NTNU
Norwegian University of
Science and Technology

Modelling risk in multi asset-class portfolios

Anders Schmelck

Master of Science in Physics and Mathematics
Submission date: June 2010
Supervisor: Jacob Laading, MATH

Norwegian University of Science and Technology
Department of Mathematical Sciences

Problem Description

This thesis is a study of risk in multi asset-class portfolios. A simulation based model, using the Black-Scholes framework for equities and the LIBOR Market Model for interest rates will be implemented. The validation of models and risk measures will be performed, using backtesting against historical data with particular emphasis on data from the turbulent period of 2007 to the present.

Assignment given: 22. January 2010
Supervisor: Jacob Laading, MATH

Preface

The work on this thesis was carried out at the Department of Mathematical Sciences at the Norwegian University of Science and Technology (NTNU), Trondheim, during the spring semester 2010. It represent a semester load of work and leads to the degree Master of Science.

I would like to express my gratitude to my supervisor Associate Professor Jacob Laading for excellent guidance and constructive feedback.

Trondheim, June 2010.

Anders Schmelck

Abstract

Using a simulation based model, with the Black-Scholes framework for equity and The LIBOR Market Model for interest rates, we study market risk in multi asset-class portfolios, with static and dynamic weighting. The risk measures considered are Value-at-Risk and Expected-Tail-Loss. The theoretical foundation is introduced and imperfections in the models and their assumptions are pointed out. The validity of the models and risk measures is tested using a backtesting procedure against data ranging from September 1999 to September 2009, with particular emphasis on the turbulent period of 2007 to September 2009. The results indicate that the models perform slightly worse on the portfolio with the added complexity of a dynamic weighting regime. No evidence of the models performing less satisfactory under the latest financial turbulence is found.

Contents

Chapter 1

Introduction

1.1 Background

The current financial crisis began with the breakdown in the US subprime mortgage market in July 2007. It started as a credit crisis, with the interbank credit market drying up. And until September 2008 it was mainly a credit crisis, but the collapse of the investment bank Lehman Brothers, 15^{th} September was the start of a downturn for the whole world-wide economy. During the past two years a large number of banks has either been nationalized or filed for bankruptcy. The crisis quickly evolved from an US mortgage crisis to a global financial crisis. And now the markets are concerned about potential governmental bankruptcies in the EU resulting in a collapse of the Euro. It is an ongoing crisis that we do not yet truly understand the consequences of.

The area of financial risk management rapidly grew larger after the series of losses due to derivatives in the 1990s. The concerns were that the technology behind even more complex financial instruments had developed faster than the procedures controlling them. In modern risk management today, risk measures has evolved from just describing worst case scenarios of the risk held by traders and business units, to be central in decisions made at every level of a financial institution. Risk measures as Value-at-Risk plays a pivotal role in the regulatory framework laid down by the Basel Committee, determining the amount of buffer capital financial institutions are obligated to set aside. Earlier, portfolio management has been held back by the lack of satisfactory risk measures. Hence, risk measures as Value-at-Risk and Expected-Tail-Loss can be used to optimize portfolios and provide the best tradeoff between risk and reward. In the light of the current financial crisis, modern risk management can get a central part in dealing with moral-hazard problems in financial institutions. E.g. the performance-bonus systems for traders, the convex profit pattern in the payoff can lead to risky positions that are not in the best interest of the institution. The system is similar to buying an option, where the worst case is large losses, but only a lost job and a damaged reputation as a consequence for the trader. As the best case, is huge profits leading to life long wealth for the trader.[1] The bonus systems has gotten a lot of attention by the financial press and politicians, and a risk adjusted bonus system has been suggested. Having satisfactory risk measures is the cornerstone in determining the relationship between risk and reward.

1.2 Thesis Outline

This thesis consists of eight chapters. Chapter 2 describes financial risk and the theoretical foundation of the two most common risk measures Value-at-Risk and Expected-Tail-Loss. In chapter 3 and 4 we look at mathematical models to describe equity, interest rates and financial products belonging to equity and interest rates. Chapter 6 describes the data used in this thesis and check some of the model assumptions that are made in chapter 3 and 4 against historical data.

To evaluate our model we want to perform a backtesting. To do this we need to define portfolios on which we test our model. In this thesis we are concerned with market-risk resulting from changes in equity and interest rates. To keep our work relevant in terms of real life markets, we want to analyze portfolios that resemble those of real market participants. Given the products we analyze, it is natural to take a closer look at the portfolios carried by life insurance companies. Chapter 5 describes how we implement our risk estimations procedure and gives a detailed description of the multi asset-class portfolios we consider in this thesis. Our results are given in chapter 7 and in chapter 8 we present a conclusion and possible extensions of the thesis.

[1] This in contrast to the system for risk managers which can be compared to selling an option. At best, nothing happens, and the worst case is that they fail to detect a problem, and loose their job.

Chapter 2

Financial Risk Management

The design and implementation of systems that identifies, measures and manage financial risk are the core of financial risk management. Without satisfactory routines in risk management, firms can fail to identify risks that can lead to huge losses and potential bankruptcy. To get an impression of what insufficient risk management can lead to, one should look up the examples of *Metallgesellschaft*, *Orange County* and *Barings PLC*. Financial risk is usually divided into four categories: market risk, credit risk, liquidity risk and operational risk. It should be noted that these often overlap. Let us consider a rogue trader who holds large risky positions, and then suffer large losses on the positions. The losses of the large risky positions comes from market risk, but on the second hand some internal process must have failed to let him take these positions. Hence, in this situation there is overlap between market risk and operational risk. In this thesis we will consider market risk. But for the sake of completeness we will give a short introduction to the other types of risk.

Market Risk

Market risk, is risk resulting from movements in market prices or the volatility of market prices. Market risk is measured as absolute risk, i.e. loss measured in the relevant currency. Or as relative risk, i.e. loss relative to a benchmark index. Market risk is often classified as directional and nondirectional risk. Directional risk is the exposure to the direction of movements in financial variables, e.g. stock prices and interest rates. Nondirectional risk involves nonlinear exposures and exposures to hedged positions or to volatilities. Of the models for risk estimation, the models for estimating market risk are the most developed.

Credit Risk

Credit risk arises when counterparties may not be willing, or able, to fulfill their obligations. The effect is measured by the cost of replacing cash flows if the other party defaults. Losses associated with credit risk can occur before an actual default. This because the mark-to-market value of debt will change with credit events, e.g. debt downgrades. This creates some overlap between market risk and credit risk. One form of credit risk is settlement risk. This occurs when cash flows are exchanged, but not simultaneously. Then it is a chance for one counterparty to default, before both have made its payments. There are some examples of credit risk, leading to legal risk. Because investors, that have lost money on a transaction, often will try to turn to a court to make the trade invalid. However the most important form of credit risk comes from lending out capital and holding bonds. This situation arises when counterparties are not able to handle their debt. Here the counterparty can take many forms, from homeowners (e.g. as in the U.S. 2008-2009) to governments (e.g. as for Argentina 2002). In recent years much work has been put into developing satisfactory models to estimate credit risk. Famous models are CreditMetrics (by J.P. Morgan), the KMV model (now owned by Moody's) and CreditRisk+ (by Credit Suisse Financial Products).

Liquidity Risk

Liquidity risk takes two forms, asset liquidity risk and funding risk. Asset liquidity risk arises when a transaction fails to be executed at prevailing market prices, due to the size of the position relative to normal trading volume. This can be factored loosely into VaR, by choosing the risk horizon greater than an orderly liquidation period. Funding risk arises when one fails to meet obligated payments, which may force early liquidation, and transforming paper losses into realized losses. This is highly relevant for leveraged portfolios, which are subject to margin calls.

Operational Risk

Operational risk, is risk relating to unforeseen failures such as fraudulent activities, internal systems failures, natural phenomena or unexpected changes in legal and regulatory environments. Some experts claims that operational risk is just as significant as market risk. If we investigate some of the most recent and important banking problems, we find that operational risk often plays an important role. However, to quantify operational risk is challenging and requires routines for collection of internal data of losses due to operational risk. It can be impossible for small institutions to collect sufficient data. Quantifying operational risk is still in its infancy and the best protection against operational risk consists of strong internal processes.

2.1 Value-at-Risk (VaR)

VaR is a statistical measure of downside risk, based on current positions. It has grown to be the industry standard in the field of Risk Management, since J.P. Morgen introduced the new measure in the early 1990s. The greatest advantage of VaR is that it summarizes risk in one easily understood number. VaR answers the question: *what is the greatest loss that can occur at a given confidence level within a given time horizon?* Let us define VaR in a formal fashion

$$\mathrm{VaR}_\alpha = \inf\{l \in \mathbb{R} : P(L > l) \leq 1 - \alpha\} = \inf\{l \in \mathbb{R} : F_L(l) \geq \alpha\}, \tag{2.1.1}$$

where α is the given confidence level and L is the loss not exceeding l. According to Jorion (2007), [10], these steps are required to calculate VaR

- Mark-to-market the current portfolio (e.g., $100 million).
- Measure the variability of the risk factor (e.g., 15% per annum).
- Set the time horizon, or the holding period (e.g., adjust to 10 trading days).
- Set the confidence level (e.g., 99%, which yields a 2.33 factor assuming a normal distribution).
- Report the worst potential loss by processing all the preceding information into a probability distribution of revenues, which is summarized by VaR (e.g., the worst loss over 10 trading days is $ 7 million at the 99 percent confidence level).

Marking the positions to market should be done regardless of risk-measure and accounting practice. It is the only technique that measures the current value of assets and liabilities. There are many ways to measure the variability of the risk factor, and it will be discussed later. A rule of thumb is that the time horizon should represent the time needed to liquidate the portfolio, or hedge away the market risk. Confidence levels are usually set to be between 95% and 99%. We are most concerned of the tails of the loss distribution, i.e. relative high α values. Seeing that it is challenging to estimate extreme rare events, it should not be chosen too high either. Another approach that can be used by financial institutions is to match confidence level to a wished credit rating.

2.2 Expected Tail Loss (ETL)

Value-at-Risk does not give us any information of how severe the losses that occur with probability $(1-\alpha)$ can be. Let us consider two portfolios with the same VaR estimates but different loss distributions, where one has heavier tails than the other. It is then clear that these portfolios carry different risk, even with equal VaR estimates. Expected tail loss (ETL) (also known as expected shortfall (ES), conditional loss or conditional Value-at-Risk (CVaR)) answers the question: *how much can we loose if we are "hit" beyond VaR?* ETL is thus concerned with the distribution of the tails in the loss distribution. For a loss L with $E(|L|) \leq \infty$ and cumulative distribution F_L the expected tail loss at a given confidence level $\alpha \in (0,1)$ is formally defined as

$$\mathrm{ETL}_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 q_u(F_L)du, \tag{2.2.1}$$

where $q_u(F_L)$ is the quantile function of the loss distribution F_L. By noticing that $q_u(F_L) = \mathrm{VaR}_u(L)$ we can write equation (2.2.1) as

$$\mathrm{ETL}_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_u(L)du. \tag{2.2.2}$$

ETL can be interpreted as the expected loss given that it exceeds VaR,

$$\mathrm{ETL}_\alpha = E(L|L \geq \mathrm{VaR}_\alpha) = \frac{1}{1-\alpha}E(L; L \geq q_\alpha(L)). \tag{2.2.3}$$

2.3 Calculating the Risk Measures

There are several ways to calculate VaR, and we will only be implementing Monte-Carlo valuations. But for the sake of completeness we will mention other possible solutions.

