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Simone Cifani

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Thesis for the degree of Philosophiae Doctor

Trondheim, December 2011

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On nonlinear fractional convection - diffusion equations

Simone Cifani, NTNU, PhD thesis

PREFACE

This thesis contains my research work as a member of the project *Integro-PDEs: Numerical Methods, Analysis, and Applications to Finance* (grant no. 176877/V30) sponsored by the eVita program of the Research Council of Norway.

I would like to take some time here to thank all those who helped me during the time I spent at NTNU. First and foremost, I would like to thank my principal advisor Prof. Espen R. Jakobsen. Then, I would like to thank my subsidiary advisor Prof. Kenneth H. Karlsen (UiO). Special thanks are also due to Prof. Nathaël Alibaud (UFC) with whom I have collaborated. Finally, I would like to mention those at NTNU who I am grateful to for discussions and suggestions: Hilde Sande, Prof. Helge Holden, Prof. Peter Lindqvist, Prof. Sigmund Selberg, Prof. Harald Hanche-Olsen, Prof. Harald Krogstad, Prof. Einar Rønquist.

Simone Cifani
August 2, 2011, Oslo

Til Julie

Introduction

INTRODUCTION

In the following few introductory pages we will briefly sketch our main contributions to the field of nonlocal nonlinear partial differential equations of the form (1.1). We will start by presenting the setting we found when we started our study; then we will point out the direction of research we chose, the difficulties related to the questions we tried to answer, and a few issues which we consider relevant but leave unanswered. A full detailed presentation of all the results which we will just touch upon herein can be found in the following attached publications/preprints [2, 17, 18, 19, 20, 21]. Such papers contain the whole of our research, and we therefore invite the reader to look therein for detailed answers. This brief introduction has the sole purpose of presenting and framing for reader the problem under study.

1. The initial value problem

In this thesis we study the following nonlinear nonlocal partial differential equation (or *integro-PDE* in short)

$$\begin{cases} \partial_t u(x,t) + \operatorname{div}\left(f(u)\right)(x,t) = \mathcal{L}^\mu[A(u(\cdot,t))](x) & (x,t)\in\mathbb{R}^d\times(0,T),\\ u(x,0) = u_0(x), & x\in\mathbb{R}^d, \end{cases} \tag{1.1}$$

where $u = u(x,t) : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is the unknown solution to be found, the shorthand "div" denotes the divergence operator with respect to the space variable, and, for all bounded functions $\phi \in C^2(\mathbb{R}^d)$, the nonlocal operator $\mathcal{L}^\mu[\cdot]$ takes the form

$$\mathcal{L}^\mu[\phi](x) = \int_{|z|>0} \phi(x+z) - \phi(x) - z\cdot D\phi(x)\mathbf{1}_{|z|\le 1}\, \mathrm{d}\mu(z). \tag{1.2}$$

Here we have used the notation $D\phi$ for the gradient of ϕ, while $\mathbf{1}_{|z|\le 1}$ stands for the indicator function of the interval $\{z : |z| \le 1\}$. For reasons that will become clear in the course of this introduction, equations of the form (1.1) are often referred to as *nonlinear fractional convection-diffusion equations*.

For each different set-up in the thesis [2, 17, 18, 19, 20, 21], the regularity of the data (f, A, u_0, μ) will be suitably chosen depending on the results which we will be striving to establish. In particular:

(i) The function $f(\cdot)$ will always be at least Lipschitz, $f = (f_1, \ldots, f_d) \in W^{1,\infty}(\mathbb{R};\mathbb{R}^d)$, with additional regularity at times required. Moreover, some results will be proven under the technical assumption $f(0) = 0$.

(ii) The function $A(\cdot)$ will always be nondecreasing and Lipschitz, $A \in W^{1,\infty}(\mathbb{R})$, with $A(0) = 0$. This assumption is very natural and widely used in the literature dedicated to local nonlinear diffusion equations [14, 15, 41, 51].

(iii) Some our results will be proved using initial data $u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ (with bounded total variation at times required), others will be proved using initial data in $L^2(\mathbb{R}^d)$.

(iv) Throughout the thesis $\mu(\cdot)$ will always be a nonnegative Radon measure which is defined on $\{x : x \in \mathbb{R}^d \setminus \{0\}\}$ and satisfies the condition

$$\int_{\mathbb{R}^d \setminus \{0\}} |z|^2 \wedge 1 \, \mathrm{d}\mu(z) < \infty, \tag{1.3}$$

where $a \wedge b$ stands for $\min\{a, b\}$.

Detailed discussions around the optimality of the various assumptions we make on the data are included in each paper [2, 17, 18, 19, 20, 21]. Here let us just briefly mention that, since almost everywhere in this thesis the solutions we work with are going to bounded, both nonlinearities $f(\cdot)$ and $A(\cdot)$ do not need to be globally Lipschitz; locally Lipschitz would do as well. Moreover, let us also point out that there is no real loss of generality connected with the technical assumptions $f(0) = 0$ and $A(0) = 0$ since, to treat the case $f(0)$ and $A(0)$ general, one could replace f by $f - f(0)$ and A by $A - A(0)$.

Nonlinear nonlocal integro-PDEs like (1.1) have recently attracted great attention due to their applications in several different areas of interest such as, to mention only a few, mathematical finance [25], flow in porous media [27], and radiation hydrodynamics [52, 53] - for several other applications of interest let us refer the reader to [1, 2, 19]. Needless to say, this (increasingly growing) number of applications has spawned a great deal of theoretical research on such equations aimed at settling core issues like well-posedness and regularity of solutions. Most of the research so far has focused on linear nonlocal operators - i.e., $A(\cdot)$ linear - and particular Lévy measures $\mu(\cdot)$ - mainly those underlying the so-called fractional Laplace operator [33]. As we will see more in details in a short while, for such "linear" equations it is known that shocks discontinuities can occur in finite time [4, 29, 39, 40, 43, 48, 56], that weak solutions can be non-unique [5], and that the initial value problem (1.1) is well-posed with the notion of entropy solutions in the sense of Kruzkov [1, 47, 56]. The Kruzkov entropy solution theory has been recently extended in [20] to cover the full problem (1.1) for nonlinear functions $A(\cdot)$ and general Lévy measures $\mu(\cdot)$.

2. The nonlocal operator

In order to present the reader with the main difficulties which arise when trying to solve the initial value problem (1.1), we will first have to introduce what will be the leading character of this introduction, the nonlocal operator $\mathcal{L}^{\mu}[\cdot]$ in (1.2).

In the literature, nonnegative Radon measures like $\mu(\cdot)$ in (1.3) are also referred to as *Lévy measures*. This is because their associated nonlocal operators $\mathcal{L}^{\mu}[\cdot]$ turn out to be generators of so-called pure jump *Lévy processes* [7, 54]. As an example, let us choose the measure $\mu(\cdot)$ as $\pi(\cdot)$ where, for all $\lambda \in (0, 2)$,

$$\pi(z) = \frac{1}{|z|^{d+\lambda}} \text{ (up to a positive multiplicative constant).} \tag{2.1}$$

The nonlocal operator $\mathcal{L}^{\pi}[\cdot]$ associated with the Lévy measure $\pi(\cdot)$ turns out to be the generator of so-called *α-stable diffusion processes* and is often referred to as *fractional Laplacian* [33, 46], in symbols

$$\mathcal{L}^{\pi}[\cdot] = -\left(-\triangle\right)^{\lambda/2}.$$

Such a name comes from the fact that the fractional Laplacian is a nonlocal generalization of the classical Laplacian, as its own definition via the Fourier transform immediately shows:

$$\left(-\triangle\right)^{\lambda/2} \phi = \mathcal{F}^{-1}\left(|\cdot|^{\lambda} \mathcal{F}\phi\right).$$

That is to say, as the exponent lambda approaches two, the nonlocal fractional Laplace operator reduces to the local Laplace operator. It is also worth recalling that, as the exponent lambda approaches zero, the fractional Laplacian reduces to the identity operator.

As already mentioned, the nonlocal operator $\mathcal{L}^\mu[\cdot]$ in (1.2) is well defined for any bounded twice differentiable function ϕ. Indeed, for all $x \in \mathbb{R}^d$,

$$|\mathcal{L}^\mu[\phi](x)| \leq \max_{x\in\mathbb{R}^d, |z|\leq 1} |D^2\phi(x+z)| \int_{0<|z|\leq 1} |z|^2 \, \mathrm{d}\mu(z) + 2 \, \|\phi\|_{L^\infty(\mathbb{R}^d)} \int_{|z|>1} \mathrm{d}\mu(z)$$

where $D^2\phi$ is the Hessian of ϕ. Such a consideration takes us to a starting question: whenever they exist, can we expect solutions of the initial value problem (1.1) to be so regular (twice differentiable)? The answer is of course no since $A'(\cdot)$ is allowed to degenerate on intervals of positive measure. To give an example, it is well-known that so-called conservation laws, that is to say the case $(f, A \equiv 0, u_0, \mu)$ (or equivalently $\mu \equiv 0$),

$$\partial_t u(x,t) + \operatorname{div}\left(f(u)\right)(x,t) = 0, \tag{2.2}$$

develop shock discontinuities in finite time even when smooth initial data $u_0(\cdot)$ is considered [37]. It has long being known that *weak* (or *distributional*) solutions of (2.2) are not unique. Indeed, as shown by Kruzkov in [44], uniqueness can only be proved for those weak solutions which satisfy an additional entropy inequality [37]. That is the reason why conservation laws solutions are often referred to as *entropy* solutions. The method employed by Kruzkov in [44] is the by now famous *doubling of variables* method.

2.1. **Fractal (fractional) conservation laws.** It is clear that, in general, the initial value problem (1.1) does not admit classical solutions. But what happens when the nonlocal operator $\mathcal{L}^\mu[\cdot]$ in (1.2) is not disregarded as done in (2.2)? Does it have some sort of regularizing influence on the possibly discontinuous entropy solution of the conservation law (2.2)? After all, the nonlocal operator $\mathcal{L}^\mu[\cdot]$ in (1.2) is of diffusive nature (at least for symmetric Lévy measures), and such questions are thus perfectly natural. As an illustrative example, let us recall what can be said whenever a nonlocal operator like the fractional Laplacian is added to the conservation law (2.2). An interesting result on this subject has been formally proved in [4]: equations of the form

$$\partial_t u(x,t) + \operatorname{div}\left(f(u)\right)(x,t) = \mathcal{L}^\pi[u(\cdot,t)](x), \tag{2.3}$$

which are referred to in the literature both as *fractal* or *fractional* conservation laws (we will use those names interchangeably throughout the whole thesis), does not admit in general classical (smooth) solutions (at least for values of lambda less than one). If we cannot have smooth solutions, how can we generalize the concept of solution in order to establish well-posedness for the initial value problem (1.1)? We will look closer at this issue in the following section.

3. Entropy formulation

The main issues at stake in this thesis are indeed classical for partial differential equations [34, 36]. We would like to answer the following questions. Does the initial value problem (1.1) admit solutions at all? And if solutions exist, how many are they? Are they stable with respect to the data (f, A, u_0, μ)? And if so, can they be captured through some numerical approximation? As we will see, our research has produced answers for all such questions.

3.1. **Alibaud's entropy formulation.** How to proceed then to give sense to the general initial value problem (1.1)? How to define its weak (entropy) solutions? A promising starting point may lie in a groundbreaking result derived by Alibaud [1, 5] for a less general family of nonlocal equations: fractal (fractional) conservation laws like (2.3).

Alibaud's idea for fractal (fractional) conservation laws is to split the nonlocal operator $\mathcal{L}^\pi[\cdot]$ into the sum of two sub-operators: for all $r > 0$ we write

$$\mathcal{L}^\pi[\phi](x) = \mathcal{L}_r^\pi[\phi](x) + \mathcal{L}^{\mu,r}[\phi](x),$$

where

$$\begin{aligned}\mathcal{L}_r^\mu[\phi](x) &= \int_{0<|z|\leq r} \phi(x+z) - \phi(x) - z \cdot D\phi(x)\,\mathrm{d}\mu(z),\\ \mathcal{L}^{\mu,r}[\phi](x) &= \int_{|z|>r} \phi(x+z) - \phi(x)\,\mathrm{d}\mu(z).\end{aligned}$$

Here it is important to note that such a decomposition is only valid for symmetric Lévy measures like $\pi(\cdot)$ in (2.1) since, away from the singularity ($z_1 > 0$),

$$\int_{z_1<|z|<z_2} z\,\mathrm{d}\pi(z) = 0.$$

For later use, let us also remember the so-called Kruzkov entropies [44], $\eta(u,k) = |u-k|$, $\eta'(u,k) = \mathrm{sgn}(u-k)$ and the entropy fluxes

$$q_f(u,k) = \mathrm{sgn}\,(u-k)\,(f(u) - f(k)) \in \mathbb{R}^d.$$

With this notation at hand, we can now recall Alibaud's entropy formulation [1] for so-called fractal (fractional) conservation laws like (2.3).

Definition 3.1. *A function $u \in L^\infty(\mathbb{R}^d \times (0,T))$ is an entropy solution of* (2.3) *provided that, for all $k \in \mathbb{R}$, all $r > 0$, and all nonnegative $\phi \in C_c^\infty(\mathbb{R}^{d+1})$,*

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^d} \eta(u,k)\,\partial_t\phi + q_f(u,k)\cdot D\phi\,\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad\qquad + \int_0^T \int_{\mathbb{R}^d} \eta(u,k)\,\mathcal{L}_r^\pi[\phi] + \eta'(u,k)\,\mathcal{L}^{\pi,r}[u]\,\phi\,\,\mathrm{d}x\,\mathrm{d}t \geq 0.\end{aligned}$$

The crucial point here is that the possibly discontinuous entropy solution $u(\cdot)$ is left inside the operator $\mathcal{L}^{\mu,r}[\cdot]$, while the entire weight of the singularity lying inside the operator $\mathcal{L}_r^\mu[\cdot]$ is unloaded onto the test function $\phi(\cdot)$. Furthermore, the entropy inequality is required to hold *for any* $r > 0$.

Remark 3.1. Borrowing some well-established ideas and techniques from the theory of viscosity solutions for second-order elliptic integro-PDEs [8, 9, 38, 57], the main intuition behind Alibaud's entropy formulation is that, given their different features, the operators $\mathcal{L}_r^\mu[\cdot]$ and $\mathcal{L}^{\mu,r}[\cdot]$ must play different roles in the uniqueness (contraction) proof. Loosely speaking, the idea is that the operator $\mathcal{L}^{\mu,r}[\cdot]$ is the one which is best suited for the Kruzkov's doubling of variables argument [44], and thus should support the whole structure of the proof. On the other end, the operator $\mathcal{L}_r^\mu[\cdot]$ is less attractive to work with, and thus should be simply carried along, until the possibility of choosing $r > 0$ arbitrary small (and the fact that the operator $\mathcal{L}_r^\mu[\cdot]$ is applied onto a test function) is used to let it vanish in the limit. All the details can be found in [1, 19].

4. Uniqueness for more general equations

From the very beginning of our research we convinced ourselves that the intuitions behind Alibaud's entropy formulation were extremely powerful, and that several more results could be proven from them. More precisely, we asked ourselves the following questions.

(i) How to establish well-posedness for general nonlinear nonlocal equations of the form (1.1)?

(ii) How to prove convergence for suitably devised numerical methods, and how to measure their rates of convergence?

To point out to reader the relevance of such issues, let us now quickly discuss the former of the two questions: generalization to equations of the form (1.1). A discussion of the latter, convergence of numerical methods, will take the whole of next section.

4.1. **Cifani/Jakobsen's entropy formulation.** The breakthrough when it comes to well-posedness for general initial value problems of the form (1.1) is contained in [19]. Therein the authors show how to combine Alibaud's insights [1] and the Kruzkov's doubling of variables machinery [41, 44] in order to cope with a nonlocal operator $\mathcal{L}^{\mu}[\cdot]$ which is no longer linear,

$$\mathcal{L}^{\mu}[A(u+v)] \neq \mathcal{L}^{\mu}[A(u)] + \mathcal{L}^{\mu}[A(v)]$$

and no longer symmetric,

$$\mathcal{L}^{\mu}[A(u)] \neq \mathcal{L}^{\mu}_{r}[A(u)] + \mathcal{L}^{\mu,r}[A(u)].$$

This is significant leap forward, and indeed the most surprising thing about this generalization is the fact that uniqueness (contraction) can still be established by using a strategy which is essentially similar to the one pioneered by Alibaud for equations featuring a linear symmetric nonlocal operator (the fractional Laplacian): roughly speaking, we will split the nonlocal operator $\mathcal{L}^{\mu}[\cdot]$ in (1.2) in a proper fashion, unload the weight of the singularity onto a test function, and demand the so derived entropy inequality to hold for any $r > 0$.

Remark 4.1. As we will see more in details in a short while, the family of all solutions of the initial value problem (1.1) is much richer than the the family of all solutions of, for example, fractal (fractional) conservation laws like (2.3). Loosely speaking, opposite to equations like the ones treated in [18, 42], equation (1.1) has now finally become genuinely nonlinear since, for the first time, the nonlinearity $A(\cdot)$ has been taken inside the nonlocal operator $\mathcal{L}^{\mu}[\cdot]$. As explained in the following section, this fact gives way to a whole new spectra of phenomena which are not present in simpler equations like (2.2) or (2.3).

4.2. **Genuinely nonlinear nonlocal equations.** To stress the reach of the generalization achieved in [19], let us recall a well-know regularity result [30, 32]: solutions of nonlocal equations of the form

$$\partial_t u(x,t) = -\left(-\triangle\right)^{\lambda/2} u(x,t), \tag{4.1}$$

a nonlocal generalization of the heat equation, are smooth for any $\lambda \in (0,2)$. We mention this here to point out the fact that linear symmetric nonlocal operators like the fractional Laplacian do have a regularizing effect of their own. Such a smoothing effect wanes as the exponent lambda decreases toward zero, but for higher lambdas is still strong enough to guarantee existence of classical solutions for fractal (fractional) conservation laws [30, 32]. Loosely speaking, even though linear symmetric operators like the fractional Laplacian do have a regularizing effect of their own, such an effect may sometimes be to weak for preventing shock discontinuities from developing [4].

Of course, this is no longer the case in general whenever genuinely nonlinear nonlocal asymmetric operators like $\mathcal{L}^{\mu}[A(\cdot)]$ are considered instead. To give an example, let us consider the equation

$$\partial_t u(x,t) = -\left(-\triangle\right)^{\lambda/2} A(u(x,t)). \tag{4.2}$$

The point here is that the first derivative of the nonlinear nondecreasing function $A(\cdot)$ can degenerate on intervals of positive measure, something which eliminates any possible (global) regularizing effect. In other words, the family of all solutions of (4.2) has a much richer structure than the family of all solutions of (4.1) since the diffusion can be switched on and off by using the shape of the nonlinear function $A(\cdot)$ at hand. As numerical simulations seem to suggest, such a difference is even more pronounced whenever general asymmetric measures $\mu(\cdot)$ are also allowed. Indeed, the solutions of the equation

$$\partial_t u(x,t) = \mathcal{L}^{\mu}[A(u(\cdot,t))](\cdot) \tag{4.3}$$

do not only feature shock discontinuities whenever their initial condition does. They may also develop such discontinuities on their own from perfectly smooth initial data as a result of the mixed convective/diffusive nature of the nonlinear nonlocal asymmetric operator $\mathcal{L}^{\mu}[A(\cdot)]$.

Remark 4.2. One has to be careful when dealing with discontinuous solutions of nonlinear nonlocal equations like (4.2). Their behavior may indeed be very different from that of (local) classical nonlinear diffusion equations like

$$\partial_t u(x,t) = \Delta A(u(x,t)), \tag{4.4}$$

where shocks discontinuities can develop *exactly* in the same region where $A'(\cdot)$ degenerates [15, 35, 41]. For nonlocal equations things are not so straightforward since contributions from regions further afield can still be relevant enough to smooth shock discontinuities out in the region where $A'(\cdot)$ degenerates. However such a regularizing effect could take some time to kick in, giving shock discontinuities the possibility to develop (at least for a finite period of time) [19].

More generally, it is correct to state that equation (4.2) is a natural nonlocal generalization of equation (4.4) since the family of λ-indexed entropy solutions of (4.2), $\{u_\lambda(\cdot)\}_{\lambda\in(0,2)}$, reduces to the unique entropy solution $u(\cdot)$ of (4.4) as the exponent lambda, $\lambda \in (0,2)$, approaches the value two [3]. See also [13, 27, 59].

5. Stability with respect to the data and convergence rates of numerical approximations

As it is often the case for nonlinear partial differential equations, once a strategy for proving uniqueness (contraction) is made available, that very same strategy can be further developed to prove more refined results, like *continuous dependence estimates with respect to the data*. In perfect Kruzkov's doubling of variables style [44], the uniqueness (contraction) proof can be repeated up to the point where the the test function is still not specified. This intermediate stage in the proof is often referred to as *Kuznetsov's lemma*, after Kuznetsov's work on conservation laws [45]. From this point on the proof departs from the uniqueness (contraction) one. Instead of using two functions u and v which are assumed to be entropy solutions of (1.1) with different initial conditions, the goal now is to measure the distance between the entropy solution u of (1.1) and an arbitrary function v which may be chosen as the entropy solution of a similar initial value problem with slightly different data (f, A, u_0, μ). Of course, if it is only the initial datum that differs between the two, we end up with no less than the uniqueness (contraction) proof itself. This whole machinery is indeed a generalization of the uniqueness (contraction) proof. For the sake of concreteness, let us mention here one the main results in [2] where continuous dependence estimates for nonlinear nonlocal equations like (1.1) are derived.

Theorem 5.1. (Continuous dependence estimates on the data)
Let u and v be the entropy solutions of two different initial value problems of the form (1.1) with, respectively, data sets (f, A, u_0, μ) and (g, B, v_0, ν). Then, we have that

$$\|u - v\|_{C([0,T];L^1(\mathbb{R}^d))} \le \|u_0 - v_0\|_{L^1(\mathbb{R}^d)} + C\left(\|f' - g'\|_{L^\infty(\mathbb{R})} + \|A' - B'\|^{\frac{1}{2}}_{L^\infty(\mathbb{R})} + \left(\int_{\mathbb{R}^d\setminus\{0\}} |z|^2 \wedge 1 \,\mathrm{d}|\mu - \nu|(z)\right)^{\frac{1}{2}}\right)$$

where the "constant" C only depends on the final time $t = T$ chosen, the dimension d, and the data.

Let us mention that, as shown in [3], optimal continuous dependence estimates can be derived whenever the Lévy operator (1.2) is chosen as the fractional Laplacian.

A very useful application for the original Kuznetsov's lemma for conservation laws [45] was to produce error estimates for numerical approximations [37]. To this end, the groundbreaking idea behind the whole Kuznetsov's construction was to choose the above-mentioned arbitrary function v as the solution of a difference method, and then work out the method's rate of convergence by estimating each error contribution in the eponymous lemma. Is it possible to adapt such a strategy to numerical approximations of nonlocal equations like (1.1)? Again, the answer to this question turns out to be yes [21]. More precisely we will show that, by choosing the arbitrary function v in the generalized Kuznetsov's lemma derived in [2] as the solution $\bar{u}$ of a properly defined difference method [21], we are able to measure the error between the unique entropy solution u of (1.1) and its numerical approximation $\bar{u}$ as described in the following theorem.

Theorem 5.2. (Convergence rate for numerical approximations)
Let $\sigma_\lambda(\cdot)$ be the modulus of continuity

$$\sigma_\lambda(\tau) = \begin{cases} \tau^{\frac{1}{2}} & \lambda \in (0,1), \\ \tau^{\frac{1}{2}} \log(\tau) & \lambda = 1, \\ \tau^{\frac{2-\lambda}{2}} & \lambda \in (1,2). \end{cases} \tag{5.1}$$

Then, for all $\lambda \in (0, 2)$, we have that

$$\|u - \bar{u}\|_{C(0,T;L^1(\mathbb{R}^d))} \le C\,\sigma_\lambda(\Delta x), \tag{5.2}$$

where Δx is the so-called discretization parameter, the "constant" C only depends on the final time $t = T$ chosen, the dimension d, and the data.

Let us point out here that the fractional Laplacian exponent $\lambda \in (0, 2)$ appears in the error estimate (5.2) due to the fact that the theorem has been proved in [21] for all Lévy measures $\mu(\cdot)$ such that

$$\mu(z) \le \frac{c}{|z|^{d+\lambda}}$$

for some constant $c > 0$.

6. Existence

After uniqueness and stability, the last challenge to face in order to establish well-posedness for the initial value problem (1.1) is existence. How to prove that a weak (entropy) solution of (1.1) actually exists? Numerical approximation is the main focus of this thesis, and existence for the general problem (1.1) has been established in [21] via a finite volume method (and compactness). For more details, cf. the following section.

Let us also mention here that a somewhat different approach has been used in [20]. Herein the existence proof is divided into two steps: in the first one, a (Fourier) numerical method and compactness is used to prove convergence to a viscous equivalent of the original equations. Then, the classical vanishing viscosity method is used to prove convergence to the original equation itself. In a few words, the vanishing viscosity method [37] consists in adding to the original equation artificial viscosity (i.e., $\epsilon\,\Delta u$) which is vanishing as the multiplicative constant $\epsilon \to 0$. The advantage in doing so is that it is well-known how to prove existence of (smooth) solutions u_ϵ of the new viscous equation. If one also has some uniform control on the so derived viscous solutions, the sequence u_ϵ can be proved to be compact and its limit u reveals to be the original weak (entropy) solution one was looking for [20]. The rate of convergence for a new (generalized) vanishing viscosity method for equations like (1.1) can even be measured as shown in [2, 20].

7. Numerical approximation

Once well-posedness for the initial value problem (1.1) is established and in absence of closed-form solutions, one is left with the task of actually producing reliable numerical approximations of such solutions to be used in applications [2, 31]. Here the possibilities for research are abundant given the numerous questions to be answered: from how to devise a method which guarantees convergence under some reasonable general assumptions, to how to devise a method which converges spectrally (exponentially) fast under stricter assumptions. Furthermore, how to measure the distance between such numerical approximations and the original entropy solutions (error)? Being the equations at hand both nonlocal and nonlinear, we are guaranteed to find plenty of difficulties on our way.

The numerical literature available for local convection equations like (2.2) is of course immense [37], but far less so is the amount of research devoted to numerical approximations of linear/nonlinear equations which also feature some form of nonlocal diffusion. For example, some literature is available on numerical methods for nonlocal *linear* equations which find application in mathematical finance [6, 10, 11, 12, 24, 28, 50], but only a few numerical methods have been devised for nonlocal nonlinear equations: Dedner *et al.* introduced in [26] a general class of differences methods for a nonlinear nonlocal equations coming from a specific problem in radiative hydrodynamics, while Droniou [30] was the first to analyze a general class of difference methods for fractional (fractal) conservation laws and prove convergence toward Alibaud's entropy solution [1]. Issues like high-order convergence and error analysis had been left unanswered until the works [17, 18, 20, 21] appeared.

The first issues we would like to touch upon herein are the following. How to devise a numerical method which is general enough to capture the whole family of solutions of (1.1)? And if such a method exist, how can we measure its rate of convergence? Given the whole set of well-established techniques for treating divergence-form operators like $\operatorname{div}(f(\cdot))$, our strategy for retrieving solutions of (1.1) reduces to the following three-steps approximation of the nonlocal operator

$\mathcal{L}^\mu[\cdot]$ in (1.2):

(i) We cut off the singularity by suitably using the size of the discretization grid.

(ii) Replace the gradient operator in (1.2) with an upwind difference operator whose base is suitably linked to the Lévy measure $\mu(\cdot)$ at hand.

(iii) We do a finite volume approximation of the semi-discrete equation resulting from *(i)* and *(ii)*: that is to say, we multiply both sides of the semi-discrete equation by a test function [17], integrate over each grid cell, and replace the original solution u in (1.1) by a piecewise constant approximation. Different type of solution spaces will be used in [17, 18, 20].

This three-steps procedure returns a numerical method [17, 18, 21] which is general enough to guarantee convergence for all initial value problems of the form (1.1). Furthermore, the method's convergence rate can be explicitly measured by using the generalized Kuznetsov's lemma established in [2]. Such a method takes the form

$$U_\alpha^{n+1} = U_\alpha^n + \Delta t \sum_{l=1}^{d} D_l^- \hat{f}(U_\alpha^n, U_{\alpha+e_l}^n) + \frac{\Delta t}{\Delta x^d} \sum_{\beta\in\mathbb{Z}} G_\beta^\alpha A(U_\beta^n)$$

where $\hat{f}(\cdot)$ is a suitable numerical flux and $\{G_\beta^\alpha\}_{\alpha,\beta\in\mathbb{Z}^d}$ a set of opportunely chosen numerical weights [17, 21].

Needless to say, the biggest advantage of this briefly sketched numerical method is at the same time its biggest drawback: its scope. The fact that the above mentioned method ensures convergence for such a wide family of equations, is also the reason why the method is slow and inefficient [21]. By reducing the scope of our investigation, we can come up with numerical approximations which are less general but more efficient and therefore faster. In particular, we would like to do the following:

(i) Find a way to approximate pathological (discontinuous) solutions with higher speed of convergence (convergence rates higher than one) [17, 18].

(ii) Devise a method which is able to converge spectrally (exponentially) fast toward smooth solutions, but at the same time is also able to converge (at a slower speed) toward pathological (discontinuous) solutions [20].

We will see that for this to happen we need to reduce the scope or our investigation and consider a subset of all problems of the form (1.1). Moreover, more refined solutions spaces will be needed for improving speed of convergence.

7.1. **Discontinuous Galerkin methods.** Even though they do converge in almost all situations, it is well-known that piecewise constant difference methods for nonlinear convection equations converge with a rate which is at most one [37] (they are, so to say, slow). How to improve their performance without loosing their ability to retrieve all solutions, even the pathological (discontinuous) ones? One of the most promising method in the literature which is able to do so is the so-called *discontinuous Galerkin method* [22, 23]. In a nutshell, the method secret consists in the use of numerical solutions selected from the space of all possibly discontinuous high-order polynomials. More precisely, let $x_i = i\Delta x$ and $I_i = (x_i, x_{i+1})$, we will work with Legendre polynomials $\{\varphi_{0,i}, \varphi_{1,i}, \ldots, \varphi_{k,i}\}$ of degree at most $k \in \{0, 1, 2, \ldots\}$ and with support on the interval I_i [17, 22]. The method solutions will be linear combinations of such discontinuous polynomials.

Is it possible to generalize such a method to nonlocal equations like (1.1)? The authors show in [17, 18] that this is fully possible whenever the function $A(\cdot)$ is linear and the measure $\mu(\cdot)$ is symmetric. However, such assumptions on both the nonlinearity and the measure are essential for the method to work, and it is not at all clear how they could be further relaxed. A (limited in scope) attempt toward a generalization has been done in [18] where special (simpler) nonlocal nonlinear convection-diffusion equations are considered. However, it seems to be difficult to stretch the results derived therein to treat genuinely nonlinear nonlocal diffusion equations like (1.1).

7.2. **Spectral vanishing viscosity methods.** The discontinuous Galerkin method is not the sole high-order method at our disposal. Indeed, it turns out that the most promising numerical method up to date for nonlocal nonlinear equations like (1.1) is the *spectral vanishing viscosity method* [16, 49, 55, 58]. This method reads

$$\partial_t u_N + \partial_x \cdot P_N f(u_N) = \mathcal{L}^\mu[u_N] + \epsilon_N \sum_{j,k=1}^{d} \partial^2_{jk} Q_N^{j,k} * u_N, \tag{7.1}$$

where the approximate solutions u_N are N-trigonometric polynomials,

$$u_N(x,t) = \sum_{|\xi| \leq N} \hat{u}_\xi(t)\, e^{i\xi\cdot x},$$

P_N is the Fourier projection operator, and $\epsilon_N \sum_{j,k=1}^{d} \partial^2_{jk} Q_N^{j,k} * u_N$ is the so-called spectral diffusion which will opportunely vanish as N increases [20, 16].

As shown in [20], this method performs extremely well for periodic solutions of (1.1) with $A(\cdot)$ linear and general Lévy measure $\mu(\cdot)$. The main reason behind such a good showing is the fact that this method manages to diagonalize the nonlocal operator (1.2). Indeed, the nonlocal operator can be rewritten as

$$\mathcal{L}^\mu[u_N] = \sum_{|\xi| \leq N} G^\mu(\xi)\, \hat{u}_\xi(t)\, e^{i\xi\cdot x}$$

with weights

$$G^\mu(\xi) = \int_{|z|>0} e^{i\xi\cdot z} - 1 - i\xi \cdot z\, \mathbf{1}_{|z|<1}\, \mathrm{d}\mu(z).$$

Loosely speaking, this means that method (7.1) will reduce to a system of ordinary differential equations (for the numerical solution's coefficients $\hat{u}_\xi$) deprived of the full matrices which are common to nearly all nonlocal numerical approximations. This feature, of course, incredibly speeds up computational time.

7.3. **A possible direction for future research.** As for the discontinuous Galerkin approximations in [17, 18], it seems to be very difficult to establish convergence for the spectral vanishing viscosity method in [20] whenever genuinely nonlinear nonlocal operators are considered; furthermore, genuinely nonlinear nonlocal operators get no longer diagonalized by such a method. It is by no means clear how such difficulties could be overcome, however this is surely the most promising direction of investigation for future researchers interested in our work.

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Paper I

The discontinuous Galerkin method for fractal conservation laws

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Paper II

The discontinuous Galerkin method for fractional degenerate convection-diffusion equations

S. Cifani, E. R. Jakobsen, and K. H. Karlsen

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THE DISCONTINUOUS GALERKIN METHOD FOR FRACTIONAL DEGENERATE CONVECTION-DIFFUSION EQUATIONS

SIMONE CIFANI, ESPEN R. JAKOBSEN, AND KENNETH H. KARLSEN

Abstract. We propose and study discontinuous Galerkin methods for strongly degenerate convection-diffusion equations perturbed by a fractional diffusion (Lévy) operator. We prove various stability estimates along with convergence results toward properly defined (entropy) solutions of linear and nonlinear equations. Finally, the qualitative behavior of solutions of such equations are illustrated through numerical experiments.

1. Introduction

We consider degenerate convection-diffusion equations perturbed by a fractional diffusion (Lévy) operator; more precisely, problems of the form

$$\begin{cases} u_t + f(u)_x = (a(u)u_x)_x + b\mathcal{L}[u] & (x,t) \in Q_T = \mathbb{R} \times (0,T), \\ u(x,0) = u_0(x) & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $f, a : \mathbb{R} \to \mathbb{R}$ ($a \geq 0$ and bounded) are Lipschitz continuous functions, $b \geq 0$ is a constant, and $\mathcal{L}$ is a nonlocal operator whose singular integral representation reads (cf. [27, 12])

$$\mathcal{L}[u(x,t)] = c_\lambda \int_{|z|>0} \frac{u(x+z,t) - u(x,t)}{|z|^{1+\lambda}}\, dz, \ \ \lambda \in (0,1) \text{ and } c_\lambda > 0.$$

For sake of simplicity, we assume $f(0) = 0$. The initial datum $u_0 : \mathbb{R} \to \mathbb{R}$ is chosen in different spaces (cf. Theorems 4.2, 4.4 and 5.8) depending on whether the equations are linear or nonlinear.

The operator $\mathcal{L}$ is known as the fractional Laplacian (a nonlocal generalization of the Laplace operator) and can also be defined in terms of its Fourier transform:

$$\widehat{\mathcal{L}[u(\cdot,t)]}(\xi) = -|\xi|^\lambda \hat{u}(\xi,t). \tag{1.2}$$

As pointed out in [2, 12, 27], $u(\cdot,t)$ has to be rather smooth with suitable growth at infinity for the quantity $\mathcal{L}[u]$ to be pointwise well defined. However, smooth solutions of (1.1) do not exist in general (shocks may develop), and weak entropy solutions have to be considered, cf. Definition 5.1 and Lemma A.1 below.

Nonlocal equations like (1.1) appear in different areas of research. For instance, in mathematical finance, option pricing models based on jump processes (cf. [8]) give rise to linear partial differential equations with nonlocal terms. Nonlinear equations appear in dislocation dynamics, hydrodynamics and molecular biology

Key words and phrases. Convection-diffusion equations, degenerate parabolic, conservation laws, fractional diffusion, entropy solutions, direct/local discontinuous Galerkin methods.

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[13]; applications to semiconductors devices and explosives can also be found [29]. For more information about the possible applications of such equations we refer the reader to the detailed discussions in [2], [3], and [11].

Equation (1.1) consists of three different terms: nonlinear convection $f(u)_x$, nonlinear diffusion $(a(u)u_x)_x$, and fractional diffusion $\mathcal{L}[u]$. It is expected that the effect of a diffusion operator is that solutions become smoother than the prescribed initial data. In our case, however, a can be strongly degenerate (i.e., vanish on intervals of positive length), and hence solutions can exhibit shocks. We refer to [14, 13] for the case when $b = 0$, and to [3, 5] for the case when $\lambda \in (0, 1)$ and $a \equiv 0$. The issue at stake here is that the fractional diffusion operator may not be strong enough to prevent solutions of (1.1) from developing discontinuities. However, and as expected, in the linear case ($f(u) = cu$, $a(u) = au$ with $c \in \mathbb{R}$, $a > 0$), some regularity can be proved (cf. Lemma 4.1).

An ample literature is available on numerical methods for computing entropy solutions of degenerate convection-diffusion equations, cf. [7, 13, 14, 15, 17, 18, 23, 24, 20]. To the best of our knowledge, there are no works on nonlocal versions of these equations. However, for the special case of fractional conservation laws ($a \equiv 0$) there are a few recent works [10, 11, 5]. Dedner and Rohde [10] introduced a general class of difference methods for equations appearing in radiative hydrodynamics. Droniou [11] devised a classs difference method for (1.1) ($a = 0$) and proved convergence. Cifani *et al.* [5] applied the discontinuous Galerkin method to (1.1) ($a = 0$) and proved error estimates. Finally, let us mention that the discontinuous Galerkin method has also been used to numerically solve nonlinear convection problems appended with possibly nonlocal dissipative terms in [21, 22].

The discontinuous Galerkin (DG hereafter) method is a well established method for approximating solutions of convection [6] and convection-diffusion equations [7, 20]. To obtain a DG approximation of a nonlinear equation, one has to pass to the weak formulation, do integration by parts, and replace the nonlinearities with suitable numerical fluxes (fluxes which enforce numerical stability and convergence). Available DG methods for convection-diffusion equations are the local DG (LDG hereafter) [7] and the direct DG (DDG hereafter) [20]. In the LDG method, the convection-diffusion equation is rewritten as a first order system and then approximated by the DG method for conservation laws. In the DDG method, the DG method is applied directly to the convection-diffusion equation after a suitable numerical flux has been derived for the diffusion term.

This paper is a continuation of our previous work on DG methods for fractional conservation laws [5]. We devise and study DDG and LDG approximations of (1.1), we prove that both approximations are L^2-stable and, whenever linear equations are considered, high-order accurate. In the nonlinear case, we work with an entropy formulation for (1.1) which generalizes the one in [30, 14], and we show that the DDG method converges toward an entropy solution when piecewise constant elements are used. To do so, we extend the results in [14] to our nonlocal setting. Finally, we present numerical experiments shedding some light on the qualitative behavior of solutions of fractional, strongly degenerate convection-diffusion equations.

2. A semi-discrete method

Let us choose a spatial grid $x_i = i\Delta x$ ($\Delta x > 0$, $i \in \mathbb{Z}$), and label $I_i = (x_i, x_{i+1})$. We denote by $P^k(I_i)$ the space of all polynomials of degree at most k with support on I_i, and let

$$V^k = \{v : v|_{I_i} \in P^k(I_i),\ i \in \mathbb{Z}\}.$$

Let us introduce the Legendre polynomials $\{\varphi_{0,i}, \varphi_{1,i}, \dots, \varphi_{k,i}\}$, where $\varphi_{j,i} \in P^j(I_i)$. Each function in $P^k(I_i)$ can be written as a linear combination of these polynomials.

We recall the following well known properties of the Legendre polynomials: for all $i \in \mathbb{Z}$,

$$\int_{I_i} \varphi_{p,i}\varphi_{q,i}\,dx = \begin{cases} \frac{\Delta x}{2q+1} & \text{for } p = q \\ 0 & \text{otherwise} \end{cases}, \; \varphi_{p,i}(x_{i+1}^-) = 1 \text{ and } \varphi_{p,i}(x_i^+) = (-1)^p,$$

where $\varphi(x_i^\pm) = \lim_{s\to x_i^\pm} \varphi(s)$.

The following fractional Sobolev space is also needed in what follows (see, e.g., [1] or [16, Section 6]):

$$\|u\|^2_{H^{\lambda/2}(\mathbb{R})} = \|u\|^2_{L^2(\mathbb{R})} + |u|^2_{H^{\lambda/2}(\mathbb{R})},$$

with semi-norm $|u|^2_{H^{\lambda/2}(\mathbb{R})} = \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(u(z)-u(x))^2}{|z-x|^{1+\lambda}}\,dz\,dx$. Finally, let us introduce the operators

$$[p(x_i)] = p(x_i^+) - p(x_i^-), \quad \overline{p(x_i)} = \frac{1}{2}(p(x_i^+) + p(x_i^-)).$$

From now on we split our exposition into two parts, one dedicated to the DDG method and another one dedicated to the LDG method.

2.1. **DDG method.** Let us multiply (1.1) by an arbitrary $v \in P^k(I_i)$, integrate over I_i, and use integration by parts, to arrive at

$$\begin{aligned} \int_{I_i} u_t v - \int_{I_i} f(u)v_x + f(u_{i+1})v_{i+1}^- - f(u_i)v_i^+ \\ + \int_{I_i} a(u)u_x v_x - h(u_{i+1}, u_{x,i+1})v_{i+1}^- + h(u_i, u_{x,i})v_i^+ = b\int_{I_i} \mathcal{L}[u]v, \end{aligned} \tag{2.1}$$

where $f(u_i) = f(u(x_i))$, $h(u, u_x) = a(u)u_x$ and $(u_i, u_{x,i}) = (u(x_i), u_x(x_i))$. Let us introduce the Lipschitz continuous E-flux (a consistent and monotone flux),

$$\hat{f}(u_i) = \hat{f}(u(x_i^-), u(x_i^+)). \tag{2.2}$$

Note that since $\hat{f}$ is consistent ($\hat{f}(u,u) = f(u)$) and monotone (increasing w.r.t. its first variable and decreasing w.r.t its second variable),

$$\int_{u_i^-}^{u_i^+} \left[f(x) - \hat{f}(u_i^-, u_i^+)\right] dx \geq 0. \tag{2.3}$$

Following Jue and Liu [20], let us also introduce the flux

$$\begin{aligned} \hat{h}(u_i) &= \hat{h}(u(x_i^-), \dots, \partial_x^k u(x_i^-), u(x_i^+), \dots, \partial_x^k u(x_i^+)) \\ &= \beta_0 \frac{[A(u_i)]}{\Delta x} + \overline{A(u_i)_x} + \sum_{m=1}^{\lfloor k/2 \rfloor} \beta_m \Delta x^{2m-1}[\partial_x^{2m} A(u_i)], \end{aligned}$$

where $A(u) = \int^u a$ and the weights $\{\beta_0, \dots, \beta_{\lfloor k/2\rfloor}\}$ fulfill the following admissibility condition: there exist $\gamma \in (0,1)$ and $\alpha \geq 0$ such that

$$\sum_{i\in\mathbb{Z}} \hat{h}(u_i)[u_i] \geq \alpha \sum_{i\in\mathbb{Z}} \frac{[A(u_i)]}{\Delta x}[u_i] - \gamma \sum_{i\in\mathbb{Z}} \int_{I_i} a(u)(u_x)^2. \tag{2.4}$$

Note that the numerical flux $\hat{h}$ is an approximation of $A(u_i)_x = a(u(x_i))u_x(x_i)$ involving the average $\overline{A(u_i)_x}$ and the jumps of even order derivatives of $A(u_i)$ up to $m = k/2$. For example, if $k = 0$ and $\beta_0 = 1$, then

$$\hat{h}(u_i) = \frac{1}{\Delta x}[A(u_i)] = \frac{A(u(x_i^+)) - A(u(x_i^-))}{\Delta x},$$

and this function satisfies condition (2.4). In this case ($k = 0$),

$$\overline{A(u_i)_x} = \overline{a(u(x_i))\partial_x u(x_i)} = \frac{1}{2}\Big(a(u(x_i^+))\partial_x u(x_i^+) + a(u(x_i^-))\partial_x u(x_i^-)\Big) = 0.$$

When $k \geq 2$, some extra differentiability on a is required. For example, with $k = 2$,

$$\sum_{m=1}^{\lfloor k/2 \rfloor} \beta_m \Delta x^{2m-1}[\partial_x^{2m} A(u_i)] = \beta_1 \Delta x[\partial_x^2 A(u_i)] = \beta_1 \Delta x[a'(u_i)(\partial_x u_i)^2 + a(u_i)\partial_x^2 u_i].$$

We see that the flux $\hat{h}$ is locally Lipschitz if a is sufficently regular, and that $\hat{h}(0) = 0$ for all k. Let us rewrite (2.1) as

$$\begin{aligned}\int_{I_i} u_t v - \int_{I_i} f(u)v_x + \hat{f}(u_{i+1})v_{i+1}^- - \hat{f}(u_i)v_i^+ \\ + \int_{I_i} a(u)u_x v_x - \hat{h}(u_{i+1})v_{i+1}^- + \hat{h}(u_i)v_i^+ = b\int_{I_i} \mathcal{L}[u]v,\end{aligned} \tag{2.5}$$

and use the initial condition

$$\int_{I_i} u(x,0)v(x)\,dx = \int_{I_i} u_0(x)v(x)\,dx. \tag{2.6}$$

The DDG method consists of finding functions $\hat{u} : Q_T \to \mathbb{R}$, $\hat{u}(\cdot,t) \in V^k$, and

$$\hat{u}(x,t) = \sum_{i\in\mathbb{Z}} \sum_{p=0}^{k} U_{p,i}(t)\varphi_{p,i}(x), \tag{2.7}$$

which satisfy (2.5)-(2.6) for all $v \in P^k(I_i)$, $i \in \mathbb{Z}$.

2.2. **LDG method.** Let us write $a(u)u_x = \sqrt{a(u)}g(u)_x$, where $g(u) = \int^u \sqrt{a}$, and turn equation (1.1) into the following system of equations

$$\begin{cases} u_t + (f(u) - \sqrt{a(u)}q)_x = b\mathcal{L}[u], \\ q - g(u)_x = 0. \end{cases} \tag{2.8}$$

Let us introduce the notation $\mathbf{w} = (u,q)'$ (here $'$ denotes the transpose), and write

$$\mathbf{h}(\mathbf{w}) = \mathbf{h}(u,q) = \begin{pmatrix} h_u(\mathbf{w}) \\ h_q(u) \end{pmatrix} = \begin{pmatrix} f(u) - \sqrt{a(u)}q \\ -g(u) \end{pmatrix}.$$

Let us multiply each equation in (2.8) by arbitrary $v_u, v_q \in P^k(I_i)$, integrate over the interval I_i, and use integration by parts, to arrive at

$$\begin{aligned}&\int_{I_i} \partial_t u v_u - \int_{I_i} h_u(\mathbf{w})\partial_x v_u + h_u(\mathbf{w}_{i+1})v_{u,i+1}^- - h_u(\mathbf{w}_i)v_{u,i}^+ = b\int_{I_i} \mathcal{L}[u]v_u,\\ &\int_{I_i} q v_q - \int_{I_i} h_q(u)\partial_x v_q + h_q(u_{i+1})v_{q,i+1}^- - h_q(u_i)v_{q,i}^+ = 0,\end{aligned}$$

where $h_u(\mathbf{w}_i) = h_u(u_i,q_i)$, $u_i = u(x_i)$, $q_i = q(x_i)$, $v_{u,i}^- = v_u(x_i^-)$ and $v_{u,i}^+ = v_u(x_i^+)$. Following Cockburn and Shu [7], we introduce the numerical flux

$$\hat{\mathbf{h}}(\mathbf{w}_i^-, \mathbf{w}_i^+) = \begin{pmatrix} \hat{h}_u(\mathbf{w}_i^-, \mathbf{w}_i^+) \\ \hat{h}_q(u_i^-, u_i^+) \end{pmatrix} = \begin{pmatrix} \frac{[F(u_i)]}{[u_i]} - \frac{[g(u_i)]}{[u_i]}\overline{q_i} \\ -\overline{g(u_i)} \end{pmatrix} - \mathbb{C}[\mathbf{w}_i], \tag{2.9}$$

where $F(u) = \int^u f$, $\mathbb{C} = \begin{pmatrix} c_{11} & c_{12} \\ -c_{12} & 0 \end{pmatrix}$,

$$c_{11} = \frac{1}{[u_i]}\left(\frac{[F(u_i)]}{[u_i]} - \hat{f}(u_i^-, u_i^+)\right),$$

$c_{12} = c_{12}(\mathbf{w}_i^-, \mathbf{w}_i^+)$ is Lipschitz continuous in all its variables, and $c_{12} = 0$ whenever $a = 0$ or $\mathbf{w}_i^-, \mathbf{w}_i^+ = 0$. Note that $c_{11} \geq 0$ since $\hat{f}$ is an E-flux and, thus, the matrix $\mathbb{C}$ is semipositive definite.

The LDG method consists of finding $\tilde{\mathbf{w}} = (\tilde{u}, \tilde{q})'$, where

$$\tilde{u}(x,t) = \sum_{i\in\mathbb{Z}} \sum_{p=0}^{k} U_{p,i}(t)\varphi_{p,i}(x) \quad \text{and} \quad \tilde{q}(x,t) = \sum_{i\in\mathbb{Z}} \sum_{p=0}^{k} Q_{p,i}(t)\varphi_{p,i}(x)$$

are functions satisfying

$$\begin{aligned} &\int_{I_i} \partial_t u v_u - \int_{I_i} h_u(\mathbf{w})\partial_x v_u + \hat{h}_u(\mathbf{w}_{i+1})v_{u,i+1}^- - \hat{h}_u(\mathbf{w}_i)v_{u,i}^+ = b\int_{I_i} \mathcal{L}[u]v_u, \\ &\int_{I_i} q v_q - \int_{I_i} h_q(u)\partial_x v_q + \hat{h}_q(u_{i+1})v_{q,i+1}^- - \hat{h}_q(u_i)v_{q,i}^+ = 0, \end{aligned} \tag{2.10}$$

for all $v_u, v_q \in P^k(I_i)$, $i \in \mathbb{Z}$, and initial conditions for u and q given by (2.6).

3. L^2-STABILITY FOR NONLINEAR EQUATIONS

We will show that in the semidiscrete case (no time discretization) both the DDG and LDG methods are L^2-stable, for linear and nonlinear equations.

In this section and the subsequent one, we assume the existence of solutions $\hat{u}$ and $\tilde{\mathbf{w}} = (\tilde{u}, \tilde{q})'$ of the DDG and LDG methods (2.5) and (2.10), respectively, satisfying $\hat{u}, \tilde{u}, \tilde{q} \in C^1([0,T]; V^k \cap L^2(\mathbb{R}))$, in which case the integrals containing the nonlocal operator $\mathcal{L}[\cdot]$ are all well defined. Indeed, by Lemma A.3, $V^k \cap L^2(\mathbb{R}) \subseteq H^{\lambda/2}(\mathbb{R})$, and hence all integrals of the form

$$\int_{\mathbb{R}} \varphi_1 \, \mathcal{L}[\varphi_2] \quad \text{for} \quad \varphi_1, \varphi_2 \in V^k \cap L^2(\mathbb{R}),$$

can be interpreted as the pairing between $\varphi_1 \in H^{\lambda/2}(\mathbb{R})$ and $\mathcal{L}[\varphi_2] \in H^{-\lambda/2}(\mathbb{R})$. Here $H^{-\lambda/2}(\mathbb{R})$ is the dual space of $H^{\lambda/2}(\mathbb{R})$, and $\mathcal{L}[\varphi] \in H^{-\lambda/2}(\mathbb{R})$ whenever $\varphi \in H^{\lambda/2}(\mathbb{R})$ (cf. Corollary A.3 and proof in [5]).

Remark 3.1. The existence and uniqueness of solutions in $C^1([0,T]; V^k \cap L^2(\mathbb{R}))$ can be proved using the Picard-Cauchy-Lipschitz theorem. The argument outlined in [5, Section 3], can be adapted to the current setting since all numerical fluxes are (locally) Lipschitz (cf. [7] for the LDG case). For the DDG method with $k > 2$, additional differentiability on a is needed for this proof to work.

3.1. **DDG method.** Let us sum over all $i \in \mathbb{Z}$ in (2.5), integrate over $t \in (0,T)$, and introduce the functional

$$\begin{aligned} M_{DDG}[u,v] = &\int_0^T \int_{\mathbb{R}} u_t v - \int_0^T \sum_{i\in\mathbb{Z}} \left[\hat{f}(u_i)[v_i] + \int_{I_i} f(u)v_x\right] \\ &+ \int_0^T \sum_{i\in\mathbb{Z}} \left[\hat{h}(u_i)[v_i] + \int_{I_i} a(u)u_x v_x\right] - b\int_0^T \int_{\mathbb{R}} \mathcal{L}[u]v. \end{aligned} \tag{3.1}$$

Let us define

$$\Gamma_T[u] = (1-\gamma)\int_0^T \sum_{i\in\mathbb{Z}} \int_{I_i} a(u)(u_x)^2 + \alpha \int_0^T \sum_{i\in\mathbb{Z}} \frac{[A(u_i)]}{\Delta x}[u_i],$$

where $\gamma \in (0,1)$ and $\alpha > 0$. Note that $\Gamma_T \geq 0$ since $a \geq 0$ and, using the Taylor's formula, $[A(u_i)][u_i] = a(\xi_i)[u_i]^2 \geq 0$ where and $\xi_i \in [u(x_i^-), u(x_i^+)]$, $i \in \mathbb{Z}$.

Theorem 3.2. (Stability) *Let $\hat{u}$ be a solution of (2.5) such that both $\hat{u}, A(\hat{u})$ and their first k derivatives are sufficiently integrable. Then*

$$\|\hat{u}(\cdot,T)\|^2_{L^2(\mathbb{R})} + 2\Gamma_T[\hat{u}] + bc_\lambda \int_0^T |\hat{u}(\cdot,t)|^2_{H^{\lambda/2}(\mathbb{R})}\,dt \le \|u_0\|^2_{L^2(\mathbb{R})}.$$

Remark 3.3. Since $\tilde{u} \in C^1([0,T]; V^k \cap L^2(\mathbb{R}))$ and $f(0) = 0$, all terms in (3.2) below are well defined – except for

$$\int_0^T \Big[\sum_{i\in\mathbb{Z}} \hat{h}(\hat{u}_i)[\hat{u}_i] + \int_{I_i} a(\hat{u})(\hat{u}_x)^2\Big].$$

When $k \ge 2$, additional integrability of $\hat{u}, A(\hat{u})$, and their first k derivatives, is required in order to give meaning to the $\hat{h}$-term.

Proof. By construction, $M_{DDG}[\hat{u}, v] = 0$ for all $v \in V^k \cap L^2(\mathbb{R})$. If we set $v = \hat{u}$, we obtain

$$\begin{aligned}\int_0^T\int_{\mathbb{R}} \hat{u}_t\hat{u} - \int_0^T \sum_{i\in\mathbb{Z}} \Big[\hat{f}(\hat{u}_i)[\hat{u}_i] + \int_{I_i} f(\hat{u})\hat{u}_x\Big]&\\ + \int_0^T \sum_{i\in\mathbb{Z}} \Big[\hat{h}(\hat{u}_i)[\hat{u}_i] + \int_{I_i} a(\hat{u})(\hat{u}_x)^2\Big] - b\int_0^T\int_{\mathbb{R}} \mathcal{L}[\hat{u}]\hat{u} = 0.&\end{aligned} \tag{3.2}$$

Next, as a direct consequence of (2.3) and a change of variables, we see that

$$\int_0^T \sum_{i\in\mathbb{Z}} \Big[\hat{f}(\hat{u}_i)[\hat{u}_i] + \int_{I_i} f(\hat{u})\hat{u}_x\Big] \le 0. \tag{3.3}$$

Since $\hat{h}$ satisfies the expression (2.4),

$$\int_0^T \sum_{i\in\mathbb{Z}} \hat{h}_i(\hat{u}_i)[\hat{u}_i] \ge \alpha\int_0^T \sum_{i\in\mathbb{Z}} \frac{[A(\hat{u}_i)]}{\Delta x}[\hat{u}_i] - \gamma\int_0^T\sum_{i\in\mathbb{Z}}\int_{I_i} a(\hat{u})(\hat{u}_x)^2. \tag{3.4}$$

Finally, using Lemma A.1,

$$\int_{\mathbb{R}} \mathcal{L}[\hat{u}]\hat{u} = -\frac{c_\lambda}{2}|\hat{u}|^2_{H^{\lambda/2}(\mathbb{R})}. \tag{3.5}$$

We conclude by inserting (3.3), (3.4), and (3.5) into (3.2). □

3.2. **LDG method.** By summing over all $i \in \mathbb{Z}$, we can rewrite (2.10) as

$$\int_{\mathbb{R}} \partial_t u v_u - \sum_{i\in\mathbb{Z}}\Big(\hat{h}_u(\mathbf{w}_i)[v_{u,i}] + \int_{I_i} h_u(\mathbf{w})\partial_x v_u\Big) = b\int_{\mathbb{R}}\mathcal{L}[u]v_u,$$

$$\int_{\mathbb{R}} qv_q - \sum_{i\in\mathbb{Z}}\Big(\hat{h}_q(u_i)[v_{q,i}] + \int_{I_i} h_q(u)\partial_x v_q\Big) = 0.$$

We add the two equations and integrate over $t \in (0,T)$ to find $M_{LDG}[\mathbf{w},\mathbf{v}] = 0$ for

$$\begin{aligned} M_{LDG}[\mathbf{w},\mathbf{v}] &= \int_0^T\int_{\mathbb{R}} u_t v_u + \int_0^T\int_{\mathbb{R}} qv_q\\ &\quad - \int_0^T\sum_{i\in\mathbb{Z}}\Big(\hat{\mathbf{h}}(\mathbf{w}_i)'[\mathbf{v}_i] + \int_{I_i}\mathbf{h}(\mathbf{w})'\partial_x\mathbf{v}\Big) - b\int_0^T\int_{\mathbb{R}}\mathcal{L}[u]v_u,\end{aligned} \tag{3.6}$$

where $\hat{\mathbf{h}}(\mathbf{w}_i) = (\hat{h}_u(\mathbf{w}_i), \hat{h}_q(u_i))'$, $\mathbf{v} = (v_u, v_q)'$ and $\mathbf{v}_i = (v_{u,i}, v_{q,i})'$. Moreover, let (remember that, as noted earlier, the matrix $\mathbb{C}$ is semipositive definite)

$$\Theta_T[\mathbf{w}] = \int_0^T \sum_{i\in\mathbb{Z}} [\mathbf{w}_i]'\mathbb{C}[\mathbf{w}_i]\ (\ge 0).$$

Theorem 3.4. (Stability) *If* $\tilde{\mathbf{w}} = (\tilde{u}, \tilde{q})'$ *is a* $C^1([0,T]; (V^k \cap L^2)^2)$ *solution of* (2.10), *then*

$$\|\tilde{u}(\cdot,T)\|^2_{L^2(\mathbb{R})} + 2\|\tilde{q}\|^2_{L^2(Q_T)} + 2\Theta_T(\tilde{\mathbf{w}}) + bc_\lambda \int_0^T |\tilde{u}(\cdot,t)|^2_{H^{\lambda/2}(\mathbb{R})}\,dt \le \|u_0\|^2_{L^2(\mathbb{R})}.$$

Here, as opposed to Theorem 3.2, no further integrability of the first k derivatives of the numerical solution $\tilde{\mathbf{w}} = (\tilde{u}, \tilde{q})'$ is needed. The reason is that the numerical flux $\hat{\mathbf{h}}$ has been built without the use of derivatives of $\tilde{\mathbf{w}} = (\tilde{u}, \tilde{q})'$. Each term in expression (3.7) below is well defined thanks to (3.8), the fact that $f(0) = 0$ (which implies that $c_{11}(0) = 0$), $c_{12}(0) = 0$, and $\tilde{u}, \tilde{q} \in C^1([0,T]; V^k \cap L^2(\mathbb{R}))$.

Proof. By construction, $M_{LDG}(\hat{\mathbf{w}}, \mathbf{v}) = 0$ for all $\mathbf{v} = (v_u, v_q)'$, $v_u, v_q \in V^k \cap L^2(\mathbb{R})$. We set $\mathbf{v} = \hat{\mathbf{w}}$ and find that

$$\begin{aligned}\int_0^T\int_{\mathbb{R}} \tilde{u}_t\tilde{u} + \int_0^T\int_{\mathbb{R}} \tilde{q}^2 - \int_0^T \sum_{i\in\mathbb{Z}} \left(\hat{\mathbf{h}}(\tilde{\mathbf{w}}_i)'[\tilde{\mathbf{w}}_i] + \int_{I_i} \mathbf{h}(\tilde{\mathbf{w}})'\partial_x\tilde{\mathbf{w}}\right)&\\ -b\int_0^T\int_{\mathbb{R}} \mathcal{L}[\tilde{u}]\tilde{u} = 0.&\end{aligned} \tag{3.7}$$

Here we also used the fact that

$$-\int_0^T \sum_{i\in\mathbb{Z}} \left(\hat{\mathbf{h}}(\tilde{\mathbf{w}}_i)'[\tilde{\mathbf{w}}_i] + \int_{I_i} \mathbf{h}(\tilde{\mathbf{w}})'\partial_x\tilde{\mathbf{w}}\right) = \int_0^T \sum_{i\in\mathbb{Z}} [\tilde{\mathbf{w}}_i]\mathbb{C}[\tilde{\mathbf{w}}_i], \tag{3.8}$$

see [7] for a proof. To conclude, insert (3.8) and (3.5) into (3.7). □

4. High-order convergence for linear equations

In this section we consider the linear problem

$$\begin{cases} u_t + cu_x = u_{xx} + b\mathcal{L}[u] & (x,t) \in Q_T, \\ u(x,0) = u_0(x) & x \in \mathbb{R}, \end{cases} \tag{4.1}$$

with the aim of proving that the DDG and LDG methods converge to a regular solution of (4.1) with high-order accuracy.

Lemma 4.1. *Let* $u_0 \in H^{k+1}(\mathbb{R})$, *with* $k \ge 0$. *There exists a unique function* $u \in H^{k+1}_{\text{par}}(Q_T)$ *solving* (4.1), *where*

$$H^{k+1}_{\text{par}}(Q_T) := \left\{\phi \in L^2(Q_T) : \|\partial_t^m \partial_x^r u\|_{L^2(Q_T)} < \infty \text{ for all } 0 \le r + 2m \le k+1\right\}.$$

Moreover, $\|u(\cdot,t)\|_{H^{k+1}(\mathbb{R})} \le \|u_0\|_{H^{k+1}(\mathbb{R})}$.

Proof. Since the equation is linear, we can pass to the Fourier space. In view of (1.2), the Fourier transform of (4.1) is $\hat{u}_t + i\xi c\hat{u} = -\xi^2\hat{u} - b|\xi|^\lambda\hat{u}$. It follows that

$$\hat{u}(\xi,t) = \hat{u}_0(\xi)e^{-(i\xi c + \xi^2 + b|\xi|^\lambda)t}.$$

By the properties of the Fourier transform, the above expression implies the existence of a unique L^2-stable weak solution of (4.1). The L^2-stability for higher derivatives can be obtained by iteration as follows: take the derivative of (4.1), use the Fourier transform to get stability, and iterate up to the kth derivative. Regularity in time follows from the regularity in space since equation (4.1) implies that $\partial_t^k u = (-c\partial_x + \partial_x^2 + b\mathcal{L})^k u$. □

In the following two theorems we obtain L^2-type error estimates for the DDG and LDG methods in the case that equation (4.1) has H^{k+1}_{par}-regular solutions. (Note that the time regularity does not play any role here). To do so, we combine estimates for the local terms derived in [7, 20] with estimates for the nonlocal term derived by the authors in [5]. In [6] it was observed that most relevant numerical $\hat{f}$ fluxes reduce to

$$\hat{f}(u_i^-, u_i^+) = c\overline{u_i} - |c|\frac{[u_i]}{2}$$

in the linear case. In this section we only consider this $\hat{f}$ flux.

4.1. DDG method.

Theorem 4.2. (Convergence) *Let $u \in H^{k+1}_{\text{par}}(Q_T)$, $k \geq 0$, be a solution of* (4.1) *and $\hat{u} \in C^1([0,T]; V^k \cap L^2(\mathbb{R}))$ be a solution of* (2.5). *With $e = u - \hat{u}$,*

$$\int_{\mathbb{R}} e^2(x,T) + \frac{|c|}{2}\int_0^T \sum_{i\in\mathbb{Z}} [e_i]^2 + (1-\gamma)\int_0^T \int_{\mathbb{R}} (e_x)^2 + \alpha \int_0^T \sum_{i\in\mathbb{Z}} \frac{[e_i]^2}{\Delta x}$$
$$+ bc_\lambda \int_0^T |e|^2_{H^{\lambda/2}(\mathbb{R})} = \mathcal{O}(1)\Delta x^{2k}.$$

Remark 4.3. The error $\mathcal{O}(1)\Delta x^{2k}$ is due to the diffusion term u_{xx}. The errors from the convection term cu_x and the fractional diffusion term $b\mathcal{L}[u]$ are of the form $\mathcal{O}(1)\Delta x^{2k+1}$ and $\mathcal{O}(1)\Delta x^{2k+2-\lambda}$ respectively.

Proof. Let us set

$$M_a[u,v] = \int_0^T \int_{\mathbb{R}} u_t v + \int_0^T \int_{\mathbb{R}} u_x v_x + \int_0^T \sum_{i\in\mathbb{Z}} \hat{h}(u_i)[v_i],$$
$$M_f[u,v] = -\int_0^T \sum_{i\in\mathbb{Z}} \Big[\hat{f}(u_i)[v_i] + \int_{I_i} cuv_x\Big],$$
$$M_{\mathcal{L}}[u,v] = -b\int_0^T \int_{\mathbb{R}} \mathcal{L}[u]v.$$

With this notation in hand, we can write (3.1) as

$$M_{DDG}[u,v] = M_a[u,v] + M_f[u,v] + M_{\mathcal{L}}[u,v].$$

Let $\mathbb{P}e$ be the L^2-projection of e into V^k, i.e., $\mathbb{P}e$ is the $V^k \cap L^2(\mathbb{R})$ function satisfying

$$\int_{I_i} \big(\mathbb{P}e(x) - e(x)\big)\varphi_{ji}(x)\,dx = 0 \text{ for all } i \in \mathbb{Z} \text{ and } j = \{0, \dots, k\}.$$

Note that $\mathbb{P}e \in H^{\lambda/2}(\mathbb{R})$ since $V^k \cap L^2(\mathbb{R}) \subset H^{\lambda/2}(\mathbb{R})$ by Lemma A.3. For all $v \in V^k \cap L^2(\mathbb{R})$, we have $M_{DDG}[\hat{u}, v] = 0$ since $\hat{u}$ is a DDG solution of (4.1), while $M_{DDG}[u,v] = 0$ since u is a continuous (by Sobolev imbedding) solution of (1.1) and hence a solution of (4.1). Thus $M_{DDG}[e,v] = 0$, and by bilinearity ($\hat{h}$ is linear since $a \equiv 1$),

$$M_{DDG}[\mathbb{P}e, \mathbb{P}e] = M_{DDG}[\mathbb{P}e - e, \mathbb{P}e]. \tag{4.2}$$

One can proceed as in [20] (in that paper, combine the last inequality of the proof of Lemma 3.3 with Lemma 3.2 and (3.5)) to obtain

$$M_a[\mathbb{P}e - e, \mathbb{P}e] = \frac{1}{2}\int_0^T \int_{\mathbb{R}} (\mathbb{P}e_x)^2 + \frac{1}{2}\int_0^T \sum_{i\in\mathbb{Z}} \hat{h}(\mathbb{P}e_i)[\mathbb{P}e_i] + \mathcal{O}(1)\Delta x^{2k}. \tag{4.3}$$

Moreover, proceeding as in [6, Lemma 2.17],

$$M_f[\mathbb{P}e - e, \mathbb{P}e] = \frac{|c|}{4}\int_0^T \sum_{i\in\mathbb{Z}} [\mathbb{P}e_i]^2 + \mathcal{O}(1)\Delta x^{2k+1}. \tag{4.4}$$

As shown by the authors in [5],

$$
\begin{aligned}
&M_{\mathcal{L}}[\mathbb{P}e-e,\mathbb{P}e]-M_{\mathcal{L}}[\mathbb{P}e,\mathbb{P}e]=b\int_0^T\int_{\mathbb{R}}\mathcal{L}[e]\mathbb{P}e\\
&=\frac{b}{2}\int_0^T\int_{\mathbb{R}}\mathcal{L}[\mathbb{P}e]\mathbb{P}e+\frac{b}{2}\int_0^T\int_{\mathbb{R}}\mathcal{L}[e]e-\frac{b}{2}\int_0^T\int_{\mathbb{R}}\mathcal{L}[e-\mathbb{P}e](e-\mathbb{P}e)\\
&\leq-\frac{bc_\lambda}{4}\int_0^T|\mathbb{P}e|^2_{H^{\lambda/2}(\mathbb{R})}-\frac{bc_\lambda}{4}\int_0^T|e|^2_{H^{\lambda/2}(\mathbb{R})}+\frac{bc_\lambda}{4}\int_0^T\|e-\mathbb{P}e\|^2_{H^{\lambda/2}(\mathbb{R})},
\end{aligned}
\tag{4.5}
$$

where $M_{\mathcal{L}}[\mathbb{P}e,\mathbb{P}e]=\frac{bc_\lambda}{2}\int_0^T|\mathbb{P}e|^2_{H^{\lambda/2}(\mathbb{R})}$ (Lemma A.1) and

$$
\|e-\mathbb{P}e\|^2_{H^{\lambda/2}(\mathbb{R})}\leq\mathcal{O}(1)\Delta x^{2k+2-\lambda}.
\tag{4.6}
$$

By (3.1), Lemma A.1, and the definition of $\hat{f}$,

$$
\begin{aligned}
M_{DDG}[\mathbb{P}e,\mathbb{P}e]=&\int_{\mathbb{R}}(\mathbb{P}e^2)_t+\frac{|c|}{2}\int_0^T\sum_{i\in\mathbb{Z}}[\mathbb{P}e_i]^2+\int_0^T\int_{\mathbb{R}}(\mathbb{P}e_x)^2\\
&+\int_0^T\sum_{i\in\mathbb{Z}}\hat{h}(\mathbb{P}e_i)[\mathbb{P}e_i]+\frac{bc_\lambda}{2}\int_0^T|\mathbb{P}e|^2_{H^{\lambda/2}(\mathbb{R})}.
\end{aligned}
$$

Inserting this equation along with (4.3), (4.4), and (4.5) into (4.2) then shows that

$$
\begin{aligned}
\int_0^T\int_{\mathbb{R}}(\mathbb{P}e^2)_t+\frac{|c|}{4}\int_0^T\sum_{i\in\mathbb{Z}}[\mathbb{P}e_i]^2+\frac{1}{2}\int_0^T\int_{\mathbb{R}}(\mathbb{P}e_x)^2+\frac{1}{2}\int_0^T\sum_{i\in\mathbb{Z}}\hat{h}(\mathbb{P}e_i)[\mathbb{P}e_i]&\\
+\frac{bc_\lambda}{4}\int_0^T|\mathbb{P}e|^2_{H^{\lambda/2}(\mathbb{R})}+\frac{bc_\lambda}{4}\int_0^T|e|^2_{H^{\lambda/2}(\mathbb{R})}=\mathcal{O}(1)\Delta x^{2k},&
\end{aligned}
$$

and, using the admissibility condition (2.4),

$$
\begin{aligned}
\int_0^T\int_{\mathbb{R}}(\mathbb{P}e^2)_t+\frac{|c|}{4}\int_0^T\sum_{i\in\mathbb{Z}}[\mathbb{P}e_i]^2+\frac{1-\gamma}{2}\int_0^T\int_{\mathbb{R}}(\mathbb{P}e_x)^2+\frac{\alpha}{2}\int_0^T\sum_{i\in\mathbb{Z}}\frac{[\mathbb{P}e_i]^2}{\Delta x}&\\
+\frac{bc_\lambda}{4}\int_0^T|\mathbb{P}e|^2_{H^{\lambda/2}(\mathbb{R})}+\frac{bc_\lambda}{4}\int_0^T|e|^2_{H^{\lambda/2}(\mathbb{R})}=\mathcal{O}(1)\Delta x^{2k}.&
\end{aligned}
$$

To conclude, we need to pass form $\mathbb{P}e$ to e in the above expression. This has already been done for the diffusion term in Section 3 in [20] and for the convection term in the proof of Lemma 2.4 in [7]. For the nonlocal term, we see that by (4.6)

$$
|\mathbb{P}e|^2_{H^{\lambda/2}(\mathbb{R})}=|e|^2_{H^{\lambda/2}(\mathbb{R})}-\mathcal{O}(1)\Delta x^{2k+2-\lambda},
$$

and the conclusion follows. □

4.2. **LDG method.**

Theorem 4.4. (Convergence) *Let $u\in H^{k+1}_{\mathrm{par}}(Q_T)$, $k\geq 0$, be a solution of (4.1) and $\tilde{\mathbf{w}}=(\tilde{u},\tilde{q})'\in C^1([0,T];V^k\cap L^2)$ be a solution of (2.5). With $e_u=u-\tilde{u}$ and $e_q=q-\tilde{q}$,*

$$
\int_{\mathbb{R}}e_u^2(x,T)+\int_0^T\int_{\mathbb{R}}e_q^2+\Theta_T[\mathbf{e}]+bc_\lambda\int_0^T|e_u|^2_{H^{\lambda/2}(\mathbb{R})}=\mathcal{O}(1)\Delta x^{2k}.
$$

Proof. Let us choose a test function $\mathbf{v}=(v_u,v_q)'$, $v_u,v_q\in V^k\cap L^2(\mathbb{R})$, and define

$$
M_l[\mathbf{w},\mathbf{v}]=\int_0^T\int_{\mathbb{R}}u_tv_u+\int_0^T\int_{\mathbb{R}}qv_q-\int_0^T\sum_{i\in\mathbb{Z}}\Big(\hat{\mathbf{h}}(\mathbf{w}_i)'[\mathbf{v}_i]+\int_{I_i}\mathbf{h}(\mathbf{w})'\partial_x\mathbf{v}\Big).
$$

With this notation at hand, we can write (3.6) as

$$M_{LDG}[\mathbf{w},\mathbf{v}] = M_l[\mathbf{w},\mathbf{v}] + M_{\mathcal{L}}[\mathbf{w},\mathbf{v}],$$

where $M_{\mathcal{L}}$ is defined in the previous proof. Proceeding as in the proof of Theorem 4.2, we find that

$$M_{LDG}[\mathbb{P}\mathbf{e},\mathbb{P}\mathbf{e}] = M_{LDG}[\mathbb{P}\mathbf{e}-\mathbf{e},\mathbb{P}\mathbf{e}]. \tag{4.7}$$

In [7] (Lemma 2.4) it is proved that

$$M_l(\mathbb{P}\mathbf{e}-\mathbf{e},\mathbb{P}\mathbf{e}) = \frac{1}{2}\Theta_T[\mathbb{P}\mathbf{e}] + \frac{1}{2}\int_0^T\int_{\mathbb{R}} \mathbb{P}e_q^2 + \mathcal{O}(1)\Delta x^{2k}. \tag{4.8}$$

By (3.6), (3.8), and Lemma A.1,

$$M_{LDG}[\mathbb{P}\mathbf{e},\mathbb{P}\mathbf{e}] = \int_0^T\int_{\mathbb{R}}(\mathbb{P}e_u^2)_t + \int_0^T\int_{\mathbb{R}}\mathbb{P}e_q^2 + \Theta_T[\mathbb{P}\mathbf{e}] + \frac{bc_\lambda}{2}\int_0^T |\mathbb{P}e_u|^2_{H^{\lambda/2}(\mathbb{R})}.$$

By inserting this inequality along with (4.8) and (4.5) into (4.7), we find that

$$\begin{aligned}\int_0^T\int_{\mathbb{R}}(\mathbb{P}e_u^2)_t + \frac{1}{2}\int_0^T\int_{\mathbb{R}}\mathbb{P}e_q^2 + \frac{1}{2}\Theta_T[\mathbb{P}\mathbf{e}]\\ + \frac{bc_\lambda}{4}\int_0^T |\mathbb{P}e_u|^2_{H^{\lambda/2}(\mathbb{R})} + \frac{bc_\lambda}{4}\int_0^T |e_u|^2_{H^{\lambda/2}(\mathbb{R})} = \mathcal{O}(1)\Delta x^{2k}.\end{aligned}$$

The conclusion now follows as in the proof of Theorem 4.2. □

5. Convergence for nonlinear equations

In the nonlinear case we will show that the DDG method converges towards an appropriately defined entropy solution of (1.1) whenever piecewise constant elements are used. In what follows we need the functions

$$\begin{aligned}\eta_k(s) &= |s-k|,\\ \eta_k'(s) &= \operatorname{sgn}(s-k),\\ q_k(s) &= \eta_k'(s)(f(s)-f(k)),\\ r_k(s) &= \eta_k'(s)(A(s)-A(k)).\end{aligned}$$

Remember that $A(u) = \int^u a$, and let $C^{1,\frac{1}{2}}(Q_T)$ denote the Hölder space of bounded functions $\phi : Q_T \to \mathbb{R}$ for which there is a constant $c_\phi > 0$ such that

$$|\phi(x,t)-\phi(y,\tau)| \le c_\phi\left[|x-y| + \sqrt{|t-\tau|}\right] \quad \text{for all} \quad (x,t),(y,\tau)\in Q_T.$$

We now introduce the entropy formulation for (1.1).

