarrows}} & N \end{array} \qquad \pi'_M\colon D(\nu_M \oplus 1) \to M,\quad \pi'_N\colon D(\nu_N \oplus 1) \to N$$

For $t \in I$ and $x \in S(\nu_M \oplus 1)$, by tx we mean multiplication of x by t in the fiber over $\pi_M(x)$ in $\nu_M \oplus 1$. Hence we have $tx \in D(\nu_M \oplus 1) \subseteq D(\nu_N \oplus 1)$. Also, $\pi'_M(tx) = \pi_M(x)$ for all $t \in I$. Defining the map

$$H\colon S(\nu_M \oplus 1) \times I \to M, \quad (x, t) \mapsto \pi\pi'_N(tx),$$

we get

$$H(x, 0) = \pi\pi'_N(0x) = \pi'_M(0x) = \pi_M(x), \quad H(x, 1) = \pi\pi'_N(x) = \pi\pi'_N i(x).$$

H being a homotopy, we have

$$\pi_M \simeq \pi\pi'_N i\colon S(\nu_M \oplus 1) \to M.$$

Thus $(S(\nu_M \oplus 1) \times I, p^* s_{M!}(x), fH)$ is a bordism giving

$$(S(\nu_M \oplus 1), s_{M!}(x), f\pi_M) \sim_{\mathcal{B}} (S(\nu_M \oplus 1), s_{M!}(x), f\pi\pi'_N i).$$

We now show there is another bordism,

$$(S(\nu_M \oplus 1), s_{M!}(x), f\pi\pi'_N i) \sim_{\mathcal{B}} (S(\nu_N \oplus 1), s_{N!}(y), g\pi_N).$$

Define

$$K := D(\nu_N \oplus 1) - \big(D(\nu_M \oplus 1) - S(\nu_M \oplus 1)\big).$$

Since $D(\nu_M \oplus 1) \subseteq D(\nu_N \oplus 1) - S(\nu_N \oplus 1)$, we have

$$S(\nu_N \oplus 1) \sqcup S(\nu_M \oplus 1)^- \subseteq \partial K.$$

Also, $\partial K - \big(S(\nu_N \oplus 1) \sqcup S(\nu_M \oplus 1)\big) \subseteq D(\nu_{\partial N} \oplus 1)$. So for the map $f\pi\pi'_N|_K\colon K \to X$, we have

$$\begin{aligned} &f\pi\pi'_N|_K(\partial K - S(\nu_N \oplus 1) \sqcup S(\nu_M \oplus 1)) \\ \subseteq &f\pi\pi'_N(D(\nu_{\partial N} \oplus 1)) \cup f\pi'_M(D(\nu_{\partial M} \oplus 1)) \\ \subseteq &f\pi(\partial N) \cup f(\partial M) \subseteq A. \end{aligned}$$

We see that $f\pi\pi'_N|_K$ restricts to $g\pi_N = f\pi\pi_N$ on $S(\nu_N \oplus 1)$ and to $f\pi\pi'_N i$ on $S(\nu_M \oplus 1)$, as required.

Finally, we show there is an element of $h^*(K)$ which restricts to $s_{M!}(x) \in h^*(S(\nu_M \oplus 1))$ and to $(s_N s)_!(x) = s_{N!}(y) \in h^*(S(\nu_N \oplus 1))$: We define a map

$$S\colon (M \times I, \partial(M \times I)) \to (K, \partial K)$$

given by $(p,t) \mapsto t \cdot s_M(p) + (1-t) \cdot (s_N s)(p)$ (addition and multiplication in $\nu_N \oplus 1$). Note that S restricts to $s_N s$ on $M \times \{0\}$ and to s_M on $M \times \{1\}$. We have the diagram

$$\begin{array}{ccccc} h^*(M \times I) & \longrightarrow & h^*(\partial(M \times I)) & \longrightarrow & h^*(M \sqcup M) \\ \downarrow{\scriptstyle S_!} & & \downarrow{\scriptstyle (\partial S)_!} & & \downarrow{\scriptstyle (s_N s \sqcup s_M)_!} \\ h^*(K) & \longrightarrow & h^*(\partial K) & \longrightarrow & h^*(S(\nu_N \oplus 1) \sqcup S(\nu_M \oplus 1)) \end{array},$$

the unlabeled arrows being restriction maps induced by the inclusions. The right-hand square commutes by Proposition 1.16. Since the left-hand square commutes up to sign on each component of M (Proposition 1.10), so does the diagram. The inclusion $i\colon M \sqcup M \hookrightarrow M \times I$ induces the map $i^*\colon h^*(M \times I) \to h^*(M \sqcup M)$. The diagram

$$\begin{array}{ccc} M \times \{0\} \cup M \times \{1\} & \longrightarrow & M \times \{1\} \cup M \times \{0\} \\ & \searrow \quad \swarrow & \\ & M \times I & \end{array}$$

commutes up to homotopy, and we thus get

$$\operatorname{im} i^* = \{u \sqcup u \in h^*(M \sqcup M) \mid u \in h^*(M)\}.$$

From this we see there is an element $x' \in h^*(M \times I)$ such that

$$(s_N s \sqcup s_M)_!(i^*(x')) = (s_N s)_!(x) \sqcup s_{M!}(x) = s_{N!}(y) \sqcup s_{M!}(x).$$

To x', there is an associated element $x'' \in h^*(M \times I)$ (by adjusting signs) such that the restriction of $S_!(x'') \in h^*(K)$ is $s_{N!}(y) \sqcup s_{M!}(x)$.

This makes the triple $(K, S_!(x''), f\pi\pi'_N|_K)$ a bordism, giving

$$(S(\nu_M \oplus 1), s_{M!}(x), f\pi\pi'_N i) \sim_{\mathcal{B}} (S(\nu_N \oplus 1), s_{N!}(y), g\pi_N).$$

Connecting the two bordisms, we finally have

$$\begin{aligned}(S(\nu_M \oplus 1), s_{M!}(x), f\pi_M) &\sim_{\mathcal{B}} (S(\nu_M \oplus 1), s_{M!}(x), f\pi\pi'_N i) \\ &\sim_{\mathcal{B}} (S(\nu_N \oplus 1), s_{N!}(y), g\pi_N).\end{aligned}$$ □

2.4 PROPOSITION. *Let $(M, x, f) \sim_{\mathcal{BS}} (M', x', f')$. Then there are normal bundles ν_M and $\nu_{M'}$ and bordant sphere triples*

$$(S(\nu_M \oplus 1), s_!(x), f\pi) \sim_{\mathcal{B}} (S(\nu_{M'} \oplus 1), s'_!(x'), f'\pi')$$

of (M, x, f) and (M', x', f'), respectively.

PROOF. First, let (W, y, F) be a bordism for $(M, x, f) \sim_{\mathcal{B}} (M', x', f')$. For simplicity, we assume W, M and M' to be single-dimensional. This gives $n := \dim M = \dim M'$. By the collaring theorem, there is an imbedding $(W, \partial W) \hookrightarrow (\mathbb{H}^{k+n+1}, \mathbb{R}^{k+n})$ such that the restriction of its normal bundle $\nu_W \downarrow W$ to ∂W is the normal bundle of the imbedding $\partial W \hookrightarrow \mathbb{R}^{k+n}$. Hence we have the bundle inclusion maps $\nu_{M \sqcup M'^-} \to \nu_{\partial W} \to \nu_W$:

$$\begin{array}{ccccc} \nu_{M \sqcup M'^-} & \hookrightarrow & \nu_{\partial W} & \hookrightarrow & \nu_W \\ \downarrow & & \downarrow & & \downarrow \\ M \sqcup M'^- & \hookrightarrow & \partial W & \hookrightarrow & W \end{array}$$

Using instead the imbedding

$$(W, \partial W) \hookrightarrow (\mathbb{H}^{k+n+1}, \mathbb{R}^{k+n}) \hookrightarrow (\mathbb{H}^{k+n+2}, \mathbb{R}^{k+n+1}),$$

we get the bundle inclusion maps of normal bundles $\nu_{M \sqcup M'^-} \oplus 1 \to \nu_{\partial W} \oplus 1 \to \nu_W \oplus 1$. Thus we have

$$\begin{aligned} S(\nu_M \oplus 1) \sqcup S(\nu_{M'} \oplus 1)^- &= S(\nu_M \oplus 1) \sqcup S(\nu_{M'^-} \oplus 1) \\ &= S(\nu_{M \sqcup M'^-} \oplus 1) \\ &\subseteq S(\nu_{\partial W} \oplus 1) \\ &= \partial S(\nu_W \oplus 1) \\ &\subseteq S(\nu_W \oplus 1). \end{aligned}$$

We see that $(S(\nu_W \oplus 1), s_{W!}(y), F\pi)$ is a bordism between $(S(\nu_M \oplus 1), s_{M!}(x), f\pi)$ and $(S(\nu_{M'} \oplus 1), s_{M'!}(x'), f'\pi)$, since

$$\begin{aligned} &\partial S(\nu_W \oplus 1) - (S(\nu_M \oplus 1) \sqcup S(\nu_{M'} \oplus 1)) \\ &= S(\nu_{\partial W} \oplus 1) - (S(\nu_M \oplus 1) \sqcup S(\nu_{M'} \oplus 1)) \\ &= S(\nu_{\partial W - (M \sqcup M')} \oplus 1) \end{aligned}$$

gives

$$F\pi\big(\partial S(\nu_W \oplus 1) - (S(\nu_M \oplus 1) \sqcup S(\nu_{M'} \oplus 1))\big) = F(\partial W - (M \sqcup M')) \subseteq A.$$

So when (M, x, f) and (M', x', f') are bordant—with bordism (W, y, F)—there is an imbedding of W in Euclidean half-space whose normal bundle $\nu_W \oplus 1$ again gives rise to a bordism between sphere triples of (M, x, f) and (M', x', f'). For simplicity, we have shown this when M and M' are n-manifolds. The same argument applies componentwise in the multi-dimensional case.

Now to the general setting of this proposition: Let $(M, x, f) \sim_{\mathcal{BS}} (M', x', f')$. There is a finite sequence of triples between (M, x, f) and (M', x', f'), each pair of consecutive triples being bordant or sphere related. As bordant and sphere related triples both give rise to *bordant* sphere triples (cf. Lemma 2.3), we imbed all the manifolds participating in the sequence into the same Euclidean half-space to obtain a sequence of bordant sphere triples from their normal bundles. The imbeddings must be done in the appropriate way:

For each pair of consecutive bordant triples, we imbed the manifold of a *bordism triple* as described here. For a pair of consecutive sphere related triples, we imbed the manifold of the sphere triple and imbed its base space as described in the proof of the previous lemma (i.e. via the section). For each pair of consecutive triples in the chain, the imbedding of its manifold must be disjoint from the other imbeddings. Except for M and M', each manifold participating in the chain is therefore imbedded *twice*. Now the sphere triples associated to the normal bundles of these imbeddings are all bordant. □

REMARK. Note that the triple $(\varnothing, 0, \mathrm{ef})$ only has one sphere triple, namely $(\varnothing, 0, \mathrm{ef})$ itself. Thus any triple which is $\mathcal{BS}$-related to $(\varnothing, 0, \mathrm{ef})$, has a sphere triple *bordant* to $(\varnothing, 0, \mathrm{ef})$. □

2.5 LEMMA. *For triples (M, x, f), (M, y, f) in (X, A), we have*

$$[M \sqcup M, x \sqcup y, f \sqcup f] = [M, x + y, f].$$

PROOF. Let $(M, \partial M) \overset{i}{\hookrightarrow} (\mathbb{H}^{k-2}, \mathbb{R}^{k-3}) \hookrightarrow (\mathbb{H}^{k-1}, \mathbb{R}^{k-2}) \hookrightarrow (\mathbb{H}^k, \mathbb{R}^{k-1})$ be an imbedding followed by the two standard inclusions. The normal bundle of i we denote by ν_i, and by ν_M the normal bundle of the composition of i with the first inclusion. The normal bundle of the total imbedding then becomes $\nu_M \oplus 1$, and we have $\nu_M = \nu_i \oplus 1$.

We define two sections,

$$s,\ s'\colon M \to \nu_M \oplus 1:$$

In 1, we let s and s' both be the zero-section, and in ν_M they are the respective sections

$$M \to \nu_i \oplus 1 = \nu_M, \quad x \mapsto (0, \pm\frac{1}{2}).$$

s and s' are disjoint imbeddings of M in (the total space of) $\nu_M \oplus 1$, giving normal bundles $\nu_s \oplus 1$ and $\nu_{s'} \oplus 1$ by the construction of s and s'. Their disk bundles we take small enough to be disjoint and to get

$$D(\nu_s \oplus 1),\ D(\nu_{s'} \oplus 1) \subseteq D(\nu_M \oplus 1) - S(\nu_M \oplus 1).$$

Note that since the sections are

$$s,\ s'\colon M \to \nu_i \oplus 1 \oplus 1, \quad x \mapsto (0, \pm\frac{1}{2}, 0),$$

the projection $D(\nu_M \oplus 1) \to M$ restricts to $D(\nu_s \oplus 1) \to M$ and $D(\nu_{s'} \oplus 1) \to M$ on the respective subsets. We have the h-manifold

$$W := D\big(\nu_M \oplus 1\big) - \Big(\big(D(\nu_s \oplus 1) - S(\nu_s \oplus 1)\big) \sqcup \big(D(\nu_{s'} \oplus 1) - S(\nu_{s'} \oplus 1)\big)\Big),$$

with

$$S(\nu_M \oplus 1) \sqcup \big(S(\nu_s \oplus 1) \sqcup S(\nu_{s'} \oplus 1)\big)^- \subseteq \partial W$$

and

$$\partial W - \big(S(\nu_M \oplus 1) \sqcup S(\nu_s \oplus 1) \sqcup S(\nu_{s'} \oplus 1)\big) \subseteq (\nu_M \oplus 1)|_{\partial M}.$$

There are sections

$$\begin{aligned}
\sigma\colon M &\to S(\nu_M \oplus 1), \quad x \mapsto (0_{\pi^{-1}(x)}, 1),\\
\sigma_s\colon M &\to S(\nu_s \oplus 1), \quad x \mapsto (0_{\pi_s^{-1}(x)}, 1),\\
\sigma_{s'}\colon M &\to S(\nu_{s'} \oplus 1), \quad x \mapsto (0_{\pi_{s'}^{-1}(x)}, 1).
\end{aligned}$$

We write $\pi_W\colon W \to M$ for the restriction of $\pi\colon \nu_M \oplus 1 \to M$ to W. We are about to show there is a bordism

$$\begin{aligned}
&(S(\nu_M \oplus 1), \sigma_!(x + y), f\pi)\\
\sim_{\mathcal{B}}\ &(S(\nu_s \oplus 1) \sqcup S(\nu_{s'} \oplus 1), \sigma_{s!}(x) \sqcup \sigma_{s'!}(y), f\pi_s \sqcup f\pi_{s'}).
\end{aligned}$$

The bordism triple is $(W, z, f\pi_W)$, where $z \in h^*(W)$ an element which we now describe. We define two maps,

$$S, S'\colon (M \times I, \partial(M \times I)) \to (W, \partial W),$$

where S is given by $(x, t) \mapsto t \cdot \sigma(x) + (1 - t) \cdot \sigma_s(x)$, and S' in the similar way, replacing σ_s with $\sigma_{s'}$. The multiplication and addition are vector space operations taking place in $\nu_M \oplus 1$ in the fiber over x. We have

$$S(x, 0) = 0 \cdot \sigma(x) + 1 \cdot \sigma_s(x) = \sigma_s(x),$$

etc. Note that S avoids $S(\nu_{s'} \oplus 1)$ and that S' avoids $S(\nu_s \oplus 1)$. In the following diagrams, we will write σ_s—resp. σ—for the restriction of S to $M \times \{0\}$—resp. to $M \times \{1\}$—and similarly for the restrictions of S'.

In the following diagram, the vertical arrows are maps induced by inclusions, while the remaining two unlabeled arrows to the left are induced by projections.

$$
\begin{array}{ccccc}
h^*(M) & \longrightarrow & h^*(M \times I) & \xrightarrow{S_! + S'_!} & h^*(W) \\
& \searrow & \downarrow & & \downarrow \\
& & h^*(\partial(M \times I)) & \xrightarrow{(\partial S)_! + (\partial S')_!} & h^*(\partial W) \\
& & \downarrow & & \downarrow \\
& & h^*(M \times \{1\}) & \xrightarrow{\sigma_!} & h^*(S(\nu_M \oplus 1))
\end{array}
$$

The diagram commutes up to sign by Proposition 1.10 and Proposition 1.16. The composition with the diagonal arrow from $h^*(M)$ is the actual $\sigma_!$, and hence there is an element $z \in h^*(W)$ which restricts to $\sigma_!(x + y)$ in $h^*(S(\nu_M \oplus 1))$.