2.3.1 Nonparametric Calculations

This is the most general method to calculate the risk measures. It makes no prior assumption about the shape of the distribution of the risk factor. But instead we use the empirical distribution of historical data. We then simulate losses by drawing changes in the risk factor from our empirical distribution. Let us define $\hat{L}_m$ to be the resulting loss of a change in a risk factor in period m.

$$\hat{L}_m = -(f(t+1, z_t + x_m) - f(t, z_t)) \tag{2.3.1}$$

If this is repeated n times we get a distribution of the portfolio loss. Then both VaR_α and ETL_α can be calculated by empirical quantile estimation. This can be summarized as:

- Draw n possible changes in the risk factor from the empirical distribution.
- Estimate the portfolio losses $\hat{L}_{t+1,1}, \dots, \hat{L}_{t+1,n}$, sort them.
- VaR_α is then $\hat{L}_{t+1,\alpha \cdot n}$.
- ETL_α is then found by averaging $\hat{L}_{t+1,\alpha \cdot n}, \dots, \hat{L}_{t+1,\cdot n}$.

The greatest advantage of this model is that it does not assume anything about the future distribution of returns. But this is also its largest drawback since it demands a lot from the data set available. There is also no guarantee that historical returns are satisfactory predictors of future returns.

2.3.2 Parametric Calculation

By assuming a distribution from the parametric family, e.g. the normal distribution, the VaR computation can be simplified considerably. The VaR can then be derived directly from the standard deviation of the portfolio, by using a multiplicative factor that corresponds to a given confidence level. The ETL can be measured directly by equation (2.2.1). The name parametric derives from the involvement of parameter estimation, e.g. the standard deviation and the mean of a probability distribution. This method is simple and yields quite accurate measures of VaR and ETL. Its largest drawbacks is whether the distribution assumption is realistic, and the difficulties with parameter estimation.

2.3.3 Monte Carlo Simulation

Monte Carlo simulation is an extension of the parametric approach. We simulate possible paths of market returns using stochastic models. This to build an empirical distribution of future losses. Then the VaR quantile can be estimated by sorting the M simulated losses, and pick element $\alpha \cdot M$. ETL can be estimated by finding the mean of the VaR quantile. With this approach we can deal with non-linear securities. Flexibility is what makes Monte Carlo simulation the most powerful approach in measuring VaR and ETL. A major drawback is the fact that it can be computationally expensive.

2.3.4 VaR and ETL as risk measures

According to Artzner et al (1999), [1], a risk measure, $\rho(\cdot)$ should satisfy these properties:

- Monotonicity: For all X and Y with $X \leq Y$, we have $\rho(Y) \leq \rho(X)$. If a portfolio has systematically lower returns than another, then its risk must be higher.
- Translation invariance: For all real numbers α we have $\rho(X+\alpha) = \rho(X) - \alpha$. Adding α cash to a portfolio should reduce the risk by α.
- Positive Homogeneity: For all $\alpha \geq 0$, $\rho(\alpha X) = \alpha\rho(X)$. Increasing the size of a portfolio by a factor α should scale its risk by the same factor.
- Subadditivity: $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$. Merging portfolios cannot increase risk.

A risk measure satisfying these four properties is coherent. Artzner et al (1999) shows that general VaR fails to satisfy the last property, while ETL is a coherent risk measure. We note that if VaR is calculated under a distribution for which all prices are jointly normally distributed, then VaR do satisfy subadditivity.

Chapter 3

Equity Modeling

3.1 Equity

Equity represent the ownership of a small part of a company. This small part can be a share, or any other security, that ensures its owner a part of the company. This is a usual way for companies to raise capital. The owner of a share is referred to as a shareholder. The company is in reality selling off future profits. A company's purpose is often seen as to maximize the profit to its shareholders. By paying dividends and reinvesting profit, one can achieve this. The latter to increase the value of the company and its shares. If we take a look at charts describing the history of share prices, we immediately see the resemblance to a random walk. This is formalized by the efficient market hypothesis(EMH), first given by Fama (1970), [5].

The EMH states that: *the past history is fully reflected in the present price of an asset, which does not hold any further information and that the market respond immediately to new information about an asset.*

A consequence of the EMH is that we cannot predict future asset prices, but that does not mean that our models cannot tell us anything. If we want to model the return of an asset, we would expect that investors want a time dependent return on their investment. And from the EMH we would expect a random term with some level of volatility. If we put this together in a mathematical sense we get

$$\frac{dS}{S} = \mu dt + \sigma dW \tag{3.1.1}$$

where μ is the drift, σ the volatility and dW our random term represented by a Wiener process. Now we have a simple random walk for modeling the return on an asset.

Fundamental Theorem of Asset Pricing

The fundamental theorem of asset pricing states that the opportunity of risk-free instantaneous profit does not exist in the financial market. Also known as the NFLVR Condition (No Free Lunch with Vanishing Risk). A good intuitive explanation of this is given in Wilmott (1995), [21]. For a more formal proof see F. Delbaen and W. Schachermayer (1994), [4].

3.2 Derivatives

Derivatives, or contingent claims, refers to a contract where one agrees on a future exchange of cash or an underlying security. Thus, it is a financial instrument that is derived from some underlying, e.g. an asset, index or interest rate. The simplest examples of derivatives are the forward and futures contracts. A forward contract is a contract where two parties agrees on exchange of an underlying at a predetermined price at a future-date. In this derivative both parties has an obligation to fulfill the terms given in the contract. There is also a counterparty risk, i.e. it exists a risk of one party to default. A futures contract is basically the same as a forward contract with some minor technical differences. Futures are normally traded on an exchange and the change in a futures value is paid every day. This eliminates the counterparty risk. But it can be shown, Shreve (2004), [18], that when interest rates are nonrandom the futures price and the forward price agrees. An option is a contract where its holder has the right, but not the obligation, to take some action. E.g. sell or buy the underlying in the future. On the other hand, the writer of an option have a potential obligation, e.g. must buy or sell the underlying in the future. The writer must be compensated for this obligation. This compensation is referred to as the price of the option. The simplest forms of options comes in European and American versions. In the European version the holder has the right to take some action at a future predetermined date. As for the American version the holder has the right to take some action at any time before a predetermined date. A swap is a contract on exchanging cash flows in the future.

3.3 Hedging

Hedging is a term that one often runs into when dealing with finance. Most of the time one wants to eliminate, or at least minimize the risk of an investment. One can do this by making another investment that offsets your original risk exposure. This is called hedging.

Let us consider a simple example. The small transport company *FastFish Logistics* has made a three year deal with one of the largest supermarket chains in Norway, to deliver fresh fish from northern Norway to the chain stores all around the country. The payments are determined for all the three years the contract is valid. Because of fierce competition from East European transport companies, their margins are low. They are concerned about rising fuel prices leading to large losses. Since they are not in the business of speculating in fuel prices, they want to offset this risk. To do this, they assume that the price of fuel is correlated with the oil price. Thus hedge by buying a call option on the oil price. If the prices of oil and fuel then rises above a given level, they loose money in their business, but gain money from the option contract. If the hedge is done correctly they will not have lost (or gained) money. If the opposite happens, that oil and fuel prices decrease, they will make a greater profit.

The above mentioned example is an example of using derivatives as an insurance against unwanted risk. And that is the main purpose of derivatives, even though they are also used for speculation.

3.4 The Black-Scholes Financial Market

In this thesis we will assume that equity evolves through time as described in the Black-Scholes framework.

In the Black-Scholes framework we make some assumptions about the financial market. Here we will present the most important ones mentioned in Wilmott (2006), [19].

- The underlying follows a log-normal random walk.
- The risk-free interest rate and the volatility are deterministic.
- The underlying pays no dividends.
- Delta hedging is done continuously.
- There are no transaction costs.
- The market satisfy the NFLVR condition.

In this framework the financial market consists of only two traded assets; a numéraire and one further asset. These follow the price process

$$dS_0(t) = rS_0(t)dt, \quad S_0(0) = 1,$$
$$dS_1(t) = \mu S_1(t)dt + \sigma S_1(t)dW, \quad S_1(0) > 0.$$

And the market contains no arbitrage opportunities. The price process of any European derivative with payoff $D = D(T)$ and maturity T is then given by

$$V_D(t) = E_{\widetilde{Q}}(\exp(-r(T-t))D(T)), \quad t \in [0,T].$$

Let us consider an European call option. An European call option gives its holder the right, but not the obligation to buy the underlying equity with current price $S(t)$ at a given date in the future known as the expiry date T to a given price, known as the strike price X. Hence, the payoff function for an European call option with expiry date T and strike price X is given by

$$D(T) := (S(T) - X)^+ \tag{3.4.1}$$

An European put option gives its holder the right, but not the obligation to sell the underlying equity with current price $S(t)$ at a given date in the future, known as the expiry date T, to a given price, known as the strike price X. Hence the payoff function for an European put option with expiry date T and strike price X is given by

$$D(T) := (X - S(T))^+ \tag{3.4.2}$$

Equation (3.4) is solved for an European call option with strike price X and expiry date T by Black-Scholes equation

$$Call^{BS}(t,T,X) = S(t)N(d_1) - e^{-r(T-t)}XN(d_2), \tag{3.4.3}$$

where

$$d_1 = \frac{\ln\left(\frac{S(t)}{X}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 := d_1 - \sigma\sqrt{T-t},$$

and $N(\cdot)$ is the cumulative normal distribution.

The corresponding price of an European put option is given by

$$Put^{BS}(t,T,X) = e^{-r(T-t)}XN(-d_2) - S(t)N(-d_1). \tag{3.4.4}$$

3.5 Simulation

When pricing options we know that we can do this under the risk-neutral measure. Under the Equivalent Martingale Measure $\widetilde{Q}$, let the following Black-Scholes model be given

$$S_0(t) = e^{rt}, \quad S_1(t) = s\exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right), \quad 0 \le t \le T \in (0,\infty), \tag{3.5.1}$$

for a Wiener process W and positive constants r, s and σ. If we are interested in how the price of an equity will evolve in the future, we cannot simulate under the risk-neutral measure. We can no longer assume that the drift equals the risk-free interest rate, so we assume that the equity has a non-zero drift μ. Thus, when simulating equity prices under the real measure we need to replace r with μ in equation (3.5.1).

Valuation of derivatives using Monte-Carlo simulation can be summarized by these steps

1. Simulate one random path of the underlying.
2. Calculate the cash flow(s)/payoff(s).
3. Discount each cash flow/payoff.
4. Repeat step 1-3 to get many sample cash flows/payoffs.
5. Calculate the mean of the sample cash flows/payoffs to get an estimate of the expected value of the cash flow(s)/payoff(s).

Monte-Carlo simulation can be used to price a wide range of derivatives with complex payoff functions. Including path-depended derivatives, derivatives with more than one payoff through time and derivatives that depend on more than one underlying. The largest drawbacks of valuation with Monte-Carlo simulation are that it can be time consuming and hardware demanding and the convergence is often slow.

3.6 Convergence

Let us consider a simple example of using Monte-Carlo simulations to price an European put option. From equation (3.4.4) we can check how fast our simulated solution converge to the analytical. This is done in figure 3.6.1. We observe that they converge fast and reaches a relative constant level at approximately $15,000$ iterations. As we would expect the At-the-Money option converges faster than the Deep-Out-of-the-Money option. This is because when the option is deep out of the money we need large changes in the value of the underlying to get a non-zero payoff, as we see of the deltas in table 3.6.2. Since large changes have a lower probability, the proportion of non-zero payoffs in the simulation is small. We know from the Law of Large Numbers [3], that the convergence, $\bar{X}_N \to \mu$ is slower. In table 3.6.1 the simulated and analytical prices of an European put option are given, and we see that the error is small at 10^5 iterations.