Definition 5.1. *A function $u \in L^\infty(Q_T)$ is a BV* entropy solution *of the initial value problem* (1.1) *provided that the following conditions hold:*

(**D**.1) $u \in L^1(Q_T)\cap BV(Q_T)$;
(**D**.2) $A(u) \in C^{1,\frac{1}{2}}(Q_T)$;
(**D**.3) *for all non-negative test functions* $\varphi \in C_c^\infty(\mathbb{R}\times[0,T))$ *and all* $k\in\mathbb{R}$,

$$\begin{aligned}\int_{Q_T} \eta_k(u)\varphi_t + q_k(u)\varphi_x + r_k(u)\varphi_{xx} + \eta_k'(u)\mathcal{L}[u]\varphi \; dx\,dt&\\ + \int_{\mathbb{R}}\eta_k(u_0(x))\varphi(0,x)\,dx \ge 0.&\end{aligned}$$

This definition is a straightforward combination of the one of Wu and Yin [30] (cf. also [14]) for degenerate convection-diffusion equations ($b = 0$) and the one of Cifani *et al.* [5] for fractional conservation laws ($a \equiv 0$). By the regularity of φ and u and Lemma A.1, each term in the entropy inequality (**D**.3) is well defined.

Remark 5.1. The L^1-contraction property (uniqueness) for BV entropy solutions follows along the lines of [26], since the BV-regularity of u and the L^∞-bound on $A(u)_x$ makes it possible to recover from (**D**.3) the more precise entropy inequality utilized in [26] for $L^1 \cap L^\infty$ entropy solutions.

We will now prove, under some additional assumptions, that the explicit DDG method with piecewise constant elements (i.e., $k = 0$) converges to the BV entropy solution of (1.1). In addition to convergence for the numerical method, this also gives the first existence result for entropy solutions of (1.1).

5.1. **The explicit DDG method with piecewise constant elements.** When piecewise constant elements are used ($k = 0$ in (2.7)), equation (2.5) takes the form

$$\int_{I_i} \hat{u}_t + \hat{f}(\hat{u}_{i+1}) - \hat{f}(\hat{u}_i) - \hat{h}(\hat{u}_{i+1}) + \hat{h}(\hat{u}_i) = b \int_{I_i} \mathcal{L}[\hat{u}].$$

Since $\hat{u}(x,t) = \sum_{i\in\mathbb{Z}} U_i(t)\mathbf{1}_i(x)$ (i.e., $\varphi_{0,i} = \mathbf{1}_i$, the indicator function of the interval I_i), we can and will use the admissible flux $\hat{h}(u_i) = \frac{1}{\Delta x}[A(u_i)]$ (which satisfies (2.4) with $k = 0$ and $\beta_0 = 1$) to rewrite the above equation as

$$\Delta x \frac{d}{dt} U_i + \hat{f}(U_i, U_{i+1}) - \hat{f}(U_{i-1}, U_i) - \frac{[A(U_{i+1})]}{\Delta x} + \frac{[A(U_i)]}{\Delta x} = b \sum_{j\in\mathbb{Z}} U_j \int_{I_i} \mathcal{L}[\mathbf{1}_{I_j}].$$

For $\Delta t > 0$ we set $t_n = n\Delta t$ for $n = \{0, \dots, N\}$, $T = t_N$, and $\phi_i^n = \phi(x_i, t_n)$ for any function ϕ. By a forward difference approximation in time, we obtain the explicit numerical method

$$\begin{aligned} \frac{U_i^{n+1} - U_i^n}{\Delta t} &+ \frac{\hat{f}(U_i^n, U_{i+1}^n) - \hat{f}(U_{i-1}^n, U_i^n)}{\Delta x} \\ &- \frac{A(U_{i+1}^n) - A(U_i^n)}{\Delta x^2} + \frac{A(U_i^n) - A(U_{i-1}^n)}{\Delta x^2} = \frac{b}{\Delta x} \sum_{j\in\mathbb{Z}} G_j^i U_j^n, \end{aligned} \tag{5.1}$$

where the weights $G_j^i = \int_{I_i} \mathcal{L}[\mathbf{1}_{I_j}]$ for all $(i,j) \in \mathbb{Z}\times\mathbb{Z}$. All relevant properties of these weights are collected in Lemma A.2. Next we define

$$D_\pm U_i = \pm \frac{1}{\Delta x}(U_{i\pm1} - U_i) \quad \text{and} \quad \mathcal{L}\langle U^n\rangle_i = \frac{1}{\Delta x}\int_{I_i} \mathcal{L}[\bar{U}^n]\,dx = \frac{1}{\Delta x}\sum_{j\in\mathbb{Z}} G_j^i U_j^n,$$

where $\bar{U}^n$ is the piecewise constant interpolant of U^n:

$$\bar{U}^n(x) = U_i^n, \qquad x \in [x_i, x_{i+1}).$$

The explicit numerical method we study can then be written as

$$\begin{cases} \frac{U_i^{n+1} - U_i^n}{\Delta t} + D_-\Big[\hat{f}(U_i^n, U_{i+1}^n) - D_+A(U_i^n)\Big] = b\mathcal{L}\langle U^n\rangle_i, \\ U_i^0 = \frac{1}{\Delta x}\int_{I_i} u_0(x)\,dx. \end{cases} \tag{5.2}$$

As we will see in what follows, the low-order difference method (5.2) allows for a complete convergence analysis for general nonlinear equations of the form (1.1).

Let us now prove that the difference scheme (5.2) is conservative (**P**.1), monotone (**P**.2), and translation invariant (**P**.3).

(**P**.1) Assume $\bar{U}^n \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$. By Lemma A.1

$$\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} |G_j^i U_j^n| \leq \int_{\mathbb{R}} |\mathcal{L}[\bar{U}^n(x)]|\,dx \leq c_\lambda C \|\bar{U}^n\|_{L^1(\mathbb{R})}^{1-\lambda} |\bar{U}^n|_{BV(\mathbb{R})}^{\lambda}, \tag{5.3}$$

and hence we can revert the order of summation to obtain

$$\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} G_j^i U_j^n = \sum_{j\in\mathbb{Z}} U_j^n \sum_{i\in\mathbb{Z}} G_j^i = 0$$

since $\sum_{i\in\mathbb{Z}} G_j^i = 0$ by Lemma (A.2). By summing over all $i \in \mathbb{Z}$ on each side of (5.2), we then find that

$$\sum_{i\in\mathbb{Z}} U_i^{n+1} = \sum_{i\in\mathbb{Z}} \Big(U_i^n + \frac{\Delta t}{\Delta x} \sum_{j\in\mathbb{Z}} G_j^i U_j^n \Big) = \sum_{i\in\mathbb{Z}} U_i^n.$$

(**P**.2) We show that U_i^{n+1} is an increasing function of all $\{U_i^n\}_{i\in\mathbb{Z}}$. First note that

$$\frac{\partial U_i^{n+1}}{\partial U_j^n} \geq 0 \quad \text{for} \quad i \neq j,$$

since $\hat{f}$ is monotone and $G_j^i \geq 0$ for $i \neq j$ by Lemma A.2. By Lemma A.2 we also see that $G_i^i = -d_\lambda \Delta x^{1-\lambda} \leq 0$, and hence

$$\begin{aligned}\frac{\partial U_i^{n+1}}{\partial U_i^n} &= 1 - \frac{\Delta t}{\Delta x}\Big[\partial_{u_1}\hat{f}(U_i^n, U_{i+1}^n) - \partial_{u_2}\hat{f}(U_{i-1}^n, U_i^n)\Big] \\ &\quad - 2\frac{\Delta t}{\Delta x^2} a(U_i^n) - \frac{\Delta t}{\Delta x^\lambda} d_\lambda.\end{aligned}$$

Here $\partial_{u_i}\hat{f}$ denotes the derivative of $\hat{f}(u_1, u_2)$ w.r.t. u_i for $i = 1, 2$. Therefore the following CFL condition makes the explicit method (5.2) monotone:

$$\frac{\Delta t}{\Delta x}\Big(\|\partial_{u_1}\hat{f}\|_{L^\infty(\mathbb{R})} + \|\partial_{u_2}\hat{f}\|_{L^\infty(\mathbb{R})} \Big) + \frac{2\Delta t}{\Delta x^2}\|a\|_{L^\infty(\mathbb{R})} + d_\lambda \frac{\Delta t}{\Delta x^\lambda} \leq 1. \tag{5.4}$$

(**P**.3) Translation invariance ($V_i^0 = U_{i+1}^0$ implies $V_i^n = U_{i+1}^n$) is straightforward since (5.2) does not depend explicitly on a grid point x_i.

Remark 5.2. For several well known numerical fluxes $\hat{f}$ (i.e. Godunov, Engquist-Osher, Lax-Friedrichs, *etc.*), we may replace

$$\|\partial_{u_1}\hat{f}\|_{L^\infty(\mathbb{R})} + \|\partial_{u_2}\hat{f}\|_{L^\infty(\mathbb{R})}$$

in the above CFL condition by the Lipschitz constant of the original flux f.

In the following, we always assume that the CFL condition (5.4) holds.

5.2. **Further properties of the explicit DDG method** (5.2)**.** Define

$$\|U\|_{L^1(\mathbb{Z})} = \sum_{i\in\mathbb{Z}} |U_i|, \quad \|U\|_{L^\infty(\mathbb{Z})} = \sup_{i\in\mathbb{Z}} |U_i|, \quad \text{and} \quad |U|_{BV(\mathbb{Z})} = \sum_{i\in\mathbb{Z}} |U_{i+1} - U_i|.$$

Lemma 5.3.

i) $\|U^n\|_{L^1(\mathbb{Z})} \leq \|u_0\|_{L^1(\mathbb{R})}$,
ii) $\|U^n\|_{L^\infty(\mathbb{Z})} \leq \|u_0\|_{L^\infty(\mathbb{R})}$,
iii) $|U^n|_{BV(\mathbb{Z})} \leq |u_0|_{BV(\mathbb{R})}$.

Proof. Since the numerical method (5.2) is conservative monotone and translation invariant, the results due to Crandall-Tartar [9, 14] and Lucier [28, 14] apply. □

For all $(x,t) \in R_i^n = [x_i, x_{i+1}) \times [t_n, t_{n+1})$, let $\hat{u}_{\Delta x}(x,t)$ be the time-space bilinear interpolation of U_i^n, i.e.

$$\begin{aligned} \hat{u}_{\Delta x}(x,t) = U_i^n + (U_{i+1}^n - U_i^n)\left(\frac{x - i\Delta x}{\Delta x}\right) + (U_i^{n+1} - U_i^n)\left(\frac{t - n\Delta t}{\Delta t}\right) \\ + (U_{i+1}^{n+1} - U_i^{n+1} - U_{i+1}^n + U_i^n)\left(\frac{x - i\Delta x}{\Delta x}\right)\left(\frac{t - n\Delta t}{\Delta t}\right). \end{aligned} \tag{5.5}$$

Note that $\hat{u}_{\Delta x}$ is continuous and a.e. differentiable on Q_T. We need the above bilinear interpolation – rather than a piecewise constant one – to prove the Hölder regularity in (**D**.2). We will show that the functions $A(\hat{u}_{\Delta x})$ enjoy Hölder regularity as in (**D**.2), and then via an Ascoli-Arzelà type of argument, so does the limit $A(u)$.

The following lemmas which are needed in the proof of Theorem 5.8, are nonlocal generalizations of the ones proved in [14]. In what follows we assume $f \in C^1(\mathbb{R})$, and note that the general case follows by approximation as in [14].

Lemma 5.4.

$$\begin{aligned} &\left\| \hat{f}(U_i^n, U_{i+1}^n) - D_+A(U_i^n) - \sum_{k=-\infty}^{i} \sum_{j\in\mathbb{Z}} G_j^k U_j^n \right\|_{L^\infty(\mathbb{Z})} \\ &\qquad \le \left\| \hat{f}(U_i^0, U_{i+1}^0) - D_+A(U_i^0) - \sum_{k=-\infty}^{i} \sum_{j\in\mathbb{Z}} G_j^k U_j^0 \right\|_{L^\infty(\mathbb{Z})}, \end{aligned} \tag{5.6}$$

$$\begin{aligned} &\left| \hat{f}(U_i^n, U_{i+1}^n) - D_+A(U_i^n) - \sum_{k=-\infty}^{i} \sum_{j\in\mathbb{Z}} G_j^k U_j^n \right|_{BV(\mathbb{Z})} \\ &\qquad \le \left| \hat{f}(U_i^0, U_{i+1}^0) - D_+A(U_i^0) - \sum_{k=-\infty}^{i} \sum_{j\in\mathbb{Z}} G_j^k U_j^0 \right|_{BV(\mathbb{Z})}. \end{aligned} \tag{5.7}$$

Proof. Inequality (5.6). Let us start by defining $V_i^n = \frac{\Delta x}{\Delta t}\sum_{k=-\infty}^{i}(U_k^n - U_k^{n-1})$. This sum is finite since $U^n \in L^1(\mathbb{Z})$ for all $n \ge 0$. If we use (5.2), we can write

$$V_i^{n+1} = -\left[\hat{f}(U_i^n, U_{i+1}^n) - D_+A(U_i^n)\right] + \sum_{k=-\infty}^{i} \sum_{j\in\mathbb{Z}} G_j^k U_j^n. \tag{5.8}$$

Here we have used that $U^n \in L^1(\mathbb{Z}) \cap BV(\mathbb{Z})$, f and A are Lipschitz continuous, and $f(0) = 0$ to conclude that the sum $\sum_{k=-\infty}^{i} D_-[\hat{f}(U_j^n, U_{j+1}^n) - D_+A(U_j^n)]$ is finite and has value $[\hat{f}(U_i^n, U_{i+1}^n) - D_+A(U_i^n)]$. Next we rewrite the right-hand side of (5.8) in terms of $\{V_i^n\}_{i\in\mathbb{Z}}$. By (5.8),

$$\begin{aligned} V_i^{n+1} = V_i^n - \left[\hat{f}(U_i^n, U_{i+1}^n) - \hat{f}(U_i^{n-1}, U_{i+1}^{n-1}) - D_+(A(U_i^n) - A(U_i^{n-1}))\right] \\ + \sum_{k=-\infty}^{i} \sum_{j\in\mathbb{Z}} G_j^k (U_j^n - U_j^{n-1}). \end{aligned} \tag{5.9}$$

We prove that

$$\sum_{k=-\infty}^{i} \sum_{j\in\mathbb{Z}} G_j^k (U_j^n - U_j^{n-1}) = \frac{\Delta t}{\Delta x} \sum_{j\in\mathbb{Z}} G_j^i V_j^n. \tag{5.10}$$

Indeed, note that $D_-V_j^n = \frac{1}{\Delta t}\left(U_j^n - U_j^{n-1}\right)$ and

$$\sum_{j\in\mathbb{Z}} G_j^k V_{j-1}^n = \sum_{j\in\mathbb{Z}} G_{j+1}^k V_j^n = \sum_{j\in\mathbb{Z}} G_j^{k-1} V_j^n$$

since $G^k_{j+1} = G^{k-1}_j$. Thus,

$$\begin{aligned}
\sum_{k=-\infty}^{i}\sum_{j\in\mathbb{Z}} G^k_j (U^n_j - U^{n-1}_j) &= \Delta t \sum_{k=-\infty}^{i}\sum_{j\in\mathbb{Z}} G^k_j D_- V^n_j \\
&= \frac{\Delta t}{\Delta x}\sum_{k=-\infty}^{i}\sum_{j\in\mathbb{Z}} G^k_j (V^n_j - V^n_{j-1}) \\
&= \frac{\Delta t}{\Delta x}\sum_{k=-\infty}^{i}\sum_{j\in\mathbb{Z}} G^k_j V^n_j - \frac{\Delta t}{\Delta x}\sum_{k=-\infty}^{i}\sum_{j\in\mathbb{Z}} G^k_j V^n_{j-1} \\
&= \frac{\Delta t}{\Delta x}\sum_{k=-\infty}^{i}\sum_{j\in\mathbb{Z}} G^k_j V^n_j - \frac{\Delta t}{\Delta x}\sum_{k=-\infty}^{i}\sum_{j\in\mathbb{Z}} G^{k-1}_j V^n_j \\
&= \frac{\Delta t}{\Delta x}\sum_{k=-\infty}^{i}\sum_{j\in\mathbb{Z}} G^k_j V^n_j - \frac{\Delta t}{\Delta x}\sum_{k=-\infty}^{i-1}\sum_{j\in\mathbb{Z}} G^k_j V^n_j \\
&= \frac{\Delta t}{\Delta x}\sum_{j\in\mathbb{Z}} G^i_j V^n_j.
\end{aligned}$$

Using Taylor expansions, we can replace the nonlinearities $\hat{f}, A$ with linear approximations as follows. We write

$$\hat{f}(U^n_i, U^n_{i+1}) - \hat{f}(U^{n-1}_i, U^{n-1}_{i+1}) = \Delta t \hat{f}^n_{1,i} D_- V^n_i + \Delta t \hat{f}^n_{2,i} D_- V^n_{i+1}, \tag{5.11}$$

where $\hat{f}^n_{1,i} = \partial_1 \hat{f}(\alpha^n_i, U^n_{i+1})$, $\hat{f}^n_{2,i} = \partial_2 \hat{f}(U^{n-1}_i, \tilde{\alpha}^n_{i+1})$ and $\alpha^n_i, \tilde{\alpha}^n_i \in (U^{n-1}_i, U^n_i)$. Similarly, we write

$$A(U^n_i) - A(U^{n-1}_i) = a(\beta^n_i)(U^n_i - U^{n-1}_i) = \Delta t a^n_i D_- V^n_i, \tag{5.12}$$

where $a^n_i = a(\beta^n_i)$ and $\beta^n_i \in (U^{n-1}_i, U^n_i)$. Inserting (5.10) and (5.11)-(5.12) into expression (5.9) returns

(5.13)
$$V^{n+1}_i = V^n_i - \Delta t(\hat{f}^n_{1,i} D_- V^n_i + \hat{f}^n_{2,i} D_- V^n_{i+1}) + \Delta t D_+(a^n_i D_- V^n_i) + \frac{\Delta t}{\Delta x}\sum_{j\in\mathbb{Z}} G^i_j V^n_j$$

or

$$V^{n+1}_i = A^n_i V^n_{i-1} + B^n_i V^n_i + C^n_i V^n_{i+1} + \frac{\Delta t}{\Delta x}\sum_{j\in\mathbb{Z}} G^i_j V^n_j, \tag{5.14}$$

where

$$\begin{aligned}
A^n_i &= \left[\frac{\Delta t}{\Delta x}\hat{f}^n_{1,i} + \frac{\Delta t}{\Delta x^2} a^n_i\right], \\
B^n_i &= \left[1 - \frac{\Delta t}{\Delta x}(\hat{f}^n_{1,i} - \hat{f}^n_{2,i}) - \frac{\Delta t}{\Delta x^2}(a^n_i + a^n_{i+1})\right], \\
C^n_i &= \left[\frac{\Delta t}{\Delta x^2} a^n_{i+1} - \frac{\Delta t}{\Delta x}\hat{f}^n_{2,i}\right].
\end{aligned}$$

Since $\hat{f}$ is monotone and $a \geq 0$, $A^n_i, C^n_i \geq 0$. Moreover, $B^n_i + \frac{\Delta t}{\Delta x} G^i_i \geq 0$ since the CFL condition (5.4) holds true. Thus, since (5.14) is conservative, monotone, and translation invariant (cf. the proof of Lemma 5.3), $\|V^n\|_{L^\infty(\mathbb{Z})} \leq \ldots \leq \|V^1\|_{L^\infty(\mathbb{Z})}$, and the conclusion follows from (5.8).

Inequality (5.7). Let us introduce $Z_i^n = V_i^n - V_{i-1}^n$. Note that, since $G_{j-1}^{i-1} = G_j^i$ for all $(i,j) \in \mathbb{Z} \times \mathbb{Z}$,

$$\sum_{j\in\mathbb{Z}} \left(G_j^i V_j^n - G_j^{i-1} V_j^n\right) = \sum_{j\in\mathbb{Z}} \left(G_j^i V_j^n - G_j^i V_{j-1}^n\right) = \sum_{j\in\mathbb{Z}} G_j^i Z_j^n.$$

Thus, (5.13) can be rewritten as

$$Z_i^{n+1} = Z_i^n - \Delta t D_-(\hat{f}_{1,i}^n Z_i^n + \hat{f}_{2,i}^n Z_{i+1}^n) + \Delta t D_- D_+(a_i^n Z_i^n) + \frac{\Delta t}{\Delta x} \sum_{j\in\mathbb{Z}} G_j^i Z_j^n$$

or

$$Z_i^{n+1} = \bar{A}_i^n Z_{i-1}^n + \bar{B}_i^n Z_i^n + \bar{C}_i^n Z_{i+1}^n + \frac{\Delta t}{\Delta x} \sum_{j\in\mathbb{Z}} G_j^i Z_j^n, \tag{5.15}$$

where $\bar{A}_i^n, \bar{B}_i^n, \bar{C}_i^n$ have similar properties as A_i^n, B_i^n, C_i^n. Proceeding as in the first part of the proof, (5.15) can be shown to be conservative, monotone, and translation invariant. Thus $\|Z^n\|_{L^1(\mathbb{Z})} \leq \ldots \leq \|Z^1\|_{L^1(\mathbb{Z})}$, and the conclusion follows from (5.8). We refer to [14] for the precise details concerning $\bar{A}_i^n, \bar{B}_i^n, \bar{C}_i^n$. □

The next lemma ensures that the numerical solutions are uniformly L^1-Lipschitz in time (and hence BV in both space and time by Lemma 5.3).

Lemma 5.5.

$$\sum_{i\in\mathbb{Z}} |U_i^m - U_i^n| \leq \left|\hat{f}(U_i^0, U_{i+1}^0) - D_+ A(U_i^0) - \sum_{k=-\infty}^{i} \sum_{j\in\mathbb{Z}} G_j^k U_j^0\right|_{BV(\mathbb{R})} \frac{\Delta t}{\Delta x} |m-n|.$$

Proof. Let us assume that $m > n$, the case $m < n$ is analogous. Note that

$$\begin{aligned}
\sum_{i\in\mathbb{Z}} |U_i^m - U_i^n| &\leq \sum_{l=n}^{m-1} \sum_{i\in\mathbb{Z}} |U_i^{l+1} - U_i^l| \\
&\leq \Delta t \sum_{l=n}^{m-1} \sum_{i\in\mathbb{Z}} \left| D_- \left[\hat{f}(U_i^l, U_{i+1}^l) - D_+ A(U_i^l)\right] - \frac{1}{\Delta x} \sum_{j\in\mathbb{Z}} G_j^i U_j^l \right|.
\end{aligned}$$

Since $D_- \left(\sum_{k=-\infty}^{i} \sum_{j\in\mathbb{Z}} G_j^k U_j\right) = \frac{1}{\Delta x} \sum_{j\in\mathbb{Z}} G_j^i U_j$,

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}} |U_i^m - U_i^n| \\
&\leq \Delta t \sum_{l=n}^{m-1} \sum_{i\in\mathbb{Z}} \left| D_- \left[\hat{f}(U_i^l, U_{i+1}^l) - D_+ A(U_i^l) - \sum_{k=-\infty}^{i} \sum_{j\in\mathbb{Z}} G_j^k U_j^l\right]\right|.
\end{aligned}$$

To conclude, use (5.7). □

We now show that the numerical solutions satisfy a discrete version of (**D**.2).

Lemma 5.6. *If* $|\hat{f}(U_i^0, U_{i+1}^0) - D_+ A(U_i^0) - \sum_{k=-\infty}^{i} \sum_{j\in\mathbb{Z}} G_j^k U_j^0|_{BV(\mathbb{Z})} < \infty$, *then*

$$|A(U_i^m) - A(U_j^n)| = \mathcal{O}(1) \left[|i-j|\Delta x + \sqrt{|m-n|\Delta t}\right].$$

Proof. Let us write

$$\left|A(U_i^m) - A(U_j^n)\right| \leq \left|A(U_i^m) - A(U_j^m)\right| + \left|A(U_j^m) - A(U_j^n)\right| = I_1 + I_2.$$

We first estimate the term I_1, then the term I_2.

Estimate of I_1. Using (5.6), (5.3), Lemma 5.3 ii), and the fact that f is Lipschitz continuous,

$$\|D_+A(U_i^m)\|_{L^\infty(\mathbb{Z})} \le \left\|\hat{f}(U_i^0, U_{i+1}^0) - D_+A(U_i^0) - \sum_{k=-\infty}^{i}\sum_{j\in\mathbb{Z}} G_j^k U_j^0\right\|_{L^\infty(\mathbb{Z})} + \left\|\hat{f}(U_i^m, U_{i+1}^m)\right\|_{L^\infty(\mathbb{Z})} + \left\|\sum_{k=-\infty}^{i}\sum_{j\in\mathbb{Z}} G_j^k U_j^m\right\|_{L^\infty(\mathbb{Z})} = \mathcal{O}(1).$$

Hence $I_1 = \mathcal{O}(1)|i-j|\Delta x$.

Estimate of I_2. Take a test function $\phi \in C_c^1(\mathbb{R})$, and let $\phi_i = \phi(i\Delta x)$. Let us assume $m > n$ (the case $m < n$ is analogous). Using (5.13) we find that

$$\begin{aligned}
&\left|\Delta x \sum_{i\in\mathbb{Z}} \phi_i \left(V_i^m - V_i^n\right)\right| \\
&= \Delta x \left|\sum_{l=n}^{m-1}\sum_{i\in\mathbb{Z}} \phi_i \left(V_i^{l+1} - V_i^l\right)\right| \\
&= \Delta x \sum_{l=n}^{m-1}\sum_{i\in\mathbb{Z}} \phi_i \left|(\hat{f}_{1,i}^n D_- V_i^l + \hat{f}_{2,i}^n D_- V_{i+1}^l) + D_+(a_i^n D_- V_i^l)\right| \\
&\qquad + \Delta t \left|\sum_{l=n}^{m-1}\sum_{i\in\mathbb{Z}} \phi_i \sum_{j\in\mathbb{Z}} G_j^i V_j^l\right| = C_1 + C_2.
\end{aligned}$$

We use summation by parts to move D_+ onto ϕ_i and the fact that $\hat{f}_{1,i}^n$ $\hat{f}_{2,i}^n$ a_i^n and $|V^l|_{BV(\mathbb{Z})}$ are uniformly bounded to arrive that

$$C_1 = \mathcal{O}(1)\Delta t(m-n)\left(\|\phi\|_{L^\infty(\mathbb{R})} + \|\phi'\|_{L^\infty(\mathbb{R})}\right).$$

For more details, see [14]. Then by (5.3), $\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} |G_j^i V_j^l| = \mathcal{O}(1)$, and hence

$$C_2 \le \Delta t \|\phi\|_{L^\infty(\mathbb{R})} \sum_{l=n}^{m-1}\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} |G_j^i V_j^l| = \mathcal{O}(1)\Delta t(m-n)\|\phi\|_{L^\infty(\mathbb{R})}. \tag{5.16}$$

Therefore,

$$\left|\Delta x \sum_{i\in\mathbb{Z}} \phi_i \left(V_i^m - V_i^n\right)\right| = \mathcal{O}(1)\Delta t(m-n)\Big[\|\phi\|_{L^\infty(\mathbb{R})} + \|\phi'\|_{L^\infty(\mathbb{R})}\Big]. \tag{5.17}$$

The above inequality is exactly expression (40) in [14]. From now on the proof continues as in [14]. Loosely speaking we take an appropriate sequence of test functions $\phi_\varepsilon \in C_c^1(\mathbb{R})$ to deduce from (5.17) that

$$\Delta x \sum_{i\in\mathbb{Z}} |V_i^m - V_i^n| = \mathcal{O}(1)\sqrt{(m-n)\Delta t}.$$

By (5.8), Lemma 5.5, and inequality (5.16) we also find that

$$\Delta x \sum_{i\in\mathbb{Z}} |V_i^m - V_i^n| = \mathcal{O}(1)(m-n)\Delta t + \Delta x \sum_{i\in\mathbb{Z}} |D_+A(U_i^m) - D_+A(U_i^n)|,$$

and hence $\Delta x \sum_{i\in\mathbb{Z}} |D_+A(U_j^m) - D_+A(U_j^n)| = \mathcal{O}(1)\sqrt{(m-n)\Delta t}$. We conclude by noting that

$$\begin{aligned}
I_2 = |A(U_j^m) - A(U_j^n)| &= \Delta x \left| \sum_{i=-\infty}^{j} D_+A(U_i^m) - \sum_{i=-\infty}^{j} D_+A(U_i^n) \right| \\
&\leq \Delta x \sum_{i\in\mathbb{Z}} |D_+A(U_i^m) - D_+A(U_i^n)| = \mathcal{O}(1)\sqrt{(m-n)\Delta t}.
\end{aligned}$$

□

Next we show that the numerical method (5.2) satisfies a cell entropy inequality, which is a discrete version of (**D**.3).

Lemma 5.7. *Let $k \in \mathbb{R}$ and $\eta_i^n = |U_i^n - k|$. Then*

$$\eta_i^{n+1} - \eta_i^n + \Delta t D_- Q_i^n - \Delta t D_- D_+ |A(U_i^n) - A(k)| \leq \Delta t \eta_k'(U_i^{n+1})\mathcal{L}\langle U^n\rangle_i, \tag{5.18}$$

where $Q_i^n = \hat{f}(U_i^n \vee k, U_{i+1}^n \vee k) - \hat{f}(U_i^n \wedge k, U_{i+1}^n \wedge k)$.

Proof. Let us introduce the notation $a \wedge b = \min\{a,b\}$ and $a \vee b = \max\{a,b\}$. Note that $\eta_i^n = (U_i^n \vee k) - (U_i^n \wedge k)$. Since the numerical method (5.2) is monotone,

$$\begin{aligned}
&\frac{(U_i^{n+1} \vee k) - (U_i^n \vee k)}{\Delta t} + \frac{\hat{f}(U_i^n \vee k, U_{i+1}^n \vee k) - \hat{f}(U_{i-1}^n \vee k, U_i^n \vee k)}{\Delta x} \\
&- \frac{A(U_{i+1}^n \vee k) - A(U_i^n \vee k)}{\Delta x^2} + \frac{A(U_i^n \vee k) - A(U_{i-1}^n \vee k)}{\Delta x^2} \\
&\qquad\qquad \leq \Delta t \mathbf{1}_{(k,+\infty)}(U_i^{n+1})\mathcal{L}\langle U^n\rangle_i
\end{aligned}$$

and

$$\begin{aligned}
&\frac{(U_i^{n+1} \wedge k) - (U_i^n \wedge k)}{\Delta t} + \frac{\hat{f}(U_i^n \wedge k, U_{i+1}^n \wedge k) - \hat{f}(U_{i-1}^n \wedge k, U_i^n \wedge k)}{\Delta x} \\
&- \frac{A(U_{i+1}^n \wedge k) - A(U_i^n \wedge k)}{\Delta x^2} + \frac{A(U_i^n \wedge k) - A(U_{i-1}^n \wedge k)}{\Delta x^2} \\
&\qquad\qquad \geq \Delta t \mathbf{1}_{(-\infty,k)}(U_i^{n+1})\mathcal{L}\langle U^n\rangle_i.
\end{aligned}$$

To conclude, subtract the above inequalities. □

5.3. **Convergence of the DDG method.** We are now in position to prove convergence of the fully explicit numerical method (5.2) to a BV entropy solution of (1.1). Let us introduce $\mathcal{B}$ (cf. [14]), the space of all functions $z : \mathbb{R} \to \mathbb{R}$ such that

$$\left| f(z) - \partial_x A(z) - \int^x \mathcal{L}[z] \right|_{BV(\mathbb{R})} < \infty.$$

In the following theorem we choose the initial datum to be in $L^1(\mathbb{R}) \cap BV(\mathbb{R}) \cap \mathcal{B}$, which is done to make sense to the right-hand side of (5.7). Note that whenever $z \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, $\mathcal{L}[z] \in L^1(\mathbb{R})$ by Lemma A.1, and hence

$$\left| \int^x \mathcal{L}[z] \right|_{BV(\mathbb{R})} = \left\| \frac{d}{dx} \int^x \mathcal{L}[z] \right\|_{L^1(\mathbb{R})} = \|\mathcal{L}[z]\|_{L^1(\mathbb{R})} < \infty.$$

Theorem 5.8 (Convergence for DDG)**.** *Suppose $u_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R}) \cap \mathcal{B}$, and let $\hat{u}_{\Delta x}$ be the interpolant* (5.5) *of the solution of the explicit DGG scheme* (5.2)*. Then there is a subsequence of $\{\hat{u}_{\Delta x}\}$ and a function $u \in L^1(Q_T) \cap BV(Q_T)$ such that (a) $\hat{u}_{\Delta x} \to u$ in $L^1_{loc}(Q_T)$ as $\Delta x \to 0$; (b) u is a BV entropy solution of* (1.1)*.*

Corollary 5.9 (Existence)**.** *If $u_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R}) \cap \mathcal{B}$, then there exists a BV entropy solution of* (1.1)*.*

Proof of Theorem 5.8. We will prove strong L^1_{loc} compactness, and hence we need the following estimates uniformly in $\Delta x > 0$:

i) $\|\hat{u}_{\Delta x}\|_{L^\infty(Q_T)} \leq C$,
ii) $\|\hat{u}_{\Delta x}\|_{BV(Q_T)} \leq C$.

Estimate *i)* is a consequence of Lemma 5.3 and (5.5), while estimate *ii)* comes from the following computations (cf. [14] for more details). Using the interpolation (5.5), we find that

$$\begin{aligned}\int_{Q_T} |\hat{u}_x| &\leq \frac{\Delta t}{2}\sum_{n=0}^{N-1}\sum_{i\in\mathbb{Z}}\left|U^n_{i+1}-U^n_i\right| + \frac{\Delta t}{2}\sum_{n=0}^{N-1}\sum_{i\in\mathbb{Z}}\left|U^{n+1}_{i+1}-U^{n+1}_i\right|\\ &\leq T|U^0|_{BV(\mathbb{Z})}.\end{aligned}$$

Note that Lemma 5.3 has been used in the second inequality. Similarly,

$$\begin{aligned}\int_{Q_T} |\hat{u}_t| &\leq \frac{\Delta x}{2}\sum_{n=0}^{N-1}\sum_{i\in\mathbb{Z}}\left|U^{n+1}_i-U^n_i\right| + \frac{\Delta x}{2}\sum_{n=0}^{N-1}\sum_{i\in\mathbb{Z}}\left|U^{n+1}_{i+1}-U^n_{i+1}\right|\\ &\leq T\left|\hat{f}(U^0_i,U^0_{i+1}) - D_+A(U^0_i) - \sum_{h=-\infty}^{i}\sum_{j\in\mathbb{Z}} G^h_j U^0_j\right|_{BV(\mathbb{Z})},\end{aligned}$$

where Lemma 5.5 has been used in the second inequality. Hence, there exists a sequence $\{\hat{u}_{\Delta x_i}\}_{i\in\mathbb{N}}$ which converges in $L^1_{loc}(Q_T)$ to a limit

$$u \in L^1(Q_T) \cap BV(Q_T).$$

Next we check that the limit u satisfies ($\mathbf{D}$.2). We define $w_{\Delta x} = A(\hat{u}_{\Delta x})$. Note that $A(\hat{u}_{\Delta x}) \to A(u)$ a.e. since $\hat{u}_{\Delta x} \to u$ a.e. (up to a subsequence) and A is continuous. Now choose $(x,t),(y,\tau),(j,n),(i,m)$ such that $(x,t) \in R^n_j$ and $(y,\tau) \in R^m_i$ for $R^n_i = [x_i, x_{i+1}) \times [t_n, t_{n+1})$. Then,

$$\begin{aligned}|w_{\Delta x}(y,\tau) - w_{\Delta x}(x,t)| &\leq |w_{\Delta x}(y,\tau) - w_{\Delta x}(i\Delta x, m\Delta t)|\\ &\quad + |w_{\Delta x}(i\Delta x, m\Delta t) - w_{\Delta x}(j\Delta x, n\Delta t)|\\ &\quad + |w_{\Delta x}(j\Delta x, n\Delta t) - w_{\Delta x}(x,t)|\\ &= I_1 + I_2 + I_3.\end{aligned}$$

Note that by Lemma 5.6, $I_2 = \mathcal{O}(1)(|i-j|\Delta x + \sqrt{|m-n|\Delta t})$, while by Lemma 5.6 again, (5.5), and $A' = a \in L^\infty$, $I_1 + I_3 = \mathcal{O}(1)(\Delta x + \sqrt{\Delta t})$. Thus

$$|w_{\Delta x}(y,\tau) - w_{\Delta x}(x,t)| = \mathcal{O}(1)\left[|y-x| + \sqrt{|\tau - t|} + \Delta x + \sqrt{\Delta t}\right].$$

We also have that $w_{\Delta x} = A(\hat{u}_{\Delta x})$ is uniformly bounded since A is Lipschitz and $\hat{u}_{\Delta x}$ is uniformly bounded. By essentially repeating the proof of the Ascoli-Arzelà compactness theorem, we can now deduce the existence of a subsequence $\{w_{\Delta x}\}$ converging locally uniformly towards the limit $A(u)$. By the estimates on $w_{\Delta x}$, it then follows that

$$A(u) \in C^{1,\frac{1}{2}}(Q_T). \tag{5.19}$$

Finally, let us check that the limit u satisfies ($\mathbf{D}$.3) in Definition 5.1. Here we need to introduce a piecewise constant inteporlation of our data points U^n_i. We call

$$\bar{u}_{\Delta x}(x,t) = U^n_i \text{ for all } (x,t) \in [x_i, x_{i+1}) \times [t_n, t_{n+1}).$$

We do this since the discontinuous sign function η'_k makes it difficult to work with the bilinear interpolant $\hat{u}_{\Delta x}$ in what follows. The need for the piecewise linear interpolation was dictated by the condition ($\mathbf{D}$.2): continuity of the functions $A(\hat{u}_{\Delta x})$ were needed to prove Hölder space-time regularity for the limit $A(u)$ (cf. the proof of

(5.19)). To verify that the limit u also satisfies ($\mathbf{D}$.3) the piecewise constant interpolation $\bar{u}_{\Delta x}$ suffices since, as we already have strong convergence for the piecewise linear interpolation, strong convergence toward the same limit u for the piecewise constant interpolation is ensured thanks to the fact that

$$\|\bar{u}_{\Delta x}(\cdot,t) - \hat{u}_{\Delta x}(\cdot,t)\|_{L^1(Q_T)} \leq c|U^n|_{BV(\mathbb{Z})}\Delta x.$$

We now take a positive test function $\varphi \in C_c^\infty(\mathbb{R}\times[0,T))$, and let $\varphi_i^n = \varphi(x_i,t_n)$. We multiply both sides of (5.18) by φ_i^n, and sum over all (i,n). Using summation by parts, we obtain

$$\begin{aligned}\Delta x\Delta t \sum_{n=0}^{N-1}\sum_{i\in\mathbb{Z}} \eta_i^n \frac{\varphi_i^{n+1}-\varphi_i^n}{\Delta t}& \\ + \Delta x\Delta t \sum_{n=0}^{N-1}\sum_{i\in\mathbb{Z}} Q_i^n D_+\varphi_i^n& \\ + \Delta x\Delta t \sum_{n=0}^{N-1}\sum_{i\in\mathbb{Z}} |A(U_i^n)-A(k)|D_-(D_+\varphi_i^n)& \\ + \Delta x\Delta t \sum_{n=0}^{N-1}\sum_{i\in\mathbb{Z}} \eta_k'(U_i^{n+1})\mathcal{L}\langle U^n\rangle_i\varphi_i^n& \\ + \Delta x\sum_{i\in\mathbb{Z}}\varphi_i^0\eta_i^0 \geq 0.&\end{aligned} \tag{5.20}$$

A standard argument shows that all the local terms in the above expression converge to the ones appearing in the entropy inequality ($\mathbf{D}$.3), see e.g. [19, 14]. Let us look the term containing the nonlocal operator $\mathcal{L}\langle\cdot\rangle$. We can rewrite it as

$$\int_{\Delta t}^{T+\Delta t}\int_{\mathbb{R}} \eta_k'(\bar{u}_{\Delta x})\mathcal{L}[\bar{u}_{\Delta x}]\bar{\varphi}\,dx\,dt + R,$$

where $R \overset{\Delta x\to 0}{\longrightarrow} 0$ and $\bar{\varphi}$ is the piecewise constant interpolant of φ_i^n. Indeed, let us write $\eta_k'(U_i^{n+1})\mathcal{L}\langle U^n\rangle_i = \eta_k'(U_i^{n+1})\mathcal{L}\langle U^n - U^{n+1}\rangle_i + \eta_k'(U_i^{n+1})\mathcal{L}\langle U^{n+1}\rangle_i$. Note that

$$\Delta x\Delta t\sum_{n=0}^{N-1}\sum_{i\in\mathbb{Z}}|\mathcal{L}\langle U^n-U^{n+1}\rangle_i|\varphi_i^n \leq \Delta t\|\bar{\varphi}\|_{L^\infty(Q_T)}\sum_{n=0}^{N-1}\int_{\mathbb{R}}|\mathcal{L}[\bar{U}^n(x)-\bar{U}^{n+1}(x)]|\,dx,$$

where the last quantity vanishes as $\Delta x \to 0$ by L^1-Lipschitz continuity in time (cf. Lemma 5.5, and also Lemmas 5.3 and A.1). Next,

$$\begin{aligned}&\sum_{n=0}^{N}\sum_{i\in\mathbb{Z}}\eta_k'(U_i^{n+1})\mathcal{L}\langle U^{n+1}\rangle_i\varphi_i^n \\ &= \sum_{n=0}^{N}\sum_{i\in\mathbb{Z}}\eta_k'(U_i^{n+1})\mathcal{L}\langle U^{n+1}\rangle_i(\varphi_i^n-\varphi_i^{n+1}) + \sum_{n=0}^{N}\sum_{i\in\mathbb{Z}}\eta_k'(U_i^{n+1})\mathcal{L}\langle U^{n+1}\rangle_i\varphi_i^{n+1},\end{aligned}$$

where the first term on the right-hand side vanishes as $\Delta x \to 0$ since there exists a constant $c_\varphi > 0$ such that $|\varphi_i^n - \varphi_i^{n+1}| \leq c_\varphi \Delta x$ for all (i,n). To conclude, we prove that up to a subsequence and for a.e. $k\in\mathbb{R}$,

$$\int_{\Delta t}^{T+\Delta t}\int_{\mathbb{R}}\eta_k'(\bar{u}_{\Delta x})\mathcal{L}[\bar{u}_{\Delta x}]\bar{\varphi}\,dx\,dt \overset{\Delta x\to 0}{\longrightarrow} \int_{Q_T}\eta_k'(u)\mathcal{L}[u]\varphi\,dx\,dt. \tag{5.21}$$

This is a consequence of the dominated convergence theorem since the left hand side integrand converges pointwise a.e. to the right hand side integrand. Indeed, first note that $\bar{\varphi}\to\varphi$ pointwise on Q_T, while a.e. up to a subsequence, $\bar{u}_{\Delta x}\to u$

on Q_T. We also have $\eta_k'(\bar{u}_{\Delta x}) \to \eta_k'(u)$ a.e. in Q_T since for a.e. $k \in \mathbb{R}$ the measure of $\{(x,t) \in Q_T : u(x,t) = k\}$ is zero and η_k' is continuous on $\mathbb{R}\backslash\{k\}$. Finally if the (compact) support of φ is containd in $[-R,R] \times [0,T]$, $R > 0$, then a trivial extension of Lemma A.1 implies that

$$\int_{[-R,R]\times[0,T]} |\mathcal{L}[\bar{u}_{\Delta x} - u]|\,dx\,dt \leq c_\lambda C \int_0^T \|\bar{u}_{\Delta x} - u\|_{L^1(-R,R)}^{1-\lambda} |\bar{u}_{\Delta x} - u|_{BV(-R,R)}^{\lambda},$$

where the last quantity vanishes as $\Delta x \to 0$ since $\bar{u}_{\Delta x} \to u$ in $L^1_{loc}(Q_T)$. Then $\mathcal{L}[\bar{u}_{\Delta x}] \to \mathcal{L}[u]$ a.e. in $[-R,R] \times (0,T)$ up to a subsequence. Convergence for all $k \in \mathbb{R}$ can be proved along the lines of [25, Lemmas 4.3 and 4.4]. □

5.4. **Remarks on the LDG method.** The derivation of the LDG method in the piecewise constant case is not as straightforward as the one for the DDG method. Indeed, the numerical fluxes introduced in (2.9) depend on the choice of the function c_{12}, and computations cannot be performed until this function has been defined. Our aim now is to show that the LDG method reduces to a numerical method similar to (5.1) for a suitable choice of the function c_{12}.

Let us for the time being ignore the nonlinear convection and fractional diffusion terms and focus on the problem

$$\begin{cases} u_t - \partial_x \sqrt{a(u)}q = 0, \\ q - \partial_x g(u) = 0, \\ u(x,0) = u_0(x). \end{cases}$$

The LDG method (2.10) then takes the form

$$\begin{cases} \int_{I_i} \tilde{u}_t + \hat{h}_u(\tilde{\mathbf{w}}_{i+1}) - \hat{h}_u(\tilde{\mathbf{w}}_i) = 0, \\ \int_{I_i} \tilde{q} + \hat{h}_q(\tilde{u}_{i+1}) - \hat{h}_q(\tilde{u}_i) = 0, \end{cases} \tag{5.22}$$

where $\tilde{u}(x,t) = \sum_{i\in\mathbb{Z}} U_i(t)\mathbf{1}_{I_i}(x)$, $\tilde{q}(x,t) = \sum_{i\in\mathbb{Z}} Q_i(t)\mathbf{1}_{I_i}(x)$, and the fluxes $(\hat{h}_u, \hat{h}_q)$ are defined in (2.9). Let us insert $\tilde{u}$ and $\tilde{q}$ into the system (5.22), and use the flux (2.9) to get

$$\begin{cases} \frac{d}{dt}U_i\Delta x - \frac{g(U_{i+1})-g(U_i)}{U_{i+1}-U_i}\frac{Q_{i+1}+Q_i}{2} - c_{12}(Q_{i+1} - Q_i) \\ \qquad + \frac{g(U_i)-g(U_{i-1})}{U_i-U_{i-1}}\frac{Q_i+Q_{i-1}}{2} + c_{12}(Q_i - Q_{i-1}) = 0, \\ Q_i\Delta x - \frac{g(U_{i+1})+g(U_i)}{2} + c_{12}(U_{i+1} - U_i) \\ \qquad + \frac{g(U_i)+g(U_{i-1})}{2} - c_{12}(U_i - U_{i-1}) = 0. \end{cases} \tag{5.23}$$

Let us choose the function c_{12} to be

$$c_{12}(U_i, U_{i-1}) = \frac{1}{2}\frac{g(U_i) - g(U_{i-1})}{U_i - U_{i-1}}. \tag{5.24}$$

Inserting (5.24) into (5.23) then leads to

$$\begin{cases} \frac{d}{dt}U_i\Delta x - \frac{g(U_{i+1})-g(U_i)}{U_{i+1}-U_i}Q_{i+1} + \frac{g(U_i)-g(U_{i-1})}{U_i-U_{i-1}}Q_i = 0, \\ Q_i = \frac{g(U_i)-g(U_{i-1})}{\Delta x}, \end{cases}$$

or

$$\frac{d}{dt}U_i\Delta x - \frac{1}{\Delta x}\frac{(g(U_{i+1}) - g(U_i))^2}{U_{i+1} - U_i} + \frac{1}{\Delta x}\frac{(g(U_i) - g(U_{i-1}))^2}{U_i - U_{i-1}} = 0.$$

For the full equation (1.1), this choice of c_{12} along with a forward difference approximation in time, lead to the following piecewise constant LDG approximation:

$$\begin{aligned}&\frac{U_i^{n+1}-U_i^n}{\Delta t}+\frac{\hat{f}(U_i^n,U_{i+1}^n)-\hat{f}(U_{i-1}^n,U_i^n)}{\Delta x}\\&-\frac{1}{\Delta x^2}\frac{(g(U_{i+1}^n)-g(U_i^n))^2}{U_{i+1}^n-U_i^n}+\frac{1}{\Delta x^2}\frac{(g(U_i^n)-g(U_{i-1}^n))^2}{U_i^n-U_{i-1}^n}=\frac{1}{\Delta x}\sum_{j\in\mathbb{Z}}G_j^iU_j^n.\end{aligned}\tag{5.25}$$

Remark 5.10. We do not prove convergence for the numerical method (5.25). However, we note that

$$\left(\frac{dg}{du}\right)^2=\frac{dA}{du},$$

since $g=\int^u\sqrt{a}$ and $A=\int^u a$. Roughly speaking this means that

$$\frac{(g(U_{i+1}^n)-g(U_i^n))^2}{U_{i+1}-U_i}\approx A(U_{i+1}^n)-A(U_i^n),$$

and hence that (5.1) and (5.25) are closely related. Experiments indicates that the two methods produce similar solutions (cf. Figure 3).

6. Numerical experiments

We conclude this paper by presenting some experimental results obtained using the fully explicit (piecewise constant) numerical methods (5.2) and (5.25), and the DDG method (2.5) with fully explicit third order Runge-Kutta time discretization and piecewise constant, linear, and quadratic elements. In the computations we have imposed a zero Dirichlet boundary condition on the whole exterior domain $\{|x|>1\}$. In all the plots, the dotted line represents the initial datum while the solid one (or the dashed-dotted one in Figure 3) the numerical solution at $t=T$.

Remark 6.1. The operator $\mathcal{L}[\hat{u}]$ requires the evaluation of the discrete solution $\hat{u}$ on the whole real axis, thus making necessary the use of some localization procedure. In our numerical experiments we have confined the nonlocal operator $\mathcal{L}[\cdot]$ to the domain $\Omega=\{|x|\le 1\}$. That is to say, for each grid point $(x_i,t_n)\in\Omega\times(0,T)$ we have computed the value of $\hat{u}$ at time t_{n+1} by using only the values $\hat{u}(x_i,t_n)$ with $x_i\in\Omega$.

We consider two different sets of data taken from [14]. In Example 1 we take

$$\begin{aligned}&f_1(u)=u^2,\\&a_1(u)=\begin{cases}0 & \text{for } u\le 0.5\\ 2.5u-1.25 & \text{for } 0.5<u\le 0.6\\ 0.25 & \text{for } u>0.6,\end{cases}\\&u_{0,1}(x)=\begin{cases}0 & \text{for } x\le -0.5\\ 5x+2.5 & \text{for } -0.5<x\le -0.3\\ 1 & \text{for } -0.3<x\le 0.3\\ 2.5-5x & \text{for } 0.3<x\le 0.5\\ 0 & \text{for } x>0.5.\end{cases}\end{aligned}\tag{\textbf{Ex}.1}$$

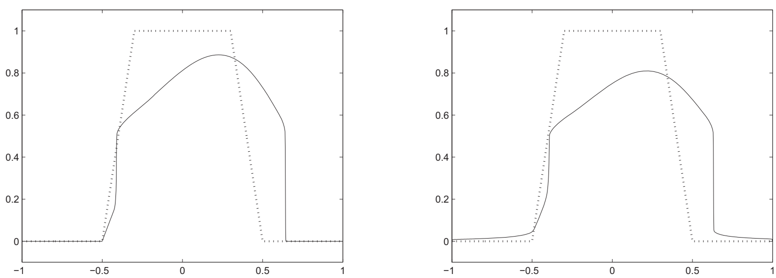
In Example 2 we choose

$$
\text{(Ex.2)}\qquad
\begin{aligned}
f_2 &= \frac{1}{4}\, f_1,\\
a_2 &= 4\, a_1,\\
u_{0,2}(x) &= \begin{cases} 1 & \text{for } x \le -0.4\\ -2.5x & \text{for } -0.4 < x \le 0\\ 0 & \text{for } x > 0.\end{cases}
\end{aligned}
$$

Furthermore, in Example 3 we use

$$
\text{(Ex.3)}\qquad
\begin{aligned}
f_3(u) &= u,\\
a_3(u) &= 0.1,\\
u_{0,3}(x) &= \exp\left(-\left(\frac{x}{0.1}\right)^2\right).
\end{aligned}
$$

The numerical results are presented in Figure 1, 2, 3, and 4. The results confirm what we expected: the solutions of the initial value problem (1.1) can develop shocks in finite time (this feature has been proved in [3] for the case $a = 0$). In Figure 1 and 2 you can see how the presence of the fractional diffusion $\mathcal{L}$ influences the shock's size and speed. In Figure 4 you can see how the accuracy of DDG method (2.5) improves when high-order polynomials are used ($k = 0, 1, 2$).



(a) $u_t + f(u)_x = (a(u)u_x)_x$

(b) Equation (1.1) with $\lambda = 0.5$

FIGURE 1. (**Ex**.1): $T = 0.15$ and $\Delta x = 1/640$.

In Figure 3, the dashed-dotted curve represents method (5.2), while the solid one represents method (5.25). The two numerical solutions stay close, and numerical convergence has been observed for finer grids. Note that here we have set $b = 0$ (no fractional diffusion) in order to stress the differences between the two methods.

The numerical rate of convergence for the solutions in Figure 1 (b), 2 (b), and 4 (b) are presented in Table 1. We have measured the L^p-error

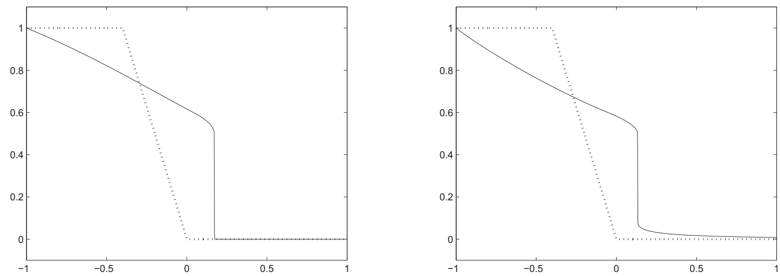
$$E_{\Delta x,p} = \|\hat{u}_{\Delta x}(\cdot,T) - \hat{u}_e(\cdot,T)\|^p_{L^p(\mathbb{R})},$$

where $\hat{u}_e$ is the numerical solution which has been computed using a very fine grid ($\Delta x = 1/640$), the relative error

$$R_{\Delta x,p} = \left(\frac{1}{\|\hat{u}_e(\cdot,T)\|^p_{L^p(\mathbb{R})}}\right) E_{\Delta x,p},$$

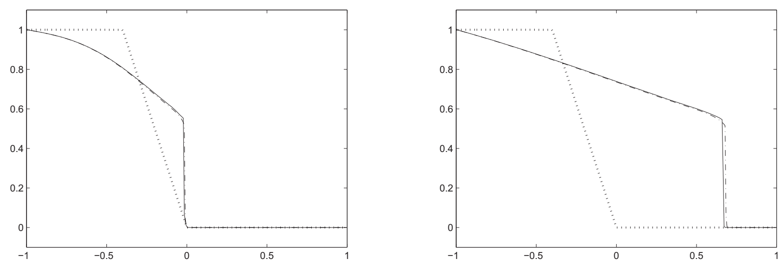
and the approximate rate of convergence

$$\alpha_{\Delta x,p} = \left(\frac{1}{\log 2}\right)\Big(\log E_{\Delta x,p} - \log E_{\Delta x/2,p}\Big).$$



(a) $u_t + f(u)_x = (a(u)u_x)_x$ (b) Equation (1.1) with $\lambda = 0.5$

FIGURE 2. (**Ex**.2): $T = 0.25$ and $\Delta x = 1/640$.



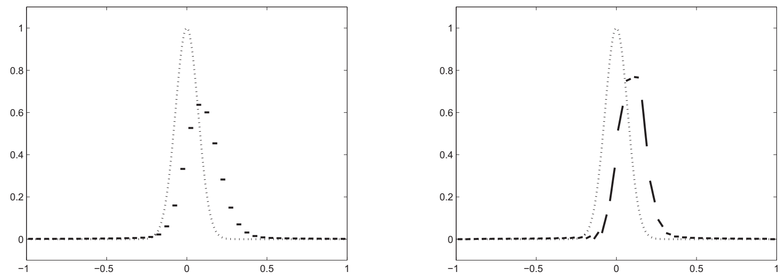
(a) $T = 0.0625$ (b) $T = 1$

FIGURE 3. (**Ex**.2): solutions of $u_t + f(u)_x = (a(u)u_x)_x$ at different times using methods (5.2) and (5.25) ($\Delta x = 1/160$).

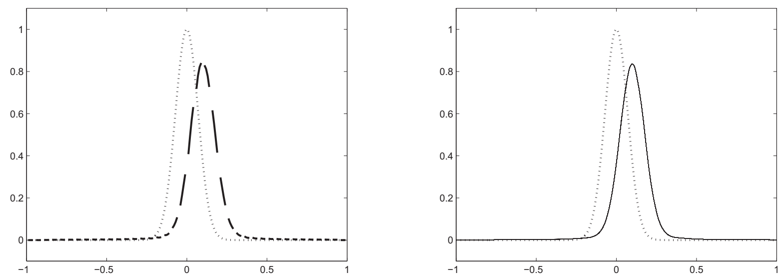
TABLE 1. Error, relative error, and numerical rate of convergence for the solutions in Figure 1 (b), 2 (b), and 4 (b).

	Figure 1 (b)			Figure 2 (b)			Figure 4 (b)		
Δx	$E_{\Delta x,1}$	$R_{\Delta x,1}$	$\alpha_{\Delta x,1}$	$E_{\Delta x,1}$	$R_{\Delta x,1}$	$\alpha_{\Delta x,1}$	$E_{\Delta x,2}$	$R_{\Delta x,2}$	$\alpha_{\Delta x,2}$
1/10	0.0706	0.0942	0.97	0.0474	0.0550	0.86	0.009000	0.093595	2.00
1/20	0.0361	0.0482	0.92	0.0261	0.0302	0.49	0.002300	0.023493	1.85
1/40	0.0191	0.0255	0.57	0.0186	0.0216	0.52	0.000626	0.006518	1.54
1/80	0.0128	0.0171	0.60	0.0130	0.0150	0.42	0.000216	0.002248	1.10
1/160	0.0084	0.0113	0.76	0.0097	0.0112	0.77	0.000101	0.001052	1.04
1/320	0.0050	0.0066	-	0.0057	0.0066	-	0.000049	0.000510	-

Our simulations seem to indicate numerical convergence of order less than one for the solutions depicted in Figure 1 (b) and 2 (b) (nonlinear equations and piecewise constant elements), and numerical convergence of order higher than one for the solution depicted in Figure 4 (b) (linear equation and piecewise linear elements). In the last case we do not seem to reach the expected value 2 (cf. the statement of Theorem 4.2). This deterioration of the numerical order of convergence for high-order polynomials has already been observed by the authors in [5]. The reasons behind this deterioration are still not clear.



(a) Piecewise constant ($k = 0$) with $\Delta x = 1/20$ (b) Piecewise linear ($k = 1$) with $\Delta x = 1/20$



(c) Piecewise quadratic ($k = 2$) with $\Delta x = 1/20$ (d) Solution computed using $\Delta x = 1/640$

FIGURE 4. (**Ex.**3): solutions at $T = 0.1$ using $k = 0, 1, 2$.

Finally, let us remind the reader that no general results concerning the rate of convergence of numerical methods for nonlinear equations like (1.1) have been produced so far. For more details, cf. [4].

Appendix A. Technical lemmas

In this appendix we state some technical results from [5] that are needed in this paper. All proofs can be found in [5].

Lemma A.1. *Let $\varphi, \phi \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$. Then there exists $C > 0$ such that*

$$\int_{\mathbb{R}} |\mathcal{L}[\varphi]| \leq c_\lambda C \|\varphi\|_{L^1(\mathbb{R})}^{1-\lambda} |\varphi|_{BV(\mathbb{R})}^{\lambda}, \tag{A.1}$$

$$\int_{\mathbb{R}} \phi \mathcal{L}[\varphi] = \int_{\mathbb{R}} \varphi \mathcal{L}[\phi], \tag{A.2}$$

$$\int_{\mathbb{R}} \varphi \mathcal{L}[\varphi] = -\frac{c_\lambda}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(\varphi(z) - \varphi(x))^2}{|z - x|^{1+\lambda}} \, dz \, dx. \tag{A.3}$$

Moreover, the last two identities also hold for all functions $\phi, \varphi \in H^{\lambda/2}(\mathbb{R})$.

To prove inequality (A.1) one can split the nonlocal operator $\mathcal{L}[\cdot]$, using an auxiliary parameter $\epsilon > 0$, into the sum of $\mathcal{L}_\epsilon[\cdot]$, the operator containing the singularity, and $\mathcal{L}^\epsilon[\cdot]$, the remaining part of the original operator. The operator $\mathcal{L}_\epsilon[\cdot]$ can then be treated using the control on the bounded variation, while the control on the L^1-norm is needed for the operator $\mathcal{L}^\epsilon[\cdot]$. To obtain exactly estimate (A.1) the optimal value of ϵ must be chosen. The proof of (A.2) - and thus of (A.3) - is essentially a change of variables.

Lemma A.2. *For all* $(i,j) \in \mathbb{Z} \times \mathbb{Z}$,

$$\sum_{k\in\mathbb{Z}} |G_k^i| < \infty, \ \sum_{k\in\mathbb{Z}} G_k^i = 0, \ G_j^i = G_i^j \ \text{and}\ G_{j+1}^{i+1} = G_j^i.$$

Moreover, $G_j^i \geq 0$ *whenever* $i \neq j$, *while*

$$G_i^i = -c_\lambda \left(\int_{|z|<1} \frac{dz}{|z|^\lambda} + \int_{|z|>1} \frac{dz}{|z|^{1+\lambda}} \right) \Delta x^{1-\lambda} \leq 0. \tag{A.4}$$

Lemma A.2 is essentially a consequence of the form of the operator $\mathcal{L}[\cdot]$ itself, and properties (A.1) and (A.2). Property (A.4) comes from a precise evaluation of the integral G_i^i.

Lemma A.3. *If* $\phi \in V^k \cap L^2(\mathbb{R})$, *then* $\phi \in H^{\frac{\lambda}{2}}(\mathbb{R})$ *for all* $\lambda \in (0,1)$, *and*

$$\|\phi\|^2_{H^{\frac{\lambda}{2}}(\mathbb{R})} \leq \frac{C}{\Delta x} \|\phi\|^2_{L^2(\mathbb{R})}. \tag{A.5}$$

Lemma A.3 is essentially a consequence of the fact that ϕ is a piecewise polynomial. The control on the L^2-norm together with the piecewise structure of ϕ ensure that its quadratic variation is bounded. Then, the finite quadratic variation plus the fact that ϕ is differentiable inside each interval I_i return (A.5).

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(Simone Cifani), Department of Mathematics, Norwegian University of Science and Technology (NTNU), N-7491 Trondheim, Norway

E-mail address: `simone.cifani@math.ntnu.no`

URL: `http://www.math.ntnu.no/~cifani/`

(Espen R. Jakobsen), Department of Mathematics, Norwegian University of Science and Technology (NTNU), N-7491 Trondheim, Norway

E-mail address: `erj@math.ntnu.no`

URL: `http://www.math.ntnu.no/~erj/`

(Kenneth H. Karlsen), Centre of Mathematics for Applications (CMA), Department of Mathematics, University of Oslo, P.O. Box 1053, Blindern, N-0316 Oslo, Norway

E-mail address: `kennethk@math.uio.no`

URL: `http://folk.uio.no/kennethk/`

Paper III

Entropy solution theory for fractional degenerate convection-diffusion equations

S. Cifani and E. R. Jakobsen

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ENTROPY SOLUTION THEORY FOR FRACTIONAL DEGENERATE CONVECTION-DIFFUSION EQUATIONS

SIMONE CIFANI AND ESPEN R. JAKOBSEN

Abstract. We study a class of degenerate convection diffusion equations with a fractional non-linear diffusion term. This class is a new, but natural, generalization of local degenerate convection diffusion equations, and include anomalous diffusion equations, fractional conservations laws, fractional Porous medium equations, and new fractional degenerate equations as special cases. We define weak entropy solutions and prove well-posedness under weak regularity assumptions on the solutions, e.g. uniqueness is obtained in the class of bounded integrable solutions. Then we introduce a new monotone conservative numerical scheme and prove convergence toward the entropy solution in the class of bounded integrable BV functions. The well-posedness results are then extended to non-local terms based on general Lévy operators, connections to some fully non-linear HJB equations are established, and finally, some numerical experiments are included to give the reader an idea about the qualitative behavior of solutions of these new equations.

1. Introduction

In this paper we study well-posedness and approximation of a Cauchy problem for the possibly degenerate non-linear non-local integral partial differential equation

$$\begin{cases} \partial_t u + \nabla \cdot f(u) = -(-\Delta)^{\lambda/2} A(u) & \text{in } Q_T = \mathbb{R}^d \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $f = (f_1, \dots, f_d) : \mathbb{R} \to \mathbb{R}$ and $A : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous with Lipschitz constants L_f and L_A, $A(\cdot)$ non-decreasing with $A(0) = 0$, and the non-local operator $-(-\Delta)^{\lambda/2}$ (or $g[\cdot]$ in shorthand notation) is the fractional Laplacian defined as

$$-(-\Delta)^{\lambda/2}\phi(x) = c_\lambda \ P.V. \int_{|z|>0} \frac{\phi(x+z,t) - \phi(x,t)}{|z|^{d+\lambda}} \,\mathrm{d}z$$

for some constants $c_\lambda > 0$, $\lambda \in (0,2)$, and a sufficiently regular function ϕ. Note that $A(\cdot)$ can be *strongly degenerate*, i.e. it may vanish on a set of positive measure.

Equation (1.1) is a fractional degenerate convection diffusion equation, and this class of equations has received considerable interest recently thanks to the wide variety of applications. They encompass various linear anomalous diffusion equations ($f \equiv 0$ and $A(u) \equiv u$), scalar conservation laws [16, 26, 34, 36, 38] ($A \equiv 0$), fractional (or *fractal*) conservation laws [1, 21] ($A(u) \equiv u$), and some (but not all!) fractional Porous medium equations [17] ($f \equiv 0$ and $A(u) = |u|u^m$, $m \geq 1$), but

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see also [5, 7]. Equation (1.1) is an extension to the fractional diffusion setting of the degenerate convection-diffusion equation [8, 31]

$$\partial_t u + \nabla \cdot f(u) = \Delta A(u). \tag{1.2}$$

When $A(\cdot)$ is strongly degenerate, equation (1.1) has never been analyzed before as far as we know.

The literature concerning the type of equations mentioned above is immense. We will only give a partial and incomplete survey of some parts we feel are more relevant for this paper. For a more complete discussion and many more references, we refer the reader to the nice papers [1] and [32]. But before we continue, we would like to mention actual and potential applications. A large variety of phenomena in physics and finance are modeled by linear anomalous diffusion equations, see e.g. [41, 4, 14]. Fractional conservation laws are generalizations of convection-diffusion equations ((1.2) with $A(u) \equiv u$), and appear in some physical models for over-driven detonation in gases [12] and semiconductor growth [41], and in areas like dislocation dynamics, hydrodynamics, and molecular biology, cf. [1, 3, 19]. Similar equations, but with slightly different non local term, also appear in radiation hydrodynamics [37]. Equations like (1.2) are used to model a vast variety of phenomena, including porous media flow [39], reservoir simulation [22], sedimentation processes [6], and traffic flow [40]. Finally, we mention [29] where degenerate elliptic-parabolic equations with fractional time derivatives are considered.

In the non-linear and degenerate setting of (1.1), we can not expect to have classical solutions and it is well-known that weak solutions are not unique in general. In the setting of fractional conservation laws this is proved in e.g. [2, 3, 33]. To get uniqueness we impose extra conditions, called entropy conditions. In this paper we will introduce a Kruzkov type entropy formulation for equation (1.1). This type of formulation was introduced by Kruzkov in [34], and used along with a doubling of variables device, to obtain general uniqueness results for scalar conservation laws. Much later, Carrillo in [8] extended these results to cover second order equations like (1.2), see also [31] for more general results and a presentation and proof which is more like our own. More recently, Alibaud [1] extended the Kruzkov formulation and uniqueness result to the fractional setting. He obtained general results for fractional conservation laws. In a new work by Karlsen and Ulusoy [32], a unified formulation is given that essentially includes the results of Alibaud and Carrillo as special cases. In [1, 32] the fractional diffusion is always linear and non-degenerate.

The entropy formulation we use is an extension of the formulation of Alibaud, and it allows us to prove a general L^1-contraction and uniqueness result for bounded integrable solutions of the initial value problem (1.1). Our uniqueness proof relies on some new observations and estimates along with ideas from [8, 31]. From a technical point of view, our proof for $\lambda \in (0, 2)$ is more related to the conservation law (or fractional conservation law) proof than the more technical proof of Carrillo for $\lambda = 2$ (equation (1.2)). E.g. we do not need a "weak chain rule" and hence do not need to assume any extra a priori regularity on the term $A(u)$.

In practice to solve (1.1) we must resort to numerical computations. But since the equation is non-linear and degenerate, many numerical methods will fail to converge or converge to false (non-entropy) solutions. The solution is to construct "good" numerical methods that insure convergence to entropy solutions. In the conservation law community, it is well known that monotone, conservative, and consistent methods will do the job for you. There is a vast literature on such methods, we refer the reader e.g. to [26] and references therein. For non-linear fractional equations there exist very few methods and results so far. Dedner and

Rhode [18] introduced a convergent finite volume method for a non-local conservation laws from radiation hydrodynamics. Droniou [19] was the first to define and prove convergence for approximations of fractional conservations laws. Karlsen and the authors then introduced and proved convergence for Discontinuous Galerkin methods for fractional conservation laws and fractional convection-diffusion equations in [10, 11]. After that, the authors introduced a convergent spectral vanishing viscosity method for fractional conservations laws in [9]. Kuznetzov type error estimates were also obtained in [9, 10]. In this paper, we discretize for the first time (1.1) in its general form. We introduce a new difference quadrature approximation that we prove converges to the entropy solution. The convergence holds for bounded integrable BV solutions, and hence we also have existence of solutions in this class. Finally, existence of solutions in the wider class of bounded integrable function is obtained through approximation via bounded integrable BV solutions (cf. Theorem 4.7).