In the following diagram, the unlabeled arrows are maps induced by inclusions. The isomorphism to the lower right is excision, and the pentagon thus commutes by Corollary 1.15. The map

$$\widehat{\partial S} \colon \partial(M \times I) \to \partial W - \operatorname{int} S(\nu_{s'} \oplus 1)$$

is the co-restriction of ∂S. This exists since S avoids $S(\nu_{s'} \oplus 1)$. The long exact sequence of the pair $(\partial W, \partial W - \operatorname{int} S(\nu_{s'} \oplus 1))$ gives the zero-map in the triangle. The diagram commutes, and so the composition map on the top is zero.

$$
\begin{array}{ccccc}
h^*(\partial(M \times I)) & \xrightarrow{(\partial S)_!} & h^*(\partial W) & \longrightarrow & h^*(S(\nu_{s'} \oplus 1)) \\
\cong \downarrow D_{\partial(M \times I)} & & \cong \downarrow D_{\partial W} & & \cong \downarrow D_{S(\nu_{s'} \oplus 1)} \\
h_*(\partial(M \times I)) & \xrightarrow{(\partial S)_*} & h_*(\partial W) & & h_*(S(\nu_{s'} \oplus 1), \partial S(\nu_{s'} \oplus 1)) \\
\searrow (\widehat{\partial S})_* & \nearrow & & \searrow & \cong \downarrow \\
h_*(\partial W - \operatorname{int} S(\nu_{s'} \oplus 1)) & & \xrightarrow{0} & & h_*(\partial W, \partial W - \operatorname{int} S(\nu_{s'} \oplus 1))
\end{array}
$$

Likewise, the map

$$h^*(\partial(M \times I)) \xrightarrow{(\partial S')_!} h^*(\partial W) \to h^*(S(\nu_s \oplus 1))$$

is zero. Together with Proposition 1.16, this makes the lower square in the following diagram commutative.

$$\begin{array}{ccccc}
h^*(M) & \longrightarrow & h^*(M \times I) & \xrightarrow{S_! + S'_!} & h^*(W) \\
 & \searrow & \downarrow & & \downarrow \\
 & & h^*(\partial(M \times I)) & \xrightarrow{(\partial S)_! + (\partial S')_!} & h^*(\partial W) \\
 & & \downarrow & & \downarrow \\
 & & h^*(M \times \{0\}) \oplus h^*(M \times \{0\}) & \xrightarrow{\sigma_{s!} \oplus \sigma_{s'!}} & h^*(S(\nu_s \oplus 1)) \oplus h^*(S(\nu_{s'} \oplus 1))
\end{array}$$

Again, it is really the composition from $h^*(M)$ which is the map $\sigma_{s!} \oplus \sigma_{s'!}$. Above we saw there was an element $z \in h^*(W)$ coming from $h^*(M)$ restricting to $\sigma_!(x + y) \in h^*(S(\nu_M \oplus 1))$. The commutativity of the lower square in the current diagram shows that the same $z \in h^*(W)$ also restricts to

$$(\sigma_{s!}(x), \sigma_{s'!}(y)) \in h^*(S(\nu_s \oplus 1)) \oplus h^*(S(\nu_{s'} \oplus 1)),$$

and therefore to

$$\sigma_{s!}(x) \sqcup \sigma_{s'!}(y) \in h^*(S(\nu_s \oplus 1) \sqcup S(\nu_{s'} \oplus 1)).$$

The bordism

$$\begin{aligned} &(S(\nu_M \oplus 1), \sigma_!(x + y), f\pi) \\ \sim_{\mathcal{B}}\ &(S(\nu_s \oplus 1) \sqcup S(\nu_{s'} \oplus 1), \sigma_{s!}(x) \sqcup \sigma_{s'!}(y), f\pi_s \sqcup f\pi_{s'}) \end{aligned}$$

is now evident. □

2.6 PROPOSITION. *On $h_d^{geo}(X, A)$, the operation*

$$[M, x, f] + [N, y, g] := [M \sqcup N, x \sqcup y, f \sqcup g]$$

is well-defined.

PROOF. We have to check that

$$(M', x', f') \sim_{\mathcal{BS}} (M, x, f), \quad (N', y', g') \sim_{\mathcal{BS}} (N, y, g),$$

yield

$$(M' \sqcup N', x' \sqcup y', f' \sqcup g') \sim_{\mathcal{BS}} (M \sqcup N, x \sqcup y, f \sqcup g).$$

This follows from the following three cases.

- Suppose we have

  $$(M', x', f') \sim_{\mathcal{B}} (M, x, f), \quad (N', y', g') \sim_{\mathcal{B}} (N, y, g),$$

  with respective bordisms (V, a, F) and (W, b, G).

  Now $(V \sqcup W, a \sqcup b, F \sqcup G)$ is a bordism giving

  $$(M' \sqcup N', x' \sqcup y', f' \sqcup g') \sim_{\mathcal{B}} (M \sqcup N, x \sqcup y, f \sqcup g).$$

- Suppose we have

$$(M', x', f') \sim_{\mathcal{B}} (M, x, f), \quad (N', y', g') \sim_{\mathcal{S}} (N, y, g).$$

By Lemma 2.3, there are sphere triples

$$\begin{aligned}(S(\nu_{N'} \oplus 1), s'_!(y'), g'\pi') &\sim_{\mathcal{S}} (N', y', g'),\\ (S(\nu_N \oplus 1), s_!(y), g\pi) &\sim_{\mathcal{S}} (N, y, g),\end{aligned}$$

and a bordism

$$(S(\nu_{N'} \oplus 1), s'_!(y'), g'\pi') \sim_{\mathcal{B}} (S(\nu_N \oplus 1), s_!(y), g\pi).$$

Over M, there is the product vector bundle of rank zero, $0 \downarrow M$. We make the identification $S(0 \oplus 1) = M \sqcup M$ and thus obtain the sphere triple

$$(M \sqcup M, x \sqcup 0, f\pi \sqcup f\pi) \sim_{\mathcal{S}} (M, x, f).$$

The vector bundle $0 \sqcup \nu_N \downarrow M \sqcup N$ then gives rise to the sphere triple

$$\begin{aligned}&(M \sqcup M \sqcup S(\nu_N \oplus 1), x \sqcup 0 \sqcup s_!(y), f\pi \sqcup f\pi \sqcup g\pi)\\ \sim_{\mathcal{S}}&(M \sqcup N, x \sqcup y, f \sqcup g).\end{aligned}$$

By the identity bordism on $(M \sqcup M, x \sqcup 0, f\pi \sqcup f\pi)$ together with the bordism given by the lemma, we have a bordism

$$\begin{aligned}&(M \sqcup M \sqcup S(\nu_{N'} \oplus 1), x \sqcup 0 \sqcup s'_!(y'), f\pi \sqcup f\pi \sqcup g'\pi')\\ \sim_{\mathcal{B}}\ &(M \sqcup M \sqcup S(\nu_N \oplus 1), x \sqcup 0 \sqcup s_!(y), f\pi \sqcup f\pi \sqcup g\pi).\end{aligned}$$

Reversing our steps, we now have the sphere triple

$$\begin{aligned}&(M \sqcup M \sqcup S(\nu_{N'} \oplus 1), x \sqcup 0 \sqcup s'_!(y'), f\pi \sqcup f\pi \sqcup g'\pi')\\ \sim_{\mathcal{S}}\ &(M \sqcup N', x \sqcup y', f \sqcup g').\end{aligned}$$

By the bordism $(M', x', f') \sim_{\mathcal{B}} (M, x, f)$ and the identity bordism on (N', y', g'), we have

$$\begin{aligned}&(M \sqcup N', x \sqcup y', f \sqcup g')\\ \sim_{\mathcal{B}}\ &(M' \sqcup N', x' \sqcup y', f' \sqcup g').\end{aligned}$$

In summary, we obtain

$$\begin{aligned}&(M \sqcup N, x \sqcup y, f \sqcup g)\\ \sim_{\mathcal{S}}\ &(M \sqcup M \sqcup S(\nu_N \oplus 1), x \sqcup 0 \sqcup s_!(y), f\pi \sqcup f\pi \sqcup g\pi)\\ \sim_{\mathcal{B}}\ &(M \sqcup M \sqcup S(\nu_{N'} \oplus 1), x \sqcup 0 \sqcup s'_!(y'), f\pi \sqcup f\pi \sqcup g'\pi')\\ \sim_{\mathcal{S}}\ &(M \sqcup N', x \sqcup y', f \sqcup g')\\ \sim_{\mathcal{B}}\ &(M' \sqcup N', x' \sqcup y', f' \sqcup g').\end{aligned}$$

- Suppose we have sphere triples

$$(M', x', f') \sim_{\mathcal{S}} (M, x, f), \quad (N', y', g') \sim_{\mathcal{S}} (N, y, g).$$

By Lemma 2.3, there are sphere triples

$$(S(\nu_{M'} \oplus 1), s_{M'!}(x'), f'\pi) \sim_{\mathcal{S}} (M', x', f'),$$
$$(S(\nu_M \oplus 1), s_{M!}(x), f\pi) \sim_{\mathcal{S}} (M, x, f),$$

resp.

$$(S(\nu_{N'} \oplus 1), s_{N'!}(y'), g'\pi) \sim_{\mathcal{S}} (N', y', g'),$$
$$(S(\nu_N \oplus 1), s_{N!}(y), g\pi) \sim_{\mathcal{S}} (N, y, g),$$

and bordisms

$$(S(\nu_{M'} \oplus 1), s_{M'!}(x'), f'\pi) \sim_{\mathcal{B}} (S(\nu_M \oplus 1), s_{M!}(x), f\pi),$$

resp.

$$(S(\nu_{N'} \oplus 1), s_{N'!}(y'), g'\pi) \sim_{\mathcal{B}} (S(\nu_N \oplus 1), s_{N!}(y), g\pi).$$

By the evident sections

$$s_{M'} \sqcup s_{N'} \colon M' \sqcup N' \to S((\nu_{M'} \sqcup \nu_{N'}) \oplus 1),$$
$$s_M \sqcup s_N \colon M \sqcup N \to S((\nu_M \sqcup \nu_N) \oplus 1),$$

we thus obtain

$$\begin{aligned}
&(M' \sqcup N', x' \sqcup y', f' \sqcup g') \\
\sim_{\mathcal{S}}\ & (S((\nu_{M'} \sqcup \nu_{N'}) \oplus 1), (s_{M'} \sqcup s_{N'})_!(x' \sqcup y'), (f' \sqcup g')\pi) \\
=\ & (S(\nu_{M'} \oplus 1) \sqcup S(\nu_{N'} \oplus 1), s_{M'!}(x') \sqcup s_{N'!}(y'), f'\pi \sqcup g'\pi) \\
\sim_{\mathcal{B}}\ & (S(\nu_M \oplus 1) \sqcup S(\nu_N \oplus 1), s_{M!}(x) \sqcup s_{N!}(y), f\pi \sqcup g\pi) \\
=\ & (S((\nu_M \sqcup \nu_N) \oplus 1), (s_M \sqcup s_N)_!(x \sqcup y), (f \sqcup g)\pi) \\
\sim_{\mathcal{S}}\ & (M \sqcup N, x \sqcup y, f \sqcup g).
\end{aligned}$$

Alternating the apostrophe marks in the six past lines, yield

$$\begin{aligned}
&(M' \sqcup N, x' \sqcup y, f' \sqcup g) \\
\sim_{\mathcal{BS}}\ & (M \sqcup N', x \sqcup y', f \sqcup g')
\end{aligned}$$

as well. □

Having established that $[M, x, f] + [N, y, g] := [M \sqcup N, x \sqcup y, f \sqcup g]$ is well-defined, we proceed to show that this makes $h_d^{geo}(X, A)$ an abelian group:

2.7 Proposition. *$h_d^{geo}(X, A)$ is an abelian group under the operation $+$.*

Proof. First of all, it is clear that $+$ is commutative. Next, we note that $[\varnothing, 0, \mathrm{ef}] \in h_d^{geo}(X, A)$ for any d. If we write $p\colon M \times \{0\} \to M$ for the projection, we have the equality $(M \sqcup \varnothing, x \sqcup 0, f \sqcup \mathrm{ef}) = (M \times \{0\}, p^*(x), fp)$. This we naturally identify with the triple (M, x, f), and we obtain

$$[M, x, f] + [\varnothing, 0, \mathrm{ef}] = [M \sqcup \varnothing, x \sqcup 0, f \sqcup \mathrm{ef}] = [M, x, f].$$

In other words, $[\varnothing, 0, \mathrm{ef}]$ is a neutral element with respect to $+$.

Now let $[M, x, f] \in h_d^{geo}(X, A)$. The triple $(M \times I, \pi^*(x \sqcup x), (f \sqcup f)\pi)$ in (X, X) is a bordism between the triples $(M \sqcup M^-, x \sqcup x, f \sqcup f)$ and $(\varnothing, 0, \mathrm{ef})$ in (X, A). With respect to $+$, $[M^-, x, f]$ is thus an inverse element of $[M, x, f]$, as we have

$$[M, x, f] + [M^-, x, f] = [M \sqcup M^-, x \sqcup x, f \sqcup f] = [\varnothing, 0, \mathrm{ef}]. \qquad \square$$

2.8 COROLLARY. *$h_*^{geo}(X, A)$ is a graded, abelian group by formally extending $+$ on each $h_d^{geo}(X, A)$.* □

Being the neutral element, we write

$$0 := [\varnothing, 0, \mathrm{ef}]$$

for the **zero-class** in $h_*^{geo}(X, A)$. Because of this, we shall speak of $(\varnothing, 0, \mathrm{ef})$ as the **null-triple**, and any triple bordant to it we will say is **null-bordant**. We emphasize that this term only applies to *bordism*, i.e. a triple may represent the zero-class without being null-bordant.

In fact, *any* triple $(N, 0, g)$ in (X, A) represents the zero-class: Let $\nu_N \downarrow N$ be a normal bundle. We have $S(\nu_N \oplus 1) \subseteq \partial D(\nu_N \oplus 1)$. Now,

$$\partial D(\nu_N \oplus 1) = S(\nu_N \oplus 1) \cup D(\nu_N|_{\partial N} \oplus 1)$$

gives

$$g\pi\big(\partial D(\nu_N \oplus 1) - S(\nu_N \oplus 1)\big) \subseteq g\pi\big(D(\nu_N|_{\partial N} \oplus 1)\big) = g(\partial N) \subseteq A.$$

Hence $(D(\nu_N \oplus 1), 0, g\pi)$ is a bordism between $(S(\nu_N \oplus 1), 0, g\pi)$ and $(\varnothing, 0, \mathrm{ef})$. This gives

$$(N, 0, g) \sim_{\mathcal{S}} (S(\nu_N \oplus 1), 0, g\pi) \sim_{\mathcal{B}} (\varnothing, 0, \mathrm{ef}),$$

i.e. $[N, 0, g] = 0$.

The following example is a short digression. It gives the affirmative answer to a previously formulated question.

EXAMPLE. Let H^* be singular cohomology with integral coefficients. $\mathbb{C}P^2$ is closed and (H-)orientable. Choosing an orientation of $\mathbb{C}P^2$ and a map $g\colon \mathbb{C}P^2 \to X$, we get the triple $(\mathbb{C}P^2, 0, g)$ in $(X, \varnothing)$. Since $\mathbb{C}P^2$ is not the boundary of any compact manifold, $(\mathbb{C}P^2, 0, g)$ is not null-bordant in $(X, \varnothing)$. Hence $(\mathbb{C}P^2, 0, g)$ and $(\varnothing, 0, \mathrm{ef})$ *are* $\mathcal{BS}$-related, but *not* $\mathcal{B}$-related. This shows that the two equivalence relations $\mathcal{B}$ and $\mathcal{BS}$ in general are different. □

Looking back to the proof of the proposition above, it is also natural to define

$$-[M, x, f] := [M^-, x, f].$$

Lemma 2.5 gives

$$[M, x, f] + [M, -x, f] = [M, x - x, f] = 0,$$

yielding $[M, -x, f] = -[M, x, f]$ as well.

Definition. A **single-dimensional** triple is a triple (M, x, f) such that M is single-dimensional. □

Remark. The difference in dimension of the manifold and the cohomology element of a triple is by definition constant between components. So for any *single-dimensional* triple, the cohomology class is *homogeneous.* □

2.9 Lemma. *Each element in $h_d^{geo}(X, A)$ is the class of a single-dimensional triple.*

Proof. Let $[N, y, g] \in h_d^{geo}(X, A)$. If $y = 0$, then $[N, y, g] = 0 = [\varnothing, 0, \mathrm{ef}]$ and we are done. So we may assume $y \neq 0$. We imbed N in some Euclidean n-space to obtain the normal bundle $\nu \downarrow N$. Then $S(\nu \oplus 1)$ is an h-manifold of dimension n. By the standard section $s\colon N \to S(\nu \oplus 1)$, we obtain $[N, y, g] = [S(\nu \oplus 1), s_!(y), g\pi]$. □

This shows that any element in $h_*^{geo}(X, A)$ can be written as a sum

$$[M_1, x_1, f_1] + \ldots + [M_r, x_r, f_r]$$

such that each M_i is single-dimensional, the x_i are homogeneous and no two $[M_i, x_i, f_i]$ are of the same dimension.

Remark. At this point, a natural question arises regarding triples. It seems possible that it would be sufficient to only consider the subset of $\Lambda(X, A)$ containing *single-dimensional* triples. Addition of two classes of triples could be defined in a similar way as above, by first choosing representative triples with manifolds being of the same dimension. Regretfully, this idea will not be pursued. □

2.4 The homology theory h_*^{geo}

We now go on to show that we have the covariant functor $h_*^{geo}\colon \mathsf{Top}^2 \to \mathsf{Ab}_*$.

2.10 Lemma. *If $\varphi\colon (X, A) \to (Y, B)$ is a map of pairs, then the induced map*

$$\varphi_*\colon h_d^{geo}(X, A) \to h_d^{geo}(Y, B),$$

given by $[M, x, f] \mapsto [M, x, \varphi f]$, is a well-defined group homomorphism.