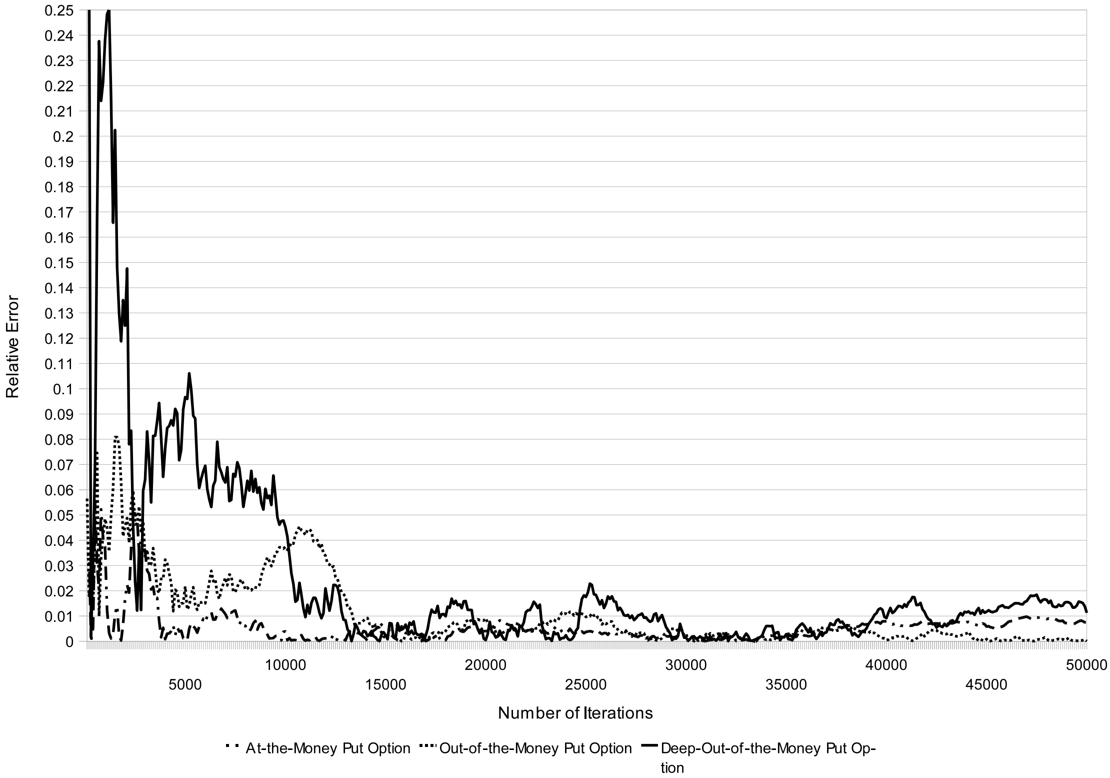

Figure 3.6.1: Relative error versus number of iterations for an European put option. At-the-Money($S = X$), Out-of-the-Money($S = 1.1X$) and Deep-Out-of-the-Money($S = 1.3 \cdot X$). Assuming r = 5%, T = 1 year and σ = 20%.

Table 3.6.1: Simulated and analytical values of an European Put Option, with strike price $X = 50$, time to maturity $T - t = 1$ year, interest rate $r = 5\%$ and volatility $\sigma = 20\%$. 1. At-the-Money, 2. Out-of-the-Money, 3. Deep-Out-of-the-Money.

	1.(S=X)	2.($S = 1.1X$)	3.($S = 1.3X$)
Analytical	2.786	1.392	0.281
10^4 Sim.	2.797	1.340	0.294
10^5 Sim.	2.794	1.395	0.275
10^6 Sim.	2.786	1.392	0.283

Table 3.6.2: The analytical deltas of the options simulated in figure 3.6.1 and table 3.6.1.

Derivative	Delta
European Put At-the-Money($S = X$)	-0.363
European Put Out-of-the-Money($S = 1.1X$)	-0.204
European Put Deep-Out-of-the-Money($S = 1.3X$)	-0.048

Chapter 4

Interest Rate Modeling

4.1 Interest Rates

Interest rates can be seen as the cost of borrowing money. This cost varies with several factors: time to maturity, the credit worthiness of the borrower and denominated currency. Interest rates are usually higher for longer maturities. This is largely explained by the fact that investors are unwilling to lock capital for long periods of time. Modeling interest rates is more complex than modeling equity. Some of the reasons for this are that interest rates with different maturities are highly correlated, they cannot become negative and they are often mean reverting. Another problem that arises when we want to price interest rate derivatives, is the fact that one cannot buy "one interest rate", like we can buy one asset. This complicates how one can hedge away risk attached to interest rate derivatives.

4.1.1 The Term Structure of Interest rates

The "term structure of interest rates" or the "yield curve" is the dependence of interest rates on time to maturity. In figure 4.1.1 the most common forms of the yield curve are shown. The increasing, which is the most common form of the yield curve, long term interest rates are higher than the short term rate, since it should be more rewarding to tie money up for a long period of time than for a short period of time. The decreasing and the humped yield curve are both typical for periods when the short rate is high, but expected to fall. Let us define the zero-coupon bond pricing formula

$$B(t,T) = \widetilde{E}\left[e^{-\int_t^T R(s)ds}|F(t)\right]. \tag{4.1.1}$$

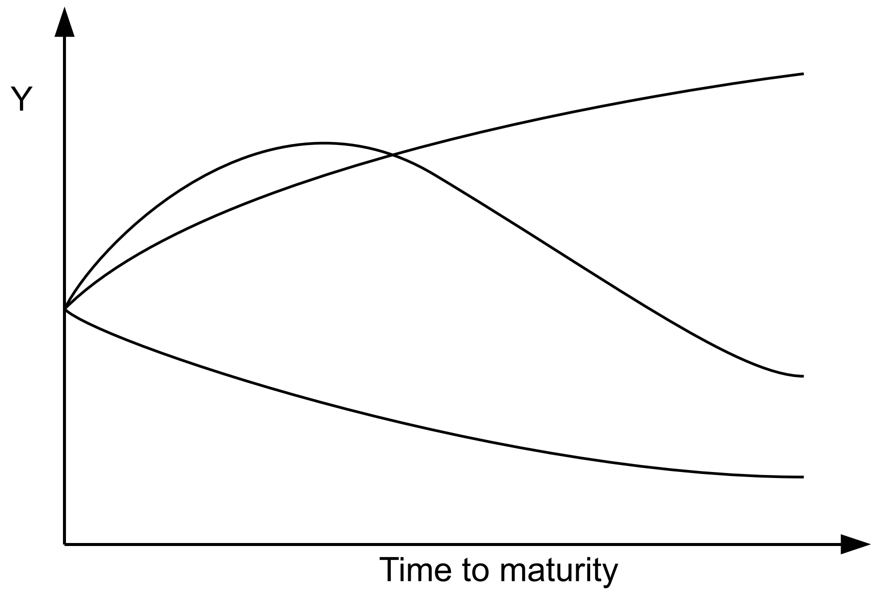

Figure 4.1.1: The most common shapes of the yield curve: increasing, decreasing and humped.

Once the zero-coupon bond price has been computed, we can define the yield between times t and T to be

$$Y(t,T) = -\frac{1}{T-t}\log B(t,T). \tag{4.1.2}$$

From these equations we can determine the long rate once we have determined the short rate.

4.2 One-Factor Interest Rate Models

The simplest models for fixed income markets begin with a stochastic differential equation for the interest rate. e.g.,

$$dR(t) = \beta(t,R(t))dt + \gamma(t,R(t))d\widetilde{W}(t), \tag{4.2.1}$$

where $\widetilde{W}$ is a Brownian motion under a risk-neutral probability measure $\widetilde{Q}$. When the interest rate is described by only one stochastic differential equation, as in this section, the model is called an one-factor model. One important property of the one-factor models is the mean reversion property; over time, interest rates seems to be drawn to a long-run average. The largest drawback of one-factor models is that they cannot capture complicated yield curve behavior. They produce deterministic shifts in the yield curve, but not changes in its slope or curvature.

4.2.1 Vasicek Model

In the model of Vasicek there is no time dependency in the parameters and the resulting stochastic differential equation is,

$$dR_t = \lambda(\mu - R_t)dt + \sigma d\widetilde{W}_t. \tag{4.2.2}$$

This simple model is mean reverting, which is sensible. However, a significant drawback is that interest rates can become negative in the model. Since the model is relative simple, there exists an explicit formula for many interest rate derivatives. For example the price of a zero-coupon bond is given by

$$B(t,T) = e^{\alpha(t,T) - R\beta(t,T)} \tag{4.2.3}$$

with

$$\begin{aligned} \beta(t,T) &= \frac{1}{\lambda}(1 - e^{-\lambda(T-t)}) \\ \alpha(t,T) &= \frac{1}{\lambda^2}(\beta(t,T) - T + t)(\lambda^2\mu - \sigma^2/2) - \frac{\sigma^2}{2\lambda}\beta(t,T)^2 \end{aligned}$$

4.3 Heath, Jarrow & Morton Model

In this section, we will summarize the Heath, Jarrow & Morton (HJM) model, which is described in Wilmott (2006), [20]. Instead of modeling the short rate, and then deriving the forward rates, the HJM is modeling the whole forward rate curve. Since forward rates are observed in the markets, yield-curve consistency is naturally contained in this model.

Forward Rate

Let $F(t;T)$ be the forward rate curve at time t. The price of a zero-coupon bond at time t, with maturity T, is then given by

$$Z(t;T) = e^{\int_t^T F(t;s)ds}. \tag{4.3.1}$$

In the HJM framework, we assume that a zero-coupon bond behaves according to the SDE

$$dZ(t;T) = \mu(t;T)Z(t;T)dt + \sigma(t,T)Z(t;T)dW. \tag{4.3.2}$$

From equation (4.3.1) we get

$$F(t;T) - \frac{\partial}{\partial T} \log Z(t;T).$$

If we differentiate this with respect to t and substituting (4.3.2) into (4.3.1) we get an equation for the behavior of the forward curve

$$dF(t;T) = \frac{\partial}{\partial T}(\frac{1}{2}\sigma^2(t,T) - \mu(t,T))dt - \frac{\partial}{\partial T}\sigma(t,T)dW. \tag{4.3.3}$$

We get the spot interest rate from the forward rate with a maturity equal the current date

$$r(t) = F(t;t).$$

Assume that today is t^* and that we know the whole forward curve, $F(t^*;T)$. We can then write the spot rate for any time in the future as

$$r(t) = F(t^*;t) + \int_{t^*}^{t} dF(s;t). \tag{4.3.4}$$

If we substitute equation (4.3.3) into equation (4.3.4) and differentiate with respect to time t we get a stochastic differential equation for the spot interest rate r. This process turns out to be non-Markov, which means that the future process is not just dependent on the current state, but it is path dependent. This has some unfortunate consequences; the general HJM model needs an infinite number of state variables to define the present state. I.e. if we write the HJM model as a partial differential equation, we need an infinite number of independent variables. This means that if we wish to price derivatives under the HJM model, we are left with only two options: to build a tree structure, or to estimate the necessary expectations by simulating the risk-neutral forward rates.