In many applications, especially in finance, the non-local term is not a fractional Laplacian, but rather a Lévy type operator g_μ:

$$g_\mu[\phi](x) = \int_{|z|>0} \phi(x+z) - \phi(x) - z \cdot \nabla\phi(x) 1_{|z|<1} \, \mathrm{d}\mu(z),$$

where the Lévy measure μ is a positive Radon measure satisfying

$$\int_{|z|>0} |z|^2 \wedge 1 \; \mu(\mathrm{d}z) < \infty.$$

These operators are the infinitesimal generators of pure jump Lévy processes. We refer to [4, 14] for the theory and applications of such processes and to [32] for a very relevant and nice discussion and many more references. The entropy solution theory related to such operators is very similar to the one for fractional Laplacians, and the first well-posedness results were obtained in [32]. In this paper we extend the entropy theory for (1.1) to this Lévy setting (cf. equation (5.1)). Our formulation is an extension of Alibaud's formulation and is different from the one given in [32]. We also treat completely general Lévy measures, i.e. our Lévy operators are slightly more general than the ones in [32].

We also discuss the fact that (1.1) is related to fully non-linear HJB equations, see Section 6. We first show an easy extension of results from [35]: In one space dimension the gradient of a viscosity solution of a fractional HJB equation is an entropy solution of a fractional conservation law. Then we show a new correspondence for any space dimension: If u is a viscosity solution of

$$u_t - A(g_\mu[u]) = 0,$$

then $v = g_\mu[u]$ is the entropy solution of

$$v_t - g_\mu[A(v)] = 0.$$

The relevance of these results are discussed in Section 6. The final part of the paper is devoted to numerical simulations to give the reader an idea about the qualitative behavior of the solutions of these new equations.

Here is the content of the paper section by section. The entropy formulation is introduced and discussed in Section 2. In Section 3, we state and prove L^1-contraction and uniqueness for entropy solutions of (1.1). The monotone conservative numerical method is then introduced and analyzed in Section 4. In Section 5, we extend the well-posedness results proved for solutions of (1.1) to a wider class of equations where the fractional Laplacian has been replaced by a general Lévy operator. In Section 6 we show how solutions of equations of the type (1.1) are related to solutions of fully non-linear HJB equations, and in the last section, we provide several numerical simulations of problems of the form (1.1).

2. Entropy formulation

In this section we introduce an entropy formulation for the initial value problem (1.1) which generalizes Alibaud's formulation in [1]. To this end, let us split the non-local operator g into two terms: for each $r>0$, we write $g[\varphi]=g_r[\varphi]+g^r[\varphi]$ where

$$g_r[\varphi](x)=c_\lambda\ P.V.\int_{|z|<r}\frac{\varphi(x+z)-\varphi(x)}{|z|^{d+\lambda}}\ \mathrm{d}z,$$
$$g^r[\varphi](x)=c_\lambda\int_{|z|>r}\frac{\varphi(x+z)-\varphi(x)}{|z|^{d+\lambda}}\ \mathrm{d}z.$$

The Cauchy principal value is defined as

$$P.V.\int_{|z|>0}\varphi(z)\ \mathrm{d}z=\lim_{b\to0}\int_{b<|z|}\varphi(z)\ \mathrm{d}z.$$

Note that, by symmetry,

$$P.V.\int_{|z|<r}\frac{z}{|z|^{d+\lambda}}\ \mathrm{d}z=0$$

and hence

$$g_r[\varphi](x)=c_\lambda\ P.V.\int_{|z|<r}\frac{\varphi(x+z)-\varphi(x)-z\cdot\nabla\varphi(x)}{|z|^{d+\lambda}}\ \mathrm{d}z. \tag{2.1}$$

Whenever φ is smooth enough, the principal value in (2.1) is well defined by the dominated convergence theorem since

$$|g_r[\varphi](x)|\leq\left\{\begin{array}{ll} c_\lambda\|D\varphi\|_{L^\infty(B(x,r))}\int_{|z|<r}\frac{|z|}{|z|^{d+\lambda}}\ \mathrm{d}z & \text{when } \lambda\in(0,1)\\ \frac{c_\lambda}{2}\|D^2\varphi\|_{L^\infty(B(x,r))}\int_{|z|<r}\frac{|z|^2}{|z|^{d+\lambda}}\ \mathrm{d}z & \text{when } \lambda\in[1,2)\end{array}\right\}<\infty.$$

The above integrals are finite because in polar coordinates they are proportional to

$$\int_0^r\frac{s^1}{s^{d+\lambda}}\ s^{d-1}\ \mathrm{d}s \text{ for } \lambda\in(0,1) \quad\text{and}\quad \int_0^r\frac{s^2}{s^{d+\lambda}}\ s^{d-1}\ \mathrm{d}s \text{ for } \lambda\in[1,2).$$

This estimate also shows that the integral in (2.1) exists and this leads to an alternative definition of the operator g_r avoiding the principal value (i.e. (2.1) without *P.V.*). This second definition is used e.g. in [1].

Let us introduce the functions $\eta_k(u)=|u-k|$, $\eta_k'(u)=\operatorname{sgn}(u-k)$, and $q_k(u)=\eta_k'(u)(f(u)-f(k))$ where the sign function is defined as

$$\operatorname{sgn}(s)=\begin{cases}1 & \text{for } s>0\\ 0 & \text{for } s=0\\ -1 & \text{for } s<0.\end{cases}$$

The entropy formulation we use is the following:

Definition 2.1. *A function u is an entropy solution of the initial value problem* (1.1) *provided that*

i) $u\in L^\infty(Q_T)\cap C([0,T];L^1(\mathbb{R}^d))$;

ii) for all $k\in\mathbb{R}$, all $r>0$, and all nonnegative test functions $\varphi\in C_c^\infty(Q_T)$,

$$\iint_{Q_T}\eta_k(u)\partial_t\varphi+q_k(u)\cdot\nabla\varphi+\eta_{A(k)}(A(u))\,g_r[\varphi]+\eta_k'(u)\,g^r[A(u)]\,\varphi\ \mathrm{d}x\mathrm{d}t\geq0;$$

iii) $u(\cdot,0)=u_0(\cdot)$ *a.e.*

Remark 2.1. By $C([0,T];L^1(\mathbb{R}^d))$ we mean the Banach space where the norm is given by $\|\phi\|_{C([0,T];L^1(\mathbb{R}^d))}=\max_{t\in[0,T]}\left\{\int_{\mathbb{R}^d}|\phi(x,t)|\ \mathrm{d}x\right\}$.

Remark 2.2. In view of *i)* and the properties of f and A, $\eta_k(u), q_k(u), \eta_{A(k)}(A(u)) \in L^\infty(Q_T)$ while $A(u) \in L^\infty(Q_T) \cap C([0,T]; L^1(\mathbb{R}^d))$. It immediately follows that the local terms in *ii)* are well-defined. Since $g_r[\phi] \in C_c^\infty(Q_T)$ for $\phi \in C_c^\infty(Q_T)$, also the g_r-term in *ii)* is well-defined. Finally we note that $g^r[\psi](x)$ is well-defined and belongs to $L^\infty(\mathbb{R}^d)$ for $\psi \in L^\infty(\mathbb{R}^d)$, and to $L^1(\mathbb{R}^d)$ for $\psi \in L^1(\mathbb{R}^d)$ by Fubini (integrating first w.r.t. x). It follows that $g^r[A(u)] \in L^\infty(Q_T) \cap C([0,T]; L^1(\mathbb{R}^d))$, and hence that the g^r-term in *ii)* is well-defined.

Remark 2.3. Since $u \in C([0,T]; L^1(\mathbb{R}^d))$ by *i)*, part *iii)* implies that the initial condition is imposed in the strong L^1-sense:

$$\lim_{t\to 0} \|u(\cdot,t) - u_0\|_{L^1(\mathbb{R}^d)} = 0.$$

A more traditional approach where initial values $u(\cdot,0)$ are included in the entropy inequality *ii)* would also work, cf. e.g. [26, Chapter 2].

Let us point out that, in the case $\lambda \in (0,1)$ and whenever the entropy solutions are sought in the BV-class, Definition 2.1 can be simplified to the following one:

Definition 2.2. *A function u is an entropy solution of the initial value problem* (1.1) *provided that*

i) $u \in L^\infty(Q_T) \cap L^\infty(0,T; BV(\mathbb{R}^d)) \cap C([0,T]; L^1(\mathbb{R}^d))$;

ii) for all $k \in \mathbb{R}$ and all nonnegative test functions $\varphi \in C_c^\infty(Q_T)$,

$$\iint_{Q_T} \eta_k(u)\partial_t\varphi + q_k(u)\cdot\nabla\varphi + \eta_k'(u)\, g[A(u)]\, \varphi \,\mathrm{d}x\mathrm{d}t \geq 0;$$

iii) $u(\cdot,0) = u_0(\cdot)$ *a.e.*

Note that the non-local term $g[A(u)]$ in the integral in *ii)* is well defined as shown in the following lemma.

Lemma 2.4. *If $\lambda \in (0,1)$, then there is a constant $C > 0$ such that*

$$\|g[A(u)]\|_{L^1(\mathbb{R}^d)} \leq c_\lambda C L_A \|u\|_{L^1(\mathbb{R}^d)}^{1-\lambda} |u|_{BV(\mathbb{R}^d)}^{\lambda}.$$

Proof. We split the integral in two parts, use Fubini and the estimate

$$\int_{\mathbb{R}^d} |u(x+z) - u(x)| \,\mathrm{d}x \leq \sqrt{d}|z||u|_{BV(\mathbb{R}^d)}$$

(cf. Lemma A.1), and change to polar coordinates ($z = ry$ for $r \geq 0$ and $|y| = 1$) to find that:

$$\int_{|z|<\epsilon}\int_{\mathbb{R}^d} \frac{|A(u(x+z)) - A(u(x))|}{|z|^{d+\lambda}} \,\mathrm{d}x\mathrm{d}z \leq L_A\sqrt{d}|u|_{BV(\mathbb{R}^d)} \int_{|z|<\varepsilon} \frac{|z|}{|z|^{d+\lambda}} \,\mathrm{d}z$$

$$= L_A\sqrt{d}|u|_{BV(\mathbb{R}^d)} \int_{|y|=1} \mathrm{d}S_y \int_0^\varepsilon \frac{\mathrm{d}r}{r^\lambda} = L_A\sqrt{d}|u|_{BV(\mathbb{R}^d)}\varepsilon^{1-\lambda} \int_{|y|=1} \mathrm{d}S_y \int_0^1 \frac{\mathrm{d}r}{r^\lambda}$$

and

$$\int_{|z|>\epsilon}\int_{\mathbb{R}^d} \frac{|A(u(x+z)) - A(u(x))|}{|z|^{d+\lambda}} \,\mathrm{d}x\mathrm{d}z \leq \frac{2L_A}{\epsilon^\lambda}\|u\|_{L^1(\mathbb{R}^d)} \int_{|y|=1} \mathrm{d}S_y \int_1^\infty \frac{\mathrm{d}r}{r^{1+\lambda}}.$$

To conclude, we choose $\epsilon = \|u\|_{L^1(\mathbb{R}^d)}|u|_{BV(\mathbb{R}^d)}^{-1}$. □

The following result shows how the two definitions of entropy solutions are interrelated and how they relate to weak and classical solutions of (1.1).

Theorem 2.5.

i) Definition 2.1 and Definition 2.2 are equivalent whenever $\lambda \in (0,1)$ and $u \in L^\infty(Q_T) \cap L^\infty(0,T; BV(\mathbb{R}^d)) \cap C([0,T]; L^1(\mathbb{R}^d))$.

ii) Any entropy solution u of (1.1) *is a weak solution: for all $\varphi \in C_c^\infty(Q_T)$,*

$$\iint_{Q_T} u\partial_t\varphi + f(u)\cdot\nabla\varphi + A(u)\, g[\varphi]\; \mathrm{d}x\mathrm{d}t = 0.$$

iii) If $A \in C^2(\mathbb{R})$, then any classical solution $u \in L^\infty(Q_T)\cap C([0,T];L^1(\mathbb{R}^d))$ of (1.1) *is an entropy solution.*

Remark 2.6. In *iii*) we need additional regularity of A to give a pointwise sense to the equation and hence also to define classical solutions. When $\lambda \in [1,2)$ it suffices to assume that $A \in C^2$, and when $\lambda \in (0,1)$ $A \in C^1$ is enough.

Proof.

i) Repeated use of the dominated convergence theorem and Lemma 2.4 first shows that, when $r \to 0$,

$$g_r[\varphi] \to 0 \quad \text{and} \quad g^r[A(u)] \to g[A(u)] \quad \textit{a.e.},$$

and then combined with this convergence result and Hölder's inequality, that Definition 2.1 implies Definition 2.2 when u is BV. To go the other way, let us note that since $A(\cdot)$ is non-decreasing,

$$\operatorname{sgn}(u-k)(A(u)-A(k)) = |A(u)-A(k)|. \tag{2.2}$$

Thus, if we write

$$\begin{aligned} g[A(u)] &= g_\epsilon[A(u)] \\ &\quad + c_\lambda \int_{\epsilon<|z|<r} \frac{(A(u(x+z,t))-A(k)) - (A(u(x,t))-A(k))}{|z|^{1+\lambda}}\,\mathrm{d}z \\ &\quad + g^r[A(u)], \end{aligned}$$

multiply each side by $\eta_k'(u)\varphi$ and integrate over Q_T, we end up with

$$\begin{aligned} \iint_{Q_T} \eta_k'(u)\, g[A(u)]\,\varphi\;\mathrm{d}x\mathrm{d}t \le \iint_{Q_T} \Bigg\{ &\eta_k'(u)\, g_\epsilon[A(u)]\,\varphi \\ &+ c_\lambda\varphi \int_{\epsilon<|z|<r} \frac{|A(u(x+z,t))-A(k)| - |A(u(x,t))-A(k)|}{|z|^{1+\lambda}}\,\mathrm{d}z \\ &+ \eta_k'(u)\, g^r[A(u)]\,\varphi \Bigg\}\;\mathrm{d}x\mathrm{d}t. \end{aligned}$$

We now use the change of variables $(z,x) \to (-z, x+z)$ to pass the test function φ inside the integral $\epsilon < |z| < r$, and obtain

$$\begin{aligned} &\iint_{Q_T} \varphi(x,t) \int_{\epsilon<|z|<r} \frac{|A(u(x+z,t))-A(k)| - |A(u(x,t))-A(k)|}{|z|^{1+\lambda}}\,\mathrm{d}z\mathrm{d}x\mathrm{d}t \\ &= \iint_{Q_T} |A(u(x,t))-A(k)| \int_{\epsilon<|z|<r} \frac{\varphi(x+z,t)-\varphi(x,t)}{|z|^{1+\lambda}}\,\mathrm{d}z\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{2.3}$$

The entropy inequality in Definition 2.1 is finally recovered in the limit as $\epsilon \to 0$.

ii) Using (2.2) and the change of variables $(z,x) \to (-z,x+z)$,

$$\begin{aligned} &\iint_{Q_T} \eta_k'(u(x,t))\, g^r[A(u(x,t))]\,\varphi(x,t)\;\mathrm{d}x\mathrm{d}t \\ &\le c_\lambda \iint_{Q_T} \varphi(x,t) \int_{|z|>r} \frac{|A(u(x+z,t))-A(k)| - |A(u(x,t))-A(k)|}{|z|^{1+\lambda}}\,\mathrm{d}z\mathrm{d}x\mathrm{d}t \\ &= \iint_{Q_T} |A(u(x,t))-A(k)|\, g^r[\varphi(x,t)]\,\mathrm{d}x\mathrm{d}t. \end{aligned}$$

Thus, since $g = g_r + g^r$, we have produced the inequality

$$\iint_{Q_T} \eta_k(u)\partial_t\varphi + q_k(u)\cdot\nabla\varphi + \eta_{A(k)}(A(u))\, g[\varphi]\ \mathrm{d}x\mathrm{d}t \geq 0.$$

By this inequality and the definitions of η and q, if $\pm k \geq \|u\|_{L^\infty(\mathbb{R})}$, then

$$\mp \iint_{Q_T} (u-k)\partial_t\phi + (f(u)-f(k))\cdot\nabla\phi + (A(u)-A(k))g[\phi]\ \mathrm{d}x\mathrm{d}t \geq 0.$$

By the Divergence theorem and a computation like in (2.3), all the k-terms are zero and hence u is a weak solution as defined in *ii)*.

iii) Since u solves equation (1.1) point-wise, for each $(x,t) \in Q_T$ and all $k \in \mathbb{R}$, we can write

$$\begin{aligned}
&\partial_t(u-k) + \nabla\cdot(f(u)-f(k)) = g_\epsilon[A(u)]\\
&+ c_\lambda \int_{\epsilon<|z|<r} \frac{(A(u(x+z,t))-A(k)) - (A(u(x,t))-A(k))}{|z|^{d+\lambda}}\ \mathrm{d}z\\
&+ g^r[A(u)].
\end{aligned}$$

If we multiply both sides of this equation by $\eta_k'(u)$ and use (2.2), we obtain

$$\begin{aligned}
&\eta_k'(u)\,\partial_t(u-k) + \eta_k'(u)\,\nabla\cdot(f(u)-f(k)) \leq \eta_k'(u)\, g_\epsilon[A(u)]\\
&+ c_\lambda \int_{\epsilon<|z|<r} \frac{|A(u(x+z,t))-A(k)| - |A(u(x,t))-A(k)|}{|z|^{d+\lambda}}\ \mathrm{d}z\\
&+ \eta_k'(u)\, g^r[A(u)].
\end{aligned}$$

Let us now multiply both sides of this inequality by a nonnegative test function φ, and integrate over Q_T to obtain

$$\begin{aligned}
&-\iint_{Q_T} \eta_k(u)\,\partial_t\varphi + q_k(u)\cdot\nabla\varphi\ \mathrm{d}x\mathrm{d}t\\
&\leq \iint_{Q_T} \Bigg\{ \eta_k'(u)\, g_\epsilon[A(u(x,t))]\ \varphi\\
&\quad + c_\lambda\varphi \int_{\epsilon<|z|<r} \frac{|A(u(x+z,t))-A(k)| - |A(u(x,t))-A(k)|}{|z|^{d+\lambda}}\ \mathrm{d}z\\
&\quad + \eta_k'(u)\, g^r[A(u(x,t))]\,\varphi \Bigg\}\ \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Thanks to (2.3), we can pass the test function φ inside the integral $\epsilon < |z| < r$, and so recover the entropy inequality in Definition 2.1 in the limit as $\epsilon \to 0$. □

3. L^1-contraction and Uniqueness

We now establish L^1-contraction and uniqueness for entropy solutions of the initial value problem (1.1) using the Kružkov's doubling of variables device [34]. This technique has already been extended to fractional conservation laws (i.e., $A(u) = u$) by Alibaud [1]. The first part of our proof builds on the ideas developed by Alibaud (and Kružkov!), but in the rest of the proof different ideas have to be used in our non-linear and possibly degenerate setting.

Theorem 3.1. *Let u and v be two entropy solutions of the initial value problem* (1.1) *with initial data u_0 and v_0. Then, for all $t \in (0,T)$,*

$$\|u(\cdot,t)-v(\cdot,t)\|_{L^1(\mathbb{R}^d)} \leq \|u_0 - v_0\|_{L^1(\mathbb{R}^d)}.$$

Uniqueness for entropy solutions of (1.1) immediately follows from the above L^1-contraction: if $u_0 = v_0$, then $u = v$ a.e. on Q_T.

Corollary 3.2. (Uniqueness) *There is at most one entropy solution of* (1.1).

Proof of Theorem 3.1.

1) We take $u = u(x,t)$ and $v = v(y,s)$, let $\psi = \psi(x,y,t,s)$ be a nonnegative test function, and denote by $\eta(u,k)$, $q(u,k)$, $\eta'(u,k)$ the quantities $\eta_k(u)$, $q_k(u)$, $\eta_k'(u)$. After integrating the entropy inequality for $u = u(x,t)$ with $k = v(y,s)$ over $(y,s) \in Q_T$, we find that

$$
\begin{aligned}
\iint_{Q_T} \iint_{Q_T} & \eta(u(x,t), v(y,s))\, \partial_t \psi(x,y,t,s) \\
& + q(u(x,t), v(y,s)) \cdot \nabla_x \psi(x,y,t,s) \\
& + \eta(A(u(x,t)), A(v(y,s)))\ g_r[\psi(\cdot, y, t, s)](x) \\
& + \eta'(u(x,t), v(y,s))\ g^r[A(u(\cdot,t))](x)\ \psi(x,y,t,s)\ \mathrm{d}x\mathrm{d}t\mathrm{d}y\mathrm{d}s \geq 0.
\end{aligned} \tag{3.1}
$$

Similarly, since $\eta(u,k) = \eta(k,u)$, $q(u,k) = q(k,u)$, and $\eta'(u,k) = -\eta'(k,u)$, integrating the entropy inequality for $v = v(y,s)$ with $k = u(x,t)$ leads to

$$
\begin{aligned}
\iint_{Q_T} \iint_{Q_T} & \eta(u(x,t), v(y,s))\, \partial_s \psi(x,y,t,s) \\
& + q(u(x,t), v(y,s)) \cdot \nabla_y \psi(x,y,t,s) \\
& + \eta(A(u(x,t)), A(v(y,s)))\ g_r[\psi(x, \cdot, t, s)](y) \\
& - \eta'(u(x,t), v(y,s))\ g^r[A(v(\cdot,s))](y)\ \psi(x,y,t,s)\ \mathrm{d}y\mathrm{d}s\mathrm{d}x\mathrm{d}t \geq 0.
\end{aligned} \tag{3.2}
$$

Let us now introduce the operator

$$
\tilde{g}^r[\varphi(\cdot,\cdot)](x,y) = \int_{|z|>r} \frac{\varphi(x+z, y+z) - \varphi(x,y)}{|z|^{d+\lambda}}\ \mathrm{d}z.
$$

Since all the terms in (3.1)–(3.2) are integrable, we are are free to change the order of integration, and hence add up inequalities (3.1)–(3.2) to find that (from now on $\mathrm{d}w = \mathrm{d}x\, \mathrm{d}t\, \mathrm{d}y\, \mathrm{d}s$)

$$
\begin{aligned}
\iint_{Q_T} \iint_{Q_T} & \eta(u(x,t), v(y,s))\, (\partial_t + \partial_s) \psi(x,y,t,s) \\
& + q(u(x,t), v(y,s)) \cdot (\nabla_x + \nabla_y) \psi(x,y,t,s) \\
& + \eta(A(u(x,t)), A(v(y,s)))\ g_r[\psi(\cdot, y, t, s)](x) \\
& + \eta(A(u(x,t)), A(v(y,s)))\ g_r[\psi(x, \cdot, t, s)](y) \\
& + \eta'(u(x,t), v(y,s))\ \tilde{g}^r[A(u(\cdot,t)) - A(v(\cdot,s))](x,y)\ \psi(x,y,t,s)\ \mathrm{d}w \geq 0.
\end{aligned} \tag{3.3}
$$

In the following we will manipulate the operator $\tilde{g}^r$, while the operators g_r will simply be carried along to finally vanish in the limit as $r \to 0$.

Let us use (2.2) to obtain the (Kato type of) inequality

$$
\begin{aligned}
& \eta'(u(x,t), v(y,s)) \Big[\Big(A(u(x+z,t)) - A(v(y+z,s)) \Big) - \Big(A(u(x,t)) - A(v(y,s)) \Big) \Big] \\
& \leq |A(u(x+z,t)) - A(v(y+z,s))| - |A(u(x,t)) - A(v(y,s))|,
\end{aligned}
$$

which implies that

$$
\eta'(u(x,t), v(y,s))\ \tilde{g}^r[A(u(\cdot,t)) - A(v(\cdot,s))](x,y) \leq \tilde{g}^r\Big[|A(u(\cdot,t)) - A(v(\cdot,s))| \Big](x,y). \tag{3.4}
$$

Furthermore, we use Fubini's Theorem and the change of variables $(z, x, y) \to (-z, x + z, y + z)$ to see that

$$\begin{aligned}
&\iint_{Q_T}\iint_{Q_T} \psi(x,y,t,s)\ \tilde{g}^r\Big[|A(u(\cdot,t)) - A(v(\cdot,s))|\Big](x,y)\ \mathrm{d}w \\
&= \iint_{Q_T}\iint_{Q_T} |A(u(x,t)) - A(u(y,s))|\ \tilde{g}^r[\psi(\cdot,\cdot,t,s)](x,y)\ \mathrm{d}w.
\end{aligned} \tag{3.5}$$

To sum up, when used in (3.3), (3.4)–(3.5) produce the inequality

$$\begin{aligned}
\iint_{Q_T}\iint_{Q_T} &\eta(u(x,t),v(y,s))\,(\partial_t + \partial_s)\psi(x,y,t,s) \\
&+ q(u(x,t),v(y,s))\cdot(\nabla_x + \nabla_y)\psi(x,y,t,s) \\
&+ \eta(A(u(x,t)),A(v(y,s)))\ g_r[\psi(\cdot,y,t,s)](x) \\
&+ \eta(A(u(x,t)),A(v(y,s)))\ g_r[\psi(x,\cdot,t,s)](y) \\
&+ \eta(A(u(x,t)),A(v(y,s)))\ \tilde{g}^r[\psi(\cdot,\cdot,t,s)](x,y)\ \mathrm{d}w \geq 0.
\end{aligned} \tag{3.6}$$

Thanks to the regularity of the test function ψ, we can now take the limit as $r \to 0$ in (3.6), and end up with

$$\begin{aligned}
\iint_{Q_T}\iint_{Q_T} &\eta(u(x,t),v(y,s))\,(\partial_t + \partial_s)\psi(x,y,t,s) \\
&+ q(u(x,t),v(y,s))\cdot(\nabla_x + \nabla_y)\psi(x,y,t,s) \\
&+ \eta(A(u(x,t)),A(v(y,s)))\ \tilde{g}[\psi(\cdot,\cdot,t,s)](x,y)\ \mathrm{d}w \geq 0,
\end{aligned} \tag{3.7}$$

where

$$\tilde{g}[\varphi(\cdot,\cdot)](x,y) = P.V.\int_{|z|>0} \frac{\varphi(x+z,y+z) - \varphi(x,y)}{|z|^{d+\lambda}}\ \mathrm{d}z.$$

Inequality (3.7) concludes the first part of the proof.

2) We now specify the test function ψ in order to derive the L^1-contraction from inequality (3.7):

$$\psi(x,t,y,s) = \hat{\omega}_\rho\left(\frac{x-y}{2}\right)\omega_\rho\left(\frac{t-s}{2}\right)\phi\left(\frac{x+y}{2},\frac{t+s}{2}\right),$$

for $\rho > 0$ and some $\phi \in C_c^\infty(Q_T)$ to be chosen later. Here $\hat{\omega}_\rho(x) = \omega_\rho(x_1)\cdots\omega_\rho(x_d)$ and $\omega_\rho(s) = \frac{1}{\rho}\omega(\frac{s}{\rho})$ for a nonnegative $\omega \in C_c^\infty(\mathbb{R})$ satisfying

$$\omega(-s) = \omega(s), \quad \omega(s) = 0 \text{ for all } |s| \geq 1, \quad \text{and} \quad \int_{\mathbb{R}} \omega(s)\ \mathrm{d}s = 1.$$

The reader can easily check that

$$\begin{aligned}
(\partial_t + \partial_s)\psi(x,y,t,s) &= \hat{\omega}_\rho\Big(\frac{x-y}{2}\Big)\omega_\rho\Big(\frac{t-s}{2}\Big)(\partial_t + \partial_s)\phi\Big(\frac{x+y}{2},\frac{t+s}{2}\Big), \\
(\nabla_x + \nabla_y)\psi(x,y,t,s) &= \hat{\omega}_\rho\Big(\frac{x-y}{2}\Big)\omega_\rho\Big(\frac{t-s}{2}\Big)(\nabla_x + \nabla_y)\phi\Big(\frac{x+y}{2},\frac{t+s}{2}\Big), \\
\tilde{g}[\psi(\cdot,\cdot,t,s)](x,y) &= \hat{\omega}_\rho\Big(\frac{x-y}{2}\Big)\omega_\rho\Big(\frac{t-s}{2}\Big)g\Big[\phi\Big(\cdot,\frac{t+s}{2}\Big)\Big]\left(\frac{x+y}{2}\right).
\end{aligned}$$

Note that with this choice of test function ψ, expressions involving $\tilde{g}$ naturally transform into expressions involving g.

We now show that, in the limit $\rho \to 0$, inequality (3.7) reduces to

$$\begin{aligned}\iint_{Q_T} &\eta(u(x,t), v(x,t))\partial_t\phi(x,t)\\ &+ q(u(x,t), v(x,t)) \cdot \nabla\phi(x,t)\\ &+ \eta(A(u(x,t)), A(v(x,t))\ g[\phi(\cdot,t)](x)\ \mathrm{d}x\mathrm{d}t \geq 0.\end{aligned} \tag{3.8}$$

Loosely speaking the reason for this is that the function ω_δ converges to the δ-measure. A proof concerning the local terms can be found in e.g. [31]. It remains to prove that

$$\begin{aligned}M := \Bigg| \iint_{Q_T} \iint_{Q_T} &|A(u(x,t)) - A(v(y,s))|\\ &\hat{\omega}_\rho\left(\frac{x-y}{2}\right) \omega_\rho\left(\frac{t-s}{2}\right) g\Big[\phi\Big(\cdot, \frac{t+s}{2}\Big)\Big]\left(\frac{x+y}{2}\right)\ \mathrm{d}w\\ &- \iint_{Q_T} |A(u(x,t)) - A(v(x,t))|\ g[\phi(\cdot,t)](x)\ \mathrm{d}x\mathrm{d}t\Bigg| \stackrel{\rho\to 0}{\longrightarrow} 0.\end{aligned}$$

To see this, we add and subtract

$$\begin{aligned}\iint_{Q_T} \iint_{Q_T} &|A(u(x,t)) - A(v(x,t))|\\ &\hat{\omega}_\rho\left(\frac{x-y}{2}\right) \omega_\rho\left(\frac{t-s}{2}\right) g\Big[\phi\Big(\cdot, \frac{t+s}{2}\Big)\Big]\left(\frac{x+y}{2}\right)\ \mathrm{d}w,\end{aligned}$$

use the fact that $\iint_{Q_T} \hat{\omega}_\rho\left(\frac{x-y}{2}\right) \omega_\rho\left(\frac{t-s}{2}\right) \mathrm{d}y\mathrm{d}s = 1$ for any fixed $t \in (0,T)$ for ρ small enough, and that ϕ has compact support in $(0,T)$ to find that

$$\begin{aligned}M \leq \iint_{Q_T} \iint_{Q_T} &\Big||A(u(x,t)) - A(v(y,s))| - |A(u(x,t)) - A(v(x,t))|\Big|\\ &\hat{\omega}_\rho\left(\frac{x-y}{2}\right) \omega_\rho\left(\frac{t-s}{2}\right) g\Big[\phi\Big(\cdot, \frac{t+s}{2}\Big)\Big]\left(\frac{x+y}{2}\right)\ \mathrm{d}w\\ + \iint_{Q_T} \iint_{Q_T} &\left|g\Big[\phi\Big(\cdot, \frac{t+s}{2}\Big)\Big]\left(\frac{x+y}{2}\right) - g[\phi(\cdot,t)](x)\right|\\ &\hat{\omega}_\rho\left(\frac{x-y}{2}\right) \omega_\rho\left(\frac{t-s}{2}\right) |A(u(x,t)) - A(v(x,t))|\ \mathrm{d}w.\end{aligned}$$

Let M_1 and M_2 denote the two integrals on the right hand side of the expression above. By the inequality $||a-c| - |b-c|| \leq |a-b|$ we see that

$$M_1 \leq K_\phi \iint_{Q_T} \iint_{Q_T} |A(v(x,t)) - A(v(y,s))|\ \hat{\omega}_\rho\left(\frac{x-y}{2}\right) \omega_\rho\left(\frac{t-s}{2}\right) \mathrm{d}w,$$

since, for all $(x,t), (y,s) \in Q_T \times Q_T$,

$$\begin{aligned}&\left|g\Big[\phi\Big(\cdot, \frac{t+s}{2}\Big)\Big]\left(\frac{x+y}{2}\right)\right|\\ &\leq K_\phi := \frac{c_\lambda}{2}\|D^2\phi\|_{L^\infty(\mathbb{R}^d)} \int_{|z|<1} \frac{|z|^2}{|z|^{d+\lambda}}\ \mathrm{d}z + 2c_\lambda\|\phi\|_{L^\infty(\mathbb{R})} \int_{|z|>1} \frac{\mathrm{d}z}{|z|^{d+\lambda}}.\end{aligned} \tag{3.9}$$

Note that both integrals in (3.9) are finite (use polar coordinates to see this). Using the change of variables $x-y=h$ and $t-s=\tau$, we obtain

$$
\begin{aligned}
&M_1\\
&\le K_\phi \iint_{Q_T}\iint_{Q_T} |A(v(x,t))-A(v(x+h,t+\tau))|\,\hat{\omega}_\rho\left(\frac{h}{2}\right)\omega_\rho\left(\frac{\tau}{2}\right)\mathrm{d}x\mathrm{d}t\mathrm{d}h\mathrm{d}\tau\\
&\le K_\phi \iint_{Q_T}\hat{\omega}_\rho\left(\frac{h}{2}\right)\omega_\rho\left(\frac{\tau}{2}\right)\left(\iint_{Q_T}|A(v(x,t))-A(v(x+h,t+\tau))|\,\mathrm{d}x\mathrm{d}t\right)\mathrm{d}h\mathrm{d}\tau\\
&\le K_\phi \sup_{|h|,|\tau|\le\rho}\left(\iint_{Q_T}|A(v(x,t))-A(v(x+h,t+\tau))|\,\mathrm{d}x\mathrm{d}t\right)\overset{\rho\to\infty}{\longrightarrow} 0
\end{aligned}
$$

by continuity of translations in L^1. We refer to Lemma 2.7.2 in [38] for a similar proof. A similar argument using the fact that $g[\phi]\in C([0,T];L^1(\mathbb{R}^d))$ (cf. Remark 2.2) shows that $M_2\to 0$ as $\rho\to 0$, and we can therefore conclude that $M\le M_1+M_2\to 0$ as $\rho\to 0$. The proof of (3.8) is now complete.

3) We now show that inequality (3.8) can be reduced to

$$
\iint_{Q_T}|u(x,t)-v(x,t)|\,\chi'(t)\;\mathrm{d}x\mathrm{d}t\ge 0, \tag{3.10}
$$

if we take $\phi=\varphi_r(x)\chi(t)$ and send $r\to\infty$ for $r>1$, $\chi\in C_c^\infty(0,T)$ (with derivative χ') to be specified later, and

$$
\varphi_r(x)=\int_{\mathbb{R}^d}\hat{\omega}(x-y)\mathbf{1}_{|y|<r}\;\mathrm{d}y.
$$

All derivatives of φ_r are bounded uniformly in r and vanish for all $||x|-r|>1$. Concerning the flux-term in (3.8), we find that

$$
\begin{aligned}
&\iint_{Q_T}\mathrm{sgn}(u(x,t)-v(x,t))(f(u(x,t))-f(v(x,t)))\cdot\nabla\phi(x,t)\;\mathrm{d}x\mathrm{d}t\\
&\le L_f\|\chi\|_{L^\infty}\iint_{Q_T}\Big(|u(x,t)|+|v(x,t)|\Big)\mathbf{1}_{||x|-r|<1}\;\mathrm{d}x\mathrm{d}t\overset{r\to\infty}{\longrightarrow}0
\end{aligned}
$$

by the dominated convergence theorem since u and v belong to L^1 and $\mathbf{1}_{||x|-r|<1}\to 0$ as $r\to\infty$ for all $x\in\mathbb{R}^d$. The term in (3.8) containing the non-local operator also tends to zero as $r\to\infty$. To see this note that $|g[\varphi_r](x)|$ is uniformly bounded in r, cf. (3.9), so by integrability of u and v and Hölder's inequality,

$$
\begin{aligned}
&\iint_{Q_T}|A(u(x,t))-A(v(x,t))|\;|g[\varphi_r](x)|\;\mathrm{d}x\mathrm{d}t\\
&\le L_A\Big(\|u\|_{L^1(Q_T)}+\|v\|_{L^1(Q_T)}\Big)\sup_{r>1}\|g[\varphi_r]\|_{L^\infty(Q_T)}<\infty.
\end{aligned}
$$

Hence we find that the integrand is bounded by an L^1-function uniformly for $r>1$:

$$
|A(u(x,t))-A(v(x,t))||g[\varphi_r](x)|\le L_A|(u(x,t)-v(x,t)|\sup_{r>1}\|g[\varphi_r]\|_{L^\infty(Q_T)}.
$$

Then for any $x,z\in\mathbb{R}^d$ fixed and $r>|x|+1$, $\varphi_r(x)=1$ and

$$
|\varphi_r(x+z)-\varphi_r(x)|\le|\mathbf{1}_{|x+z|<r-1}-1|\le\mathbf{1}_{|z|>r-1-|x|}.
$$

With this in mind we find that

$$
|g[\varphi_r](x)|\le\int_{|z|>0}\frac{\mathbf{1}_{|z|>r-1-|x|}}{|z|^{d+\lambda}}\;\mathrm{d}z\overset{r\to\infty}{\longrightarrow}0,
$$

and hence we can conclude by the dominated convergence theorem that

$$
\lim_{r\to\infty}\iint_{Q_T}|A(u(x,t))-A(v(x,t))|\;|g[\varphi_r](x)|\;\mathrm{d}x\mathrm{d}t=0.
$$

4) To conclude the proof, we now take $\chi = \chi_\mu$ for

$$\chi_\mu(t) = \int_{-\infty}^{t} \left(\omega_\mu(\tau - t_1) - \omega_\mu(\tau - t_2)\right) \, \mathrm{d}\tau,$$

where $r > 1$ and $0 < t_1 < t_2 < T$. Loosely speaking, the function χ_μ is a smooth approximation of the indicator function $\mathbf{1}_{(t_1,t_2)}$ which is zero near $t = 0$ and $t = T$ when $\mu > 0$ is small enough. Since $\chi_\mu'(t) = \omega_\mu(t-t_1) - \omega_\mu(t-t_2)$, inequality (3.10) reduces to

$$\iint_{Q_T} |u(x,t) - v(x,t)| \, \omega_\mu(t - t_2) \, \mathrm{d}x\mathrm{d}t \leq \iint_{Q_T} |u(x,t) - v(x,t)| \, \omega_\mu(t - t_1) \, \mathrm{d}x\mathrm{d}t.$$

By taking μ small enough and using Fubini's theorem, we can rewrite this inequality as

$$\Phi * \omega_\mu(t_2) \leq \Phi * \omega_\mu(t_1) \qquad \text{for} \qquad \Phi(t) = \int_{\mathbb{R}^d} |u(x,t) - v(x,t)| \, \mathrm{d}x, \tag{3.11}$$

where $\phi_1 * \phi_2(t) = \int_{\mathbb{R}} \phi_1(s) \, \phi_2(t-s) \, \mathrm{d}s$. Since $u, v \in C([0,T]; L^1(\mathbb{R}^d))$, we see that $\Phi \in C([0,T])$, and hence by standard properties of convolutions,

$$\Phi * \omega_\mu(t) \to \Phi(t) \quad \text{as} \quad \mu \to 0.$$

for all $t \in (0,T)$. Hence we can send $\mu \to 0$ in (3.11) to obtain

$$\|(u-v)(\cdot,t_2)\|_{L^1(\mathbb{R}^d)} \leq \|(u-v)(\cdot,t_1)\|_{L^1(\mathbb{R}^d)}.$$

Finally, the theorem follows from renaming t_2 and sending $t_1 \to 0$ using *iii)* and $C([0,T]; L^1(\mathbb{R}^d))$ regularity of u and v. □

4. A convergent numerical method

In this section we introduce a numerical method for the initial value problem (1.1) which is monotone and conservative. Then we prove that the limit of any convergent sequence of solutions of the method (as $\Delta x \to 0$) is an entropy solution of (1.1). Finally we prove that any sequence of solutions of the method is relatively compact whenever the initial datum is a bounded integrable function of bounded variation, and hence we establish the existence of an entropy solution of (1.1) in this case. Some numerical simulations based on this method are presented in the last section.

4.1. **Definition and properties of the numerical method.** For simplicity we only consider uniform space/time grids and we start by the one dimensional case. The spatial grid then consists of the points $x_i = i\Delta x$ for $i \in \mathbb{Z}$ and the temporal grid of $t_n = n\Delta t$ for $n = 0, \dots, N$ and $N\Delta t = T$. The explicit numerical method we consider then takes the form

$$\begin{cases} U_i^{n+1} = U_i^n - \Delta t \, D^- F(U_i^n, U_{i+1}^n) + \Delta t \displaystyle\sum_{j \neq 0} G_j(A(U_{i+j}^n) - A(U_i^n)), \\[2ex] U_i^0 = \dfrac{1}{\Delta x} \displaystyle\int_{x_i + \Delta x[0,1)} u_0(x) \, \mathrm{d}x, \end{cases}$$

where $D^- U_i = \frac{1}{\Delta x}(U_i - U_{i-1})$, $F : \mathbb{R}^2 \to \mathbb{R}$ is a numerical flux satisfying

a) F is Lipschitz continuous with Lipschitz constant L_F,
b) F is consistent, $F(u,u) = f(u)$ for all $u \in \mathbb{R}$,
c) $F(u_1, u_2)$ is non-decreasing w.r.t. u_1 and non-increasing w.r.t. u_2,

and G_i is defined by

$$G_i = c_\lambda \int_{x_i + \frac{\Delta x}{2}[-1,1)} \frac{\mathrm{d}z}{|z|^{1+\lambda}} \quad \text{for } i \neq 0.$$

In the multi dimensional case the spatial grid is $\Delta x\,\mathbb{Z}^d$ ($\Delta x > 0$) with points

$$x_\alpha = \Delta x\,\alpha \qquad \text{where } \alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d.$$

Let e_l be the d-vector with l-component 1 and the other components 0 and define the two box domains

$$R = \Delta x[0,1)^d \quad \text{and} \quad R_0 = \frac{\Delta x}{2}[-1,1)^d,$$

noting that $\cup_\alpha(x_\alpha + R) = \cup_\alpha(x_\alpha + R_0) = \mathbb{R}^d$. The explicit numerical method we consider now takes the form

$$\text{(4.1)} \quad \begin{cases} U_\alpha^{n+1} = U_\alpha^n - \Delta t \sum_{l=1}^d D_l^- F_l(U_\alpha^n, U_{\alpha+e_l}^n) + \Delta t \sum_{\beta\neq 0} G_\beta(A(U_{\alpha+\beta}^n) - A(U_\alpha^n)), \\ U_\alpha^0 = \dfrac{1}{\Delta x^d} \displaystyle\int_{x_\alpha + R} u_0(x)\,\mathrm{d}x, \end{cases}$$

where $D_l^- U_\alpha = \frac{1}{\Delta x}(U_\alpha - U_{\alpha-e_l})$, $F_l : \mathbb{R}^2 \to \mathbb{R}$ is a numerical flux satisfying *a) – c)* above with f_l replacing f, and G_α is defined by

$$G_\alpha = c_\lambda \int_{x_\alpha + R_0} \frac{\mathrm{d}z}{|z|^{d+\lambda}} \quad \text{for } \alpha \neq 0.$$

Note that G_α is positive and finite since $0 \notin x_\alpha + R_0$ unless $\alpha = 0$.

Remark 4.1. An admissible numerical flux F_l is e.g. the Lax-Friedrichs flux,

$$F_l(U_\alpha^n, U_{\alpha+e_l}^n) = \frac{1}{2}\left(f(U_\alpha^n) + f(U_{\alpha+e_l}) - \frac{\Delta x}{\Delta t}(U_{\alpha+e_l} - U_\alpha^n) \right).$$

We refer the reader to [23] or [26, Chapter 3] for a detailed presentation of more numerical fluxes which fulfill assumptions *a) – c)*.

Let us introduce the piecewise constant space/time interpolation

$$\bar{u}(x,t) = U_\alpha^n \quad \text{for all } (x,t) \in (x_\alpha + R) \times [t_n, t_{n+1}).$$

In the following we often need the relation

$$\text{(4.2)} \quad \sum_{\beta\neq 0} G_\beta(A(U_{\alpha+\beta}^n) - A(U_\alpha^n)) = c_\lambda \int_{\mathbb{R}^d \setminus R_0} \frac{A(\bar{u}(y_\alpha + z, t_n)) - A(\bar{u}(y_\alpha, t_n))}{|z|^{d+\lambda}}\,\mathrm{d}z,$$

where $y_\alpha = x_\alpha + \frac{\Delta x}{2}(1, \dots, 1)$. Note that this is an approximation of the principal value of the integral since $R_0 \to 0$ as $\Delta x \to 0$ in a symmetric way.

We now check that the numerical method (4.1) is conservative and monotone.

Lemma 4.2. *The numerical method* (4.1) *is conservative, i.e.*

$$\sum_{\alpha\in\mathbb{Z}^d} U_\alpha^{n+1} = \sum_{\alpha\in\mathbb{Z}^d} U_\alpha^n.$$

Proof. First we show that $\sum_{\alpha\in\mathbb{Z}^d}|U_\alpha^n|<\infty$ for all $n=0,\dots,N$. By (4.1), (4.3)

$$\begin{aligned}\sum_{\alpha\in\mathbb{Z}^d}\left|U_\alpha^{n+1}\right| &\le \sum_{\alpha\in\mathbb{Z}^d}\Bigg\{|U_\alpha^n|+\Delta t\sum_{l=1}^d\left|D_l^-F_l(U_\alpha^n,U_{\alpha+e_l}^n)\right| \\ &\qquad\qquad + \Delta t\sum_{\beta\neq0}G_\beta\left|A(U_{\alpha+\beta}^n)-A(U_\alpha^n)\right|\Bigg\} \\ &\le \sum_{\alpha\in\mathbb{Z}^d}\Bigg\{|U_\alpha^n|+\frac{\Delta t}{\Delta x}\sum_{l=1}^d\Big(L_F\left|U_\alpha^n-U_{\alpha-e_l}^n\right|+L_F\left|U_{\alpha+e_l}^n-U_\alpha^n\right|\Big) \\ &\qquad\qquad + \Delta t\sum_{\beta\neq0}G_\beta\Big(|A(U_{\alpha+\beta}^n)|+|A(U_\alpha^n)|\Big)\Bigg\} \\ &\le \left(1+4dL_F\frac{\Delta t}{\Delta x}+2L_A\Delta t\sum_{\beta\neq0}G_\beta\right)\sum_{\alpha\in\mathbb{Z}^d}|U_\alpha^n|,\end{aligned}$$

where, using that $\{z:|z|<\frac{\Delta x}{2}\}\subseteq R_0$,

$$\sum_{\beta\neq0}G_\beta=c_\lambda\int_{\mathbb{R}^d\setminus R_0}\frac{\mathrm{d}z}{|z|^{d+\lambda}}\le c_\lambda\int_{|z|>\frac{\Delta x}{2}}\frac{\mathrm{d}z}{|z|^{d+\lambda}}=c_\lambda\left(\frac{2}{\Delta x}\right)^\lambda\int_{|z|>1}\frac{\mathrm{d}z}{|z|^{d+\lambda}}.$$

Since $\Delta x^d\sum_{\alpha\in\mathbb{Z}^d}|U_\alpha^0|=\|\bar u_0\|_{L^1(\mathbb{R}^d)}<\infty$, we can iterate estimate (4.3) to find that $\sum_{\alpha\in\mathbb{Z}^d}|U_\alpha^n|<\infty$ and hence $\lim_{|\alpha|\to\infty}|U_\alpha^n|=0$ for all $n=0,\dots,N$.

Now we sum (4.1) over α to find that

$$\begin{aligned}\sum_{\alpha\in\mathbb{Z}^d}U_\alpha^{n+1}=\sum_{\alpha\in\mathbb{Z}^d}U_\alpha^n-\Delta t\sum_{\alpha\in\mathbb{Z}^d}\sum_{l=1}^d D_l^-F_l(U_\alpha^n,U_{\alpha+e_l}^n)& \\ +\Delta t\sum_{\alpha\in\mathbb{Z}^d}\sum_{\beta\neq0}G_\beta(A(U_{\alpha+\beta}^n)-A(U_\alpha^n)).&\end{aligned}$$

The proof is now complete if we can show that the F and G sums are equal to zero. The F-sum is telescoping and since $\lim_{|\alpha|\to\infty}|U_\alpha^n|=0$,

$$\sum_{\alpha\in\mathbb{Z}^d}D_l^-F_l(U_\alpha^n,U_{\alpha+e_l}^n)=\sum_{\alpha\in\mathbb{Z}^d}\frac{F(U_\alpha^n,U_{\alpha+e_l}^n)-F(U_{\alpha-e_l}^n,U_\alpha^n)}{\Delta x}=0.$$

To treat the G-sum, note that we have found above that

$$\sum_{\alpha\in\mathbb{Z}^d}\sum_{\beta\neq0}G_\beta\left|A(U_{\alpha+\beta}^n)-A(U_\alpha^n)\right|\le 2L_A\Delta t\sum_{\beta\neq0}G_\beta\sum_{\alpha\in\mathbb{Z}^d}|U_\alpha^n|<\infty,$$

and we also have that $\sum_\alpha|A(U_\alpha^n)|\le L_A\sum_\alpha|U_\alpha^n|<\infty$. In view of this we can now change the order of summation, and split the sums to find that

$$\begin{aligned}\sum_{\alpha\in\mathbb{Z}^d}\sum_{\beta\neq0}G_\beta(A(U_{\alpha+\beta}^n)-A(U_\alpha^n)) &= \sum_{\beta\neq0}G_\beta\sum_{\alpha\in\mathbb{Z}^d}\Big(A(U_{\alpha+\beta}^n)-A(U_\alpha^n)\Big) \\ &= \sum_{\beta\neq0}G_\beta\Big(\sum_{\alpha\in\mathbb{Z}^d}A(U_\alpha^n)-\sum_{\alpha\in\mathbb{Z}^d}A(U_\alpha^n)\Big)=0.\end{aligned}$$

The proof is now complete. □

Next, we check monotonicity by showing that the right-hand side of the numerical method (4.1) is a non-decreasing function of all its variables U_β^n. This is clear for all U_β^n such that $\beta\neq\alpha$ since the numerical flux F_l is increasing w.r.t. its first variable, non-increasing w.r.t. its second one, the function A is non-decreasing, and the weights G_β are all positive. Then we differentiate the right hand side of (4.1) w.r.t. U_α^n and find that it is non-negative provided the following the CFL condition holds,

$$2dL_F\frac{\Delta t}{\Delta x}+\Big(c_\lambda 2^\lambda L_A\int_{|z|>1}\frac{\mathrm{d}z}{|z|^{d+\lambda}}\Big)\frac{\Delta t}{\Delta x^\lambda}\leq 1. \tag{4.4}$$

We have thus proved the following result:

Lemma 4.3. *The numerical method* (4.1) *is monotone provided that the CFL condition* (4.4) *is assumed to hold.*

In what follows, the CFL condition (4.4) is always assumed to hold, and monotonicity is thus always ensured.

4.2. **Convergence toward the entropy solution.** We prove that any limit of a uniformly bounded sequence of solutions of the numerical method (4.1) is an entropy solution of (1.1).

Theorem 4.4. *If $\{\bar u\}$ is a sequence of solutions of* (4.1)*, uniformly bounded in $L^\infty(Q_T)$, and there exists $u\in L^\infty(Q_T)\cap C([0,T];L^1(\mathbb{R}^d))$ such that $\bar u\to u$ in $C([0,T];L^1(\mathbb{R}^d))$ as $\Delta x\to 0$, then u is an entropy solution of* (1.1)*.*

Proof. Note that part $i)$ in the definition of entropy solution (Definition 2.1) is already satisfied. Part $iii)$ follows since $\|\bar u(\cdot,0)-u_0\|_{L^1(\mathbb{R}^d)}\to 0$ as $\Delta x\to 0$ by the definition of $\bar u$. What remains to prove is part $ii)$.

First we prove that the numerical method (4.1) satisfies a discrete entropy inequality which resembles the one in $ii)$, Definition 2.1. To this end, let us introduce the notation $a\wedge b=\min\{a,b\}$ and $a\vee b=\max\{a,b\}$, choose an $r>0$, and exploit monotonicity to obtain the inequalities

$$\begin{aligned}U_\alpha^{n+1}\vee k\leq U_\alpha^n\vee k&-\Delta t\sum_{l=1}^d D_l^- F_l(U_\alpha^n\vee k,U_{\alpha+e_l}^n\vee k)\\&+\Delta t\sum_{0<\Delta x|\beta|\leq r}G_\beta\Big(A(U_{\alpha+\beta}^n\vee k)-A(U_\alpha^n\vee k)\Big)\\&+\Delta t\,\mathbf{1}_{(k,+\infty)}(U_\alpha^{n+1})\sum_{\Delta x|\beta|>r}G_\beta\Big(A(U_{\alpha+\beta}^n)-A(U_\alpha^n)\Big)\end{aligned}$$

and

$$\begin{aligned}U_\alpha^{n+1}\wedge k\leq U_\alpha^n\wedge k&-\Delta t\sum_{l=1}^d D_l^- F_l(U_\alpha^n\wedge k,U_{\alpha+e_l}^n\wedge k)\\&+\Delta t\sum_{0<\Delta x|\beta|\leq r}G_\beta\Big(A(U_{\alpha+\beta}^n\wedge k)-A(U_\alpha^n\wedge k)\Big)\\&+\Delta t\,\mathbf{1}_{(-\infty,k)}(U_\alpha^{n+1})\sum_{\Delta x|\beta|>r}G_\beta\Big(A(U_{\alpha+\beta}^n)-A(U_\alpha^n)\Big).\end{aligned}$$

Note that the polygonal set

$$P_r:=\bigcup_{0<\Delta x|\beta|\leq r}(x_\beta+R_0)$$

($x_\beta=\Delta x\beta$) does not include points from the box R_0, and converges to the punctured ball $\{z:0<|z|\leq r\}$ as $\Delta x\to 0$ in the sense that $\mathbf{1}_{P_r}(z)\to\mathbf{1}_{0<|z|\leq r}(z)$ a.e. as $\Delta x\to 0$.

Remember that $\eta_k(U_\alpha^n)=|U_\alpha^n-k|$, and let

$$Q_{h,l}(U_\alpha^n)=F_l(U_\alpha^n\vee k,U_{\alpha+e_l}^n\vee k)-F_l(U_\alpha^n\wedge k,U_{\alpha+e_l}^n\wedge k).$$

Thanks to the relations

$$\begin{aligned}|u-k| &= u\vee k - u\wedge k,\\ |A(u)-A(k)| &= A(u\vee k) - A(u\wedge k),\end{aligned}$$

we can subtract the above two inequalities to obtain that

$$\begin{aligned}\eta_k(U_\alpha^{n+1}) - \eta_k(U_\alpha^n) + \frac{\Delta t}{\Delta x}\sum_{l=1}^{d}\Big(Q_{h,l}(U_\alpha^n) - Q_{h,l}(U_{\alpha-e_l}^n)\Big)&\\ - \Delta t \sum_{0<\Delta x|\beta|\le r} G_\beta\Big(\eta_{A(k)}(A(U_{\alpha+\beta}^n)) - \eta_{A(k)}(A(U_\alpha^n))\Big)&\\ - \Delta t\ \eta_k'(U_\alpha^{n+1}) \sum_{\Delta x|\beta|>r} G_\beta\Big(A(U_{\alpha+\beta}^n) - A(U_\alpha^n)\Big) &\le 0.\end{aligned}$$

Let us take a nonnegative function $\varphi \in C_c^\infty(Q_T)$, and define $\varphi_\alpha^n = \varphi(x_\alpha, t_n)$. If we multiply both sides of the above inequality by φ_α^n, sum over all $\alpha \in \mathbb{Z}^d$ and all $n \in \{0,\dots,N\}$, and use summation by parts for the local terms, we end up with the cell entropy inequality

$$\begin{aligned}&\Delta x^d \Delta t \sum_{n=1}^{N}\sum_{\alpha\in\mathbb{Z}^d} \eta_k(U_\alpha^n)\ \frac{\varphi_\alpha^n - \varphi_\alpha^{n-1}}{\Delta t}\\ &+ \Delta x^d\Delta t \sum_{n=0}^{N}\sum_{\alpha\in\mathbb{Z}^d}\sum_{l=1}^{d} Q_{h,l}(U_\alpha^n)\ \frac{\varphi_{\alpha+e_l}^n - \varphi_\alpha^n}{\Delta x}\\ &+ \Delta x^d\Delta t \sum_{n=0}^{N}\sum_{\alpha\in\mathbb{Z}^d} \eta_{A(k)}(A(U_\alpha^n)) \sum_{0<\Delta x|\beta|\le r} G_\beta\Big(\varphi_{\alpha+\beta}^n - \varphi_\alpha^n\Big)\\ &+ \Delta x^d\Delta t \sum_{n=0}^{N}\sum_{\alpha\in\mathbb{Z}^d} \eta_k'(U_\alpha^{n+1})\ \varphi_\alpha^n \sum_{\Delta x|\beta|>r} G_\beta\Big(A(U_{\alpha+\beta}^n) - A(U_\alpha^n)\Big) \ge 0.\end{aligned} \tag{4.5}$$

To derive this inequality we have used the change of indices $(\beta,\alpha)\to(-\beta,\alpha+\beta)$ to see that

$$\begin{aligned}&\Delta x^d\Delta t \sum_{n=0}^{N}\sum_{\alpha\in\mathbb{Z}^d} \varphi_\alpha^n \sum_{0<\Delta x|\beta|\le r} G_\beta\Big(\eta_{A(k)}(A(U_{\alpha+\beta}^n)) - \eta_{A(k)}(A(U_\alpha^n))\Big)\\ &= \Delta x^d\Delta t \sum_{n=0}^{N}\sum_{\alpha\in\mathbb{Z}^d} \eta_{A(k)}(A(U_\alpha^n)) \sum_{0<\Delta x|\beta|\le r} G_\beta\Big(\varphi_{\alpha+\beta}^n - \varphi_\alpha^n\Big).\end{aligned}$$

Let $R_\alpha = x_\alpha + R$. We now claim that for each fixed $\Delta x > 0$, inequality (4.5) implies

(4.6)
$$\begin{aligned}
&\sum_{n=0}^{N-1}\sum_{\alpha\in\mathbb{Z}^d}\eta_k(U_\alpha^n)\int_{t_n}^{t_{n+1}}\int_{R_\alpha}\frac{\varphi(x,t)-\varphi(x,t-\Delta t)}{\Delta t}\,\mathrm{d}x\mathrm{d}t\\
&+\sum_{n=0}^{N-1}\sum_{\alpha\in\mathbb{Z}^d}\sum_{l=1}^{d}Q_{h,l}(U_\alpha^n)\int_{t_n}^{t_{n+1}}\int_{R_\alpha}\frac{\varphi(x+\Delta x\,e_l,t)-\varphi(x,t)}{\Delta x}\,\mathrm{d}x\mathrm{d}t\\
&+\sum_{n=0}^{N-1}\sum_{\alpha\in\mathbb{Z}^d}\eta_{A(k)}(A(U_\alpha^n))\int_{t_n}^{t_{n+1}}\int_{R_\alpha}\int_{P_r}\frac{\varphi(x+z,t)-\varphi(x,t)}{|z|^{d+\lambda}}\,\mathrm{d}z\mathrm{d}x\mathrm{d}t\\
&+\sum_{n=0}^{N-1}\sum_{\alpha\in\mathbb{Z}^d}\eta_k'(U_\alpha^{n+1})\sum_{\Delta x|\beta|>r}G_\beta\Big(A(U_{\alpha+\beta}^n)-A(U_\alpha^n)\Big)\int_{t_n}^{t_{n+1}}\int_{R_\alpha}\varphi(x,t)\,\mathrm{d}x\mathrm{d}t\geq 0.
\end{aligned}$$

To see this we proceed by contradiction, and assume that (4.6) is strictly negative. We then sum together several inequalities of the form (4.5) where, instead of $\varphi_\alpha^n = \varphi(x_\alpha,t_n)$ which are computed on the original space/time grid (x_α,t_n), we use the values $\varphi_\alpha^n = \varphi(\hat{x}_\alpha,\hat{t}_n)$ computed on the finer grid $(\hat{x}_\alpha,\hat{t}_n)$ where $\hat{x}_\alpha = (\Delta x/M)\alpha$ while $\hat{t}_n = n(\Delta t/M)$ for some $M > 0$. Note that, since all these inequalities of the form (4.5) share the same underlying numerical solution (U_i^n), they can be rearranged as one inequality, i.e.

(4.7)
$$\begin{aligned}
&\sum_{n=1}^{d}\sum_{\alpha\in\mathbb{Z}^d}\eta_k(U_\alpha^n)\left(\left(\frac{\Delta x}{M}\right)^d\frac{\Delta t}{M}\sum_{m:\,\hat{t}_m\in[t_n,t_{n+1})}\ \sum_{\gamma:\,\hat{x}_\gamma\in R_\alpha}\frac{\varphi_\gamma^m-\varphi_\gamma^{m-1}}{\Delta t}\right)\\
&+\sum_{n=0}^{d}\sum_{\alpha\in\mathbb{Z}^d}\sum_{l=1}^{d}Q_{h,l}(U_\alpha^n)\left(\left(\frac{\Delta x}{M}\right)^d\frac{\Delta t}{M}\sum_{m:\,\hat{t}_m\in[t_n,t_{n+1})}\ \sum_{\gamma:\,\hat{x}_\gamma\in R_\alpha}\frac{\varphi_{\gamma+e_l}^m-\varphi_\gamma^m}{\Delta x}\right)\\
&+\sum_{n=0}^{d}\sum_{\alpha\in\mathbb{Z}^d}\eta_{A(k)}(A(U_\alpha^n))\\
&\qquad\left(\left(\frac{\Delta x}{M}\right)^d\frac{\Delta t}{M}\sum_{m:\,\hat{t}_m\in[t_n,t_{n+1})}\ \sum_{\gamma:\,\hat{x}_\gamma\in R_\alpha}\ \sum_{0<\Delta x|\beta|\leq r}G_\beta\Big(\varphi_{\gamma+\beta}^m-\varphi_\gamma^m\Big)\right)\\
&+\sum_{n=0}^{d}\sum_{\alpha\in\mathbb{Z}^d}\eta_k'(U_\alpha^{n+1})\sum_{\Delta x|\beta|>r}G_\beta\Big(A(U_{\alpha+\beta}^n)-A(U_\alpha^n)\Big)\\
&\qquad\left(\left(\frac{\Delta x}{M}\right)^d\frac{\Delta t}{M}\sum_{m:\,\hat{t}_m\in[t_n,t_{n+1})}\ \sum_{\gamma:\,\hat{x}_\gamma\in R_\alpha}\varphi_\gamma^m\right)\geq 0
\end{aligned}$$

(loosely speaking, by summing all these inequalities of the form (4.5) together we are filling the mesh-sets $R_\alpha\times[t_n,t_{n+1})$ with several samples of the test function φ; this has been done in order to recreate in each mesh-set a Riemann sum approximation which gets closer and closer to its respective integral as the value of the control parameter M increases). The Riemann sum approximations in the first, second, and fourth term of (4.7) are arbitrarily close to their respective terms in (4.6) as

M increases. For the third term in (4.7) note that, cf. (4.2),

$$\begin{aligned}&\left(\frac{\Delta x}{M}\right)^d \frac{\Delta t}{M} \sum_{m:\,\hat{t}_m\in[t_n,t_{n+1})} \sum_{\gamma:\,\hat{x}_\gamma\in R_\alpha} \sum_{0<\Delta x|\beta|\le r} G_\beta\left(\varphi^m_{\gamma+\beta}-\varphi^m_\gamma\right)\\ &= \left(\frac{\Delta x}{M}\right)^d \frac{\Delta t}{M} \sum_{m:\,\hat{t}_m\in[t_n,t_{n+1})} \sum_{\gamma:\,\hat{x}_\gamma\in R_\alpha} \int_{z\in P_r} \frac{\bar{\varphi}(y_\gamma+z,\hat{t}_m)-\bar{\varphi}(y_\gamma,\hat{t}_m)}{|z|^{d+\lambda}}\,\mathrm{d}z\end{aligned}\tag{4.8}$$

(the definitions of $\bar{\varphi}, y_\gamma$ are analogous to those of $\bar{u}, y_\alpha$) and so the Riemann sum approximation on the right-hand side of (4.8) is, as M increases, arbitrarily close to

$$\int_{t_n}^{t_{n+1}} \int_{R_\alpha} \int_{z\in P_r} \frac{\varphi(x+z,t)-\varphi(x,t)}{|z|^{d+\lambda}}\,\mathrm{d}z\mathrm{d}x\mathrm{d}t.$$

This is due to the fact that, since we are integrating away from the singularity, the right-hand side of (4.8) is well defined, and the sum over all $(\hat{x}_\gamma, \hat{t}_m)$ can be moved inside the integral $z \in P_r$. Therefore, since (4.7) is arbitrarily close to the left-hand side of (4.6), the left-hand side of (4.6) cannot be negative, and we have produced a contradiction.

Using the piecewise constant space/time interpolation $\bar{u}$, we can now rewrite inequality (4.6) as

$$\begin{aligned}&\iint_{Q_T} \Bigg\{ \eta_k(\bar{u}(x,t))\,\partial_t\varphi(x,t) + \sum_{l=1}^d Q_{h,l}(\bar{u}(x,t))\,\partial_{x_l}\varphi(x,t)\\ &+ \eta_{A(k)}(A(\bar{u}(x,t)))\int_{P_r} \frac{\varphi(x+z,t)-\varphi(x,t)}{|z|^{d+\lambda}}\,\mathrm{d}z\\ &+ \eta_k'(\bar{u}(x,t+\Delta t))\,\varphi(x,t)\int_{\mathbb{R}^d\setminus P_r} \frac{A(\bar{u}(x+z,t))-A(\bar{u}(x,t))}{|z|^{d+\lambda}}\,\mathrm{d}z\Bigg\}\,\mathrm{d}x\mathrm{d}t\\ &\ge O(\Delta x)+O(\Delta t).\end{aligned}\tag{4.9}$$

Convergence up to a subsequence for the first three terms in (4.9) is immediate thanks to the a.e. convergence of $\bar{u}$ toward u. For the local terms this is already well known, cf. [26, Theorem 3.9]. For the term containing the inner integral P_r, convergence follows thanks to the convergence of $\mathbf{1}_{P_r} \to \mathbf{1}_{0<|z|<r}$ a.e., the properties of φ ($\int_{P_r} \frac{\varphi(x+z,t)-\varphi(x,t)}{|z|^{d+\lambda}}\,\mathrm{d}z$ is uniformly bounded and compactly supported), uniform boundedness of $\bar{u}$, and the fact that the function $\eta_k(\cdot)$ is continuous.

To conclude, we need to establish convergence for the term containing the discontinuous sign function $\eta_k'(\cdot)$, and we argue as in [19] (p. 109). First note that since $\bar{u} \to u$ in $C([0,T];L^1(\mathbb{R}^d))$, also $\bar{u}(\cdot,\cdot+\Delta t) \to u$ in $C([0,T];L^1(\mathbb{R}^d))$ and a.e. for a subsequence. Then note that $\eta_k'(s)$ is continuous for $s \ne k$, and that the measure of the set

$$\mathcal{U}_k = \{(x,t)\in Q_T:\ u(x,t)=k\}$$

is 0 for a.e. $k \in \mathbb{R}$. For such k, $\eta_k'(\bar{u}(\cdot+\Delta t,\cdot)) \to \eta_k'(u)$ a.e., and we can go to the limit in the term involving η_k' in (4.9) using the dominated convergence theorem, $|\eta_k'| \le 1$, and uniform boundedness of $\bar{u}$ and $A(\bar{u})$.

For the remaining k, we use an approximating sequence made of those k for which convergence holds true. To be more precise, let a_m, b_m be sequence of values such that $\mathrm{meas}(\mathcal{U}_{a_m}) = \mathrm{meas}(\mathcal{U}_{b_m}) = 0$, where $a_m \nearrow k$ and $b_m \searrow k$. Note that the mean value

$$\frac{1}{2}(\eta_{a_m}'(u)+\eta_{b_m}'(u)) \to \eta_k'(u) \qquad \text{as } a_m, b_m \to k.$$

Thus we can use the entropy inequality for the sequence a_m and the entropy inequality for the sequence b_m, take the average, and go to the limit to prove the entropy inequality for every critical value k. Convergence for the whole sequence $\bar{u}$ is a consequence of uniqueness for entropy solutions of (1.1). $\square$

4.3. **BV initial data: Compactness and existence.** We now show that the sequence of solutions of the method, $\{\bar{u} : \Delta x > 0\}$, is relatively compact whenever

$$u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \cap BV(\mathbb{R}^d).$$

Using this result and Theorem 4.4, we then obtain existence of an entropy solution of the initial value problem (1.1). We start by the following a priori estimates.

Lemma 4.5. *If $u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$, then, for all $t, s \geq 0$,*

i) $\|\bar{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)}$,
ii) $\|\bar{u}(\cdot,t)\|_{L^1(\mathbb{R}^d)} \leq \|u_0\|_{L^1(\mathbb{R}^d)}$,
iii) $|\bar{u}(\cdot,t)|_{BV(\mathbb{R}^d)} \leq |u_0|_{BV(\mathbb{R}^d)}$,
iv) $\|\bar{u}(\cdot,s) - \bar{u}(\cdot,t)\|_{L^1(\mathbb{R}^d)} \leq \sigma(|s-t| + \Delta t)$ *where, for some $c > 0$,*

$$\sigma(s) = \begin{cases} c\,|s| & \text{if } \lambda \in (0,1), \\ c\,|s \ln s| & \text{if } \lambda = 1, \\ c\,|s|^{\frac{1}{\lambda}} & \text{if } \lambda \in (1,2). \end{cases}$$

Lemma 4.5 along with a Kolmogorov type of compactness theorem, cf. Theorem A.8 in [26], yields the existence of a subsequence $\{\bar{u}\}$ which converges in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^d))$ (and hence a.e. up to a further subsequence) toward a limit u as $\Delta x \to 0$. Moreover, the limit u inherits all the a priori estimates *i)-iv)* in Lemma 4.5 (with $\Delta t = 0$). Moreover, by *ii)* and the dominated convergence theorem, we see that $\bar{u} \to u$ also in $C([0,T]; L^1(\mathbb{R}^d))$. In short, we have the following result:

Lemma 4.6. *The numerical solutions $\{\bar{u} : \Delta x > 0\}$ converge, up to a subsequence, toward a limit u in $C([0,T]; L^1(\mathbb{R}^d))$ as $\Delta x \to 0$. Moreover,*

$$u \in L^\infty(Q_T) \cap C([0,T]; L^1(\mathbb{R}^d)) \cap L^\infty(0,T; BV(\mathbb{R}^d)).$$

Lemma 4.6 and Theorem 4.4 imply the following existence result:

Theorem 4.7. (Existence) *If $u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, then there exists an entropy solution of the initial value problem* (1.1).

Proof. For initial data $u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$ existence is granted by the numerical method (4.1) (Lemma 4.6). For more general initial data $u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$, we consider approximations $u_{0,n} \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$ such that

$$\|u_0 - u_{0,n}\|_{L^1(\mathbb{R}^d)} \to 0 \text{ as } n \to \infty.$$

Let u_m, u_n denote the entropy solutions corresponding to $u_{0,n}, u_{0,m}$ respectively, and use the L^1-contraction (Theorem 3.1) to see that

$$\|u_n - u_m\|_{C([0,T];L^1(\mathbb{R}^d))} \leq \|u_{0,n} - u_{0,m}\|_{L^1(\mathbb{R}^d)} \to 0 \text{ as } n, m \to \infty.$$

Therefore, the sequence of entropy solutions $\{u_n\}$ is Cauchy in $C([0,T]; L^1(\mathbb{R}^d))$ and admits a limit u. To prove that u is also an entropy solution of (1.1), one can pass to the limit $n \to \infty$ in the entropy inequality for u_n. $\square$

Proof of Lemma 4.5. The maximum principle *i)* is a direct consequence of monotonicity. To see this let $s = \sup_{\alpha \in \mathbb{Z}^d} |U^n_\alpha|$, and choose $U^n \equiv s$ to obtain that

$$U^{n+1}_\alpha \leq s - \Delta t \sum_{l=1}^{d} D^-_l F_l(s,s) + \Delta t \sum_{\beta \neq 0} G_\beta(A(s) - A(s)) = s.$$

Similarly, choosing $U^n \equiv -s$, one obtains $U_\alpha^{n+1} \geq -s$. Furthermore, since the numerical method (4.1) is conservative, monotone, and translation invariant (translation invariance is a consequence of the fact that the numerical method does not explicitly depend on the variables x_α, t_n), inequalities *ii)-iii)* are consequences of the results due to Crandall-Tartar [15].