Proof. First we show that the assignment is well-defined: Let (M, x, f) and (M', x', f') be two triples representing the same class in $h_d^{geo}(X, A)$. As usual, we can assume that either (M, x, f) and (M', x', f') are bordant or that (M', x', f') is a sphere triple of (M, x, f):

- Let $(M, x, f) \sim_{\mathcal{B}} (M', x', f')$. Then there exists a bordism (W, y, F) between them, and clearly $(W, y, \varphi F)$ is a bordism between $(M, x, \varphi f)$ and $(M', x', \varphi f')$.
- Let $(M', x', f') = (S(E \oplus 1), s_!(x), f\pi)$ be a sphere triple of (M, x, f). Then $(M', x', \varphi f') = (S(E \oplus 1), s_!(x), \varphi f\pi)$ is a sphere triple of $(M, x, \varphi f)$.

Hence $(M, x, \varphi f)$ and $(M', x', \varphi f')$ are $\mathcal{BS}$-related whenever (M, x, f) and (M', x', f') are, and so the assignment is well-defined.

Now

$$\begin{aligned}\varphi_*([M,x,f]+[M',x',f']) &= \varphi_*[M \sqcup M', x \sqcup x', f \sqcup f'] \\ &= [M \sqcup M', x \sqcup x', \varphi(f \sqcup f')] \\ &= [M,x,\varphi f] + [M',x',\varphi f'] \\ &= \varphi_*[M,x,f] + \varphi_*[M',x',f']\end{aligned}$$

shows that φ_* is a group homomorphism. □

The $\varphi_*\colon h_d^{geo}(X,A) \to h_d^{geo}(Y,B)$ extend linearly to become a graded (degree zero) group homomorphism $\varphi_*\colon h_*^{geo}(X,A) \to h_*^{geo}(Y,B)$. It is clear that we have $(\varphi\gamma)_* = \varphi_*\gamma_*$ for composition maps and that $1_* = 1$. With the assignment $\varphi \mapsto \varphi_*$, h_*^{geo} becomes a covariant functor.

Before showing that h_*^{geo} is a homology theory, we recall the axioms. We exclude the dimension axiom of Eilenberg and Steenrod, but include additivity.

First, let $R\colon \mathsf{Top}^2 \to \mathsf{Top}^2$ be the **restriction functor** defined by

$$\begin{gathered} R(X,A) = (A,\varnothing), \\ R(f\colon (X,A)\to(Y,B)) = f|_{(A,\varnothing)}\colon (A,\varnothing) \to (B,\varnothing). \end{gathered}$$

DEFINITION. A **homology theory** is a covariant functor

$$k_*\colon \mathsf{Top}^2 \to \mathsf{Ab}_*$$

together with a degree -1 natural transformation

$$\partial\colon k_* \to k_* \circ R$$

satisfying the following:

- HOMOTOPY INVARIANCE.

  $\varphi_1 \simeq \varphi_2\colon (X,A) \to (Y,B)$ implies $\varphi_{1*} = \varphi_{2*}\colon k_*(X,A) \to k_*(Y,B)$.

- LONG EXACT SEQUENCE.

  For every pair (X,A), the inclusions $(A,\varnothing) \overset{i}{\hookrightarrow} (X,\varnothing) \overset{j}{\hookrightarrow} (X,A)$ induce a long exact sequence

  $$\dots \xrightarrow{\partial_{(X,A)}} k_n(A,\varnothing) \xrightarrow{i_*} k_n(X,\varnothing) \xrightarrow{j_*} k_n(X,A) \xrightarrow{\partial_{(X,A)}} k_{n-1}(A,\varnothing) \xrightarrow{i_*} \dots .$$

- EXCISION.

  Let (X,A) be a pair of spaces and $U \subseteq X$ a subspace such that $\bar{U} \subseteq \operatorname{int} A$. Then the inclusion of pairs induces an isomorphism

  $$i_*\colon k_*(X-U, A-U) \xrightarrow{\cong} k_*(X,A).$$

- ADDITIVITY.
  For any collection of pairs $(X_\alpha, A_\alpha)_\alpha$, the inclusions
  $$i_\alpha\colon (X_\alpha, A_\alpha) \hookrightarrow (\bigsqcup X_\alpha, \bigsqcup A_\alpha)$$
  give an isomorphism
  $$\coprod_\alpha i_{\alpha *}\colon \coprod_\alpha k_*(X_\alpha, A_\alpha) \xrightarrow{\cong} k_*(\bigsqcup X_\alpha, \bigsqcup A_\alpha).$$ □

REMARK. There is also the notion of homology theories on CW pairs, being functors
$$k_*\colon \mathsf{CW}^2 \to \mathsf{Ab}_*$$
satisfying axioms similar to those above. We shall thus be referring to homology theories on Top^2 and CW^2, respectively. □

We write $x|_{\partial M} \in h^*(\partial M)$ for the restriction of $x \in h^*(M)$. When $f\colon (M, \partial M) \to (X, A)$ is a map, we take its restriction from M to ∂M *and* its co-restriction from X to A to obtain the map $\partial f\colon (\partial M, \varnothing) \to (A, \varnothing)$. In the case of $x = 0$, $\dim x$ is not defined, and we shall then understand $(-1)^{\dim x} x|_{\partial M}$ to be zero.

DEFINITION. For each pair $(X, A) \in \mathsf{Top}^2$ and $n \in \mathbb{Z}$, the **boundary homomorphism** in geometric homology is the map
$$\partial\colon h_n^{geo}(X, A) \to h_{n-1}^{geo}(A, \varnothing),$$
$$[M, x, f] \mapsto [\partial M, (-1)^{\dim x} x|_{\partial M}, \partial f],$$
where (M, x, f) is any single-dimensional triple representing the class $[M, x, f]$. □

2.11 PROPOSITION. *The boundary homomorphism is a well-defined homomorphism.*

PROOF. First of all, Lemma 2.9 and its preceding remark ensures that every class can be represented by a single-dimensional triple, and that the cohomology class of such a triple always is homogeneous. Also, it is clear that we have $(\partial M, (-1)^{\dim x} x|_{\partial M}, \partial f) \in \Lambda(A, \varnothing)$, and $n = \dim M - \dim x$ implies
$$\dim \partial M - \dim(-1)^{\dim x} x|_{\partial M} = (\dim M - 1) - \dim x = n - 1.$$
Hence we have
$$[\partial M, (-1)^{\dim x} x|_{\partial M}, \partial f] \in h_{n-1}^{geo}(A, \varnothing).$$

We must show that any two $\mathcal{BS}$-related, single-dimensional triples remain $\mathcal{BS}$-related under the assignment $(M, x, f) \mapsto (\partial M, (-1)^{\dim x} x|_{\partial M}, \partial f)$. The two following cases show that this is true for single-dimensional triples which are bordant—resp. related by sphere bundle modification. As soon as these two cases have been established, the proof is completed by a short argument showing that the same is true for $\mathcal{BS}$-related triples.

- Let (W, y, F) be a bordism between two single-dimensional triples (M, x, f) and (M', x', f'). We view $M \sqcup M'^-$ as imbedded in ∂W. We define
  $$K := \partial W - \operatorname{int}(M \sqcup M'^-),$$

which is seen to be an h-manifold with boundary $\partial K = \partial M \sqcup \partial M'^{-}$. From $F\colon (W, \partial W) \to (X, A)$, we obtain the map

$$F|_K\colon (K, \partial K) \to (A, A).$$

Further, it may very well be the case that we have $\dim x \neq \dim x'$, i.e. y non-homogeneous. We denote the components of ∂K by $(\partial K)_i$ and the corresponding components of ∂M and $\partial M'$ by $(\partial M)_i$ and $(\partial M')_i$. This gives

$$(\partial M)_i \sqcup (\partial M')_i^- \subseteq (\partial K)_i.$$

For each i, $y|_K \in h^*(K)$ restricts to the homogeneous element $y|_{K_i} \in h^*(K_i)$. We may thus define

$$\widehat{y|_K} \coloneqq \bigsqcup (-1)^{\dim y|_{K_i}} y|_{K_i} \in h^*\big(\bigsqcup K_i\big).$$

This element restricts to

$$(-1)^{\dim x|_{(\partial M)_i}} x|_{(\partial M)_i} \in h^*\big((\partial M)_i\big), \quad (-1)^{\dim x'|_{(\partial M')_i}} x'|_{(\partial M')_i} \in h^*\big((\partial M')_i\big).$$

Since x and x' by assumption are homogeneous, we have $\dim x|_{(\partial M)_i} = \dim x$ and $\dim x'|_{(\partial M')_i} = \dim x'$ for all i. Hence $(K, \widehat{y|_K}, F|_K)$ gives the bordism

$$(\partial M, (-1)^{\dim x} x|_{\partial M}, \partial f) \sim_{\mathcal{B}} (\partial M', (-1)^{\dim x'} x'|_{\partial M'}, \partial f').$$

- Let $(S(E \oplus 1), s_!(x), f\pi)$ be a sphere triple of (M, x, f), E and M of dimensions n and m. We want to show

$$\big(\partial S(E \oplus 1), (-1)^{\dim s_!(x)} s_!(x)|_{\partial S(E\oplus 1)}, \partial(f\pi)\big) \sim_{\mathcal{S}} \big(\partial M, (-1)^{\dim x} x|_{\partial M}, \partial f\big).$$

Restricting $E \downarrow M$—resp. $s\colon M \to S(E \oplus 1)$—gives the vector bundle $\partial E \downarrow \partial M$—resp. the section $\partial s\colon \partial M \to S(\partial E \oplus 1)$. Thus we have the sphere triple

$$\big(S(\partial E \oplus 1), (\partial s)_!((-1)^{\dim x} x|_{\partial M}), (\partial f)\pi\big) \sim_{\mathcal{S}} \big(\partial M, (-1)^{\dim x} x|_{\partial M}, \partial f\big).$$

It is clear that $S(\partial E \oplus 1) = \partial S(E \oplus 1)$ with $(\partial f)\pi = \partial(f\pi)$. By Proposition 1.10, the diagram

$$\begin{array}{ccc} h^k(M) & \xrightarrow{i_M^*} & h^k(\partial M) \\ \downarrow{\scriptstyle s_!} & & \downarrow{\scriptstyle (\partial s)_!} \\ h^{k+n-m}(S(E\oplus 1)) & \xrightarrow{i_N^*} & h^{k+n-m}(\partial S(E \oplus 1)) \end{array}$$

commutes up to sign, specifically $(\partial s)_! i_M^* = (-1)^{n-m} i_N^* s_!$.

$k \coloneqq \dim x$ gives $\dim s_!(x) = k + n - m$, and so

$$\begin{aligned} (\partial s)_!((-1)^{\dim x} x|_{\partial M}) &= (\partial s)_!((-1)^k i_M^*(x)) \\ &= (-1)^{k+n-m} i_N^* s_!(x) \\ &= (-1)^{\dim s_!(x)} s_!(x)|_{\partial S(E\oplus 1)}. \end{aligned}$$

Hence we have shown

$$\begin{aligned}&(\partial S(E\oplus 1), (-1)^{\dim s_!(x)} s_!(x)|_{\partial S(E\oplus 1)}, \partial(f\pi))\\ =& (S(\partial E\oplus 1), (\partial s)_!((-1)^{\dim x} x|_{\partial M}), (\partial f)\pi)\\ \sim_{\mathcal{S}}& (\partial M, (-1)^{\dim x} x|_{\partial M}, \partial f).\end{aligned}$$

Now let $(M,x,f) \sim_{\mathcal{BS}} (M',x',f')$, both triples single-dimensional. By Proposition 2.4, we have the sequence

$$(M,x,f) \sim_{\mathcal{S}} (S(E\oplus 1), s_!(x), f\pi) \sim_{\mathcal{B}} (S(E'\oplus 1), s'_!(x'), f'\pi') \sim_{\mathcal{S}} (M',x',f'),$$

where E and E' are normal bundles. All of these triples are then single-dimensional. The two cases we have just shown thus apply successively to the sequence. We obtain

$$\begin{aligned}&(\partial M, (-1)^{\dim x} x|_{\partial M}, \partial f)\\ \sim_{\mathcal{S}}& (\partial S(E\oplus 1), (-1)^{\dim s_!(x)} s_!(x)|_{\partial S(E\oplus 1)}, \partial(f\pi))\\ \sim_{\mathcal{B}}& (\partial S(E'\oplus 1), (-1)^{\dim s'_!(x')} s'_!(x')|_{\partial S(E'\oplus 1)}, \partial(f'\pi'))\\ \sim_{\mathcal{S}}& (\partial M', (-1)^{\dim x'} x'|_{\partial M'}, \partial f').\end{aligned}$$

The assignment

$$[M,x,f] \mapsto [\partial M, (-1)^{\dim x} x|_{\partial M}, \partial f]$$

is thus seen to be well-defined.

Finally, we show that ∂ is a homomorphism: Given two triples representing two classes, we may imbed their two manifolds in the same Euclidean n-space. The two normal bundles are then both of dimension n. Associated sphere triples represent the two initial classes, *and* both of their manifolds have dimension n. If in addition the two classes are of equal dimension, then so are the homogeneous cohomology classes of the new triples. As both $+$ and ∂ are independent of choices of representing triples, we may assume that our triples have the mentioned properties. That is, M and M' are n-manifolds, and thus $\dim x = \dim x'$:

$$\begin{aligned}&\partial([M,x,f] + [M',x',f'])\\ =&\partial[M\sqcup M', x\sqcup x', f\sqcup f']\\ =&[\partial(M\sqcup M'), (-1)^{\dim x\sqcup x'}(x\sqcup x')|_{\partial(M\sqcup M')}, \partial(f\sqcup f')]\\ =&[\partial M\sqcup \partial M', (-1)^{\dim x} x|_{\partial M} \sqcup (-1)^{\dim x'} x'|_{\partial M'}, \partial f\sqcup \partial f']\\ =&[\partial M, (-1)^{\dim x} x|_{\partial M}, \partial f] + [\partial M', (-1)^{\dim x'} x'|_{\partial M'}, \partial f']\\ =&\partial[M,x,f] + \partial[M',x',f'].\end{aligned}$$

□

REMARK. From now on, we will implicitly assume (M,x,f) to be single-dimensional whenever a class $[M,x,f]$ is arbitrarily chosen. □

For a map $\varphi\colon (X,A) \to (Y,B)$, we have

$$\begin{aligned}\partial(\varphi_*[M,x,f]) &= [\partial M, (-1)^{\dim x} x|_{\partial M}, \partial(\varphi f)]\\ &= [\partial M, (-1)^{\dim x} x|_{\partial M}, \varphi|_A \partial f]\\ &= (\varphi|_A)_* \partial[M,x,f],\end{aligned}$$

and so the following diagram commutes.

$$\begin{array}{ccc} h_n^{geo}(X,A) & \xrightarrow{\partial} & h_{n-1}^{geo}(A,\varnothing) \\ \downarrow{\scriptstyle \varphi_*} & & \downarrow{\scriptstyle (\varphi|_A)_*} \\ h_n^{geo}(Y,B) & \xrightarrow{\partial} & h_{n-1}^{geo}(B,\varnothing) \end{array}$$

Let n be fixed. We have defined ∂ for each pair (X,A). Referring to the axioms of a homology theory, ∂ should properly have been denoted $\partial_{(X,A)}$. The collection of all these boundary homomorphisms constitutes a transformation of functors $h_n^{geo} \to h_{n-1}^{geo} \circ R$. These we will also denote by ∂. As the diagram above commutes for each map φ, ∂ is a natural transformation.

We are now ready to verify the four axioms which make $h_*^{geo}\colon \mathsf{Top}^2 \to \mathsf{Ab}_*$ (together with ∂) a homology theory.

2.12 PROPOSITION (HOMOTOPY INVARIANCE). *Let*

$$\varphi_0 \simeq \varphi_1\colon (X,A) \to (Y,B)$$

be homotopic maps. Then they induce the same map

$$\varphi_{0*} = \varphi_{1*}\colon h_*^{geo}(X,A) \to h_*^{geo}(Y,B)$$

in geometric homology.

PROOF. We show $\varphi_{0*} = \varphi_{1*}\colon h_d^{geo}(X,A) \to h_d^{geo}(Y,B)$. Then the general result follows by linearity.

Let $H\colon (X \times I, A \times I) \to (Y,B)$ be a homotopy between φ_0 and φ_1. Given a triple (M,x,f) in (X,A), we get the triples $(M,x,\varphi_0 f)$ and $(M,x,\varphi_1 f)$ in (Y,B). We show how the homotopy gives rise to a bordism between them.

From the composition

$$H \circ (f \times 1)\colon (M \times I, \partial M \times I) \xrightarrow{f\times 1} (X \times I, A \times I) \xrightarrow{H} (Y,B),$$

we obtain the map

$$F\colon (M \times I, \partial(M \times I)) \to (Y,Y).$$

Letting $\pi\colon M \times I \to M$ denote the projection, we have the triple

$$(M \times I, \pi^*(x), F) \in \Lambda(Y,Y).$$

For $i = 0,1$, we have

$$F(x,i) = H(f(x),i) = \varphi_i f(x),$$

and we identify M with $M \times \{i\}$ to obtain $F|_{M\times\{i\}} = \varphi_i f$. Thus $(M \times I, \pi^*(x), F)$ is a bordism between $(M, x, \varphi_0 f)$ and $(M, x, \varphi_1 f)$, noting that

$$F(\partial(M \times I) - M \sqcup M^-) \subseteq F(\partial M \times I) = H(f(\partial M) \times I) \subseteq B.$$

The result follows. □

We proceed to show that any pair of spaces gives rise to a long exact sequence in geometric homology. First, we shall have three lemmas.