4.4 LIBOR Market Model

In this section we are going to take a closer look at the LIBOR market model (LMM) which is described in Glasserman (2004), [7]. The LMM can be seen as the discrete version of the HJM model. The strength of the LMM lies in the fact that it models simple forward rates. These are observed in the markets. Hence, yield curve fitting is naturally contained in this model. Actually, the term "market model" refers to modeling based on observable market rates. An important benchmark in the world of interest rates is the London Inter-Bank Offered Rates (LIBOR). LIBOR is calculated by an average of rates offered by banks in the London wholesale money market and it is also available with different maturities and currencies. Let $L(0,T)$ be the forward LIBOR rate, which is the rate set at time 0 for the interval $[T, T+\delta]$. The relationship between forward LIBOR rates and bond prices is given by

$$L(0,T) = \frac{B(0,T) - B(0,T+\delta)}{\delta B(0,T+\delta)}. \tag{4.4.1}$$

It should be noted that we assume that the LIBOR rate is risk-free. If this is not the case, (4.4.1) may not hold exactly. In the LMM we assume a fixed set of maturities

$$0 = T_0 < T_1 < \cdots < T_M < T_{M+1}.$$

This because many derivatives tied to LIBOR and swap rates are only sensitive to a finite set of maturities, and we do not need to introduce a continuum to price and hedge these derivatives. The lengths between tenor dates are given by

$$\delta_i = T_{i+1} - T_i, \quad i = 0, \dots, M.$$

To simplify notation we let $B_n(t) = B(t, T_n)$ with $n \in \{1, 2, \dots, M+1\}$. $B_n(t)$ is then the price of a bond at time t, with maturity T_n. And $L_n(t)$ is the forward rate at time t for the accrual period $[T_n, T_{n+1}]$. Then the relationship between forward LIBOR rates and bond prices are given by

$$L_n(t) = \frac{B_n(t) - B_{n+1}(t)}{\delta_n B_{n+1}(t)}, \quad 0 \le t \le T_n, \quad n = 0, 1, \dots, M. \tag{4.4.2}$$

By inverting this relationship we get

$$B_n(T_i) = \prod_{j=i}^{n-1} \frac{1}{1 + \delta_j L_j(T_i)}, \quad n = i+1, \dots, M+1. \tag{4.4.3}$$

But this equation does not determine the bond prices for a date t, that is not a tenor date. This because the forward LIBOR rates cannot determine the discount factor for intervals shorter than the accrual periods. We get bond prices for all dates by defining a function $\eta : [0, T_{M+1}) \to \{1, \dots, M+1\}$, where $\eta(t)$ is the unique integer satisfying $T_{\eta(t)-1} \le t \le T_{\eta(t)}$. I.e. $\eta(t)$ gives us the next tenor date at time t. Using this notation we get

$$B_n(t) = B_{\eta(t)}(t) \prod_{j=\eta(t)}^{n-1} \frac{1}{1 + \delta_j L_j(t)}, \quad 0 \le t \le T_n. \tag{4.4.4}$$

$B_{\eta(t)}(t)$ is the price of the bond associated with the shortest maturity, today.

Spot Measure

We want a model where the behavior of the forward LIBOR rates are described by a system of SDEs on the form

$$\frac{dL_n(t)}{L_n(t)} = \mu_n(t)dt + \sigma_n(t)^\top dW(t), \quad 0 \le t \le T_n, \quad n = 1, \dots, M, \tag{4.4.5}$$

where W is a d-dimensional Brownian motion. In the HJM setting the risk-neutral numéraire, $\beta(t) = \exp(\int_0^t r(u)du)$, is used. But this will not be useful in the LMM setting since we are trying to build a model based on simple observed interest rates. Therefore a discrete spot measure numéraire is given by

$$B^*(t) = B_{\eta(t)}(t) \prod_{j=0}^{\eta(t)-1} [1 + \delta_j L_j(T_j)].$$

We call the bond price divided by the numéraire the deflated bond price

$$D_n(t) = \left(\prod_{j=0}^{\eta(t)-1} \frac{1}{1 + \delta_j L_j(T_j)} \right) \prod_{j=\eta(t)}^{n-1} \frac{1}{1 + \delta_j L_j(t)}, \quad 0 \le t \le T_n. \tag{4.4.6}$$

We notice that the spot measure numéraire B^* cancels $B_{\eta(t)}(t)$ and we have an expression defined only by the LIBOR rates. Since we require that the D_ns are positive martingales, some restrictions apply for equation (4.4.5). As shown in Glasserman (2004), [7], the required drift parameter is

$$\mu_n(t) = \sum_{j=\eta(t)}^{n} \frac{\delta_j L_j(t) \sigma_n(t)^\top \sigma_j(t)}{1 + \delta_j L_j(t)} \tag{4.4.7}$$

Forward Measure

We can formulate the LMM under the forward measure P_{M+1} for maturity T_{M+1} and use the bond B_{M+1} as the numéraire. The deflated bond price then becomes

$$D_n(t) = \prod_{j=n+1}^{M} (1 + \delta_j L_j(t)). \tag{4.4.8}$$

We see that this expression only depends on the forward LIBOR rates. This leaves us with the systems of SDEs given by

$$\frac{dL_n(t)}{L_n(t)} = - \sum_{j=n+1}^{M} \frac{\delta_j L_j(t) \sigma_n(t)^\top \sigma_j(t)}{1 + \delta_j L_j(t)} dt + \sigma_n(t)^\top dW^{M+1}(t), \quad 0 \le t \le T_n, \tag{4.4.9}$$

where W^{M+1} is a standard d-dimensional Brownian motion under P_{M+1}.

4.5 Interest Rate Derivatives

4.5.1 Interest Rate Swap

An interest rate swap is a contract where two parties agrees on exchanging cash flows that are represented by the interest on a notional principal, N. One side pays fixed interest rate and the other floating interest rate. The exchange of the fixed and floating interest payments normally occur every six months. One important factor is that the principal is not exchanged at maturity. In figure 4.5.1 we see an

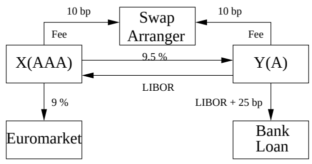

Figure 4.5.1: Illustration of an example swap with a swap dealer. Where two parties with different credit ratings exchange fixed against floating interest rate.

illustration of an example swap. The fixed interest payments can be seen as a sum of zero-coupon bonds. If the fixed rate is R_S and δ is the time between payments, the fixed payments becomes

$$R_S \delta \sum_{i=1}^{N} Z(t; T_i). \tag{4.5.1}$$

Usually the fixed rate in the swap is chosen such that the swap has zero value to both parties when the contract is set up. The fixed rate is then given by

$$R_S = \frac{1 - Z(t; T_N)}{\delta \sum_{i=1}^{N} Z(t; T_i)}. \tag{4.5.2}$$

Swaps are very popular and extremely liquid instruments.

4.5.2 Swaptions

A swaption is an option on a swap. It has a strike rate R_E, this is the fixed rate that will be swapped against floating rate if the option is exercised. They exist in both European and American versions. If one is long a call swaption,(also known as a payer swaption) one has the right to become the fixed rate payer. And if one is long a put swaption, (also known as a receiver option) one has the right to become the payer of the floating leg.

4.5.3 Caps and Floors

A cap is a contract that guarantees its holder that the otherwise floating rate will not exceed a given level. A typical cap involves a given number of cash flows a year. Each of these cash flows is called a caplet. One can analyze a caplet as an European call option on the interest rate. Then the payoff is given by

$$N \cdot \delta(R_{float} - R_{cap})^+$$

The value of a caplet under the forward measure P_{n+1}, with maturity T_{n+1} is given by

$$V_n(0) = B_{n+1}E_{n+1}\left[\frac{\delta_n(L_n(T_n) - K)^+}{B_{n+1}(T_{n+1})}\right], \tag{4.5.3}$$

with E_{n+1} denoting expectation under the forward measure, P_{n+1}. Since $B_{n+1}(T_{n+1}) \equiv 1$, does equation (4.5.3) only depend on the marginal distribution of $L_n(T_N)$. And this distribution is the log-normal distribution when assuming deterministic volatility. Then from Glasserman (2004), [7] the Black caplet formula is given by

$$V_n(0) = \delta_n B_{n+1}(0)\left[L_n(0)\Phi\left(d_1\right) - R_{cap}\Phi\left(d_2\right)\right] \tag{4.5.4}$$

where

$$d_1 = \frac{\log\left(\frac{L_n(0)}{R_{cap}}\right) + \frac{\sigma^2 T_n}{2}}{\sigma\sqrt{T_n}} \text{and} d_2 = \frac{\log\left(\frac{L_n(0)}{R_{cap}}\right) - \frac{\sigma^2 T_n}{2}}{\sigma\sqrt{T_n}} \tag{4.5.5}$$

where Φ is the cumulative normal distribution. Equation (4.5.4) is useful to calibrate the LIBOR market model, by finding the implied volatilities. A floor is analogue to a cap, except that the floor guarantees its owner that the interest rate will not fall below a given level. A floor consists of a sum of floorlets. Hence the payoff of a floorlet is given by

$$N \cdot \delta(R_{floor} - R_{float})^+ \tag{4.5.6}$$

A floorlet is similar to an European put option. And the Black formula for a floorlet is given by

$$V_n(0) = \delta_n B_{n+1}(0)\left[-L_n(0)\Phi\left(-d_1\right) + R_{cap}\Phi\left(-d_2\right)\right] \tag{4.5.7}$$

An interest rate collar is a derivative that guarantees its owner that interest rates will be between a lower and an upper bound. One can construct a collar by combining a long position in a cap and a short position in a floor.

4.6 Simulation

The power of the LIBOR market model lies in its practical implementation through simulations. In the LMM we only consider a fixed set of maturities and we only need to discretize the time axis. The time grid to simulate over becomes

$$0 = t_0 < t_1 < \cdots < t_m < t_{m+1}.$$

For simplicity and computational speed we will choose the time steps to be the tenor dates, i.e. we will simulate from one tenor date to another. We will also choose constant volatilities σ_n. From equation (4.4.9) we know that simulation of the forward LIBOR rate is a special case of simulation of a system of SDEs. Using an Euler scheme to discretize equation (4.4.9) under the forward measure we get

$$\hat{L}_n(t_{i+1}) = \hat{L}_n(t_i) \exp\left(\left(\mu_n(\hat{L}(t_i), t_i) - \frac{1}{2}\sigma_n^2\right)\Delta t + \sqrt{\Delta t}\sigma_n \widetilde{Z}_{i+1}\right) \tag{4.6.1}$$

where $\Delta t = t_{i+1} - t_i$, $\widetilde{Z} \sim N(0, \boldsymbol{\rho})$ and

$$\mu_n(\hat{L}(t_i), t_i) = -\sum_{j=n+1}^{M} \frac{\delta_j \hat{L}_j(t_i)\sigma_n\sigma_j}{1 + \delta_j \hat{L}_j(t_i)} \tag{4.6.2}$$

note that $\mu_M \equiv 0$. This means that we simulate L_M without discretization error under the forward measure P_{M+1}. Another thing to notice is that L_n is close to log-normal, when we have chosen σ_n to be constant. And equation (4.6.1) keeps interest rates positive, which is preferable in an interest rate model. It is important to realize that we do not intend to do perfect simulations; discretization errors will be made. One consequence of this is that if we price caplets, the simulated price will not converge exactly to the Black price. The size of this error is further discussed in Glasserman (2004), [7]. To initialize the simulation we assume that we are given bond prices or interest rates for maturities $1, \ldots, M$. In the case of given bond prices the initial forward LIBOR rate is given through

$$\hat{L}_n(0) = \frac{B_n(0) - B_{n+1}(0)}{\delta_n B_{n+1}(0)}, \quad n = 1, \ldots M. \tag{4.6.3}$$

In the Euler scheme we have chosen, we approximate L_n by a geometric Brownian motion over one time step. Since we have chosen constant volatilities, we would expect L_n to be close to a log-normal distribution. In figure 4.6.1, we have investigated this. And all the simulated forward rates seems to come from a log-normal distribution. We observe that the density gets wider, i.e. the standard deviation increase with time to maturity. This corresponds to the real world, since the rates with longer time to maturity are more uncertain.