We now prove *iv)*. By (4.1) and Lipschitz continuity of F_l,

$$\begin{aligned}&\left|U_\alpha^{n+1} - U_\alpha^n\right|\\ &\leq \frac{\Delta t L_F}{\Delta x}\sum_{l=1}^d \Big(|U^n_{\alpha+e_l} - U^n_\alpha| + |U^n_\alpha - U^n_{\alpha-e_l}|\Big) + \Delta t \sum_{\beta\neq 0} G_\beta |A(U^n_{\alpha+\beta}) - A(U^n_\alpha)|.\end{aligned}$$

Let us multiply by Δx^d in the above inequality, and sum over $\alpha \in \mathbb{Z}^d$ to see that

$$\begin{aligned}&\Delta x^d \sum_{\alpha\in\mathbb{Z}^d} \left|U_\alpha^{n+1} - U_\alpha^n\right|\\ &\leq 2L_F \Delta x^{d-1}\Delta t \sum_{l=1}^d \sum_{\alpha\in\mathbb{Z}^d} |U^n_{\alpha+e_l} - U^n_\alpha| + \Delta x^d \Delta t \sum_{\alpha\in\mathbb{Z}^d}\sum_{\beta\neq 0} G_\beta |A(U^n_{\alpha+\beta}) - A(U^n_\alpha)|.\end{aligned}$$

Let $\bar{u}^n(\cdot) = \bar{u}(\cdot, t_n)$ and note that the first term then is equal to

$$2L_F\Delta t \sum_{l=1}^d \int_{\mathbb{R}^{d-1}} |\bar{u}^n(\cdot, x')|_{BV_{x_l}(\mathbb{R})}\ \mathrm{d}x' \leq 2dL_F \Delta t |\bar{u}^n|_{BV(\mathbb{R}^d)} = O(\Delta t),$$

while the second term can be estimated by (cf. (4.2))

$$\begin{aligned}&\Delta x^d \sum_{\alpha\in\mathbb{Z}^d}\sum_{\beta\neq 0} G_\beta \left|A(U^n_{\alpha+\beta}) - A(U^n_\alpha)\right|\\ &\leq c_\lambda \sum_{\alpha\in\mathbb{Z}^d} \int_{|z|>\frac{\Delta x}{2}} \frac{|A(\bar{u}^n(y_\alpha+z)) - A(\bar{u}^n(y_\alpha))|}{|z|^{d+\lambda}}\ \mathrm{d}z \Delta x^d\\ &\leq c_\lambda L_A \bigg(\int_{\frac{\Delta x}{2}<|z|<1} + \int_{|z|>1}\bigg) \sum_{\alpha\in\mathbb{Z}^d} \frac{|\bar{u}^n(y_\alpha+z) - \bar{u}^n(y_\alpha)|}{|z|^{d+\lambda}}\ \Delta x^d \mathrm{d}z\\ &\leq c_\lambda L_A \bigg(|\bar{u}^n|_{BV(\mathbb{R}^d)} \int_{\frac{\Delta x}{2}<|z|<1} \frac{|z|}{|z|^{d+\lambda}}\ \mathrm{d}z + 2\|\bar{u}^n\|_{L^1(\mathbb{R}^d)} \int_{|z|>1} \frac{\mathrm{d}z}{|z|^{d+\lambda}}\bigg).\end{aligned}$$

Easy computations in polar coordinates show that the second integral is $O(1)$ while

$$I_{\Delta x} = \int_{\frac{\Delta x}{2}<|z|<1} \frac{|z|}{|z|^{d+\lambda}}\ \mathrm{d}z = \begin{cases} O(1) & \text{if } \lambda \in (0,1),\\ O(|\ln \Delta x|) & \text{if } \lambda = 1,\\ O(\Delta x^{1-\lambda}) & \text{if } \lambda \in (1,2).\end{cases}$$

Adding all the above estimates yields

$$\Delta x^d \sum_{\alpha\in\mathbb{Z}^d} |U_\alpha^{n+1} - U_\alpha^n| = O(\Delta t) + O(\Delta t I_{\Delta x}) + O(\Delta t).$$

By the CFL condition (4.4), $\Delta t I_{\Delta x} = \sigma(\Delta t)$, and the result follows. □

5. Extension to general Lévy operators

The ideas developed in this paper can also be used to establish well-posedness for entropy solutions of a more general class of fractional equations of the form

$$\begin{cases} \partial_t u + \nabla\cdot f(u) = g_\mu[A(u)] & \text{in } Q_T = \mathbb{R}^d \times (0,T),\\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^d,\end{cases} \tag{5.1}$$

where the fractional Laplacian g has been replaced with a more general Lévy operator g_μ:

$$g_\mu[\phi](x) = \int_{|z|>0} \phi(x+z) - \phi(x) - z \cdot \nabla\phi(x)\mathbf{1}_{|z|<1} \, \mathrm{d}\mu(z),$$

where the Lévy measure μ is a positive Radon measure satisfying

$$\int_{|z|>0} |z|^2 \wedge 1 \ \mu(\mathrm{d}z) < \infty. \tag{5.2}$$

Note that g_μ is self-adjoint if and only if μ is symmetric: $\mu(-B) = \mu(B)$ for all open sets B. The adjoint g_μ^* (defined through $\int u g_\mu[v] = \int g_\mu^*[u]v$) equals

$$g_\mu^*[\phi](x) = \int_{|z|>0} \phi(x-z) - \phi(x) + z \cdot \nabla\phi(x)\mathbf{1}_{|z|<1} \, \mathrm{d}\mu(z).$$

A Taylor expansion shows that both $g_\mu[\phi]$ and $g_\mu^*[\phi]$ are well defined whenever ϕ is C^2 and bounded. The gradient term is needed when μ is not radially symmetric, in the radially symmetric case g_μ $(= g_\mu^*)$ can be defined as before as a principal value and no gradient term. The operator g_μ is the generator of a pure jump Lévy process and these processes have many applications in Physics and Finance, cf. e.g. [4].

We need a modified definition of Entropy solutions. Remember the notation η_k and q_k introduced in Section 2, and define for $r > 0$,

$$g_\mu[\varphi] = g_{\mu,r}[\varphi] + g_\mu^r[\varphi] - \gamma_\mu^r \cdot \nabla\varphi$$

where $g_{\mu,r}[\varphi](x) = g_\mu[\varphi(\cdot)\mathbf{1}_{|z|\leq r}](x)$,

$$g_\mu^r[\varphi](x) = \int_{|z|>r} \varphi(x+z) - \varphi(x) \ \mu(\mathrm{d}z), \quad \text{and} \quad \gamma_\mu^r = \int_{r<|z|<1} z \ \mu(\mathrm{d}z).$$

We also use the notation $g_{\mu,r}^*$ and $g_\mu^{r,*}$ for the adjoint operators, and note that

$$g_\mu^*[\phi] = g_{\mu,r}^*[\phi] + g_\mu^{r,*}[\phi] + \gamma_\mu^r \cdot \nabla\phi.$$

Let us point out that the adjoint operator g_μ^* could have also been defined as g_ν with $\nu(B) = \mu(-B)$. From this equivalent definition it is clear that the adjoint operator g_μ^* is still a Lévy operator.

Definition 5.1. *A function u is an entropy solution of the initial value problem* (5.1) *provided that*

i) $u \in L^\infty(Q_T) \cap C([0,T]; L^1(\mathbb{R}^d))$*;*

ii) for all $k \in \mathbb{R}$, all $r > 0$, and all nonnegative test functions $\varphi \in C_c^\infty(Q_T)$,

$$\iint_{Q_T} \eta_k(u)\partial_t\varphi + q_k(u) \cdot \nabla\varphi + \eta_{A(k)}(A(u))\, g_{\mu,r}^*[\varphi] \\ + \eta_k'(u)\, g_\mu^r[A(u)]\,\varphi + \eta_{A(k)}(A(u))\ \gamma_\mu^r \cdot \nabla\varphi \ \mathrm{d}x\mathrm{d}t \geq 0;$$

iii) $u(x,0) = u_0(x)$ *a.e.*

Remark 5.1. All terms in *ii*) are well-defined in view of *i*). Except for the g_μ^r-term, this follows from the discussion proceeding Definition 2.1 – see Remark 2.2. Note that the integrand of $g_\mu^r[A(u)]$ is measurable w.r.t. the product measure $\mathrm{d}\mu(z)\mathrm{d}x\mathrm{d}t$ since since it is the $\mathrm{d}\mu(z)\mathrm{d}x\mathrm{d}t$-a.e. limit of continuous functions. This follows readily from the fact that u is the $\mathrm{d}x\mathrm{d}t$-a.e. limit of smooth functions. Integrability then follows by Fubini's theorem, integrate first w.r.t. to $\mathrm{d}x\mathrm{d}t$ and then w.r.t. $\mathrm{d}\mu(z)$ using (5.2). By Fubini we also see that $g_\mu^r[A(u)] \in C([0,T]; L^1(\mathbb{R}^d))$ and it easily follows that the g_μ^r-term is well-defined.

Again classical solutions are entropy solutions and entropy solutions are weak solutions. The proof is essentially the same as the one given in Section 2 with the additional information that whenever $A(u)$ is smooth

$$\eta_k'(u(x))\nabla[A(u(x))] = \eta_{A(k)}'(A(u(x))\nabla[A(u(x))] = \nabla[\eta_{A(k)}(A(u(x)))] \ a.e.$$

We also have a L^1-contraction and hence uniqueness result:

Theorem 5.2. *Let u and v be two entropy solutions of the initial value problem* (5.1) *with initial data u_0 and v_0. Then, for all $t \in (0, T)$,*

$$\|u(\cdot, t) - v(\cdot, t)\|_{L^1(\mathbb{R}^d)} \leq \|u_0 - v_0\|_{L^1(\mathbb{R}^d)}.$$

Proof. We proceed as in the proof of Theorem 3.1: let us take the entropy inequality for $u = u(x, t)$ and the one for $v = v(y, s)$, integrate both in space/time, and sum the resulting inequalities together to obtain an expression equivalent to (3.3). At this point we use the change of variables $(x, y) \to (x - z, y - z)$ to obtain the inequality

$$\begin{aligned}\iint_{Q_T} \iint_{Q_T} & \eta(u(x,t), v(y,s))\, (\partial_t + \partial_s)\psi(x,y,t,s) \\ & + q(u(x,t), v(y,s)) \cdot (\nabla_x + \nabla_y)\psi(x,y,t,s) \\ & + \eta(A(u(x,t)), A(v(y,s)))\ g_{\mu,r}^*[\psi(\cdot,y,t,s)](x) \\ & + \eta(A(u(x,t)), A(v(y,s)))\ g_{\mu,r}^*[\psi(x,\cdot,t,s)](y) \\ & + \eta(A(u(x,t)), A(v(y,s)))\ \gamma_\mu^r \cdot (\nabla_x + \nabla_y)\psi(x,y,t,s) \\ & + \eta(A(u(x,t)), A(v(y,s)))\ \tilde{g}_\mu^{r,*}[\psi(\cdot,\cdot,t,s)](x,y)\ \mathrm{d}w \geq 0,\end{aligned}$$

where

$$\tilde{g}_\mu^{r,*}[\varphi(\cdot,\cdot)](x,y) = \int_{|z|>r} \varphi(x-z, y-z) - \varphi(x,y)\ \mu(\mathrm{d}z).$$

We can now send $r \to 0$ and recover the equivalent of expression (3.7) in the present setting,

$$\begin{aligned}\iint_{Q_T} \iint_{Q_T} & \eta(u(x,t), v(y,s))\, (\partial_t + \partial_s)\psi(x,y,t,s) \\ & + q(u(x,t), v(y,s)) \cdot (\nabla_x + \nabla_y)\psi(x,y,t,s) \\ & + \eta(A(u(x,t)), A(v(y,s)))\ \tilde{g}_\mu^*[\psi(\cdot,\cdot,t,s)](x,y)\ \mathrm{d}w \geq 0,\end{aligned}$$

where

$$\begin{aligned}&\tilde{g}_\mu^*[\varphi(\cdot,\cdot)](x,y) \\ &= \int_{|z|>0} \varphi(x-z,y-z) - \varphi(x,y) + z \cdot (\nabla_x + \nabla_y)\varphi(x,y)\mathbf{1}_{|z|<1}\ \mu(\mathrm{d}z).\end{aligned}$$

From now on, the proof follows the one of Theorem 3.1 (just replace the operator g therein with the operator g_μ^*). □

Existence of solutions can be obtained e.g. by the vanishing viscosity method and a compensated compactness argument, but we do not give the details here. We just remark that the vanishing viscosity equations have smooth solutions since the principle term is the (linear 2nd order) Laplace term.

Theorem 5.3. *There exists a unique entropy solution of the initial value problem* (5.1).

6. Connections to HJB equations

In one space dimension it is well known that the gradient of the (viscosity) solution of a HJB equation is the (entropy) solution of a conservation law, see e.g. [35]. Variants of this result are still true in the current fractional setting as we will explain now. First we consider the following two initial value problems in one space dimension:

$$\text{(HJB)} \qquad \begin{cases} u_t + f(\partial_x u) + g[u] = \varepsilon \partial_x^2 u & \text{in } Q_T, \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}, \end{cases}$$

and

$$\text{(FCL)} \qquad \begin{cases} v_t + \partial_x f(v) + g[v] = \varepsilon \partial_x^2 v & \text{in } Q_T, \\ v(x,0) = \partial_x u_0(x) & \text{in } \mathbb{R}, \end{cases}$$

for any $\varepsilon \geq 0$. The first equation is a HJB equation and the second one a fractional conservation law. To simplify, let us consider the following strong but rather standard regularity assumptions:

(a1) $f \in C^2(\mathbb{R})$,
(a2) u_0 is bounded and Lipschitz continuous, and
(a3) $\partial_x u_0$ is bounded and belongs to $L^1(\mathbb{R}) \cap BV(\mathbb{R})$.

Standard results then show that:

(i) there is a unique bounded Hölder continuous (viscosity) solution u^ε of (HJB) for any $\varepsilon \geq 0$ [28],
(ii) there is a unique bounded (entropy) solution $v^\varepsilon \in L^1(0,T; BV \cap L^1)$ of the (fractional) conservation law for any $\varepsilon \geq 0$ [1, 21],
(iii) when $\varepsilon > 0$ both u_ε and v_ε are C^2,
(iv) $u^\varepsilon \to u^0$ uniformly [28] and $v^\varepsilon \to v^0$ in L^1 [1] as $\varepsilon \to 0$.

By differentiating (HJB) and using uniqueness for (FCL), we find that

$$v^\varepsilon = \partial_x u^\varepsilon$$

for any $\varepsilon > 0$, and hence

$$\iint v^\varepsilon \phi = -\iint u^\varepsilon \partial_x \phi \quad \text{for any} \quad \phi \in C_c^\infty(Q_T).$$

Sending $\varepsilon \to 0$ in the above inequality using dominated convergence theorem and (iv) then leads to

$$\iint v^0 \phi = -\iint u^0 \partial_x \phi \quad \text{for any} \quad \phi \in C_c^\infty(Q_T),$$

and we have the following result:

Theorem 6.1. *The distributional x-derivative of the viscosity solution of (HJB) is equal to the unique entropy solution of (FCL).*

The only part missing in the proof of this theorem, is the proof of (iii). This result follows e.g. from energy estimates and standard parabolic compactness results (yields $L^2(0,T; H^1)$ solutions) combined with regularity theory for the Heat equation, interpolation, and bootstrapping arguments (yields smooth solutions). We skip the long and fairly standard details.

If we drop the convection term, we get a similar correspondence in any space dimension. Consider the following two initial value problems:

$$\text{(HJB2)} \qquad \begin{cases} u_t - A(g_\mu[u]) = \varepsilon \Delta u & \text{in } Q_T, \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^d, \end{cases}$$

and

$$\text{(FDE)} \qquad \begin{cases} v_t - g_\mu[A(u)] = \varepsilon\Delta v & \text{in } Q_T, \\ v(x,0) = g_\mu[u_0](x) & \text{in } \mathbb{R}^d, \end{cases}$$

for any $\varepsilon \geq 0$. The first equation is still a HJB equation while the second one is a degenerate fractional diffusion equation. To simplify, let us consider the following rather strong regularity assumptions:

(b1) $A \in C^2(\mathbb{R})$ is non-decreasing and Lipschitz continuous,
(b2) u_0 is bounded and Lipschitz continuous, and
(b3) $g_\mu[u_0]$ is bounded, BV, and belongs to L^1.

Again we have the following of properties:

(i) there is a unique bounded Hölder continuous (viscosity) solution u^ε of (HJB2) for any $\varepsilon \geq 0$,
(ii) there is a unique bounded (entropy) solution $v^\varepsilon \in L^1(0,T;BV \cap L^1)$ of (FDE) for any $\varepsilon \geq 0$,
(iii) when $\varepsilon > 0$ both u_ε and v_ε are C^2,
(iv) $u^\varepsilon \to u^0$ uniformly and $v^\varepsilon \to v^0$ in L^1 as $\varepsilon \to 0$.

By applying g_μ to (HJB2) and using uniqueness for (FDE), we find that

$$v^\varepsilon = g_\mu[u^\varepsilon]$$

for any $\varepsilon > 0$, and hence

$$\iint v^\varepsilon \phi = \iint u^\varepsilon g_\mu^*[\phi] \quad \text{for any} \quad \phi \in C_c^\infty(Q_T).$$

Sending $\varepsilon \to 0$ in the above inequality using the dominated convergence theorem and (iv) then leads to

$$\iint v^0 \phi = \iint u^0 g_\mu^*[\phi] \quad \text{for any} \quad \phi \in C_c^\infty(Q_T),$$

and we have the following result:

Theorem 6.2. *If u is the unique viscosity solution of (HJB2), then $v = g_\mu[u]$ (where g_μ is taken in the sense of distributions) is the unique entropy solution of (FDE).*

Proof. For the HJB equation well-posedness of viscosity solutions for $\varepsilon \geq 0$ and the uniform convergence $u^\varepsilon \to u^0$ is fairly standard and can be found e.g. as a simple special case of results in [28].

Existence and uniqueness in (ii) follow from this paper for $\varepsilon = 0$. The arguments in this paper can easily be extended to include the $\varepsilon\Delta v$-term (this is standard) and hence we have (ii) for any $\varepsilon > 0$. The limit $v^\varepsilon \to v$ can be obtained through a standard Kuznetzov type argument, cf. [1, 10] for the case when A is linear. We will give the result for the non-linear case in a future paper. The regularity for $\varepsilon > 0$ is clear since the $\varepsilon\Delta v$-term is the principal term in the equation. It follows e.g. from (i) energy estimates and a classical parabolic compactness argument (yields $L^2(0,T;H^1(\mathbb{R}^d))$-solutions) and (ii) regularity theory for the Heat equation combined with bootstrapping (yields smooth solutions). The fractional term is always related to integer order derivatives through interpolation estimates. The detailed proof is long and rather classical and is best left to the interested reader. □

Remark 6.3. Such correspondences between HJB equations and degenerate convection diffusion equation can be useful for at least two reasons.

1) They allow for integral representation formulas for the solutions of the degenerate convection diffusion equations via representation formulas for the solutions of the HJB equations. See e.g. chapter 3.4 in [24] for the case of one dimensional scalar conservation laws.
2) They allow for efficient numerical methods for the non-divergence form HJB equation, by solving the divergence form degenerate convection diffusion equation by finite elements or spectral methods and then using the correspondence (and the HJB equation) to find the HJB solution.

The solutions of the above HJB equations are value functions of suitably defined stochastic differential games (see e.g. [27]), i.e. they have integral representation formulas. Since HJB equations are fully non-linear non-divergence form equations, it is not natural or easy to solve them directly by well-established, flexible, and efficient methods like the finite element and spectral methods. Such methods do apply to divergence form equations like the degenerate convection diffusion equations (cf. e.g. [13, 30, 11]).

7. Numerical experiments

We conclude this paper by presenting some experimental results obtained using the numerical method (4.1) with $d = 1$. We simulate fractional strongly degenerate equations and compare them to fractional conservation laws and local convection diffusion equations. Our simulations give some insight into how the solutions of these new equations behave. Note that this type of fractional equations have never been simulated (or analyzed) before.

In our computations, we restrict ourselves to the bounded region $\Omega = \{x : |x| \leq 2\}$ and impose zero Dirichlet boundary conditions on the whole exterior domain $\{x : |x| > 2\}$. We consider the degenerate fractional convection-diffusion equations with Burgers type convection ($f(u) = u^2/2$),

$$\partial_t u + u\partial_x u = g[A(u)], \tag{7.1}$$

and fractional degenerate diffusion equations ($f \equiv 0$),

$$\partial_t u = g[A(u)], \tag{7.2}$$

for two different strongly degenerate diffusions, defined through two different A's:

$$A_1(u) = \max(u, 0)$$

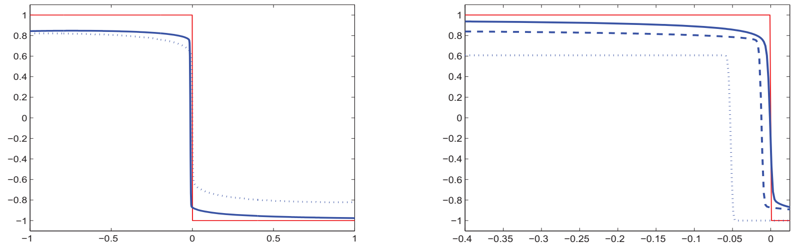
and

$$A_2(u) = \begin{cases} 0 & u \leq 0.5, \\ 5(2.5u - 1.25)(u - 0.5) & 0.5 < u \leq 0.6, \\ 1.25 + 2.5(u - 0.6) & u > 0.6. \end{cases}$$

The numerical experiments below show e.g. how solutions of (1.1) can develop shock discontinuities in finite time for all $\lambda \in (0, 2)$. Furthermore, they show that, contrary to the linear case, equation (7.2) does not have smooth solutions for $t > 0$ when the initial data is non-smooth. We also observe that for $\lambda \approx 2$, solutions are very close to solutions of the corresponding local problem with $\lambda = 2$.

In figure Figure 1 (a)–(b) we plot the solutions of (7.1) with linear and non-linear fractional diffusion ($A(u) = u$ and $A = A_1$) to show how the non-linearity influences both the shock size and speed.

Figure 2 (a) shows that a shock discontinuity develops in finite time in the region where A_2 is zero. This phenomenon is well known for degenerate convection-diffusion equations (1.2) as shown in Figure 2 (b). Here and in what follows, we



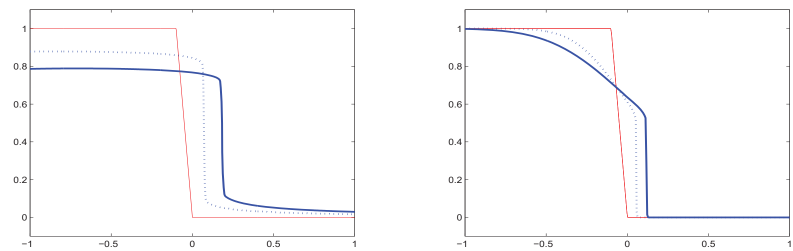
(a) (b)

FIGURE 1. Numerical solutions of (7.1) at $T = 0.5$ with $\Delta x = 1/500$ and piecewise constant initial data: (a) $A = A_1$ (solid) and $A(u) = u$ (dotted) for $\lambda = 0.5$; (b) $A = A_1$ with $\lambda = 0.001$ (dotted), $\lambda = 0.5$ (dashed), and $\lambda = 0.999$ (solid).

have used the convergent numerical scheme (cf. [25])

$$U_i^{n+1} = U_i^n - \Delta t D^- F(U_i^n, U_{i+1}^n) + (2\pi)^2 \Delta t D^- \left(\frac{A(U_{i+1}^n) - A(U_i^n)}{\Delta x} \right). \tag{7.3}$$

to compute the solutions of degenerate convection-diffusion equations (1.2).

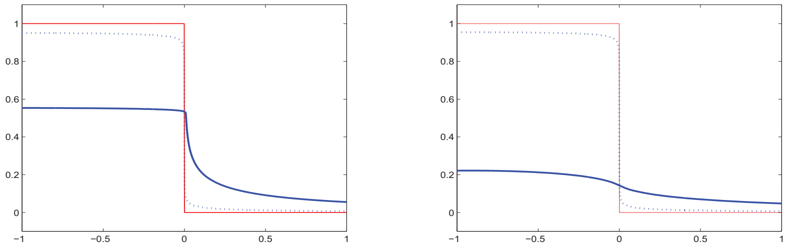


(a) (b)

FIGURE 2. Burgers's flux and $A = A_2$ with $\Delta x = 1/500$ and piecewise linear initial data: (a) solutions of (4.1) with $\lambda = 0.3$ at $T = 0.25$ (dotted) and $T = 0.5$ (solid); (b) solutions of (7.3) at $T = 0.01$ (dotted) and $T = 0.025$ (solid).

FIGURE 3 (a) displays the solutions of (7.2) with $A(u) = u$ and $A = A_2$. Note that, when $A = A_2$, the initially discontinuous solution becomes continuous in finite time but not differentiable. In the non-degenerate case, $\partial_t u = g[u]$, the initially discontinuous solution becomes smooth immediately for all values of λ, cf. FIGURE 3 (b). This behavior agrees with results from [20].

In FIGURE 4 we compare the solutions of (7.2) for $\lambda \approx 2$, with the solutions of a properly scaled equation (1.2) ($\lambda = 2$). We use our scheme (4.1) to compute the first set of solutions, while scheme (7.3) is used to compute the second. Again, we have restricted our computational domain to Ω. As expected, the solutions of the two equations are very close since $-(-\Delta)^{\frac{\lambda}{2}}\phi \to \Delta\phi$ as $\lambda \to 2$ for regular enough ϕ. The two methods are however fundamentally different: (7.3) uses a three-points stencil, while (4.1) uses a whole-domain stencil.



(a) (b)

FIGURE 3. Solutions of (7.2) with $\Delta x = 1/500$, $\lambda = 0.3$, and piecewise constant initial data: (a) $A = A_1$ with $T = 0.1$ (dotted) and $T = 3$ (solid); (b) $A(u) = u$ with $T = 0.1$ (dotted) and $T = 3$ (solid).

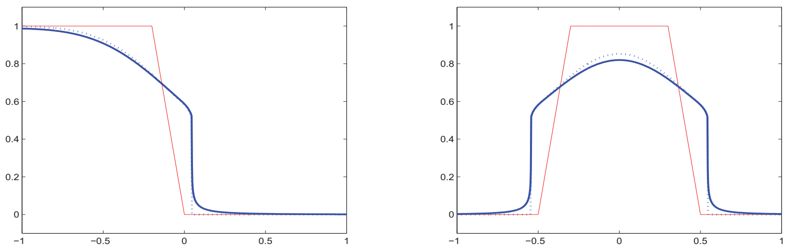


FIGURE 4. $A(u) = A_2$ with $T = 0.005$, $\Delta x = 1/500$, and piecewise constant initial data: (a)–(b) solutions of the non-local numerical method (4.1) (solid) with $\lambda \approx 2$ compared with solutions of the local numerical method (7.3) (dotted).

APPENDIX A. A TECHNICAL RESULT

In this section, we prove a technical result used in the proof of Lemma 2.4.

Lemma A.1. *Let $u \in BV(\mathbb{R}^d)$, then*

$$\int_{\mathbb{R}^d} |u(x+z) - u(x)| \, \mathrm{d}x \leq \sqrt{d}\,|z||u|_{BV(\mathbb{R}^d)}. \tag{A.1}$$

Note that a more refined argument would give a factor 1 instead of $\sqrt{d}$ in (A.1). This is unimportant in this paper and we skip it. We now give a proof for (A.1) in the case $d = 2$, analogous ideas can then be used in higher dimensions.

Proof. We define the total variation $|u|_{BV(\mathbb{R}^2)}$ as, cf. [26, expression A.19],

$$|u|_{BV(\mathbb{R}^2)} = \int_{\mathbb{R}} |u(x_1, \cdot)|_{BV(\mathbb{R})} \, \mathrm{d}x_1 + \int_{\mathbb{R}} |u(\cdot, x_2)|_{BV(\mathbb{R})} \, \mathrm{d}x_2. \tag{A.2}$$

Then, since $\int_{\mathbb{R}} |u(x+z) - u(x)| \, \mathrm{d}x \leq |z||u|_{BV(\mathbb{R})}$, we write

$$\int_{\mathbb{R}^2} |u(x+z) - u(x)| \, \mathrm{d}x = \int_{\mathbb{R}^2} |u(x_1 + z_1, x_2 + z_2) - u(x_1, x_2)| \, \mathrm{d}x_1 \mathrm{d}x_2$$

which, by triangle inequality, is less than or equal to

$$\begin{aligned}
&\int_{\mathbb{R}^2} |u(x_1+z_1,x_2+z_2)-u(x_1,x_2+z_2)|\ \mathrm{d}x_1\mathrm{d}x_2\\
&\qquad+\int_{\mathbb{R}^2} |u(x_1,x_2+z_2)-u(x_1,x_2)|\ \mathrm{d}x_1\mathrm{d}x_2\\
&\le |z_1|\int_{\mathbb{R}} |u(\cdot,x_2+z_2)|_{BV(\mathbb{R})}\ \mathrm{d}x_2\\
&\qquad+|z_2|\int_{\mathbb{R}} |u(x_1,\cdot)|_{BV(\mathbb{R})}\ \mathrm{d}x_1\\
&\le \sqrt{2}|z||u|_{BV(\mathbb{R}^2)},
\end{aligned}$$

thanks to (A.2) and inequality $|z_1|+|z_2|\le\sqrt{2}\,|z|$. □

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(Simone Cifani), Department of Mathematics, Norwegian University of Science and Technology (NTNU), N-7491 Trondheim, Norway
E-mail address: simone.cifani@math.ntnu.no
URL: http://www.math.ntnu.no/~cifani/

(Espen R. Jakobsen), Department of Mathematics, Norwegian University of Science and Technology (NTNU), N-7491 Trondheim, Norway
E-mail address: erj@math.ntnu.no
URL: http://www.math.ntnu.no/~erj/

Paper IV

On the spectral vanishing viscosity method for periodic fractional conservation laws

S. Cifani and E. R. Jakobsen

Submitted for publication, 2010

ON THE SPECTRAL VANISHING VISCOSITY METHOD FOR PERIODIC FRACTIONAL CONSERVATION LAWS

SIMONE CIFANI AND ESPEN R. JAKOBSEN

ABSTRACT. We introduce and analyze a spectral vanishing viscosity approximation of periodic fractional conservation laws. The fractional part of these equations can be a fractional Laplacian or other non-local operators that are generators of pure jump Lévy processes. To accommodate for shock solutions, we first extend to the periodic setting the Kružkov-Alibaud entropy formulation and prove well-posedness. Then we introduce the numerical method, which is a non-linear Fourier Galerkin method with an additional spectral viscosity term. This type of approximation was first introduced by Tadmor for pure conservation laws. We prove that this *non-monotone* method converges to the entropy solution of the problem, that it retains the spectral accuracy of the Fourier method, and that it diagonalizes the fractional term reducing dramatically the computational cost induced by this term. We also derive a robust L^1-error estimate, and provide numerical experiments for the fractional Burgers' equation.

1. INTRODUCTION

In this paper we are concerned with a spectral vanishing viscosity (henceforth SVV) approximation for periodic solutions of non-local or fractional conservation laws of the form

$$\begin{cases} \partial_t u + \partial_x \cdot f(u) = -(-\Delta)^{\lambda/2} u, & (x,t) \in D_T \\ u(x,0) = u_0(x), & x \in \Lambda, \end{cases}$$

for $\lambda \in (0,2)$, or more generally, for

$$\begin{cases} \partial_t u + \partial_x \cdot f(u) = \mathcal{L}^\mu[u], & (x,t) \in D_T \\ u(x,0) = u_0(x), & x \in \Lambda, \end{cases} \tag{1.1}$$

where $D_T = \Lambda \times (0,T)$ and $\Lambda = (0,2\pi)^d$, and $\mathcal{L}^\mu[\cdot]$ is a non-local (Lévy type) operator defined as

$$\mathcal{L}^\mu[\phi(\cdot)](x) = \int_{|z|>0} \phi(x+z) - \phi(x) - z \cdot \partial_x \phi(x)\, \mathbf{1}_{|z|<1}\, \mathrm{d}\mu(z), \tag{1.2}$$

where $\mathbf{1}(\cdot)$ is the indicator function. Throughout the paper we assume that

($\mathbf{A}$.1) $f = (f_1, \ldots, f_d)$ with $f_j \in C^s(\mathbb{R})$ for all $j = 1, \ldots, d$ (s to be defined);

($\mathbf{A}$.2) $\mu \geq 0$ is a Radon measure such that $\int_{|z|>0} |z|^2 \wedge 1\, \mathrm{d}\mu(z) < \infty$;

($\mathbf{A}$.3) $u_0 \in L^\infty(\Lambda) \cap BV(\Lambda)$, u_0 is Λ-periodic.

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Key words and phrases. Fractional/fractal conservation laws, entropy solutions, Fourier spectral methods, spectral vanishing viscosity, convergence, error estimate.

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Here and in the rest of the paper, $a \wedge b = \min(a, b)$,

$$\partial_t = \frac{\partial}{\partial t}, \quad \partial_j = \frac{\partial}{\partial x_j} \quad \text{and} \quad \partial_x = (\partial_1, \partial_2, \dots, \partial_d).$$

Integro-PDEs like (1.1) typically model anomalous convection-diffusion phenomena. When $\mu = \pi_\lambda$ is defined by

$$d\pi_\lambda(z) = \frac{c_\lambda}{|z|^{d+\lambda}} dz, \qquad c_\lambda > 0 \text{ and } \lambda \in (0, 2), \tag{1.3}$$

then $\mathcal{L} = -(-\Delta)^{\lambda/2}$ and the equation finds applications in e.g. over-driven detonation in gases [14] and anomalous diffusion in semiconductor growth [38]. Applications in dislocation dynamics, hydrodynamics and molecular biology can also be found, see e.g. the references in [1, 18]. Many more applications can be found if asymmetric measures μ are allowed. For example, we cover the (linear) option pricing equations for all Lévy models used in mathematical finance [15, 34], if

$$d\mu(z) = g(z)\, d\pi_\lambda(z), \tag{1.4}$$

for some possibly asymmetric locally Lipschitz continuous function $g(\cdot)$. An example is the one-dimensional ($d = 1$) CGMY model where

$$g(z) = \begin{cases} Ce^{-G|z|} & \text{for } z > 0, \\ Ce^{-M|z|} & \text{for } z < 0, \end{cases}$$

for positive constants C, G, M (and $Y = \lambda$). In general the non-local operator $\mathcal{L}$ is the generator of a pure jump Lévy process, and conversely, any Lévy process will have generator like $\mathcal{L}$ when $(\mathbf{A}.2)$ is satisfied. We refer to the books [3, 15, 32] for more information about Lévy processes and their many applications. The most general Lévy measures for which the results of this paper applies, are Lévy measures μ that can be decomposed as

$$\mu = \mu_s + \mu_n, \tag{1.5}$$

where

$$\mu_s, \mu_n \geq 0, \quad \mu_s \text{ is symmetric} \quad \text{and} \quad \int_{|z|>0} |z| \wedge 1 \; \mathrm{d}\mu_n(z) < \infty. \tag{1.6}$$

See Section 8 for statements of results and remarks. This class possibly includes all Lévy measures, but we have so far not found a proof of this. At least it includes all the Lévy measures found in finance, see Remark 8.3, and also many singular measures like e.g. delta-measures.

It is important to note that non-linear equations like (1.1) do not admit classical solutions in general, and that shock discontinuities can develop even from regular initial conditions. This is well known for pure conservations laws (where $\mathcal{L} = 0$), see e.g. [23]. For fractional conservation laws where $\mathcal{L} = -(-\Delta)^{\lambda/2}$, it is shown in recent works that solutions are smooth for $\lambda \in [1, 2)$ [8, 18, 26]. However, when $\lambda \in (0, 1)$, the fractional diffusion is too weak to prevent shock discontinuities from forming, see [1, 10, 26]. In some cases however, these shocks are smoothed out over time [9]. When shocks form, weak solutions become non-unique and entropy conditions are needed to select the physically correct solution – the entropy solution. The well known Kružkov entropy solution theory for conservation laws was extended to fractional conservation laws in [1]. This extension relies on new ideas for the fractional term and is strongly influenced by the viscosity solution theory for fractional Hamilton-Jacobi-Bellman equations. Extensions of the Kružkov-Alibaud theory to general Lévy operators and even non-linear fractional terms can be found in [12, 25].

In this paper we deal with a SVV (spectral vanishing viscosity) approximation of Λ-periodic entropy solutions of (1.1). The method is a Fourier Galerkin method with an additional spectral viscosity term. Because of the formation of shocks in the solutions of (1.1), it is very difficult to devise a convergent *and* spectrally accurate numerical approximation of this equation. This has to do with the fact that Fourier spectral methods support spurious Gibbs oscillations, and thus fails to converge strongly toward discontinuous solutions. It is well known that such methods need to be augmented by some kind of vanishing viscosity in order to achieve convergence. But the standard vanishing viscosity method is not spectrally accurate. To overcome these problems, we use the SVV approximation developed by Tadmor in [35], cf. also [11, 30, 33, 36] and the books [4, 6]. To suppress spurious oscillations without sacrificing the overall spectral accuracy of the method, Tadmor adds a modified viscosity term, which in Fourier space only affects high frequencies. There are two parameters involved in this approach, the coefficient of the viscosity term ε and the size m of the viscosity free spectrum. Spectral accuracy and convergence toward the unique, possibly discontinuous, entropy solution, then follows by imposing appropriate conditions on ϵ and m. We also like to mention another important feature of the method. In all cases, it diagonalizes the fractional term and hence reduces dramatically the computational cost induced by this term. In our rather naive implementation for the fractional Burgers' equation, the SVV method turned out to be orders of magnitude faster than a Discontinuous Galerkin approximation of the same equation where the fractional term gives full matrices.

When equation (1.1) is linear, $f(u) = u$, or when it is local $\mathcal{L} = 0$, there is a vast literature on numerical methods and analysis, some methods and many references can be found e.g. in [4, 15, 23, 34]. In the general case however, there is not much work on numerical methods, we only know of the papers [13, 17, 21]. Difference methods are introduced in [17] for equation (1.1), and in [21] for an equation similar to (1.1) from radiation hydrodynamics. In [17], the first general convergence result for monotone schemes is obtained. Finally, in [13], a Discontinuous Galerkin approximation of (1.1) is analyzed and a Kuznetsov type of theory is established and used to derive error estimates. A periodic extension of this theory will be used to find error estimates in this paper.

Throughout the paper we will use the following additional notation. A subscript p indicates Λ-periodicity in the space variables (i.e. in L^∞_p or C^∞_p). Here Λ-periodic means 2π-periodic in each coordinate direction. As a generic constant we use C. Note that the value of C may change from line to line and expression to expression. We also need notation for high order derivatives and their norms. Let $\alpha = (\alpha_1, \ldots, \alpha_d)$ be a multi index, then

$$\partial_x^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_1}\cdots\partial_d^{\alpha_d}, \quad \partial_x^s = \bigcup_{|\alpha|=s}\left\{\partial_x^\alpha\right\}, \quad \text{and} \quad \|\partial_x^s\phi\|_{L^p}^p = \sum_{|\alpha|=s}\|\partial_x^\alpha\phi\|_{L^p}^p.$$

Remember that $\alpha_j \geq 0$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$, and that $x^\alpha = x_1^{\alpha_1}\cdots x_d^{\alpha_d}$ for any $x \in \mathbb{R}^d$.

The rest of this paper is organized as follows. In Section 2 we introduce an entropy formulation for periodic solutions (1.1), and give a L^1-contraction and uniqueness result. In the same section we introduce the classical vanishing viscosity approximation of (1.1), and show convergence towards (1.1) with optimal L^1 error estimate. As a corollary we get existence for (1.1). The proofs rely on the Kružkov's doubling of variables device [1, 27] and Kuznetsov type of arguments [13, 28], and are given in the Appendix. The SVV approximation of (1.1) is introduced in Section 3, and we show that it is spectrally accurate and that it diagonalizes the non-local operator. In sections 4–6, we assume that the measure μ is symmetric. In Section 4 we prove an energy estimate for the SVV method. Along with results from [11], this allows us to control the so-called "truncation error", the spectral projection error coming from the non-linear term. In Section 5 we prove a priori L^∞, BV, and time regularity estimates for the SVV method, and obtain compactness. In Section 6 we prove that the SVV method converge to the classical vanishing viscosity method from Section 2. Combined with the results of that section, it follows that the SVV method converges to the entropy solution of (1.1). In the process, we also prove the optimal L^1-rate of convergence for our SVV approximation. We solve numerically, using our SVV method, the the fractional Burgers' equation in Section 7. Finally, in Section 8 we extend the results in the previous sections to allow for asymmetric measures μ.

2. Entropy formulation for periodic solutions

In this section we introduce an entropy formulation for Λ-periodic solutions of the initial value problem (1.1). To this end, we write the operator $\mathcal{L}^\mu[\cdot]$ as

$$\mathcal{L}^\mu[\phi] = \mathcal{L}^\mu_r[\phi] + \mathcal{L}^{\mu,r}[\phi] - \gamma^r_\mu \cdot \partial_x \phi,$$

where

$$\begin{aligned}
\mathcal{L}^\mu_r[\phi(\cdot)](x) &= \int_{|z|\leq r} \phi(x+z) - \phi(x) - z\cdot\partial_x\phi(x)\,\mathbf{1}_{|z|<1}\ \mathrm{d}\mu(z),\\
\mathcal{L}^{\mu,r}[\phi(\cdot)](x) &= \int_{|z|>r} \phi(x+z) - \phi(x)\ \mathrm{d}\mu(z),\\
\gamma^r_\mu &= \int_{r<|z|<1} z\ \mathrm{d}\mu(z).
\end{aligned}$$

If $r>1$, we take $\gamma^r_\mu = 0$. The adjoint of $\mathcal{L}^\mu[\cdot]$ takes the form

$$\mathcal{L}^{*,\mu}_r[\phi] = \mathcal{L}^{*,\mu}_r[\phi] + \mathcal{L}^{*,\mu,r}[\phi] + \gamma^r_\mu \cdot \partial_x \phi,$$

where

$$\begin{aligned}
\mathcal{L}^{*,\mu}_r[\phi(\cdot)](x) &= \int_{|z|\leq r} \phi(x-z) - \phi(x) + z\cdot\partial_x\phi(x)\,\mathbf{1}_{|z|<1}\ \mathrm{d}\mu(z),\\
\mathcal{L}^{*,\mu,r}[\phi(\cdot)](x) &= \int_{|z|>r} \phi(x-z) - \phi(x)\ \mathrm{d}\mu(z).
\end{aligned}$$

We also let η, η', and q denote the functions

$$\eta(u,k) = |u-k|, \quad \eta'(u,k) = \operatorname{sgn}(u-k), \quad q_j(u,k) = \eta'(u,k)\,(f_j(u) - f_j(k)).$$

We now define the solution concept we will use in this paper.

Definition 2.1. (Periodic entropy solutions) *A function u is a periodic entropy solution of the initial value problem* (1.1) *provided that*

i) $u \in C([0,T]; L^\infty_p(\mathbb{R}^d))$;

ii) for all $k\in\mathbb{R}$, all $r>0$, and all nonnegative test functions $\varphi \in C^\infty_p(\mathbb{R}^d \times (0,T))$,

$$\begin{aligned}
\iint_{D_T} &\eta(u,k)\,\partial_t\varphi + q(u,k)\cdot\partial_x\varphi \\
&+ \eta(u,k)\,\mathcal{L}^{*,\mu}_r[\varphi] + \eta'(u,k)\,\mathcal{L}^{\mu,r}[u]\,\varphi + \eta(u,k)\,\gamma^r_\mu\cdot\partial_x\varphi\ \mathrm{d}x\,\mathrm{d}t \geq 0;
\end{aligned} \tag{2.1}$$

iii) $\operatorname{esslim}_{t\to 0}\|u(\cdot,t) - u_0(\cdot)\|_{L^1(\Lambda)} = 0$.

Remark 2.1. In the entropy inequality (2.1) it is easy to see that all the terms, except possibly the $\mathcal{L}^{\mu,r}$-term, are well defined and Λ-periodic in view of *i)*. The problem with the $\mathcal{L}^{\mu,r}$-term is that we integrate a Lebesgue measurable function w.r.t. a Radon measure μ. But the term is still well defined because the integrand of $\mathcal{L}^{\mu,r}[u]$ is measurable w.r.t. the product measure $\mathrm{d}\mu(z)\mathrm{d}x\mathrm{d}t$. This is true because the integrand is the $\mathrm{d}\mu(z)\mathrm{d}x\mathrm{d}t$-a.e. limit of continuous functions, a fact which readily follows from the fact that u is the $\mathrm{d}x\mathrm{d}t$-a.e. limit of smooth functions. By *i)*, (**A**.3), and Fubini, we then find that $\mathcal{L}^{\mu,r}[u] \in C([0,T]; L^\infty_p(\mathbb{R}^d))$.

We now state the following central result:

Theorem 2.2. (L^1-contraction) *Let u and v be two entropy solutions of the initial value problem* (1.1) *with initial data u_0 and v_0. Then, for a.e. $t \in (0,T)$,*

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^1(\Lambda)} \leq \|u_0 - v_0\|_{L^1(\Lambda)}. \tag{2.2}$$

The proof will be given in Appendix A. Uniqueness for periodic entropy solutions of (1.1) immediately follows by setting $u_0 = v_0$.

Corollary 2.3. (Uniqueness) *There is at most one entropy solution of* (1.1).

We now consider the vanishing viscosity approximation of (1.1),

$$\begin{cases} \partial_t u_\epsilon + \partial_x \cdot f(u_\epsilon) = \mathcal{L}^\mu[u_\epsilon] + \epsilon\,\Delta u_\epsilon & (x,t) \in D_T, \\ u_\epsilon(x,0) = u_0(x) & x \in \Lambda. \end{cases} \tag{2.3}$$

In this paper we always assume that this problem admits a unique classical solution u_ϵ. This is of course true, but a proof lays outside the scope of this paper. Remark 2.6 below provides some ideas on how to prove this result. We now give an estimate on the rate of convergence of u_ε toward the entropy solution u of (1.1).

Theorem 2.4 (Convergence rate I)**.** *Let u be the periodic entropy solution of* (1.1)*, and u_ϵ be a smooth solution of* (2.3)*. Then,*

$$\|u(\cdot,t) - u_\epsilon(\cdot,t)\|_{L^1(\Lambda)} \leq C\,\sqrt{\epsilon}. \tag{2.4}$$

The proof is given in Appendix B. This result generalizes to periodic fractional conservation laws Kuznetsov's well known result for scalar conservation laws [28]. As a by-product of the well-posedness of (2.3) and Theorem 2.4, we have the existence of entropy solutions of (1.1).

Corollary 2.5. (Existence) *There exists an entropy solution of* (1.1).

Remark 2.6. Uniqueness of solutions of (2.3) can be proved using an entropy formulation (see the start of Appendix B) and a standard adaptation of the proof of Theorem 2.2 incorporating ideas of Carrillo [7] to handle the Laplace term. Existence of an entropy solution can be proven e.g. by appropriately modifying our spectral approximation, compactness, and convergence analysis, see the following sections. The solution of (2.3) will also be smooth. To see this, note that the principal term in (2.3) is the $\epsilon\Delta$-term while the $\mathcal{L}^\mu$-term is of lower order, and hence regularity proofs for viscous conservation laws ((2.3) with $\mu \equiv 0$ and $\varepsilon > 0$) should still work after some modifications. We refer to e.g. [31] for regularity of viscous conservation laws, and note that the modifications typically consist of using interpolation inequalities for the $\mathcal{L}^\mu$-term, see e.g. Lemma 2.2.1 in [22].

3. The spectral vanishing viscosity method

We introduce a Fourier spectral method for the d-periodic initial value problem (1.1). The approximate solutions will be N-trigonometric polynomials,

$$u_N(x,t) = \sum_{|\xi| \leq N} \hat{u}_\xi(t)\, e^{i\xi\cdot x},$$

which solve the semi-discrete spectral vanishing viscosity (SVV) approximation

$$\partial_t u_N + \partial_x \cdot P_N f(u_N) = \mathcal{L}^\mu[u_N] + \epsilon_N \sum_{j,k=1}^{d} \partial^2_{jk} Q_N^{j,k} * u_N \tag{3.1}$$

with

$$u_N(x,0) = P_N u_0(x), \tag{3.2}$$

where the Fourier projection P_N is defined as

$$P_N\phi(x) = \sum_{|\xi|\leq N} \hat{\phi}_\xi \, e^{i\xi\cdot x} \qquad \text{for} \qquad \hat{\phi}_\xi = \frac{1}{(2\pi)^d}\int_\Lambda \phi(x)\, e^{-i\xi\cdot x}\,\mathrm{d}x.$$

The (spectral) vanishing viscosity term has the following three ingredients:

(**A**.4) a vanishing viscosity amplitude $\epsilon_N \sim N^{-\theta}$ with $0<\theta<1$;

(**A**.5) a viscosity-free spectrum $m_N \sim N^{\frac{\theta}{2}}(\log N)^{-\frac{d}{2}}$;

(**A**.6) a family of viscosity kernels

$$Q_N^{j,k}(x,t) = \sum_{p=m_N}^{N} \hat{Q}_p^{j,k}(t) \sum_{|\xi|=p} e^{i\xi\cdot x}$$

satisfying

- $\hat{Q}_p^{j,k}$ is monotonically p-increasing,
- $\hat{Q}_p^{j,k}$ spherically symmetric, $\hat{Q}_\xi^{j,k} = \hat{Q}_p^{j,k}$ for all $|\xi| = p$,
- $|\hat{Q}_p^{j,k} - \delta_{jk}| \leq C\, m_N^2\, p^{-2}$ for all $p \geq m_N$.

Such kernels can be conveniently implemented in Fourier space,

$$\sum_{j,k=1}^{d} \partial^2_{jk} Q_N^{j,k} * u_N = -\sum_{|\xi|=m_N}^{N} \left(\sum_{j,k=1}^{d} \hat{Q}_\xi^{j,k}(t)\,\xi_j\,\xi_k \right) \hat{u}_\xi(t)\, e^{i\xi\cdot x}.$$

Combined with one's favorite ODE solver (e.g. Euler, Runge-Kutta, etc.), (3.1) and (3.2) give a fully discrete numerical approximation method for (1.1).

With left-hand sides set to zero ($\mu \equiv 0$ and $\varepsilon_N = 0$), (3.1) becomes the standard Fourier approximation of (1.1). It is well known that this approximation is spectrally accurate but, as opposed to the equation, it lacks entropy dissipation. The approximation supports spurious Gibbs oscillations which prevent strong convergence toward solutions containing shock discontinuities. If the $\mathcal{L}^\mu$-term is present in the equations, shock solutions are still possible in some situations [2], and the problem of the Gibbs oscillations remains. In order to suppress such oscillations without sacrificing the overall spectral accuracy of the method, we have followed Tadmor [35] and added a vanishing spectral viscosity term to the scheme, $\epsilon_N \sum_{j,k=1}^d \partial^2_{jk} Q_N^{j,k} * u_N$.

An important feature of Fourier method (3.1) is that it *diagonalizes*, and hence *localizes*, the non-local operator $\mathcal{L}^\mu[\cdot]$! This leads to dramatically reduced computational cost for this term. Indeed,

$$\mathcal{L}^\mu[u_N] = \sum_{|\xi|\leq N} G^\mu(\xi)\, \hat{u}_\xi(t)\, e^{i\xi\cdot x}, \tag{3.3}$$

where

$$G^\mu(\xi) = \int_{|z|>0} e^{i\xi\cdot z} - 1 - i\xi\cdot z\,\mathbf{1}_{|z|<1}\,\mathrm{d}\mu(z). \tag{3.4}$$

Furthermore, when the measure μ is *symmetric*,

$$\mu(-B) = \mu(B) \quad \text{for all Borel sets} \quad B \in \mathbb{R}^d \setminus \{0\}, \tag{3.5}$$

the weights (3.4) are all *real and non-positive*. This follows since the imaginary part of the integrand is odd and the real part is even and non-positive ($e^{i\xi\cdot z} = \cos(\xi\cdot z) + i\,\sin(\xi\cdot z)$). Finally, we stress that the approximation of the non-local operator (1.2) is *spectrally accurate* since, by Taylor's formula,

$$\begin{aligned}&\|\mathcal{L}^\mu[u_N(\cdot,t)] - \mathcal{L}^\mu[u(\cdot,t)]\|_{L^2(\Lambda)}\\ &\qquad\le C\left(\sup_{j,k}\|\partial_j\partial_k(u_N-u)(\cdot,t)\|_{L^2(\Lambda)} + \|(u_N-u)(\cdot,t)\|_{L^2(\Lambda)}\right).\end{aligned}$$

Now we define

$$\hat{R}^{j,k}_\xi(t) = \begin{cases}\delta_{jk} & |\xi|\le m_N,\\ \delta_{jk} - \hat{Q}^{j,k}_\xi(t) & |\xi|>m_N,\end{cases}\qquad R^{j,k}_N(x,t) = \sum_{|\xi|\le N}\hat{R}^{j,k}_\xi(t)\,e^{i\xi\cdot x},$$

and note that

$$\Delta u_N(\cdot,t) = \sum_{j,k=1}^d \partial_j\partial_k Q^{j,k}_N(\cdot,t) * u_N(\cdot,t) + \sum_{j,k=1}^d \partial_j\partial_k R^{j,k}_N(\cdot,t) * u_N(\cdot,t). \tag{3.6}$$

To conclude this section, we recall that by Lemma 3.1 and Corollary 3.2 of [11], the spectral vanishing viscosity term is an L^p-bounded perturbation of the standard vanishing viscosity $\epsilon_N\Delta u_N$:

Lemma 3.1. *For $0\le r\le s\le 2$,*

$$\left\|\sum_{j,k=1}^d \partial_j^r\partial_k^{s-r}R^{j,k}_N(\cdot,t)\right\|_{L^1(\Lambda)} \le C\,m_N^s\,(\log N)^d. \tag{3.7}$$

Moreover, if $c_N \le C\epsilon_N\,m_N^2\,(\log N)^d \le \hat{C}$, then for all $p\ge 1$, $\varphi\in L^p(\Lambda)$,

$$\epsilon_N\left\|\sum_{j,k=1}^d \partial_j\partial_k R^{j,k}_N(\cdot,t) * \varphi(\cdot)\right\|_{L^p(\Lambda)} \le c_N\,\|\varphi\|_{L^p(\Lambda)}. \tag{3.8}$$

4. Spectrally small truncation error for symmetric μ

In this section we assume that the measure μ is symmetric, cf. (3.5). In the SVV approximation (3.1), the convection term $\partial_x\cdot f(u)$ is replaced by $\partial_x\cdot P_N f(u_N)$ which leads to the (truncation) term error

$$\partial_x\cdot(I-P_N)f(u_N).$$

We will now show that this error is spectrally small due to the presence of the spectral vanishing viscosity term.

Let us start by noting that a straightforward estimate leads to

$$\|\partial_x^\alpha(I-P_N)f(u_N)\|_{L^2(\Lambda)} = \left(\sum_{j=1}^d\sum_{|\xi|>N}|\xi^\alpha|^2|\widehat{f_j(u_N)}(\xi)|^2\right)^{\frac12} \le \frac{\|\partial_x^{\alpha+\beta}f(u_N)\|_{L^2(\Lambda)}}{N^{|\beta|}}$$

for all multi-indices α,β. Note that there is no divergence in this estimate, so $\partial_x^\alpha f$ is a vector. By Theorem 7.1 in [11], there is a constant $\mathcal{K}_s$ such that

$$\|\partial_x^s f(u_N)\|_{L^2(\Lambda)} \le \mathcal{K}_s\|\partial_x^s u_N\|_{L^2(\Lambda)} \quad \text{for} \quad \mathcal{K}_s \le C\sum_{k=1}^s |f|_{C^k}\|u_N\|^{k-1}_{L^\infty(\Lambda)} \tag{4.1}$$

and $s = 1, 2, \ldots$, where $|f|_{C^k} = \|\partial_x^k f(\cdot)\|_{L^\infty(\Omega_N)}$ and $\Omega_N = \{u : |u| \leq \|u_N\|_{L^\infty(\Lambda)}\}$. This inequality is a type of Gagliardo-Nirenberg-Moser estimate, and similar results can be found in page 22 in Taylor [37]. By these two inequalities we can conclude that, for all $0 \leq r \leq s$,

$$\|\partial_x^r (I - P_N) f(u_N)\|_{L^2(\Lambda)} \leq \frac{\mathcal{K}_s}{N^{s-r}} \|\partial_x^s u_N\|_{L^2(\Lambda)}. \tag{4.2}$$

Inequality (4.2) states that the r-derivative of the truncation error decays as rapidly as the s-smoothness of u_N permits. Of course the s-derivatives of an arbitrary N-trigonometric polynomial u_N may grow as fast as N^s, in which case nothing is gained from (4.2). However, if u_N is solves our VVS approximation (3.1), we can have the better bound ϵ_N^{-s} in L^2. This will be a consequence of the following energy estimate:

Theorem 4.1. *Consider the SVV approximation* (3.1) *with ϵ_N and m_N such that*

$$\begin{cases} \epsilon_N > \dfrac{8\, d^{\frac{s}{2}} \mathcal{K}_{s+1}}{N}, \\ \epsilon_N \, m_N^2 \, (\log N)^d \leq C. \end{cases} \tag{$\mathbf{A}$.7}$$

Then there is a constant $\mathcal{B}_s$ (proportional to $\Pi_{k=1}^s \mathcal{K}_s$ for $s \geq 1$ and to $\|u_N\|_{L^\infty}$ for $s = 0$) such that

$$\begin{aligned} \epsilon_N^s \|\partial_x^s u_N(\cdot, t)\|_{L^2(\Lambda)} + \epsilon_N^s \left(- \sum_{|\alpha|=s} \sum_{|\xi| \leq N} G^\mu(\xi) |\xi^\alpha|^2 \int_0^t |\hat{u}_\xi(\tau)|^2 \, \mathrm{d}\tau \right)^{\frac{1}{2}} \\ + \epsilon_N^{s+\frac{1}{2}} \|\partial_x^{s+1} u_N\|_{L^2(D_T)} \leq \mathcal{B}_s + 3\epsilon_N^s \|\partial_x^s u_N(\cdot, 0)\|_{L^2(\Lambda)}. \end{aligned} \tag{4.3}$$

Remember that in this section μ is symmetric and hence G^μ is real and non-positive. Now if

($\mathbf{A}$.8) $|f|_{C^s} < \infty$ for sufficiently large s, cf. (4.7) below, and

($\mathbf{A}$.9) u_0 is such that $\epsilon_N^s \, \|\partial_x^s u_N(\cdot, 0)\|_{L^2(\Lambda)} \leq C$,

then Theorem 4.1 implies that

$$\|\partial_x^s u_N(\cdot, t)\|_{L^2(\Lambda)} \leq C \, \epsilon_N^{-s} \quad \text{and} \quad \|\partial_x^{s+1} u_N\|_{L^2(D_T)} \leq C \, \epsilon_N^{-(s+\frac{1}{2})}.$$

Taking into account (4.2), we then find that

$$\|\partial_x^r (I - P_N) f(u_N(\cdot, t))\|_{L^2(\Lambda)} \leq C \, \mathcal{B}_s \, N^{-s_r}, \quad s_r = s(1-\theta) - r, \tag{4.4}$$

$$\|\partial_x^r (I - P_N) f(u_N)\|_{L^2(D_T)} \leq C \, \mathcal{B}_s \, N^{-(s_r + \frac{\theta}{2})}, \quad \forall \, s \geq 1. \tag{4.5}$$

We can now turn these inequalities into spectral decay estimates in the uniform norm using the Sobolev inequality (cf. Theorem 6, Chapter 5, in [20])

$$\|\partial_x^r \varphi\|_{L^\infty} \leq C \, \|\partial_x^{r + [\frac{d}{2}] + 1} \varphi\|_{L^2}.$$

For example, inequality (4.5) becomes

$$\|\partial_x^r (I - P_N) f(u_N)\|_{L^\infty(D_T)} \leq C \, \mathcal{B}_s \, N^{-s_r + [\frac{d}{2}] + 1 - \frac{\theta}{2}} \leq C \, \mathcal{B}_s \, N^{-s_r + [\frac{d}{2}] + 1}. \tag{4.6}$$

Note that the polynomial decay rate in (4.6) can be made as large as the C^s-smoothness of $f(\cdot)$ permits. Taking $r = 2$, we can find the following result.

Theorem 4.2. *If $f \in C^s$ with*

$$s \geq \frac{4+[\frac{d}{2}]}{1-\theta}, \tag{4.7}$$

then

$$\|\partial_x(I-P_N)f(u_N)\|_{L^\infty(D_T)} + \|\partial_x^2(I-P_N)f(u_N)\|_{L^\infty(D_T)} \leq \frac{C\,\mathcal{B}_s}{N}. \tag{4.8}$$

The smoothness requirement (4.7) will be sufficient for all the estimates derived throughout the paper.

Proof of Theorem 4.1. For sake of brevity, we will write $\|\cdot\|$ instead of $\|\cdot\|_{L^2(\Lambda)}$. With (3.6) in mind, we rewrite the SVV approximation (3.1) in the two equivalent forms

$$\partial_t u_N + \partial_x \cdot P_N f(u_N) - \mathcal{L}^\mu[u_N] - \epsilon_N \Delta u_N = -\epsilon_N \sum_{j,k=1}^{d} \partial_j\partial_k R_N^{j,k} * u_N, \tag{4.9}$$

$$\begin{aligned}&\partial_t u_N + \partial_x \cdot f(u_N) - \mathcal{L}^\mu[u_N] - \epsilon_N \Delta u_N \\ &\qquad = -\epsilon_N \sum_{j,k=1}^{d} \partial_j\partial_k R_N^{j,k} * u_N + \partial_x \cdot (I-P_N) f(u_N).\end{aligned} \tag{4.10}$$

Since $G^\mu(\xi) \leq 0$ (μ is symmetric) and $u_N(x)$ and $\mathcal{L}^\mu[u_N]$ are real,

$$\int_\Lambda \mathcal{L}^\mu[u_N]u_N \,\mathrm{d}x = \sum_{|\xi|\leq N} G^\mu(\xi)|\hat{u}_\xi(t)|^2 \leq 0,$$

and hence spatial integration of (4.10) against u_N yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_N\|^2 - \sum_{|\xi|\leq N} G^\mu(\xi)|\hat{u}_\xi(t)|^2 + \epsilon_N\|\partial_x u_N\|^2 \\ &\leq \epsilon_N \|u_N\| \left\| \sum_{j,k=1}^{d} \partial_j\partial_k R_N^{j,k} * u_N \right\| + \sum_{j=1}^{d} \|\partial_j u_N\| \|(I-P_N)f_j(u_N)\|.\end{aligned}$$

Using (3.8) with $p=2$ for the first term on the right and (4.2) with $(r,s)=(0,1)$ for the second term, we find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_N\|^2 - \sum_{|\xi|\leq N} G^\mu(\xi)|\hat{u}_\xi(t)|^2 + \left(\epsilon_N - \frac{\mathcal{K}_1}{N}\right)\|\partial_x u_N\|^2 \leq c_N\|u_N\|^2$$

with $c_N \leq C\epsilon_N\, m_N^2\, (\log N)^d \leq \hat{C}$. Hence (4.3) follows for $s=0$ since by (**A**.7),

$$\left(\epsilon_N - \frac{\mathcal{K}_1}{N}\right) > \frac{\epsilon_N}{2},$$

and $c_N\|u_N\|^2 \leq C\,\|u_N\|^2_{L^\infty(\Lambda)} = \mathcal{B}_0^2$.

The general case follows by induction on s. Spatial integration of (4.9) against $\partial_x^{2\alpha} u_N$ for some multi-index α yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^\alpha u_N\|^2 - \sum_{|\xi|\leq N} G^\mu(\xi)|\xi^\alpha|^2|\hat{u}_\xi(t)|^2 + \epsilon_N\|\partial_x^\alpha \partial_x u_N\|^2 \\ &\leq \epsilon_N\|\partial_x^\alpha u_N\| \left\| \sum_{j,k=1}^{d} \partial_j\partial_k R_N^{j,k} * \partial_x^\alpha u_N \right\| + \|\partial_x^\alpha\partial_x u_N\| \|\partial_x^{|\alpha|-1}\partial_x \cdot P_N f(u_N)\|.\end{aligned} \tag{4.11}$$

After having used (3.8) and Young's inequality to bound the first and second term on the right hand side, we find that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^\alpha u_N\|^2 - \sum_{|\xi|\le N} G^\mu(\xi)|\xi^\alpha|^2|\hat{u}_\xi(t)|^2 + \frac{\epsilon_N}{2}\|\partial_x^\alpha\partial_x u_N\|^2 \\ \le C\,\|\partial_x^\alpha u_N\|^2 + \frac{1}{2\epsilon_N}\|\partial_x^{|\alpha|}P_N f(u_N)\|^2.\end{aligned}$$

Now we sum over all $|\alpha| = s$ to find that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^s u_N\|^2 - \sum_{|\alpha|=s}\sum_{|\xi|\le N} G^\mu(\xi)|\xi^\alpha|^2|\hat{u}_\xi(t)|^2 + \frac{\epsilon_N}{2}\|\partial_x^{s+1} u_N\|^2 \\ \le C\,\|\partial_x^s u_N\|^2 + \frac{d^s}{2\epsilon_N}\|\partial_x^s P_N f(u_N)\|^2.\end{aligned} \tag{4.12}$$

By (4.1) and (4.2),

$$\begin{aligned}\|\partial_x^s P_N f(u_N)\| &\le \|\partial_x^s f(u_N)\| + \|\partial_x^s (I - P_N) f(u_N)\| \\ &\le \mathcal{K}_s\,\|\partial_x^s u_N\| + \frac{\mathcal{K}_{s+1}}{N}\,\|\partial_x^{s+1} u_N\|,\end{aligned}$$

and hence by inequality (4.12) we see that

(4.13)

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^s u_N\|^2 - \sum_{|\alpha|=s}\sum_{|\xi|\le N} G^\mu(\xi)|\xi^\alpha|^2|\hat{u}_\xi(t)|^2 + \left(\frac{\epsilon_N}{2} - \frac{d^s\mathcal{K}_{s+1}^2}{N^2\epsilon_N}\right)\|\partial_x^{s+1} u_N\|^2 \\ \le \left(C + \frac{d^s\mathcal{K}_s^2}{\epsilon_N}\right)\,\|\partial_x^s u_N\|^2 \le \frac{2\,d^s\mathcal{K}_s^2}{\epsilon_N}\,\|\partial_x^s u_N\|^2,\end{aligned}$$

where the last inequality holds for N big enough. By (**A**.7) and integration in time, we then find that

(4.14)

$$\begin{aligned}\frac{1}{2}\|\partial_x^s u_N(\cdot,t)\|^2 - \sum_{|\alpha|=s}\sum_{|\xi|\le N} G^\mu(\xi)|\xi^\alpha|^2\int_0^t |\hat{u}_\xi(\tau)|^2\,\mathrm{d}\tau + \frac{\epsilon_N}{4}\|\partial_x^{s+1} u_N\|^2_{L^2(D_T)} \\ \le \frac{2d^s\mathcal{K}_s^2}{\epsilon_N}\,\|\partial_x^s u_N\|^2_{L^2(D_T)} + \frac{1}{2}\|\partial_x^s u_N(\cdot,0)\|^2.\end{aligned}$$

At this point (4.3) follows by the induction assumption on s since

$$\|\partial_x^s u_N\|^2_{L^2(D_T)} \le C\,\mathcal{B}_{s-1}^2\epsilon_N^{-(2s-1)}.$$

The proof is now complete. □

5. A priori estimates and compactness

In this section we prove uniform

$$L^\infty(D_T),\ L^\infty(0,T;BV(\Lambda)),\text{ and } C^{0,\frac{1}{2}}([0,T];L^1(\Lambda))$$

bounds on the solutions $\{u_N : N \in \mathbb{N}\}$ of the SVV approximation (3.1). As a consequence we obtain compactness in L^1.

5.1. **Regularity in space.**

Lemma 5.1. (L^∞-stability) *Let* ($\mathbf{A}$.1)–($\mathbf{A}$.9) *and* (3.5) *hold and* u_N *be the solution of the SVV approximation* (3.1)*. Then for* $t < C\ln N$*,*

$$\|u_N(\cdot,t)\|_{L^\infty(\Lambda)} \le C\,\|u_N(\cdot,0)\|_{L^\infty(\Lambda)}.$$

Proof. For sake of brevity, we write just $\|\cdot\|_\infty$ instead of $\|\cdot\|_{L^\infty(\Lambda)}$. First we note that, for any smooth convex function $\eta(\cdot)$ with derivative $\eta'(\cdot)$, we have that

$$\eta'(u_N)\,\mathcal{L}^\mu[u_N] \le \mathcal{L}^\mu[\eta(u_N)]. \tag{5.1}$$

This is a consequence of the inequality $\eta'(b)(a-b) \le \eta(a)-\eta(b)$ which holds for all smooth convex functions $\eta(\cdot)$. Moreover,

$$\int_\Lambda \mathcal{L}^\mu[\eta(u_N(\cdot,t))](x)\;\mathrm{d}x = 0. \tag{5.2}$$

To see this note that

$$\begin{aligned}&\int_\Lambda\int_{|z|>0}\left|\eta(u_N(x+z))-\eta(u_N(x))+z\cdot\partial_x\eta(u_N(x))\,\mathbf{1}_{|z|<1}\right|\;\mathrm{d}\mu(z)\,\mathrm{d}x\\&\qquad\le\|\partial_x^2\eta(u_N)\|_\infty\int_{|z|<1}|z|^2\;\mathrm{d}\mu(z)+\|\eta(u_N)\|_\infty\int_{|z|>1}\mathrm{d}\mu(z)<\infty,\end{aligned}$$

since u_N is smooth and periodic. By Fubini we then find that

$$\begin{aligned}&\int_\Lambda \mathcal{L}^\mu[\eta(u_N(\cdot,t))](x)\;\mathrm{d}x\\&=\int_{|z|>0}\int_\Lambda \eta(u_N(x+z))-\eta(u_N(x))+z\cdot\partial_x\eta(u_N(x))\,\mathbf{1}_{|z|<1}\;\mathrm{d}x\,\mathrm{d}\mu(z).\end{aligned}$$

By Λ-periodicity of u_N, (5.2) now follows since

$$\int_\Lambda \eta(u_N(x+z))\,\mathrm{d}x = \int_\Lambda \eta(u_N(x))\,\mathrm{d}x$$

for every z, and

$$\begin{aligned}&\int_\Lambda \partial_{x_i}\eta(u_N(x',x_i))\,\mathrm{d}x'\mathrm{d}x_i\\&=\int_{(0,2\pi)^{d-1}}\eta(u_N(x',2\pi))\,\mathrm{d}x'-\int_{(0,2\pi)^{d-1}}\eta(u_N(x',0))\,\mathrm{d}x'=0.\end{aligned}$$

Let us now integrate (4.10) against the function $p\,u_N^{p-1}$ (with p even), and use (5.1) and (5.2) to get rid of the non-local operator $\mathcal{L}^\mu[\cdot]$. We then find that

$$\begin{aligned}p\,\|u_N(\cdot,t)\|_{L^p(\Lambda)}^{p-1}\frac{\mathrm{d}}{\mathrm{d}t}\|u_N(\cdot,t)\|_{L^p(\Lambda)}&=\frac{\mathrm{d}}{\mathrm{d}t}\|u_N(\cdot,t)\|_{L^p(\Lambda)}^p=\int_\Lambda u_N^{p-1}(x,t)\partial_t u_N(x,t)dx\\&\le p\int_\Lambda u_N^{p-1}(x,t)\left(\epsilon_N\sum_{j,k=1}^d\partial_j\partial_k R_N^{j,k}*u_N(x,t)+\partial_x\cdot(I-P_N)f(u_N(x,t))\right)\mathrm{d}x\end{aligned}$$

which by the Hölder inequality (with p and $q=\frac{p}{p-1}$) is less than or equal to

$$\begin{aligned}&p\,\|u_N(\cdot,t)^{p-1}\|_{L^{\frac{p}{p-1}}(\Lambda)}\\&\quad\left(\epsilon_N\left\|\sum_{j,k=1}^d\partial_j\partial_k R_N^{j,k}*u_N(\cdot,t)\right\|_{L^p(\Lambda)}+\|\partial_x\cdot(I-P_N)f(u_N(\cdot,t))\|_{L^p(\Lambda)}\right).\end{aligned}$$

Since $\|\phi^{p-1}\|_{L^{\frac{p}{p-1}}} = \|\phi\|_{L^p}^{p-1}$, we may divide both sides by $p\,\|u_N(\cdot,t)\|_{L^p(\Lambda)}^{p-1}$ and send $p\to\infty$ to discover that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_N(\cdot,t)\|_\infty \le \epsilon_N \Big\| \sum_{j,k=1}^d \partial_j\partial_k R_N^{j,k} * u_N(\cdot,t)\Big\|_\infty + \|\partial_x\cdot(I-P_N)f(u_N(\cdot,t))\|_\infty.$$

By (4.8), (3.8), the definitions of $\mathcal{B}_s$, $\mathcal{K}_s$ and c_N, and (**A**.7), it follows that

$$\|\partial_x\cdot(I-P_N)f(u_N(\cdot,t))\|_\infty \le \frac{\mathcal{B}_s}{N} \le \frac{C}{N}\prod_{k=1}^s \mathcal{K}_s \le \frac{\hat{C}}{N}\|u_N\|_\infty^{\frac{s^2}{2}},$$

$$\epsilon_N\Big\| \sum_{j,k=1}^d \partial_j\partial_k R_N^{j,k}(\cdot,t) * u_N(\cdot,t)\Big\|_\infty \le c_N\|u_N\|_\infty \le C\|u_N\|_\infty,$$

and hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_N(\cdot,t)\|_\infty \le c_N\|u_N(\cdot,t)\|_\infty + \frac{C}{N}\,\|u_N(\cdot,t)\|_\infty^{\frac{s^2}{2}}.$$

Letting $y(t)=e^{-c_Nt}\|u_N(\cdot,t)\|_\infty$, and multiplying by the integrating factor e^{-c_Nt}, we find that

$$\frac{\mathrm{d}y}{\mathrm{d}t}(t) \le \frac{C}{N}\,y^{\frac{s^2}{2}}(t)\,e^{c_N(\frac{s^2}{2}-1)t}$$

which implies that

$$y(t)\le y(0)\left(1-\frac{C\left(e^{c_N(\frac{s^2}{2}-1)t}-1\right)y^{\frac{s^2}{2}-1}(0)}{Nc_N}\right)^{-\frac{1}{\frac{s^2}{2}-1}}.$$

Going back to $\|u_N(\cdot,t)\|_\infty$, we can conclude that

$$\|u_N(\cdot,t)\|_\infty \le e^{c_Nt}\|u_N(\cdot,0)\|_\infty\left(1-\frac{Ce^{c_N(\frac{s^2}{2}-1)t}\|u_0\|_\infty^{\frac{s^2}{2}-1}}{Nc_N}\right)^{-\frac{2}{2-s^2}},$$

where the last factor is bounded for $t\le C\ln N$ for some C. □

We also have the following result:

Lemma 5.2. (BV-stability) *Let* (**A**.1)–(**A**.9) *and* (3.5) *hold, and u_N be the solution of the SVV approximation* (3.1). *Then*

$$\|u_N(\cdot,T)\|_{BV(\Lambda)} \le e^{c_NT}\left(\|u_N(\cdot,0)\|_{BV(\Lambda)} + C\,N^{-s_2}\right)$$

with $c_N=\epsilon_N\,m_N^2\,(\log N)^d\le C$ and $s_2=s(1-\theta)-2>0$.