2.13 Lemma. *Let (M, x, f) be a triple in (X, A). If $M' \subseteq M$ is a compact, codimension zero submanifold such that*

$$f(M - \operatorname{int} M') \subseteq A,$$

then there is a bordism

$$(M, x, f) \sim_{\mathcal{B}} (M', x|_{M'}, f|_{M'}).$$

Proof. $\partial M' \subseteq M - \operatorname{int} M'$ makes $(M', x|_{M'}, f|_{M'})$ a triple in (X, A). We see that $(M \times I, \pi^*(x), f\pi)$ gives the bordism

$$(M', x|_{M'}, f|_{M'}) \sim_{\mathcal{B}} (M, x, f):$$

We have $M' \sqcup M^- \subseteq M \sqcup M^- \subseteq \partial(M \times I)$. $\pi^*(x)$ restricts to $x|_{M'}$ and x, and $f\pi$ restricts to $f|_{M'}$ and f. Also, $\partial(M \times I) = (\partial M \times I) \cup (M \sqcup M^-)$ gives

$$\begin{aligned} f\pi\big(\partial(M \times I) - (M' \sqcup M^-)\big) &\subseteq f\pi\big(\partial M \times I\big) \cup f\pi\big((M \sqcup M^-) - (M' \sqcup M^-)\big) \\ &= f(\partial M) \cup f(M - M') \subseteq A. \end{aligned}$$

□

2.14 Lemma. *Let $\varphi_*\colon h_n^{geo}(X, A) \to h_n^{geo}(Y, B)$ be an induced map. Then any element of $\ker \varphi_*$ is the class of a triple (M, x, f) in (X, A) such that $(M, x, \varphi f)$ in (Y, B) is null-bordant.*

Proof. Let $[N, y, g] \in \ker \varphi_*$. Since $(N, y, \varphi g) \sim_{\mathcal{BS}} (\varnothing, 0, \mathrm{ef})$, the remark succeeding Proposition 2.4 states there then is a sphere triple $(S(E \oplus 1), s_!(y), \varphi g\pi)$ of $(N, y, \varphi g)$ in (Y, B) which is null-bordant.

$$(M, x, f) \coloneqq (S(E \oplus 1), s_!(y), g\pi) \sim_{\mathcal{S}} (N, y, g)$$

yield $[M, x, f] = [N, y, g]$. □

2.15 Lemma. *Any element in the kernel of $\partial\colon h_n^{geo}(X, A) \to h_{n-1}^{geo}(A, \varnothing)$ is the class of a triple (M, x, f) in (X, A) such that $(\partial M, (-1)^{\dim x} x|_{\partial M}, \partial f)$ in $(A, \varnothing)$ is null-bordant.*

Proof. Let $[N, y, g] \in \ker \partial \subseteq h_n^{geo}(X, A)$, i.e.

$$\partial[N, y, g] = [\partial N, (-1)^{\dim y} y|_{\partial N}, \partial g] = 0 \in h_{n-1}^{geo}(A, \varnothing).$$

By the remark succeeding Proposition 2.4, there is a null-bordant sphere triple

$$\big(S(\nu_{\partial N} \oplus 1), (\partial s)_!((-1)^{\dim y} y|_{\partial N}), (\partial g)\pi\big) \sim_{\mathcal{S}} \big(\partial N, (-1)^{\dim y} y|_{\partial N}, \partial g\big)$$

in $(A, \varnothing)$, with $\nu_{\partial N} \downarrow \partial N$ the normal bundle of an imbedding $\partial N \hookrightarrow \mathbb{R}^k$. By the collaring theorem, this can be taken to be the restriction of an imbedding $(N, \partial N) \hookrightarrow (\mathbb{H}^{k+1}, \mathbb{R}^k)$ such that $\nu_N \downarrow N$ restricts to $\nu_{\partial N} \downarrow \partial N$. Now we can identify $S(\nu_{\partial N} \oplus 1)$ and $\partial S(\nu_N \oplus 1)$, and we have $(\partial g)\pi = \partial(g\pi)$. Also, $\partial s\colon \partial N \to S(\nu_{\partial N} \oplus 1)$ can be taken to be the restriction of $s\colon N \to S(\nu_N \oplus 1)$. The sphere triple

$$\big(M, x, f\big) \coloneqq \big(S(\nu_N \oplus 1), s_!(y), g\pi\big) \sim_{\mathcal{S}} \big(N, y, g\big)$$

gives

$$\begin{aligned}\big(\partial M, (-1)^{\dim x} x|_{\partial M}, \partial f\big) &= \big(\partial S(\nu_N \oplus 1), (-1)^{\dim s_!(y)} s_!(y)|_{\partial S(\nu_N \oplus 1)}, \partial(g\pi)\big) \\ &= \big(S(\nu_{\partial N} \oplus 1), (\partial s)_!((-1)^{\dim y} y|_{\partial N}), (\partial g)\pi\big),\end{aligned}$$

where we have

$$(-1)^{\dim s_!(y)} s_!(y)|_{\partial S(\nu_N \oplus 1)} = (\partial s)_!((-1)^{\dim y} y|_{\partial N})$$

from Proposition 1.10. Hence $\big(\partial M, (-1)^{\dim x} x|_{\partial M}, \partial f\big)$ is null-bordant. □

2.16 PROPOSITION (LONG EXACT SEQUENCE). *For every pair (X, A), the inclusions*

$$(A, \varnothing) \overset{i}{\hookrightarrow} (X, \varnothing) \overset{j}{\hookrightarrow} (X, A)$$

induce a long exact sequence

$$\dots \xrightarrow{\partial} h_n^{geo}(A, \varnothing) \xrightarrow{i_*} h_n^{geo}(X, \varnothing) \xrightarrow{j_*} h_n^{geo}(X, A) \xrightarrow{\partial} h_{n-1}^{geo}(A, \varnothing) \xrightarrow{i_*} \dots \ .$$

PROOF. We start by showing that the sequence is a complex.

- $j_* i_*$ factors through $h_n^{geo}(A, A)$, which is seen to be zero:

  Any triple (M, x, f) in (A, A) is null-bordant, with bordism $(M \times I, \pi^*(x), f\pi)$. This is evident from $f\pi(M \times I - (M \sqcup \varnothing)) \subseteq f\pi(M \times I) \subseteq A$.

- Let $[M, x, f] \in h_n^{geo}(X, \varnothing)$. $f(\partial M) \subseteq \varnothing$ implies $\partial M = \varnothing$. Thus we have

  $$(\partial M, (-1)^{\dim x} x|_{\partial M}, \partial(jf)) = (\varnothing, 0, \mathrm{ef}),$$

  as the null-triple is the only triple with the empty set as manifold. And so

  $$\partial j_*[M, x, f] = [\partial M, (-1)^{\dim x} x|_{\partial M}, \partial(jf)] = [\varnothing, 0, \mathrm{ef}] = 0 \in h_{n-1}^{geo}(A, \varnothing).$$

  Hence $\partial j_* = 0$.

- Let $[M, x, f] \in h_n^{geo}(X, A)$. Then

  $$i_*\partial[M, x, f] = [\partial M, (-1)^{\dim x} x|_{\partial M}, i(\partial f)] \in h_{n-1}^{geo}(X, \varnothing).$$

  We view f as a map $(M, \partial M) \to (X, X)$ and get that $(M, (-1)^{\dim x} x, f)$ gives the bordism $(\partial M, (-1)^{\dim x} x|_{\partial M}, i(\partial f)) \sim_{\mathcal{B}} (\varnothing, 0, \mathrm{ef})$.

  Hence $i_*\partial = 0$.

We finish the proof by showing the three opposite inclusions, i.e. image containing kernel.

- Let $[M,x,f] \in \ker i_* \subseteq h_n^{geo}(A,\varnothing)$, i.e.

$$i_*[M,x,f] = [M,x,if] = 0 \in h_n^{geo}(X,\varnothing).$$

By Lemma 2.14, we may assume that (M,x,if) is null-bordant. Let $(W,y,F) \in \Lambda(X,X)$ be such a bordism. $F(\partial W - M) \subseteq \varnothing$ gives $\partial W = M$. Hence

$$F(\partial W) = F(M) = if(M) \subseteq i(A) = A,$$

which gives us the map $F\colon (W,\partial W) \to (X,A)$. Furthermore,

$$\dim W - \dim(-1)^{\dim x} y = 1 + \dim M - \dim x = n+1.$$

Evidently, we have

$$[W,(-1)^{\dim x} y, F] \in h_{n+1}^{geo}(X,A).$$

$(\partial W, (-1)^{2\dim x} y|_{\partial W}, \partial F) = (M,x,f)$ now gives

$$\partial[W,(-1)^{\dim x} y, F] = [\partial W, (-1)^{2\dim x} y|_{\partial W}, \partial F] = [M,x,f] \in h_n^{geo}(A,\varnothing).$$

Hence $\operatorname{im} \partial \supseteq \ker i_*$.

- Let $[M,x,f] \in \ker j_* \subseteq h_n^{geo}(X,\varnothing)$, i.e.

$$j_*[M,x,f] = [M,x,jf] = 0 \in h_n^{geo}(X,A).$$

By Lemma 2.14, we may assume that (M,x,jf) is null-bordant. Let $(W,y,F) \in \Lambda(X,X)$ be such a bordism. Then we have the triple

$$(\partial W - M, y|_{\partial W - M}, F|_{\partial W - M}) \in \Lambda(A,\varnothing):$$

$f\colon (M,\partial M) \to (X,\varnothing)$ yields $\partial M = \varnothing$. Since $M \hookrightarrow \partial W$ is a codimension zero imbedding of closed manifolds, M is the union of components of ∂W. The same is then true for its compliment, making $\partial W - M$ a closed manifold with the canonical orientation from ∂W. (W,y,F) being a bordism, we have $F(\partial W - M) \subseteq A$. Hence we have the map $F|_{\partial W - M}\colon (\partial W - M, \varnothing) \to (A,\varnothing)$. Also,

$$\dim(\partial W - M) - \dim y|_{\partial W - M} = \dim M - \dim y|_M = \dim M - \dim x = n.$$

This shows that $(\partial W - M, y|_{\partial W - M}, F|_{\partial W - M})$ is a triple in $(A,\varnothing)$, as asserted. It has dimension n, and so

$$[\partial W - M, y|_{\partial W - M}, F|_{\partial W - M}] \in h_n^{geo}(A,\varnothing).$$

From $(\partial W - M) \sqcup M = \partial W \subseteq \partial W$ and with F agreeing with $iF|_{\partial W - M}$ on $\partial W - M$ and with f on M—it is clear that (W,y,F) also gives a bordism

$$(\partial W - M, y|_{\partial W - M}, iF|_{\partial W - M}) \sim_{\mathcal{B}} (M,x,f).$$

Now we have

$$i_*[\partial W - M, y|_{\partial W - M}, F|_{\partial W - M}] = [M,x,f] \in h_n^{geo}(X,\varnothing).$$

Hence $\operatorname{im} i_* \supseteq \ker j_*$.

- Let $[M, x, f] \in \ker \partial \subseteq h_n^{geo}(X, A)$, i.e.

$$\partial[M, x, f] = [\partial M, (-1)^{\dim x} x|_{\partial M}, \partial f] = 0 \in h_{n-1}^{geo}(A, \varnothing).$$

By Lemma 2.15, we can assume that $(\partial M, (-1)^{\dim x} x|_{\partial M}, \partial f)$ is null-bordant, and we let $(W, y, F) \in \Lambda(A, A)$ be a null-bordism. $F(\partial W - \partial M) \subseteq \varnothing$ then gives $B \coloneqq \partial M = \partial W$. We take M and W^- to have collars $\partial M \times [0, 1)$ and $\partial W^- \times (-1, 0]$ and glue M and W^- together along their common boundary, B. We obtain the smooth, closed manifold

$$M \cup_B W^-.$$

(Cf. [Hir76].) Since M and W^- have orientations agreeing on B, they impose an orientation on $M \cup_B W^-$ by requiring the natural imbeddings to be orientation preserving.

We identify B, M and W with their respective images in $M \cup_B W^-$ via the natural imbeddings. The union of the collars of M and W is the open neighborhood $B \times (-1, 1)$ of B in $M \cup_B W^-$. In $M \cup_B W^-$, we also have the open neighborhoods

$$M \cup (\partial W \times (-1, 0]) \supseteq M, \qquad W \cup (\partial M \times [0, 1)) \supseteq W,$$

which cover $M \cup_B W^-$, and their intersection reads

$$\big(M \cup (\partial W \times (-1, 0])\big) \cap \big(W \cup (\partial M \times [0, 1))\big) = B \times (-1, 1).$$

This gives the Mayer-Vietoris sequence (cf. [May99])

$$\ldots \to h^n(M \cup_B W^-) \xrightarrow{\alpha} h^n(M \cup (\partial W \times (-1, 0])) \oplus h^n(W \cup (\partial M \times [0, 1))) \xrightarrow{\beta} h^n(B \times (-1, 1)) \to \ldots ,$$

where the maps are $\alpha(z) = (i^*(z), j^*(z))$, $\beta(x, y) = k^*(x) - l^*(y)$, with i, j, k, l being the obvious inclusions. We have the diagram

$$\begin{array}{ccc} h^n(M \cup (\partial W \times (-1, 0])) \oplus h^n(W \cup (\partial M \times [0, 1))) & \xrightarrow{\beta} & h^n(B \times (-1, 1)) \\ \downarrow \cong & & \downarrow \cong \\ h^n(M) \oplus h^n(W) & \xrightarrow{\widehat{\beta}} & h^n(B \times \{0\}) \end{array} ,$$

with $\widehat{\beta}$ defined like β, and the isomorphisms being induced by inclusions. The underlying diagram of inclusion maps commutes, hence so does this. We shall identify $B \times \{0\} \approx B$. (W, y, F) being a null-bordism for $(\partial M, (-1)^{\dim x} x|_{\partial M}, \partial f)$, we have $y|_{\partial W} = y|_{\partial M} = x|_{\partial M}$, i.e.

$$\widehat{\beta}(x, y) = x|_B - y|_B = 0.$$

The elements

$$x' \in h^*\big(M \cup (\partial W \times (-1, 0])\big), \qquad y' \in h^*\big(W \cup (\partial M \times [0, 1))\big),$$

corresponding to $x \in h^*(M)$, $y \in h^*(W)$, thus give

$$\beta(x', y') = 0.$$

By exactness of the Mayer-Vietoris sequence and commutativity of the diagram above, there is an element $z' \in h^*(M \cup_B W^-)$ such that $\alpha(z') = (x', y')$. Clearly, the restriction of z' to $h^*(M) \oplus h^*(W)$ factors via α, and so z' restricts to $x \in h^*(M)$, $y \in h^*(W)$.

The maps f and F agree on B, and so they define a map

$$(f, F) \colon (M \cup_B W^-, \varnothing) \to (X, \varnothing).$$

This gives us $[M \cup_B W^-, z', (f, F)] \in h_n^{geo}(X, \varnothing)$.

$M \subseteq M \cup_B W^-$ is a compact, codimension zero submanifold with

$$j(f, F)(M \cup_B W^- - \operatorname{int} M) = jF(W) = F(W) \subseteq A.$$

Since $(M, x, f) = (M, z'|_M, j(f, F)|_M)$, Lemma 2.13 gives

$$j_*[M \cup_B W^-, z', (f, F)] = [M \cup_B W^-, z', j(f, F)] = [M, x, f] \in h_n^{geo}(X, A).$$

Hence $\operatorname{im} j_* \supseteq \ker \partial$.

We have shown exactness at $h_n^{geo}(A, \varnothing)$, $h_n^{geo}(X, \varnothing)$ and $h_n^{geo}(X, A)$, and so we have the long exact sequence of geometric homology groups. □

2.17 PROPOSITION (EXCISION). *Let (X, A) be a pair of spaces and let $U \subseteq X$ be a subspace such that $\bar{U} \subseteq \operatorname{int} A$. Then the inclusion of pairs induces an isomorphism*

$$i_* \colon h_*^{geo}(X - U, A - U) \xrightarrow{\cong} h_*^{geo}(X, A).$$

PROOF. For a fixed n, we show that

$$i_* \colon h_n^{geo}(X - U, A - U) \xrightarrow{\cong} h_n^{geo}(X, A)$$

is surjective and injective, respectively. We comment that each case immediately reduces to a pure problem of bordism, and there is little parting this proof from the standard proof of excision in classical bordism.

- Let $[M, x, f] \in h_n^{geo}(X, A)$. We use the triple (M, x, f) in (X, A) to find a triple in $(X - U, A - U)$ which becomes bordant to (M, x, f) when included in (X, A).