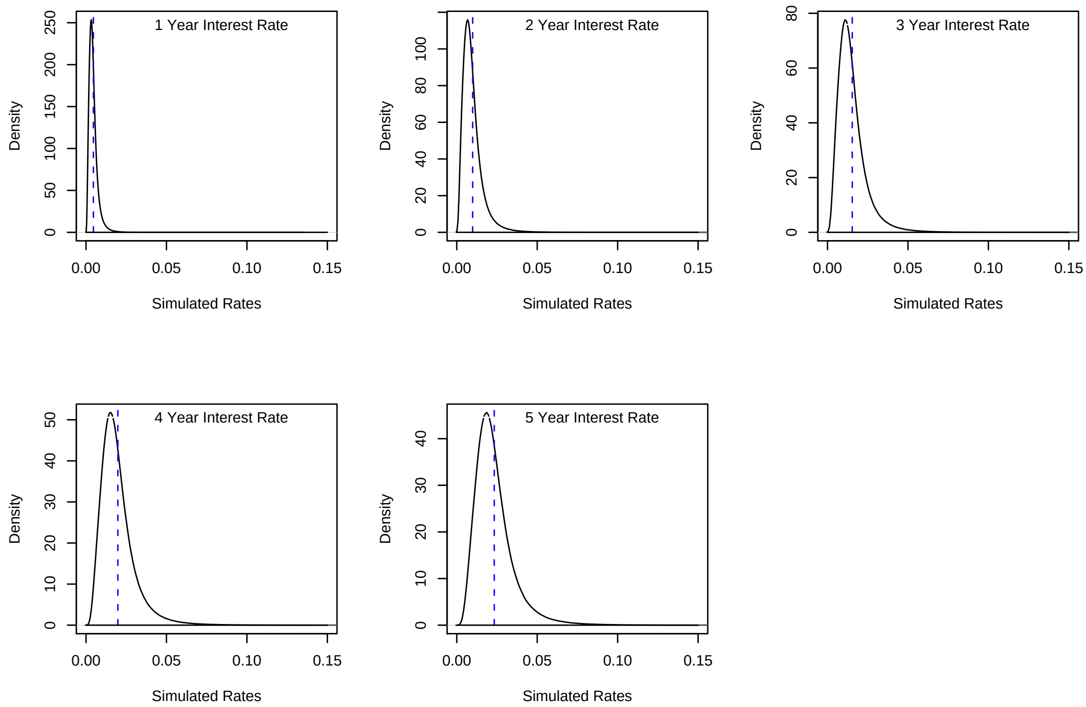

Figure 4.6.1: Smoothed histograms of 10^6 simulations of the Forward Rates $\hat{L}_n(t)$, $t = [1, \ldots, 5]$, $n = [1, \ldots, 5]$. The dashed line represent the initial forward rate, $L_n(0)$.

4.7 Model Calibration

There are several ways to calibrate the model. One way is to calculate the volatilities and covariances from historical bond prices or interest rates. Another way is to calculate the implied volatilities from the market price of At-the-Money caps, and the covariances from swaption prices. Combining implied volatilities and historical covariances, or the other way around, is also possible. However it is important to notice that if we want the model to be market consistent, it should be calibrated against derivatives. We should also notice that when pricing derivatives in illiquid markets, it can be difficult to get a market consistent model. In the latest financial crisis the trading volume of many interest rate derivatives plunged, making market consistent pricing difficult.

Drawing From a Multivariate Normal Distribution

To simulate from a multivariate normal distribution with mean **0** and correlation matrix $\hat{\boldsymbol{\rho}}$ we do as follows:

1. Require $\hat{\boldsymbol{\rho}}$ to be positive definite. It is symmetrical by definition.
2. Calculate the Cholesky decomposition of $\hat{\boldsymbol{\rho}}$, that is, find an unique lower triangular matrix that satisfies $\hat{\boldsymbol{\rho}} = LL^T$.
3. Draw a vector of independent normally distributed random variables, $Z = (Z_1, \ldots, Z_M)^T$.
4. Then $LZ \sim N(\mathbf{0}, \boldsymbol{\rho})$.

4.8 Pricing Derivatives

To price derivatives under the LIBOR market model we simulate paths of the forward rates. Then, we sum the discounted payoffs according to

$$\sum_{n=1}^{M} g(\hat{L}(T_n)) \cdot B_{M+1}(0) \prod_{j=n}^{M} (1 + \delta_j \hat{L}_j(T_j)), \tag{4.8.1}$$

where $B_{M+1}(0)$ is the current price of a bonds maturing at T_{M+1} and chosen as the numéraire asset. In our case it is given by equation (4.4.3). Let us consider the payoff functions to these derivatives

Swap

$$g(\hat{L}(T_n)) = (\hat{L}(T_n) - K), \quad n = 1, \ldots, M, \tag{4.8.2}$$

Cap

$$g(\hat{L}(T_n)) = (\hat{L}(T_n) - K)^+, \quad n = 1, \ldots, M, \tag{4.8.3}$$

Floor

$$g(\hat{L}(T_n)) = (K - \hat{L}(T_n))^+, \quad n = 1, \ldots, M, \tag{4.8.4}$$

where K is the fixed rate.

If we want to price all the three types of derivatives, we can use the cap-floor parity to price one of the derivatives, to reduce the simulations time.

$$\text{Value of Cap} - \text{Value of Floor} = \text{Value of Swap}$$

By averaging independent replications of these equations we get an estimate for the price of the derivatives.

4.9 Convergence

Let us use the Libor Market Model to price a floorlet. We calibrate the model to historical rates only. Since the analytical price of a floorlet is given by equation (4.5.7) we can check how fast our simulation converges, as we did for the European put option in chapter 3. In figure 4.9.1 the relative error of a simulation of different floorlets are plotted against the number of iterations. We observe a large drop in the relative errors during the first $1,000$ iterations and that they reach a relative constant level at approximately $20,000$ iterations. We notice that the At-the-Money floorlet converges faster than the Deep-Out-of-the-Money floorlet. This is explained by the log-normal properties of the forward rate. The simulations does not converge exactly to the Black-price, as previously noted, the size of this error is discussed in Glasserman (2004), [7].

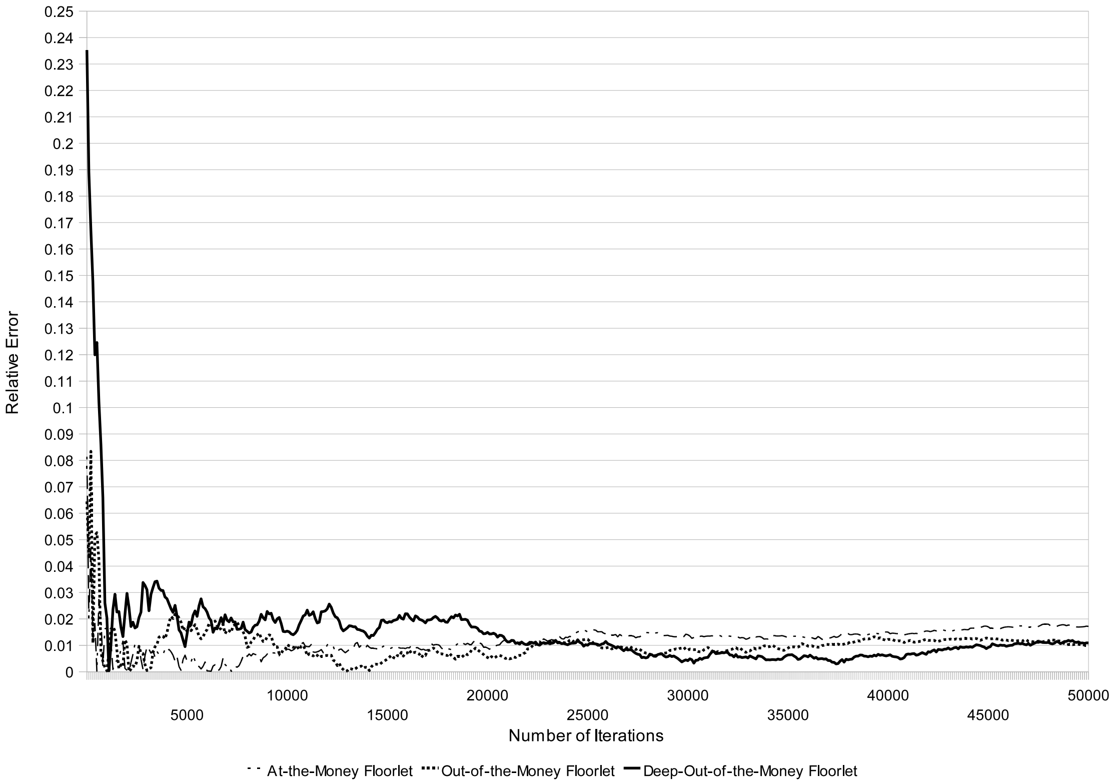

Figure 4.9.1: Relative error versus number of iterations of floorlets At-the-Money($R_{floor} = L_5(0)$), Out-of-the-Money($R_{floor} = 0.9L_5(0)$) and Deep-Out-of-the-Money($R_{floor} = 0.7L_5(0)$). With five years to maturity on the five year LIBOR forward rate.

Chapter 5

Implementation

5.1 Volatilities and Correlations

There are several ways to calibrate our models, depending on the setting, as mentioned briefly for interest rates in section 4.7. In this thesis we are going to use the method of Exponential Weighted Moving Average(EWMA), suggested by RiskMetrics™ - Technical Document (1996), [11]. But for the sake of completeness we will mention other possibilities.

Simple Moving Average (SMA)

This is the traditional way to forecasting volatilities and correlations and relies on that every historical return is equally weighted. The correlation matrix $\boldsymbol{\rho}$, is estimated from the correlation of the daily log-returns of our historical data. Let $L_i(t_k)$ denote the quoted price of asset i in our portfolio at t_k. Let $r_i(t_k) = \log\left(\frac{L_i(t_{k+1})}{L_i(t_K)}\right)$. Assuming 252 trading days per year, we calculate the covariance matrix from the historical data

$$\hat{\Sigma}_{ij} = 252\frac{1}{N}\sum_{k=1}^{N-1}\left(r_i(t_k) - \hat{\mu}_i\right)\left(r_j(t_k) - \hat{\mu}_j\right), \tag{5.1.1}$$

where N is the size of the data set and

$$\hat{\mu}_i = \frac{1}{N-1}\sum_{k=1}^{N-1} r_i(t_k) = \frac{1}{N-1}\log\left(\frac{L_i(t_N)}{L_i(t_1)}\right). \tag{5.1.2}$$

Then the correlations are given by

$$\hat{\rho}_{ij} = \frac{\hat{\Sigma}_{ij}}{\sqrt{\hat{\Sigma}_{ii}\hat{\Sigma}_{jj}}}. \tag{5.1.3}$$

The historical volatilities are given by

$$\hat{\sigma}_i = \sqrt{\Sigma_{ii}}. \tag{5.1.4}$$

Exponential Weighted Moving Average (EWMA)

This is the method used by RiskMetrics™ and the method we are going to use in this thesis. The method has two important advantages over the equally weighted method. First, volatility react faster to shocks in the market as recent data carry more weight than data in the distant past. Second, following a shock, i.e. a large return, the volatility declines exponentially as the weight of the large return falls. As for the equally weighted we let $L_i(t_k)$ denote the quoted price of asset i in our portfolio at t_k. Let $r_i(t_k) = \log\left(\frac{L_i(t_{k+1})}{L_i(t_k)}\right)$. We calculate the 1-day covariance matrix forecast from the historical data

$$\hat{\Sigma}_{ij} = (1-\lambda)\sum_{k=1}^{N} \lambda^{k-1}(r_i(t_k) - \hat{\mu}_i)(r_i(t_k) - \hat{\mu}_j) \tag{5.1.5}$$

where

$$\hat{\mu}_i = \frac{1}{N-1}\sum_{k=1}^{N-1} r_i(t_k) = \frac{1}{N-1}\log\left(\frac{L_i(t_N)}{L_i(t_1)}\right). \tag{5.1.6}$$

An attractive feature of the EWMA estimator is that it can be updated recursively

$$\hat{\Sigma}_{ij,t+1|t} = \lambda\hat{\Sigma}_{ij,t|t-1} + (1-\lambda) r_i(t) \cdot r_j(t) \tag{5.1.7}$$