Proof. Spatial differentiation of (4.10) yields

$$\partial_t\partial_i u_N + \partial_x\cdot(f'(u_N)\partial_i u_N) - \mathcal{L}^\mu[\partial_i u_N] - \epsilon_N\,\Delta\partial_i u_N$$
$$= \partial_i\partial_x\cdot(I-P_N)f(u_N) + \epsilon_N\sum_{j,k=1}^d \partial_j\partial_k R_N^{j,k} * \partial_i u_N.$$

If we integrate this expression against $\mathrm{sgn}_\varrho(\partial_i u_N)$, where $\mathrm{sgn}_\varrho(\cdot)$ is a smooth approximation of the sign function, we can get rid of the non-local operator $\mathcal{L}^\mu[\cdot]$ as in the proof of Lemma 5.1. If we also use (3.8) with $p=1$ and take the limit as $\varrho\to0$, a standard computations reveal that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_i u_N(\cdot,t)\|_{L^1(\Lambda)} \le C\,\|\partial_i\partial_x\cdot(I-P_N)f(u_N)\|_{L^1(\Lambda)} + c_N\|\partial_i u_N(\cdot,t)\|_{L^1(\Lambda)}.$$

Since $\|u_N(\cdot,t)\|_{BV(\Lambda)} \le \sum_{i=1}^d \|\partial_i u_N(\cdot,t)\|_{L^1(\Lambda)}$, we integrate this inequality in time to see that

$$\|u_N(\cdot,t)\|_{BV(\Lambda)} \le e^{c_N t}\left(\|u_N(\cdot,0)\|_{BV(\Lambda)} + C\,\|\partial_x^2(I-P_N)f(u_N)\|_{L^1(D_T)}\right).$$

But by (4.4),

$$\|\partial_x^2(I-P_N)f(u_N)\|_{L^1(D_T)} \le C\,\|\partial_x^2(I-P_N)f(u_N)\|_{L^2(D_T)} \le C\,\mathcal{B}_s\sqrt{T}\,N^{-s_2},$$

and the proof is complete. □

5.2. **Regularity in time.**

Lemma 5.3. (Regularity in time) *Let* (**A**.1)–(**A**.9) *and* (3.5) *hold and, u_N be the solution of the SVV approximation* (3.1). *Then*

$$\|u_N(\cdot,t_1)-u_N(\cdot,t_2)\|_{L^1(\Lambda)} \le C\,\sqrt{|t_1-t_2|}.$$

Proof. Let $u_N^\epsilon(\cdot,t) = u_N(\cdot,t)*\omega_\epsilon(\cdot)$ for an approximate unit ω_ϵ (cf. the proof of Theorem 2.2). By the triangle inequality we see that

$$\begin{aligned}\|u_N(\cdot,t_1)-u_N(\cdot,t_2)\|_{L^1(\Lambda)} &\le \|u_N(\cdot,t_1)-u_N^\epsilon(\cdot,t_1)\|_{L^1(\Lambda)}\\ &\quad+\|u_N^\epsilon(\cdot,t_1)-u_N^\epsilon(\cdot,t_2)\|_{L^1(\Lambda)}+\|u_N^\epsilon(\cdot,t_2)-u_N(\cdot,t_2)\|_{L^1(\Lambda)}.\end{aligned}\tag{5.3}$$

The first and the third term on the right-hand side of (5.3) are bounded by $\epsilon|u|_{BV}$:

$$\begin{aligned}\|u_N(\cdot,t)-u_N^\epsilon(\cdot,t)\|_{L^1(\Lambda)} &= \int_\Lambda\left|\int_{\mathbb{R}^d}\omega_\epsilon(y-x)\Big(u_N(x,t)-u_N(y,t)\Big)\,\mathrm{d}y\right|\mathrm{d}x\\ &\le \int_\Lambda\int_{\mathbb{R}^d}\omega_\epsilon(s)\Big|u_N(x,t)-u_N(s+x,t)\Big|\,\mathrm{d}s\,\mathrm{d}x\\ &\le \sqrt{d}\,|u(\cdot,t)|_{BV(\Lambda)}\int_{\mathbb{R}^d}|s|\,\omega_\epsilon(s)\,\mathrm{d}s\\ &\le \sqrt{d}\,\epsilon\,|u(\cdot,t)|_{BV(\Lambda)}.\end{aligned}$$

Let us estimate the second term. By Taylor's formula with integral remainder,

$$\begin{aligned}&\|u_N^\epsilon(\cdot,t_1)-u_N^\epsilon(\cdot,t_2)\|_{L^1(\Lambda)}\\ &\qquad\le |t_1-t_2|\int_\Lambda\int_0^1|\partial_t u_N^\epsilon(x,t_1+\tau(t_2-t_1))|\,\mathrm{d}\tau\,\mathrm{d}x.\end{aligned}$$

We now derive a bound for $\|\partial_t u_N\|_{L^1}$ (and hence also for $\|\partial_t u_N^\varepsilon\|_{L^1}$) by using the SVV approximation (3.1) itself. To this end, we take the convolution product of both sides of (4.9) with ω_ϵ to obtain

$$\begin{aligned}\|\partial_t u_N^\epsilon\|_{L^1(\Lambda)} &\le \|\partial_x\cdot P_N f(u_N)*\omega_\epsilon\|_{L^1(\Lambda)}+\|\mathcal{L}^\mu[u_N]*\omega_\epsilon\|_{L^1(\Lambda)}\\ &\quad+\epsilon_N\,\|\Delta u_N*\omega_\epsilon\|_{L^1(\Lambda)}+\epsilon_N\left\|\left(\sum_{j,k=1}^d\partial_j\partial_k R_N^{j,k}*u_N\right)*\omega_\epsilon\right\|_{L^1(\Lambda)}\\ &= I_1+I_2+I_3+I_4.\end{aligned}$$

By the triangle inequality and Young's inequality for convolutions,

$$\begin{aligned}I_1 &= \|\partial_x\cdot P_N f(u_N)*\omega_\epsilon\|_{L^1(\Lambda)}\\ &\le \|\partial_x\cdot f(u_N)*\omega_\epsilon\|_{L^1(\Lambda)}+\|\partial_x\cdot(I-P_N)f(u_N)*\omega_\epsilon\|_{L^1(\Lambda)}\\ &\le \|\partial_x\cdot f(u_N)\|_{L^1(\Lambda)}+\|\partial_x\cdot(I-P_N)f(u_N)\|_{L^1(\Lambda)}.\end{aligned}$$

Therefore, by the regularity of f and u_N ((**A**.8), Lemmas 5.1-5.2) and (4.8), we find that

$$I_1 \leq C \left(|u(\cdot,t)|_{BV(\Lambda)} + \frac{1}{N} \right).$$

For the term containing the non-local operator we write

$$\begin{aligned} I_2 \leq & \int_\Lambda \left| \left(\int_{\mathbb{R}^d} \int_{|z|<1} u_N(x+z) - u_N(x) - z \cdot \partial_x u_N(x) \, \mathrm{d}\mu(z) \right) \omega_\epsilon(x-y) \, \mathrm{d}y \right| \mathrm{d}x \\ & + \int_\Lambda \left| \left(\int_{\mathbb{R}^d} \int_{|z|>1} u_N(x+z) - u_N(x) \, \mathrm{d}\mu(z) \right) \omega_\epsilon(x-y) \, \mathrm{d}y \right| \mathrm{d}x \end{aligned}$$

The second term on the right hand side of the inequality above is easily seen to be bounded by $C\|u_N(\cdot,t)\|_{L^1}$, while Taylor's formula with integral reminder and integration by parts reveals that the first term is bounded by

$$\begin{aligned} \int_\Lambda \int_{\mathbb{R}^d} \int_{|z|<1} \int_0^1 (1-\tau)\, |z|^2 \, |\partial_x u_N(x,t)| \, |\partial_x \omega_\epsilon(x-y)| \; \mathrm{d}\tau \, \mathrm{d}\mu(z) \, \mathrm{d}y \, \mathrm{d}x \\ \leq C \, \epsilon^{-1} \, |u|_{BV(\Lambda)}. \end{aligned}$$

For the Laplace term we have

$$I_3 \leq \|\partial_x u * \partial_x \omega_\epsilon\|_{L^1(\Lambda)} \leq \epsilon^{-1} \, |u|_{BV(\Lambda)},$$

and finally, using Young's inequality for convolutions and (3.8),

$$I_4 = \epsilon_N \left\| \left(\sum_{j,k=1}^d \partial_j \partial_k R_N^{j,k} * u_N \right) * \omega_\epsilon \right\|_{L^1(\Lambda)} \leq C \, \|u_N\|_{L^1(\Lambda)}.$$

To sum up we have

$$\|\partial_t u_N^\epsilon\|_{L^1(\Lambda)} \leq \|\partial_t u_N\|_{L^1(\Lambda)} \leq C \, \left(1 + \frac{1}{\epsilon}\right),$$

and inequality (5.3) and the above estimates then implies that

$$\|u_N(\cdot,t_1) - u_N(\cdot,t_2)\|_{L^1(\Lambda)} \leq C \Big(\epsilon + |t_1 - t_2| \, \left(1 + \epsilon^{-1}\right) \Big).$$

Take $\varepsilon = \sqrt{|t_1 - t_2|}$ and the proof is complete. □

5.3. **Compactness.** Thanks to the space/time a priori estimates in Lemmas 5.1 – 5.3 and a Helly like compactness theorem, cf. Theorem A.8 in [23], the family $\{u_N : N \in \mathbb{N}\}$ of solutions of the SVV approximation (3.1) is compact.

Theorem 5.4 (Compactness)**.** *Let* (**A**.1)–(**A**.9) *and* (3.5) *hold, and* u_N *be the solution of the SVV approximation* (3.1)*. Then there exists a subsequence* u_N *converging in* $C([0,T];L^1(\Lambda))$ *to a limit* $u \in C([0,T];L^1(\Lambda)) \cap L^\infty(D_T) \cap L^\infty(0,T;BV(\Lambda))$.

6. Convergence and error estimate

The solution v_{ϵ_N} of the vanishing viscosity method (2.3) converges to the unique entropy solution u of (1.1), and by Theorem 2.4,

$$\|u(\cdot,t) - v_{\epsilon_N}(\cdot,t)\|_{L^1(\Lambda)} \leq C \, \sqrt{\epsilon_N}.$$

In this section we prove a similar error estimate between v_{ϵ_N} and the SVV approximation u_N.

Theorem 6.1. *Let* ($\mathbf{A}$.1)–($\mathbf{A}$.9) *and* (3.5) *hold,* u_N *be the solution of the SVV method* (2.3)*, and* v_{ϵ_N} *be the solution of* (3.1)*. Then*

$$\|u_N(\cdot,T) - v_{\epsilon_N}(\cdot,T)\|_{L^1(\Lambda)} \le C\sqrt{\epsilon_N}.$$

A direct consequence of Theorems 2.4 and 6.1, is the following convergence and error estimate for the SVV method.

Corollary 6.2. (Convergence with rate) *Let* ($\mathbf{A}$.1)–($\mathbf{A}$.9) *and* (3.5) *hold,* u_N *be the solution of the SVV method* (3.1)*, and* u *be an entropy solution of* (1.1)*. Then*

$$\|u(\cdot,T) - u_N(\cdot,T)\|_{L^1(\Lambda)} \le C\sqrt{\epsilon_N}.$$

Proof of Theorem 6.1. Since v_{ϵ_N} is smooth, we can subtract equation (2.3) from equation (3.1) to obtain

$$\begin{aligned}\partial_t(u_N - v_{\epsilon_N}) + \partial_x\cdot(f(u_N) - f(v_{\epsilon_N})) - \mathcal{L}^\mu[u_N - v_{\epsilon_N}] - \epsilon_N\Delta(u_N - v_{\epsilon_N})\\ = -\epsilon_N\sum_{j,k=1}^d \partial_j R_N^{j,k} * \partial_k u_N + \partial_x(I - P_N)f(u_N).\end{aligned}$$

As explained in the proof of Lemma 5.2, we can integrate such an inequality against (a smooth approximation of) $\operatorname{sgn}(u_N - v_{\epsilon_N})$, to find that (after going to the limit)

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|u_N - v_{\epsilon_N}\|_{L^1(\Lambda)}\\ &\le \epsilon_N\Big\|\sum_{j,k=1}^d \partial_j R_N^{j,k}(\cdot,t) * \partial_k u_N(\cdot,t)\Big\|_{L^1(\Lambda)} + \|\partial_x\cdot(I - P_N)f(u_N(\cdot,t))\|_{L^1(\Lambda)}.\end{aligned}$$

By (3.7) with $r = s = 1$, ($\mathbf{A}$.4), ($\mathbf{A}$.5), and Lemma 5.2,

$$\begin{aligned}\Big\|\sum_{j,k=1}^d \partial_j R_N^{j,k}(\cdot,t) * \partial_k u_N(\cdot,t)\Big\|_{L^1(\Lambda)} &\le \Big\|\sum_{j,k=1}^d \partial_j R_N^{j,k}(\cdot,t)\Big\|_{L^1(\Lambda)}\Big\|\partial_k u_N(\cdot,t)\Big\|_{L^1(\Lambda)}\\ &\le C\, m_N\,(\log N)^d\|u_N(\cdot,t)\|_{BV(\Lambda)} \le C\,\epsilon_N^{-\frac{1}{2}},\end{aligned}$$

so we can integrate in time to obtain

$$\begin{aligned}\|u_N(\cdot,t) - v_{\epsilon_N}(\cdot,t)\|_{L^1(\Lambda)} &\le C\sqrt{\epsilon_N} + \|\partial_x\cdot(I - P_N)f(u_N(\cdot,T))\|_{L^1(D_T)}\\ &\le C\Big(\sqrt{\epsilon_N} + \|\partial_x\cdot(I - P_N)f(u_N(\cdot,T))\|_{L^2(D_T)}\Big).\end{aligned}$$

By (4.5),

$$\|\partial_x\cdot(I - P_N)f(u_N(\cdot,T))\|_{L^2(D_T)} \le C\,\mathcal{K}_s\,N^{-(s_1+\frac{\theta}{2})} \le C\,\mathcal{K}_s\,N^{-\frac{\theta}{2}} = C\sqrt{\epsilon_N},$$

since $s_1 = s(1-\theta) - 1 > 0$, cf. (4.7). The proof is now complete. □

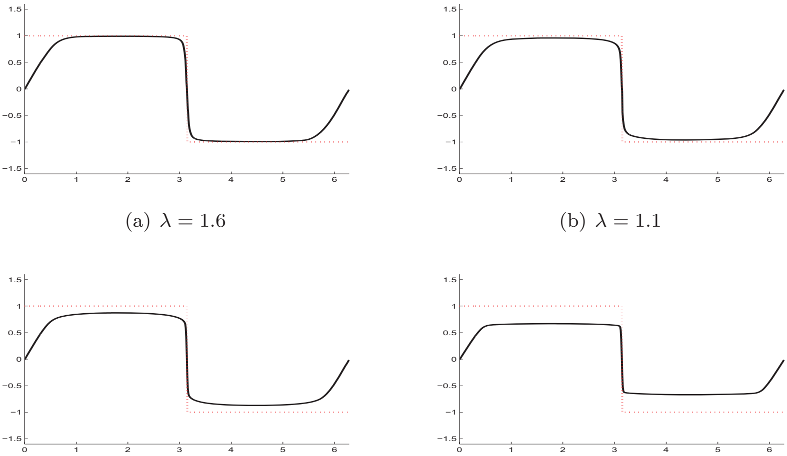
7. An application: the fractional Burgers' equation

In this section we apply the results of the previous sections to numerically solve the fractional (or fractal) Burgers' equation in $\mathbb{R}^d$,

$$\begin{cases}\partial_t u + u\sum_{j=1}^d \partial_{x_j} u = -(-\Delta)^{\lambda/2}u, & (x,t)\in D_T,\\ u(x,0) = u_0(x), & x\in\Lambda,\end{cases} \tag{7.1}$$

where the fractional Laplacian term $-(-\Delta)^{\lambda/2}u_N = \mathcal{L}^{\pi_\lambda}[u_N]$ and π_λ has been defined in (1.3). In this setting expression (3.4) becomes

$$G^{\pi_\lambda}(\xi) = c_\lambda\int_{|z|>0} e^{i\xi\cdot z} - 1 - i\xi\cdot z\,\mathbf{1}_{|z|<1}\,\frac{\mathrm{d}z}{|z|^{d+\lambda}},$$



(a) $\lambda = 1.6$ (b) $\lambda = 1.1$

(c) $\lambda = 0.6$ (d) $\lambda = 0.1$

FIGURE 1. Solutions of system (7.3) with $N = 256$ and $T = 0.5$. The piecewise constant initial datum is $u_0(x) = \mathrm{sgn}(\pi - x)$.

with $c_\lambda = \lambda\,\Gamma(\frac{d+\lambda}{2})\left(2\pi^{\frac{d}{2}+\lambda}\,\Gamma(1-\frac{\lambda}{2})\right)^{-1}$, cf. [19]. We have the following result:

Proposition 7.1.

$$G^{\pi_\lambda}(\xi) = \begin{cases} -\,C_\lambda\,|\xi|^\lambda & \text{for } d = 1, \\ -\,C_\lambda\,|\xi|^\lambda \displaystyle\int_{|y|=1} \mathrm{d}S_y & \text{for } d > 1, \end{cases} \tag{7.2}$$

where $C_\lambda = 2\,c_\lambda\,\lambda^{-1}\int_0^\infty x^{-\lambda}\sin x\,\mathrm{d}x > 0$ *and* $\int_{|y|=1}\mathrm{d}S_y = 2\pi^{d/2}\,\Gamma^{-1}(\frac{d}{2})$.

The proof is given at the end of this section. In the above result and in the following, dS_y will denote the surface area measure of the unit sphere $|y| = 1$. Expression (7.2) is the "Fourier symbol" of the fractional Laplace operator in our periodic setting. When $\lambda \in (0, 1)$, the integral $\Theta_\lambda = \int_0^\infty x^{-\lambda}\sin x\,\mathrm{d}x$ is a *generalized Fresnel integral* [29] with value

$$\Theta_\lambda = \Gamma(1-\lambda)\sin\left(\frac{\pi(1-\lambda)}{2}\right).$$

When $\lambda = 1$, Θ_λ is a *Dirichlet integral* [24] and has value $\frac{\pi}{2}$. For $\lambda \in (1, 2)$, the integral Θ_λ has to be evaluated numerically since explicit formulas are not available.

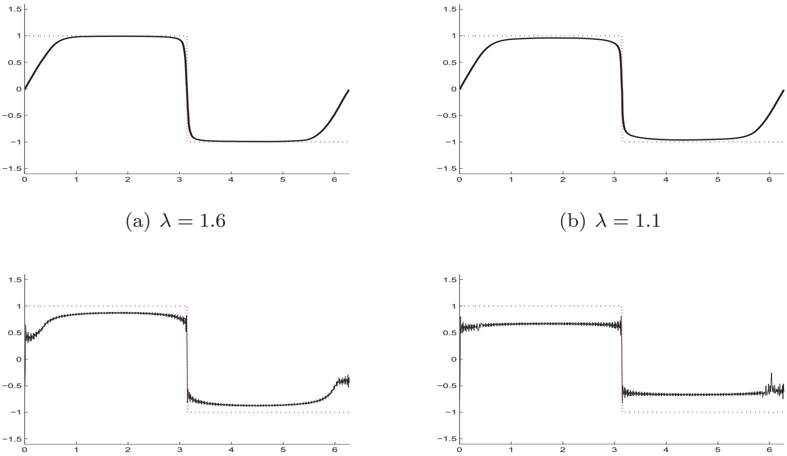
Remark 7.2. By Proposition 7.1 there is a positive constant such that

$$\int_\Lambda \mathcal{L}^{\pi_\lambda}[u_N(\cdot,t)]\,u_N(\cdot,t)\;\mathrm{d}x = -C\sum_{|\xi|\le N}|\xi|^\lambda|\hat{u}_\xi(t)|^2,$$

where right-hand side is a fractional Sobolev semi-norm [4]

$$\sum_{|\xi|\le N}|\xi|^\lambda|\hat{u}_\xi(t)|^2 = |u_N(\cdot,t)|^2_{H^{\lambda/2}(\Lambda)}.$$

Simple energy estimates can then be used to show that the solutions of (7.1) belong to $H^{\lambda/2}(\Lambda)$, which is more regularity than what can be expected for general solutions of the pure Burgers' equation ($\mu = 0$).



(a) $\lambda = 1.6$ (b) $\lambda = 1.1$

(c) $\lambda = 0.6$ (d) $\lambda = 0.1$

FIGURE 2. Solutions of system (7.3) with $N = 256$, $T = 0.5$, and $\epsilon_N = 0$. The piecewise constant initial datum is $u_0(x) = \mathrm{sgn}(\pi - x)$.

We now use the SVV method (3.1) to work out some approximate solutions of the fractional Burgers' equation (7.1) with $d = 1$. Hence $f(u) = u^2/2$ and $\mu = \pi_\lambda$ in (3.1). We multiply both sides of (3.1) by $e^{-i\xi x}$, and integrate over $(0, 2\pi)$ to obtain the following system of ODEs

(7.3)
$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_\xi(t) + \frac{i\xi}{2}\sum_{\substack{|p|,|q|\le N\\ p+q=\xi}} \hat{u}_p(t)\,\hat{u}_q(t) + C_\lambda\,|\xi|^\lambda \hat{u}_\xi(t) + \epsilon_N \mathbf{1}_{m_N\le|\xi|\le N}|\xi|^2\,\hat{Q}_\xi(t)\,\hat{u}_\xi(t) = 0,$$

where the Fourier coefficients $\hat{Q}_\xi$ satisfy the assumptions listed in Section 3 and are chosen as in [30] (they vary continuously between zero and one). In our simulations we have used a fourth order Runge-Kutta solver for (7.3).

The results of our numerical simulations can be found in Figure 1 and 2. The results in Figure 1 confirm the convergence of our SVV approximation (7.3) for all all values of $\lambda \in (0, 2)$. In Figure 2 we have solved the (7.3) with $\epsilon_N = 0$ (no spectral vanishing viscosity). For $\lambda > 1$, convergence continues to hold, while for $\lambda < 1$, convergence fails and spurious Gibbs oscillations appear. This is consistent with the theoretical results for fractional conservation laws [2, 18]: These equations admit smooth solutions for $\lambda > 1$ (the strong diffusion case), while shock discontinuities may appear for $\lambda < 1$ (the weak diffusion case).

Proof of Proposition 7.1. Let us prove the case $d = 1$ first. By Euler's formula, $e^{i\xi z} = \cos(\xi z) + i\sin(\xi z)$, we find that

$$\int_{|z|<1} \frac{e^{i\xi z} - 1 - i\xi z}{|z|^{1+\lambda}}\,\mathrm{d}z = \int_{|z|<1} \frac{\cos(\xi z) - 1}{|z|^{1+\lambda}}\,\mathrm{d}z + i\int_{|z|<1} \frac{\sin(\xi z) - \xi z}{|z|^{1+\lambda}}\,\mathrm{d}z.$$

Taylor expansions show that these integrals are finite. In fact, the sin-integral is zero since the its integrand is odd. Integration by parts then leads to

$$\begin{aligned}\int_{|z|<1} \frac{\cos(\xi z)-1}{|z|^{1+\lambda}}\,\mathrm{d}z &= 2\int_0^1 \frac{\cos(\xi z)-1}{z^{1+\lambda}}\,\mathrm{d}z\\ &= -\frac{2}{\lambda z^\lambda}(\cos(\xi z)-1)\Big|_0^1 - \frac{2\xi}{\lambda}\int_0^1 \frac{\sin(\xi z)}{z^\lambda}\,\mathrm{d}z\\ &= -\frac{2}{\lambda}(\cos(\xi)-1) - \frac{2\xi}{\lambda}\int_0^1 \frac{\sin(\xi z)}{z^\lambda}\,\mathrm{d}z.\end{aligned}$$

Now we consider the integral over $|z|>r$. Again the imaginary part (the sine part) is zero, and a computation like the one we performed above reveals that

$$\int_{|z|>1} \frac{e^{i\xi z}-1}{|z|^{1+\lambda}}\,\mathrm{d}z = \frac{2}{\lambda}(\cos(\xi)-1) - \frac{2\xi}{\lambda}\int_1^\infty \frac{\sin(\xi z)}{z^\lambda}\,\mathrm{d}z.$$

Note the + sign of the cosine-term! We add these two equations and find that

$$G^{\pi_\lambda}(\xi) = -\frac{2\xi c_\lambda}{\lambda}\int_0^\infty \frac{\sin(\xi z)}{z^\lambda}\,\mathrm{d}z.$$

The integral $\int_0^\infty z^{-\lambda}\sin(\xi z)\,\mathrm{d}z$ is finite and positive for all $\lambda\in(0,2)$ (cf. [16] for details). Whenever $\xi>0$, we can use the change of variable $\xi z\to x$ to deduce that

$$\int_0^\infty \frac{\sin(\xi z)}{z^\lambda}\,\mathrm{d}z = \xi^{\lambda-1}\int_0^\infty \frac{\sin x}{x^\lambda}\,\mathrm{d}x$$

and thus

$$G^{\pi_\lambda}(\xi) = -\frac{2c_\lambda}{\lambda}\,\xi^\lambda\int_0^\infty \frac{\sin x}{x^\lambda}\,\mathrm{d}x.$$

When $\xi<0$, we use the relation $\sin(-\xi x) = -\sin(\xi x)$ to obtain

$$G^{\pi_\lambda}(\xi) = -\frac{2c_\lambda}{\lambda}\,|\xi|^\lambda\int_0^\infty \frac{\sin x}{x^\lambda}\,\mathrm{d}x,$$

and the conclusion for $d=1$ follows.

When $d>1$ we use polar coordinates $x=ry$ for $r>0$ and $|y|=1$, and we find that

$$\begin{aligned}\int_{|z|<1} \frac{e^{i\xi\cdot z}-1-i\xi\cdot z}{|z|^{d+\lambda}}\,\mathrm{d}z &= \int_{|y|=1}\int_0^1 \frac{\cos(\xi\cdot y\,r)-1}{r^{d+\lambda}}\,r^{d-1}\mathrm{d}r\,\mathrm{d}S_y,\\ \int_{|z|>1} \frac{e^{i\xi\cdot z}-1}{|z|^{d+\lambda}}\,\mathrm{d}z &= \int_{|y|=1}\int_1^\infty \frac{\cos(\xi\cdot y\,r)-1}{r^{1+\lambda}}\,\mathrm{d}r\,\mathrm{d}S_y.\end{aligned}$$

Proceeding as in the $d=1$ case for the r-integral with y fixed, we find that

$$\begin{aligned}G^{\pi_\lambda}(\xi) &= -\frac{2c_\lambda}{\lambda}\int_{|y|=1}|\xi\cdot y|^\lambda\,\mathrm{d}S_y\int_0^\infty \frac{\sin x}{x^\lambda}\,\mathrm{d}x\\ &= -\frac{2c_\lambda}{\lambda}\,|\xi|^\lambda\int_{|y|=1}\left|\frac{\xi}{|\xi|}\cdot y\right|^\lambda\,\mathrm{d}S_y\int_0^\infty \frac{\sin x}{x^\lambda}\,\mathrm{d}x.\end{aligned}$$

By symmetry, the value of the y-integral is the same for any ξ. Therefore,

$$\int_{|y|=1}\left|\frac{\xi}{|\xi|}\cdot y\right|^\lambda\,\mathrm{d}S_y = \int_{|y|=1}|y\cdot y|^\lambda\,\mathrm{d}S_y = \int_{|y|=1}\mathrm{d}S_y.$$

The proof for the case $d>1$ is now complete. $\square$

8. Extension to asymmetric measures μ

In this section we show how to modify the arguments of the previous sections to obtain results for a large class of non-symmetric measures μ including all the Lévy measures used in finance. A careful look at the previous arguments shows that symmetry of μ is used for the sole purpose of having a sign of the fractional term in the energy inequality (see (4.14)) in order to prove Theorems 4.1 and 4.2. This fractional term is

$$\iint_{D_T} \mathcal{L}^\mu[u_N]\, \partial_x^{2\alpha} u_N \,\mathrm{d}x\,\mathrm{d}t = \sum_{|\xi|\leq N} G^\mu(\xi)\, |\xi^\alpha|^2 \int_0^T |\hat{u}_\xi(t)|^2 \,\mathrm{d}t, \tag{8.1}$$

and it is non-positive when μ is symmetric. In the general case the sign of the fractional term (8.1) is unknown, but everything still works if we assume that

$$\mu = \mu_s + \mu_n,$$

for μ_s, μ_n satisfying (1.6) (i.e. we assume (1.5) and (1.6)). Note that in this case, we may split the weights in (3.4) into their symmetric and non-symmetric parts,

$$G^\mu(\xi) = G^{\mu_s}(\xi) + G^{\mu_n}(\xi),$$

where $G^{\mu_s}(\xi)$ is again real and non-positive, and by (1.6),

$$|G^{\mu_n}(\xi)| = \left| \int_{|z|>0} e^{i\xi\cdot z} - 1 - i\xi\cdot z\,\mathbf{1}_{|z|<1}\,\mathrm{d}\mu_n(z)\right| \leq C_n\Big(1+|\xi|\Big). \tag{8.2}$$

The main result of this section is the following:

Theorem 8.1. (Convergence with rate) *Let* ($\mathbf{A}$.1)–($\mathbf{A}$.9), (1.5) *and* (1.6) *hold,* u_N *be the solution of the SVV method* (3.1)*, and* u *be an entropy solution of* (1.1)*. Then,*

$$\|u(\cdot,T) - u_N(\cdot,T)\|_{L^1(\Lambda)} \leq C\sqrt{\epsilon_N}.$$

To prove this result, we have to modify the arguments of the previous sections. In view of the above discussion the key result to obtain is a version of Theorem 4.1 for measures μ satisfying (1.5) and (1.6):

Theorem 8.2. *Assume (**A**.1)–(**A**.7),* (1.5), (1.6) *hold, and let* u_N *be the solution of the SVV approximation* (3.1)*. Then there exists a constant* $\tilde{\mathcal{B}}_s$ *(proportional to* $1+\Pi_{k=1}^s \mathcal{K}_s$ *for* $s\geq 1$ *and to* $\|u_N\|_{L^\infty}$ *for* $s=0$*, see Theorem 4.1) such that*

$$\epsilon_N^s \|\partial_x^s u_N(\cdot,t)\|_{L^2(\Lambda)} + \epsilon_N^{s+\frac{1}{2}} \|\partial_x^{s+1} u_N\|_{L^2(D_T)} \leq \tilde{\mathcal{B}}_s + 4\epsilon_N^s \|\partial_x^s u_N(\cdot,0)\|_{L^2(\Lambda)}.$$

We prove this result at the end of this section. Now if we also assume that ($\mathbf{A}$.8) and ($\mathbf{A}$.9) hold, then it easily follows that Theorem 4.2 still holds if we replace $\mathcal{B}_s$ by $\tilde{\mathcal{B}}_s$. At this point the reader may easily check that *all* the other results also hold if we everywhere replace $\mathcal{B}_s$ by $\tilde{\mathcal{B}}_s$ – and hence Theorem 8.1 follows.

Remark 8.3. A Lévy measure μ defined by

$$\mathrm{d}\mu = g(z)\,\mathrm{d}\pi_\lambda(z),$$

(see (1.4)) can be written as $\mu = \mu_s + \mu_n$ where

$$\mathrm{d}\mu_s = g(z)\wedge g(-z)\,\mathrm{d}\pi_\lambda \quad \text{and} \quad \mathrm{d}\mu_n = [g(z) - g(z)\wedge g(-z)]\,\mathrm{d}\pi_\lambda.$$

Note that $\mu_s, \mu_n \geq 0$, μ_s is symmetric, and that μ_n satisfies the integrability condition in (1.6) if g is locally Lipschitz: Let $g_n(z) = g(z) - g(z)\wedge g(-z)$ and note that $g_n(0) = 0$, hence $g_n(z) = |g_n(z) - g_n(0)| \leq C\,|z|$ for $|z|<1$.

We now show how to modify the proof of Theorem 4.1 to prove Theorem 8.2.

Proof of Theorem 8.2. Once again we use the shorthand $\|\cdot\|$ instead of $\|\cdot\|_{L^2(\Lambda)}$, and rewrite the SVV approximation (3.1) as in (4.9) and (4.10). Note that (5.1) and (5.2) holds for general measures μ, so we find that

$$\int_\Lambda \mathcal{L}^\mu[u_N]\, u_N \ \mathrm{d}x \le \int_\Lambda \mathcal{L}^\mu[u_N^2]\ \mathrm{d}x = 0.$$

Hence, spatial integration of (4.10) against u_N yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_N\|^2 + \epsilon_N\,\|\partial_x u_N\|^2\\ &\le \epsilon_N\|u_N\|\left\|\sum_{j,k=1}^d \partial_j\partial_k R_N^{j,k} * u_N\right\| + \sum_{j=1}^d \|\partial_j u_N\|\|(I-P_N)f_j(u_N)\|,\end{aligned}$$

and the conclusion in the case $s=0$ follows exactly as in the first part of the proof of Theorem 4.1.

Now let $s>0$, and note that by (8.2) and Young's inequality,

$$\begin{aligned}\int_\Lambda \partial_x^{2\alpha} u_N\ \mathcal{L}^{\mu_n}[u_N]\ \mathrm{d}x &= \sum_{|\xi|\le N} (-i\xi)^{2\alpha}\, G^{\mu_n}(\xi)\,|\hat{u}_\xi(t)|^2\\ &\le \sum_{|\xi|\le N} C_n\Big(1+|\xi|\Big)|\xi^\alpha|^2\,|\hat{u}_\xi(t)|^2\\ &\le \sum_{|\xi|\le N}\left(C_n + \frac{\epsilon_N}{4}\,|\xi|^2 + \frac{C_n^2}{\epsilon_N}\right)|\xi^\alpha|^2|\hat{u}_\xi(t)|^2.\end{aligned}$$

If we take this into account and perform spatial integration of (4.9) against $\partial_x^{2\alpha}u_N$ for some multi-index α, we find the following modified version of (4.11),

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^\alpha u_N\|^2 - \sum_{|\xi|\le N} G^{\mu_s}(\xi)|\xi^\alpha|^2|\hat{u}_\xi(t)|^2 + \frac{3\,\epsilon_N}{4}\|\partial_x^\alpha\partial_x u_N\|^2\\ &\le \epsilon_N\|\partial_x^\alpha u_N\|\left\|\sum_{j,k=1}^d \partial_j\partial_k R_N^{j,k} * \partial_x^\alpha u_N\right\|\\ &\quad + \|\partial_x^\alpha\partial_x u_N\|\|\partial_x^{|\alpha|-1}\partial_x\cdot P_N f(u_N)\| + \frac{2\,C_n^2}{\epsilon_N}\|\partial_x^\alpha u_N\|^2.\end{aligned}$$

As in the proof of Theorem 4.1, we now use (3.8) and Young's inequality to bound the first and second term on the right hand side. The result is that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^\alpha u_N\|^2 - \sum_{|\xi|\le N} G^{\mu_s}(\xi)|\xi^\alpha|^2|\hat{u}_\xi(t)|^2 + \frac{\epsilon_N}{2}\|\partial_x^\alpha\partial_x u_N\|^2\\ &\qquad\le C\,\|\partial_x^\alpha u_N\|^2 + \frac{1}{\epsilon_N}\|\partial_x^{|\alpha|}P_N f(u_N)\|^2 + \frac{2\,C_n^2}{\epsilon_N}\|\partial_x^\alpha u_N\|^2.\end{aligned}$$

Now we sum over all $|\alpha|=s$ to find that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^s u_N\|^2 - \sum_{|\alpha|=s}\sum_{|\xi|\le N} G^{\mu_s}(\xi)|\xi^\alpha|^2|\hat{u}_\xi(t)|^2 + \frac{\epsilon_N}{2}\|\partial_x^{s+1}u_N\|^2\\ &\qquad\le C\,\|\partial_x^s u_N\|^2 + \frac{d^s}{\epsilon_N}\|\partial_x^s P_N f(u_N)\|^2 + \frac{2\,C_n^2}{\epsilon_N}\|\partial_x^s u_N\|^2.\end{aligned}$$

Thanks to (4.1) and (4.2),

$$\|\partial_x^s P_N f(u_N)\| \le \mathcal{K}_s\,\|\partial_x^s u_N\| + \frac{\mathcal{K}_{s+1}}{N}\,\|\partial_x^{s+1}u_N\|,$$

and hence

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_x^s u_N\|^2 - \sum_{|\alpha|=s}\sum_{|\xi|\le N} G^{\mu_s}(\xi)|\xi^\alpha|^2|\hat{u}_\xi(t)|^2 + \left(\frac{\epsilon_N}{2} - \frac{2d^s\mathcal{K}_{s+1}^2}{N^2\epsilon_N}\right)\|\partial_x^{s+1}u_N\|^2$$
$$\le \left(C + \frac{2\,C_n^2 + 2d^s\mathcal{K}_s^2}{\epsilon_N}\right)\|\partial_x^s u_N\|^2 \le \frac{2\,C_n^2 + 3d^s\mathcal{K}_s^2}{\epsilon_N}\|\partial_x^s u_N\|^2,$$

where the last inequality holds for N big enough.

To conclude, we use (**A**.7) to obtain

$$\frac{1}{2}\|\partial_x^s u_N(\cdot,t)\|^2 - \sum_{|\alpha|=s}\sum_{|\xi|\le N} G^{\mu_s}(\xi)|\xi^\alpha|^2\int_0^t |\hat{u}_\xi(\tau)|^2\,\mathrm{d}\tau + \frac{\epsilon_N}{4}\|\partial_x^{s+1}u_N\|^2_{L^2(D_T)}$$
$$\le \frac{2\,C_n^2 + 3d^s\mathcal{K}_s^2}{\epsilon_N}\|\partial_x^s u_N\|^2_{L^2(D_T)} + \frac{1}{2}\|\partial_x^s u_N(\cdot,0)\|^2.$$

The proof is now complete since by induction on s,

$$\|\partial_x^s u_N\|^2_{L^2(D_T)} \le C\,\tilde{\mathcal{B}}_{s-1}^2\epsilon_N^{-(2s-1)}.$$

□

Appendix A. Proof of Theorem 2.2

Let us take $\varphi = \psi(x,y,t,s)$, $u = u(x,t)$ and $v = v(y,s)$. We set $k = v(y,s)$ in the entropy inequality for $u(x,t)$, and integrate over all $(y,s) \in Q_T$ to obtain

$$\begin{aligned}\iint_{D_T}\iint_{D_T} & \eta(u(x,t),v(y,s))\,\partial_t\psi(x,y,t,s)\\ & + q(u(x,t),v(y,s))\cdot\partial_x\psi(x,y,t,s)\\ & + \eta(u(x,t),v(y,s))\,\mathcal{L}_r^{*,\mu}[\psi(\cdot,y,t,s)](x)\\ & + \eta'(u(x,t),v(y,s))\,\mathcal{L}^{\mu,r}[u(\cdot,t)](x)\,\psi(x,y,t,s)\\ & + \eta(u(x,t),v(y,s))\,\gamma_\mu^r\cdot\partial_x\psi(x,y,t,s)\;\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s \ge 0.\end{aligned}$$

In the entropy inequality for $v(y,s)$, we set $k = u(x,t)$ and integrate with respect to (x,t) to find that

$$\begin{aligned}\iint_{D_T}\iint_{D_T} & \eta(u(x,t),v(y,s))\,\partial_s\psi(x,y,t,s)\\ & + q(u(x,t),v(y,s))\cdot\partial_y\psi(x,y,t,s)\\ & + \eta(u(x,t),v(y,s))\,\mathcal{L}_r^{*,\mu}[\psi(x,\cdot,t,s)](y)\\ & - \eta'(u(x,t),v(y,s))\,\mathcal{L}^{\mu,r}[v(\cdot,s)](y)\,\psi(x,y,t,s)\\ & + \eta(u(x,t),v(y,s))\,\gamma_\mu^r\cdot\partial_y\psi(x,y,t,s)\;\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t \ge 0.\end{aligned}$$

In the following we need the $\mathbb{R}^{2d}$-operators

$$\tilde{\mathcal{L}}^{\mu,r}[\phi(\cdot,\cdot)](x,y) = \int_{|z|>r}\phi(x+z,y+z) - \phi(x,y)\;\mathrm{d}\mu(z),$$
$$\tilde{\mathcal{L}}^{*,\mu,r}[\phi(\cdot,\cdot)](x,y) = \int_{|z|>r}\phi(x-z,y-z) - \phi(x,y)\;\mathrm{d}\mu(z).$$

With these definitions in mind, we add the two inequalities above and change the order of integration to find that

$$\begin{aligned}\iint_{D_T}\iint_{D_T} &\eta(u(x,t),v(y,s))\,(\partial_t+\partial_s)\psi(x,y,t,s)\\ &+q(u(x,t),v(y,s))\cdot(\partial_x+\partial_y)\psi(x,y,t,s)\\ &+\eta(u(x,t),v(y,s))\,\mathcal{L}_r^{*,\mu}[\psi(\cdot,y,t,s)](x)\\ &+\eta(u(x,t),v(y,s))\,\mathcal{L}_r^{*,\mu}[\psi(x,\cdot,t,s)](y)\\ &+\eta'(u(x,t),v(y,s))\,\tilde{\mathcal{L}}^{\mu,r}[u(\cdot,t)-v(\cdot,s)](x,y)\,\psi(x,y,t,s)\\ &+\eta(u(x,t),v(y,s))\,\gamma_\mu^r\cdot(\partial_x+\partial_y)\psi(x,y,t,s)\;\mathrm{d}w\geq 0.\end{aligned}$$

Here and in the following we use the shorthand $\mathrm{d}w=\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s$. Note that

$$\eta'(u(x,t),v(y,s))\,\tilde{\mathcal{L}}^{\mu,r}[u(\cdot,t)-v(\cdot,s)](x,y)\leq\tilde{\mathcal{L}}^{\mu,r}[\eta(u(\cdot,t),v(\cdot,s))](x,y).$$

Moreover, using the change of variables $(x,y)\to(x-z,y-z)$,

$$\begin{aligned}&\iint_{D_T}\iint_{D_T}\psi(x,y,t,s)\,\tilde{\mathcal{L}}^{\mu,r}[\eta(u(\cdot,t),v(\cdot,s))](x,y)\;\mathrm{d}w\\ &=\int_{|z|>r}\int_0^T\int_{z+\Lambda}\int_0^T\int_{z+\Lambda}\eta(u(x,t),v(y,s))\,\psi(x-z,y-z,t,s)\;\mathrm{d}w\,\mathrm{d}\mu(z)\\ &\quad-\int_{|z|>r}\iint_{D_T}\iint_{D_T}\eta(u(x,t),v(y,s))\,\psi(x,y,t,s)\;\mathrm{d}w\,\mathrm{d}\mu(z),\end{aligned}$$

which by periodicity and the definition of $\tilde{\mathcal{L}}^{*,\mu,r}$ equals to

$$\begin{aligned}&\int_{|z|>r}\int_0^T\int_{\Lambda}\int_0^T\int_{\Lambda}\eta(u(x,t),v(y,s))\,\psi(x-z,y-z,t,s)\;\mathrm{d}w\,\mathrm{d}\mu(z)\\ &\quad-\int_{|z|>r}\iint_{D_T}\iint_{D_T}\eta(u(x,t),v(y,s))\,\psi(x,y,t,s)\;\mathrm{d}w\,\mathrm{d}\mu(z)\\ &=\iint_{D_T}\iint_{D_T}\eta(u(x,t),v(y,s))\,\tilde{\mathcal{L}}^{*,\mu,r}[\psi(\cdot,\cdot,t,s)](x,y)\;\mathrm{d}w.\end{aligned}$$

Therefore we have proved so far that

$$\begin{aligned}\iint_{D_T}\iint_{D_T} &\eta(u(x,t),v(y,s))\,(\partial_t+\partial_s)\psi(x,y,t,s)\\ &+q(u(x,t),v(y,s))\cdot(\partial_x+\partial_y)\psi(x,y,t,s)\\ &+\eta(u(x,t),v(y,s))\,\mathcal{L}_r^{*,\mu}[\psi(\cdot,y,t,s)](x)\\ &+\eta(u(x,t),v(y,s))\,\mathcal{L}_r^{*,\mu}[\psi(x,\cdot,t,s)](y)\\ &+\eta(u(x,t),v(y,s))\,\tilde{\mathcal{L}}^{*,\mu,r}[\psi(\cdot,\cdot,t,s)](x,y)\\ &+\eta(u(x,t),v(y,s))\,\gamma_\mu^r\cdot(\partial_x+\partial_y)\psi(x,y,t,s)\;\mathrm{d}w\geq 0.\end{aligned}$$

We now send $r\to 0$, remembering the definition of γ_μ^r and defining

$$\begin{aligned}&\tilde{\mathcal{L}}^{*,\mu}[\phi(\cdot,\cdot)](x,y)\\ &=\int_{|z|>0}\phi(x-z,y-z)-\phi(x,y)+z\cdot(\partial_x+\partial_y)\phi(x,y)\,\mathbf{1}_{|z|<1}\;\mathrm{d}\mu(z).\end{aligned}$$

The result is

$$\begin{aligned}\iint_{D_T}\iint_{D_T} &\eta(u(x,t),v(y,s))\,(\partial_t+\partial_s)\psi(x,y,t,s)\\ &+q(u(x,t),v(y,s))\cdot(\partial_x+\partial_y)\psi(x,y,t,s)\\ &+\eta(u(x,t),v(y,s))\,\tilde{\mathcal{L}}^{*,\mu}[\psi(\cdot,\cdot,t,s)](x,y)\;\mathrm{d}w\geq 0.\end{aligned}\tag{A.1}$$

To conclude, we show how to derive the L^1-contraction (2.2) from this inequality by choosing the test function ψ as

$$\psi(x,y,t,s) = \hat{\omega}_\rho\left(\frac{x-y}{2}\right)\omega_\delta\left(\frac{t-s}{2}\right)\phi(t), \quad \rho,\delta>0, \tag{A.2}$$

where $\omega_\delta(\tau) = \frac{1}{\delta}\,\omega(\frac{\tau}{\delta})$ for a nonnegative $\omega \in C_c^\infty(\mathbb{R})$ satisfying

$$\omega(-\tau) = \omega(\tau), \quad \omega(\tau) = 0 \text{ for all } |\tau| \geq 1, \quad \text{and} \quad \int_{\mathbb{R}} \omega(\tau)\,\mathrm{d}\tau = 1,$$

while $\hat{\omega}_\rho(x) = \bar{\omega}_\rho(x_1)\cdots\bar{\omega}_\rho(x_d)$ with $\bar{\omega}_\rho(\cdot)$ such that

$$\bar{\omega}_\rho(\tau) = \sum_{k\in\mathbb{Z}} \omega_\rho(\tau + 2\pi k).$$

Note that $\hat{\omega}_\rho$ is periodic in each coordinate direction. By a direct computation,

$$\begin{aligned}
&(\partial_t + \partial_s)\psi(x,y,t,s) = \hat{\omega}_\rho\left(\frac{x-y}{2}\right)\omega_\delta\left(\frac{t-s}{2}\right)\phi'(t),\\
&(\partial_x + \partial_y)\psi(x,y,t,s) = 0,\\
&\tilde{\mathcal{L}}^*[\psi(\cdot,\cdot,t,s)](x,y) = 0.
\end{aligned}$$

Thus, with this test function ψ at hand, inequality (A.1) becomes

$$\iint_{D_T} |u(x,t) - v(y,s)|\,\hat{\omega}_\rho\left(\frac{x-y}{2}\right)\omega_\delta\left(\frac{t-s}{2}\right)\phi'(t)\;\mathrm{d}w \geq 0. \tag{A.3}$$

We then go to the limit as $(\rho,\delta)\to 0$ to find that

$$\iint_{D_T} |u(x,t) - v(x,t)|\,\phi'(t)\;\mathrm{d}x\,\mathrm{d}t \geq 0. \tag{A.4}$$

To conclude the proof we now take $\phi = \chi_\mu$ for

$$\chi_\mu(t) = \int_{-\infty}^{t} (\omega_\mu(\tau - t_1) - \omega_\mu(\tau - t_2))\;\mathrm{d}\tau, \qquad 0 < t_1 < t_2 < T. \tag{A.5}$$

Loosely speaking, the function χ_μ is a smooth approximation of the indicator function $\mathbf{1}_{(t_1,t_2)}$ which is zero near $t=0$ and $t=T$ for $\mu>0$ small. Since

$$\chi_\mu'(t) = \omega_\mu(t-t_1) - \omega_\mu(t-t_2),$$

inequality (A.4) reduces to

$$\iint_{Q_T} |u(x,t) - v(x,t)|\,\omega_\mu(t-t_2)\;\mathrm{d}t\,\mathrm{d}x \leq \iint_{Q_T} |u(x,t) - v(x,t)|\,\omega_\mu(t-t_1)\;\mathrm{d}t\,\mathrm{d}x.$$

By the integrability of u and v and Fubini's theorem, the function

$$\Phi(t) = \int_\Lambda |u(x,t) - v(x,t)|\;\mathrm{d}x \in L^1(0,T),$$

and we may write the above inequality as a convolution

$$\Phi * \omega_\mu(t_2) \leq \Phi * \omega_\mu(t_1).$$

By standard properties of convolutions, $\Phi * \omega_\mu(t) \to \Phi(t)$ a.e. t as $\mu \to 0$. Hence,

$$\|(u-v)(\cdot,t_2)\|_{L^1(\Lambda)} \leq \|(u-v)(\cdot,t_1)\|_{L^1(\Lambda)} \quad \text{for a.e. } t_1,t_2 \in (0,T).$$

Finally, the theorem follows from renaming t_2 and using part *iii)* in Definition 2.1 to send $t_1 \to 0$.

Appendix B. Proof of Theorem 2.4

The vanishing viscosity problem (2.3) has a unique classical solution u_ϵ for $\varepsilon > 0$, see Remark 2.6. If we multiply (2.3) by $\eta'(u_\epsilon)$ for any smooth convex function η, use standard manipulations on the conservation law part combined with the inequalities

$$\eta'(u_\epsilon)\mathcal{L}^\mu[u_\epsilon] = \eta'(u_\epsilon)\bigg(\mathcal{L}^\mu_r[u_\epsilon] + \mathcal{L}^{\mu,r}[u_\epsilon]\bigg) \leq \mathcal{L}^\mu_r[\eta(u_\epsilon)] + \eta'(u_\epsilon)\mathcal{L}^{\mu,r}[u_\epsilon],$$
$$\eta'(u_\epsilon)\,\Delta u_\epsilon = \Delta\eta(u_\epsilon) - \epsilon\,\eta''(u_\epsilon)|\partial_x u_\epsilon|^2 \leq \Delta\eta(u_\epsilon),$$

we find, after integration against any nonnegative test function ϕ, that u_ϵ satisfies the (entropy) inequality

$$\iint_{D_T} \eta(u_\epsilon,k)\,\partial_t\varphi + q(u_\epsilon,k)\cdot\partial_x\varphi + \eta(u_\epsilon,k)\,\mathcal{L}^{*,\mu}_r[\varphi] + \eta'(u_\epsilon,k)\,\mathcal{L}^{\mu,r}[u_\epsilon]$$
$$+\,\eta(u_\epsilon,k)\,\gamma^r_\mu\cdot\partial_x\varphi + \epsilon\,\eta(u_\epsilon,k)\,\Delta\varphi\;\mathrm{d}x\,\mathrm{d}t \geq 0.$$

From this inequality we proceed as in the proof of the L^1-contraction (Theorem 2.2). We take $u = u(x,t)$, $u_\epsilon = u_\epsilon(y,s)$, and find the inequalities

$$\begin{aligned}\iint_{D_T}\iint_{D_T} &\eta(u(x,t),u_\epsilon(y,s))\,\partial_t\psi(x,y,t,s)\\ &+ q(u(x,t),u_\epsilon(y,s))\cdot\partial_x\psi(x,y,t,s)\\ &+ \eta(u(x,t),u_\epsilon(y,s))\,\mathcal{L}^{*,\mu}_r[\psi(\cdot,y,t,s)](x)\\ &+ \eta'(u(x,t),u_\epsilon(y,s))\,\mathcal{L}^{\mu,r}[u(\cdot,t)](x)\,\psi(x,y,t,s)\\ &+ \eta(u(x,t),u_\epsilon(y,s))\,\gamma^r_\mu\cdot\partial_x\psi(x,y,t,s)\;\mathrm{d}w \geq 0.\end{aligned}$$

and

$$\begin{aligned}\iint_{D_T}\iint_{D_T} &\eta(u(x,t),u_\epsilon(y,s))\,\partial_s\psi(x,y,t,s)\\ &+ q(u(x,t),u_\epsilon(y,s))\cdot\partial_y\psi(x,y,t,s)\\ &+ \eta(u(x,t),u_\epsilon(y,s))\,\mathcal{L}^{*,\mu}_r[\psi(x,\cdot,t,s)](y)\\ &- \eta'(u(x,t),u_\epsilon(y,s))\,\mathcal{L}^{\mu,r}[u_\epsilon(\cdot,s)](y)\,\psi(x,y,t,s)\\ &+ \eta(u(x,t),u_\epsilon(y,s))\,\gamma^r_\mu\cdot\partial_y\psi(x,y,t,s)\;\mathrm{d}w\\ &+ \epsilon\,\eta(u(x,t),u_\epsilon(y,s))\,\Delta_y\psi(x,y,t,s)\;\mathrm{d}w \geq 0.\end{aligned}$$

As in the proof of Theorem 2.2, we add and manipulate these to get (see (A.1))

$$\begin{aligned}\iint_{D_T}\iint_{D_T} &\eta(u(x,t),u_\epsilon(y,s))\,(\partial_t+\partial_s)\psi(x,y,t,s)\\ &+ q(u(x,t),u_\epsilon(y,s))\cdot(\partial_x+\partial_y)\psi(x,y,t,s)\\ &+ \eta(u(x,t),u_\epsilon(y,s))\,\tilde{\mathcal{L}}^{*,\mu}[\psi(\cdot,\cdot,t,s)](x,y)\\ &+ \epsilon\,\eta(u(x,t),u_\epsilon(y,s))\,\Delta_y\psi(x,y,t,s)\;\mathrm{d}w \geq 0.\end{aligned}$$

We now take the test function ψ as in (A.2) and find that (see (A.3))

$$\begin{aligned}&-\iint_{D_T}\iint_{D_T} |u(x,t)-v(y,s)|\,\hat{\omega}_\rho\left(\frac{x-y}{2}\right)\omega_\delta\left(\frac{t-s}{2}\right)\phi'(t)\;\mathrm{d}w\\ &\qquad\leq \epsilon\iint_{D_T}\iint_{D_T}\eta(u(x,t),u_\epsilon(y,s))\,\Delta_y\psi(x,y,t,s)\;\mathrm{d}w.\end{aligned} \tag{B.1}$$

After an integration by parts, the right-hand side (R.H.S.) is bounded by

$$\begin{aligned}\text{R.H.S.} &\leq \epsilon \iint_{D_T} \iint_{D_T} \Big|\partial_y |u(x,t) - u_\epsilon(y,s)|\Big| \Big|\partial_y \psi(x,y,t,s)\Big| \,\mathrm{d}w \\ &\leq \epsilon \iint_{D_T} \iint_{D_T} |\partial_y u_\epsilon(y,s)| \, |\partial_y \psi(x,y,t,s)| \,\mathrm{d}w \\ &\leq CT\, |u_0|_{BV(\Lambda)} \frac{\epsilon}{\rho},\end{aligned}$$

where the last inequality is a consequence of the estimate $|u_\varepsilon(\cdot,t)|_{BV(\Lambda)} \leq |u_0|_{BV(\Lambda)}$ and (A.2).

To estimate the left hand side (L.H.S.) of (B.1), note that

$$\begin{aligned}&- |u_\epsilon(y,s) - u(x,t)|\phi'(t) \\ &\geq -|u_\epsilon(x,t) - u(x,t)|\phi'(t) - |u_\epsilon(x,s) - u_\epsilon(x,t)||\phi'(t)| - |u_\epsilon(y,s) - u_\epsilon(x,s)||\phi'(t)|,\end{aligned}$$

and that

$$\iint_{D_T} \iint_{D_T} |u_\epsilon(x,s) - u_\epsilon(x,t)|\, \hat{\omega}_\rho \left(\frac{x-y}{2}\right) \omega_\delta \left(\frac{t-s}{2}\right) |\phi'(t)| \,\mathrm{d}w \overset{\delta\to 0}{\longrightarrow} 0$$

and

$$\iint_{D_T} \iint_{D_T} |u_\epsilon(y,s) - u_\epsilon(x,s)|\, \hat{\omega}_\rho \left(\frac{x-y}{2}\right) \omega_\delta \left(\frac{t-s}{2}\right) |\phi'(t)| \,\mathrm{d}w \leq C\,T |u_0|_{BV}\, \rho.$$

Hence we conclude after sending $\delta \to 0$ that

$$- \iint_{D_T} |u_\epsilon(x,t) - u(x,t)|\, \phi'(t) dx dt - C\rho \leq L.H.S.\ (\leq R.H.S.).$$

The results then follows by setting $\rho = \sqrt{\epsilon}$ and $\phi = \chi_\mu$ as in (A.5), and conclude as in the proof of Theorem 2.2: Sending $\mu \to 0$, setting $t_2 = t$, and using part *iii)* in Definition 2.1 to send $t_1 \to 0$.

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(Simone Cifani), Department of Mathematics, Norwegian University of Science and Technology (NTNU), N-7491 Trondheim, Norway

E-mail address: `simone.cifani@math.ntnu.no`

URL: `http://www.math.ntnu.no/~cifani/`

(Espen R. Jakobsen), Department of Mathematics, Norwegian University of Science and Technology (NTNU), N-7491 Trondheim, Norway

E-mail address: `erj@math.ntnu.no`

URL: `http://www.math.ntnu.no/~erj/`

Paper V

Continuous dependence estimates for nonlinear fractional convection-diffusion equations

N. Alibaud, S. Cifani, and E. R. Jakobsen

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CONTINUOUS DEPENDENCE ESTIMATES FOR NONLINEAR FRACTIONAL CONVECTION-DIFFUSION EQUATIONS

NATHAËL ALIBAUD, SIMONE CIFANI, AND ESPEN R. JAKOBSEN

Abstract. We develop a general framework for finding error estimates for convection-diffusion equations with nonlocal, nonlinear, and possibly degenerate diffusion terms. The equations are nonlocal because they involve fractional diffusion operators that are generators of pure jump Lévy processes (e.g. the fractional Laplacian). As an application, we derive continuous dependence estimates on the nonlinearities and on the Lévy measure of the diffusion term. Estimates of the rates of convergence for general nonlinear nonlocal vanishing viscosity approximations of scalar conservation laws then follow as a corollary. Our results both cover, and extend to new equations, a large part of the known error estimates in the literature.

1. Introduction

This paper is concerned with the following Cauchy problem:

$$\begin{cases} \partial_t u(x,t) + \operatorname{div}\left(f(u)\right)(x,t) = \mathcal{L}^\mu[A(u(\cdot,t))](x) & \text{in} \quad Q_T := \mathbb{R}^d \times (0,T), \\ u(x,0) = u_0(x), & \text{in} \quad \mathbb{R}^d, \end{cases} \tag{1.1}$$

where u is the scalar unknown function, div denotes the divergence with respect to (w.r.t.) x, and the operator $\mathcal{L}^\mu$ is defined for all $\phi \in C_c^\infty(\mathbb{R}^d)$ by

$$\mathcal{L}^\mu[\phi](x) := \int_{\mathbb{R}^d\setminus\{0\}} \phi(x+z) - \phi(x) - z \cdot D\phi(x)\mathbf{1}_{|z|\leq 1} \, \mathrm{d}\mu(z), \tag{1.2}$$

where $D\phi$ denotes the gradient of ϕ w.r.t. x and $\mathbf{1}_{|z|\leq 1} = 1$ for $|z| \leq 1$ and $= 0$ otherwise. Throughout the paper, the data (f, A, u_0, μ) is assumed to satisfy the following assumptions:

$$f \in W^{1,\infty}(\mathbb{R}, \mathbb{R}^d) \text{ with } f(0) = 0, \tag{1.3}$$

$$A \in W^{1,\infty}(\mathbb{R}) \text{ is nondecreasing with } A(0) = 0, \tag{1.4}$$

$$u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \cap BV(\mathbb{R}^d), \tag{1.5}$$

and

(1.6) μ is a nonnegative Radon measure on $\mathbb{R}^d \setminus \{0\}$ satisfying

$$\int_{\mathbb{R}^d\setminus\{0\}} |z|^2 \wedge 1 \, \mathrm{d}\mu(z) < +\infty,$$

where we use the notation $a \wedge b = \min\{a, b\}$. The measure μ is a Lévy measure.

Remark 1.1.

(1) Subtracting constants to f and A if necessary, there is no loss of generality in assuming that $f(0) = 0$ and $A(0) = 0$.

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(2) Our results also hold for locally Lipschitz-continuous nonlinearities f and A since solutions will be bounded; see Remark 2.3 for more details.
(3) Assumption (1.6) and Taylor expansion reveals that $\mathcal{L}^\mu[\phi]$ is well-defined for e.g. bounded C^2 functions ϕ:

$$|\mathcal{L}^\mu[\phi](x)| \leq \max_{|z|\leq 1} |D^2\phi(x+z)| \int_{0<|z|\leq 1} \frac{1}{2}|z|^2 \mathrm{d}\mu(z) + 2\|\phi\|_{L^\infty(\mathbb{R})} \int_{|z|>1} \mathrm{d}\mu(z)$$

where $D^2\phi$ is the Hessian of ϕ. If in addition $D^2\phi$ is bounded on $\mathbb{R}^d$, then so is $\mathcal{L}^\mu[\phi]$.

Under (1.6), $\mathcal{L}^\mu$ is the generator of a pure jump Lévy process, and reversely, any pure jump Lévy process has a generator of like $\mathcal{L}^\mu$ (see e.g. [6, 56]). This class of diffusion processes contains e.g. the α-stable process whose generator is the fractional Laplacian $-(-\triangle)^{\frac{\alpha}{2}}$ with $\alpha \in (0,2)$. It can be defined for all $\phi \in C_c^\infty(\mathbb{R}^d)$ via the Fourier transform as

$$(-\triangle)^{\frac{\alpha}{2}} \phi = \mathcal{F}^{-1}\left(|\cdot|^\alpha \mathcal{F}\phi\right),$$

or in the form (1.2) with the following Lévy measure (see e.g. [6, 32]):

$$\mathrm{d}\mu(z) = \frac{\mathrm{d}z}{|z|^{d+\alpha}} \text{ (up to a positive multiplicative constant).} \tag{1.7}$$

Many other Lévy processes/operators of practical interest can be found in e.g. [6, 24]. Under assumption (1.4), $\mathcal{L}^\mu[A(\cdot)]$ is an example of a nonlinear nonlocal diffusion operator. For recent studies of this and similar type of operators, we refer the reader to [8, 9, 15, 19, 27] and the references therein.

Equation (1.1) appears in many different contexts such as overdriven gas detonations [22], mathematical finance [24], flow in porous media [27], radiation hydrodynamics [53, 54], and anomalous diffusion in semiconductor growth [59]. Equations of the form (1.1) constitute a large class of nonlinear degenerate parabolic integro-differential equations (integro-PDEs). Let us give some representative examples.

When $A = 0$ or $\mu = 0$, (1.1) is the well-known scalar conservation law (see e.g. [25] and references therein):

$$\partial_t u + \operatorname{div} f(u) = 0. \tag{1.8}$$

When $A(u) = u$ and $\mathcal{L}^\mu$ is the fractional Laplacian, (1.1) is the so-called fractal/fractional conservation law:

$$\partial_t u + \operatorname{div} f(u) = -(-\triangle)^{\frac{\alpha}{2}} u. \tag{1.9}$$

Equation (1.9) has been extensively studied since the nineties [1, 2, 3, 4, 5, 7, 10, 11, 12, 16, 17, 20, 21, 28, 29, 30, 31, 32, 35, 38, 39, 40, 41, 43, 46, 51, 52]. When A is nonlinear, (1.1) can be seen as a generalization of the following classical convection-diffusion equation (possibly degenerate):

$$\partial_t u + \operatorname{div} f(u) = \triangle A(u); \tag{1.10}$$

see e.g. [13, 14, 18, 23, 42] for precise references on (1.10). Equations combining non-linear local diffusion and more general Lévy diffusion,

$$\partial_t u + \operatorname{div} f(u) = \operatorname{div}(a(u)\nabla u) + \mathcal{L}^\mu[u], \tag{1.11}$$

have been considered in [43].

The case of nonlinear nonlocal diffusions has been studied in [27] in the setting of nonlocal porous media equations, and in [19] where a general $L^\infty \cap L^1$-theory for (1.1) is developed along with connections to Hamilton-Jacobi-Bellman equations of stochastic control theory. Other interesting examples concern the class of non-singular Lévy measures satisfying $\int_{\mathbb{R}^d\setminus\{0\}} \mathrm{d}\mu(z) < +\infty$. In that case, (1.1) can be seen as a generalization of Rosenau's models [44, 45, 49, 50, 57, 58] and nonlinear radiation hydrodynamics models [53] of the form

$$\partial_t u + \mathrm{div} f(u) = g * A(u) - A(u), \tag{1.12}$$

where $*$ denotes the convolution product w.r.t. x and $g \in L^1(\mathbb{R}^d)$ is nonnegative with $\int_{\mathbb{R}^d} g(z) \mathrm{d}z = 1$.

Most of the results on such nonlocal convection-diffusion equations concern Equation (1.9) whose diffusion is linear. It is known that shocks can occur in finite time [4, 28, 44, 45, 46, 50, 57], that weak solutions can be nonunique [2], and that the Cauchy problem is well-posed with the notion of entropy solutions in the sense of Kruzhkov [1, 43, 49, 57]; see also [37] for the related topic of time fractional derivatives. Entropy solutions theories can also be found in [53] for nonlinear but non-singular nonlocal diffusions and in [43] for linear but more general singular Lévy diffusions along with nonlinear local diffusions. Very recently, the entropy solution theory has been extended in [19] to cover the full problem (1.1) for general singular Lévy measures and nonlinear A.

The purpose of the present paper is to develop an abstract framework for finding error estimates for entropy solutions of (1.1). As applications, we focus in this paper on continuous dependence estimates and convergence rates for vanishing viscosity approximations. We refer the reader to [13, 18, 23, 42, 48] and the references therein for similar analysis on (1.10) and related local equations. As far as nonlocal equations are concerned, continuous dependence estimates for fully nonlinear integro-PDEs have already been derived in [36] in the context of viscosity solutions of Bellman-Isaacs equations; see also [32, 34, 36] for error estimates on nonlocal vanishing viscosity approximations.

To the best of our knowledge, the only continuous dependence estimate for nonlocal conservation laws can be found in [43]; see also [1, 29, 32, 49, 57] for convergence rates for vanishing viscosity approximations of Equations (1.9) and (1.12). The general estimate in [43] is given for Equation (1.11) in the case of self-adjoint Lévy operators. Inspired by the present paper, a formal discussion on possible extensions to nonlinear nonlocal diffusions is also given. On the technical level, [43] employs so-called entropy defect measures while we do not.

To finish with the bibliography, let us also refer the reader to [20, 21, 26, 30, 53] for the related topic of error estimates for numerical approximations.

Our main result is stated in Lemma 3.1, and it compares the entropy solution u of (1.1) with a general function v. Our main application consists in comparing u with the entropy solution v of

$$\begin{cases} \partial_t v + \mathrm{div} g(v) = \mathcal{L}^\nu[B(v)], \\ v(x,0) = v_0, \end{cases} \tag{1.13}$$

where the data set (g, B, v_0, ν) is assumed to satisfy (1.3)–(1.6). We obtain explicit continuous dependence estimates on the data stated in Theorems 3.3–3.4. Let us recall that when $B = 0$ or $\nu = 0$, (1.13) is the pure scalar conservation law in (1.8). Equation (1.1) can thus be seen as a nonlinear nonlocal vanishing viscosity approximation of (1.8) if A or μ vanishes. The rate of convergence is then obtained as a consequence of Theorems 3.3–3.4, see Theorem 3.9.

It is natural to compare Theorems 3.3–3.4 and Theorem 3.9 with the known error estimates for Equations (1.9) and (1.12). One can see that a quite important part of them are particular cases of our general results. We discuss this point in Section 3 by giving precise examples. Let us mention that we also give a simple example of Hamilton-Jacobi equations suggesting that Theorems 3.3–3.4 are in some sense the "conservation laws' versions" of the results in [36]; see Example 3.2.

To finish, let us mention that in the case of fractional Laplacians of order $\alpha \geq 1$, Theorems 3.3–3.4 can be improved by taking advantage of the homogeneity of the measures in (1.7). In order not to make this paper too long, this special case (including $\alpha < 1$) is investigated in a second paper [3].

The rest of this paper is organized as follows. In Section 2 we list the notation used throughout the paper; we also recall the notion of entropy solution to (1.1). In Section 3, we state and discuss our main results. Sections 4–5 are devoted to the proofs of our main results; Section 4 states some preliminary results on the nonlocal operator.

2. Preliminaries

In this section we explain most of the notation used in the paper, and we give the definition of entropy solutions of (1.1) along with a well-posedness result.

2.1. **Notation.** Throughout the paper $d \in \mathbb{N}$ is a fixed dimension, $T > 0$ a time, and $(x, t) \in Q_T := \mathbb{R}^d \times (0, T)$ the generic space-time variable. We let $a \wedge b := \min\{a, b\}$, $a \vee b := \max\{a, b\}$, $a^+ := a \vee 0$, and $a^- := (-a) \vee 0$, while $\cdot$ and $|\cdot|$ denote the Euclidean inner product and norm of $\mathbb{R}^m$. For matrices $A \in \mathbb{R}^{m\times m}$, we use the norm $|A| = \max\{A\,w : w \in \mathbb{R}^m,\ |w| \leq 1\}$. We let $-E := \{-w \in \mathbb{R}^m : w \in E\}$, and denote the characteristic function of the set E by $\mathbf{1}_E$.

By C^∞ and C_c^∞ we denote the spaces of infinitely differentiable functions and infinitely differentiable functions with compact support. Moreover, for $p \in [1, +\infty]$, L^p, $W^{k,p}$, L^1_{loc}, BV and $\mathcal{D}'$ denote the Lebesgue and Sobolev spaces, the locally integrable functions, the functions of bounded variation, and the Schwartz distributions respectively. The symbols $\|\cdot\|$ and $|\cdot|$ are used to denote norms and semi-norms respectively. The support of a function (or a distribution) u is denoted by $\operatorname{supp} u$. We let $\partial_t u$, $D_x u$, and $D_x^2 u$ denote the partial derivative in time, the spatial gradient, and the spatial Hessian matrix of $u = u(x, t)$ respectively. If there is no confusion, we write D instead of D_x.

The positive and negative parts and total variation of a Radon measure μ are denoted by $\mu^\pm$ and $|\mu|$. Its tensor product with the Lebesgue measure is denoted by $\mathrm{d}\mu(z)\,\mathrm{d}w$ where z is the variable of μ and w of the Lebesgue measure.

2.2. **Entropy formulation and well-posedness.** Let us recall the formal computations leading to the entropy formulation of (1.1). First we split $\mathcal{L}^\mu$ into 3 parts:

$$\mathcal{L}^\mu[\phi](x) = \mathcal{L}^\mu_r[\phi](x) + \operatorname{div}\left(b^\mu_r\,\phi\right)(x) + \mathcal{L}^{\mu,r}[\phi](x) \tag{2.1}$$

for $\phi \in C_c^\infty(\mathbb{R}^d)$, $r > 0$, and $x \in \mathbb{R}^d$, where

$$\mathcal{L}^\mu_r[\phi](x) := \int_{0<|z|\leq r} \phi(x+z) - \phi(x) - z \cdot D\phi(x)\,\mathbf{1}_{|z|\leq 1}\,\mathrm{d}\mu(z), \tag{2.2}$$

$$b^\mu_r := -\int_{|z|>r} z\mathbf{1}_{|z|\leq 1}\,\mathrm{d}\mu(z), \tag{2.3}$$

$$\mathcal{L}^{\mu,r}[\phi](x) := \int_{|z|>r} \phi(x+z) - \phi(x)\,\mathrm{d}\mu(z). \tag{2.4}$$

Consider then the Kruzhkov [47] entropies $|\cdot - k|$, $k \in \mathbb{R}$, and entropy fluxes

$$q_f(u, k) := \operatorname{sgn}(u-k)\,(f(u) - f(k)) \in \mathbb{R}^d, \tag{2.5}$$

where we always use the following everywhere representative of the sign function:

$$\operatorname{sgn}(u) := \begin{cases} \pm 1 & \text{if } \pm u > 0, \\ 0 & \text{if } u = 0. \end{cases} \tag{2.6}$$

By (1.4) it is readily seen that for all $u, k \in \mathbb{R}$,

$$\operatorname{sgn}(u-k)\,(A(u)-A(k)) = |A(u)-A(k)|, \tag{2.7}$$

and we formally deduce from (2.1), (2.7), and the nonnegativity of μ that

$$\begin{aligned}&\operatorname{sgn}(u-k)\,\mathcal{L}^{\mu}[A(u)]\\ &\leq \mathcal{L}^{\mu}_{r}[|A(u)-A(k)|] + \operatorname{div}(b^{\mu}_{r}\,|A(u)-A(k)|) + \operatorname{sgn}(u-k)\,\mathcal{L}^{\mu,r}[A(u)].\end{aligned}$$

Let u be a solution of (1.1), and multiply (1.1) by $\operatorname{sgn}(u-k)$. Formal computations then reveals that

$$\begin{aligned}&\partial_t|u-k| + \operatorname{div}\,(q_f(u,k) - b^{\mu}_{r}\,|A(u)-A(k)|)\\ &\qquad\leq \mathcal{L}^{\mu}_{r}[|A(u)-A(k)|] + \operatorname{sgn}(u-k)\,\mathcal{L}^{\mu,r}[A(u)].\end{aligned}$$

The entropy formulation in Definition 2.1 below consists in asking that u satisfies this inequality for all entropy-flux pairs (i.e. for all $k \in \mathbb{R}$) and all $r > 0$. Roughly speaking one can give a sense to $\operatorname{sgn}(u-k)\,\mathcal{L}^{\mu,r}[A(u)]$ for bounded discontinuous u thanks to (1.6). But since μ may be singular at $z = 0$, see Remark 1.1 (3), the other terms have to be interpreted in the sense of distributions: Multiply by test functions ϕ and integrate by parts to move singular operators onto test functions. For the nonlocal terms this can be done by change of variables: First take $(z,x,t) \to (-z,x,t)$ to see (formally) that

$$\int_{Q_T} \phi \operatorname{div}\,(b^{\mu}_{r}\,|A(u)-A(k)|)\,\mathrm{d}x\mathrm{d}t = \int_{Q_T} D\phi \cdot b^{\mu^*}_{r}\,|A(u)-A(k)|\,\mathrm{d}x\mathrm{d}t,$$

where μ^* is the Lévy measure (i.e. it satisfies (1.6)) defined by

$$\mu^*(B) := \mu\,(-B) \quad \text{for all Borelian } B \subseteq \mathbb{R}^d \setminus \{0\}. \tag{2.8}$$

In view of (2.2), we can take $(z,x,t) \to (-z,x+z,t)$ to find that

$$\int_{Q_T} \phi\,\mathcal{L}^{\mu}_{r}[|A(u)-A(k)|]\,\mathrm{d}x\mathrm{d}t = \int_{Q_T} |A(u)-A(k)|\,\mathcal{L}^{\mu^*}_{r}[\phi]\,\mathrm{d}x\mathrm{d}t.$$

This leads to the following definition introduced in [19].