  $\bar{U} \subseteq \operatorname{int} A$ gives disjoint, closed subsets $f^{-1}(\bar{U})$ and $f^{-1}(X - \operatorname{int} A)$ of M. There is a smooth, real-valued function $\gamma \colon M \to [0, 1]$ and a $t \in [0, 1]$ such that

  $$\sup \gamma(f^{-1}(X - \operatorname{int} A)) < t < \inf \gamma(f^{-1}(\bar{U}))$$

  and making $\widehat{M} := \gamma^{-1}([0, t]) \subseteq M$ a compact, *topological* submanifold of codimension zero. $\widehat{M}$ can be given a smooth structure by straightening the angle (cf. Lemma 3.1. in [Con79]). This makes $\widehat{M} \subseteq M$ a (smooth) submanifold (cf. [Buc]), and it thus inherits a canonical orientation from M. We have

  $$f^{-1}(X - \operatorname{int} A) \subseteq \widehat{M}, \qquad f(\widehat{M}) \subseteq X - \bar{U}, \qquad \partial\widehat{M} \subseteq \gamma^{-1}(\{t\}) \cup \partial M.$$

  Thus

  $$f(\partial\widehat{M}) \subseteq f(\gamma^{-1}(\{t\})) \cup f(\partial M) \subseteq (\operatorname{int} A) \cup A \subseteq A,$$

and so $f(\widehat{M}) \subseteq X - \bar{U}$ gives $f(\partial \widehat{M}) \subseteq A - \bar{U} \subseteq A - U$. From this, there is the map

$$f|_{\widehat{M}} \colon (\widehat{M}, \partial \widehat{M}) \to (X - U, A - U),$$

which gives

$$[\widehat{M}, x|_{\widehat{M}}, f|_{\widehat{M}}] \in h_n^{geo}(X - U, A - U).$$

Since $f^{-1}(X - \operatorname{int} A) \subseteq \widehat{M} \subseteq M$ and $f(\partial \widehat{M}) \subseteq A$, we have $f(M - \operatorname{int} \widehat{M}) \subseteq A$. So by Lemma 2.13,

$$i_*[\widehat{M}, x|_{\widehat{M}}, f|_{\widehat{M}}] = [\widehat{M}, x|_{\widehat{M}}, if|_{\widehat{M}}] = [M, x, f] \in h_n^{geo}(X, A).$$

Hence i_* is an epimorphism.

- Let $[M, x, f] \in \ker i_*$. By Lemma 2.14, we can assume that (M, x, if) in (X, A) is null-bordant. Write (W, y, F) for such a null-bordism. We use (W, y, F) in (X, X) to find a new triple in $(X - U, A - U)$ which is bordant to (M, x, f). The new triple is then seen to be null-bordant, completing the proof.

  $\bar{U} \subseteq \operatorname{int} A$ gives disjoint, closed subsets $F^{-1}(\bar{U})$ and $F^{-1}(X - \operatorname{int} A)$ of W. By the same argument as above, there is a compact, codimension zero submanifold $\widehat{W}$ of W such that

  $$F^{-1}(X - \operatorname{int} A) \subseteq \widehat{W}, \qquad \widehat{W} \cap F^{-1}(\bar{U}) = \varnothing.$$

  This gives us the map $F|_{\widehat{W}} \colon (\widehat{W}, \partial \widehat{W}) \to (X - U, X - U)$ and thus the triple

  $$(\widehat{W}, y|_{\widehat{W}}, F|_{\widehat{W}}) \in \Lambda(X - U, X - U).$$

  $M \cap \partial \widehat{W}$ and $f^{-1}(\bar{U})$ are disjoint, closed subsets of M. Again, there is a compact, codimension zero submanifold $\widehat{M}$ of M satisfying $M \cap \partial \widehat{W} \subseteq \widehat{M} \subseteq M$ as well as $\widehat{M} \cap f^{-1}(\bar{U}) = \varnothing$.

  We have

  $$M \cap F^{-1}(X - A) = \partial W \cap F^{-1}(X - A) = \partial \widehat{W} \cap F^{-1}(X - A),$$

  so $M \cap \partial \widehat{W} \subseteq \widehat{M} \subseteq M$ implies $\widehat{M} \cap F^{-1}(X - A) = M \cap F^{-1}(X - A)$. The explicit construction of $\widehat{M}$ (cf. [Con79]) now ensures

  $$\partial \widehat{M} \cap F^{-1}(X - A) = \partial M \cap F^{-1}(X - A) = \varnothing,$$

  and hence we have $f(\partial \widehat{M}) = F(\partial \widehat{M}) \subseteq A$. This gives the map

  $$f|_{\widehat{M}} \colon (\widehat{M}, \partial \widehat{M}) \to (X - U, A - U).$$

  Also, $f(M - \widehat{M}) \subseteq A - U$ follows from $M \cap F^{-1}(X - A) = \widehat{M} \cap F^{-1}(X - A)$ together with $\widehat{M} \subseteq M \subseteq f^{-1}(X - U)$. Hence the triples (M, x, f) and $(\widehat{M}, x|_{\widehat{M}}, f|_{\widehat{M}})$ are bordant in $(X - U, A - U)$ by Lemma 2.13.

  We have essentially replaced (M, x, f) with $(\widehat{M}, x|_{\widehat{M}}, f|_{\widehat{M}})$ in $(X - U, A - U)$, the important difference being that the image of $\widehat{M}$ is disjoint from $\bar{U}$, not just from U.

  Now we have the disjoint, closed subsets $\widehat{M} \cup F^{-1}(X - \operatorname{int} A)$ and $F^{-1}(\bar{U})$ of W. Once again, there is a compact, codimension zero submanifold $\widetilde{W}$ of W such that

$\widehat{M} \cup F^{-1}(X - \operatorname{int} A) \subseteq \widetilde{W} \subseteq W$ and $\widetilde{W} \cap F^{-1}(\bar{U}) = \varnothing$. The latter gives

$$(\widetilde{W}, y|_{\widetilde{W}}, F|_{\widetilde{W}}) \in \Lambda(X - U, X - U).$$

This is a null-bordism for $(\widehat{M}, x|_{\widehat{M}}, f|_{\widehat{M}})$, seen as follows:
A point lying on the boundary of a manifold must necessarily also lie on the boundary of any codimension zero submanifold which contains that point. So the inclusions $\widehat{M} \subseteq \widetilde{W}$ and $\widehat{M} \subseteq M \subseteq \partial W$ give

$$\widehat{M} \subseteq \widetilde{W} \cap \partial W \subseteq \partial \widetilde{W}.$$

$\widetilde{W} \cap F^{-1}(X - A) = W \cap F^{-1}(X - A)$, together with the construction of $\widetilde{W}$ as an inverse image of a real-valued function (cf. [Con79]), ensures

$$\partial \widetilde{W} \cap F^{-1}(X - A) = \partial W \cap F^{-1}(X - A).$$

Hence $F(\partial W - \widehat{M}) \subseteq A$ gives $F|_{\widetilde{W}}(\partial \widetilde{W} - \widehat{M}) \subseteq A$. Moreover, $F(\widetilde{W}) \subseteq X - U$, and so we have $F|_{\widetilde{W}}(\partial \widetilde{W} - \widehat{M}) \subseteq A - U$. Also, $(F|_{\widetilde{W}})|_{\widehat{M}} = f|_{\widehat{M}}$ and $(y|_{\widetilde{W}})|_{\widehat{M}} = x|_{\widehat{M}}$.
In summary,

$$(M, x, f) \sim_{\mathcal{B}} (\widehat{M}, x|_{\widehat{M}}, f|_{\widehat{M}}) \sim_{\mathcal{B}} (\varnothing, 0, \mathrm{ef}).$$

Thus $\ker i_* = 0$, making i_* a monomorphism. □

2.18 PROPOSITION (ADDITIVITY). *Let $(X_\alpha, A_\alpha)_\alpha$ be a collection of pairs of spaces. The inclusions $i_\alpha \colon (X_\alpha, A_\alpha) \hookrightarrow (\sqcup X_\alpha, \sqcup A_\alpha)$ give an isomorphism*

$$\coprod_\alpha i_{\alpha *} \colon \coprod_\alpha h_*^{geo}(X_\alpha, A_\alpha) \xrightarrow{\cong} h_*^{geo}(\sqcup X_\alpha, \sqcup A_\alpha),$$
$$([M_\alpha, x_\alpha, f_\alpha])_\alpha \mapsto [\sqcup M_\alpha, \sqcup x_\alpha, \sqcup i_\alpha f_\alpha].$$

PROOF. The assignment is described only for a sum of homogeneous elements in some fixed degree, but is understood to be extended linearly. Any element of $\coprod_\alpha h_n^{geo}(X_\alpha, A_\alpha)$ is on the form $([M_\alpha, x_\alpha, f_\alpha])_\alpha$—with $[M_\alpha, x_\alpha, f_\alpha]$ non-zero only for finitely many α—and we may therefore assume that the union $\sqcup M_\alpha$ is finite and thus compact (each class being zero may be represented by the null-triple). Since addition in $h_*^{geo}(\sqcup X_\alpha, \sqcup A_\alpha)$ is well-defined, the same is therefore true for the assignment above. This is clearly a homomorphism. We define the inverse map:

Let $[M, x, f] \in h_n^{geo}(\sqcup X_\alpha, \sqcup A_\alpha)$. Since M is compact, it can be written as a finite union of its components, $M = \sqcup M_k$. We define $(X_k, A_k) \in (X_\alpha, A_\alpha)_\alpha$ by $f(M_k) \subseteq X_k$. This gives the decomposition $(M, x, f) = (\sqcup M_k, \sqcup x_k, \sqcup f_k)$, where $x_k \in h^*(M_k)$ and $f_k \colon (M_k, \partial M_k) \to (X_k, A_k)$ are the natural restrictions. Finiteness now gives

$$([M_k, x_k, f_k])_k \in \coprod_k h_n^{geo}(X_k, A_k).$$

Hence the assignment

$$[\sqcup M_k, \sqcup x_k, \sqcup f_k] \mapsto ([M_k, x_k, f_k])_k$$

followed by the natural inclusion

$$\coprod_k h_n^{geo}(X_k, A_k) \hookrightarrow \coprod_\alpha h_n^{geo}(X_\alpha, A_\alpha)$$

is readily seen to describe an inverse homomorphism to $\coprod_\alpha i_{\alpha *}$. □

REMARK. A pair (K, L), with K compact and $L \subseteq K$ closed, is referred to as a **compact pair**. A homology theory k_* on Top^2 is said to have **compact support** if it satisfies the following: For every pair (X, A) and for each element $z \in k_n(X, A)$, there is a compact pair $(K, L) \subseteq (X, A)$ such that z is in the image of $k_*(K, L) \to k_*(X, A)$.

Referring to [Spa66], this is equivalent to the following: Let $\{(K_\alpha, L_\alpha)\}_\alpha$ be the system of compact pairs contained in (X, A), directed by inclusions. This gives the system $\{k_*(K_\alpha, L_\alpha)\}_\alpha$, directed by the maps induced by inclusions. Then

$$\operatorname*{colim}_\alpha k_*(K_\alpha, L_\alpha) \cong k_*(X, A).$$

By construction, so to speak, we see that h_*^{geo} has compact support: Suppose given a pair (X, A) and an element $[M, x, f] \in h_n^{geo}(X, A)$. Let us write f as the composition map

$$f\colon (M, \partial M) \xrightarrow{\widehat{f}} (f(M), f(\partial M)) \xhookrightarrow{i} (X, A).$$

M and ∂M being compact Hausdorff spaces, it is clear that $(f(M), f(\partial M)) \subseteq (X, A)$ is a compact pair. Now $[M, x, f]$ is in the image of i_*, as we have

$$[M, x, \widehat{f}] \in h_n^{geo}(f(M), f(\partial M)),$$

with $i_*[M, x, \widehat{f}] = [M, x, f] \in h_n^{geo}(X, A)$. □

3 NATURALLY EQUIVALENT HOMOLOGY THEORIES

From the cohomology theory h^* on CW^2, we have constructed the homology theory h_*^{geo} on Top^2. We explain how the spectrum representing h^* gives rise to h_*—the dual homology theory that we already have encountered several times. In parts of this chapter, we restrict h_*^{geo} from Top^2 to being a homology theory on CW^2. We shall see that the two homology theories h_*^{geo} and h_* then coincide. That is, on CW pairs, the homology groups correspond in a well-behaved way with respect to induced maps and boundary homomorphisms.

3.1 Spectra and (co)homology theories

Definition. Let k_* and k'_* be homology theories on CW^2 with boundary homomorphisms ∂ and ∂'. We say that

$$T\colon k_* \to k'_*$$

is a **natural transformation of homology theories** if T is a natural transformation such that the diagram

$$\begin{array}{ccc} k_n(X,A) & \xrightarrow{\partial_{(X,A)}} & k_{n-1}(A,\varnothing) \\ \big\downarrow T_{(X,A)} & & \big\downarrow T_{(A,\varnothing)} \\ k'_n(X,A) & \xrightarrow{\partial'_{(X,A)}} & k'_{n-1}(A,\varnothing) \end{array}$$

commutes for each n and for every pair (X,A). □

When a natural transformation of homology theories is an equivalence, we say that the two homology theories are **equivalent**.

3.1 Theorem. *Let k_* and k'_* be homology theories on CW^2 and let*

$$T\colon k_* \to k'_*$$

be a natural transformation of homology theories. If

$$T_{(\mathrm{pt},\varnothing)}\colon k_*(\mathrm{pt},\varnothing) \xrightarrow{\cong} k'_*(\mathrm{pt},\varnothing)$$

is an isomorphism, then T is an equivalence. □

DEFINITION. A **spectrum** is a collection $E = (E_n, \sigma_n)_{n\in\mathbb{Z}}$ of pointed CW complexes E_n and basepoint preserving maps $\sigma_n\colon \Sigma E_n \to E_{n+1}$. □

In the next definition, we will use the following, where C and Σ are the reduced *cone* and *suspension* functors, respectively: When (X, A) is a *pointed* CW pair, we have the projections

$$p\colon X \cup CA \to (X \cup CA)/X$$
$$q\colon X \cup CA \to (X \cup CA)/CA,$$

the latter being a homotopy equivalence. We make the two identifications $\Sigma A = (X \cup CA)/X$ and $X/A = (X\cup CA)/CA$. Choosing a homotopy inverse q^{-1} then gives the map

$$pq^{-1}\colon X/A \to \Sigma A.$$

DEFINITION. Let $E = (E_r, \sigma_r)_r$ be a spectrum. Then for each n and for each CW pair (X, A), we define the group

$$k_n(X, A) \coloneqq \operatorname*{colim}_r [\mathbb{S}^{n+r}, E_r \wedge (X/A)],$$

where the colimit is taken over the direct system

$$\begin{aligned}&\big[\mathbb{S}^{n+r} \xrightarrow{f} E_r \wedge (X/A)\big]\\ \mapsto\ &\big[\mathbb{S}^{n+r+1} \xrightarrow{1} \mathbb{S}^1 \wedge \mathbb{S}^{n+r} \xrightarrow{1\wedge f} \mathbb{S}^1 \wedge E_r \wedge (X/A) \xrightarrow{\sigma_r\wedge 1} E_{r+1} \wedge (X/A)\big].\end{aligned}$$

For each r, there is a homomorphism

$$\partial_r\colon [\mathbb{S}^{n+r}, E_r \wedge (X/A)] \to [\mathbb{S}^{n+r}, E_{r+1} \wedge (A/\varnothing)]$$

given by

$$\begin{aligned}&\big[\mathbb{S}^{n+r} \xrightarrow{f} E_r \wedge (X/A)\big]\\ \mapsto\ &\big[\mathbb{S}^{n+r} \xrightarrow{f} E_r \wedge (X_+/A_+) \xrightarrow{1\wedge pq^{-1}} E_r \wedge \Sigma A_+ \xrightarrow{1} \Sigma E_r \wedge A_+ \xrightarrow{\sigma_r\wedge 1} E_{r+1} \wedge A_+\big].\end{aligned}$$

Here we have identified $X/A = X_+/A_+$ and $A/\varnothing = A_+$. The diagram

$$\begin{array}{ccc}[\mathbb{S}^{n+r}, E_r \wedge (X/A)] & \longrightarrow & [\mathbb{S}^{n+r+1}, E_{r+1} \wedge (X/A)]\\ \downarrow{\scriptstyle\partial_r} & & \downarrow{\scriptstyle\partial_{r+1}}\\ [\mathbb{S}^{n+r}, E_{r+1} \wedge (A/\varnothing)] & \longrightarrow & [\mathbb{S}^{n+r+1}, E_{r+2} \wedge (A/\varnothing)]\end{array}$$

commutes for each r. By the universal property of direct limits, the collection of ∂_r then induces a map, the **boundary homomorphism**

$$\partial\colon k_n(X, A) \to k_{n-1}(A, \varnothing).$$ □

This makes $k_* \coloneqq (k_n)_n$ (together with the boundary homomorphisms) a homology theory on the category of CW pairs (cf. [May99]). We refer to it as the **homology theory associated to** E.

When Y is a based space, ΩY denotes the space of loops in Y, i.e. the function space of based maps $\mathbb{S}^1 \to Y$ given the compact-open topology. In the category of based spaces, every map $f\colon \Sigma X \to Y$ has an *adjoint map*, $\mathrm{adj}(f)\colon X \to \Omega Y$. When identifying $\Sigma X = \mathbb{S}^1 \wedge X$, this map is given by $\mathrm{adj}(f)(x)(t) := f(t \wedge x)$.

DEFINITION. An **Ω-spectrum** is a spectrum $(E_n, \sigma_n)_n$ such that for each n, the adjoint map $\mathrm{adj}(\sigma_n)\colon E_n \to \Omega E_{n+1}$ is a homotopy equivalence. □

There is also k^*, the **co**homology theory associated to a spectrum, now on the category of *finite* CW pairs. The construction is similar to the one above. When E is not only a spectrum, but an Ω-spectrum, k^* extends to a homology theory on *all* CW pairs. By a theorem due to E. H. Brown, every cohomology theory k^* on CW pairs is represented by an Ω-spectrum. Such a spectrum is unique in the following sense: Two Ω-spectra representing k^* give rise to naturally equivalent homology theories.