Then the correlation forecasts are given by

$$\hat{\rho}_{ij,t+1|t} = \frac{\hat{\Sigma}_{ij,t+1|t}}{\sqrt{\hat{\Sigma}_{ii,t+1|t}\hat{\Sigma}_{jj,t+1|t}}}. \tag{5.1.8}$$

The volatility forecasts are given by

$$\hat{\sigma}_{i,t+1|t} = \sqrt{\Sigma_{ii,t+1|t}} \tag{5.1.9}$$

Implied Volatilities

There is no guarantee that the historic volatilities and correlations will predict future volatilities and correlations. In some settings we would like to have a forward looking measure that accounts for the markets expectations. This is where implied volatilities come to play. For equity we derive the implied volatility from options, by inverting the Black-Scholes equation. For interest rates we derive the volatility from at-the-money caps, by inverting the Black formula. Covariances can be

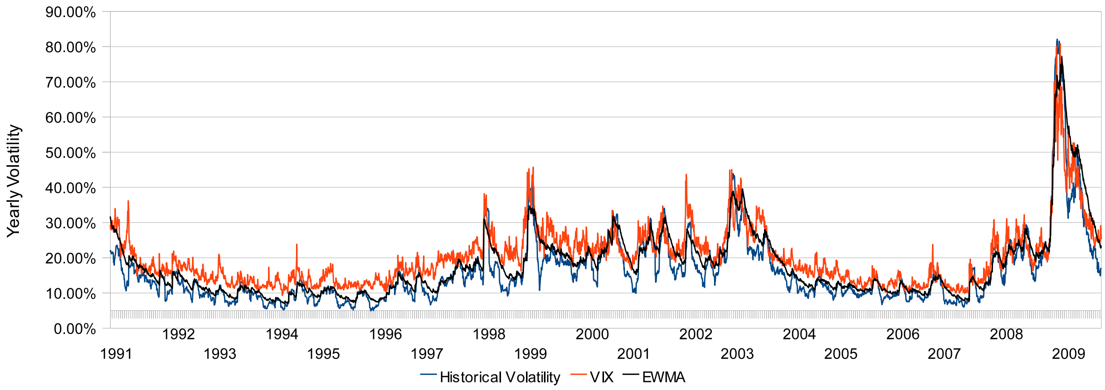

Figure 5.1.1: Implied volatility and EWMA volatility measures compared to the historical volatility.

derived from swaptions. It is important to notice that if we want our models to be market consistent, they should be calibrated against derivatives. The largest drawback with implied volatilities is that we need highly liquid markets to get satisfactory estimates. In the latest financial crisis the trading volume of many interest rate derivatives plunged, making market consistent pricing difficult.

We compare the EWMA and implied volatility measures by considering the S&P-500 index. For the EWMA we use the equations above, with $\lambda = 0.97$, which is recommended for monthly estimates. For the implied volatilities we consider the Chicago Board Options Exchange Volatility Index (VIX), a popular measure of the implied volatility of the S&P-500. From figure 5.1.1 we see that the estimates have similar behavior, but that the implied volatility overestimates the volatility most of the time. This can be seen as one way the market handles imperfections in the Black-Scholes framework.

5.2 Weighting the Portfolio

Determine portfolio weights and which assets the portfolio should consist of is the "more-than-full-time" job of extremely well paid fund managers. So it should be clear that we need to limit our active management of the portfolio, to simple and automated procedures. There are many ways to do this, and among the most famous models are Modern Portfolio Theory (MPT), introduced by Harry Markowitz (1952), [15] and the Capital Asset Pricing Model (CAPM) introduced by Treynor, Sharpe, Lintner and Mossin[1] which largely builds on the work done

[1]For a summary of the development of CAPM see French (2003), [6].

by Markowitz. Since the asset classes that we will consider are very different i nature, and the returns are not directly comparable, we need another model to set the weights between asset classes.

5.2.1 Asset Class Weighting

Since we have a portfolio that in many ways resemblance a portfolio of a life insurance company we have chosen to consider how Norway's largest privately owned life and pension insurance company weight their portfolio. And we have build a model for weighting the different asset classes in our portfolio based on our observations. For a more comprehensive description of the model see appendix A. The proportion of equity in the portfolio is determined by

$$\hat{w}_{eq}(t) = 11.5440 - 3.5784 \cdot \sigma_{eq}(t) + 4.1051 \cdot L_1(t) \quad \% \tag{5.2.1}$$

and the proportion of bonds is determined by

$$\hat{w}_b = 65.2476 + 5.2213 \cdot \sigma_{eq}(t) - 2.4027 \cdot L_1(t) \quad \%, \tag{5.2.2}$$

where $\sigma_{eq}(t)$ is the EWMA estimate of the volatility of the S&P-500 at time t, and $L_1(t)$ is the 1-Year US interest rate at time t. The proportion of the portfolio invested in money market positions is given by

$$\hat{w}_{mm} = 100\% - \hat{w}_{eq} - \hat{w}_b. \tag{5.2.3}$$

In figure 5.3.1 we see the weighting of our portfolio for our ten years of data described in chapter 6.

5.2.2 Modern Portfolio Theory(MPT)

The idea behind Modern Portfolio Theory is that we cannot select an asset in a portfolio based on the history and expectation of the single asset alone. Rather we have to consider how each asset performs relative to the rest of the portfolio. Key concepts in MPT are risk adjusted return and correlations. Let the expected return for asset i, $\hat{\mu}_i$ be given by equation (5.1.2) and the covariance between asset i and j, $\hat{\Sigma}_{ij}$ is estimated by equation (5.1.1). Let us also assume that the risk-free rate for the given investment horizon is known. Now, consider a market consisting of n risky assets, that we assign with weights $w_1, w_2, \ldots, w_n$ such that $\sum_{i=1}^{n} w_i = 1$. This results in an expected return of the portfolio of

$$r_p = \sum_{i=1}^{n} w_i \hat{\mu}_i, \tag{5.2.4}$$

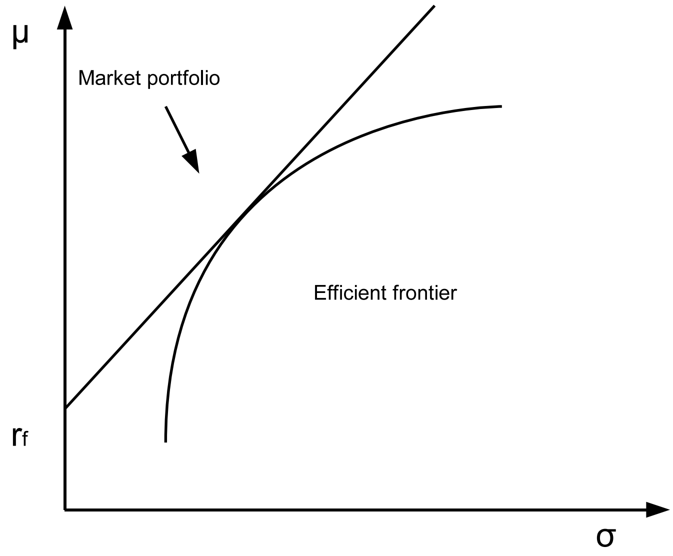

Figure 5.2.1: Capital market line.

and the standard deviation of the portfolio given by

$$\sigma_p = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \Sigma_{ij}}. \tag{5.2.5}$$

In figure 5.2.1 we see the possible combinations of risk and expected return. If we invest in risky assets alone, the efficient frontier is the optimal portfolio, given a certain expected return $\hat{\mu}$. When we have the opportunity to borrow and lend money at the risk-free rate r_f, the optimal portfolio will be along the line, which is a tangent to the efficient frontier. We are interested in the market portfolio which is shown in the figure.

The problem of finding the market portfolio corresponds to maximizing the Sharp Ratio $(r_p - r_f)/\sigma_p$. As shown by Luenberger (1998), [14] when negative weights are allowed, i.e. short sale is permitted, this corresponds to solving a system of linear equations. Define vector of weights $v = (v_1, \ldots, v_n)^T$ and the excess return vector $\hat{r}_e = (\hat{\mu}_1 - r_f, \ldots, \hat{\mu}_n - r_f)^T$. We then solve

$$\hat{\Sigma} v = \hat{r}_e \tag{5.2.6}$$

for v. Then to get the normalized weights

$$w_i = \frac{v_i}{\sum_{i=1}^{n} v_i}. \tag{5.2.7}$$

To avoid shorting and unlimited lending we will set all negative weights to zero and normalize the non negative weights accordingly.

The Markowitz portfolio allocations are often found to be unrealistically extreme and unstable. Because of this, we are going to consider an exponentially weighted allocation, discussed by Maller et. al. (2005), [17]. I.e. we smooth out the changes of the Markowitz weights by exponentially averaging them.

$$\mathbf{W}_t = \mathbf{w}_t \cdot (1 - \lambda) + \mathbf{W}_{t-1} \cdot \lambda \tag{5.2.8}$$

where $\mathbf{W}_t$ is a vector of the exponential weighted allocations at time t and $\mathbf{w}_t$ is a vector of the Markowitz allocations given from equation (5.2.7) at time t. The resulting weighting of the equity described in section 6.1 is given in figure 5.2.2

The largest drawback of MPT is that for estimating weights for N assets we need $N + N + N(N-1)/2$ input parameters, which all needs to be estimated. It is therefore slow for large portfolios, and not necessarily accurate, since it depends on how we estimate the parameters. Here the CAPM has a clear advantage over MPT, fewer inputs makes it computationally faster and potential more accurate.

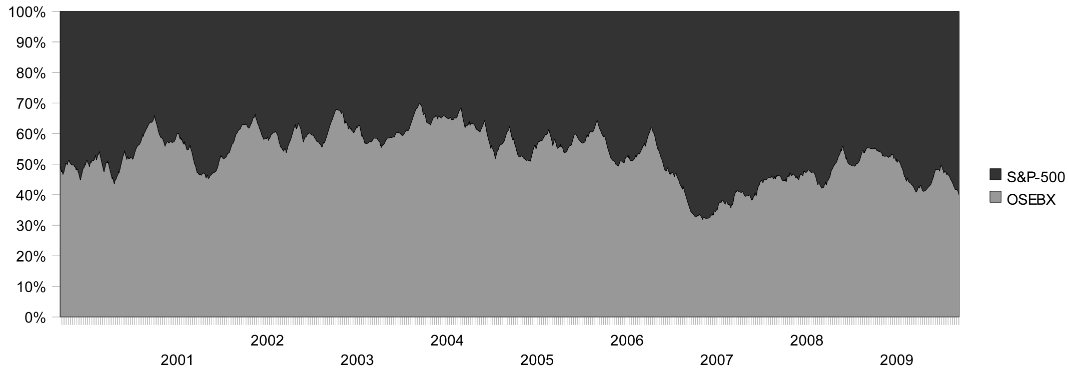

Figure 5.2.2: Simulated weighting of the equity during the last ten years.

5.3 Resulting Portfolios

We will evaluate multi asset-class portfolios consisting of bonds, equity and money market positions that are hedged with interest rate floors. Since bonds and derivatives change their characteristics as they move to maturity we want to update the portfolio after each time-period considered. We re-balance the portfolio after each time period considered, both to keep the predetermined weights of the portfolio and to keep equal time to maturity on bonds as well as equal time to maturity on the derivatives. The interest rate floors will initially be Out-of-the-Money. We assume that there are no transaction costs and that all assets are divisible.