Definition 2.1. (Entropy solutions) *Assume* (1.3)–(1.6). *We say that a function $u \in L^\infty(Q_T) \cap C\left([0,T];L^1\right)$ is an entropy solution of* (1.1) *provided that for all $k \in \mathbb{R}$, all $r > 0$, and all nonnegative $\phi \in C_c^\infty(\mathbb{R}^{d+1})$,*

$$\begin{aligned}&\int_{Q_T} |u-k|\,\partial_t\phi + \Big(q_f(u,k) + b^{\mu^*}_{r}\,|A(u)-A(k)|\Big)\cdot D\phi\,\mathrm{d}x\mathrm{d}t\\ &\qquad + \int_{Q_T} |A(u)-A(k)|\,\mathcal{L}^{\mu^*}_{r}[\phi] + \operatorname{sgn}(u-k)\,\mathcal{L}^{\mu,r}[A(u)]\,\phi\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad\qquad - \int_{\mathbb{R}^d} |u(x,T)-k|\,\phi(x,T)\,\mathrm{d}x + \int_{\mathbb{R}^d} |u_0(x)-k|\,\phi(x,0)\,\mathrm{d}x \geq 0.\end{aligned} \tag{2.9}$$

Remark 2.1.

(1) Under assumptions (1.3)–(1.6), the entropy inequality (2.9) is well-defined independently of the a.e. representative of u. To see this note that μ^* obviously satisfies (1.6), and hence it easily follows that $\mathcal{L}^{\mu^*}_{r}[\phi] \in C_c^\infty(\mathbb{R}^{d+1})$. Since $\operatorname{sgn}(u-k)$, $q_f(u,k)$, and $A(u)$ belong to L^∞ by (2.6) and (1.3)–(1.4), it is then clear that all terms in (2.9) are well-defined except possibly the $\mathcal{L}^{\mu,r}$-term. Here it may look like we are integrating Lebesgue measurable functions w.r.t. a Radon measure μ. However, the integrand does have the right measurability by a classical approximation procedure, see Remark 5.1 in [19]. We therefore find that since $A(u)$ belongs to $C([0,T];L^1)$, so does also $\mathcal{L}^{\mu,r}[A(u)]$ and we are done.

(2) Another way to understand the measurablitiy issue in (1), is simply to consider that all the spaces L^1, BV, etc correspond to Borel measurable functions. This does not change the statement of our L^1-continuous dependence estimates in the next section.
(3) In the definition of entropy solutions, it is possible to consider functions u only defined for a.e. $t \in [0,T]$ by taking test functions with compact support in Q_T and adding an explicit initial condition, see e.g. [19].
(4) One can check that classical solutions are entropy solutions, thus justifying the formal computations leading to Definition 2.1. Moreover entropy solution are weak solutions and hence smooth entropy solutions are classical solutions. We refer the reader to [19] for the proofs.

Here is a well-posedness result from [19].

Theorem 2.2. (Well-posedness) *Assume* (1.3)–(1.6). *There exists a unique entropy solution u of* (1.1)*. This entropy solution belongs to* $L^\infty(Q_T) \cap C\left([0,T];L^1\right) \cap L^\infty\left(0,T;BV\right)$ *and*

$$\begin{cases} \|u\|_{L^\infty(Q_T)} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)}, \\ \|u\|_{C([0,T];L^1)} \leq \|u_0\|_{L^1(\mathbb{R}^d)}, \\ |u|_{L^\infty(0,T;BV)} \leq |u_0|_{BV(\mathbb{R}^d)}. \end{cases} \tag{2.10}$$

Moreover, if v is the entropy solution of (1.1) *with $v(0) = v_0$ for another initial data v_0 satisfying* (1.5)*, then*

$$\|u - v\|_{C([0,T];L^1)} \leq \|u_0 - v_0\|_{L^1(\mathbb{R}^d)}. \tag{2.11}$$

Remark 2.3. By the L^∞-estimate in (2.10), all the results of this paper also holds for locally Lipschitz-continuous nonlinearities (f, A). Simply replace the data (f, A) by $(f, A)\,\psi_M$, where $\psi_M \in C_c^\infty(\mathbb{R})$ is such that $\psi_M = 1$ in $[-M, M]$ for $M = \|u_0\|_{L^\infty(\mathbb{R}^d)}$.

3. Main results

Our first main result is a Kuznetsov type of lemma that measures the distance between the entropy solution u of (1.1) and an arbitrary function v.

Let $\epsilon, \delta > 0$ and $\phi^{\epsilon,\delta} \in C^\infty(Q_T^2)$ be the test function

$$\phi^{\epsilon,\delta}(x,t,y,s) := \theta_\delta(t-s)\,\bar{\theta}_\epsilon(x-y), \tag{3.1}$$

where $\theta_\delta(t) := \frac{1}{\delta}\,\tilde{\theta}_1\left(\frac{t}{\delta}\right)$ and $\bar{\theta}_\epsilon(x) := \frac{1}{\epsilon^d}\tilde{\theta}_d\left(\frac{x}{\epsilon}\right)$ are, respectively, time and space approximate units with kernel $\tilde{\theta}_n$ with $n = 1$ and $n = d$ satisfying

$$\tilde{\theta}_n \in C_c^\infty(\mathbb{R}^n), \quad \tilde{\theta}_n \geq 0, \quad \operatorname{supp}\tilde{\theta}_n \subseteq \{|x| < 1\}, \quad \text{and} \quad \int_{\mathbb{R}^n} \tilde{\theta}_n(x)\,\mathrm{d}x = 1. \tag{3.2}$$

We also let $\omega_u(\delta)$ be the modulus of continuity of $u \in C\left([0,T];L^1\right)$.

Lemma 3.1 (Kuznetsov type Lemma)**.** *Assume* (1.3)–(1.6)*. Let u be the entropy solution of* (1.1) *and* $v \in L^\infty(Q_T) \cap C\left([0,T];L^1\right)$ *with $v(0) = v_0$. Then for all $r > 0$,*

$\epsilon > 0$, *and* $0 < \delta < T$,

(3.3)

$$
\begin{aligned}
&\|u(T) - v(T)\|_{L^1(\mathbb{R}^d)} \\
&\leq \|u_0 - v_0\|_{L^1(\mathbb{R}^d)} + \epsilon\, C_{\tilde\theta}\, |u_0|_{BV(\mathbb{R}^d)} + 2\,\omega_u(\delta) \vee \omega_v(\delta) \\
&\quad - \iint_{Q_T^2} |v(x,t) - u(y,s)|\, \partial_t \phi^{\epsilon,\delta}(x,t,y,s)\, \mathrm{d}w \\
&\quad - \iint_{Q_T^2} \Big(q_f(v(x,t), u(y,s)) + b_r^{\mu^*} |A(v(x,t)) - A(u(y,s))| \Big) \cdot D_x \phi^{\epsilon,\delta}(x,t,y,s)\, \mathrm{d}w \\
&\quad + \iint_{Q_T^2} |A(v(x,t)) - A(u(y,s))|\, \mathcal{L}_r^{\mu^*}[\phi^{\epsilon,\delta}(x,t,\cdot,s)](y)\, \mathrm{d}w \\
&\quad - \iint_{Q_T^2} \operatorname{sgn}(v(x,t) - u(y,s))\, \mathcal{L}^{\mu,r}[A(u(\cdot,s))](y)\, \phi^{\epsilon,\delta}(x,t,y,s)\, \mathrm{d}w \\
&\quad + \iint_{\mathbb{R}^d \times Q_T} |v(x,T) - u(y,s)|\, \phi^{\epsilon,\delta}(x,T,y,s)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}s \\
&\quad - \iint_{\mathbb{R}^d \times Q_T} |v_0(x) - u(y,s)|\, \phi^{\epsilon,\delta}(x,0,y,s)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}s
\end{aligned}
$$

where $\mathrm{d}w := \mathrm{d}x\, \mathrm{d}t\, \mathrm{d}y\, \mathrm{d}s$, *and* $C_{\tilde\theta} := 2\int_{\mathbb{R}^d} |x| \tilde\theta_d(x)\, \mathrm{d}x$.

Remark 3.2.

(1) The error in time only depends on the moduli of continuity of u and v at $t = 0$ and $t = T$. Here we simply take the global-in-time moduli of continuity $\omega_u(\delta)$ and $\omega_v(\delta)$, since this is sufficient in our settings.
(2) When $A = 0$ or $\mu = 0$ this lemma reduces to the well-known Kuznetsov lemma [48] for multidimensional scalar conservation laws.
(3) Notice that the $\mathcal{L}_r^{\mu^*}$-term vanishes when $r \to 0$, see Lemma 4.4.
(4) Lemma 3.1 has many applications. In this paper and in [3] we focus on continuous dependence results and error estimates for the vanishing viscosity method. In a future paper, we will use the lemma to obtain error estimates for numerical approximations of (1.1).

In this paper we apply Lemma 3.1 to compare the entropy solution u of (1.1) with the entropy solution v of (1.13). This is our second main result, and we present it in the two theorems below. The first focuses on the dependence on the nonlinearities (with $\mu = \nu$) and the second one on the Lévy measure (with $A = B$).

Theorem 3.3. (Continuous dependence on the nonlinearities) *Let u and v be the entropy solutions of* (1.1) *and* (1.13) *respectively with data sets* (f, A, u_0, μ) *and* $(g, B, v_0, \nu = \mu)$ *satisfying* (1.3)–(1.6). *Then for all* $T, r > 0$,

$$
\begin{aligned}
\|u - v\|_{C([0,T];L^1)} &\leq \|u_0 - v_0\|_{L^1(\mathbb{R}^d)} + |u_0|_{BV(\mathbb{R}^d)}\, T\, \|f' - g'\|_{L^\infty(\mathbb{R},\mathbb{R}^d)} \\
&\quad + |u_0|_{BV(\mathbb{R}^d)} \sqrt{c_d\, T \int_{0<|z|\leq r} |z|^2\, \mathrm{d}\mu(z)\, \|A' - B'\|_{L^\infty(\mathbb{R})}} \\
&\quad + |u_0|_{BV(\mathbb{R}^d)}\, T \left| \int_{r\wedge 1<|z|\leq r\vee 1} z\, \mathrm{d}\mu(z) \right| \|A' - B'\|_{L^\infty(\mathbb{R})} \\
&\quad + T \int_{|z|>r} \|u_0(\cdot + z) - u_0\|_{L^1(\mathbb{R}^d)}\, \mathrm{d}\mu(z)\, \|A' - B'\|_{L^\infty(\mathbb{R})},
\end{aligned}
\tag{3.4}
$$

where $c_d = \frac{4d^2}{d+1}$.

Theorem 3.4. (Continuous dependence on the Lévy measure) *Let u and v be the entropy solutions of* (1.1) *and* (1.13) *respectively with data sets* (f, A, u_0, μ) *and* $(g, B = A, v_0, \nu)$ *satisfying* (1.3)–(1.6). *Then for all* $T, r > 0$,

$$\begin{aligned}
\|u - v\|_{C([0,T];L^1)} \leq\ & \|u_0 - v_0\|_{L^1(\mathbb{R}^d)} + |u_0|_{BV(\mathbb{R}^d)}\, T\, \|f' - g'\|_{L^\infty(\mathbb{R},\mathbb{R}^d)} \\
& + |u_0|_{BV(\mathbb{R}^d)} \sqrt{c_d\, T\, \|A'\|_{L^\infty(\mathbb{R})} \int_{0<|z|\leq r} |z|^2 \, \mathrm{d}|\mu - \nu|(z)} \\
& + |u_0|_{BV(\mathbb{R}^d)}\, T\, \|A'\|_{L^\infty(\mathbb{R})} \left| \int_{r\wedge 1<|z|\leq r\vee 1} z \, \mathrm{d}(\mu - \nu)(z) \right| \\
& + T\, \|A'\|_{L^\infty(\mathbb{R})} \int_{|z|>r} \|u_0(\cdot + z) - u_0\|_{L^1(\mathbb{R}^d)} \, \mathrm{d}|\mu - \nu|(z),
\end{aligned} \tag{3.5}$$

where $c_d = \frac{4d^2}{d+1}$.

Remark 3.5. In the error estimates of Theorems 3.3 and 3.4, there are 3 terms accounting for the dependence on the fractional diffusion term in (1.1): One term accounts for the behaviour near the singularity of μ at $z = 0$ (the integral over $0 < |z| \leq r$), an other term accounts for the behaviour near inifinity (the integral over $|z| \geq r$), and the last term (the integral over $r \wedge 1 < |z| \leq r \vee 1$) is a drift term that is only present for non-symmetric measures μ.

Remark 3.6. Since the initial data is $BV \cap L^1$, an application of Fubini's theorem shows that for any $\hat{r} > r > 0$,

$$\begin{aligned}
& \int_{|z|>r} \|u_0(\cdot + z) - u_0\|_{L^1(\mathbb{R}^d)} \, \mathrm{d}\mu(z) \\
& \leq |u_0|_{BV(\mathbb{R}^d)} \int_{r<|z|\leq \hat{r}} |z| \, \mathrm{d}\mu(z) + 2\|u_0\|_{L^1(\mathbb{R}^d)} \int_{|z|>\hat{r}} \mathrm{d}\mu(z).
\end{aligned}$$

From Theorems 3.3 and 3.4 we can easily find a general continuous dependence estimate when both A and μ are different from B and ν, respectively. E.g. we can take an intermediate solution w of $w_t + \operatorname{div} f(w) = \mathcal{L}^\mu[B(w)]$ and $w(0) = u_0$, and use the triangle inequality. Using this idea we can show that the following estimates always have to hold:

Corollary 3.7. *Let u and v be the entropy solutions of* (1.1) *and* (1.13) *respectively with data sets* (f, A, u_0, μ) *and* (g, B, v_0, ν) *satisfying* (1.3)–(1.6). *Then*

$$\begin{aligned}
\|u - v\|_{C([0,T];L^1)} \leq\ & \|u_0 - v_0\|_{L^1(\mathbb{R}^d)} + |u_0|_{BV(\mathbb{R}^d)}\, T\, \|f' - g'\|_{L^\infty(\mathbb{R},\mathbb{R}^d)} \\
& + C\,(T^{\frac{1}{2}} \vee T) \left(\sqrt{\|A' - B'\|_{L^\infty(\mathbb{R})}} + \sqrt{\int_{\mathbb{R}^d\setminus\{0\}} |z|^2 \wedge 1 \, \mathrm{d}|\mu - \nu|(z)} \right)
\end{aligned} \tag{3.6}$$

where C only depends on d and the data. Moreover, if in addition

$$\int_{\mathbb{R}^d\setminus\{0\}} |z| \wedge 1 \, \mathrm{d}\mu(z) + \int_{\mathbb{R}^d\setminus\{0\}} |z| \wedge 1 \, \mathrm{d}\nu(z) < +\infty,$$

then we have the better estimate

$$\begin{aligned}
\|u - v\|_{C([0,T];L^1)} \leq\ & \|u_0 - v_0\|_{L^1(\mathbb{R}^d)} + |u_0|_{BV(\mathbb{R}^d)}\, T\, \|f' - g'\|_{L^\infty(\mathbb{R},\mathbb{R}^d)} \\
& + CT \left(\|A' - B'\|_{L^\infty(\mathbb{R})} + \int_{\mathbb{R}^d\setminus\{0\}} |z| \wedge 1 \, \mathrm{d}|\mu - \nu|(z) \right),
\end{aligned} \tag{3.7}$$

where C only depends on the data.

Outline of proof. To prove (3.6), we use Theorems 3.3 and 3.4 with $r = 1$ and the triangle inequality. We also use estimates like $|a - b| \leq \sqrt{|a| + |b|}\sqrt{|a - b|}$, $|\mu - \nu| \leq |\mu| + |\nu|$ etc. To prove (3.7), we also use Remark 3.6 and set $r = 0$ and $\hat{r} = 1$. □

Remark 3.8.

(1) All these estimates hold for arbitrary Lévy measures μ, ν and even for strongly degenerate diffusions where A, B may vanish on large sets. They are consistent (at least for the $|\mu-\nu|$ term) with general results for nonlocal Hamilton-Jacobi-Bellman equations in [36]. When μ, ν have the special form (1.7) (with possibly different α's), then it is possible to use the extra symmetry and homogeneity properties to obtain better estimates, see [3].

(2) The optimal choice of the $r, \hat{r}$ in Remark 3.6 depends on the behavior of the Lévy measures at zero and infinity, see the discussion above and at the end of this section for more details.

Let us now consider the nonlocal vanishing viscosity problem

$$\begin{cases} \partial_t u^\epsilon + \operatorname{div} f(u^\epsilon) = \epsilon\, \mathcal{L}^\mu[A(u^\epsilon)], \\ u^\epsilon(0) = u_0, \end{cases} \tag{3.8}$$

i.e. problem (1.8) with a perturbation term $\epsilon\, \mathcal{L}^\mu[A(u^\epsilon)]$. When $\epsilon > 0$ tend to zero, u^ϵ is expected to converge toward the solution u of (1.8). As an immediate application of Theorem 3.3 or 3.4, we have the following result:

Theorem 3.9 (Vanishing viscosity)**.** *Assume* (1.3)–(1.6)*. Let u and u^ϵ be the entropy solutions of* (1.8) *and* (3.8) *respectively. Then*

$$\begin{aligned} \|u - u^\epsilon\|_{C([0,T];L^1)} \leq C \min_{\hat{r}>r>0} \Bigg\{ & d^{\frac{1}{2}} T^{\frac{1}{2}} \epsilon^{\frac{1}{2}} \sqrt{\int_{0<|z|\leq r} |z|^2 \,\mathrm{d}\mu(z)} \\ & + T\epsilon \left[\int_{r<|z|\leq \hat{r}} |z| \,\mathrm{d}\mu(z) + \left| \int_{r\wedge 1<|z|\leq r\vee 1} z \,\mathrm{d}\mu(z) \right| + \int_{|z|>\hat{r}} \mathrm{d}\mu(z) \right] \Bigg\}, \end{aligned} \tag{3.9}$$

where C only depends on $\|u_0\|_{L^1(\mathbb{R}^d)\cap BV(\mathbb{R}^d)}$ and $\|A'\|_{L^\infty(\mathbb{R})}$.

Outline of proof. Note that u can be seen as the entropy solution of (1.1) with $A = 0$ and μ as Lévy measure. Hence we can estimate $\|u - u^\epsilon\|_{C([0,T];L^1)}$ from Theorem 3.3. The error coming from the difference of the derivatives of the nonlinearities is equaled to $\epsilon \|A'\|_{L^\infty(\mathbb{R})}$. Inequality (3.9) then follows from (3.4) and Remark 3.6. □

Corollary 3.10. *Assume* (1.3)–(1.6)*. Let u and u^ϵ be the entropy solutions of* (1.8) *and* (3.8) *respectively. Then*

$$\|u - u^\epsilon\|_{C([0,T];L^1)} \leq C\, (T^{\frac{1}{2}} \vee T)\, \epsilon^{\frac{1}{2}},$$

where C only depends on d and the data. Moreover, if in addition

$$\int_{\mathbb{R}^d\setminus\{0\}} |z| \wedge 1 \,\mathrm{d}\mu(z) < +\infty,$$

then we have the better estimate

$$\|u - u^\epsilon\|_{C([0,T];L^1)} \leq CT\epsilon,$$

where C depends on the data.

This corollary follows immediately from Theorem 3.9 or Corollary 3.7.

Remark 3.11.

(1) Our estimates are just as good or better than the standard $\mathcal{O}(\epsilon^{\frac{1}{2}})$ estimate for the classical vanishing viscosity method ((1.10) with $A(u) = \epsilon\, u$).
(2) Our estimates hold for arbitrary Lévy measures μ and even for strongly degenerate diffusions where A may vanish on a large set! This is consistent with general results for nonlocal Hamilton-Jacobi-Bellman equations [36].
(3) If the solutions u are smoother, it is possible to obtain better estimates. E.g. it is straight-forward to prove that the error estimate of Theorem 3.9 becomes $O(\epsilon)$ if u also belongs to $W^{2,1}$. Also if μ is as in (1.7), the additional symmetry and homogeneity can be used to obtain better estimates which can be proved to be optimal. See Example 3.3 below.
(4) Corollary 3.10 contains less information than Theorem 3.9 and is not strong enough to get optimal results in all cases, e.g. in Example 3.3 with $\alpha \geq 1$.
(5) The error estimates above trivially also holds for the more general vanishing viscosity equation

$$\begin{cases} \partial_t u^\epsilon + \operatorname{div} f(u^\epsilon) = \mathcal{L}^\nu[B(u^\epsilon)] + \epsilon\, \mathcal{L}^\mu[A(u^\epsilon)], \\ u^\epsilon(0) = u_0. \end{cases}$$

Further discussion. We now make a more precise comparison of the results above with known estimates from the literature. We begin with continuous dependence estimates and finish with convergence rates for vanishing viscosity approximations.

Let u and v denote the entropy solutions of (1.1) and (1.13), respectively. To simplify, we take the same data sets $(f, A, u_0) = (g, B, v_0)$ and we only allow the Lévy measures μ and ν to be different. We also let C denote a constant only depending on T, d and the data.

Example 3.1. Let us consider Equation (1.9), i.e. $A(u) = u$. Let us also consider the class of Lévy operators satisfying

$$\begin{cases} \int_{\mathbb{R}^d \setminus \{0\}} |z|^2 \wedge |z| \,\mathrm{d}\mu(z) < +\infty, \\ \mu = \mu^*. \end{cases}$$

For such kind of equations, the following continuous dependence estimate on the Lévy measure has been established in [43]:

$$\|u - v\|_{C([0,T];L^1)} \leq C \int_{0<|z|\leq 1} |z|^2 \,\mathrm{d}|\mu - \nu|(z) + C \int_{|z|>1} |z| \,\mathrm{d}|\mu - \nu|(z).$$

This estimate follows from Theorem 3.4 and Remark 3.6 by taking $r = 1$ and $\hat{r} = +\infty$ in (3.5).

Example 3.2. Consider the following one-dimensional Hamilton-Jacobi-Bellman equation

$$U_t + f(U_x) = \mathcal{L}^\mu[U]$$

with initial data $U_0(x) := \int_{-\infty}^x u_0(y)\,\mathrm{d}y$. This particular equation is related to the nonlocal conservation law (1.8), its solution $U(x,t) = \int_{-\infty}^x u(y,t)\,\mathrm{d}y$ where u solves (1.8), see [19]. It is also an example of an integro-PDE for which the general theory of [36] applies, and this theory allows us to establish the following continuous dependence estimate on the Lévy measure:

$$\sup_{\mathbb{R}\times[0,T]} |U - V| \leq C \sqrt{\int_{\mathbb{R}\setminus\{0\}} |z|^2 \wedge 1 \,\mathrm{d}|\mu - \nu|(z)},$$

where $V(x,t) := \int_{-\infty}^x v(y,t)\,\mathrm{d}y$. (This result is a version of Theorem 4.1 (in [36]) which follows from Theorem 3.1 by setting $p_0, \ldots, p_4, p_s = 0$ and $\rho = |z| \wedge 1$ in

(A0)). Since

$$\sup_{\mathbb{R}\times[0,T]} |U-V| \leq \|u-v\|_{C([0,T];L^1)},$$

this estimate also follows from (3.6) in Corollary 3.7 when $(A,f,u_0)=(B,g,v_0)$.

Let us now compare Theorem 3.9 with known convergence rates. We keep the same notation for u, u^ϵ and $\mathcal{O}(\cdot)$ as in Theorem 3.9.

Example 3.3. Let us consider the case where $A(u^\epsilon)=u^\epsilon$ and $\mathcal{L}^\mu=-(-\triangle)^{\frac{\alpha}{2}}$, $\alpha\in(0,2)$. Then the following optimal rates have been derived in [1, 29]:

$$\|u-u^\epsilon\|_{C([0,T];L^1)} = \begin{cases} \mathcal{O}\left(\epsilon^{\frac{1}{\alpha}}\right) & \text{if } \alpha>1, \\ \mathcal{O}\left(\epsilon\,|\ln\epsilon|\right) & \text{if } \alpha=1, \\ \mathcal{O}\left(\epsilon\right) & \text{if } \alpha<1. \end{cases} \tag{3.10}$$

Let us explain how these results can be deduced from (3.9). First we use (1.7) to explicitly compute the integrals in (3.9) and obtain

$$\|u-u^\epsilon\|_{C([0,T];L^1)} = \mathcal{O}\left(\min_{\hat{r}>r>0}\left\{\sqrt{\epsilon\,\frac{r^{2-\alpha}}{2-\alpha}}+\epsilon\int_r^{\hat{r}}\frac{\mathrm{d}\tau}{\tau^\alpha}+\epsilon\,\hat{r}^{-\alpha}\right\}\right).$$

We then deduce (3.10) by taking $r=\epsilon^{\frac{1}{\alpha}}$ and $\hat{r}=+\infty$ if $\alpha>1$, $r=\epsilon$ and $\hat{r}=1$ if $\alpha=1$, and $r=0$ and $\hat{r}=1$ if $\alpha<1$.

Example 3.4. Let us consider the class of Lévy operators where $\mathrm{d}\mu(z)=g(z)\,\mathrm{d}z$ for $0\leq g\in L^1(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} g(z)\,\mathrm{d}z=1$. This corresponds to problem (1.12) since we may write

$$\mathcal{L}^\mu[u^\epsilon]=g*u^\epsilon-u^\epsilon \quad (* \text{ is the convolution in space}).$$

The following optimal rate of convergence has been derived in [49, 57]:

$$\|u-u^\epsilon\|_{C([0,T];L^1)}=\mathcal{O}(\epsilon).$$

This result also follows from (3.9) by taking $\hat{r}=r=0$.

4. Auxiliary results concerning $\mathcal{L}^\mu$

Before proving our main results in the next section, we state an auxilliary lemma.

Lemma 4.1. *Assume* (1.6) *and* $r>0$. *Then for all* $\phi\in C_c^\infty(\mathbb{R}^d)$,

$$\|\mathcal{L}_r^\mu[\phi]\|_{L^1(\mathbb{R}^d)} \leq \int_{0<|z|\leq r}|z|^2\,\mathrm{d}\mu(z)\;\|\phi\|_{W^{2,1}(\mathbb{R}^d)}.$$

The proof easily follows from a Taylor expansion and Fubini's theorem. In the next result, we establish a Kato type inequality for $\mathcal{L}^{\mu,r}[A(u)]$.

Lemma 4.2. *Assume* (1.4) *and* (1.6). *Then for all* $u\in L^1(\mathbb{R}^d)$, $k\in\mathbb{R}$, $r>0$, *and all* $0\leq\phi\in C_c^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d}\operatorname{sgn}(u-k)\,\mathcal{L}^{\mu,r}[A(u)]\,\phi\,\mathrm{d}x \leq \int_{\mathbb{R}^d}|A(u)-A(k)|\,\mathcal{L}^{\mu^*,r}[\phi]\,\mathrm{d}x.$$

Proof. Note first that $A(u)$ is L^1 by (1.4), and hence $\mathcal{L}^{\mu,r}[A(u)]$ is well-defined in L^1 by Remark 3.6 with $\hat{r}=r$. Easy computations then reveal that

$$\begin{aligned}
&\int_{\mathbb{R}^d} \operatorname{sgn}(u-k)\,\mathcal{L}^{\mu,r}[A(u)]\,\phi\,\mathrm{d}x,\\
&= \int_{\mathbb{R}^d}\int_{|z|>r} \operatorname{sgn}(u(x)-k)\,\Big(A(u(x+z))-A(u(x))\Big)\,\phi(x)\,\mathrm{d}\mu(z)\,\mathrm{d}x,\\
&= \int_{\mathbb{R}^d}\int_{|z|>r} \operatorname{sgn}(u(x)-k)\\
&\qquad\qquad \Big\{(A(u(x+z))-A(k))-(A(u(x))-A(k))\Big\}\,\phi(x)\,\mathrm{d}\mu(z)\,\mathrm{d}x,\\
&\le \int_{\mathbb{R}^d}\int_{|z|>r} \Big(|A(u(x+z))-A(k)|-|A(u(x))-A(k)|\Big)\,\phi(x)\,\mathrm{d}\mu(z)\,\mathrm{d}x \quad \text{by (2.7)},\\
&= \underbrace{\int_{\mathbb{R}^d}\int_{|z|>r} |A(u(x+z))-A(k)|\,\phi(x)\,\mathrm{d}\mu(z)\,\mathrm{d}x}_{=:I}\\
&\qquad\qquad - \underbrace{\int_{\mathbb{R}^d}\int_{|z|>r} |A(u(x))-A(k)|\,\phi(x)\,\mathrm{d}\mu(z)\,\mathrm{d}x}_{=:J}.
\end{aligned}$$

Let us notice that all the integrands above are $\mathrm{d}\mu(z)\,\mathrm{d}x$–integrable, thanks to (1.6) and Fubini's theorem[1].

By the respective changes of variable $(z,x)\to(-z,x+z)$ and $(z,x)\to(-z,x)$, we find that

$$\begin{aligned}
I &= \int_{\mathbb{R}^d}\int_{|z|>r} \phi(x+z)\,|A(u(x))-A(k)|\,\mathrm{d}\mu^*(z)\,\mathrm{d}x,\\
J &= \int_{\mathbb{R}^d}\int_{|z|>r} \phi(x)\,|A(u(x))-A(k)|\,\mathrm{d}\mu^*(z)\,\mathrm{d}x.
\end{aligned}$$

Here the measure μ^* in (2.8) appears because of the relabelling of z. This measure has the same properties as μ. Hence we can conclude that

$$\int_{\mathbb{R}^d} \operatorname{sgn}(u-k)\,\mathcal{L}^{\mu,r}[A(u)]\,\phi\,\mathrm{d}x \le I-J = \int_{\mathbb{R}^d} |A(u)-A(k)|\,\mathcal{L}^{\mu^*,r}[\phi]\,\mathrm{d}x,$$

and the proposition follows. □

The next lemma is a consequence of the Kato inequality, and it plays a key role in the doubling of variables arguments throughout this paper and in the uniqueness proof of [1, 19].

Lemma 4.3. *Assume* (1.4) *and* (1.6), *and let* $u,v\in L^\infty(Q_T)\cap C([0,T];L^1)$, $0\le\psi\in L^1(\mathbb{R}^d\times(0,T)^2)$, *and* $r>0$. *Then*

$$\begin{aligned}
&\iint_{Q_T^2} \operatorname{sgn}(u(y,s)-v(x,t))\\
&\qquad \cdot\Big(\mathcal{L}^{\mu,r}[A(u(\cdot,s))](y)-\mathcal{L}^{\mu,r}[A(v(\cdot,t))](x)\Big)\,\psi(x-y,t,s)\,\mathrm{d}w \le 0
\end{aligned}$$

(where $\mathrm{d}w=\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s$*).*

---

[1] Recall that the measurability is immediate if the reader only consider Borel measurable representants of u as suggested in the first item of Remark 2.1.

Proof. Note that

$$\begin{aligned}
&\operatorname{sgn}(u(y,s)-v(x,t))\Big(A(u(y+z,s))-A(u(y,s))\Big)\\
&\qquad-\operatorname{sgn}(u(y,s)-v(x,t))\Big(A(v(x+z,t))-A(v(x,t))\Big)\\
&=\operatorname{sgn}(u(y,s)-v(x,t))\\
&\qquad\cdot\Big\{\Big(A(u(y+z,s))-A(v(x+z,t))\Big)-\Big(A(u(y,s))-A(v(x,t))\Big)\Big\}\\
&\leq |A(u(y+z,s))-A(v(x+z,t))|-|A(u(y,s))-A(v(x,t))|
\end{aligned}$$

where these functions are both defined. By an integration w.r.t. $\mathbf{1}_{|z|>r}\,\mathrm{d}\mu(z)$, we find that for all $(t,s)\in(0,T)^2$ and a.e. $(x,y)\in\mathbb{R}^{2d}$,

$$\begin{aligned}
&\operatorname{sgn}(u(y,s)-v(x,t))\Big(\mathcal{L}^{\mu,r}[A(u(\cdot,s))](y)-\mathcal{L}^{\mu,r}[A(v(\cdot,t))](x)\Big)\\
&\leq\int_{|z|>r}|A(u(y+z,s))-A(v(x+z,t))|-|A(u(y,s))-A(v(x,t))|\,\mathrm{d}\mu(z).
\end{aligned}$$

After another integration, this time w.r.t. $\psi(x-y,t,s)\,\mathrm{d}w$, we then get that

$$\begin{aligned}
&\iint_{Q_T^2}\operatorname{sgn}(u(y,s)-v(x,t))\Big(\mathcal{L}^{\mu,r}[A(u(\cdot,s))](y)-\mathcal{L}^{\mu,r}[A(v(\cdot,t))](x)\Big)\,\psi\,\mathrm{d}w\\
&\leq\iint_{Q_T^2}\int_{|z|>r}|A(u(y+z,s))-A(v(x+z,t))|\;\psi(x-y,t,s)\,d\mu(z)\,\mathrm{d}w\\
&\quad-\iint_{Q_T^2}\int_{|z|>r}|A(u(y,s))-A(v(x,t))|\;\psi(x-y,t,s)\,d\mu(z)\,\mathrm{d}w,\\
&=: I+J.
\end{aligned}$$

Note that I and J are finite since $\|A(u)\|_{C([0,T];L^1)}\leq\|A'\|_{L^\infty}\|u\|_{C([0,T];L^1)}$ (A is Lipschitz-continuous and 0 at 0) and by Fubini (note the convolution integrals in x and y),

$$I,J\leq\Big(\|A(u)\|_{C([0,T];L^1)}+\|A(v)\|_{C([0,T];L^1)}\Big)\|\psi\|_{L^1(\mathbb{R}^d\times(0,T)^2)}\int_{|z|>r}\mathrm{d}\mu(x).$$

We then change variables $(z,x,t,y,s)\to(z,x-z,t,y-z,s)$ in I,

$$I=\iint_{Q_T}\int_{|z|>r}|A(u(y,s))-A(v(x,t))|\;\psi(x-z-(y-z),t,s)\,\mathrm{d}\mu(z)\,\mathrm{d}w,$$

to find that $I+J=0$ and the proof is complete. □

Lemma 4.4. *Under the assumptions of Lemma 3.1,*

$$I=\iint_{Q_T^2}|A(v(x,t))-A(u(y,s))|\,\mathcal{L}_r^{\mu^*}[\phi^{\epsilon,\delta}(x,t,\cdot,s)](y)\,\mathrm{d}w\leq C_\epsilon\int_{0<|z|\leq r}|z|^2\,\mathrm{d}\mu(z),$$

where $C_\epsilon>0$ does not depend on $r>0$.

Proof. Easy computations show that

$$\begin{aligned}
&\mathcal{L}_r^{\mu^*}[\phi^{\epsilon,\delta}(x,t,\cdot,s)](y)\\
&=\theta_\delta(t-s)\int_{0<|z|\leq r}\bar\theta_\epsilon(x-y-z)-\bar\theta_\epsilon(x-y)+z\cdot D\bar\theta_\epsilon(x-y)\,\mathbf{1}_{|z|\leq1}\,\mathrm{d}\mu^*(z)\\
&=\theta_\delta(t-s)\int_{0<|z|\leq r}\bar\theta_\epsilon(x-y+z)-\bar\theta_\epsilon(x-y)-z\cdot D\bar\theta_\epsilon(x-y)\,\mathbf{1}_{|z|\leq1}\,\mathrm{d}\mu(z)\\
&=\theta_\delta(t-s)\,\mathcal{L}_r^{\mu}[\bar\theta_\epsilon](x-y),
\end{aligned}$$

and by Fubini (there are again convolution integrals in I!),

$$
\begin{aligned}
I &\leq \iint_{Q_T^2} |A(u(y,s)) - A(v(x,t))|\, \theta_\delta(t-s) \left|\mathcal{L}_r^\mu[\bar{\theta}_\epsilon](x-y)\right| \mathrm{d}w \\
&\leq \Big(\|A(u)\|_{L^1(Q_T)} + \|A(v)\|_{L^1(Q_T)}\Big) \|\theta_\delta\, \mathcal{L}_r^\mu[\bar{\theta}_\epsilon]\|_{L^1(\mathbb{R}^{d+1})} \\
&\leq T\|A'\|_{L^\infty} \Big(\|u\|_{C([0,T];L^1)} + \|v\|_{C([0,T];L^1)}\Big) \|\theta_\delta\, \mathcal{L}_r^\mu[\bar{\theta}_\epsilon]\|_{L^1(\mathbb{R}^{d+1})}.
\end{aligned}
$$

By classical properties of approximate units and Lemma 4.1,

$$
\begin{aligned}
\|\theta_\delta\, \mathcal{L}_r^\mu[\bar{\theta}_\epsilon]\|_{L^1(\mathbb{R}^{d+1})} &= \underbrace{\|\theta_\delta\|_{L^1(\mathbb{R})}}_{=1} \|\mathcal{L}_r^\mu[\bar{\theta}_\epsilon]\|_{L^1(\mathbb{R}^d)} \\
&\leq \frac{1}{2}\|\bar{\theta}_\epsilon\|_{W^{2,1}(\mathbb{R}^d)} \int_{0<|z|\leq r} |z|^2\, \mathrm{d}\mu(z),
\end{aligned}
$$

and the proof is complete since $\bar{\theta}_\epsilon \in C_c^\infty(\mathbb{R}^d)$ in (3.1) does not depend on $r > 0$. □

5. Proofs of the main results

The proofs of this section use the so-called doubling of variables technique of Kruzhkov [47] (see also [1, 19] for nonlocal equations) along with ideas from [48]. It consists in considering u as a function of the new variables (y,s) and using the approximate units $\phi^{\epsilon,\delta}$ in (3.1) as test functions. For brevity, we do not specify anymore the variables of $u = u(y,s)$, $v = v(x,t)$ and $\phi^{\epsilon,\delta} = \phi^{\epsilon,\delta}(x,t,y,s)$ when the context is clear; recall also that $\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s$ is denoted by $\mathrm{d}w$.

5.1. **Proof of Lemma 3.1.**

Let $(x,t) \in Q_T$ be fixed and $u = u(y,s)$, $k = v(x,t)$, and $\phi(y,s) := \phi^{\epsilon,\delta}(x,t,y,s)$. The entropy inequality for u (see (2.9)) then takes the form

$$
\begin{aligned}
&\int_{Q_T} |u-v|\, \partial_s \phi^{\epsilon,\delta} + \Big(q_f(u,v) + |A(u)-A(v)|\, b_r^{\mu^*}\Big) \cdot D_y \phi^{\epsilon,\delta}\, \mathrm{d}y\, \mathrm{d}s \\
&+ \int_{Q_T} |A(u)-A(v)|\, \mathcal{L}_r^{\mu^*}[\phi^{\epsilon,\delta}(x,t,\cdot,s)](y)\, \mathrm{d}y\, \mathrm{d}s \\
&+ \int_{Q_T} \operatorname{sgn}(u-v)\, \mathcal{L}^{\mu,r}[A(u(\cdot,s))](y)\, \phi^{\epsilon,\delta}\, \mathrm{d}y\, \mathrm{d}s \\
&- \int_{\mathbb{R}^d} |u(y,T) - v(x,t)|\, \phi^{\epsilon,\delta}(x,t,y,T)\, \mathrm{d}y \\
&+ \int_{\mathbb{R}^d} |u_0(y) - v(x,t)|\, \phi^{\epsilon,\delta}(x,t,y,0)\, \mathrm{d}y \geq 0.
\end{aligned}
$$

We integrate this inequality w.r.t. $(x,t) \in Q_T$, noting that q_f in (2.5) is symmetric, and that $\partial_s \phi^{\epsilon,\delta} = -\partial_t \phi^{\epsilon,\delta}$ and $D_y \phi^{\epsilon,\delta} = -D_x \phi^{\epsilon,\delta}$ by (3.1). Consequently we find that

$$
\begin{aligned}
&I_1 + \cdots + I_5 \\
&:= -\iint_{Q_T^2} |u-v|\,\partial_t \phi^{\epsilon,\delta} + \Big(q_f(v,u) + |A(u)-A(v)|\,b_r^{\mu^*}\Big)\cdot D_x\phi^{\epsilon,\delta}\,\mathrm{d}w \\
&\quad + \iint_{Q_T^2} |A(u)-A(v)|\,\mathcal{L}_r^{\mu^*}[\phi^{\epsilon,\delta}(x,t,\cdot,s)](y)\,\mathrm{d}w \\
&\quad + \iint_{Q_T^2} \operatorname{sgn}(u-v)\,\mathcal{L}^{\mu,r}[A(u(\cdot,s))](y)\,\phi^{\epsilon,\delta}\,\mathrm{d}w \\
&\quad - \iint_{Q_T\times\mathbb{R}^d} |u(y,T)-v(x,t)|\,\phi^{\epsilon,\delta}(x,t,y,T)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y \\
&\quad + \iint_{Q_T\times\mathbb{R}^d} |u_0(y)-v(x,t)|\,\phi^{\epsilon,\delta}(x,t,y,0)\,\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y \geq 0.
\end{aligned}
\tag{5.1}
$$

Note that the terms in the inequality above are well-defined since they are all essentially of the form of convolution integrals of L^1-functions. See Lemma 4.1, Remark 3.6, and the discussions in the proofs of Lemmas 4.3 and 4.4 for more details.

A classical computation from [48] reveals that

$$
\begin{aligned}
&I_4 + I_5 - \iint_{\mathbb{R}^d\times Q_T} |u(y,s)-v(x,T)|\,\phi^{\epsilon,\delta}(x,T,y,s)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \\
&\quad + \iint_{\mathbb{R}^d\times Q_T} |u(y,s)-v_0(x)|\,\phi^{\epsilon,\delta}(x,0,y,s)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \\
&\leq -\|u(T)-v(T)\|_{L^1(\mathbb{R}^d)} + \|u_0-v_0\|_{L^1(\mathbb{R}^d)} \\
&\quad + \epsilon\, C_{\tilde\theta}\,|u_0|_{BV(\mathbb{R}^d)} + 2\omega_u(\delta)\vee\omega_v(\delta),
\end{aligned}
\tag{5.2}
$$

where $C_{\tilde\theta}$ is as in Lemma 3.1. Lemma 3.1 now follows from (5.1) and the above estimates on I_4 and I_5.

5.2. **Proof of Theorem 3.3.** The proof uses the Kuznetsov lemma, and morally speaking it amounts to doubling the variables and subtracting the $u(x,t)$ and $v(y,s)$ equations, multiplying by $\operatorname{sgn}(u-v)$ and a test function ϕ and integrating, and then applying both new and classical tricks to arrive at an L^1-estimate of $u-v$. We expect to see terms involving

$$\operatorname{sgn}(u-v)\Big(\mathcal{L}^{\mu,r}[A(u)] - \mathcal{L}^{\mu,r}[B(v)]\Big),$$

and naively we can write this as ($\mathcal{L}^{\mu,r}$ is linear!)

$$\operatorname{sgn}(u-v)\mathcal{L}^{\mu,r}[(A-B)(u)] + \operatorname{sgn}(u-v)\mathcal{L}^{\mu,r}[B(u)-B(v)].$$

These terms are estimated by Kato type inequalities (see Lemmas 4.2 and 4.3), the first term should give the dependence on $A-B$ while the second term is a non-positive term that also appears in the uniqueness proof. *The problem with this approach is that we can not apply Kato for the first term because $A-B$ then have to be monotone!*

There are different ways to overcome this monotonicity problem, and we have chosen to adapt ideas from [36] – a paper on continuous dependence estimates for fully non-linear HJB-type of equations via viscosity solution techniques. We consider the region where $A' \geq B'$ and its complementary. Let $E_\pm$ be sets satisfying:

$$
\begin{cases}
E_\pm \subseteq \mathbb{R} \text{ are Borel sets;} \\
\cup_\pm E^\pm = \mathbb{R} \text{ and } \cap_\pm E_\pm = \emptyset; \\
\mathbb{R}\setminus \operatorname{supp}(A'-B')^\mp \subseteq E_\pm.
\end{cases}
\tag{5.3}
$$

For all $u \in \mathbb{R}$, we define

$$
\begin{aligned}
A_\pm(u) &:= \int_0^u A'(\tau)\,\mathbf{1}_{E_\pm}(\tau)\,\mathrm{d}\tau,\\
B_\pm(u) &:= \int_0^u B'(\tau)\,\mathbf{1}_{E_\pm}(\tau)\,\mathrm{d}\tau,\\
C_\pm(u) &:= \pm(A_\pm(u) - B_\pm(u)).
\end{aligned}
\tag{5.4}
$$

These functions satisfy the following properties:

Lemma 5.1. *Under the assumptions of Theorem 3.3,*

(i) $A = A_+ + A_-$ *and* $B = B_+ + B_-$;
(ii) $A_\pm, B_\pm, C_\pm$ *satisfy* (1.4)*, in particular, they are monotone;*
(iii) $\sum_\pm |C_\pm(u)|_{L^1(0,T;BV)} \leq \|A' - B'\|_{L^\infty(\mathbb{R})}\,|u|_{L^1(0,T;BV)}$;
(iv) *for all* $z \in \mathbb{R}^d \setminus \{0\}$,

$$\sum_\pm \|C_\pm(u(\cdot + z, \cdot)) - C_\pm(u)\|_{L^1(Q_T)} \leq \|A' - B'\|_{L^\infty(\mathbb{R})}\,\|u(\cdot + z, \cdot) - u\|_{L^1(Q_T)}.$$

The proofs of (i) and (ii) are immediate, whereas (iii) and (iv) follow from standart arguments for Lipschitz-continuous and BV-functions (see e.g. [13, 33, 55]); the details are left to the reader.

In the proof below, $A^\pm - B^\pm$ will be the monotone functions replacing the non-monotone function $A - B$ of the formal argument above.

Proof of Theorem 3.3. Let us divide the proof into several steps.

1. We argue as in the beginning of the proof of Lemma 3.1 changing the roles of u and v. We fix (y, s) and take $k = u(y, s)$ and $\phi^{\epsilon,\delta} = \phi^{\epsilon,\delta}(x, t, y, s)$ in the entropy inequality for $v = v(x, t)$ to find that

$$
\begin{aligned}
&\iint_{Q_T^2} |v - u|\,\partial_t\phi^{\epsilon,\delta} + \Big(q_g(v, u) + |B(v) - B(u)|\,b_r^{\mu^*}\Big)\cdot D_x\phi^{\epsilon,\delta}\,\mathrm{d}w\\
&+ \iint_{Q_T^2} |B(v) - B(u)|\,\mathcal{L}_r^{\mu^*}[\phi^{\epsilon,\delta}(\cdot, t, y, s)](x)\,\mathrm{d}w\\
&+ \iint_{Q_T^2} \operatorname{sgn}(v - u)\,\mathcal{L}^{\mu,r}[B(v(\cdot, t))](x)\,\phi^{\epsilon,\delta}\,\mathrm{d}w\\
&- \iint_{\mathbb{R}^d\times Q_T} |v(x, T) - u(y, s)|\,\phi^{\epsilon,\delta}(x, T, y, s)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\
&+ \iint_{\mathbb{R}^d\times Q_T} |v_0(x) - u(y, s)|\,\phi^{\epsilon,\delta}(x, 0, y, s)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \geq 0.
\end{aligned}
$$

Then we add this inequality and inequality (3.3) in Lemma 3.1,

$$
\begin{aligned}
&\|u(T)-v(T)\|_{L^1(\mathbb{R}^d)}\\
&\leq \|u_0-v_0\|_{L^1(\mathbb{R}^d)}+\epsilon\, C_{\tilde{\theta}}\,|u_0|_{BV(\mathbb{R}^d)}+2\,\omega_u(\delta)\vee\omega_v(\delta)\\
&\quad+\underbrace{\iint_{Q_T^2}(q_g-q_f)(v,u)\cdot D_x\phi^{\epsilon,\delta}\,\mathrm{d}w}_{=:I_1}\\
&\quad+\underbrace{\iint_{Q_T^2}|B(v)-B(u)|\,\mathcal{L}_r^{\mu^*}[\phi^{\epsilon,\delta}(\cdot,y,t,s)](x)\,\mathrm{d}w}_{=:I_2}\\
&\quad+\underbrace{\iint_{Q_T^2}|A(v)-A(u)|\,\mathcal{L}_r^{\mu^*}[\phi^{\epsilon,\delta}(x,t,\cdot,s)](y)\,\mathrm{d}w}_{=:I_2'}\\
&\quad+\underbrace{\iint_{Q_T^2}\Big(|B(v)-B(u)|-|A(v)-A(u)|\Big)\,b_r^{\mu^*}\cdot D_x\phi^{\epsilon,\delta}\,\mathrm{d}w}_{=:I_3}\\
&\quad+\underbrace{\iint_{Q_T^2}\operatorname{sgn}(v-u)\Big(\mathcal{L}^{\mu,r}[B(v(\cdot,t))](x)-\mathcal{L}^{\mu,r}[A(u(\cdot,s))](y)\Big)\,\phi^{\epsilon,\delta}\,\mathrm{d}w}_{=:I_4},
\end{aligned}
\tag{5.5}
$$

where $r,\epsilon>0$, $0<\delta<T$, and $C_{\tilde{\theta}}>0$ only depends on the kernel $\tilde{\theta}_d$ from (3.2).

2. It is standard to estimate I_1 (cf. e.g. [25, 48]), and I_2+I_2' can be estimated by Lemma 4.4,

$$I_1\leq |u_0|_{BV(\mathbb{R}^d)}\,T\,\|f'-g'\|_{L^\infty(\mathbb{R},\mathbb{R}^d)},\tag{5.6}$$

$$I_2+I_2'\leq C_\epsilon\int_{0<|z|\leq r}|z|^2\,\mathrm{d}\mu(z),\tag{5.7}$$

where C_ϵ does not depend on $r>0$. Now we focus on I_3 and I_4.

3. *Cutting w.r.t.* $E_\pm$. We split I_3 and I_4 into four new terms using the sets $E_\pm$, see (5.3)–(5.4). By Lemma 5.1 (i), I_4 can be written as

$$I_4=\sum_{\pm}\iint_{Q_T^2}\operatorname{sgn}(v-u)\Big(\mathcal{L}^{\mu,r}[B_\pm(v(\cdot,t))](x)-\mathcal{L}^{\mu,r}[A_\pm(u(\cdot,s))](y)\Big)\,\phi^{\epsilon,\delta}\,\mathrm{d}w.$$

By Lemma 5.1 (ii), we can apply twice Lemma 4.3 with B_+ and A_- instead of A, followed by the definitions of $C_\pm$, see (5.4), to show that

$$
\begin{aligned}
I_4&\leq\iint_{Q_T^2}\operatorname{sgn}(v-u)\,\mathcal{L}^{\mu,r}\Big[B_+(u(\cdot,s))-A_+(u(\cdot,s))\Big](y)\,\phi^{\epsilon,\delta}\,\mathrm{d}w\\
&\quad+\iint_{Q_T^2}\operatorname{sgn}(v-u)\,\mathcal{L}^{\mu,r}\Big[B_-(v(\cdot,t))-A_-(v(\cdot,t))\Big](x)\,\phi^{\epsilon,\delta}\,\mathrm{d}w\\
&=\iint_{Q_T^2}\operatorname{sgn}(u-v)\,\mathcal{L}^{\mu,r}\big[C_+(u(\cdot,s))\big](y)\,\phi^{\epsilon,\delta}\,\mathrm{d}w\\
&\quad+\iint_{Q_T^2}\operatorname{sgn}(v-u)\,\mathcal{L}^{\mu,r}\big[C_-(v(\cdot,t))\big](x)\,\phi^{\epsilon,\delta}\,\mathrm{d}w\\
&=:I_4^++I_4^-.
\end{aligned}
\tag{5.8}
$$

Note that it is crucial to have u in the first term and v in the second – otherwise we will not be able to apply the Kato inequality later on!

We now consider I_3. By (2.7), Lemma 5.1 (i)–(ii), the formula $D_x\phi^{\epsilon,\delta} = -D_y\phi^{\epsilon,\delta}$, and the definitions $D_+ = D_y$ and $D_- = D_x$, it follows that

$$\begin{aligned}
&\Big(|B(v) - B(u)| - |A(v) - A(u)|\Big) D_x\phi^{\epsilon,\delta}\\
&= \operatorname{sgn}(u - v) \Big\{ (A(u) - B(u)) - (A(v) - B(v)) \Big\} D_y\phi^{\epsilon,\delta}\\
&= \sum_{\pm} \operatorname{sgn}(u - v) \Big\{ \pm (A_\pm(u) - B_\pm(u)) \mp (A_\pm(v) - B_\pm(v)) \Big\} D_\pm\phi^{\epsilon,\delta}\\
&= \sum_{\pm} |C_\pm(u) - C_\pm(v)|\, D_\pm\phi^{\epsilon,\delta}.
\end{aligned}$$

We can then rewrite I_3 as

$$I_3 = \sum_{\pm} \underbrace{\iint_{Q_T} |C_\pm(u) - C_\pm(v)|\, b_r^{\mu^*} \cdot D_\pm\phi^{\epsilon,\delta}\,\mathrm{d}w}_{=:I_3^\pm}. \tag{5.9}$$

4. *Cutting w.r.t. z.* We decompose $\mathcal{L}^{\mu,r}$ into two new terms using a new cutting parameter $r_1 > r$. Let $\mu = \mu_1 + \mu|_{|z|>r_1}$ for

$$\mu_1 := \mu|_{0<|z|\le r_1},$$

and note that by (2.4), $\mathcal{L}^{\mu,r} = \mathcal{L}^{\mu_1,r} + \mathcal{L}^{\mu,r_1}$. Then

$$\begin{aligned}
I_4^+ = &\underbrace{\iint_{Q_T^2} \operatorname{sgn}(u - v)\, \mathcal{L}^{\mu_1,r}[C_+(u(\cdot,s))](y)\, \phi^{\epsilon,\delta}\,\mathrm{d}w}_{=:I_5^+}\\
&+ \iint_{Q_T^2} \operatorname{sgn}(u - v)\, \mathcal{L}^{\mu,r_1}[C_+(u(\cdot,s))](y)\, \phi^{\epsilon,\delta}\,\mathrm{d}w.
\end{aligned} \tag{5.10}$$

Since C_+ satisfies (1.4) by Lemma 5.1 (ii) and μ_1 clearly satisfies (1.6), we can apply the Kato type inequality in Lemma 4.2 (with $k = v(x,t)$ and $A = C_+$) to show that

$$\begin{aligned}
I_5^+ &= \int_{Q_T}\int_{Q_T} \operatorname{sgn}(u(y,s) - v(x,t))\, \mathcal{L}^{\mu_1,r}[C_+(u(\cdot,s))](y)\, \phi^{\epsilon,\delta}\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t\\
&\le \int_{Q_T}\int_{Q_T} |C_+(u(y,s)) - C_+(v(x,t))|\, \mathcal{L}^{\mu_1^*,r}[\phi^{\epsilon,\delta}(x,t,\cdot,s)](y)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Adding I_3^+ in the form (5.9) then gives

$$I_3^+ + I_5^+ \le \iint_{Q_T^2} |C_+(u) - C_+(v)| \Big(b_r^{\mu^*} \cdot D_y\phi^{\epsilon,\delta} + \mathcal{L}^{\mu_1^*,r}[\phi^{\epsilon,\delta}(x,t,\cdot,s)](y) \Big)\,\mathrm{d}w. \tag{5.11}$$

Now easy computations show that

$$D_y\phi^{\epsilon,\delta} = -\theta_\delta(t-s)\, D\bar\theta_\epsilon(x-y), \quad \mathcal{L}^{\mu_1^*,r}[\phi^{\epsilon,\delta}(x,t,\cdot,s)](y) = \theta_\delta(t-s)\, \mathcal{L}^{\mu_1,r}[\bar\theta_\epsilon](x-y).$$

Hence by adding and subtracting $z \cdot D\bar\theta_\epsilon(x - y)$, we get that

$$\begin{aligned}
&b_r^{\mu^*} \cdot D_y\phi^{\epsilon,\delta} + \mathcal{L}^{\mu_1^*,r}[\phi^{\epsilon,\delta}(x,t,\cdot,s)](y)\\
&= \theta_\delta(t - s) \int_{r<|z|\le r_1} \bar\theta_\epsilon(x - y + z) - \bar\theta_\epsilon(x - y) - z \cdot D\bar\theta_\epsilon(x - y)\,\mathrm{d}\mu(z)\\
&\quad + \theta_\delta(t - s)\, D\bar\theta_\epsilon(x - y) \cdot \underbrace{\left(-b_r^{\mu^*} + \int_{r<|z|\le r_1} z\,\mathrm{d}\mu(z) \right)}_{=\operatorname{sgn}(r_1-1)\int_{r_1\wedge(1\vee r)<|z|\le r_1\vee 1} z\,\mathrm{d}\mu(z)},
\end{aligned} \tag{5.12}$$

where the last equality comes from (2.3) and the change of variable $z \to -z$. We insert (5.12) into (5.11) and combine the resulting inequality with (5.10),

$$
\begin{aligned}
&I_3^+ + I_4^+ \leq \\
&\iint_{Q_T^2} |C_+(u) - C_+(v)| \\
&\quad \cdot \theta_\delta(t-s) \int_{r<|z|\leq r_1} \bar\theta_\epsilon(x-y+z) - \bar\theta_\epsilon(x-y) - z\cdot D\bar\theta_\epsilon(x-y)\,\mathrm{d}\mu(z)\,\mathrm{d}w \\
&+ \iint_{Q_T^2} |C_+(u) - C_+(v)| \\
&\quad \cdot \theta_\delta(t-s)\, D\bar\theta_\epsilon(x-y)\cdot \operatorname{sgn}(r_1-1) \int_{r_1\wedge(1\vee r)<|z|\leq r_1\vee 1} z\,\mathrm{d}\mu(z)\,\mathrm{d}w \\
&+ \iint_{Q_T^2} \operatorname{sgn}(u-v)\,\mathcal{L}^{\mu,r_1}[C_+(u(\cdot,s))](y)\,\phi^{\epsilon,\delta}\,\mathrm{d}w \\
&=: J_1^+ + J_2^+ + J_3^+.
\end{aligned}
\tag{5.13}
$$

Similar arguments show that we can bound $I_3^- + I_4^-$ (see (5.8)–(5.9)) as follows,

$$
\begin{aligned}
&I_3^- + I_4^- \leq \\
&\iint_{Q_T^2} |C_-(v) - C_-(u)| \\
&\quad \cdot \theta_\delta(t-s) \int_{r<|z|\leq r_1} \bar\theta_\epsilon(x-y-z) - \bar\theta_\epsilon(x-y) + z\cdot D\bar\theta_\epsilon(x-y)\,\mathrm{d}\mu(z)\,\mathrm{d}w \\
&- \iint_{Q_T^2} |C_-(v) - C_-(u)| \\
&\quad \cdot \theta_\delta(t-s)\, D\bar\theta_\epsilon(x-y)\cdot \operatorname{sgn}(r_1-1) \int_{r_1\wedge(1\vee r)<|z|\leq r_1\vee 1} z\,\mathrm{d}\mu(z)\,\mathrm{d}w \\
&+ \underbrace{\iint_{Q_T^2} \operatorname{sgn}(v-u)\,\mathcal{L}^{\mu,r_1}[C_-(v(\cdot,t))](x)\,\phi^{\epsilon,\delta}\,\mathrm{d}w}_{\leq \iint_{Q_T^2} \operatorname{sgn}(v-u)\,\mathcal{L}^{\mu,r_1}[C_-(u(\cdot,s))](y)\,\phi^{\epsilon,\delta}\,\mathrm{d}w \text{ by Lemma 4.3}} \\
&=: J_1^- + J_2^- + J_3^-.
\end{aligned}
\tag{5.14}
$$

5. $L^1\cap BV$*-regularity.* It remains to estimate $J_i^\pm$ for $i = 1,\ldots,3$ in (5.13)–(5.14). For $J_1^\pm$ and $J_2^\pm$, we use Fubini and integrate by parts to take advantage of the BV-regularity of the entropy solution u. After some computations given in Appendix (see Lemma A.1), we find that

$$
\begin{aligned}
|J_1^\pm| &\leq \frac{1}{2\epsilon}\int_{\mathbb{R}^d} |D\tilde\theta_d|\mathrm{d}x \int_{r<|z|\leq r_1} |z|^2\,\mathrm{d}\mu(z)\,|C_\pm(u)|_{L^1(0,T;BV)}, \\
|J_2^\pm| &\leq \left|\int_{r_1\wedge(1\vee r)<|z|\leq r_1\vee 1} z\,\mathrm{d}\mu(z)\right| |C_\pm(u)|_{L^1(0,T;BV)},
\end{aligned}
$$

and hence

$$
\begin{aligned}
\sum_\pm (J_1^\pm + J_2^\pm) \leq{}& \frac{1}{2\epsilon}\int_{\mathbb{R}^d} |D\tilde\theta_d|\mathrm{d}x \int_{r<|z|\leq r_1} |z|^2\,\mathrm{d}\mu(z) \sum_\pm |C_\pm(u)|_{L^1(0,T;BV)} \\
&+ \left|\int_{r_1\wedge(1\vee r)<|z|\leq r_1\vee 1} z\,\mathrm{d}\mu(z)\right| \sum_\pm |C_\pm(u)|_{L^1(0,T;BV)}.
\end{aligned}
$$

By Lemma 5.1 (iii) and a priori estimates for u, cf. (2.10), we see that

$$\sum_{\pm}(J_1^{\pm}+J_2^{\pm}) \le \frac{1}{2\epsilon}\int_{\mathbb{R}^d}|D\tilde{\theta}_d|\mathrm{d}x\,\underbrace{|u|_{L^1(0,T;BV)}}_{\le |u_0|_{BV(\mathbb{R}^d)}T}\int_{r<|z|\le r_1}|z|^2\,\mathrm{d}\mu(z)\,\|A'-B'\|_{L^\infty(\mathbb{R})}$$

$$+\underbrace{|u|_{L^1(0,T;BV)}}_{\le |u_0|_{BV(\mathbb{R}^d)}T}\left|\int_{r_1\wedge(1\vee r)<|z|\le r_1\vee 1} z\,\mathrm{d}\mu(z)\right|\,\|A'-B'\|_{L^\infty(\mathbb{R})}. \tag{5.15}$$

Let us now estimate J_3^+ in (5.13). Easy computations (see the proofs of Lemmas 4.3–4.4) show that

$$J_3^+ \le \|\theta_\delta\,\bar{\theta}_\epsilon\|_{L^1(\mathbb{R}^{d+1})}\,\|\mathcal{L}^{\mu,r_1}[C_+(u)]\|_{L^1(Q_T)}.$$

Let us recall that $\|\theta_\delta\,\bar{\theta}_\epsilon\|_{L^1(\mathbb{R}^{d+1})}=\|\theta_\delta\|_{L^1(\mathbb{R})}\,\|\bar{\theta}_\epsilon\|_{L^1(\mathbb{R}^d)}=1$, and then

$$J_3^+ \le \int_0^T\int_{|z|>r_1}\|C_+(u(\cdot+z,s))-C_+(u(\cdot,s))\|_{L^1(\mathbb{R}^d)}\,\mathrm{d}\mu(z)\mathrm{d}s.$$

Since $C_+(u)\in L^\infty\cap C([0,T];L^1)$, $(z,s)\to\|C_+(u(\cdot+z,s))-C_+(u(\cdot,s))\|_{L^1(\mathbb{R}^d)}$ is a continuous function, hence Borel and $\mathrm{d}\mu(z)\,\mathrm{d}s$-measurable. Thus, we may change the order of the integration to find

$$J_3^+ \le \int_{|z|>r}\|C_+(u(\cdot+z,\cdot))-C_+(u)\|_{L^1(Q_T)}\,\mathrm{d}\mu(z).$$

We get a similar estimates for J_3^- and find by Lemma 5.1 (iii)–(iv) and (2.10) that

$$\sum_{\pm}J_3^{\pm} \le \int_{|z|>r_1}\sum_{\pm}\|C_\pm(u(\cdot+z,\cdot))-C_\pm(u)\|_{L^1(Q_T)}\,\mathrm{d}\mu(z),$$

$$\le \int_{|z|>r_1}\underbrace{\|u(\cdot+z,\cdot))-u\|_{L^1(Q_T)}}_{\le T\,\|u_0(\cdot+z)-u_0\|_{L^1(\mathbb{R}^d)}}\mathrm{d}\mu(z)\,\|A'-B'\|_{L^\infty(\mathbb{R}^d)}. \tag{5.16}$$

The last inequality (under the bracket) comes from (2.11) applied to the solution $u(\cdot+z,\cdot)$ of (1.1) with initial data $u_0(\cdot+z)$.

6. *Conclusion.* By (5.8)–(5.9) and (5.13)–(5.14), $I_3+I_4\le\sum_\pm\sum_{i=1}^3 J_i^\pm$. Therefore we may estimate (5.5) by (5.6)–(5.7) and (5.15)–(5.16). For all $r_1>r>0$, $\epsilon>0$, and $T>\delta>0$, we find that

$$\begin{aligned}
&\|u(T)-v(T)\|_{L^1(\mathbb{R}^d)}\\
&\le \|u_0-v_0\|_{L^1(\mathbb{R}^d)}+|u_0|_{BV(\mathbb{R}^d)}\,T\,\|f'-g'\|_{L^\infty(\mathbb{R},\mathbb{R}^d)}\\
&\quad+\epsilon\,C_{\tilde{\theta}}\,|u_0|_{BV(\mathbb{R}^d)}+2\,\omega_u(\delta)\vee\omega_v(\delta)+C_\epsilon\int_{0<|z|\le r}|z|^2\,\mathrm{d}\mu(z)\\
&\quad+\frac{1}{2\epsilon}\int_{\mathbb{R}^d}|D\tilde{\theta}_d|\mathrm{d}x\,|u_0|_{BV(\mathbb{R}^d)}\,T\int_{r<|z|\le r_1}|z|^2\,\mathrm{d}\mu(z)\,\|A'-B'\|_{L^\infty(\mathbb{R}^d)}\\
&\quad+|u_0|_{BV(\mathbb{R}^d)}\,T\left|\int_{r_1\wedge(1\vee r)<|z|\le r_1\vee 1}z\,\mathrm{d}\mu(z)\right|\,\|A'-B'\|_{L^\infty(\mathbb{R}^d)}\\
&\quad+T\int_{|z|>r_1}\|u_0(\cdot+z)-u_0\|_{L^1(\mathbb{R}^d)}\,\mathrm{d}\mu(z)\,\|A'-B'\|_{L^\infty(\mathbb{R}^d)},
\end{aligned} \tag{5.17}$$

where $C_\epsilon>0$ does not depend on $r>0$.

To finish, we first pass to the limit as $r\to 0$ in (5.17). By the dominated convergence theorem, the result is equivalent to setting $r=0$ in each term, and in particular the term $C_\epsilon\int_{0<|z|\le r}|z|^2\,\mathrm{d}\mu(z)$ vanishes. Secondly, we pass to the limit as $\delta \to 0$ to get rid of the term $2\,\omega_u(\delta) \vee \omega_v(\delta)$. Finally, we optimize the remaining terms w.r.t. $\epsilon > 0$ by using the formula $\min_{\epsilon>0}\left(\epsilon\, a + \frac{b}{\epsilon}\right) = 2\sqrt{ab}$ (for $a, b \geq 0$). This gives us the following continuous dependence estimate: For all $r_1 > 0$,

$$
\begin{aligned}
\|u-v\|_{C([0,T];L^1)} \leq{}& \|u_0 - v_0\|_{L^1(\mathbb{R}^d)} + |u_0|_{BV(\mathbb{R}^d)}\, T\, \|g' - f'\|_{L^\infty(\mathbb{R},\mathbb{R}^d)} \\
&+ 2\sqrt{\frac{1}{2}\, C_{\tilde{\theta}} \int_{\mathbb{R}^d} |D\tilde{\theta}_d| \mathrm{d}x\, |u_0|^2_{BV(\mathbb{R}^d)}\, T \int_{0<|z|\leq r_1} |z|^2\, \mathrm{d}\mu(z)\, \|A' - B'\|_{L^\infty(\mathbb{R})}} \\
&+ |u_0|_{BV(\mathbb{R}^d)}\, T \left| \int_{r_1\wedge 1<|z|\leq r_1 \vee 1} z\, \mathrm{d}\mu(z) \right| \|A' - B'\|_{L^\infty(\mathbb{R})} \\
&+ T \int_{|z|\geq r_1} \|u_0(\cdot + z) - u_0\|_{L^1(\mathbb{R}^d)}\, \mathrm{d}\mu(z)\, \|A' - B'\|_{L^\infty(\mathbb{R})},
\end{aligned}
\tag{5.18}
$$

where $\tilde{\theta}_d$ is an arbitrary approximate unit (3.2) and $C_{\tilde{\theta}} = 2\int_{\mathbb{R}^d} |x|\, \tilde{\theta}_d(x)\, \mathrm{d}x$ by Lemma 3.1.

Let $\tilde{\theta}_d = \theta_n$ where $\{\theta_n\}_{n\in\mathbb{N}}$ is a sequence of kernels s.t. θ_n satisfies (3.2), $\theta_n \to {\omega_d}^{-1}\mathbf{1}_{|\cdot|<1}$ in L^1, and $\int_{\mathbb{R}^d} |D\theta_n|\, \mathrm{d}x \to {\omega_d}^{-1}|\mathbf{1}_{|\cdot|<1}|_{BV(\mathbb{R}^d)}$. Here ω_d is the volume of the unit ball in $\mathbb{R}^d$. Note that the BV-semi-norm of the indicator function of the unit ball is equaled to the surface area of the unit sphere, i.e. $|\mathbf{1}_{|\cdot|<1}|_{BV(\mathbb{R}^d)} = d\omega_d$. Moreover, we have

$$
\int_{\mathbb{R}^d} |x||\theta_n(x)|\, \mathrm{d}x \to \frac{1}{\omega_d} \int_{|x|<1} |x|\, \mathrm{d}x = \frac{d}{d+1}.
$$

The proof of (3.4) is then complete after passing to the limit as $n \to +\infty$ in (5.18). □

5.3. **Proof of Theorem 3.4.** We argue step by step as in the proof of Theorem 3.3. This time, $E_\pm$ are taken such as

$$
\begin{cases}
E_\pm \subseteq \mathbb{R}^d \setminus \{0\} \text{ are Borel sets;} \\
\cup_\pm E^\pm = \mathbb{R}^d \setminus \{0\} \text{ and } \cap_\pm E_\pm = \emptyset; \\
\left(\mathbb{R}^d \setminus \{0\}\right) \setminus \operatorname{supp}(\mu - \nu)^\mp \subseteq E_\pm.
\end{cases}
\tag{5.19}
$$

Let $\mu_\pm$ and $\nu_\pm$ denote the restrictions of μ and ν to $E_\pm$. It is clear that

$$
\begin{cases}
\mu = \sum_\pm \mu_\pm \quad \text{and} \quad \nu = \sum_\pm \nu_\pm, \\
\pm(\mu_\pm - \nu_\pm) = (\mu - \nu)^\pm, \\
\mu_\pm, \nu_\pm, \text{ and } \pm(\mu_\pm - \nu_\pm) \text{ all satisfy (1.6).}
\end{cases}
\tag{5.20}
$$

Proof of Theorem 3.4.

1. We apply Lemma 3.1 with $A = B$, but different Lévy measures μ and ν, along with the entropy inequality for v to show that for all $r, \epsilon > 0$, $0 < \delta < T$

$$\begin{aligned}
&\|u(T) - v(T)\|_{L^1(\mathbb{R}^d)} \\
&\leq \|u_0 - v_0\|_{L^1(\mathbb{R}^d)} + \epsilon\, C_{\tilde{\theta}}\, |u_0|_{BV(\mathbb{R}^d)} + 2\,\omega_u(\delta) \vee \omega_v(\delta) \\
&\quad + \iint_{Q_T^2} (q_g - q_f)(v,u) \cdot D_x \phi^{\epsilon,\delta}\, \mathrm{d}w \\
&\quad + \iint_{Q_T^2} |A(v) - A(u)|\, \mathcal{L}_r^{\nu^*}[\phi^{\epsilon,\delta}(\cdot,y,t,s)](x)\, \mathrm{d}w \\
&\quad + \iint_{Q_T^2} |A(v)) - A(u)|\, \mathcal{L}_r^{\mu^*}[\phi^{\epsilon,\delta}(x,t,\cdot,s)](y)\, \mathrm{d}w \\
&\quad + \underbrace{\iint_{Q_T^2} |A(v) - A(u)| \left(b_r^{\nu^*} - b_r^{\mu^*}\right) \cdot D_x \phi^{\epsilon,\delta}\, \mathrm{d}w}_{=:I_3} \\
&\quad + \underbrace{\iint_{Q_T^2} \operatorname{sgn}(v-u) \Big(\mathcal{L}^{\nu,r}[A(v(\cdot,t))](x) - \mathcal{L}^{\mu,r}[A(u(\cdot,s))](y)\Big)\, \phi^{\epsilon,\delta}\, \mathrm{d}w}_{=:I_4},
\end{aligned} \tag{5.21}$$

where $C_\epsilon > 0$ does not depend on $r > 0$. Except for I_3 and I_4, the other terms were estimated in the proof of Theorem 3.3.