3.2 Identifying h_*^{geo} and h_* on CW^2

Throughout this section, we restrict h_*^{geo} to a homology theory on CW^2. We shall define a natural transformation of homology theories

$$\psi\colon h_*^{geo} \to h_*$$

which will be an equivalence, i.e. an isomorphism on each CW pair.

For an h-manifold M, we write

$$D_M\colon h^*(M) \xrightarrow{\cong} h_*(M, \partial M)$$

for the Poincaré duality isomorphism.

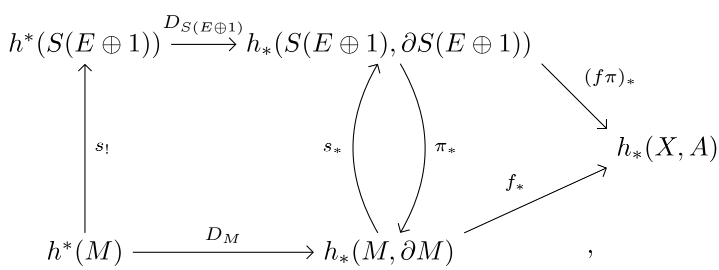
3.2 PROPOSITION. *The map*

$$\psi_{(X,A)}\colon h_*^{geo}(X,A) \to h_*(X,A), \quad [M,x,f] \mapsto f_* D_M(x),$$

is a well-defined homomorphism.

PROOF. We show the proposition for $\psi_{(X,A)}\colon h_k^{geo}(X,A) \to h_k(X,A)$. Then the general result follows by linearity. We shall abbreviate $\psi_{(X,A)}$ to ψ.

- Let $(S(E\oplus 1), s_!(x), f\pi)$ be a sphere triple of (M,x,f). In the diagram



,

the appropriate sub-diagrams commute, and a short chase readily gives

$$(f\pi)_*(D_{S(E\oplus 1)}(s_!(x))) = (f\pi)_*(s_*(D_M(x))) = f_*(D_M(x)).$$

- Let (M', x', f') and (M, x, f) be bordant of dimension k, (W, y, F) a bordism. We can assume M' and M are of dimension n, yielding x' and x of dimension $n-k$. The multi-dimensional case will follow by linear extension. We have the following diagram:

$$\begin{array}{ccccc}
h^{n-k}(W) & \longrightarrow & h^{n-k}(\partial W) & \longrightarrow & h^{n-k}(M' \sqcup M^-) \\
 & & & & \bigoplus \\
 & & & & h^{n-k}(\partial W - \mathrm{int}(M' \sqcup M^-)) \\
 & & & & \cong \big\downarrow D_{M' \sqcup M^-} \oplus D_{\partial W - \mathrm{int}(M' \sqcup M^-)} \\
\cong \big\downarrow D_W & & \cong \big\downarrow D_{\partial W} & & h_k(M' \sqcup M^-, \partial(M' \sqcup M^-)) \\
 & & & & \bigoplus \\
 & & & & h_k(\partial W - \mathrm{int}(M' \sqcup M^-), \partial(M' \sqcup M^-)) \\
 & & & & \cong \big\downarrow i_* + j_* \\
h_{k+1}(W, \partial W) & \xrightarrow{\partial} & h_k(\partial W) & \longrightarrow & h_k(\partial W, \partial(M' \sqcup M^-)) \\
\big\downarrow 0 & \swarrow & \big\downarrow (F|_{\partial W})_* & & \big\downarrow \\
h_k(W) & \xrightarrow{F_*} & h_k(X) & \longrightarrow & h_k(X, A) .
\end{array}$$

The upper left-hand square commutes up to sign, and the remaining two squares and triangles commute. Especially, the upper right-hand square commutes by Lemma 1.14. Further, the composition map

$$h_k(\partial W - \mathrm{int}(M' \sqcup M^-), \partial(M' \sqcup M^-)) \to h_k(X, A)$$

is zero. This is clear since $F(\partial W - (M' \sqcup M^-)) \subseteq A$ and $F(\partial(M' \sqcup M^-)) \subseteq A$ give a factorization through $h_k(A, A) = 0$. The vertical map on the right-hand side of the diagram is therefore

$$h^{n-k}(M' \sqcup M^-) \xrightarrow{D_{M' \sqcup M^-}} h_k(M' \sqcup M^-, \partial(M' \sqcup M^-)) \xrightarrow{(f' \sqcup f)_*} h_k(X, A).$$

The long exact sequence of the pair $(W, \partial W)$ gives the zero-map at the lower left. Since $x' \sqcup x = y|_{M' \sqcup M}$, we thus get $(f' \sqcup f)_* D_{M' \sqcup M^-}(x' \sqcup x) = 0$.

The lower horizontal map being addition, the following diagram commutes by definition:

$$\begin{array}{ccc}
h^{n-k}(M') \oplus h^{n-k}(M^-) & \xrightarrow{\cong} & h^{n-k}(M' \sqcup M^-) \\
\cong \big\downarrow D_{M'} \oplus D_{M^-} & & \cong \big\downarrow D_{M' \sqcup M^-} \\
h_k(M', \partial M') \oplus h_k(M^-, \partial M^-) & \xrightarrow{\cong} & h_k(M' \sqcup M^-, \partial(M' \sqcup M^-)) \\
\big\downarrow f'_* \oplus f_* & & \big\downarrow (f' \sqcup f)_* \\
h_k(X, A) \oplus h_k(X, A) & \longrightarrow & h_k(X, A) .
\end{array}$$

Since we have

$$D_{M^-}(x) = x \cap [M^-, \partial M^-] = x \cap -[M, \partial M] = -(x \cap [M, \partial M]) = -D_M(x),$$

the last diagram gives us

$$0 = (f' \sqcup f)_* D_{M' \sqcup M^-}(x' \sqcup x) = f'_* D_{M'}(x') + f_* D_{M^-}(x) = f'_* D_{M'}(x') - f_* D_M(x).$$

This shows that the assignment $[M, x, f] \mapsto f_* D_M(x)$ is well-defined. The following makes it a homomorphism:

Let $[M, x, f]$ and $[M', x', f']$ be elements of $h_k^{geo}(X, A)$. Now we have

$$\begin{aligned} \psi([M, x, f] + [M', x', f']) =& \psi[M \sqcup M', x \sqcup x', f \sqcup f'] \\ =& (f \sqcup f')_* D_{M \sqcup M'}(x \sqcup x') \\ =& f_* D_M(x) + f'_* D_{M'}(x') \\ =& \psi[M, x, f] + \psi[M', x', f']. \end{aligned}$$ □

The collection of $\psi_{(X,A)}$ will constitute the natural transformation of homology theories ψ. To show ψ is an equivalence, we shall be needing a *geometric* variant of Poincaré duality: For an h-manifold N, there is the isomorphism

$$D_N^{geo} \colon h^*(N) \xrightarrow{\cong} h_*^{geo}(N, \partial N)$$

given by the following theorem:

3.3 THEOREM (POINCARÉ DUALITY). *Let N be an h-manifold of dimension n. Then the maps*

$$\begin{aligned} D_N^{geo} \colon h^k(N) \to h_{m-k}^{geo}(N, \partial N), \quad & x \mapsto [N, x, 1] \\ d_N^{geo} \colon h_{m-k}^{geo}(N, \partial N) \to h^k(N), \quad & [M, x, f] \mapsto f_!(x) \end{aligned}$$

are well-defined isomorphisms, inverse to each other.

PROOF OUTLINE. The map D_N^{geo} is clearly well-defined, and it is a homomorphism by Lemma 2.5. We only check that the map d_N^{geo} is a well-defined homomorphism. It is then trivially verified that the composition $d_N^{geo} D_N^{geo}$ is the identity on $h^k(N)$. For a proof of the reversed composition being the identity on $h_{m-k}^{geo}(N, \partial N)$, cf. [Jak00].

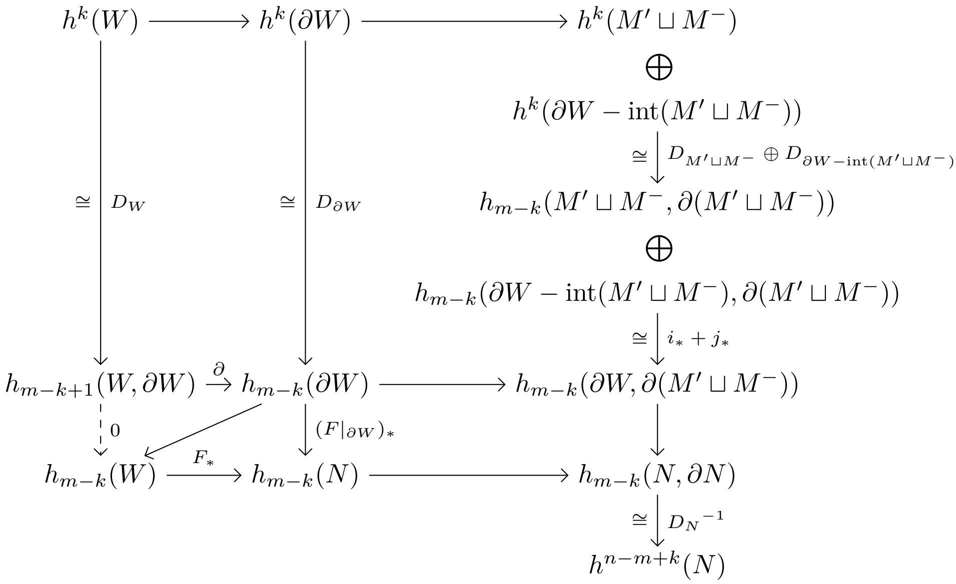
- Let $(S(E \oplus 1), s_!(x), f\pi)$ be a sphere triple of (M, x, f) in $(N, \partial N)$. $\pi s = 1 \colon M \to M$ gives

$$(f\pi)_!(s_!(x)) = (f\pi s)_!(x) = f_!(x).$$

- Let (M', x', f') and (M, x, f) be bordant, with bordism (W, y, F). As usual, we can assume M' and M m-dimensional such that x' and x are of dimension k. We must show

$$f'_!(x') - f_!(x) = 0.$$

The argument is very similar to the corresponding part of the proof of Proposition 3.2, and we reuse the diagram found there with only a few modifications at the bottom:



This setting differs from the proof of Proposition 3.2 only in that the codomain of F now is the h-manifold N. The same argument applies here to the vertical composition map (*) on the right: The homomorphism vanishes on the second summand, as the map

$$h_{m-k}(\partial W - \mathrm{int}(M' \sqcup M^-), \partial(M' \sqcup M^-)) \to h_{m-k}(N, \partial N)$$

factors through $h_{m-k}(\partial N, \partial N) = 0$. Thus (*) is the Gysin homomorphism

$$(f' \sqcup f)_! \colon h^k(M' \sqcup M^-) \to h^{n-m+k}(N).$$

Again the diagram commutes up to sign, and thus any composition map $h^k(W) \to h^{n-m+k}(N)$ in the diagram is zero. By the bordism, we have $x' \sqcup x = y|_{M' \sqcup M}$, and so

$$(f' \sqcup f)_!(x' \sqcup x) = 0.$$

Finally, the inverted Thom class orienting M^- gives rise to a change of sign in the ordinary Poincaré duality map, yielding

$$0 = (f' \sqcup f)_!(x' \sqcup x) = f'_!(x') - f_!(x).$$

The map

$$d_N^{geo} \colon h^{geo}_{m-k}(N, \partial N) \to h^k(N), \quad [M, x, f] \mapsto f_!(x),$$

is therefore well-defined.

$$(f' \sqcup f)_! \colon h^k(M' \sqcup M) \to h^{n-m+k}(N)$$

gives $(f' \sqcup f)_!(x' \sqcup x) = f'_!(x') + f_!(x)$, and d_N^{geo} is by that seen to be a homomorphism.

Now [Jak00] shows that $[M, x, f] = [N, f_!(x), 1]$ in the sense that the composition

$$\begin{aligned} &D_N^{geo} d_N^{geo} \colon h_{m-k}^{geo}(N, \partial N) \to h^k(N) \to h_{m-k}^{geo}(N, \partial N), \\ &[M, x, f] \mapsto f_!(x) \mapsto [N, f_!(x), 1], \end{aligned}$$

is the identity map. □

REMARK. We shall write D_N^{geo-1} instead of d_N^{geo} for the inverse of D_N^{geo}. □

3.4 THEOREM. *The transformation*

$$\psi \colon h_*^{geo} \to h_*,$$

given by

$$\psi_{(X,A)}[M, x, f] = f_* D_M(x),$$

is a natural transformation of homology theories.

PROOF. Proposition 3.2 shows that ψ is a well-defined transformation. Let

$$\varphi \colon (X, A) \to (Y, B)$$

be a map. We need only check that the following two diagrams commute:

$$\begin{array}{ccc} h_k^{geo}(X,A) & \xrightarrow{\varphi_*} & h_k^{geo}(Y,B) \\ \downarrow{\scriptstyle \psi_{(X,A)}} & & \downarrow{\scriptstyle \psi_{(Y,B)}} \\ h_k(X,A) & \xrightarrow{\varphi_*} & h_k(Y,B) \end{array} \qquad \begin{array}{ccc} h_k^{geo}(X,A) & \xrightarrow{\partial^{geo}} & h_{k-1}^{geo}(A) \\ \downarrow{\scriptstyle \psi_{(X,A)}} & & \downarrow{\scriptstyle \psi_{(A,\varnothing)}} \\ h_k(X,A) & \xrightarrow{\partial'} & h_{k-1}(A) \end{array}$$

The diagram to the left readily commutes by

$$\varphi_*(\psi[M, x, f]) = \varphi_* f_* D_M(x) = \psi[M, x, \varphi f] = \psi(\varphi_*[M, x, f]).$$

In the diagram to the right, we have labeled the boundary homomorphisms ∂^{geo} and ∂' to reserve ∂ for the boundary operation on manifolds. The two ways around the diagram respectively give

$$\partial' \psi_{(X,A)}[M, x, f] = \partial' f_* D_M(x) = (\partial f)_* \partial' D_M(x)$$

and

$$\psi_{(A,\varnothing)} \partial^{geo}[M, x, f] = \psi_{(A,\varnothing)}[\partial M, (-1)^{\dim x} x|_{\partial M}, \partial f] = (\partial f)_* D_{\partial M}((-1)^{\dim x} x|_{\partial M}).$$

Equality follows from the diagram found in the proof of Proposition 1.10, assuming M and x being of dimensions n and $n-k$:

$$\begin{array}{ccc} h^{n-k}(M) & \xrightarrow{i^*} & h^{n-k}(\partial M) \\ \downarrow{\scriptstyle D_M} & & \downarrow{\scriptstyle D_{\partial M}} \\ h_k(M, \partial M) & \xrightarrow{\partial'} & h_{k-1}(\partial M) \end{array}$$

The diagram commutes with sign $(-1)^{n-k}$, which gives

$$\partial' D_M(x) = (-1)^{\dim x} D_{\partial M}(x|_{\partial M}) = D_{\partial M}((-1)^{\dim x} x|_{\partial M}). \qquad \square$$

3.5 THEOREM. *The natural transformation of homology theories*

$$\psi \colon h_*^{geo} \to h_*$$

is an equivalence.

PROOF. Let N be an h-manifold. Then $\psi_{(N,\partial N)}$ is composition of the two Poincaré duality isomorphisms in the diagram

$$\begin{array}{ccc} & h^*(N) & \\ {\scriptstyle D_N^{geo-1}} \nearrow \cong & & \cong \searrow {\scriptstyle D_N} \\ h_*^{geo}(N, \partial N) & \xrightarrow{\psi_{(N,\partial N)}} & h_*(N, \partial N)\ . \end{array}$$

This is clear, since $[M, x, f] \in h_*^{geo}(N, \partial N)$ gives

$$\psi_{(N,\partial N)}[M, x, f] = f_* D_M(x) = D_N f_!(x) = D_N {D_N^{geo}}^{-1}[M, x, f]$$

by the definition of $f_!$. The special case $(N, \partial N) = (\mathrm{pt}, \varnothing)$ now yields the isomorphism

$$\psi_{(\mathrm{pt},\varnothing)} \colon h_k^{geo}(\mathrm{pt}, \varnothing) \xrightarrow{\cong} h_k(\mathrm{pt}, \varnothing).$$

By Theorem 3.1, ψ is then an equivalence. $\square$

4 THE GEOMETRIC CAP PRODUCT

Until now, we have let h^* be a multiplicative cohomology theory on CW pairs. However, we could just as well have let h^* be a multiplicative cohomology theory on *all* pairs of spaces whose restriction to CW pairs was represented by an Ω-spectrum. This shall be our setting in this chapter.

For either A or B empty, we shall see that the homology theory $h_*^{geo} \colon \mathsf{Top}^2 \to \mathsf{Ab}_*$ and the cup product

$$\cup \colon h^*(X, A) \otimes h^*(X, B) \to h^*(X, A \cup B)$$

can be used to define the *geometric cap product*,

$$h^*(X, A) \otimes h_*^{geo}(X, A \cup B) \to h_*^{geo}(X, B),$$

a pairing which very much resembles a cap product. We demonstrate some of its basic properties. When restricted to CW pairs, this essentially *is* the spectrally defined cap product.

4.1 Defining the geometric cap product

On the category of CW pairs, we have the natural equivalence of homology theories

$$\psi \colon h_*^{geo} \to h_*.$$

For each h-manifold N, ψ therefore defines an isomorphism

$$\widehat{D}_N^{geo} \colon h^*(N, \partial N) \xrightarrow{\cong} h_*^{geo}(N)$$

by imposing commutativity on the diagram

$$\begin{array}{ccc} h^k(N, \partial N) & & \\ {\scriptstyle \widehat{D}_N^{geo}} \downarrow & \overset{\widehat{D}_N}{\underset{\cong}{\searrow}} & \\ h_{n-k}^{geo}(N) & \xrightarrow[\cong]{\psi_N} & h_{n-k}(N) \end{array} .$$

This, we also refer to as Poincaré duality. To display the dimension shifts in (co)homology, we have in the diagram let N be of dimension n.