5.3.1 Portfolio 1

We start by considering a multi asset-class portfolio, consisting of bonds, equity positions and money market positions with corresponding interest rate floors. We receive floating LIBOR rates on the money market positions. We choose a static weighting of the portfolio, with resemblance an average portfolio of a Norwegian life insurance company. The weights are chosen to be 45% zero-coupon bonds, 15% equity equally distributed between the S&P-500 and OSEBX and 40% in money market positions hedged with interest rate floors. This is summarized in table 5.3.1.

Table 5.3.1: The weighting in the portfolios.

Investment	**Portfolio 1**	**Portfolio 2**
Zero Coupon Bonds	**45 %**	$\hat{\mathbf{w}}_{\mathbf{b}}(\mathbf{t})$
5 year	45 %	$w_b(t)$
Equity	**15 %**	$\hat{\mathbf{w}}_{\mathbf{eq}}(\mathbf{t})$
Standard & Poor 500 Index	7.5 %	Markowitz
Oslo Stock Exchange Benchmark Index	7.5 %	Markowitz
Money Market and Floors	**40 %**	$\hat{\mathbf{w}}_{\mathbf{mm}}(\mathbf{t})$

5.3.2 Portfolio 2

Portfolio 2 is based on the same assumptions and asset classes. Only now we will use the dynamic weighting regime between the asset classes, as described in section 5.2.1. And we will use the Markowitz weights described in section 5.2.2 to internally weight the equity. The portfolio weighting is summarized in table 5.3.1.

How the asset-class weightings change through time is shown in figure 5.3.1, and how the internal equity weightings change through time is shown in figure 5.2.2.

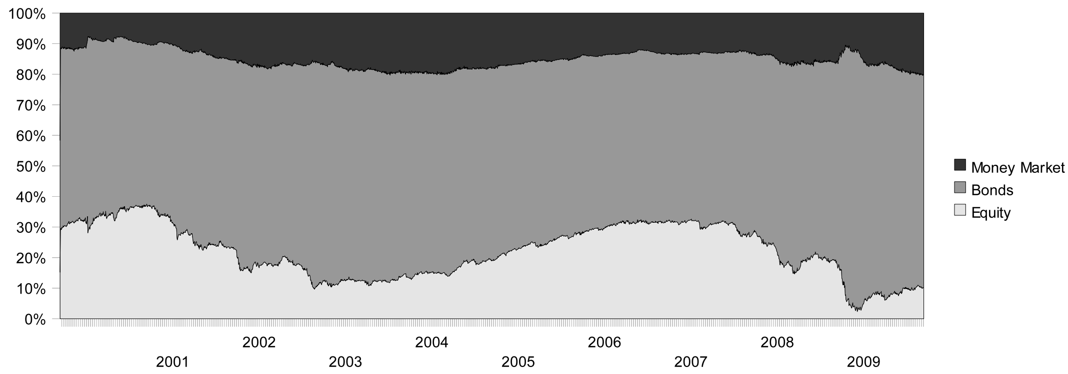

Figure 5.3.1: Simulated weighting of our portfolio during the last ten years.

5.4 Simulation and Pricing of the Portfolio

First we simulate the underlying risk factors, e.g. equity and interest rates, Δt into the future. Then we price the portfolio at time $t + \Delta t$, which can include the pricing of derivatives. In these two steps we are basically doing the same things. But there are some differences. First is the drift. When simulating one "real" time step into the future we have to use the real drift. As for derivatives we remember that we can use the risk-neutral drift. When we are pricing derivatives, we have to repeat our simulations many times to get an expected value of the price. When simulating the price of the underlying risk factor Δt into the future, we use the real drift, estimated from the historic data. When pricing derivatives we use the risk-neutral drift.

5.5 Flowchart

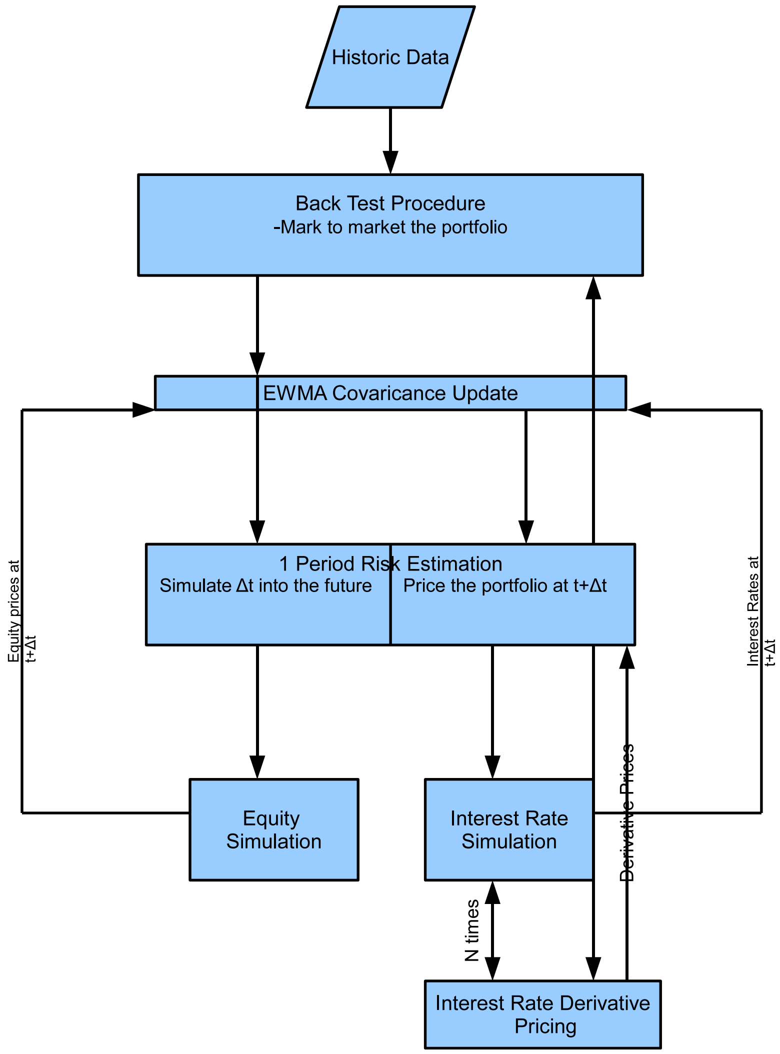

Figure 5.5.1: A flowchart describing the process.

5.6 Implementation Error

Implementation errors can come from a number of sources, such as wrong data input, making false assumptions and choices made by the professionals implementing the model. This form of system risk is difficult to quantify and is often not recognized by senior management who may take the model results as face value. A lot of research has been put into how different models perform relative to each other, [2], [8], [12], but it would be interesting to see how well different implementations of the same model perform against each other. In a paper by Marshall and Siegel (1996), [16], eight different vendors who provides VaR software based on the RiskMetrics™ approach, were given portfolios to come up with VaR estimates. The similarity of the results produced by the vendors was closely tied to the nature of the instruments in the portfolios. For FX forwards, money markets and FRAs the estimates was close. However for non linear instruments as interest rate derivatives and FX options the standard deviation as percentage of median of total VaR was 21% and 25% respectively. It should then be clear that implementation risk can be severe and should be considered when using VaR estimates.

Chapter 6

Data Analysis

Our data consists of 10 years with daily quoted equity prices of Oslo Stock Exchange Benchmark Index(OSEBX) and Standard & Poors 500 Index(S&P-500). And 10 years of US interest rates with maturities $\{1y, 2y, 3y, 4y, 5y\}$.

6.1 Equity

In this section, we are going to look at the Black-Scholes Financial Market, where we assume that the log-returns of an asset or index are realizations of a standard normal distribution. We are going to investigate this assumption. If we look at figure 6.1.1, we see that it varies how good the standard normal distribution fits the data. We see that most of the time the model fits the data, but it underestimates the density of large changes in the index prices. If we study the S&P-500-index, we see that we have extreme log-return values larger than ± 0.10, i.e. changes in the price of $\pm 10\%$ a day. If we compare this to the fact that the greatest annual change in the S&P-500-index during the last 20 years is -38.49%, we realize that these values are extreme. Further, if we investigate figure 6.1.2, we notice that the empirical distribution of the indexes has a steeper peak and "fatter tails" than the normal distribution. The latter confirms our observation in figure 6.1.1, that the normal distribution underestimates the number of extreme observations. Another interesting thing to observe is how the volatility changes over time. If we investigate figure 6.1.3, we can clearly detect that the volatility is not constant over longer periods of time. However, if we look at shorter time intervals we notice that an assumption of constant volatility can be reasonable. I.e. for short time equity contracts, an assumption of constant volatility can be reasonable. The observation of non-constant volatility leads us to think that the normality of the prices also varies over time. Figure 6.1.3 indicate volatility clustering, which is common in financial markets, i.e. large changes tends to be followed by large changes, and

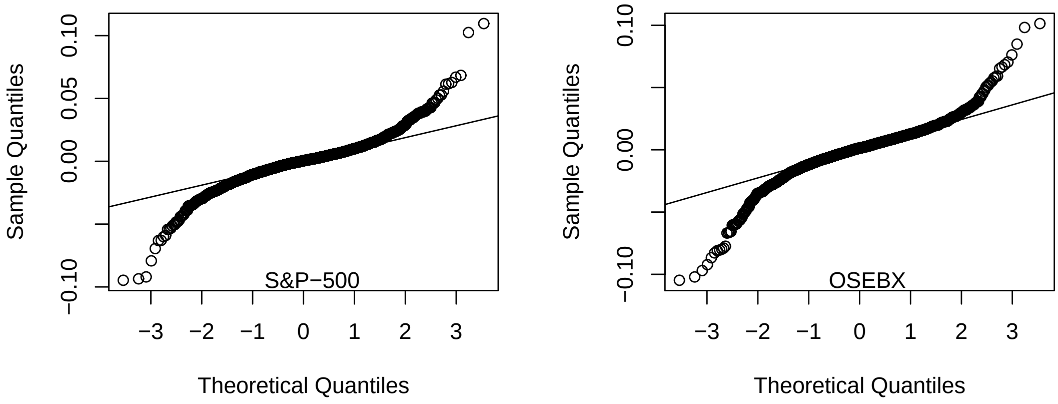

Figure 6.1.1: Quantile-quantile plots of the log-returns of the S&P-500 and OSEBX for our 10 years with data.

small changes tends to be followed by small changes. Comparing the quantile-quantile-plots in figure 6.1.4, one in the period from 09.15.2008, the day Lehman Brothers filed for bankruptcy[1], to 09.16.2009, the day Ben Bernanke, US Federal Reserve chairman, said that the US recession was likely to be over. The other is the period of 2004 to 2006. It is clear that in some periods of time the normal distribution fits the data good and in other periods, not so good. Liquidity is also a factor that should be considered, since we expect that normality is better in liquid markets. The trading volume of S&P-500 is significantly higher than the trading volume of Oslo Stock Exchange Benchmark Index(OSEBX). And if we focus on the "normal" market conditions between 2004 and 2006 we can compare the quantile-quantile-plot of S&P-500 with the quantile-quantile-plot of OSEBX in the left side of figure 6.1.4. We observe a slightly better fit for the S&P-500 than for the OSEBX. Event though we should be careful to make any conclusions, since there are other factors involved (e.g. OSEBX is heavily correlated with the oil price), it definitively support our assumption.