2. *Cutting w.r.t. $E_\pm$.* We use the notation introduced in (5.19). We apply Lemma 4.3 twice with ν_+ and μ_- instead of μ, along with linearity of $\mathcal{L}^{\mu,r}$ in μ, see (2.2), to see that

$$\begin{aligned}
I_4 &= \sum_{\pm} \iint_{Q_T^2} \operatorname{sgn}(v-u) \Big(\mathcal{L}^{\nu_\pm,r}[A(v(\cdot,t))](x) - \mathcal{L}^{\mu_\pm,r}[A(u(\cdot,s))](y)\Big)\, \phi^{\epsilon,\delta}\, \mathrm{d}w \\
&\leq \iint_{Q_T^2} \operatorname{sgn}(v-u) \Big(\mathcal{L}^{\nu_+,r}[A(u(\cdot,s))](y) - \mathcal{L}^{\mu_+,r}[A(u(\cdot,s))](y)\Big)\, \phi^{\epsilon,\delta}\, \mathrm{d}w \\
&\quad + \iint_{Q_T^2} \operatorname{sgn}(v-u) \Big(\mathcal{L}^{\nu_-,r}[A(v(\cdot,t))](x) - \mathcal{L}^{\mu_-,r}[A(v(\cdot,t))](x)\Big)\, \phi^{\epsilon,\delta}\, \mathrm{d}w \\
&= \iint_{Q_T^2} \operatorname{sgn}(u-v)\, \mathcal{L}^{\mu_+-\nu_+,r}[A(u(\cdot,s))](y)\, \phi^{\epsilon,\delta}\, \mathrm{d}w \\
&\quad + \iint_{Q_T^2} \operatorname{sgn}(v-u)\, \mathcal{L}^{-(\mu_- -\nu_-),r}[A(v(\cdot,t))](x)\, \phi^{\epsilon,\delta}\, \mathrm{d}w \\
&=: I_4^+ + I_4^-.
\end{aligned} \tag{5.22}$$

Again, it is crucial to have u in I_4^+ and v in I_4^- in order to use Kato's inequality later on.

Let us now consider I_3. By (2.3) and (2.8), b_r^μ and μ^* are linear w.r.t μ. Easy computations using (5.20) then leads to

$$\left(b_r^{\nu^*} - b_r^{\mu^*}\right) \cdot D_x \phi^{\epsilon,\delta} = \sum_{\pm} b_r^{\pm(\mu_\pm - \nu_\pm)^*} \cdot D_\pm \phi^{\epsilon,\delta}$$

where $D_+ = D_y$ and $D_- = D_x$, and hence

$$I_3 = \sum_{\pm} \iint_{Q_T} |A(u) - A(v)|\, b_r^{\pm(\mu_\pm - \nu_\pm)^*} \cdot D_\pm \phi^{\epsilon,\delta}\, \mathrm{d}w =: I_3^+ + I_3^-.$$

3. *Cutting w.r.t. z.* The computations of this step are similar to the ones in the proof of Theorem 3.3. For the reader's convenience, we estimate $I_3^- + I_4^-$, the terms that was left to the reader in the preceding proof.

For any measure $\tilde{\mu}$ we let $\tilde{\mu}_1 = \tilde{\mu}_{|_{0<|z|\leq r_1}}$ and write $\tilde{\mu} = \tilde{\mu}_1 + \tilde{\mu}_{|_{|z|>r_1}}$ for $r_1 > r$. Then

$$\begin{aligned}I_4^- \leq\ & \underbrace{\iint_{Q_T} \operatorname{sgn}(v-u)\, \mathcal{L}^{-(\mu_- - \nu_-)_1, r}[A(v(\cdot,t))](x)\, \phi^{\epsilon,\delta}\, \mathrm{d}w}_{=:I_5^-} \\ &+ \iint_{Q_T} \operatorname{sgn}(v-u)\, \mathcal{L}^{-(\mu_- - \nu_-), r_1}[A(v(\cdot,t))](x)\, \phi^{\epsilon,\delta}\, \mathrm{d}w.\end{aligned}$$

Recall that $-(\mu_- - \nu_-)_1$ is a positive Lévy measure by (5.20), so we can apply Lemma 4.2 with $-(\mu_- - \nu_-)_1$ instead of μ and $k = u(y,s)$ to find that

$$I_5^- \leq \iint_{Q_T^2} |A(v) - A(u)|\, \mathcal{L}^{-(\mu_- - \nu_-)_1^*, r}[\phi^{\epsilon,\delta}(\cdot,t,y,s)](x)\, \mathrm{d}w$$

and

$$\begin{aligned}&I_3^- + I_5^- \leq \\ &\iint_{Q_T^2} |A(v) - A(u)| \left(b_r^{-(\mu_- - \nu_-)^*} \cdot D_x \phi^{\epsilon,\delta} + \mathcal{L}^{-(\mu_- - \nu_-)_1^*, r}[\phi^{\epsilon,\delta}(\cdot,t,y,s)](x) \right) \mathrm{d}w.\end{aligned}$$

Easy computations then leads to

$$\begin{aligned}&\mathcal{L}^{-(\mu_- - \nu_-)_1^*, r}[\phi^{\epsilon,\delta}(\cdot,t,y,s)](x) \\ &= \theta_\delta(t-s) \int_{r<|z|\leq r_1} \bar{\theta}_\epsilon(x-y-z) - \bar{\theta}_\epsilon(x-y)\, \mathrm{d}(\nu_- - \mu_-)(z),\end{aligned}$$

and we can rewrite the nonlocal operator as follows,

$$\begin{aligned}&b_r^{-(\mu_- - \nu_-)^*} \cdot D_x \phi^{\epsilon,\delta} + \mathcal{L}^{-(\mu_- - \nu_-)_1^*, r}[\phi^{\epsilon,\delta}(\cdot,t,y,s)](x) \\ &= \theta_\delta(t-s) \int_{r<|z|\leq r_1} \bar{\theta}_\epsilon(x-y-z) - \bar{\theta}_\epsilon(x-y) + z \cdot D\bar{\theta}_\epsilon(x-y)\, \mathrm{d}(\nu_- - \mu_-)(z) \\ &\quad - \theta_\delta(t-s)\, D\bar{\theta}_\epsilon(x-y) \cdot \underbrace{\left(-b_r^{-(\mu_- - \nu_-)^*} + \int_{r<|z|\leq r_1} z\, \mathrm{d}(\nu_- - \mu_-)(z) \right)}_{=\operatorname{sgn}(r_1 - 1) \int_{r_1 \wedge (1 \vee r) < |z| \leq r_1 \vee 1} z\, \mathrm{d}(\nu_- - \mu_-)(z)}.\end{aligned}$$

Compare this expression with (5.12) that appear when I_3^+ and I_4^+ are considered.

We add the different estimates and find that for all $r_1 > r$,

$$
\begin{aligned}
& I_3^- + I_4^- \\
& \leq \iint_{Q_T^2} |A(u) - A(v)|\, \theta_\delta(t-s) \\
& \qquad \cdot \int_{r<|z|\leq r_1} \bar{\theta}_\epsilon(x-y-z) - \bar{\theta}_\epsilon(x-y) + z \cdot D\bar{\theta}_\epsilon(x-y)\, \mathrm{d}(\nu_- - \mu_-)(z)\, \mathrm{d}w \\
& \quad - \iint_{Q_T^2} |A(u) - A(v)|\, \theta_\delta(t-s)\, D\bar{\theta}_\epsilon(x-y) \\
& \qquad \cdot \operatorname{sgn}(r_1 - 1) \int_{r_1 \wedge (1\vee r)<|z|\leq r_1 \vee 1} z\, \mathrm{d}(\nu_- - \mu_-)(z)\, \mathrm{d}w \\
& \quad + \underbrace{\iint_{Q_T^2} \operatorname{sgn}(v-u)\, \mathcal{L}^{-(\mu_- - \nu_-), r_1}[A(v(\cdot,t))](x)\, \phi^{\epsilon,\delta}\, \mathrm{d}w}_{\leq \iint_{Q_T^2} \operatorname{sgn}(v-u)\, \mathcal{L}^{-(\mu_- - \nu_-), r_1}[A(u(\cdot,s))](y)\, \phi^{\epsilon,\delta}\, \mathrm{d}w \text{ by Lemma 4.3}} \\
& = J_1^- + J_2^- + J_3^-.
\end{aligned}
$$

Similar arguments also lead to

$$
\begin{aligned}
& I_3^+ + I_4^+ \\
& \leq \iint_{Q_T^2} |A(u) - A(v)| \theta_\delta(t-s) \\
& \qquad \cdot \int_{r<|z|\leq r_1} \bar{\theta}_\epsilon(x-y+z) - \bar{\theta}_\epsilon(x-y) - z \cdot D\bar{\theta}_\epsilon(x-y)\, \mathrm{d}(\mu_+ - \nu_+)(z)\, \mathrm{d}w \\
& \quad + \iint_{Q_T^2} |A(u) - A(v)|\, \theta_\delta(t-s)\, D\bar{\theta}_\epsilon(x-y) \\
& \qquad \cdot \operatorname{sgn}(r_1 - 1) \int_{r_1 \wedge (1\vee r)<|z|\leq r_1 \vee 1} z\, \mathrm{d}(\mu_+ - \nu_+)(z)\, \mathrm{d}w \\
& \quad + \iint_{Q_T^2} \operatorname{sgn}(u-v)\, \mathcal{L}^{(\mu_+ - \nu_+), r_1}[A(u(\cdot,s))](y)\, \phi^{\epsilon,\delta}\, \mathrm{d}w, \\
& =: J_1^+ + J_2^+ + J_3^+.
\end{aligned}
$$

4. *$L^1 \cap BV$-regularity.* We estimate $J_i^\pm$ $(i = 1, \ldots, 3)$. By (A.1) of Lemma A.1 and (2.10), it follows that

$$
\sum_\pm J_1^\pm \leq \frac{1}{2\epsilon} \int_{\mathbb{R}^d} |D\tilde{\theta}_d| \mathrm{d}x\, |u_0|_{BV(\mathbb{R}^d)}\, T\, \|A'\|_{L^\infty(\mathbb{R})} \int_{r<|z|\leq r_1} |z|^2\, \mathrm{d} \underbrace{\sum_\pm \pm(\mu_\pm - \nu_\pm)(z)}_{=|\mu-\nu| \text{ by (5.20)}}.
$$

Note now that $\sum_\pm (\mu_\pm - \nu_\pm) = \mu - \nu$, and hence

$$
\begin{aligned}
\sum_\pm J_2^\pm = & \iint_{Q_T^2} |A(u) - A(v)|\, \theta_\delta(t-s)\, D\bar{\theta}_\epsilon(x-y) \\
& \qquad \cdot \operatorname{sgn}(r_1 - 1) \int_{r_1 \wedge (1\vee r)<|z|\leq r_1 \vee 1} z\, \mathrm{d}(\mu - \nu)(z)\, \mathrm{d}w.
\end{aligned}
$$

An other application of (A.2) of Lemma A.1 and (2.10), can be used to see that

$$
\sum_\pm J_2^\pm \leq |u_0|_{BV(\mathbb{R}^d)}\, T\, \|A'\|_{L^\infty(\mathbb{R})} \left| \int_{r_1 \wedge (1\vee r)<|z|\leq r_1 \vee 1} z\, \mathrm{d}(\mu - \nu)(z) \right|.
$$

Finally,

$$\sum_{\pm} J_3^{\pm} \leq T\,\|A'\|_{L^\infty(\mathbb{R})} \int_{|z|\geq r_1} \|u_0(\cdot+z)-u_0\|_{L^1(\mathbb{R}^d)}\,\mathrm{d}|\mu-\nu|(z).$$

5. *Conclusion.* The rest of the proof is the same as for Theorem 3.3; i.e. we use the estimates on $J_i^{\pm}$ to estimate $I_3+I_4 \leq \sum_{i=1}^3 \sum_{\pm} J_i^{\pm}$ in (5.21) and pass to limit and/or optimizes w.r.t. the parameters $r,\epsilon,\delta>0$. The proof is complete. □

Appendix A. Technical computations

Lemma A.1. *Assume* (1.4) *and* (1.6). *Let* $u,v \in L^\infty(Q_T)\cap C([0,T];L^1)\cap L^\infty(0,T;BV)$, $\phi^{\epsilon,\delta}$ *be as in Lemma 3.1, and* $r_1>r>0$. *Then*

$$\begin{aligned}&\bigg|\iint_{Q_T^2} |A(v(x,t))-A(u(y,s))|\\ \text{(A.1)}\qquad &\cdot\,\theta_\delta(t-s)\int_{r<|z|\leq r_1} \bar\theta_\epsilon(x-y\pm z)-\bar\theta_\epsilon(x-y)\mp z\cdot D\bar\theta_\epsilon(x-y)\,\mathrm{d}\mu(z)\,\mathrm{d}w\bigg|\\ &\leq \frac{1}{2\epsilon}\int_{\mathbb{R}^d}|D\tilde\theta_d|\mathrm{d}x\int_{r<|z|\leq r_1}|z|^2\,\mathrm{d}\mu(z)\,|A(u)|_{L^1(0,T;BV)}.\end{aligned}$$

and

$$\begin{aligned}&\bigg|\iint_{Q_T^2} |A(v(x,t))-A(u(y,s))|\\ \text{(A.2)}\qquad &\cdot\,\theta_\delta(t-s)\,D\bar\theta_\epsilon(x-y)\cdot\operatorname{sgn}(r_1-1)\int_{r_1\wedge(1\vee r)<|z|\leq r_1\vee 1} z\,\mathrm{d}\mu(z)\,\mathrm{d}w\bigg|\\ &\leq \bigg|\int_{r_1\wedge(1\vee r)<|z|\leq r_1\vee 1} z\,\mathrm{d}\mu(z)\bigg|\,|A(u)|_{L^1(0,T;BV)}.\end{aligned}$$

Proof. We start by proving (A.1) in the + case. Similar arguments give the proof also in the − case. From Taylor's formula with integral remainder,

$$\bar\theta_\epsilon(x-y+z)-\bar\theta_\epsilon(x-y)-z\cdot D\bar\theta_\epsilon(x-y) = \int_0^1 (1-\tau)\,D^2\bar\theta_\epsilon(x-y+\tau z)\,z\cdot z\,\mathrm{d}\tau.$$

Let I denote the integral in the left-hand side of (A.1). By Fubini's theorem,

$$\begin{aligned}\text{(A.3)}\quad I = &\iint_{(0,T)^2}\int_{r<|z|\leq r_1}\int_0^1 \theta_\delta(t-s)\,(1-\tau)\\ &\cdot\underbrace{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|A(v(x,t))-A(u(y,s))|\,D^2\bar\theta_\epsilon(x-y+\tau z)\,z\cdot z\,\mathrm{d}y\,\mathrm{d}x}_{=:J}\,\mathrm{d}\tau\,\mathrm{d}\mu(z)\,\mathrm{d}t\,\mathrm{d}s.\end{aligned}$$

For any $k\in\mathbb{R}$, it is classical that $\eta_k(A(u(\cdot,s)))=|k-A(u(\cdot,s))|\in BV$ with

$$|D\eta_k(A(u(\cdot,s)))|\leq|DA(u(\cdot,s))|,$$

as composition of a Lipschitz-continuous function with a BV-function; see e.g. [13, 33, 55]. Integration by parts w.r.t. y (for fixed z,x,t,s), then leads to

$$\begin{aligned}|J| &= \bigg|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} D\bar\theta_\epsilon(x-y+\tau z)\cdot z\,z\cdot\mathrm{d}D\eta_{A(v(x,t))}(A(u(\cdot,s)))(y)\,\mathrm{d}x\bigg|,\\ &\leq |z|^2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|D\bar\theta_\epsilon(x-y+\tau z)|\,\mathrm{d}|DA(u(\cdot,s))|(y)\,\mathrm{d}x.\end{aligned}$$

Let us notice that by the definition of $\bar{\theta}_\epsilon$ (just below (3.1)), we have

$$\int_{\mathbb{R}^d} |D\bar{\theta}_\epsilon(x)|\,\mathrm{d}x = \frac{1}{\epsilon}\int_{\mathbb{R}^d} |D\tilde{\theta}_d|\mathrm{d}x.$$

Hence, we change the order of integration (using Fubini) to see that

$$|J| \le |z|^2\,|A(u(s))|_{BV(\mathbb{R}^d)} \int_{\mathbb{R}^d} |D\bar{\theta}_\epsilon(x)|\,\mathrm{d}x \le |z|^2\,|A(u(s))|_{BV(\mathbb{R}^d)}\frac{1}{\epsilon}\int_{\mathbb{R}^d} |D\tilde{\theta}_d|\mathrm{d}x,$$

and then from (A.3) that

$$\begin{aligned}|I| \le{}& \frac{1}{\epsilon}\int_{\mathbb{R}^d} |D\tilde{\theta}_d|\mathrm{d}x\\ &\cdot \iint_{(0,T)^2}\int_{r<|z|\le r_1}\int_0^1 \theta_\delta(t-s)\,(1-\tau)\,|z|^2\,|A(u(s))|_{BV(\mathbb{R}^d)}\,\mathrm{d}\tau\,\mathrm{d}\mu(z)\,\mathrm{d}t\,\mathrm{d}s.\end{aligned}$$

Let us recall that the integrand above is $\mathrm{d}\tau\,\mathrm{d}\mu(z)\,\mathrm{d}t\,\mathrm{d}s$-measurable since $s \to |u(s)|_{BV(\mathbb{R}^d)}$ is lower semi-continuous. By Fubini we then integrate first w.r.t. t and use that $\int_0^T \theta_\delta(t-s)\,\mathrm{d}t \le 1$ to see that

$$|I| \le \frac{1}{\epsilon}\int_{\mathbb{R}^d} |D\tilde{\theta}_d|\mathrm{d}x\int_0^1 (1-\tau)\,\mathrm{d}\tau\int_{r<|z|\le r_1} |z|^2\mathrm{d}\mu(z)\int_0^T |A(u(s))|_{BV(\mathbb{R}^d)}\,\mathrm{d}s,$$

and the proof of (A.1) is complete.

We prove (A.2) by similar arguments. Define

$$q(v,u) := |v-u|\,\mathrm{sgn}\,(r_1-1)\int_{r_1\wedge(1\vee r)<|z|\le r_1\vee 1} z\,\mathrm{d}\mu(z), \tag{A.4}$$

and note that it is Lipschitz-continuous. Again we denote by I the integral of the left-hand side of (A.2). By Fubini's theorem,

$$I = \iint_{(0,T)^2} \theta_\delta(t-s)\underbrace{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} D\bar{\theta}_\epsilon(x-y)\cdot q(A(v(x,t),A(u(y,s)))\,\mathrm{d}y\,\mathrm{d}x}_{=:J}\,\mathrm{d}t\,\mathrm{d}s. \tag{A.5}$$

For fixed (x,t,s), $q(A(v(x,t),\cdot)$ is Lipschitz-continuous and $A(u(\cdot,s))$ is BV; hence, the composition $q(A(v(x,t)),A(u(\cdot,s)))$ is in $BV(\mathbb{R}^d,\mathbb{R}^d)$ with

$$|\mathrm{div}_y q(A(v(x,t)),A(u(\cdot,s)))| \le |DA(u(\cdot,s))|\,\|q_u\|_{L^\infty(\mathbb{R},\mathbb{R}^d)},$$

where $\|q_u\|_{L^\infty(\mathbb{R},\mathbb{R}^d)}$ denotes the Lipschitz constant of q w.r.t. its second variable. We thus may integrate by parts in y to see that

$$|J| \le \|q_u\|_{L^\infty(\mathbb{R},\mathbb{R}^d)}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\bar{\theta}_\epsilon(x-y)\,\mathrm{d}|DA(u(\cdot,s))|(y)\,\mathrm{d}x.$$

Changing the order of integration, we find that

$$J \le |A(u(s))|_{BV(\mathbb{R}^d)}\|q_u\|_{L^\infty(\mathbb{R},\mathbb{R}^d)},$$

and hence by (A.5) and integrating first w.r.t. t, we get that

$$|I| \le \|q_u\|_{L^\infty(\mathbb{R},\mathbb{R}^d)}\int_0^T |A(u(s))|_{BV(\mathbb{R}^d)}\,\mathrm{d}s. \tag{A.6}$$

The proof of (A.2) is now complete since by (A.4),

$$\|q_u\|_{L^\infty(\mathbb{R},\mathbb{R}^d)} = \left|\int_{r_1\wedge(1\vee r)<|z|\le r_1\vee 1} z\,\mathrm{d}\mu(z)\right|.$$

□

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(Nathaël Alibaud) UMR CNRS 6623, Université de Franche-Comté, 16 route de Gray, 25 030 Besançon cedex, France and Department of Mathematics, Faculty of Science, Prince of Songkla University, Hat Yai, Songkhla, Thailand, 90112
E-mail address: nathael.alibaud@ens2m.fr
URL: http://www-math.univ-fcomte.fr/pp_Annu/NALIBAUD/

(Simone Cifani) Department of Mathematics, Norwegian University of Science and Technology (NTNU), N-7491 Trondheim, Norway
E-mail address: simone.cifani@math.ntnu.no
URL: http://www.math.ntnu.no/~cifani/

(Espen R. Jakobsen) Department of Mathematics, Norwegian University of Science and Technology (NTNU), N-7491 Trondheim, Norway
E-mail address: erj@math.ntnu.no
URL: http://www.math.ntnu.no/~erj/

Paper VI

A numerical method for nonlinear fractional convection-diffusion equations

S. Cifani, and E. R. Jakobsen

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A NUMERICAL METHOD FOR NONLINEAR FRACTIONAL CONVECTION-DIFFUSION EQUATIONS

SIMONE CIFANI AND ESPEN R. JAKOBSEN

Abstract. We introduce and analyze a numerical method for a class of nonlinear nonlocal partial differential equations given by the combination of multidimensional scalar conservation laws with generalized (possibly degenerate) diffusion operators which are generators of pure jump Lévy processes. Our numerical method is general and converges toward the relevant entropy solution for any Lipschitz nonlinearity and any measure underlying the diffusion operator which satisfies minimal integrability assumptions. The main advantage of the method is that it allows for a complete error analysis whenever the measure underlying the diffusion operator is explicitly chosen. As an illustrative example we work out the case of fractional measures like the ones underlying the fractional Laplace operator. For the very first time our error analysis produces a rate of convergence which also stretches to cover the strong diffusion setting like the case $\lambda \in [1, 2)$ for the fractional Laplacian.

1. Introduction

In this paper we introduce and analyze a numerical method for partial integro-differential equations of the form

$$\begin{cases} \partial_t u + \mathrm{div} f(u) = \mathcal{L}^\mu[A(u)], & (x,t) \in Q_T, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where Q_T represents the space/time strip $\mathbb{R}^d \times (0, T)$ and the nonlocal operator $\mathcal{L}^\mu[\cdot]$ is a generator of pure jump Lévy processes [14],

$$\mathcal{L}^\mu[\phi(\cdot)](x) = \int_{|z|>0} \phi(x+z) - \phi(x) - z \cdot \nabla\phi(x)\, \mathbf{1}_{|z|<1}(z)\, \mathrm{d}\mu(z) \tag{1.2}$$

for some smooth bounded function ϕ with bounded second derivatives. Here as in the rest of the paper the shorthand $\mathbf{1}(\cdot)$ stands for the indicator function.

Throughout the whole paper the data set (f, A, μ, u_0) is assumed to satisfy the following assumptions

(**A**.1) $f = (f_1, \ldots, f_d) \in W^{1,\infty}(\mathbb{R}; \mathbb{R}^d)$ with $f(0) = 0$,

(**A**.2) $A \in W^{1,\infty}(\mathbb{R})$, $A(\cdot)$ non-decreasing with $A(0) = 0$,

(**A**.3) $\mu \geq 0$ is a Radon measure such that $\int_{|z|>0} |z|^2 \wedge 1\, \mathrm{d}\mu(z) < \infty$,

(**A**.4) $u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$.

The measure μ is a Lévy measure. Here, as in what follows, we use the shorthand notation $a \wedge b$ for $\min(a, b)$ - equivalently, we will also use $a \vee b$ for $\max(a, b)$.

In recent years, partial integro-differential equations of the form (1.1) have been at the center of a very active field of research. A thorough description of the mathematical background for such equations, relevant bibliography, and applications to

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several disciplines of interest can be found in [1, 2, 6]. Whenever the Lévy measure μ underlying the nonlocal diffusion operator $\mathcal{L}^{\mu}[\cdot]$ is chosen as

$$\mu(z) = \frac{c_\lambda}{|z|^{d+\lambda}}, \qquad c_\lambda > 0 \text{ and } \lambda \in (0,2), \tag{1.3}$$

that is to say whenever the operator $\mathcal{L}^{\mu}[\cdot]$ is chosen as the fractional Laplacian $-(-\Delta)^{\lambda/2}$, these equations are also referred to as fractional convection-diffusion equations. Let us stress the fact that the class of all measures of the form (**A**.4) does include asymmetric measures. To give just one non-symmetric example, let us mention the measures used in the well-known CGMY model from mathematical finance, where $d = 1$, $\lambda \in (0,2)$, $(C,G,M) > 0$ and

$$\mu(z) = \begin{cases} \dfrac{C\,e^{-G|z|}}{|z|^{1+\lambda}} & \text{for } z > 0, \\[2mm] \dfrac{C\,e^{-M|z|}}{|z|^{1+\lambda}} & \text{for } z < 0. \end{cases}$$

We refer the reader to [9, 15] for more details on the CGMY model.

Remark 1.1. In the whole paper we will refer to the family of all Lévy (non necessarily simmetric) measures μ such that

$$\mu(z) \le \frac{c_\lambda}{|z|^{d+\lambda}}, \qquad c_\lambda > 0 \text{ and } \lambda \in (0,2), \tag{1.4}$$

as *fractional measures*. Such a name is non-standard and, since the nonlocal operators (1.1) associated to such measures are generators of α-stable Lévy processes [14], a more precise (but longer) name would be *α-stable like Lévy measures*.

In this paper we introduce a new numerical method for (1.1), and prove convergence toward the relevant entropy solution under assumptions (**A**.1)–(**A**.4). Apart from its ability to capture the whole family of equations of the form (1.1), the main advantage of our numerical method is that it allows for a complete error analysis. For example, as we will show in Section 7.1, whenever the measure μ is chosen as in (1.4), our error analysis stretches to cover all powers $\lambda \in (0,2)$. Previous attempts found in the literature were only able to either treat less general equations or not going further than the weak diffusion setting like the case $\lambda \in (0,1)$ for the fractional Laplacian [7].

This work is a part of a project started by the authors in [2]; therein the authors have derived a new general Kuznetsov type of lemma for equations of the form (1.1). Such a lemma is used in [2] to produce a rate of convergence for a generalized vanishing viscosity method plus continuous dependence estimates on the nonlinearities and on the measure underlying the Lévy operator. The new Kuznetsov type of lemma will be used herein to produce a rate of convergence for our numerical method. In a following paper, such lemma will be used to produce optimal continuous dependence estimate for fractional convection-diffusion equations [3]. Such works generalize to nonlocal equations of the form (1.1) the results derived for classical convection-diffusion equations in [5]. Let us also mention a recent work on the speed of convergence of a difference method for classical convection-diffusion equations [12], and a recent study on quadrature schemes for Bellman equations [4].

This paper is organized as follows. In Section 2 we recall the entropy formulation for (1.1) as introduced in [6]. In Section 3 we recall the Kuznetsov type of lemma derived in [2]. In Section 4 we introduce the numerical method without convection, $f \equiv 0$ - this has been done to simplify the exposition, leaving the generalization to the case $f \neq 0$ to Section 7.3. In Section 5 we point out some relevant features of the numerical method which will be useful in the following sections. In Section 6 we establish existence, uniqueness, and a priori estimates for the numerical method's solutions. In Section 7 we establish a general framework for deriving error estimates. In Section 7.1 we use the general framework to establish a rate of convergence for fractional type of equations, i.e. μ as in (1.4). Finally, in Section 7.3 we extend the results proved so far to all convection-diffusion equations of the form (1.1) with $f \neq 0$.

2. Entropy formulation

Let us now briefly recall the entropy formulation for equations of the form (1.1) introduced by the authors in [6]. Let us introduce the shorthand notations $\eta(u,k) = |u-k|$, $\eta'(u,k) = \operatorname{sgn}(u-k)$, and $q_l(u,k) = \eta'(u,k)\,(f_l(u)-f_l(k))$ with $l = 1,\dots,d$. Moreover, let us rewrite the nonlocal operator $\mathcal{L}^\mu[\phi]$ as $\mathcal{L}^\mu_r[\phi] + \mathcal{L}^{\mu,r}[\phi] + \gamma^{\mu,r}\cdot\nabla\phi$, where

$$\begin{aligned}
\mathcal{L}^\mu_r[\phi(\cdot)](x) &= \int_{0<|z|\le r} \phi(x+z) - \phi(x) - z\cdot\nabla\phi(x)\mathbf{1}_{|z|\le 1}\ \mathrm{d}\mu(z),\\
\mathcal{L}^{\mu,r}[\phi(\cdot)](x) &= \int_{|z|>r} \phi(x+z) - \phi(x)\ \mathrm{d}\mu(z),\\
\gamma^{\mu,r}_l &= -\int_{|z|>r} z_l\mathbf{1}_{|z|\le 1}\ \mathrm{d}\mu(z), \qquad l = 1,\dots,d.
\end{aligned}$$

We also use the notation μ^* where $\mu^*(B) = \mu(-B)$ for all Borel sets $B \not\supset \{0\}$. Let us recall that

$$\int_{\mathbb{R}^d} \varphi(x)\,\mathcal{L}^\mu[\psi(\cdot)](x)\ \mathrm{d}x = \int_{\mathbb{R}^d} \psi(x)\,\mathcal{L}^{\mu^*}[\varphi(\cdot)](x)\ \mathrm{d}x$$

for all smooth bounded functions φ,ψ with bounded second derivatives - cf. [2, 6].

According to [6], entropy solutions of (1.1) can be defined as follows:

Definition 2.1. (Entropy solutions) *A function $u \in L^\infty(Q_T)\cap L^\infty(0,T;L^1(\mathbb{R}^d))$ is an entropy solution of the initial value problem* (1.1) *provided that, for all $k\in\mathbb{R}$, all $r>0$, and all non-negative test functions $\varphi \in C^\infty_c(\mathbb{R}^d\times[0,T])$,*

$$\begin{aligned}
&\int_{Q_T} \eta(u,k)\,\partial_t\varphi + \Big(q(u,k) + \gamma^{\mu^*,r}\Big)\cdot\nabla\varphi + \eta(A(u),A(k))\,\mathcal{L}^{\mu^*}_r[\varphi]\\
&\qquad\qquad\qquad + \eta'(u,k)\,\mathcal{L}^{\mu,r}[A(u)]\,\varphi\ \mathrm{d}x\,\mathrm{d}t\\
&\quad - \int_{\mathbb{R}^d} \eta(u(x,T),k)\,\varphi(x,T)\ \mathrm{d}x + \int_{\mathbb{R}^d} \eta(u_0(x),k)\,\varphi(x,0)\ \mathrm{d}x \ge 0.
\end{aligned} \tag{2.1}$$

Let us conclude this section by noting that $\gamma^{\mu,r}_l \equiv 0$ for all $l = 1,\dots,d$ whenever the Lévy measure μ chosen is symmetric - that is to say, $\mu^*\equiv\mu$. It is also worth recalling that the entropy formulation in Definition 2.1 is well-posed:

Theorem 2.1. (Well-posedness) *There exists a unique entropy solution*

$$u \in L^\infty(Q_T)\cap C(0,T;L^1(\mathbb{R}^d))\cap L^\infty(0,T;BV(\mathbb{R}^d))$$

of the initial value problem (1.1). *Moreover,*

$$\begin{cases}
\|u\|_{L^\infty(Q_T)} \le \|u_0\|_{L^\infty(\mathbb{R}^d)},\\
\|u\|_{C(0,T;L^1(\mathbb{R}^d))} \le \|u_0\|_{L^1(\mathbb{R}^d)},\\
|u|_{L^\infty(0,T;BV(\mathbb{R}^d))} \le |u_0|_{BV(\mathbb{R}^d)}.
\end{cases}$$

Proof. Cf. [6]. □

3. A Kuznetsov type of lemma

For the reader's convenience we now recall the new Kuznetsov type of lemma established in [2]. Let us call $\varphi^{\epsilon,\delta}(x,y,t,s) = \Omega_\epsilon(x-y)\,\omega_\delta(t-s)$ with $\epsilon,\delta>0$, where $\omega_\delta(\tau) = \frac{1}{\delta}\,\omega\left(\frac{\tau}{\delta}\right)$ with $\omega(\cdot)$ such that

$$\omega \in C_c^\infty(\mathbb{R}), \quad 0\le\omega\le 1, \quad \omega(\tau)=0 \text{ for all } |\tau|>1, \quad \text{and} \quad \int_{\mathbb{R}} \omega(\tau)\,\mathrm{d}\tau = 1,$$

and $\Omega_\epsilon = \omega_\epsilon(x_1)\cdots\omega_\epsilon(x_d)$. In the following we often denote with C_T a non-negative constant whose value can depend on time and the BV-norm/L^1-norm of the initial datum $u_0(\cdot)$. Furthermore, let us call

$$\mathcal{E}_\delta(v) = \sup_{\substack{|t-s|<\delta\\ t,s\in[0,T]}} \|v(\cdot,t)-v(\cdot,s)\|_{L^1(\mathbb{R}^d)}. \tag{3.1}$$

Lemma 3.1. (Kuznetsov type of lemma) *Let u be the entropy solution of* (1.1) *and v be an arbitrary function such that $v\in L^\infty(Q_T)\cap L^\infty(0,T;L^1(\mathbb{R}^d)\cap BV(\mathbb{R}^d))$ and $v(\cdot\,,0)=v_0(\cdot)$. Then, for any $\epsilon,r>0$ and $0<\delta<T$, we have that (here as throughout the whole paper the shorthand* $\mathrm{d}w$ *stands for* $\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s$*)*

$$\begin{aligned}
\|u(\cdot,T)-v(\cdot,T)\|_{L^1(\mathbb{R}^d)} \le{}& \|u_0-v_0\|_{L^1(\mathbb{R}^d)} + C\left(\epsilon+\mathcal{E}_\delta(u)\vee\mathcal{E}_\delta(v)\right)\\
&- \iint_{Q_T}\iint_{Q_T} \eta(v(x,t),u(y,s))\,\partial_t\varphi^{\epsilon,\delta}(x,y,t,s)\;\mathrm{d}w\\
&- \iint_{Q_T}\iint_{Q_T} q(v(x,t),u(y,s))\cdot\nabla_x\varphi^{\epsilon,\delta}(x,y,t,s)\;\mathrm{d}w\\
&+ \iint_{Q_T}\iint_{Q_T} \eta(A(v(x,t)),A(u(y,s)))\,\mathcal{L}_r^{\mu^*}[\varphi^{\epsilon,\delta}(x,\cdot,t,s)](y)\;\mathrm{d}w\\
&- \iint_{Q_T}\iint_{Q_T} \eta'(v(x,t),u(y,s))\,\mathcal{L}^{\mu,r}[A(v(\cdot,t))](x)\,\varphi^{\epsilon,\delta}(x,y,t,s)\;\mathrm{d}w\\
&- \iint_{Q_T}\iint_{Q_T} \eta(A(v(x,t)),A(u(y,s)))\,\gamma^{\mu^*,r}\cdot\nabla_x\varphi^{\epsilon,\delta}(x,y,t,s)\;\mathrm{d}w\\
&+ \iint_{Q_T}\int_{\mathbb{R}^d} \eta(v(x,T),u(y,s))\,\varphi^{\epsilon,\delta}(x,T,y,s)\;\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\
&- \iint_{Q_T}\int_{\mathbb{R}^d} \eta(v_0(x),u(y,s))\,\varphi^{\epsilon,\delta}(x,0,y,s)\;\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s
\end{aligned} \tag{3.2}$$

Proof. Cf. [2]. The result for the case $\mu=0$ (or $A=0$), i.e. the multidimensional scalar conservation law case, has been originally derived by Kuznetsov in [13]. $\square$

In the following sections we will give our exposition for $f\equiv 0$. The results so derived will be generalized to the case $f\neq 0$ later in Section 7.3.

4. Derivation of the numerical method

Let us introduce the uniform space/time grids $x_\alpha = \alpha\,\Delta x$ and $t_n = n\,\Delta t$, where $\alpha\in\mathbb{Z}^d$ and $R_\alpha = x_\alpha + \Delta x\,(0,1)^d$ while $n=0,\dots,N$ and $N\Delta t = T$.

4.1. **Discretization of the nonlocal operator.** For the time being let us fix a time $s\in(0,T)$ and let $u=u(x)$ be a smooth solution of the initial value problem (1.1) at time $t=s$. Let us also consider a differentiable function $A(\cdot)$ (otherwise no smooth solutions of (1.1) would exist). We approximate the nonlocal operator

$$\mathcal{L}^\mu[A(u(\cdot))](x) = \int_{|z|>0} A(u(x+z)) - A(u(x)) - z\cdot\nabla A(u(x))\,\mathbf{1}_{|z|<1}(z)\;\mathrm{d}\mu(z)$$

as follows.

i) We replace the solution $u = u(x)$ with a piecewise constant interpolant $\bar{U} = \bar{U}(x)$, where

$$\bar{U}(x) = \sum_{\beta\in\mathbb{Z}^d} U_\beta \, \mathbf{1}_{R_\beta}(x).$$

Let us note that, with this notation at hand, we can write

$$A(\bar{U}(x)) = \sum_{\beta\in\mathbb{Z}^d} A(U_\beta) \, \mathbf{1}_{R_\beta}(x).$$

ii) We cut off the singularity by using the discretization parameter Δx. More precisely, we replace domain of integration $\{|z| > 0\}$ in (1.2) with $\{|z| > \frac{\Delta x}{2}\}$ (cf. also the discussion in Remark 7.5).

iii) We replace the gradient $\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}\right)$ with the finite difference operator

$$\hat{D}_{\Delta x} = (\hat{D}^1_{\Delta x}, \cdots, \hat{D}^d_{\Delta x}) \tag{4.1}$$

(we will precisely define the notation $\hat{D}^l_{\Delta x}$ in a short while).

Proceeding as described in *i)-iii)*, we now replace the nonlocal operator $\mathcal{L}^\mu[A(u(\cdot))](\cdot)$ with the nonlocal operator $\hat{\mathcal{L}}^\mu[A(\bar{U}(\cdot))](\cdot)$,

$$\hat{\mathcal{L}}^\mu[A(\bar{U}(\cdot))](x) = \int_{|z|>\frac{\Delta x}{2}} A(\bar{U}(x+z)) - A(\bar{U}(x)) - z \cdot \hat{D}_{\Delta x} A(\bar{U}(x)) \, \mathbf{1}_{|z|\leq 1} \, \mathrm{d}\mu(z). \tag{4.2}$$

Note that, opposite to $\mathcal{L}^\mu[\cdot]$ where the singularity needs to be integrated, the operator $\hat{\mathcal{L}}^\mu[\cdot]$ is well-defined for merely bounded piecewise constant functions $\bar{U}(\cdot)$.

Next, in order to use the nonlocal operator (4.2) for numerical approximations, let us discretize it: we take the average value of (4.2) on each cell R_α, and write

$$\hat{\mathcal{L}}^\mu\langle A(U)\rangle_\alpha = \frac{1}{\Delta x^d} \int_{R_\alpha} \hat{\mathcal{L}}^\mu[A(\bar{U}(\cdot))](x) \, \mathrm{d}x. \tag{4.3}$$

Moreover, let us write the finite difference operator $\hat{D}^l_{\Delta x}$ in (4.1), $l = 1, \dots, d$, as

$$\hat{D}^l_{\Delta x}\bar{U}(x) = \sum_{\beta\in\mathbb{Z}^d} D_l U_\beta \, \mathbf{1}_{R_\beta}(x)$$

where D_l is either a forward (D^+) or a backward (D^-) difference (we will define this precisely in a short while). With this notation at hand, let us now manipulate the right-hand side of (4.3) to obtain

$$
\begin{aligned}
\Delta x^d \hat{\mathcal{L}}^\mu \langle A(U)\rangle_\alpha &= \int_{R_\alpha} \hat{\mathcal{L}}^\mu[A(\bar{U}(\cdot))](x)\ \mathrm{d}x \\
&= \int_{R_\alpha}\int_{|z|>\frac{\Delta x}{2}} A(\bar{U}(x+z)) - A(\bar{U}(x))\ \mathrm{d}\mu(z)\,\mathrm{d}x \\
&\qquad\qquad - \sum_{l=1}^{d} \gamma_l^{\mu,\frac{\Delta x}{2}} \int_{R_\alpha} \sum_{\beta\in\mathbb{Z}^d} D_l A(U_\beta)\,\mathbf{1}_{R_\beta}(x)\ \mathrm{d}x \\
&= \int_{R_\alpha}\int_{|z|>\frac{\Delta x}{2}} A(\bar{U}(x+z)) - A(\bar{U}(x))\ \mathrm{d}\mu(z)\,\mathrm{d}x \\
&\qquad\qquad + \sum_{l=1}^{d} \gamma_l^{\mu,\frac{\Delta x}{2}} \int_{R_\alpha} \sum_{\beta\in\mathbb{Z}^d} A(U_\beta)\, D_l\mathbf{1}_{R_\beta}(x)\ \mathrm{d}x \\
&= \sum_{\beta\in\mathbb{Z}^d} \underbrace{\left(\int_{R_\alpha}\int_{|z|>\frac{\Delta x}{2}} \mathbf{1}_{R_\beta}(x+z) - \mathbf{1}_{R_\beta}(x)\ \mathrm{d}\mu(z)\,\mathrm{d}x\right)}_{G_{\alpha,\beta}} A(U_\beta) \\
&\qquad\qquad + \sum_{\beta\in\mathbb{Z}^d} \underbrace{\left(\sum_{l=1}^{d} \gamma_l^{\mu,\frac{\Delta x}{2}} \int_{R_\alpha} D_l\mathbf{1}_{R_\beta}(x)\ \mathrm{d}x\right)}_{G^{\alpha,\beta}} A(U_\beta).
\end{aligned}
$$

We now call $G_\beta^\alpha = G_{\alpha,\beta} + G^{\alpha,\beta}$ and, for each $\Delta x > 0$ and $(\alpha,\beta)\in\mathbb{Z}^d\times\mathbb{Z}^d$,

$$
\begin{aligned}
G_{\alpha,\beta} &= \int_{R_\alpha}\int_{|z|>\frac{\Delta x}{2}} \mathbf{1}_{R_\beta}(x+z) - \mathbf{1}_{R_\beta}(x)\ \mathrm{d}\mu(z)\,\mathrm{d}x, \\
G^{\alpha,\beta} &= \sum_{l=1}^{d} \gamma_l^{\mu,\frac{\Delta x}{2}} \int_{R_\alpha} D_l^\gamma \mathbf{1}_{R_\beta}(x)\ \mathrm{d}x,
\end{aligned}
\tag{4.4}
$$

where we have precisely choosen $D_l\mathbf{1}_{R_\beta}$ as

$$
D_l\mathbf{1}_{R_\beta} \equiv D_l^\gamma \mathbf{1}_{R_\beta} = \begin{cases} D_l^+\mathbf{1}_{R_\beta} := \dfrac{\mathbf{1}_{R_\beta+e_l} - \mathbf{1}_{R_\beta}}{\Delta x} & \text{if } \gamma_l^{\mu,\frac{\Delta x}{2}} > 0, \\ D_l^-\mathbf{1}_{R_\beta} := \dfrac{\mathbf{1}_{R_\beta} - \mathbf{1}_{R_\beta-e_l}}{\Delta x} & \text{otherwise.} \end{cases}
\tag{4.5}
$$

Here, as in what follows, we have denoted with e_l the d-dimensional vector with l-component 1 and 0 otherwise. To sum up, we have shown that

$$
\begin{aligned}
\hat{\mathcal{L}}^\mu \langle A(U)\rangle_\alpha &= \frac{1}{\Delta x^d}\int_{R_\alpha} \hat{\mathcal{L}}^\mu[A(\bar{U}(\cdot))](x)\ \mathrm{d}x \\
&= \frac{1}{\Delta x^d}\sum_{\beta\in\mathbb{Z}^d} G_\beta^\alpha\, A(U_\beta).
\end{aligned}
\tag{4.6}
$$

Relation (4.6) lies at the core of our analysis and will be very much used in the rest of the paper.

Remark 4.1.

- Note how the difference operator D_l^γ in (4.5) changes direction depending on the sing of the term

$$
\gamma_l^{\mu,\frac{\Delta x}{2}} = -\int_{|z|>\frac{\Delta x}{2}} z_l \mathbf{1}_{|z|\le 1}\ \mathrm{d}\mu(z).
$$

  This property is fundamental and is needed in the proof of Lemma 5.1.

- Note that $G_{\alpha,\beta}$ is a full matrix while $G^{\alpha,\beta}$ has non-zero entries only on the main diagonal and sub/super-diagonals. Furthermore, note that $G^{\alpha,\beta} = 0$ whenever the measure μ is symmetric.

4.2. **Definition of the numerical method.** In this paper we study both the implicit numerical method

$$U_\alpha^{n+1} = U_\alpha^n + \Delta t\, \hat{\mathcal{L}}^\mu \langle A(U^{n+1})\rangle_\alpha \tag{4.7}$$

and the explicit one

$$U_\alpha^{n+1} = U_\alpha^n + \Delta t\, \hat{\mathcal{L}}^\mu \langle A(U^{n})\rangle_\alpha \tag{4.8}$$

where, as derived in the previous pages, the discrete operator $\hat{\mathcal{L}}^\mu\langle\cdot\rangle_\alpha$ takes the form

$$\hat{\mathcal{L}}^\mu \langle A(U^n)\rangle_\alpha = \frac{1}{\Delta x^d} \sum_{\beta\in\mathbb{Z}} G_\beta^\alpha\, A(U_\beta^n) \tag{4.9}$$

with $G_\beta^\alpha = G_{\alpha,\beta} + G^{\alpha,\beta}$ as in (4.4).

In the rest of the paper we require the following CFL condition to be fulfilled for both the implicit method (4.7) and the explicit one (4.8),

$$4\, d\, L_A \left(\frac{\Delta t}{\Delta x^2} \int_{\frac{\Delta x}{2}<|z|\leq 1} |z|^2 \,\mathrm{d}\mu(z) + \frac{\Delta t}{\Delta x} \int_{|z|>1} \mathrm{d}\mu(z) \right) < \frac{1}{2}. \tag{4.10}$$

Moreover, we choose the initial condition for both methods as, for all $\alpha \in \mathbb{Z}^d$,

$$U_\alpha^0 = \frac{1}{\Delta x^d} \int_{R_\alpha} u_0(x)\ \mathrm{d}x.$$

Finally, let us extend the solutions of the implicit method (4.7) to each point (x,t) on the space/time strip Q_T by using the piecewise constant space/time interpolation

$$\bar{u}(x,t) = U_\alpha^{n+1} \quad \text{for all } (x,t) \in R_\alpha \times (t_n, t_{n+1}]. \tag{4.11}$$

On the other side, let us use the following space/time interpolation for the explicit method (4.8)

$$\bar{u}(x,t) = U_\alpha^{n} \quad \text{for all } (x,t) \in R_\alpha \times [t_n, t_{n+1}). \tag{4.12}$$

Remark 4.2. Let us note that, whenever the Lévy measure μ is chosen as in (1.4), the CFL condition (4.10) reduces to

$$c \left(\frac{\Delta t}{\Delta x^\lambda} + \frac{\Delta t}{\Delta x} \right) < \frac{1}{2} \tag{4.13}$$

where $\lambda \in (0,2)$ and

$$c = 4\, d\, L_A \left(\int_{|z|\leq 1} |z|^2 \,\mathrm{d}\mu(z) + \int_{|z|>1} \mathrm{d}\mu(z) \right).$$

5. Properties of the numerical method

5.1. **Properties of the discrete nonlocal operator.** We will now point out a few properties enjoyed by the weights G_β^α in (4.4) which will be used in what follows.

Lemma 5.1. *For all $\beta \in \mathbb{Z}^d$, we have that*

$$\sum_{\alpha\in\mathbb{Z}^d} G_\beta^\alpha = \sum_{\alpha\in\mathbb{Z}^d} G_\alpha^\beta = 0.$$

Furthermore, $G_\beta^\beta \leq 0$, $G_\beta^\alpha \geq 0$ whenever $\alpha \neq \beta$, and $G_\alpha^\beta = G_{\alpha+e_l}^{\beta+e_l}$ for all $\alpha, \beta \in \mathbb{Z}^d$ and $l = 1, \ldots, d$.

Proof. See Appendix A. □

Remark 5.2.

- To visualize the symmetry shared by the weights G_β^α in (4.4), let us restrict ourselves to the one dimensional case $d = 1$: loosely speaking, Lemma 5.1 says that the matrix G_β^α can be built by translating a vector each time by one position. Furthermore, note that G_β^α is symmetric whenever the Lévy measure μ is.
- Let us split both $G_{\alpha,\beta}$ and $G^{\alpha,\beta}$ as

$$\begin{aligned} G_{\alpha,\beta} &= G^r_{\alpha,\beta} + G_{\alpha,\beta,r}, \\ G^{\alpha,\beta} &= G^{\alpha,\beta,r} + G_r^{\alpha,\beta}. \end{aligned}$$

where $r > 0$,

$$\begin{aligned} G^r_{\alpha,\beta} &= \int_{R_\alpha} \int_{\frac{\Delta x}{2} < |z| \le r} \mathbf{1}_{R_\beta}(x+z) - \mathbf{1}_{R_\beta}(x)\ \mathrm{d}\mu(z)\,\mathrm{d}x, \\ G_{\alpha,\beta,r} &= \int_{R_\alpha} \int_{|z|>r} \mathbf{1}_{R_\beta}(x+z) - \mathbf{1}_{R_\beta}(x)\ \mathrm{d}\mu(z)\,\mathrm{d}x, \end{aligned}$$

and

$$\begin{aligned} G^{\alpha,\beta,r} &= \sum_{l=1}^d \gamma_{l,r}^{\mu,\frac{\Delta x}{2}} \int_{R_\alpha} D_l^{\gamma_r} \mathbf{1}_{R_\beta}(x)\ \mathrm{d}x, \\ G_r^{\alpha,\beta} &= \sum_{l=1}^d \gamma_l^{\mu,r} \int_{R_\alpha} D_l^{\gamma^r} \mathbf{1}_{R_\beta}(x)\ \mathrm{d}x. \end{aligned}$$

Here, for all $l = 1, \dots, d$, we have introduced the shorthand notation

$$\gamma_{l,r}^{\mu,\frac{\Delta x}{2}} = -\int_{\frac{\Delta x}{2} < |z| \le r} z_l \mathbf{1}_{|z|\le 1}\ \mathrm{d}\mu(z).$$

Finally, let us call

$$\begin{aligned} G_\alpha^{\beta,r} &= G^r_{\alpha,\beta} + G^{\alpha,\beta,r}, \\ G^\beta_{\alpha,r} &= G_{\alpha,\beta,r} + G_r^{\alpha,\beta}. \end{aligned}$$

The reader can easily check that, for any $r > 0$, all the properties listed in Lemma 5.1 for the weights G_α^β are also true for the weights $G_\alpha^{\beta,r}$ and $G^\beta_{\alpha,r}$.

Thanks to the results established in Lemma 5.1 we can now prove the following useful discrete Kato type of inequality for the discrete nonlocal operator (4.9).

Lemma 5.3. *(Discrete Kato inequality) Let* $\{u_\alpha, v_\alpha\}_{\alpha\in\mathbb{Z}^d}$ *be two bounded sequences. Then,*

$$\operatorname{sgn}(u_\alpha - v_\alpha) \sum_{\beta\in\mathbb{Z}^d} G_\beta^\alpha (A(u_\beta) - A(v_\beta)) \le \sum_{\beta\in\mathbb{Z}^d} G_\beta^\alpha\, |A(u_\beta) - A(v_\beta)|$$

Proof. The proof is an immediate consequence of the fact that $G_\alpha^\beta \geq 0$ whenever $\alpha \neq \beta$, while $\operatorname{sgn}(u)A(u) = |A(u)|$. Thus,

$$\begin{aligned}
&\operatorname{sgn}(u_\alpha - v_\alpha) \sum_{\beta\in\mathbb{Z}^d} G_\beta^\alpha (A(u_\beta) - A(v_\beta)) \\
&\quad = G_\alpha^\alpha \left|A(u_\alpha) - A(v_\alpha)\right| + \operatorname{sgn}(u_\alpha - v_\alpha) \sum_{\beta\neq\alpha} G_\beta^\alpha (A(u_\beta) - A(v_\beta)) \\
&\quad \leq G_\alpha^\alpha \left|A(u_\alpha) - A(v_\alpha)\right| + \sum_{\beta\neq\alpha} G_\beta^\alpha \left|A(u_\beta) - A(v_\beta)\right| \\
&\quad = \sum_{\beta\in\mathbb{Z}^d} G_\beta^\alpha \left|A(u_\beta) - A(v_\beta)\right|.
\end{aligned}$$

□

5.2. **Cell entropy inequalities.** We now prove discrete cell-entropy inequalities for both methods (4.7) and (4.8).

Theorem 5.4. (Cell-entropy inequalities)

- *Let $\bar{u}$ be a solution of the implicit method* (4.7)*. Then, for all $r > 0$,*

$$\begin{aligned}
\eta(U_\alpha^{n+1}, k) \leq \eta(U_\alpha^n, k) &+ \Delta t \sum_{\beta\in\mathbb{Z}^d} G_\beta^{\alpha,r}\, \eta(A(U_\beta^{n+1}), A(k)) \\
&+ \Delta t\, \eta'(U_\alpha^{n+1}, k) \sum_{\beta\in\mathbb{Z}^d} G_{\beta,r}^\alpha\, A(U_\beta^{n+1}).
\end{aligned} \tag{5.1}$$

- *Let $\bar{u}$ be a solution of the explicit method* (4.8)*. Then, for all $r > 0$,*

$$\begin{aligned}
\eta(U_\alpha^{n+1}, k) \leq \eta(U_\alpha^n, k) &+ \Delta t \sum_{\beta\in\mathbb{Z}^d} G_\beta^{\alpha,r}\, \eta(A(U_\alpha^n), A(k)) \\
&+ \Delta t\, \eta'(U_\alpha^{n+1}, k) \sum_{\beta\in\mathbb{Z}^d} G_{\beta,r}^\alpha\, A(U_\alpha^n).
\end{aligned} \tag{5.2}$$

Remark 5.5. The main difference between the implicit method (4.7) and the explicit one (4.8) is already evident in the cell-entropy inequalities (5.1) and (5.2). Indeed, in the cell-entropy inequality generated by the implicit method the sing term $\eta'(U_\alpha^{n+1}, k)$ is aligned in time with the associated nonlocal discrete operator $\sum_{\beta\in\mathbb{Z}^d} G_{\beta,r}^\alpha\, A(U_\beta^{n+1})$. This is however not true for the cell-entropy inequality generated by the explicit method. Such a difference is at the root of of the discrepancy between the two methods' convergence rates as we will see in Section 7.1.

In the following remark, we only briefly sketch how to prove convergence for both methods (4.7) and (4.8) starting from the cell-entropy inequalities (5.1) and (5.2). We will not go into details here since this result is an immediate consequence of the framework for error estimates which we will establish in Section 7.

Remark 5.6. The cell entropy inequalities (5.1) and (5.2) allow to establish convergence for both methods (4.7) and (4.8) in a standard fashion: multiply both sides of either (5.1) or (5.2) by a piecewise constant approximation $\bar{\varphi} = \varphi_\alpha^n$ of the test function $\varphi \in C_c^\infty(\mathbb{R}^d \times [0,T])$, sum over all α and n, and move the respective operators onto $\bar{\varphi}$ by either summations by parts (for the local operators) or change of variables (for the nonlocal one). All this will return the follow inequality

$$\begin{aligned}\iint_{Q_T} \eta(\bar u,k)\, D_{\Delta t}\bar\varphi + \eta(A(\bar u),A(k))\Big(\hat{\mathcal{L}}_r^{\mu^*}[\bar\varphi] + \hat D_{\Delta x}\bar\varphi\cdot\gamma^{\mu^*,r}\Big)&\\ + \eta'(\bar u,k)\,\mathcal{L}^{\mu,r}[A(\bar u)]\,\bar\varphi\;\mathrm{d}x\,\mathrm{d}t&\\ -\int_{\mathbb{R}^d}\eta(\bar u(x,T),k)\,\bar\varphi(x,T)\;\mathrm{d}x + \int_{\mathbb{R}^d}\eta(u_0(x),k)\,\bar\varphi(x,0)\;\mathrm{d}x \ge 0&\end{aligned}\tag{5.3}$$

for the implicit method (4.7) and

$$\begin{aligned}\iint_{Q_T} \eta(\bar u,k)\, D_{\Delta t}\bar\varphi + \eta(A(\bar u),A(k))\Big(\hat{\mathcal{L}}_r^{\mu^*}[\bar\varphi] + \hat D_{\Delta x}\bar\varphi\cdot\gamma^{\mu^*,r}\Big)&\\ + \eta'(\bar u(\cdot,t+\Delta t),k)\,\mathcal{L}^{\mu,r}[A(\bar u)]\,\bar\varphi\;\mathrm{d}x\,\mathrm{d}t&\\ -\int_{\mathbb{R}^d}\eta(\bar u(x,T),k)\,\bar\varphi(x,T)\;\mathrm{d}x + \int_{\mathbb{R}^d}\eta(u_0(x),k)\,\bar\varphi(x,0)\;\mathrm{d}x \ge 0&\end{aligned}\tag{5.4}$$

for the explicit method (4.8). At this point one could proceed as done, for example, in [6] to show that both inequalities (5.3) and (5.4) reduce to the original entropy inequality (2.1) as the discretization gets finer.

Proof of Theorem 5.4 for the implicit method (4.7). First, let us note that, for all $k\in\mathbb{R}$,

$$\begin{aligned}U_\alpha^{n+1}\vee k &\le U_\alpha^n\vee k + \Delta t\,\mathbf{1}_{(k,+\infty)}(U_\alpha^{n+1})\,\hat{\mathcal{L}}^\mu\langle A(U^{n+1})\rangle_\alpha,\\ U_\alpha^{n+1}\wedge k &\ge U_\alpha^n\wedge k + \Delta t\,\mathbf{1}_{(-\infty,k)}(U_\alpha^{n+1})\,\hat{\mathcal{L}}^\mu\langle A(U^{n+1})\rangle_\alpha.\end{aligned}$$

Remember the shorthand $\eta(u,k)=|u-k|$ and $\eta'(u,k)=\mathrm{sgn}\,(u-k)$. With this notation at hand the two inequalities above can be subtracted to yield

$$\eta(U_\alpha^{n+1},k)\le\eta(U_\alpha^n,k)+\Delta t\,\eta'(U_\alpha^{n+1},k)\,\hat{\mathcal{L}}^\mu\langle A(U^{n+1})\rangle_\alpha.\tag{5.5}$$

Next, for any $r>0$, we split the weights G_β^α into $G_\beta^{\alpha,r}$ and $G_{\beta,r}^\alpha$ - cf. Remark 5.2. Moreover, we remember that $A(\cdot)$ is non-decreasing. Then,

$$\eta'(U_\alpha^{n+1},k)(A(U_\alpha^{n+1})-A(k))=\eta(A(U_\alpha^{n+1}),A(k))$$

and, thanks to Lemma 5.3,

$$\begin{aligned}&\eta'(U_\alpha^{n+1},k)\sum_{\beta\in\mathbb{Z}^d}G_\beta^{\alpha,r}\,A(U_\beta^{n+1})\\ &\quad=\eta'(U_\alpha^{n+1},k)\sum_{\beta\in\mathbb{Z}^d}G_\beta^{\alpha,r}\,(A(U_\beta^{n+1})-A(k))\qquad\Bigg(\text{since }\sum_{\beta\in\mathbb{Z}^d}G_\beta^{\alpha,r}=0\Bigg)\\ &\quad\le\sum_{\beta\in\mathbb{Z}^d}G_\beta^{\alpha,r}\,\eta(A(U_\beta^{n+1}),A(k)).\end{aligned}\tag{5.6}$$

Thus, expression (5.5) returns the cell entropy inequality (5.2) via (5.6). □

Proof of Theorem 7.1 for the explicit method (4.8). Thanks to monotonicity we obtain the following inequalities: for all $r>0$,

$$\begin{aligned}U_\alpha^{n+1}\vee k\le U_\alpha^n\vee k+\Delta t\sum_{\beta\in\mathbb{Z}^d}G_\beta^{\alpha,r}\,A(U_\alpha^n\vee k)&\\ +\Delta t\,\mathbf{1}_{(k,+\infty)}(U_\alpha^{n+1})\sum_{\beta\in\mathbb{Z}^d}G_{\beta,r}^\alpha\,A(U_\alpha^n)&\end{aligned}\tag{5.7}$$

and

$$
\begin{aligned}
U_\alpha^{n+1} \wedge k \geq U_\alpha^n \wedge k &+ \Delta t \sum_{\beta\in\mathbb{Z}^d} G_\beta^{\alpha,r}\, A(U_\alpha^n \wedge k) \\
&+ \Delta t\, \mathbf{1}_{(-\infty,k)}(U_\alpha^{n+1}) \sum_{\beta\in\mathbb{Z}^d} G_{\beta,r}^{\alpha}\, A(U_\alpha^n).
\end{aligned} \tag{5.8}
$$

Note that, since $\eta(A(U),A(k)) = A(U\vee k) - A(U\wedge k)$, inequalities (5.7) and (5.8) can be subtracted to yield the cell entropy inequality (5.2). □

6. A priori estimates and compactness

6.1. **Regularity in space.** We prove the following result:

Lemma 6.1. (Regularity in space) *Let $\bar{u}$ be a solution of the implicit method* (4.7) *or the explicit method* (4.8)*. Then, for all $t>0$,*

$$
\|\bar{u}(\cdot,t)\|_{L^1(\mathbb{R}^d)} \leq \|u_0\|_{L^1(\mathbb{R}^d)}, \tag{6.1}
$$

$$
\|\bar{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)}, \tag{6.2}
$$

$$
|\bar{u}(\cdot,t)|_{BV(\mathbb{R}^d)} \leq |u_0|_{BV(\mathbb{R}^d)}. \tag{6.3}
$$

Proof of Lemma 6.1 for the implicit method (4.7)*.* For brevity let us rename $u_\alpha = U_\alpha^{n+1}$ and $h_\alpha = U_\alpha^n$, and rewrite method (4.7) as

$$
u_\alpha - \frac{\Delta t}{\Delta x^d} \sum_{\beta\in\mathbb{Z}^d} G_\beta^\alpha\, A(u_\beta) = h_\alpha. \tag{6.4}
$$

Moreover, let us introduce the operator

$$
T_\alpha[u] = u_\alpha - \epsilon \left(u_\alpha - \frac{\Delta t}{\Delta x^d} \sum_{\beta\in\mathbb{Z}^d} G_\beta^\alpha\, A(u_\beta) - h_\alpha \right), \tag{6.5}
$$

where ϵ is such that

$$
\epsilon \left\{ 1 + 4\, d\, L_A \left(\frac{\Delta t}{\Delta x^2}\right) \left(\int_{\frac{\Delta x}{2}<|z|\leq 1} |z|^2 \,\mathrm{d}\mu(z) + \int_{|z|>1} \mathrm{d}\mu(z) \right) \right\} < 1, \tag{6.6}
$$

$$
3\,(1-\epsilon) < 1. \tag{6.7}
$$

We can readily check that system (6.6)–(6.7) admits a solution. Indeed, while (6.7) implies that $\epsilon > \frac{2}{3}$, inequality (6.6) implies that

$$
\epsilon < \left\{ 1 + 4\, d\, L_A \left(\frac{\Delta t}{\Delta x^2}\right) \left(\int_{\frac{\Delta x}{2}<|z|\leq 1} |z|^2 \,\mathrm{d}\mu(z) + \int_{|z|>1} \mathrm{d}\mu(z) \right) \right\}^{-1} < 1,
$$

where

$$
\left\{ 1 + 4\, d\, L_A \left(\frac{\Delta t}{\Delta x^2}\right) \left(\int_{\frac{\Delta x}{2}<|z|\leq 1} |z|^2 \,\mathrm{d}\mu(z) + \int_{|z|>1} \mathrm{d}\mu(z) \right) \right\}^{-1} > \frac{2}{3}
$$

thanks to the CFL condition (4.10).

First, let us show that the operator (6.5) is monotone; that is to say, $T_\alpha[u] \geq T_\alpha[v]$ for all $\alpha\in\mathbb{Z}^d$ whenever $u\geq v$. To see this note that, since $G_\beta^\alpha \geq 0$ whenever $\alpha\neq\beta$ - cf. Lemma 5.1,

$$
\partial_{u_\beta} T_\alpha[u] \geq 0 \qquad \text{for all } \beta\neq\alpha,
$$

while, since $A(\cdot)$ non-decreasing and $G_\alpha^\alpha \leq 0$,

$$\begin{aligned}\partial_{u_\alpha} T_\alpha[u] &= 1-\epsilon+\frac{\epsilon\,\Delta t}{\Delta x^d}\,G_\alpha^\alpha\,A'(u_\alpha)\\ &\geq 1-\epsilon+\frac{\epsilon\,L_A\,\Delta t}{\Delta x^d}\,G_\alpha^\alpha.\end{aligned}$$

Moreover (cf. (A.1) and (A.2)),

$$\begin{aligned}G_\alpha^\alpha &= \int_{R_\alpha}\int_{|z|>\frac{\Delta x}{2}} \mathbf{1}_{R_\alpha}(x+z)-1\ \mathrm{d}\mu(z)\ \mathrm{d}x-\Delta x^{d-1}\sum_{l=1}^d \left|\gamma_l^{\mu,\frac{\Delta x}{2}}\right|\\ &\geq -\Delta x^d\int_{|z|>\frac{\Delta x}{2}}\mathrm{d}\mu(z)\ \mathrm{d}x-\Delta x^{d-1}\sum_{l=1}^d\left|\gamma_l^{\mu,\frac{\Delta x}{2}}\right|.\end{aligned}$$

Thus,

$$\partial_{u_\alpha}T_\alpha[u]\geq 1-\epsilon-\epsilon\,L_A\,\Delta t\int_{|z|>\frac{\Delta x}{2}}\mathrm{d}\mu(z)-\epsilon\,L_A\,\frac{\Delta t}{\Delta x}\sum_{l=1}^d\left|\gamma_l^{\mu,\frac{\Delta x}{2}}\right|,$$

where

$$\begin{aligned}\int_{|z|>\frac{\Delta x}{2}}\mathrm{d}\mu(z) &\leq \left(\int_{\frac{\Delta x}{2}<|z|\leq 1}+\int_{|z|>1}\right)\mathrm{d}\mu(z)\\ &\leq \frac{4}{\Delta x^2}\left(\int_{\frac{\Delta x}{2}<|z|\leq 1}|z|^2\,\mathrm{d}\mu(z)+\int_{|z|>1}\mathrm{d}\mu(z)\right)\end{aligned}$$

and

$$\begin{aligned}\sum_{l=1}^d\left|\gamma_l^{\mu,\frac{\Delta x}{2}}\right| &\leq \sum_{l=1}^d\int_{\frac{\Delta x}{2}<|z|\leq 1}|z_l|\,\mathrm{d}\mu(z)\\ &\leq d\int_{\frac{\Delta x}{2}<|z|\leq 1}|z|\,\mathrm{d}\mu(z)\\ &\leq \frac{2\,d}{\Delta x}\int_{\frac{\Delta x}{2}<|z|\leq 1}|z|^2\,\mathrm{d}\mu(z).\end{aligned}$$

Therefore,

$$\partial_{u_\alpha}T_\alpha[u]\geq 1-\epsilon-4\,\epsilon\,d\,L_A\left(\frac{\Delta t}{\Delta x^2}\right)\left(\int_{\frac{\Delta x}{2}<|z|\leq 1}|z|^2\,\mathrm{d}\mu(z)+\int_{|z|>1}\mathrm{d}\mu(z)\right)$$

which is positive due to our choice of ϵ in (6.6).

Thanks to the monotonicity of the operator (6.5) we can now use Banach's fixed point theorem. To this end, let us take the difference

$$T_\alpha[u]-T_\alpha[v]=(1-\epsilon)\,(u_\alpha-v_\alpha)+\frac{\epsilon\,\Delta t}{\Delta x^d}\sum_{\beta\in\mathbb{Z}^d}G_\beta^\alpha\,(A(u_\beta)-A(v_\beta)), \tag{6.8}$$

and assume that $u\geq v$. Thus, since $T[\cdot]$ is monotone and $A(\cdot)$ is non-decreasing,

$$|T_\alpha[u]-T_\alpha[v]|=(1-\epsilon)\,|u_\alpha-v_\alpha|+\frac{\epsilon\,\Delta t}{\Delta x^d}\sum_{\beta\in\mathbb{Z}^d}G_\beta^\alpha\,|A(u_\beta)-A(v_\beta)|$$

which, thanks to Fubini's theorem and the fact that $\sum_{\alpha\in\mathbb{Z}^d}G_\beta^\alpha=0$, returns that

$$\sum_{\alpha\in\mathbb{Z}^d}|T_\alpha[u]-T_\alpha[v]|\leq \underbrace{(1-\epsilon)}_{<1\text{ by (6.6)}}\sum_{\alpha\in\mathbb{Z}^d}|u_\alpha-v_\alpha|. \tag{6.9}$$

To treat the case u and v general we now use the triangle inequality to control the difference

$$\sum_{\alpha\in\mathbb{Z}^d} |T_\alpha[u] - T_\alpha[v]| \le \sum_{\alpha\in\mathbb{Z}^d} |T_\alpha[u] - T_\alpha[u\vee v]| + \sum_{\alpha\in\mathbb{Z}^d} |T_\alpha[u\vee v] - T_\alpha[u\wedge v]| + \sum_{\alpha\in\mathbb{Z}^d} |T_\alpha[u\wedge v] - T_\alpha[v]|.$$

Note that, thanks to what has just been proven in (6.9) for the case $u \ge v$,

$$\sum_{\alpha\in\mathbb{Z}^d} |T_\alpha[u] - T_\alpha[u\vee v]| \le (1-\epsilon) \sum_{\alpha\in\mathbb{Z}^d} |u_\alpha - (u\vee v)_\alpha|,$$
$$\sum_{\alpha\in\mathbb{Z}^d} |T_\alpha[u\vee v] - T_\alpha[u\wedge v]| \le (1-\epsilon) \sum_{\alpha\in\mathbb{Z}^d} |(u\vee v)_\alpha - (u\wedge v)_\alpha|,$$
$$\sum_{\alpha\in\mathbb{Z}^d} |T_\alpha[v] - T_\alpha[u\wedge v]| \le (1-\epsilon) \sum_{\alpha\in\mathbb{Z}^d} |v_\alpha - (u\wedge v)_\alpha|.$$

Thus, we obtain that

$$\sum_{\alpha\in\mathbb{Z}^d} |T_\alpha[u] - T_\alpha[v]| \le 3\,(1-\epsilon) \sum_{\alpha\in\mathbb{Z}^d} |(u\vee v)_\alpha - (u\wedge v)_\alpha| = \underbrace{3\,(1-\epsilon)}_{<1 \text{ by } (6.7)} \sum_{\alpha\in\mathbb{Z}^d} |u_\alpha - v_\alpha|.$$

At this point an application of Banach's fixed point theorem returns the existence of a unique solution of (6.4): there must be a (unique) $\bar{u}$ such that

$$T_\alpha[\bar{u}] = \bar{u}_\alpha \qquad \text{for all } \alpha\in\mathbb{Z}^d$$

or, equivalently,

$$\bar{u}_\alpha - \frac{\Delta t}{\Delta x^d} \sum_{\beta\in\mathbb{Z}^d} G^\alpha_\beta\, A(\bar{u}_\beta) - h_\alpha = 0.$$

Let us now prove (6.1) and (6.3). To this end, let us multiply both sides of (6.4) by $\operatorname{sgn}(u_\alpha)$, and use Lemma 5.3 to get

$$|u_\alpha| - \frac{\Delta t}{\Delta x^d} \sum_{\beta\in\mathbb{Z}^d} G^\alpha_\beta\, |A(u_\beta)| \le |h_\alpha|,$$

which, thanks to Fubini's theorem and the fact that $\sum_{\alpha\in\mathbb{Z}^d} G^\alpha_\beta = 0$, implies that

$$\sum_{\alpha\in\mathbb{Z}^d} |u_\alpha| \le \sum_{\alpha\in\mathbb{Z}^d} |h_\alpha|. \tag{6.10}$$

Next, let us use the fact that $G^\beta_\alpha = G^{\beta+e_l}_{\alpha+e_l}$ for all $\alpha, \beta \in \mathbb{Z}^d$ and $l = 1, \ldots, d$, to rewrite the difference

$$u_\alpha - u_{\alpha-e_l} - \frac{\Delta t}{\Delta x} \sum_{\beta\in\mathbb{Z}^d} G^\alpha_\beta\, A(u_\beta) - G^{\alpha-e_l}_\beta\, A(u_\beta) = h_\alpha - h_{\alpha-e_l}$$

into

$$u_\alpha - u_{\alpha-e_l} - \frac{\Delta t}{\Delta x} \sum_{\beta\in\mathbb{Z}^d} G^\alpha_\beta\, \left(A(u_\beta) - A(u_{\beta-e_l})\right) = h_\alpha - h_{\alpha-e_l}. \tag{6.11}$$

Thanks to Lemma 5.3, if we now multiply both sides of (6.11) by $\operatorname{sgn}(u_\alpha - u_{\alpha-e_l})$ and sum over all $\alpha\in\mathbb{Z}^d$, we end up with

$$\sum_{\alpha\in\mathbb{Z}^d} |u_\alpha - u_{\alpha-e_l}| \le \sum_{\alpha\in\mathbb{Z}^d} |h_\alpha - h_{\alpha-e_l}|$$

which returns (6.3) (the total variation in several dimensions is given by the sum of the total variation along each dimension, cf. [11, Appendix A] for details).