We thus have two Poincaré duality isomorphisms involving geometric homology:

$$D_N^{geo}\colon h^*(N) \xrightarrow{\cong} h_*^{geo}(N, \partial N),$$
$$\widehat{D}_N^{geo}\colon h^*(N, \partial N) \xrightarrow{\cong} h_*^{geo}(N).$$

Trying to imitate traditional Poincaré duality, we would like to have a product such that D_N^{geo} and $\widehat{D}_N^{geo}$ both are given by multiplication with a fixed (fundamental) class of $h_*^{geo}(N, \partial N)$. We will define the *geometric cap product* to obtain this. First we fix some notation.

DEFINITION. Let M be an h-manifold. We shall write

$$\overline{M} := M \cup_{\partial M} M^-$$

for the **double** of M, i.e. $M \sqcup M^-$ with the pairwise identification of boundary points along ∂M and ∂M^-. Collars on M and M^- give a smooth structure on $\overline{M}$, and an orientation is imposed by requiring orientation preserving imbeddings of M and M^-.

We identify M with its imbedded image onto the *first* copy of M in its double, and we write $I\colon M \hookrightarrow \overline{M}$ for the inclusion. There is also the natural projection $\overline{1}\colon \overline{M} \to M$. When $f\colon M \to X$ is a map, we compose with the projection to obtain $\overline{f}\colon \overline{M} \xrightarrow{\overline{1}} M \xrightarrow{f} X$. □

REMARK. The notation $\overline{1}$ for the projection map is chosen so that the identity map $f = 1\colon M \to M$ gives $\overline{f} = \overline{1}\colon \overline{M} \to M$.

Note that when C is a closed h-manifold and $f\colon C \to X$ is a map, we have

$$\overline{C} = C \sqcup C^-, \quad \overline{f} = f \sqcup f.$$

By construction, $\overline{M}$ is closed. This yields

$$\overline{\overline{M}} = \overline{M} \sqcup \overline{M}^-.$$

We also note that the composition $M \overset{I}{\hookrightarrow} \overline{M} \xrightarrow{\overline{1}} M$ is the identity on M. □

DEFINITION. When M is an h-manifold, we have the inclusions

$$I\colon (M, \partial M) \to (\overline{M}, M^-), \qquad J\colon (\overline{M}, \varnothing) \to (\overline{M}, M^-).$$

I being an excision map, we define the homomorphism

$$\overline{\cup}\colon h^*(M, \partial M) \otimes h^*(M) \to h^*(\overline{M})$$

to be the composition

$$h^*(M, \partial M) \otimes h^*(M) \xrightarrow{\cup} h^*(M, \partial M) \xrightarrow{I^{*-1}} h^*(\overline{M}, M^-) \xrightarrow{J^*} h^*(\overline{M}). \qquad \square$$

REMARK. One should be aware that the symbol $\overline{\cup}$ is not intended to represent a cup product. The construction is an intermediate step to what will be the *geometric cap product* defined below. The choice of notation $\overline{\cup}$ will then become clear. □

4.1 PROPOSITION. *Let the maps*

$$I\colon M \to \overline{M}, \qquad J\colon \overline{M} \to \overline{M},$$

respectively be the inclusion and the identity. I and J shall also represent the evident inclusions of pairs whose induced maps are found in the diagram. The diagram commutes.

$$\begin{array}{ccccc}
h^*(M,\partial M)\otimes h^*(M) & & \overset{\overline{\cup}}{\dashrightarrow} & & \\
\downarrow{\scriptstyle \cup} & & & & \\
h^*(M,\partial M) & \underset{\cong}{\overset{I^*}{\longleftarrow}} & h^*(\overline{M},M^-) & \overset{J^*}{\longrightarrow} & h^*(\overline{M}) \\
{\scriptstyle \cong}\downarrow{\scriptstyle -\cap[M,\partial M]} & & \downarrow{\scriptstyle -\cap J_*[\overline{M}]} & & {\scriptstyle \cong}\downarrow{\scriptstyle -\cap[\overline{M}]} \\
h_*(M) & \overset{I_*}{\longrightarrow} & h_*(\overline{M}) & \overset{J_*=1}{\longleftarrow} & h_*(\overline{M})
\end{array}$$

PROOF. By Proposition 1.8, we have

$$I_*[M,\partial M] = J_*[\overline{M}].$$

The commutativity of the two squares is therefore immediate from Proposition 1.12. The triangle at the top commutes by the definition of $\overline{\cup}$. □

The proposition gives an alternative description of $\overline{\cup}$, namely as the composition

$$h^*(M,\partial M)\otimes h^*(M) \xrightarrow{\cup} h^*(M,\partial M) \xrightarrow{\widehat{D}_M} h_*(M) \xrightarrow{I_*} h_*(\overline{M}) \xrightarrow{D_{\overline{M}}^{-1}} h^*(\overline{M}).$$

Depending on the circumstances, we shall be using either composition more convenient in computations involving the map $\overline{\cup}$.

REMARK. When C is a closed h-manifold, the following diagram commutes

$$\begin{array}{ccc}
h^*(C)\otimes h^*(C) & \overset{\overline{\cup}}{\longrightarrow} & h^*(C)\oplus h^*(C^-) \\
{\scriptstyle \cup}\searrow & & \nearrow{\scriptstyle 1\oplus 0} \\
& h^*(C) &
\end{array}$$

when identifying $h^*(\overline{C}) = h^*(C)\oplus h^*(C^-)$. □

The next proposition is known as the *projection formula*. We shall only be needing its corollary.

4.2 PROPOSITION. *Let $f\colon (M,\partial M)\to(N,\partial N)$ be a continuous map, where M and N are h-manifolds. For $x\in h^*(N)$ and $y\in h^*(M)$, we have*

$$f_!(f^*(x)\cup y) = x\cup f_!(y).$$

PROOF. The properties of the cap and cup products give

$$\begin{aligned} D_N f_!(f^*(x) \cup y) &= f_* D_M(f^*(x) \cup y) \\ &= f_*(f^*(x) \cap D_M(y)) \\ &= x \cap f_* D_M(y) \\ &= x \cap D_N f_!(y) \\ &= D_N(x \cup f_!(y)). \end{aligned}$$
□

Replacing f by s, y by 1 and x by $\pi^*(x)$, $\pi s = 1$ immediately gives the following.

4.3 COROLLARY. *Let M be an h-manifold and let $\pi\colon S(E \oplus 1) \to M$ be a the projection of a sphere bundle. If $s\colon M \to S(E \oplus 1)$ is a section, then for $x \in h^*(M)$, we have the equality*

$$s_!(x) = \pi^*(x) \cup s_!(1).$$
□

We shall soon see that the two products below coincide where they are both defined, i.e. for $A = \varnothing$. We therefore do not use distinct notations and shall write $z \cap [M, x, f]$ for both products.

DEFINITION. For $(X, A) \in \mathsf{Top}^2$, we define the **geometric cap products**

$$\begin{aligned} \cap\colon h^k(X) \otimes h_n^{geo}(X, A) &\to h_{n-k}^{geo}(X, A), \\ z \otimes [M, x, f] &\mapsto [M, f^*(z) \cup x, f] \end{aligned}$$

and

$$\begin{aligned} \cap\colon h^k(X, A) \otimes h_n^{geo}(X, A) &\to h_{n-k}^{geo}(X), \\ z \otimes [M, x, f] &\mapsto [\overline{M}, f^*(z) \mathbin{\overline{\cup}} x, \overline{f}]. \end{aligned}$$
□

4.4 PROPOSITION. *The geometric cap products are well-defined.*

PROOF. We begin with the upper product.

- Let $(S(E\oplus 1), s_!(x), f\pi)$ be a sphere triple of (M, x, f) in (X, A). Also, let $z \in h^k(X)$. Now $(S(E \oplus 1), (f\pi)^*(z) \cup s_!(x), f\pi)$ is a sphere triple of $(M, f^*(z) \cup x, f)$, since we have

$$\begin{aligned} s_!(f^*(z) \cup x) &= \pi^*(f^*(z) \cup x) \cup s_!(1) \\ &= \pi^* f^*(z) \cup \pi^*(x) \cup s_!(1) \\ &= (f\pi)^*(z) \cup s_!(x). \end{aligned}$$

- For $z \in h^k(X)$, let (M', x', f') and (M, x, f) in (X, A) be bordant, (W, y, F) a bordism. We see that $(W, F^*(z) \cup y, F)$ is a bordism between $(M', f'^*(z) \cup x', f')$ and $(M, f^*(z) \cup x, f)$.

This shows that the assignment

$$(z, [M, x, f]) \mapsto [M, f^*(z) \cup x, f]$$

is well-defined on the cross product $h^k(X) \times h_n^{geo}(X, A)$. Here, it is seen to be biadditive, yielding the homomorphism on the tensor product. We do the computations.

$$\begin{aligned} (z + z') \cap [M, x, f] &= [M, f^*(z + z') \cup x, f] \\ &= [M, f^*(z) \cup x + f^*(z') \cup x, f] \\ &= [M, f^*(z) \cup x, f] + [M, f^*(z') \cup x, f] \\ &= z \cap [M, x, f] + z' \cap [M, x, f], \end{aligned}$$

$$\begin{aligned} z \cap ([M, x, f] + [M', x', f']) &= [M \sqcup M', (f \sqcup f')^*(z) \cup (x \sqcup x'), f \sqcup f'] \\ &= [M \sqcup M', (f^*(z) \cup x) \sqcup (f'^*(z) \cup x'), f \sqcup f'] \\ &= z \cap [M, x, f] + z \cap [M', x', f']. \end{aligned}$$

Now to the second product,

$$\begin{aligned} \cap \colon h^k(X, A) \otimes h_n^{geo}(X, A) &\to h_{n-k}^{geo}(X), \\ z \otimes [M, x, f] &\mapsto [\overline{M}, f^*(z) \mathbin{\overline{\cup}} x, \overline{f}]. \end{aligned}$$

- Let $(S(E \oplus 1), s_!(x), f\pi)$ be a sphere triple of (M, x, f) in (X, A). Also, let $z \in h^k(X, A)$. We show that $(\overline{S(E \oplus 1)}, (f\pi)^*(z) \mathbin{\overline{\cup}} s_!(x), \overline{f\pi})$ is a sphere triple of $(\overline{M}, f^*(z) \mathbin{\overline{\cup}} x, \overline{f})$.

  First of all, we see that when E is an h-vector bundle over M, this makes $\overline{E}$ an h-vector bundle over $\overline{M}$. (Even though E is not compact and so not an h-manifold, we still use the notation $\overline{E}$ for the double of E.) It is a vector bundle, as $\partial E \subseteq E$ precisely is the fibers over ∂M. Now $\overline{E}$ gets an orientation from E and E^-, the inclusions of fibers $E \hookrightarrow \overline{E}$, $E^- \hookrightarrow \overline{E}$, being bundle inclusion maps. Now we identify $\overline{S(E \oplus 1)} = S(\overline{E} \oplus 1)$. Letting π also denote the projection $\overline{S(E \oplus 1)} \to \overline{M}$, we have $\overline{f\pi} = \overline{f}\pi$.

  We let $\widetilde{s} \colon \overline{M} \to \overline{S(E \oplus 1)}$ denote the evident section coming from $s \colon M \to S(E \oplus 1)$. We would like to show

$$\widetilde{s}_!\big(f^*(z) \mathbin{\overline{\cup}} x\big) = (f\pi)^*(z) \mathbin{\overline{\cup}} s_!(x).$$

  This follows from commutativity of the following diagram, mapping the element $f^*(z) \otimes x$ from the upper left-hand to the lower right-hand corner. We write S for $S(E \oplus 1)$ in the diagram.

$$\begin{array}{ccccccccc} h^*(M, \partial M) \otimes h^*(M) & \xrightarrow{\cup} & h^*(M, \partial M) & \xrightarrow{\widehat{D}_M} & h_*(M) & \xrightarrow{I_*} & h_*(\overline{M}) & \xrightarrow{D_{\overline{M}}^{-1}} & h^*(\overline{M}) \\ \downarrow{\scriptstyle \pi^* \otimes s_!} & & \downarrow{\scriptstyle \pi^*(-) \cup s_!(1)} & & \downarrow{\scriptstyle s_*} & & \downarrow{\scriptstyle \widetilde{s}_*} & & \downarrow{\scriptstyle \widetilde{s}_!} \\ h^*(S, \partial S) \otimes h^*(S) & \xrightarrow{\cup} & h^*(S, \partial S) & \xrightarrow{\widehat{D}_S} & h_*(S) & \xrightarrow{I_*} & h_*(\overline{S}) & \xrightarrow{D_{\overline{S}}^{-1}} & h^*(\overline{S}) \end{array}$$

  We begin by checking that the left-hand square commutes,

$$\pi^*(x) \cup s_!(y) = \pi^*(x) \cup \pi^*(y) \cup s_!(1) = \pi^*(x \cup y) \cup s_!(1).$$

For the next square, we have

$$\begin{aligned}\widehat{D}_S(\pi^*(x) \cup s_!(1)) &= \pi^*(x) \cap D_S s_!(1)\\ &= \pi^*(x) \cap s_* D_M(1)\\ &= \pi^*(x) \cap s_*[M, \partial M]\\ &= s_*(s^*\pi^*(x) \cap [M, \partial M])\\ &= s_*(x \cap [M, \partial M])\\ &= s_*\widehat{D}_M(x).\end{aligned}$$

Commutativity of the two rightmost squares is evident from $\widetilde{s}I = Is\colon M \to \overline{S}$ and the definition of $\widetilde{s}_!$.

- For $z \in h^k(X, A)$, let (M', x', f') and (M, x, f) in (X, A) be bordant, and let (W, y, F) be a bordism. We show there is a bordism between $(\overline{M'}, f'^*(z) \mathbin{\overline{\cup}} x', \overline{f'})$ and $(\overline{M}, f^*(z) \mathbin{\overline{\cup}} x, \overline{f})$ in $(X, \varnothing)$.

  We define
  $$B := \partial W - \operatorname{int}(M' \sqcup M^-).$$
  This makes $W \cup_B W^-$ an h-manifold by straightening the angle, with $\overline{M'} \sqcup \overline{M}^- = \partial(W \cup_B W^-)$ (cf. [Dav]). We have the commutative diagram
  $$\begin{array}{ccccc} h^*(W, \partial W) & \xrightarrow{I^{*-1}} & h^*(W \cup_B W^-, W^-) & \xrightarrow{J^*} & h^*(W \cup_B W^-)\\ \downarrow & & \downarrow & & \downarrow\\ h^*(M, \partial M) & \longrightarrow & h^*(\overline{M}, M^-) & \longrightarrow & h^*(\overline{M}) \end{array},$$
  where the horizontal maps to the left are inverse excision maps. (We have the similar diagram for M'.) Writing p for the projection $W \cup_B W^- \to W$, we see that $(W \cup_B W^-, J^*I^{*-1}(F^*(z) \cup y), Fp)$ is a bordism.

Thus the assignment is well-defined on the cross product. Showing biadditivity goes as the similar computations above, since $\overline{\cup}$ is additive in each argument of the tensor product. □

REMARK. For (M, x, f) a triple in $(X, \varnothing)$, M is closed. For any $z \in h^*(X)$, the two previous remarks then give

$$\begin{aligned}[\overline{M}, f^*(z) \mathbin{\overline{\cup}} x, \overline{f}] &= [M \sqcup M^-, (f^*(z) \cup x) \sqcup 0, f \sqcup f]\\ &= [M, f^*(z) \cup x, f] + [M^-, 0, f]\\ &= [M, f^*(z) \cup x, f].\end{aligned}$$

Thus the two geometric cap products coincide on $h^*(X) \otimes h_*^{geo}(X)$. We may therefore speak of *the* geometric cap product hereafter. □

4.2 Poincaré duality

4.5 Theorem. *For any h-manifold N, each of the Poincaré duality isomorphisms*

$$D_N^{geo}\colon h^*(N) \xrightarrow{\cong} h_*^{geo}(N, \partial N),$$
$$\widehat{D}_N^{geo}\colon h^*(N, \partial N) \xrightarrow{\cong} h_*^{geo}(N),$$

is by the geometric cap product given by

$$x \mapsto x \cap [N, 1, 1].$$

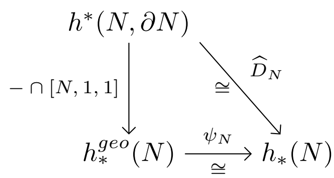
Proof. The first map is trivially verified: For $x \in h^*(N)$, we get

$$x \mapsto x \cap [N, 1, 1] = [N, 1^*(x) \cup 1, 1] = [N, x, 1] = D_N^{geo}(x).$$

The second map is defined to be

$$\widehat{D}_N^{geo} \coloneqq {\psi_N}^{-1} \widehat{D}_N.$$

Hence the result follows by showing that the diagram commutes.



We recall that ψ_N is given by $[M, y, g] \mapsto g_* D_M(y)$. For any $x \in h^*(N, \partial N)$ we thus get

$$\begin{aligned}
\psi_N(x \cap [N, 1, 1]) &= \psi_N[\overline{N}, x \mathbin{\overline{\cup}} 1, \overline{1}] \\
&= \overline{1}_* D_{\overline{N}}(x \mathbin{\overline{\cup}} 1) \\
&= \overline{1}_* D_{\overline{N}}({D_{\overline{N}}}^{-1} I_* \widehat{D}_N(x \cup 1)) \\
&= (\overline{1} I)_* \widehat{D}_N(x) = \widehat{D}_N(x).
\end{aligned}$$

□

4.3 Properties of the geometric cap product

With the geometric cap product established and having shown it solves the initial Poincaré duality problem, we close this chapter by considering properties that a cap product should have. First we give two lemmas.

4.6 Lemma. *Let M be an h-manifold, $x \in h^*(M, \partial M)$ and $y, z \in h^*(M)$. Then we have*

$$(x \cup y) \mathbin{\overline{\cup}} z = x \mathbin{\overline{\cup}} (y \cup z).$$

PROOF. Associativity of the cup product immediately gives

$$\begin{aligned}(x \cup y)\ \overline{\cup}\ z &= D_{\overline{M}}{}^{-1} I_* \widehat{D}_M((x \cup y) \cup z)\\ &= D_{\overline{M}}{}^{-1} I_* \widehat{D}_M(x \cup (y \cup z))\\ &= x\ \overline{\cup}\ (y \cup z).\end{aligned}$$ □

4.7 LEMMA. *The diagram commutes, giving the equality*

$$x\ \overline{\cup}\ y = J^* I^{*-1}(x) \cup \overline{1}^*(y).$$

$$\begin{array}{ccccc}
h^*(M,\partial M)\otimes h^*(M) & \xrightarrow{I^{*-1}\otimes \overline{1}^*} & h^*(\overline{M},M^-)\otimes h^*(\overline{M}) & \xrightarrow{J^*\otimes 1^*} & h^*(\overline{M})\otimes h^*(\overline{M})\\
\downarrow{\scriptstyle\cup} & & \downarrow{\scriptstyle\cup} & & \downarrow{\scriptstyle\cup}\\
h^*(M,\partial M) & \xrightarrow{I^{*-1}} & h^*(\overline{M},M^-) & \xrightarrow{J^*} & h^*(\overline{M})
\end{array}$$

PROOF. The right-hand square commutes by naturality of the cup product. The left-hand square is the same as the left-hand square in the following diagram.

$$\begin{array}{ccccc}
h^*(M,\partial M)\otimes h^*(M) & \xrightarrow{I^{*-1}\otimes \overline{1}^*} & h^*(\overline{M},M^-)\otimes h^*(\overline{M}) & \xrightarrow{I^*\otimes I^*} & h^*(M,\partial M)\otimes h^*(M)\\
\downarrow{\scriptstyle\cup} & & \downarrow{\scriptstyle\cup} & & \downarrow{\scriptstyle\cup}\\
h^*(M,\partial M) & \xrightarrow[\cong]{I^{*-1}} & h^*(\overline{M},M^-) & \xrightarrow[\cong]{I^*} & h^*(M,\partial M)
\end{array}$$

Also here, the right-hand square commutes by naturality of the cup product. Note that the compositions at the top and at the bottom are the respective identity maps. The left-hand square therefore commutes since the lower right-hand I^* is an isomorphism. □

4.8 PROPOSITION. *For* $1 \in h^*(X)$, $[M,x,f] \in h_*^{geo}(X,A)$, *we have*

$$1 \cap [M,x,f] = [M,x,f].$$

PROOF.

$$1 \cap [M,x,f] = [M, f^*(1) \cup x, f] = [M, 1 \cup x, f] = [M,x,f].$$ □

4.9 PROPOSITION. *Let* $\varphi\colon (X,A) \to (Y,B)$ *be a map of pairs. For any* $[M,x,f] \in h_*^{geo}(X,A)$, *the following diagram commutes*

$$\begin{array}{ccc}
h^*(X,A) & \xleftarrow{\varphi^*} & h^*(Y,B)\\
\downarrow{\scriptstyle - \cap [M,x,f]} & & \downarrow{\scriptstyle - \cap \varphi_*[M,x,f]}\\
h_*^{geo}(X) & \xrightarrow{\varphi_*} & h_*^{geo}(Y)
\end{array}.$$

That is, for any $y \in h^*(Y,B)$, we get the equality

$$\varphi_*(\varphi^*(y) \cap [M,x,f]) = y \cap \varphi_*[M,x,f].$$

Proof. Since we have written φ for the map $(X,\varnothing) \to (Y,\varnothing)$ as well, we have $\varphi\overline{f} = \overline{\varphi f}$ and thus

$$\begin{aligned}
\varphi_*(\varphi^*(y) \cap [M,x,f]) &= \varphi_*[\overline{M}, f^*\varphi^*(y) \mathbin{\overline{\cup}} x, \overline{f}] \\
&= [\overline{M}, (\varphi f)^*(y) \mathbin{\overline{\cup}} x, \varphi\overline{f}] \\
&= y \cap [M, x, \varphi f] \\
&= y \cap \varphi_*[M,x,f].
\end{aligned}$$

□

Remark. For the other variant of the geometric cap product above, the analogous result holds. The proof is essentially the same. □

4.10 Proposition. *The diagram commutes.*

$$\begin{array}{ccc}
h^k(X) \otimes h_n^{geo}(X,A) & \xrightarrow{\ \cap\ } & h_{n-k}^{geo}(X,A) \\
\Big\downarrow {\scriptstyle i^* \otimes \partial} & & \Big\downarrow {\scriptstyle (-1)^k \partial} \\
h^k(A) \otimes h_{n-1}^{geo}(A) & \xrightarrow{\ \cap\ } & h_{n-k-1}^{geo}(A)
\end{array}$$

Proof. For $z \in h^k(X)$, $[M,x,f] \in h_n^{geo}(X,A)$, let $j\colon \partial M \hookrightarrow M$ denote the inclusion. We get

$$\begin{aligned}
(\partial f)^* i^*(z) \cup x|_{\partial M} &= j^* f^*(z) \cup j^*(x) \\
&= (f^*(z) \cup x)|_{\partial M},
\end{aligned}$$

and thus

$$\begin{aligned}
i^*(z) \cap \partial[M,x,f] &= i^*(z) \cap [\partial M, (-1)^{\dim x} x|_{\partial M}, \partial f] \\
&= [\partial M, (\partial f)^* i^*(z) \cup (-1)^{\dim x} x|_{\partial M}, \partial f] \\
&= [\partial M, (-1)^{\dim x} (f^*(z) \cup x)|_{\partial M}, \partial f] \\
&= (-1)^{\dim f^*(z)} [\partial M, (-1)^{\dim f^*(z) + \dim x} (f^*(z) \cup x)|_{\partial M}, \partial f] \\
&= (-1)^{\dim z} \partial\big([M, f^*(z) \cup x, f]\big) \\
&= (-1)^k \partial\big(z \cap [M,x,f]\big).
\end{aligned}$$

□

We now consider the diagram in the next proposition. When commutative, it expresses the equality

$$(z \cup y) \cap x = z \cap (y \cap x).$$

If we replace h_*^{geo} by h_* in the diagram, the cap products are defined when the triads $(X; A \cup B, C)$, $(X; A, C)$ and $(X; B, A \cup C)$ obey certain conditions. The diagram then commutes. We have already been making use of this equality several times, e.g. in connection with Poincaré duality maps where we encounter the ordinary, non-geometric cap product.

We have only been able to define the geometric cap product

$$h^*(X,A) \otimes h_*(X, A \cup B) \to h_*(X,B)$$

for either A or B empty. The diagram below is therefore not always meaningful, so in the proposition we must restrict ourselves to spacial cases where the geometric cap product *is* defined.

4.11 PROPOSITION. *In the three respective cases*

$$B = C = \varnothing, \qquad A = C = \varnothing, \qquad A = B = \varnothing,$$

the following diagram is commutative.

$$\begin{array}{ccc} h^*(X,A) \otimes h^*(X,B) \otimes h_*^{geo}(X, A \cup B \cup C) & \xrightarrow{1 \otimes \cap} & h^*(X,A) \otimes h_*^{geo}(X, A \cup C) \\ \downarrow{\scriptstyle \cup \otimes 1} & & \downarrow{\scriptstyle \cap} \\ h^*(X, A \cup B) \otimes h_*^{geo}(X, A \cup B \cup C) & \xrightarrow{\cap} & h_*^{geo}(X, C) \end{array}$$

PROOF. In each of the three cases, we choose an element

$$z \otimes y \otimes [M,x,f] \in h^*(X,A) \otimes h^*(X,B) \otimes h_*^{geo}(X, A \cup B \cup C).$$

$B = C = \varnothing:$

$$\begin{array}{ccc} h^*(X,A) \otimes h^*(X) \otimes h_*^{geo}(X,A) & \xrightarrow{1 \otimes \cap} & h^*(X,A) \otimes h_*^{geo}(X,A) \\ \downarrow{\scriptstyle \cup \otimes 1} & & \downarrow{\scriptstyle \cap} \\ h^*(X,A) \otimes h_*^{geo}(X,A) & \xrightarrow{\cap} & h_*^{geo}(X) \end{array}$$

By Lemma 4.6, we have

$$\big(f^*(z) \cup f^*(y)\big) \mathbin{\overline{\cup}} x = f^*(z) \mathbin{\overline{\cup}} \big(f^*(y) \cup x\big).$$

This gives

$$\begin{aligned} z \cap (y \cap [M,x,f]) &= z \cap [M, f^*(y) \cup x, f] \\ &= [\overline{M}, f^*(z) \mathbin{\overline{\cup}} (f^*(y) \cup x), \overline{f}] \\ &= [\overline{M}, (f^*(z) \cup f^*(y)) \mathbin{\overline{\cup}} x, \overline{f}] \\ &= [\overline{M}, f^*(z \cup y) \mathbin{\overline{\cup}} x, \overline{f}] \\ &= (z \cup y) \cap [M,x,f]. \end{aligned}$$

$A = C = \varnothing$**:**

$$\begin{array}{ccc}
h^*(X) \otimes h^*(X,B) \otimes h_*^{geo}(X,B) & \xrightarrow{1 \otimes \cap} & h^*(X) \otimes h_*^{geo}(X) \\
\Big\downarrow {\scriptstyle \cup \otimes 1} & & \Big\downarrow {\scriptstyle \cap} \\
h^*(X,B) \otimes h_*^{geo}(X,B) & \xrightarrow{\quad \cap \quad} & h_*^{geo}(X)
\end{array}$$

The two ways around the diagram respectively give

$$\begin{aligned}
z \cap (y \cap [M,x,f]) &= z \cap [\overline{M}, f^*(y) \mathbin{\overline{\cup}} x, \overline{f}] \\
&= [\overline{M}, \overline{f}^*(z) \cup (f^*(y) \mathbin{\overline{\cup}} x), \overline{f}],
\end{aligned}$$

$$\begin{aligned}
(z \cup y) \cap [M,x,f] &= [\overline{M}, f^*(z \cup y) \mathbin{\overline{\cup}} x, \overline{f}] \\
&= [\overline{M}, (f^*(z) \cup f^*(y)) \mathbin{\overline{\cup}} x, \overline{f}].
\end{aligned}$$

Commutativity thus follows by demonstrating the equality

$$\overline{f}^*(z) \cup \big(f^*(y) \mathbin{\overline{\cup}} x\big) = f^*(z \cup y) \mathbin{\overline{\cup}} x$$

in $h^*(\overline{M})$.

We observe that the following diagram commutes, the upper and lower square each commutative by naturality of the cup product and the middle square by Lemma 4.7.

$$\begin{array}{ccc}
h^*(X) \otimes h^*(X,B) & \xrightarrow{\quad \cup \quad} & h^*(X,B) \\
\Big\downarrow {\scriptstyle f^* \otimes f^*} & & \Big\downarrow {\scriptstyle f^*} \\
h^*(M) \otimes h^*(M,\partial M) & \xrightarrow{\ \cup\ } & h^*(M,\partial M) \\
\Big\downarrow {\scriptstyle \overline{1}^* \otimes I^{*-1}} & & \Big\downarrow {\scriptstyle I^{*-1}} \\
h^*(\overline{M}) \otimes h^*(\overline{M}, M^-) & \xrightarrow{\ \cup\ } & h^*(\overline{M}, M^-) \\
\Big\downarrow {\scriptstyle 1^* \otimes J^*} & & \Big\downarrow {\scriptstyle J^*} \\
h^*(\overline{M}) \otimes h^*(\overline{M}) & \xrightarrow{\quad \cup \quad} & h^*(\overline{M})
\end{array}$$

Hence we have

$$\overline{f}^*(z) \cup J^* I^{*-1} f^*(y) = J^* I^{*-1} f^*(z \cup y).$$

This gives the asserted equality by

$$\begin{aligned}
\overline{f}^*(z) \cup \big(f^*(y) \mathbin{\overline{\cup}} x\big) &= \overline{f}^*(z) \cup \big(J^* I^{*-1}(f^*(y) \cup x)\big) \\
&= \overline{f}^*(z) \cup J^* I^{*-1} f^*(y) \cup \overline{1}^*(x) \\
&= J^* I^{*-1} f^*(z \cup y) \cup \overline{1}^*(x) \\
&= J^* I^{*-1}\big(f^*(z \cup y) \cup x\big) \\
&= f^*(z \cup y) \mathbin{\overline{\cup}} x,
\end{aligned}$$

where we have used Lemma 4.7 twice.

$A = B = \varnothing$**:**

$$\begin{array}{ccc} h^*(X) \otimes h^*(X) \otimes h_*^{geo}(X,C) & \xrightarrow{1 \otimes \cap} & h^*(X) \otimes h_*^{geo}(X,C) \\ \downarrow{\scriptstyle \cup \otimes 1} & & \downarrow{\scriptstyle \cap} \\ h^*(X) \otimes h_*^{geo}(X,C) & \xrightarrow{\cap} & h_*^{geo}(X,C) \end{array}$$

The computation is straightforward,

$$\begin{aligned} z \cap (y \cap [M,x,f]) &= z \cap [M, f^*(y) \cup x, f] \\ &= [M, f^*(z) \cup f^*(y) \cup x, f] \\ &= [M, f^*(z \cup y) \cup x, f] \\ &= (z \cup y) \cap [M,x,f]. \end{aligned}$$

□

The next theorem shows that the natural equivalence ψ is well-behaved with respect to the geometric and the non-geometric cap products. We elaborate on this in the succeeding remark.

4.12 THEOREM. *The diagrams commute.*

$$\begin{array}{ccc} h^*(X) \otimes h_*^{geo}(X,A) & \xrightarrow{\cap} & h_*^{geo}(X,A) \\ \cong \downarrow{\scriptstyle 1 \otimes \psi_{(X,A)}} & & \cong \downarrow{\scriptstyle \psi_{(X,A)}} \\ h^*(X) \otimes h_*(X,A) & \xrightarrow{\cap} & h_*(X,A) \end{array}$$

$$\begin{array}{ccc} h^*(X,A) \otimes h_*^{geo}(X,A) & \xrightarrow{\cap} & h_*^{geo}(X) \\ \cong \downarrow{\scriptstyle 1 \otimes \psi_{(X,A)}} & & \cong \downarrow{\scriptstyle \psi_{(X,A)}} \\ h^*(X,A) \otimes h_*(X,A) & \xrightarrow{\cap} & h_*(X) \end{array}$$

PROOF. We do the respective calculations,

$$\begin{aligned} \psi_{(X,A)}(y \cap [M,x,f]) &= \psi_{(X,A)}[M, f^*(y) \cup x, f] \\ &= f_* D_M(f^*(y) \cup x) \\ &= f_*\big((f^*(y) \cup x) \cap [M, \partial M]\big) \\ &= f_*\big(f^*(y) \cap (x \cap [M, \partial M])\big) \\ &= y \cap f_*(x \cap [M, \partial M]) \\ &= y \cap f_* D_M(x) \\ &= y \cap \psi_{(X,A)}[M,x,f], \end{aligned}$$

$$\begin{aligned}
\psi_{(X,A)}(y \cap [M,x,f]) &= \psi_{(X,A)}[\overline{M}, f^*(y) \mathbin{\overline{\cup}} x, \overline{f}] \\
&= \overline{f}_* D_{\overline{M}}(f^*(y) \mathbin{\overline{\cup}} x) \\
&= \overline{f}_* D_{\overline{M}} D_{\overline{M}}{}^{-1} I_* D_M(f^*(y) \cup x) \\
&= (\overline{f} I)_*\big((f^*(y) \cup x) \cap [M, \partial M]\big) \\
&= f_*\big(f^*(y) \cap (x \cap [M, \partial M])\big) \\
&= y \cap f_* D_M(x) \\
&= y \cap \psi_{(X,A)}[M,x,f].
\end{aligned}$$

□

REMARK. For a pair of spaces (X,A), $h_*^{geo}(X,A)$ may be regarded as a graded left module over the ring $h^*(X)$, with the left multiplication from $h^*(X)$ being the geometric cap product. This follows from the propositions 4.8 and 4.11. Restricting to CW pairs, we also get the similar interpretation of $h_*(X,A)$ as a graded left $h^*(X)$-module. The upper diagram in the theorem above now makes

$$\psi_{(X,A)} \colon h_*^{geo}(X,A) \xrightarrow{\cong} h_*(X,A)$$

a natural, graded $h^*(X)$-module isomorphism for any CW pair (X,A). Hence one can simply use the diagrams in the theorem to *define* the geometric cap product if confining oneself to CW pairs. □

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