To conclude, it remains to prove (6.2). First, let us note that $\sup_{\alpha\in\mathbb{Z}^d} u_\alpha < \infty$ thanks to (6.10). Next, let us assume that α_k is a sequence such that $\lim_{k\to\infty} u_{\alpha_k} = \sup_{\alpha\in\mathbb{Z}^d} u_\alpha$. Then, by going to the limit $k\to\infty$ on both sides of (6.4),

$$\sup_{\alpha\in\mathbb{Z}^d} u_\alpha \leq \lim_{k\to\infty}\left(u_{\alpha_k} - \frac{\Delta t}{\Delta x^d}\sum_{\beta\in\mathbb{Z}^d} G_\beta^{\alpha_k} A(u_\beta)\right) = \lim_{k\to\infty} h_{\alpha_k} \leq \sup_{\alpha\in\mathbb{Z}^d} h_\alpha. \tag{6.12}$$

The first inequality in (6.12) has to do with the fact that, since $\sum_{\beta\in\mathbb{Z}^d} G_\beta^\alpha = 0$,

$$\sum_{\beta\in\mathbb{Z}^d} G_\beta^{\alpha_k} A(u_\beta) = \sum_{\beta\in\mathbb{Z}^d} G_\beta^{\alpha_k}\left(A(u_\beta) - A(u_{\alpha_k})\right)$$

and thus

$$\begin{aligned}&\lim_{k\to\infty}\sum_{\beta\in\mathbb{Z}^d} G_\beta^{\alpha_k}\left(A(u_\beta) - A(u_{\alpha_k})\right)\\ &= \sum_{\beta\in\mathbb{Z}^d}\left(\lim_{k\to\infty} G_\beta^{\alpha_k}\right)\underbrace{\left(\lim_{k\to\infty}\left(A(u_\beta) - A(u_{\alpha_k})\right)\right)}_{\leq 0} \leq 0.\end{aligned} \tag{6.13}$$

Note that here we have used Lebesgue's dominated convergence theorem to take the limit inside the sum. This is justified since $\sum_{\beta\in\mathbb{Z}^d}|G_\beta^\alpha| < \infty$ for each $\Delta x > 0$. Let us also note that $\lim_{k\to\infty} G_\beta^{\alpha_k}$ is always greater or equal to zero with the only exception

$$\lim_{k\to\infty} G_\beta^{\alpha_k} = G_\beta^\beta \leq 0.$$

However, in such a situation

$$\lim_{k\to\infty}\left(A(u_\beta) - A(u_{\alpha_k}\right) = 0,$$

and inequality (6.13) is trivial. Proceeding similarly to what done in (6.12) one derives the inequality for the infimum, thus

$$\inf_{\alpha\in\mathbb{Z}^d} h_\alpha \leq \inf_{\alpha\in\mathbb{Z}^d} u_\alpha \leq \sup_{\alpha\in\mathbb{Z}^d} u_\alpha \leq \sup_{\alpha\in\mathbb{Z}^d} h_\alpha.$$

□

Proof of Theorem 6.1 for the explicit method (4.8). The statement is a consequence of the fact that the explicit method (4.8) is both conservative and monotone.

Conservative: Let us sum both sides of (4.8) over all $\alpha\in\mathbb{Z}^d$ to get

$$\begin{aligned}\sum_{\alpha\in\mathbb{Z}^d} U_\alpha^{n+1} &= \sum_{\alpha\in\mathbb{Z}^d} U_\alpha^n + \frac{1}{\Delta x^d}\sum_{\alpha\in\mathbb{Z}^d}\sum_{\beta\in\mathbb{Z}^d} G_\beta^\alpha A(U_\beta)\\ &= \sum_{\alpha\in\mathbb{Z}^d} U_\alpha^n + \frac{1}{\Delta x^d}\sum_{\beta\in\mathbb{Z}^d} A(U_\beta)\underbrace{\left(\sum_{\alpha\in\mathbb{Z}^d} G_\beta^\alpha\right)}_{=0,\text{ cf. Lemma 5.1}} = \sum_{\alpha\in\mathbb{Z}^d} U_\alpha^n.\end{aligned}$$

Monotone: We proceed as done in the proof of Lemma 6.1 for the implicit method (4.7). Let us rename the right-hand side of (4.8) as

$$T_\alpha[u] = u_\alpha + \frac{\Delta t}{\Delta x^d}\sum_{\beta\in\mathbb{Z}^d} G_\beta^\alpha A(u_\beta).$$

First, note that, since $G^\alpha_\beta \geq 0$ whenever $\alpha \neq \beta$ - cf. Lemma 5.1,

$$\partial_{u_\beta} T_\alpha[u] \geq 0 \qquad \text{for all } \beta \neq \alpha,$$

while, since $A(\cdot)$ non-decreasing and $G^\alpha_\alpha \leq 0$,

$$\begin{aligned}
\partial_{u_\alpha} T_\alpha[u] &= 1 + \frac{\Delta t}{\Delta x^d}\, G^\alpha_\alpha\, A'(u_\alpha) \\
&\geq 1 + \frac{L_A\,\Delta t}{\Delta x^d}\, G^\alpha_\alpha \\
&\geq 1 - 4\,d\,L_A \left(\frac{\Delta t}{\Delta x^2}\right)\left(\int_{\frac{\Delta x}{2}<|z|\leq 1} |z|^2\,\mathrm{d}\mu(z) + \int_{|z|>1} \mathrm{d}\mu(z)\right)
\end{aligned}$$

which is positive thanks to the CFL condition (4.10). □

6.2. **Regularity in time.** We prove the following result:

Lemma 6.2. (Regularity in time) *Let $\bar u$ be a solution of the implicit method* (4.7) *or the explicit method* (4.8). *Then, for all $s,t>0$*

$$\|\bar u(\cdot,s) - \bar u(\cdot,t)\|_{L^1(\mathbb{R}^d)} \leq \begin{cases} |s-t| + \Delta t & \textit{if } \int_{|z|>0} |z| \wedge 1\ \mathrm{d}\mu(z) < \infty, \\ \sqrt{|s-t| + \Delta t} & \textit{if } \int_{|z|>0} |z|^2 \wedge 1\ \mathrm{d}\mu(z) < \infty. \end{cases}$$

Proof of Lemma 6.2. The proof is the same for both the implicit method (4.7) and the explicit one (4.8). Let us consider the explicit method and write

$$\Delta x^d \sum_{\alpha\in\mathbb{Z}^d} \left|U^{n+1}_\alpha - U^n_\alpha\right| \leq \Delta t \sum_{\alpha\in\mathbb{Z}^d} \left| \sum_{\beta\in\mathbb{Z}^d} G^\alpha_\beta\, A(U^n_\beta)\right|. \tag{6.14}$$

Note that, thanks to (4.6),

$$\begin{aligned}
&\sum_{\alpha\in\mathbb{Z}^d} \left| \sum_{\beta\in\mathbb{Z}^d} G^\alpha_\beta\, A(U_\beta)\right| \\
&\leq \int_{\mathbb{R}^d}\int_{|z|>\frac{\Delta x}{2}} |A(\bar U(x+z)) - A(\bar U(x))| + |z\cdot \hat D A(\bar U(x))|\,\mathbf{1}_{|z|\leq 1}\ \mathrm{d}\mu(z) \\
&\leq 2\,L_A \left(|\bar U|_{BV(\mathbb{R}^d)} \int_{\frac{\Delta x}{2}<|z|\leq 1} |z|\,\mathrm{d}\mu(z) + \|\bar U\|_{L^1(\mathbb{R}^d)} \int_{|z|>1} \mathrm{d}\mu(z)\right) \qquad (6.15) \\
&\leq \frac{4\,L_A}{\Delta x} \left(|\bar U|_{BV(\mathbb{R}^d)} \int_{\frac{\Delta x}{2}<|z|\leq 1} |z|^2\,\mathrm{d}\mu(z) + \|\bar U\|_{L^1(\mathbb{R}^d)} \int_{|z|>1} \mathrm{d}\mu(z)\right).
\end{aligned}$$

Therefore, thanks to Lemma 6.1,

$$\begin{aligned}
&\Delta x^d \sum_{\alpha\in\mathbb{Z}^d} \left|U^{n+1}_\alpha - U^n_\alpha\right| \\
&\leq \frac{4\,L_A\,\Delta t}{\Delta x} \left(|u_0|_{BV(\mathbb{R}^d)} \int_{\frac{\Delta x}{2}<|z|\leq 1} |z|^2\,\mathrm{d}\mu(z) + \|u_0\|_{L^1(\mathbb{R}^d)} \int_{|z|>1} \mathrm{d}\mu(z)\right)
\end{aligned} \tag{6.16}$$

which is $O(\sqrt{\Delta t})$ thanks to the CFL condition (4.10).

Finally, let us note that whenever the measure μ is such that $\int_{|z|>0} |z| \wedge 1\ \mathrm{d}\mu(z)$ is finite, inequality (6.15) guarantees that

$$\Delta x^d \sum_{\alpha\in\mathbb{Z}^d} \left|U^{n+1}_\alpha - U^n_\alpha\right| \leq c\,\Delta t.$$

□

Let us now point out that the time regularity proved in Lemma 6.2 can be further refined for fractional measures (1.4).

Lemma 6.3. (Refined time regularity for fractional measures) *Let $\sigma_\lambda(\cdot)$ be the following modulus of continuity*

$$\sigma_\lambda(\tau) = \begin{cases} \tau & \lambda < 1, \\ \tau \ln \tau & \lambda = 1, \\ \tau^{\frac{1}{\lambda}} & \lambda > 1. \end{cases} \tag{6.17}$$

Let $\bar{u}$ be a solution of the implicit method (4.7) *or the explicit one* (4.8) *with measure μ as in* (1.4)*. Then, for all $s, t > 0$*

$$\|\bar{u}(\cdot, s) - \bar{u}(\cdot, t)\|_{L^1(\mathbb{R}^d)} \leq \sigma_\lambda(|s-t| + \Delta t).$$

Proof of Lemma 6.3. As shown in the proof of Lemma 6.2 - cf. (6.16),

$$\begin{aligned} &\|\bar{u}(\cdot, t_{n+1}) - \bar{u}(\cdot, t_n)\|_{L^1(\mathbb{R}^d)} \\ &\leq 2\, L_A\, \Delta t \left(|u_0|_{BV(\mathbb{R}^d)} \int_{\frac{\Delta x}{2} < |z| \leq 1} |z|\, \mathrm{d}\mu(z) + \|u_0\|_{L^1(\mathbb{R}^d)} \int_{|z|>1} \mathrm{d}\mu(z) \right). \end{aligned}$$

Now, since

$$\int_{\frac{\Delta x}{2} < |z| \leq 1} |z|\, \mathrm{d}\mu(z) \leq \int_{\frac{\Delta x}{2} < |z| \leq 1} \frac{|z|}{|z|^{d+\lambda}}\, \mathrm{d}z = \begin{cases} O(1) & \text{if } \lambda \in (0,1), \\ O(|\ln \Delta x|) & \text{if } \lambda = 1, \\ O(\Delta x^{1-\lambda}) & \text{if } \lambda \in (1,2), \end{cases}$$

the conclusion follows thanks to the CFL condition (4.13): for example, for $\lambda > 1$ $\Delta t = O(\Delta x^\lambda)$, and thus $O(\Delta t\, \Delta x^{1-\lambda}) = O(\Delta x) = O(\Delta t^{\frac{1}{\lambda}})$. □

6.3. **Compactness.** The a priori space/time estimates in Lemma 6.1 and 6.2 along with Kolmogorov's compactness theorem - cf. e.g. [11, Theorem 3.8] - yield convergence (up to a subsequence) toward a limit u for both methods (4.7) and (4.8). Furthermore, the limit u inherits all such a priori estimates. In short, we have the following result:

Theorem 6.4. (Compactness) *Let $\bar{u}$ be a solution of the implicit method* (4.7) *or the explicit one* (4.8)*. Then, the sequence $\{\bar{u} : \Delta x > 0\}$ converges in $C([0,T]; L^1(\mathbb{R}^d))$ as $\Delta x \to 0$ (up to a subsequence) toward a limit u such that*

$$u \in L^\infty(Q_T) \cap C([0,T]; L^1(\mathbb{R}^d)) \cap L^\infty(0,T; BV(\mathbb{R}^d)).$$

7. Framework for error estimates

We now choose v in Lemma 3.1 as the numerical solution $\bar{u}$ of either the implicit method (4.7) or the explicit method (4.8) to prove the following result (whose proof will be given in a following subsection):

Theorem 7.1. (Framework for error estimates) *Let u be the entropy solution of* (1.1)*. Then,*

(i) if $\bar{u}$ is the solution of the implicit method (4.7) *we have that, for all $\epsilon > 0$ $0 < \delta < T$ and $\frac{\Delta x}{2} < r \leq 1$,*

$$\|u(\cdot, T) - \bar{u}(\cdot, T)\|_{L^1(\mathbb{R}^d)} \leq C_T \Big(\epsilon + \mathcal{E}_\delta(u) \vee \mathcal{E}_\delta(\bar{u}) + I_1^{\epsilon, r} + I_2^{\epsilon, \delta, r} \Big), \tag{7.1}$$

where

$$I_1^{\epsilon,r} = \frac{1}{\epsilon}\int_{|z|\le r} |z|^2 \,\mathrm{d}\mu(z),$$

$$I_2^{\epsilon,\delta,r} = \left(\frac{\Delta x}{\epsilon} + \frac{\Delta t}{\delta}\right)\left(\int_{r<|z|\le 1} |z| \,\mathrm{d}\mu(z) + \int_{|z|>1} \mathrm{d}\mu(z)\right);$$

(ii) *if $\bar{u}$ is the solution of the explicit method* (4.8) *we have that, for all $\epsilon > 0$ $0 < \delta < T$ and $\frac{\Delta x}{2} < r \le 1$,*

$$\|u(\cdot,T) - \bar{u}(\cdot,T)\|_{L^1(\mathbb{R}^d)} \le C_T\Big(\epsilon + \mathcal{E}_\delta(u) \vee \mathcal{E}_\delta(\bar{u}) + I_1^{\epsilon,r} + I_2^{\epsilon,\delta,r} + I_3^r\Big), \tag{7.2}$$

where

$$I_3^r = \mathcal{E}_{\Delta t}(\bar{u}) \int_{|z|>r} \mathrm{d}\mu(z).$$

Remark 7.2. Convergence for both methods (4.7) and (4.8) follows as a by-product of Theorem 7.1, and existence of entropy solutions of the initial value problem (1.1) is so established. Let us, for instance, look closer at the implicit method (4.7).

i) Case $\int_{|z|>0} |z| \,\mathrm{d}\mu(z) < \infty$.

Convergence is immediate: just send $r \to 0$ in (7.1) and choose $\epsilon = \sqrt{\Delta x}$.

ii) Case $\int_{|z|>0} |z|^2 \,\mathrm{d}\mu(z) < \infty$.

The right-hand side of (7.1) still vanishes as $r, \epsilon \to 0$ whenever $r = r(\Delta x)$ and $\epsilon = \epsilon(\Delta x)$ are suitably chosen by using the explicit form of the Lévy measures μ under study.

It is clear that such a convergence proof yields more than convergence itself: it also produces an explicit rate of convergence. As an example, we will work out the details for the case of fractional measures (1.3) in what follows.

7.1. **Convergence rate for fractional measures.** With the time regularity granted by Lemma 6.3 and the framework developed in Theorem 7.1 we can now prove the following result:

Theorem 7.3. (Convergence rate for fractional measures) *Let $\sigma_\lambda^{IM}(\cdot)$ and $\sigma_\lambda^{EX}(\cdot)$ be the moduli of continuity*

$$\sigma_\lambda^{IM}(\tau) = \begin{cases} \tau^{\frac{1}{2}} & \lambda \in (0,1), \\ \tau^{\frac{1}{2}}\log(\tau) & \lambda = 1, \\ \tau^{\frac{2-\lambda}{2}} & \lambda \in (1,2), \end{cases} \tag{7.3}$$

and

$$\sigma_\lambda^{EX}(\tau) = \begin{cases} \tau^{\frac{1}{2}} & \lambda \in \left(0, \frac{2}{3}\right], \\ \tau^{\frac{4-3\lambda}{4}} & \lambda \in \left(\frac{2}{3}, 1\right), \\ \tau^{\frac{1}{4}}\log(\tau) & \lambda = 1, \\ \tau^{\frac{2-\lambda}{4}} & \lambda \in (1,2). \end{cases} \tag{7.4}$$

Let u be the unique entropy solution of the initial value problem (1.1) *with measure μ as in* (1.4). *Then, for all $\lambda \in (0,2)$,*

$$\|u(\cdot,T) - \bar{u}(\cdot,T)\|_{L^1(\mathbb{R}^d)} \le \begin{cases} C_T\,\sigma_\lambda^{IM}(\Delta x) & \textit{for the implicit method } (4.7), \\ \\ C_T\,\sigma_\lambda^{EX}(\Delta x) & \textit{for the explicit method } (4.8). \end{cases}$$

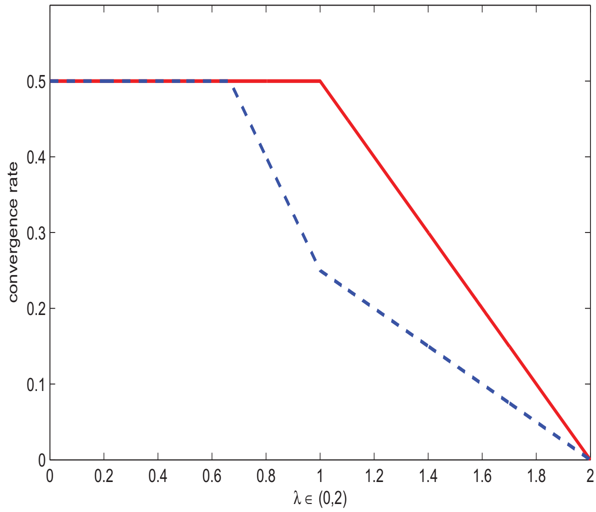

FIGURE 1. The implicit method convergence rate (7.3) (solid) and the explicit method one (7.4) (dashed) as λ varies in $(0, 2)$.

Remark 7.4. Given that we are using the same CFL condition (4.13) for both methods, it is not surprising to see that the convergence rate for the implicit method (4.7) is higher than the rate for the explicit one (4.8). We have included a snapshot of both rates (7.3) and (7.4) as λ varies in $(0, 2)$ in FIGURE 1.

Proof of Theorem 7.3. Let us first give the proof for the implicit method (4.7). First of all, let us note that in the present setting

$$\int_{|z|\le r} |z|^2 \, \mathrm{d}\mu(z) \le \int_{|z|\le r} \frac{|z|^2}{|z|^{d+\lambda}} \, \mathrm{d}z \le O\left(r^{2-\lambda}\right) \quad \text{for all } \lambda \in (0, 2)$$

while

$$\int_{r<|z|\le 1} |z| \, \mathrm{d}\mu(z) \le \int_{r<|z|\le 1} \frac{|z|}{|z|^{d+\lambda}} \, \mathrm{d}z = \begin{cases} O(1) & \text{if } \lambda \in (0, 1), \\ O(|\ln r|) & \text{if } \lambda = 1, \\ O\left(r^{1-\lambda}\right) & \text{if } \lambda \in (1, 2). \end{cases}$$

Therefore, whenever fractional measures are considered, expression (7.1) takes the form (here as in the rest of the proof we use the CFL condition (4.13) and replace the time step Δt with its respective space step $\min\{\Delta x, \Delta x^{\lambda}\}$)

$$\begin{aligned} &\|u(\cdot, T) - \bar{u}(\cdot, T)\|_{L^1(\mathbb{R}^d)} \\ &\le \begin{cases} C_T\left(\epsilon + \delta + \frac{r^{2-\lambda}}{\epsilon} + \left(\frac{\Delta x}{\epsilon} + \frac{\Delta x}{\delta}\right)\right) & \text{if } \lambda \in (0, 1), \\ C_T\left(\epsilon + \delta\,|\ln \delta| + \frac{r}{\epsilon} + |\ln r|\left(\frac{\Delta x}{\epsilon} + \frac{\Delta x}{\delta}\right)\right) & \text{if } \lambda = 1, \\ C_T\left(\epsilon + \delta^{\frac{1}{\lambda}} + \frac{r^{2-\lambda}}{\epsilon} + r^{1-\lambda}\left(\frac{\Delta x}{\epsilon} + \frac{\Delta x^{\lambda}}{\delta}\right)\right) & \text{if } \lambda \in (1, 2). \end{cases} \end{aligned} \tag{7.5}$$

The conclusion follows by choosing $r = \Delta x$ for all $\lambda \in (0, 2)$, $\epsilon = \delta = \sqrt{\Delta x}$ for $\lambda \in (0, 1]$, while $\epsilon = \Delta x^{\frac{2-\lambda}{2}}$ and $\delta = \Delta x^{\frac{\lambda}{2}}$ for $\lambda \in (1, 2)$.

On the other hand, for the explicit method (4.8) the right-hand side of (7.5) must be augmented by the error stemming from the term I_3^r in (7.2),

$$I_3^r = \sigma_\lambda(\Delta t) \underbrace{\int_{|z|>r} \mathrm{d}\mu(z)}_{O(r^{-\lambda})}.$$

In this setting, expression (7.2) takes the form

$$\|u(\cdot,T)-\bar{u}(\cdot,T)\|_{L^1(\mathbb{R}^d)} \leq \begin{cases} C_T\left(\epsilon+\delta+\frac{r^{2-\lambda}}{\epsilon}+\left(\frac{\Delta x}{\epsilon}+\frac{\Delta x}{\delta}\right)+\frac{\Delta x}{r^\lambda}\right) & \text{if } \lambda\in(0,1),\\ C_T\left(\epsilon+\delta\,|\ln\delta|+\frac{r}{\epsilon}+|\ln r|\left(\frac{\Delta x}{\epsilon}+\frac{\Delta x}{\delta}\right)+\frac{\Delta x\,|\ln(\Delta x)|}{r}\right) & \text{if } \lambda=1,\\ C_T\left(\epsilon+\delta^{\frac{1}{\lambda}}+\frac{r^{2-\lambda}}{\epsilon}+r^{1-\lambda}\left(\frac{\Delta x}{\epsilon}+\frac{\Delta x^\lambda}{\delta}\right)+\frac{\Delta x}{r^\lambda}\right) & \text{if } \lambda\in(1,2).\end{cases}$$

The conclusion follows by choosing $\epsilon=\delta=\sqrt{\Delta x}$ for $\lambda\in(0,1]$ and $\epsilon=\Delta x^{\frac{2-\lambda}{4}}$ and $\delta=\Delta x^{\frac{\lambda}{2}}$, while $r=\Delta x$ for $\lambda\in(0,\frac{1}{2}]$, $r=\Delta x^{\frac{3}{4}}$ for $\lambda\in(\frac{1}{2},1]$, and $r=\sqrt{\Delta x}$ for $\lambda\in(1,2)$. □

Remark 7.5. One could wonder if it is possible to gain speed of convergence by redefining our numerical method in such a way to cut-off the singularity at different speeds depending on the $\lambda\in(0,2)$ under consideration. In other words, one could think that instead of

$$G_{\alpha,\beta}=\int_{R_\alpha}\int_{|z|>\frac{\Delta x}{2}} \mathbf{1}_{R_\beta}(x+z)-\mathbf{1}_{R_\beta}(x)\;\mathrm{d}\mu(z)\,\mathrm{d}x$$

it would be better to use something like

$$G_{\alpha,\beta}=\int_{R_\alpha}\int_{|z|>\rho_\lambda(\Delta x)} \mathbf{1}_{R_\beta}(x+z)-\mathbf{1}_{R_\beta}(x)\;\mathrm{d}\mu(z)\,\mathrm{d}x$$

and choose the function $\rho_\lambda(\cdot)$ in order to maximize the expression

$$\epsilon+\delta+\underbrace{\frac{\rho_\lambda^{2-\lambda}(\Delta x)}{\epsilon}}_{:=A}+\underbrace{\rho_\lambda^{1-\lambda}(\Delta x)\left(\frac{\Delta x}{\epsilon}+\frac{\Delta x}{\delta}\right)}_{:=B}. \tag{7.6}$$

However, this leads to nowhere, and the best results are achieved by choosing $\rho_\lambda(\Delta x)=\frac{\Delta x}{2}$ - that is to say, a speed independent of the chosen $\lambda\in(0,2)$. To see this, note that the speed of convergence gained by squeezing the error term due to the singularity (the term A in (7.6)) results in a loss of speed of convergence in the error term due to the nonlocal nature of the operator (1.2) (the term B in (7.6)).

Remark 7.6.

i) The authors have implemented the explicit method (4.8) with measure μ as in (1.3) for different nonlinear functions $A(\cdot)$, different initial conditions $u_0(\cdot)$, and (most importantly) different $\lambda\in(0,2)$, but have found no significant variations in the (numerical) rate of convergence, which seems to be at least one-half and, needless to say, at best one (independently of the λ chosen). This fact openly contrasts with the authors' view that the convergence rates established in Theorem 7.3 are optimal, but does not refute it: that could still be some pathological $A(\cdot)$ and $u_0(\cdot)$ for which the numerical rates in Theorem 7.3 are actually fulfilled. However, it his in the authors' opinion that the connection between the nonlocal method (4.8) and the one implemented (which must be necessarily bounded, i.e. confined to a bounded subset of the space/time strip Q_T) is far from trivial. The authors think that, by reducing the nonlocal method (4.8) to a local one for numerical implementation, spurious diffusion (which improves speed of convergence) is artificially introduced.

ii) The solution of the local method we have implemented is fundamentally different from the original entropy solution of the nonlocal initial value problem (1.1). In theory, the original entropy solution of (1.1) should be retrieved by progressively increasing the size of our local numerical grid. This is like saying that there are two errors here that must be taken into account: a local error due to convergence toward an intermediate solution on a bounded domain, and a nonlocal error due to convergence toward the original entropy solution of (1.1) as the domain's volume increases. Following this explanation, the convergence rates in our experiments are incomplete: indeed, we do believe it is the additional nonlocal error that causes the global speed of convergence to decrease to the levels established in Theorem (7.3).

7.2. **Proof of Theorem 7.1.**

Proof of Theorem 7.1 for the implicit method (4.7). Let us use integration by parts on each interval (t_n, t_{n+1}) to rewrite (we refer the reader to [7] for the complete computation)

$$\begin{aligned}
(7.7)\quad & -\iint_{Q_T}\iint_{Q_T} \eta(\bar{u}(x,t),u(y,s))\,\partial_t\varphi^{\epsilon,\delta}(x,y,t,s) + \text{initial and final terms} \\
&= \iint_{Q_T} \sum_{n=0}^{N-1}\sum_{i\in\mathbb{Z}^d}\Big(\eta(U_i^{n+1},u(y,s)) - \eta(U_i^n,u(y,s))\Big)\int_{R_\alpha}\varphi^{\epsilon,\delta}(x,y,t_{n+1},s)\,\mathrm{d}x.
\end{aligned}$$

Next, let us introduce the piecewise constant function $\bar{\varphi}^{\epsilon,\delta} = \bar{\varphi}^{\epsilon,\delta}(x,y,t,s)$ which, for each $(y,s)\in Q_T$, is built from the values φ_α^n,

$$\varphi_\alpha^n = \frac{1}{\Delta x^d}\int_{R_\alpha}\varphi^{\epsilon,\delta}(x,y,t_n,s)\ \mathrm{d}x,$$

using the space/time interpolation (4.11). With the function $\bar{\varphi}^{\epsilon,\delta}$ at hand we can plug the cell entropy inequality (5.1) into (7.7), and use (3.2) to obtain that

$$\begin{aligned}
(7.8)\quad & \|u(\cdot,T)-\bar{u}(\cdot,T)\|_{L^1(\mathbb{R}^d)} \le C_T\left(\Delta x + \epsilon + \mathcal{E}_\delta(u)\vee\mathcal{E}_\delta(v)\right) \\
&+ \underbrace{\iint_{Q_T}\iint_{Q_T}\eta(A(\bar{u}(x,t)),A(u(y,s)))\,\mathcal{L}_r^{\mu^*}[\varphi^{\epsilon,\delta}(x,\cdot,t,s)](y)\ \mathrm{d}w}_{H_1} \\
&+ \underbrace{\iint_{Q_T}\iint_{Q_T}\eta(A(\bar{u}(x,t)),A(u(y,s)))\,\hat{\mathcal{L}}_r^{\mu^*}[\bar{\varphi}^{\epsilon,\delta}(\cdot,y,t,s)](x)\ \mathrm{d}w}_{H_2} \\
&+ \underbrace{\iint_{Q_T}\iint_{Q_T}\eta'(\bar{u}(x,t),u(y,s))\,\mathcal{L}^{\mu,r}[A(\bar{u}(\cdot,t))](x)\,(\bar{\varphi}^{\epsilon,\delta}-\varphi^{\epsilon,\delta})(x,y,t,s)\ \mathrm{d}w}_{H_3} \\
&+ \underbrace{\iint_{Q_T}\iint_{Q_T}\eta(A(\bar{u}(x,t)),A(u(y,s)))\,\gamma^{\mu^*,r}\cdot(\hat{D}\bar{\varphi}^{\epsilon,\delta}-\nabla_x\varphi^{\epsilon,\delta})(x,y,t,s)\ \mathrm{d}w}_{H_4}.
\end{aligned}$$

Here let us stress the fact that the discrete operator $\hat{D}_l$ is always applied onto the x-variable (the variable sitting inside the piecewise constant numerical solution $\bar{u}$). To complete the proof we need to estimate each integral H_i $(i = 1, \dots, 4)$ in (7.8).

Estimate of H_1. First, let us prove that

$$
\begin{aligned}
|H_1| &\leq C_T \, \|A'\|_{L^\infty(\mathbb{R})} \, |u_0|_{BV(\mathbb{R}^d)} \, |\varphi^{\epsilon,\delta}|_{BV(\mathbb{R}^d)} \int_{|z|\leq r} |z|^2 \, \mathrm{d}\mu(z) \\
&\leq C_T \, \|A'\|_{L^\infty(\mathbb{R})} \, |u_0|_{BV(\mathbb{R}^d)} \, \epsilon^{-1} \int_{|z|\leq r} |z|^2 \, \mathrm{d}\mu(z).
\end{aligned}
\tag{7.9}
$$

Indeed, using Taylor's formula with integral remainder and integration by parts (we refer the reader to the detailed computation in [2], Lemma B.1),

$$
\begin{aligned}
|H_1| \leq & \iint_{Q_T} \iint_{Q_T} \int_{|z|\leq r} \int_0^1 (1-\tau) \left(\sum_{i=1}^d |\partial_{y_i} \eta(A(\bar{u}(x,t)), A(u(y,s)))| \right) \\
& \qquad \omega_\delta(t-s) \left(\sum_{i=1}^d \underbrace{|\partial_{y_i} \Omega_\epsilon(x-y+\tau z)|}_{=|\partial_{x_i} \Omega_\epsilon(x-y+\tau z)|} \right) |z|^2 \, \mathrm{d}\tau \, \mathrm{d}\mu(z) \, \mathrm{d}w \\
\leq & \, L_A \left(\int_0^T \sum_{i=1}^d |u(\cdot,s)|_{BV(\mathbb{R}^i)} \, \mathrm{d}y \, \mathrm{d}s \right) \\
& \qquad \left(\int_{\mathbb{R}^d} \sum_{i=1}^d |\partial_{x_i} \Omega_\epsilon(x)| \, \mathrm{d}x \, \mathrm{d}t \right) \int_{|z|\leq r} |z|^2 \, \mathrm{d}\mu(z)
\end{aligned}
$$

which returns (7.9) thanks to the fact that the entropy solution u of (1.1) is of bounded variation.

Estimate of H_2. Next, let us show that an identical estimate can be produced for H_2. To see this, let $\bar{\varphi}^{\epsilon,\delta}_\varrho$ be a mollification in the x-variable of $\bar{\varphi}^{\epsilon,\delta}$, and note that

$$
|\bar{\varphi}^{\epsilon,\delta}_\varrho(\cdot,y,t,s)|_{BV(\mathbb{R}^d)} \leq |\bar{\varphi}^{\epsilon,\delta}(\cdot,y,t,s)|_{BV(\mathbb{R}^d)} \leq |\varphi^{\epsilon,\delta}(\cdot,y,t,s)|_{BV(\mathbb{R}^d)} = O\left(\epsilon^{-1}\right),
$$

where the first inequality holds for all ϱ small enough - cf. [16, Theorem 5.3.1] - while the second one is obvious. Let us call

$$
H_2^\varrho = \iint_{Q_T} \iint_{Q_T} \eta(A(\bar{u}(x,t)), A(u(y,s))) \, \hat{\mathcal{L}}_r^{\mu^*}[\bar{\varphi}^{\epsilon,\delta}_\varrho(\cdot,y,t,s)](x) \, \mathrm{d}w.
$$

First, let us point out that $\lim_{\varrho\to 0} H_2^\varrho = H_2$. To see this, let us note that, since we are integrating away from the singularity, we can move the limit $\varrho \to 0$ inside the integral sing (by Lebesgue's dominated convergence), and use the pointwise convergence of $\bar{\varphi}^{\epsilon,\delta}_\varrho(\cdot,y,t,s)$ to $\bar{\varphi}^{\epsilon,\delta}(\cdot,y,t,s)$. Now, since $\bar{\varphi}^{\epsilon,\delta}_\varrho(\cdot,y,t,s)$ is differentiable, we can repeat the argument used for H_1, and obtain that, for all $\varrho > 0$

$$
\begin{aligned}
|H_2^\varrho| &\leq C_T \, \|A'\|_{L^\infty(\mathbb{R})} \, |u_0|_{BV(\mathbb{R}^d)} \, |\bar{\varphi}^{\epsilon,\delta}_\varrho|_{BV(\mathbb{R}^d)} \int_{\frac{\Delta x}{2}<|z|\leq r} |z|^2 \, \mathrm{d}\mu(z) \\
&\leq C_T \, \|A'\|_{L^\infty(\mathbb{R})} \, |u_0|_{BV(\mathbb{R}^d)} \, |\varphi^{\epsilon,\delta}|_{BV(\mathbb{R}^d)} \int_{\frac{\Delta x}{2}<|z|\leq r} |z|^2 \, \mathrm{d}\mu(z) \\
&\leq C_T \, \|A'\|_{L^\infty(\mathbb{R})} \, |u_0|_{BV(\mathbb{R}^d)} \, \epsilon^{-1} \int_{|z|\leq r} |z|^2 \, \mathrm{d}\mu(z).
\end{aligned}
\tag{7.10}
$$

Therefore, in the limit $\varrho \to 0$ inequality (7.10) reduces to

$$
|H_2| \leq C_T \, \|A'\|_{L^\infty(\mathbb{R})} \, |u_0|_{BV(\mathbb{R}^d)} \, \epsilon^{-1} \int_{|z|\leq r} |z|^2 \, \mathrm{d}\mu(z). \tag{7.11}
$$

Estimate of H_3. We now prove that

$$(7.12)\quad \begin{aligned}|H_3| \le C_T \, \|A'\|_{L^\infty(\mathbb{R})} \left(\frac{\Delta x}{\epsilon} + \frac{\Delta x}{\delta}\right) \\ \left(|u_0|_{BV(\mathbb{R}^d)} \int_{r<|z|\le 1} |z| \,\mathrm{d}\mu(z) + \|u_0\|_{L^1(\mathbb{R}^d)} \int_{|z|>1} \mathrm{d}\mu(z)\right).\end{aligned}$$

To this end, let us note that, as shown in [7] in the one-dimensional case, for each $(x,t) \in Q_T$ the following inequality holds

$$(7.13)\quad \iint_{Q_T} \left|\bar{\varphi}^{\epsilon,\delta}(x,y,t,s) - \varphi^{\epsilon,\delta}(x,y,t,s)\right| \,\mathrm{d}y\,\mathrm{d}s \le O\left(\frac{\Delta x}{\epsilon} + \frac{\Delta t}{\delta}\right).$$

To see this note that

$$\bar{\varphi}^{\epsilon,\delta}(x,y,t_n,s) = \bar{\Omega}_\epsilon(x-y)\,\omega_\delta(t_n - s) = \bar{\omega}_\epsilon(x_1 - y_1)\cdots\bar{\omega}_\epsilon(x_d - y_d)\,\omega_\delta(t_n - s),$$

where $\bar{\omega}_\epsilon(\cdot) : \mathbb{R} \to \mathbb{R}$ is the stepwise constant approximation of $\omega_\epsilon(\cdot) : \mathbb{R} \to \mathbb{R}$ built using the values

$$\omega_\epsilon^i = \frac{1}{\Delta x}\int_{x_i}^{x_{i+1}} \omega_\epsilon(x)\,\mathrm{d}x.$$

Note that (let us drop the ϵ for sake of brevity)

$$\begin{aligned}\int_{\mathbb{R}} |\bar{\omega}(x) - \omega(x)|\,\mathrm{d}x &= \sum_{i\in\mathbb{Z}} \int_{x_i}^{x_{i+1}} |\bar{\omega}(x) - \omega(x)|\,\mathrm{d}x \\ &= \frac{1}{\Delta x}\sum_{i\in\mathbb{Z}} \int_{x_i}^{x_{i+1}} \left|\int_{x_i}^{x_{i+1}} \omega(y)\,\mathrm{d}y - \omega(x)\right|\,\mathrm{d}x \\ &\le \frac{1}{\Delta x}\sum_{i\in\mathbb{Z}} \int_{x_i}^{x_{i+1}}\int_{x_i}^{x_{i+1}} |\omega(y) - \omega(x)|\,\mathrm{d}y\,\mathrm{d}x \\ &\le \Delta x \underbrace{\sum_{i\in\mathbb{Z}} |\omega|_{BV(x_i,x_{i+1})}}_{=|\omega|_{BV(\mathbb{R})}}.\end{aligned}$$

Now, since

$$(7.14)\quad \begin{aligned}&\iint_{Q_T} \left|\bar{\varphi}^{\epsilon,\delta}(x,y,t,s) - \varphi^{\epsilon,\delta}(x,y,t,s)\right| \,\mathrm{d}y\,\mathrm{d}s \\ &= \iint_{Q_T} \left|\bar{\Omega}_\epsilon(x-y)\,\omega_\delta(t_n - s) - \Omega_\epsilon(x-y)\,\omega_\delta(t-s)\right|\,\mathrm{d}y\,\mathrm{d}s\end{aligned}$$

(here t_n is such that $t \in (t_n, t_{n+1})$),

$$(7.15)\quad \int_{\mathbb{R}^d} \left|\bar{\Omega}_\epsilon(x-y) - \Omega_\epsilon(x-y)\right|\,\mathrm{d}y\,\mathrm{d}s \le \sum_{l=1}^{d} \underbrace{\int_{\mathbb{R}} |\bar{\omega}_\epsilon(x_l - y_l) - \omega_\epsilon(x_l - y_l)|\,\mathrm{d}y_l}_{=O(\Delta x\,\epsilon^{-1})}$$

and

$$(7.16)\quad \int_0^T \left|\omega_\delta(t_n - s) - \omega_\delta(t-s)\right|\,\mathrm{d}s = O(\Delta t\,\delta^{-1}),$$

splitting the integral on the right-hand side of (7.14) into (7.15) and (7.16) via triangle inequality returns (7.13). With inequality (7.13) at hand we can now see that, for all $\frac{\Delta x}{2} < r \leq 1$,

$$
\begin{aligned}
H_3 &\leq \iint_{Q_T} \left|\mathcal{L}^{\mu,r}[\bar{u}(\cdot,t)](x)\right| \left(\iint_{Q_T} \left|\bar{\varphi}^{\epsilon,\delta}(x,y,t,s) - \varphi^{\epsilon,\delta}(x,y,t,s)\right| \,\mathrm{d}y\,\mathrm{d}s \right) \mathrm{d}x\,\mathrm{d}t \\
&\leq c\, L_A \left(\frac{\Delta x}{\epsilon} + \frac{\Delta t}{\delta} \right) \Bigg(\iint_{Q_T} \int_{r<|z|\leq 1} |\bar{u}(x+z,t) - \bar{u}(x,t)| \,\mathrm{d}\mu(z)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\qquad + \iint_{Q_T} \int_{|z|>1} |\bar{u}(x+z,t) - \bar{u}(x,t)| \,\mathrm{d}\mu(z)\,\mathrm{d}x\,\mathrm{d}t \Bigg)
\end{aligned}
$$

which implies (7.12).

Estimate of H_4. Finally, we now prove that

$$
|H_4| \leq C_T \,\|A'\|_{L^\infty(\mathbb{R})} \left(\frac{\Delta x}{\epsilon} + \frac{\Delta t}{\delta} \right) \int_{r<|z|\leq 1} |z| \,\mathrm{d}\mu(z). \tag{7.17}
$$

To this end, let us choose $l \in (0,\dots,d)$ and write

(7.18)

$$
\begin{aligned}
H_{4,l} &= \gamma_l^{\mu^*,r} \iint_{Q_T} \iint_{Q_T} \eta(A(\bar{u}(x,t)), A(u(y,s)))\,(\hat{D}_l \bar{\varphi}^{\epsilon,\delta} - \partial_{x_l}\varphi^{\epsilon,\delta})(x,y,t,s)\,\mathrm{d}w \\
&= \gamma_l^{\mu^*,r} \underbrace{\iint_{Q_T} \sum_{\alpha\in\mathbb{Z}^d} \sum_{n=0}^{N-1} \eta(A(U_\alpha^n), A(u(y,s))) \int_{t_n}^{t_{n+1}} \int_{R_\alpha} \hat{D}_l \bar{\varphi}^{\epsilon,\delta}(x,y,t,s)\,\mathrm{d}w}_{H_{4,l}^1} \\
&\quad - \gamma_l^{\mu^*,r} \underbrace{\iint_{Q_T} \sum_{\alpha\in\mathbb{Z}^d} \sum_{n=0}^{N-1} \eta(A(U_\alpha^n), A(u(y,s))) \int_{t_n}^{t_{n+1}} \int_{R_\alpha} \partial_{x_l}\varphi^{\epsilon,\delta}(x,y,t,s)\,\mathrm{d}w}_{H_{4,l}^2}.
\end{aligned}
$$

Thanks to integration and summation by parts,

(7.19)

$$
\begin{aligned}
H_{4,l}^1 &= -\iint_{Q_T} \sum_{\alpha\in\mathbb{Z}^d} \sum_{n=0}^{N-1} \hat{D}_l \eta(A(U_\alpha^n), A(u(y,s))) \int_{t_n}^{t_{n+1}} \int_{R_\alpha} \bar{\varphi}^{\epsilon,\delta}(x,y,t,s)\,\mathrm{d}w \\
&= -\Delta t \Delta x^d \iint_{Q_T} \sum_{\alpha\in\mathbb{Z}^d} \sum_{n=0}^{N-1} \hat{D}_l \eta(A(U_\alpha^n), A(u(y,s)))\,\bar{\varphi}^{\epsilon,\delta}(x_\alpha, y, t_{n+1}, s)\,\mathrm{d}y\,\mathrm{d}s
\end{aligned}
$$

while

(7.20)

$$
\begin{aligned}
H_{4,l}^2 = -\Delta x &\iint_{Q_T} \sum_{\alpha\in\mathbb{Z}^d} \sum_{n=0}^{N-1} \hat{D}_l \eta(A(U_\alpha^n), A(u(y,s))) \\
&\int_{t_n}^{t_{n+1}} \int \cdots \int \varphi^{\epsilon,\delta}(x, x_{\alpha_l}, y, t, s)\,\mathrm{d}x_1 \dots \mathrm{d}x_{l-1}\,\mathrm{d}x_{l+1} \dots \mathrm{d}x_d\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s.
\end{aligned}
$$

Let us point out how we have used summation by parts in (7.20): first we have integrated the partial derivative $\partial_{x_l}\varphi^{\epsilon,\delta}(\cdot,y,t,s)$ along the interval $(x_{\alpha_l}, x_{\alpha_{l+1}})$ to obtain the difference $\varphi^{\epsilon,\delta}(x, x_{\alpha_{l+1}}, y, t, s) - \varphi^{\epsilon,\delta}(x, x_{\alpha_l}, y, t, s)$; then we have used summation by parts to move this difference onto $\eta(A(U_\alpha^n), A(u(y,s)))$. Note that we now write $\varphi^{\epsilon,\delta}(x, x_{\alpha_l}, y, t, s)$ to stress that $x_l = x_{\alpha_l}$ is fixed here while the other variables

$$
x_1, \dots, x_{l-1}, x_{l+1}, \dots, x_d
$$

still span what remains of R_α. For sake of brevity, from now on we just write

$$\mathrm{d}X_l \qquad \text{instead of} \qquad \mathrm{d}x_1 \dots \mathrm{d}x_{l-1}\,\mathrm{d}x_{l+1}\dots\mathrm{d}x_d.$$

Now, by plugging both (7.19) and (7.20) into (7.18) we have that

$$\begin{aligned}
H_{4,l} &= \gamma_l^{\mu^*,r} \iint_{Q_T} \sum_{\alpha\in\mathbb{Z}^d}\sum_{n=0}^{N-1} \hat{D}_l\eta(A(U_\alpha^n), A(u(y,s)))\\
&\Big(\Delta x \int_{t_n}^{t_{n+1}} \int\cdots\int \varphi^{\epsilon,\delta}(x, x_{\alpha_l}, y, t, s)\,\mathrm{d}X_l\,\mathrm{d}t - \Delta t \Delta x^d\,\bar{\varphi}^{\epsilon,\delta}(x_\alpha, y, t_{n+1}, s)\Big)\,\mathrm{d}y\,\mathrm{d}s\\
&= \gamma_l^{\mu^*,r} \iint_{Q_T} \sum_{\alpha\in\mathbb{Z}^d}\sum_{n=0}^{N-1} \Delta x\,\hat{D}_l\eta(A(U_\alpha^n), A(u(y,s)))\\
&\Big(\int_{t_n}^{t_{n+1}} \int\cdots\int \varphi^{\epsilon,\delta}(x, x_{\alpha_l}, y, t, s)\,\mathrm{d}X_l\,\mathrm{d}t - \frac{\Delta t}{\Delta x}\int_{R_\alpha} \varphi^{\epsilon,\delta}(x, y, t_{n+1}, s)\,\mathrm{d}x\Big)\,\mathrm{d}y\,\mathrm{d}s
\end{aligned}$$

which, by adding and subtracting $\varphi^{\epsilon,\delta}(x_\alpha, y, t_n, s)$, can be rewritten as

$$\begin{aligned}
H_{4,l} &\le \gamma_l^{\mu^*,r} \sum_{\alpha\in\mathbb{Z}^d}\sum_{n=0}^{N-1} \Delta x\,\hat{D}_l\eta(A(U_\alpha^n), A(u(y,s)))\\
&\underbrace{\iint_{Q_T}\Big(\int_{t_n}^{t_{n+1}} \int\cdots\int \varphi^{\epsilon,\delta}(x, x_{\alpha_l}, y, t, s) - \varphi^{\epsilon,\delta}(x_\alpha, y, t_n, s)\,\mathrm{d}X_l\,\mathrm{d}t\Big)\,\mathrm{d}y\,\mathrm{d}s}_{I_1}\\
&+ \gamma_l^{\mu^*,r} \sum_{\alpha\in\mathbb{Z}^d}\sum_{n=0}^{N-1} \Delta x\,\hat{D}_l\eta(A(U_\alpha^n), A(u(y,s)))\\
&\qquad\underbrace{\iint_{Q_T}\Big(\frac{\Delta t}{\Delta x}\int_{R_\alpha} \varphi^{\epsilon,\delta}(x_\alpha, y, t_n, s) - \varphi^{\epsilon,\delta}(x, y, t_{n+1}, s)\,\mathrm{d}x\Big)\,\mathrm{d}y\,\mathrm{d}s}_{I_2}\\
&+ \gamma_l^{\mu^*,r} \sum_{\alpha\in\mathbb{Z}^d}\sum_{n=0}^{N-1} \Delta x\,\hat{D}_l\eta(A(U_\alpha^n), A(u(y,s)))\\
&\underbrace{\iint_{Q_T}\Big(\int_{t_n}^{t_{n+1}} \int\cdots\int \varphi^{\epsilon,\delta}(x_\alpha, y, t_n, s)\,\mathrm{d}X_l\,\mathrm{d}t - \frac{\Delta t}{\Delta x}\int_{R_\alpha} \varphi^{\epsilon,\delta}(x_\alpha, y, t_n, s)\,\mathrm{d}x\Big)\,\mathrm{d}y\,\mathrm{d}s}_{I_3}
\end{aligned}$$

Next, we need to estimate I_i, $i = 1, 2, 3$. First, note that using the triangle inequality we can write

$$\begin{aligned}
I_1 &\le \iint_{Q_T}\int_{t_n}^{t_{n+1}} \int\cdots\int |\varphi^{\epsilon,\delta}(x, x_{\alpha_l}, y, t, s) - \varphi^{\epsilon,\delta}(x_\alpha, y, t_n, s)|\,\mathrm{d}X_l\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s\\
&\le \int_{t_n}^{t_{n+1}} \int\cdots\int \underbrace{\iint_{Q_T} |\varphi^{\epsilon,\delta}(x, x_{\alpha_l}, y, t, s) - \varphi^{\epsilon,\delta}(x, x_{\alpha_l}, y, t_n, s)|\,\mathrm{d}y\,\mathrm{d}s}_{=I_{1,1}}\,\mathrm{d}X_l\,\mathrm{d}t\\
&\quad + \int_{t_n}^{t_{n+1}} \int\cdots\int \underbrace{\iint_{Q_T} |\varphi^{\epsilon,\delta}(x, x_{\alpha_l}, y, t_n, s) - \varphi^{\epsilon,\delta}(x_\alpha, y, t_n, s)|\,\mathrm{d}y\,\mathrm{d}s}_{=I_{1,2}}\,\mathrm{d}X_l\,\mathrm{d}t,
\end{aligned}$$

where

$$I_{1,1,} = \iint_{Q_T} |\varphi^{\epsilon,\delta}(x, x_{\alpha_l}, y, t, s) - \varphi^{\epsilon,\delta}(x, x_{\alpha_l}, y, t_n, s)| \,\mathrm{d}y\,\mathrm{d}s = O\left(\frac{\Delta t}{\delta}\right)$$

$$I_{1,2} = \iint_{Q_T} |\varphi^{\epsilon,\delta}(x, x_{\alpha_l}, y, t_n, s) - \varphi^{\epsilon,\delta}(x_\alpha, y, t_n, s)| \,\mathrm{d}y\,\mathrm{d}s = O\left(\frac{\Delta x}{\epsilon}\right)$$

thanks to (7.15) and (7.16). Thus,

$$I_1 = \Delta t\, \Delta x^{d-1} O\left(\frac{\Delta t}{\delta} + \frac{\Delta x}{\epsilon}\right).$$

Second, let us look closer at I_2. Note that we can use the triangle inequality to write

$$\begin{aligned}
I_2 &= \frac{\Delta t}{\Delta x}\int_{R_\alpha}\iint_{Q_T} |\varphi^{\epsilon,\delta}(x_\alpha, y, t_n, s) - \varphi^{\epsilon,\delta}(x, y, t_{n+1}, s)| \,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x \\
&\le \frac{\Delta t}{\Delta x}\int_{R_\alpha}\underbrace{\iint_{Q_T} |\varphi^{\epsilon,\delta}(x_\alpha, y, t_n, s) - \varphi^{\epsilon,\delta}(x_\alpha, y, t_{n+1}, s)| \,\mathrm{d}y\,\mathrm{d}s}_{=I_{2,1}}\,\mathrm{d}x \\
&\quad + \frac{\Delta t}{\Delta x}\int_{R_\alpha}\underbrace{\iint_{Q_T} |\varphi^{\epsilon,\delta}(x_\alpha, y, t_{n+1}, s) - \varphi^{\epsilon,\delta}(x, y, t_{n+1}, s)| \,\mathrm{d}y\,\mathrm{d}s}_{I_{2,2}}\,\mathrm{d}x,
\end{aligned}$$

and so, proceeding for $(I_{2,1}, I_{2,2})$ as done for $(I_{1,1}, I_{1,2})$, we obtain that

$$I_2 = \Delta t\, \Delta x^{d-1} O\left(\frac{\Delta t}{\delta} + \frac{\Delta x}{\epsilon}\right).$$

Finally, note that

$$\begin{aligned}
I_3 &= \int_{t_n}^{t_{n+1}}\int\cdots\int\iint_{Q_T} \varphi^{\epsilon,\delta}(x_\alpha, y, t_n, s)\,\mathrm{d}X_l\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}s \\
&\qquad\qquad - \frac{\Delta t}{\Delta x}\int_{R_\alpha}\iint_{Q_T} \varphi^{\epsilon,\delta}(x_\alpha, y, t_n, s)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x \\
&= \left(\int_{t_n}^{t_{n+1}}\int\cdots\int \mathrm{d}X_l\,\mathrm{d}t - \frac{\Delta t}{\Delta x}\int_{R_\alpha}\mathrm{d}x\right)\underbrace{\iint_{Q_T} \varphi^{\epsilon,\delta}(x_\alpha, y, t_n, s)\,\mathrm{d}y\,\mathrm{d}s}_{=1} \\
&= \Delta t \Delta x^{d-1} - \Delta t \Delta x^{d-1} \\
&= 0
\end{aligned}$$

To sum up, we have so far proved that

$$H_4 \le d\,C\left(\frac{\Delta t}{\delta} + \frac{\Delta x}{\epsilon}\right)\left(\Delta t\,\Delta x^{d-1}\sum_{n=0}^{N-1}\sum_{\alpha\in\mathbb{Z}^d}\Delta x\,\hat{D}_l\eta(A(U_\alpha^n), A(u(y,s))\right)\gamma^{\mu^*,r},$$

where (cf. the definition of functions of bounded variation in [11] Appendix A)

$$\begin{aligned}
\Delta t\,\Delta x^{d-1}&\sum_{n=0}^{N-1}\sum_{\alpha\in\mathbb{Z}^d}\Delta x\,\hat{D}_l\eta(A(U_\alpha^n), A(u(y,s))) \\
&\qquad\qquad\le L_A\,T\,|\bar{u}(\cdot, t_n)|_{BV(\mathbb{R}^d)} \le L_A\,T\,|u_0|_{BV(\mathbb{R}^d)}.
\end{aligned}$$

Therefore, $H_{4,l} = O\left(\frac{\Delta t}{\delta} + \frac{\Delta x}{\epsilon}\right)\int_{r<|z|\le 1}|z|\,\mathrm{d}\mu(z)$ which proves (7.17). □

Proof of Theorem 7.1 for the explicit method (4.8). Let us plug the cell entropy inequality (5.2) into (7.7) to obtain that

$$
\begin{aligned}
(7.21)\quad & \|u(\cdot,T)-\bar u(\cdot,T)\|_{L^1(\mathbb{R}^d)} \le C_T\,(\Delta x+\epsilon+\mathcal{E}_\delta(u)\vee\mathcal{E}_\delta(v))\\
&+\iint_{Q_T}\iint_{Q_T}\eta(A(\bar u(x,t)),A(u(y,s)))\,\mathcal{L}_r^{\mu^*}[\varphi^{\epsilon,\delta}(x,\cdot,t,s)](y)\;\mathrm{d}w\\
&+\iint_{Q_T}\iint_{Q_T}\eta(A(\bar u(x,t)),A(u(y,s)))\,\hat{\mathcal{L}}_r^{\mu^*}[\bar\varphi^{\epsilon,\delta}(\cdot,y,t,s)](x)\;\mathrm{d}w\\
&+\iint_{Q_T}\iint_{Q_T}\eta'(\bar u(x,t+\Delta t),u(y,s))\,\mathcal{L}^{\mu,r}[A(\bar u(\cdot,t))](x)\,\bar\varphi^{\epsilon,\delta}(x,y,t,s)\;\mathrm{d}w\\
&-\iint_{Q_T}\iint_{Q_T}\eta'(\bar u(x,t),u(y,s))\,\mathcal{L}^{\mu,r}[A(\bar u(\cdot,t))](x)\,\varphi^{\epsilon,\delta}(x,y,t,s)\;\mathrm{d}w\\
&+\iint_{Q_T}\iint_{Q_T}\eta(A(\bar u(x,t)),A(u(y,s)))\,\gamma^{\mu^*,r}\cdot(\hat D\bar\varphi^{\epsilon,\delta}-\nabla_x\varphi^{\epsilon,\delta})(x,y,t,s)\;\mathrm{d}w
\end{aligned}
$$

where the new term

$$
\begin{aligned}
&\iint_{Q_T}\iint_{Q_T}\eta'(\bar u(x,t+\Delta t),u(y,s))\,\mathcal{L}^{\mu,r}[A(\bar u(\cdot,t))](x)\,\bar\varphi^{\epsilon,\delta}(x,y,t,s)\;\mathrm{d}w\\
&-\iint_{Q_T}\iint_{Q_T}\eta'(\bar u(x,t),u(y,s))\,\mathcal{L}^{\mu,r}[A(\bar u(\cdot,t))](x)\,\varphi^{\epsilon,\delta}(x,y,t,s)\;\mathrm{d}w\\
&\le\underbrace{\iint_{Q_T}\iint_{Q_T}\eta'(\bar u(x,t),u(y,s))\,\mathcal{L}^{\mu,r}[A(\bar u(\cdot,t-\Delta t))-A(\bar u(\cdot,t))](x)\,\bar\varphi^{\epsilon,\delta}(x,y,t,s)\;\mathrm{d}w}_{I}\\
&\quad+\iint_{Q_T}\iint_{Q_T}\eta'(\bar u(x,t),u(y,s))\,\mathcal{L}^{\mu,r}[A(\bar u(\cdot,t))](x)\,(\bar\varphi^{\epsilon,\delta}-\varphi^{\epsilon,\delta})(x,y,t,s)\;\mathrm{d}w\\
&\qquad\qquad\qquad\qquad\qquad\qquad+\text{ terms of order }\Delta t.
\end{aligned}
$$

The terms of order Δt here stems from the fact that, when moving the Δt-shift from $\eta'(\bar u(\cdot))$ into the nonlocal operator $\mathcal{L}^{\mu,r}[A(\bar u(\cdot))](\cdot)$, we need to shift the domain of integration accordingly from the interval $(0,T)$ to $(\Delta t,T+\Delta t)$. This produces two additional error terms of the form

$$\int_{(0,\Delta t)}\cdots\qquad,\qquad\int_{(T,T+\Delta t)}\cdots$$

which are clearly $O(\Delta t)$ due to the boundedness of the integrand.

Next, the time regularity of $\bar u$ is needed in order to estimate I,

$$
\begin{aligned}
I&\le\iint_{Q_T}|\mathcal{L}^{\mu,r}[A(\bar u(\cdot,t-\Delta t))-A(\bar u(\cdot,t))](x)|\underbrace{\iint_{Q_T}\bar\varphi^{\epsilon,\delta}(x,y,t,s)\;\mathrm{d}w}_{=O(1)}\\
&\le c\,2\,L_A\left(\int_0^T\|\bar u(\cdot,t-\Delta t)-\bar u(\cdot,t)\|_{L^1(\mathbb{R}^d)}\;\mathrm{d}t\right)\int_{|z|>r}\mathrm{d}\mu(z)\\
&\le C_T\,\mathcal{E}_{\Delta t}(\bar u)\int_{|z|>r}\mathrm{d}\mu(z).
\end{aligned}
$$

(check the notation $\mathcal{E}_{\Delta t}(\bar u)$ at (3.1)). Clearly, all the remaining terms in (7.16) can be estimated as done in the proof for the explicit method (4.8). ☐

7.3. **Generalization to convection-diffusion equations.** The results established in the previous sections can be extended to the case $f \neq 0$ in a standard fashion by considering the numerical methods

$$U_\alpha^{n+1} = U_\alpha^n + \Delta t \sum_{l=1}^{d} D_l^- \hat{f}(U_\alpha^n, U_{\alpha+e_l}^n) + \Delta t\, \hat{\mathcal{L}}^\mu \langle A(U^{n+1})\rangle_\alpha, \tag{7.22}$$

$$U_\alpha^{n+1} = U_\alpha^n + \Delta t \sum_{l=1}^{d} D_l^- \hat{f}(U_\alpha^n, U_{\alpha+e_l}^n) + \Delta t\, \hat{\mathcal{L}}^\mu \langle A(U^{n})\rangle_\alpha, \tag{7.23}$$

where

(i) D_l^- stands for the backward difference operator $D_l^- U_\alpha = \frac{1}{\Delta x}(U_\alpha - U_{\alpha-e_l})$, where e_l is the d-vector with l-component 1 and 0 otherwise;

(ii) $\hat{f}_l$ is a consistent - i.e., $\hat{f}(u,u) = f(u)$ - Lipschitz continuous numerical flux which is non-decreasing w.r.t. the first variable and non-increasing w.r.t. the second one - examples of such fluxes are the well-known Lax-Friedrichs' flux, the Godunov's flux, and the Engquist-Osher's flux, cf. e.g. [8].

Since the convection term is treated explicitly in both methods (7.22) and (7.23), the CFL condition (4.10) needs to be updated in this setting to

$$2\, L_F \left(\frac{\Delta t}{\Delta x}\right) + 4\, d\, L_A \left(\frac{\Delta t}{\Delta x^2}\right)\left(\int_{\frac{\Delta x}{2}<|z|\leq 1} |z|^2 \, \mathrm{d}\mu(z) + \int_{|z|>1} \mathrm{d}\mu(z)\right) < \frac{1}{2}, \tag{7.24}$$

where L_F is the Lipschitz constant of the numerical flux $\hat{f}$. With the CFL condition (7.24) at hand the a priori estimates in Section 6 continue to hold and compactness can still be established via Kolmogorov's theorem. Convergence toward the unique entropy solution of (1.1) is also standard (we refer the reader to [11], Chapter 3, for all the details here).

To conclude, let us point out how the statement of Theorem 7.1 changes in the current setting ($f \neq 0$):

Theorem 7.7. *The statement of Theorem 7.1 can be extended to cover both methods* (7.22) *and* (7.23) *if the error term* $I_2^{\epsilon,\delta,r}$ *therein is replaced by*

$$I_2^{\epsilon,\delta,r} = \left(\frac{\Delta x}{\epsilon} + \frac{\Delta t}{\delta}\right)\left(1 + \int_{r<|z|\leq 1} |z| \, \mathrm{d}\mu(z) + \int_{|z|>1} \mathrm{d}\mu(z)\right). \tag{7.25}$$

The reader can easily check that the convergence rates in Section 7.1 are not worsened by the new error term $I_2^{\epsilon,\delta,r}$ in (7.25).

Corollary 7.8. *The convergence rates in Theorem 7.3 are also valid for both methods* (7.22) *and* (7.23).

Proof of Theorem 7.7. This has to do with the fact that in the proof of Theorem 7.1 expression (7.7) changes to (cf., the computations in [11, Example 3.14] for more details)

(7.26)

$$\begin{aligned}
&- \iint_{Q_T} \iint_{Q_T} \eta(\bar{u}(x,t), u(y,s))\, \partial_t \varphi^{\epsilon,\delta}(x,y,t,s) + \text{initial and final terms} \\
&\qquad\qquad + q(\bar{u}(x,t), u(y,s)) \cdot \nabla_x \varphi^{\epsilon,\delta} \varphi(x,y,t,s) \, \mathrm{d}w \\
&= \iint_{Q_T} \sum_{n=0}^{N-1} \sum_{\alpha\in\mathbb{Z}^d} \Bigg((\eta(U_\alpha^{n+1}, u(y,s)) - \eta(U_\alpha^n, u(y,s))) \int_{R_\alpha} \varphi^{\epsilon,\delta}(x,y,t_{n+1},s)\, \mathrm{d}x \\
&\qquad + \sum_{l=1}^{d} (q(U_\alpha^n, u(y,s)) - q(U_{\alpha-e_l}^n, u(y,s))) \int_{t_n}^{t_{n+1}} \varphi^{\epsilon,\delta}(x_\alpha, y,t,s)\, \mathrm{d}t \Bigg) \mathrm{d}y \, \mathrm{d}s.
\end{aligned}$$

Thus, since in the present setting the cell entropy inequalities also include a convection term of the form

$$\eta(U_\alpha^{n+1},k) \leq \eta(U_\alpha^n,k) - \Delta t \sum_{l=1}^d D_l^- Q_l(U_\alpha^n,k) + \text{ remaining terms},$$

where $Q_l(U_\alpha^n,k) = F_l(U_\alpha^n \vee k, U_{\alpha+e_l}^n \vee k) - F_l(U_\alpha^n \wedge k, U_{\alpha+e_l}^n \wedge k)$, we obtain that

(7.27)

$$\begin{aligned}
&\|u(\cdot,T) - \bar{u}(\cdot,T)\|_{L^1(\mathbb{R}^d)} \leq \text{ remaining terms}\\
&+ \iint_{Q_T} \sum_{n=0}^{N-1} \sum_{\alpha\in\mathbb{Z}} \Bigg(\sum_{l=1}^d D_l^- q(U_\alpha^n, u(y,s)) \int_{t_n}^{t_{n+1}} \varphi^{\epsilon,\delta}(x_\alpha,y,t,s)\,\mathrm{d}t\\
&\qquad - \frac{\Delta t}{\Delta x} \sum_{l=1}^d D_l^- Q(U_\alpha^n, u(y,s)) \int_{R_\alpha} \varphi^{\epsilon,\delta}(x,y,t_{n+1},s)\,\mathrm{d}x \Bigg) \mathrm{d}y\,\mathrm{d}s\\
&\leq \text{ remaining terms} + C_T\left(\frac{\Delta x}{\epsilon} + \frac{\Delta t}{\delta}\right)
\end{aligned}$$

(cf. again [11, Example 3.14] for the proof of this well-known estimate). $\square$

Appendix A. Proof of Lemma 5.1

Using Fubini's theorem,

$$\begin{aligned}
\sum_{\alpha\in\mathbb{Z}^d} G_{\alpha,\beta} &= \int_{\mathbb{R}^d} \int_{|z|>\frac{\Delta x}{2}} \mathbf{1}_{R_\beta}(x+z) - \mathbf{1}_{R_\beta}(x)\ \mathrm{d}\mu(z)\,\mathrm{d}x\\
&= \int_{|z|>\frac{\Delta x}{2}} \int_{\mathbb{R}^d} \mathbf{1}_{R_\beta}(x+z)\ \mathrm{d}x\,\mathrm{d}\mu(z) - \int_{|z|>\frac{\Delta x}{2}} \int_{\mathbb{R}^d} \mathbf{1}_{R_\beta}(x)\ \mathrm{d}x\,\mathrm{d}\mu(z)\\
&= \Delta x^d \left(\int_{|z|>\frac{\Delta x}{2}} \mathrm{d}\mu(z)\,\mathrm{d}x - \int_{|z|>\frac{\Delta x}{2}} \mathrm{d}\mu(z)\,\mathrm{d}x \right)\\
&= 0.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\sum_{\alpha\in\mathbb{Z}^d} G^{\alpha,\beta} &= \sum_{l=1}^d \gamma_l^{\mu,\frac{\Delta x}{2}} \int_{\mathbb{R}^d} D_l \mathbf{1}_{R_\beta}(x)\ \mathrm{d}x\\
&= \Delta x^{d-1} \left(\sum_{l=1}^d \gamma_l^{\mu,\frac{\Delta x}{2}} - \sum_{l=1}^d \gamma_l^{\mu,\frac{\Delta x}{2}} \right)\\
&= 0.
\end{aligned}$$

Therefore, $\sum_{\alpha\in\mathbb{Z}^d} G_\beta^\alpha = \sum_{\alpha\in\mathbb{Z}^d} \left(G_{\alpha,\beta} + G^{\alpha,\beta}\right) = 0$. Next,

$$\begin{aligned}
\sum_{\beta\in\mathbb{Z}^d} G_{\alpha,\beta} &= \sum_{\beta\in\mathbb{Z}^d} \int_{R_\alpha} \int_{|z|>\frac{\Delta x}{2}} \mathbf{1}_{R_\beta}(x+z) - \mathbf{1}_{R_\beta}(x)\ \mathrm{d}\mu(z)\,\mathrm{d}x\\
&= \int_{R_\alpha} \int_{|z|>\frac{\Delta x}{2}} \sum_{\beta\in\mathbb{Z}^d} \mathbf{1}_{R_\beta}(x+z) - \sum_{\beta\in\mathbb{Z}^d} \mathbf{1}_{R_\beta}(x)\ \mathrm{d}\mu(z)\,\mathrm{d}x\\
&= \int_{R_\alpha} \int_{|z|>\frac{\Delta x}{2}} (1-1)\ \mathrm{d}\mu(z)\,\mathrm{d}x\\
&= 0
\end{aligned}$$

and

$$\begin{aligned}\sum_{\beta\in\mathbb{Z}^d} G^{\alpha,\beta} &= \sum_{l=1}^{d}\gamma_l^{\mu,\frac{\Delta x}{2}}\int_{R_\alpha}\sum_{\beta\in\mathbb{Z}^d} D_l\mathbf{1}_{R_\beta}(x)\ \mathrm{d}x\\ &= \sum_{l=1}^{d}\gamma_l^{\mu,\frac{\Delta x}{2}}\int_{R_\alpha}\frac{(1-1)}{\Delta x}\ \mathrm{d}x\\ &= 0.\end{aligned}$$

Therefore, $\sum_{\beta\in\mathbb{Z}^d} G^\alpha_\beta = \sum_{\beta\in\mathbb{Z}^d}\left(G_{\alpha,\beta}+G^{\alpha,\beta}\right)=0$.

Next, note that

$$G_{\beta,\beta} = \int_{R_\beta}\int_{|z|>\frac{\Delta x}{2}} \mathbf{1}_{R_\beta}(x+z)-1\ \mathrm{d}\mu(z)\ \mathrm{d}x \leq 0, \tag{A.1}$$

while, thanks to the definition of the operator D_l^γ - cf. (4.5),

$$\begin{aligned}G^{\beta,\beta} &= \sum_{l=1}^{d}\gamma_l^{\mu,\frac{\Delta x}{2}}\int_{R_\beta} D_l^\gamma\mathbf{1}_{R_\beta}(x)\ \mathrm{d}x\\ &= -\Delta x^{-1}\sum_{l=1}^{d}\gamma_l^{\mu,\frac{\Delta x}{2}}\ \mathrm{sgn}\left(\gamma_l^{\mu,\frac{\Delta x}{2}}\right)\int_{R_\beta}\mathbf{1}_{R_\beta}(x)\ \mathrm{d}x \qquad \text{(A.2)}\\ &= -\Delta x^{d-1}\sum_{l=1}^{d}\left|\gamma_l^{\mu,\frac{\Delta x}{2}}\right| \leq 0.\end{aligned}$$

Therefore, $G^\beta_\beta = G_{\beta,\beta}+G^{\beta,\beta}\leq 0$. Moreover, note that, whenever $\alpha\neq\beta$,

$$G_{\alpha,\beta} = \int_{R_\alpha}\int_{|z|>\frac{\Delta x}{2}}\mathbf{1}_{R_\beta}(x+z)\ \mathrm{d}\mu(z)\,\mathrm{d}x \geq 0,$$

while, thanks to the definition of the operator D_l^γ - cf. (4.5),

$$G^{\alpha,\beta} = \begin{cases}\Delta x^{d-1}\sum_{l=1}^{d}\left|\gamma_l^{\mu,\frac{\Delta x}{2}}\right|\geq 0 & \text{if } \alpha=\beta\pm e_l \text{ for some } l=1,\dots,d,\\ 0 & \text{otherwise.}\end{cases}$$

Therefore, $G^\alpha_\beta = G_{\alpha,\beta}+G^{\alpha,\beta}\geq 0$ whenever $\alpha\neq\beta$.

Finally, to prove that $G^\beta_\alpha = G^{\beta+e_l}_{\alpha+e_l}$ for all $\alpha,\beta\in\mathbb{Z}^d$ and $l=1,\dots,d$ we shift the variable x_l accordingly: let us call $y=x+e_l$ and note that

$$\begin{aligned}G_{\alpha,\beta} &= \int_{R_\alpha}\int_{|z|>\frac{\Delta x}{2}}\mathbf{1}_{R_\beta}(x+z)-\mathbf{1}_{R_\beta}(x)\ \mathrm{d}\mu(z)\,\mathrm{d}x\\ &= \int_{R_{\alpha+e_l}}\int_{|z|>\frac{\Delta x}{2}}\mathbf{1}_{R_\beta}(y-e_l+z)-\mathbf{1}_{R_\beta}(y-e_l)\ \mathrm{d}\mu(z)\,\mathrm{d}y\\ &= \int_{R_{\alpha+e_l}}\int_{|z|>\frac{\Delta x}{2}}\mathbf{1}_{R_{\beta+e_l}}(y+z)-\mathbf{1}_{R_{\beta+e_l}}(y)\ \mathrm{d}\mu(z)\,\mathrm{d}y\\ &= G_{\beta+e_l,\alpha+e_l}.\end{aligned}$$

In a similar fashion we get $G^{\alpha,\beta}=G^{\beta+e_l,\alpha+e_l}$.

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(Simone Cifani), Department of Mathematics, Norwegian University of Science and Technology (NTNU), N-7491 Trondheim, Norway.
E-mail address: simone.cifani@math.ntnu.no
URL: http://www.math.ntnu.no/~cifani/

(Espen R. Jakobsen), Department of Mathematics, Norwegian University of Science and Technology (NTNU), N-7491 Trondheim, Norway.
E-mail address: erj@math.ntnu.no
URL: http://www.math.ntnu.no/~erj/