[1]this day has by many been called the start of the current financial crisis, even though one can argue that the problems started before this day

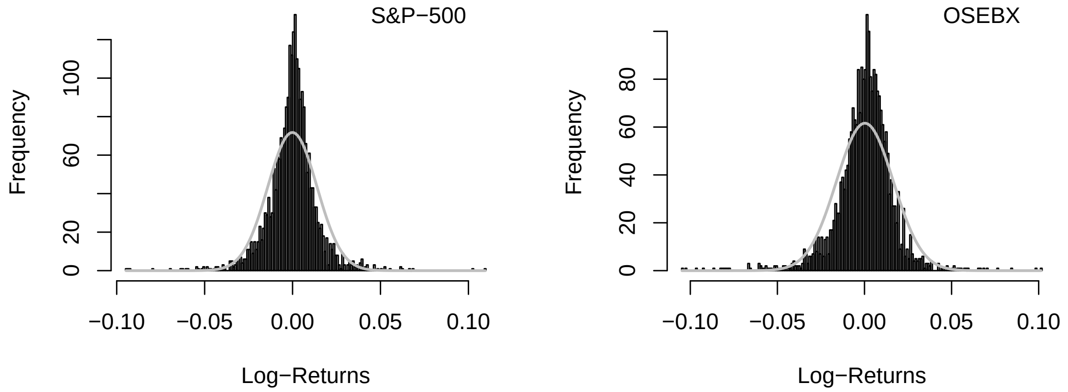

Figure 6.1.2: Histograms of the log-returns of S&P-500 and the OSEBX. Based on our 10 years with data. Together with the normal distribution in gray, with mean and variance calculated from historical data.

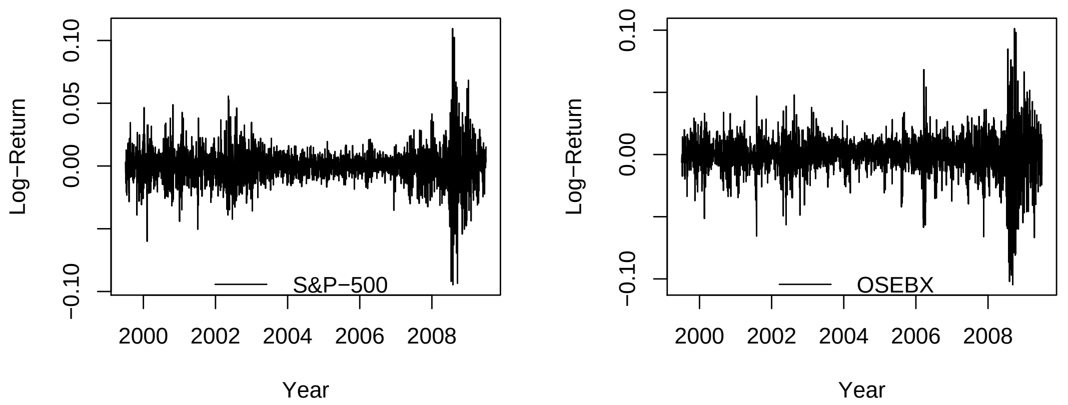

Figure 6.1.3: Daily log-returns of S&P-500 and OSEBX from our 10 years with data.

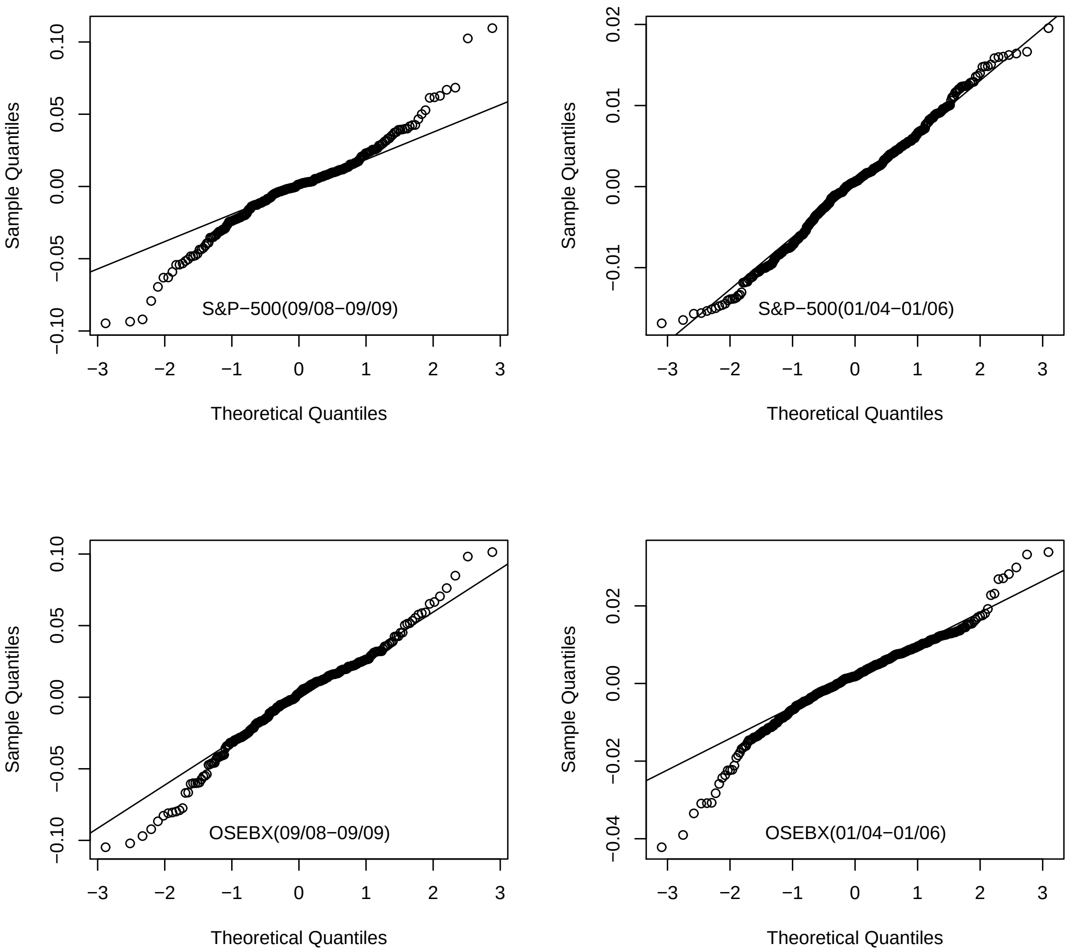

Figure 6.1.4: Quantile-quantile-plot of the log-returns of S&P-500 and OSEBX under the turbulent market conditions in the period from September 2008 to September 2009 (left) and normal market conditions, here illustrated by the period from 2004 to 2006 (right)

6.2 Interest Rates

To get a first impression of our data and how our model simulate the interest rates we consider figure 6.2.1. If we look at the simulated path[2] we notice that the simulated rates are more ragged. Even though the rates have an internal correlation structure, we do not force any constraints on how the yield curve should look like. This is not a problem when pricing derivatives, since we are only interested in the average behavior. If we are interested how the yield curve evolves through time we have to use the HJM model. As for equity, we would expect that the volatility

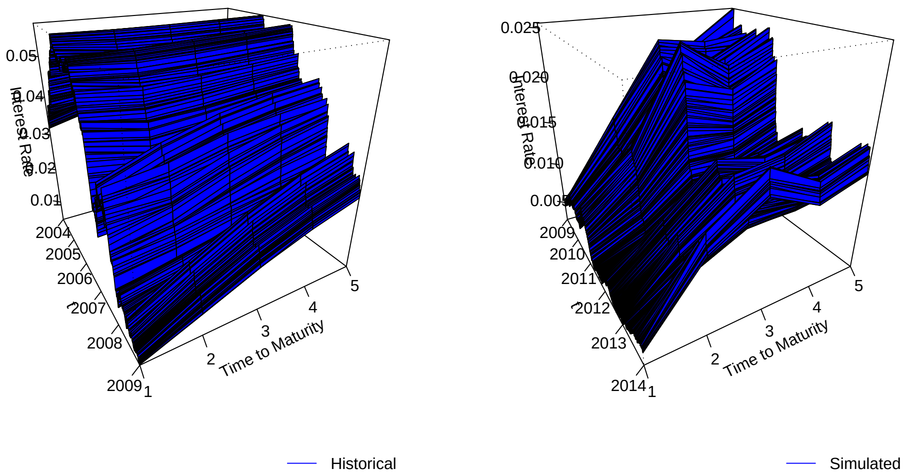

Figure 6.2.1: Historical forward rates(left) from five years back and one simulated path of forward rates(right) five years into the future plotted against time and time to maturity. NB: different scales on the interest rate axis.

only remains constant for short time intervals. This is investigated in figure 6.2.2. As for equity we observe volatility clustering. Under the LMM we expect that the residuals are multivariate normally distributed. And that each pair follows a bivariate normal distribution. We are going to test for normality as described

[2]It is important to remember that this is only one possible simulated path and should only be considered as an example of the behavior.

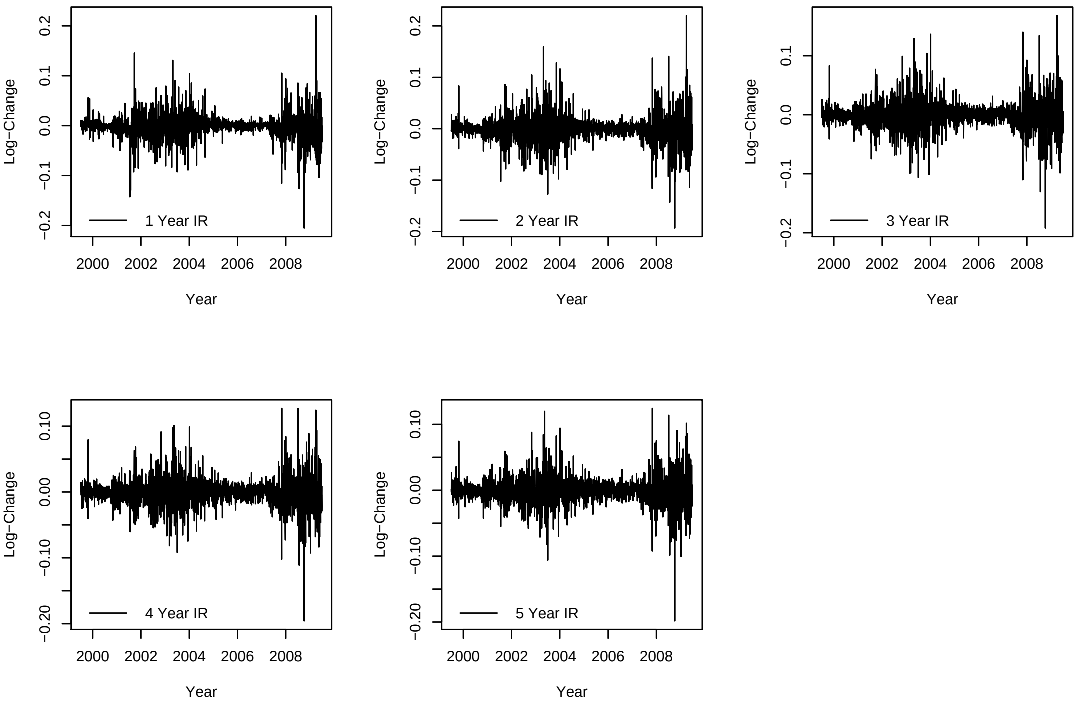

Figure 6.2.2: Daily log-changes of our interest rates.

in Johnson and Wichern (2007), [9]. When we plot the residuals from rates with different maturities against each other we would expect an elliptical form. This is confirmed in figure 6.2.3. We can also construct a confidence ellipse for the mean vector of the bivariate distributions. From the figure we observe that more points than theoretical expected falls outside the confidence ellipse. In figure 6.2.5 histograms of the residuals are shown with the normal curve, and as for equity we observe fatter tails and a steeper peak than the normal distribution suggests. Since the volatility changes over time, we would like to investigate if the model fit changes over time. In figure 6.2.4 we have investigated the residuals during the period 2005-2007. We observe a significant better fit than in figure 6.2.3. In figure 6.2.3 we have 2520 observations, and would expect approximately 25 observations to fall outside the ellipse, but observe far more. As for figure 6.2.4 we have 504 observations, and would expect approximately 5 observations to fall outside the ellipse, which is not far from what observed.

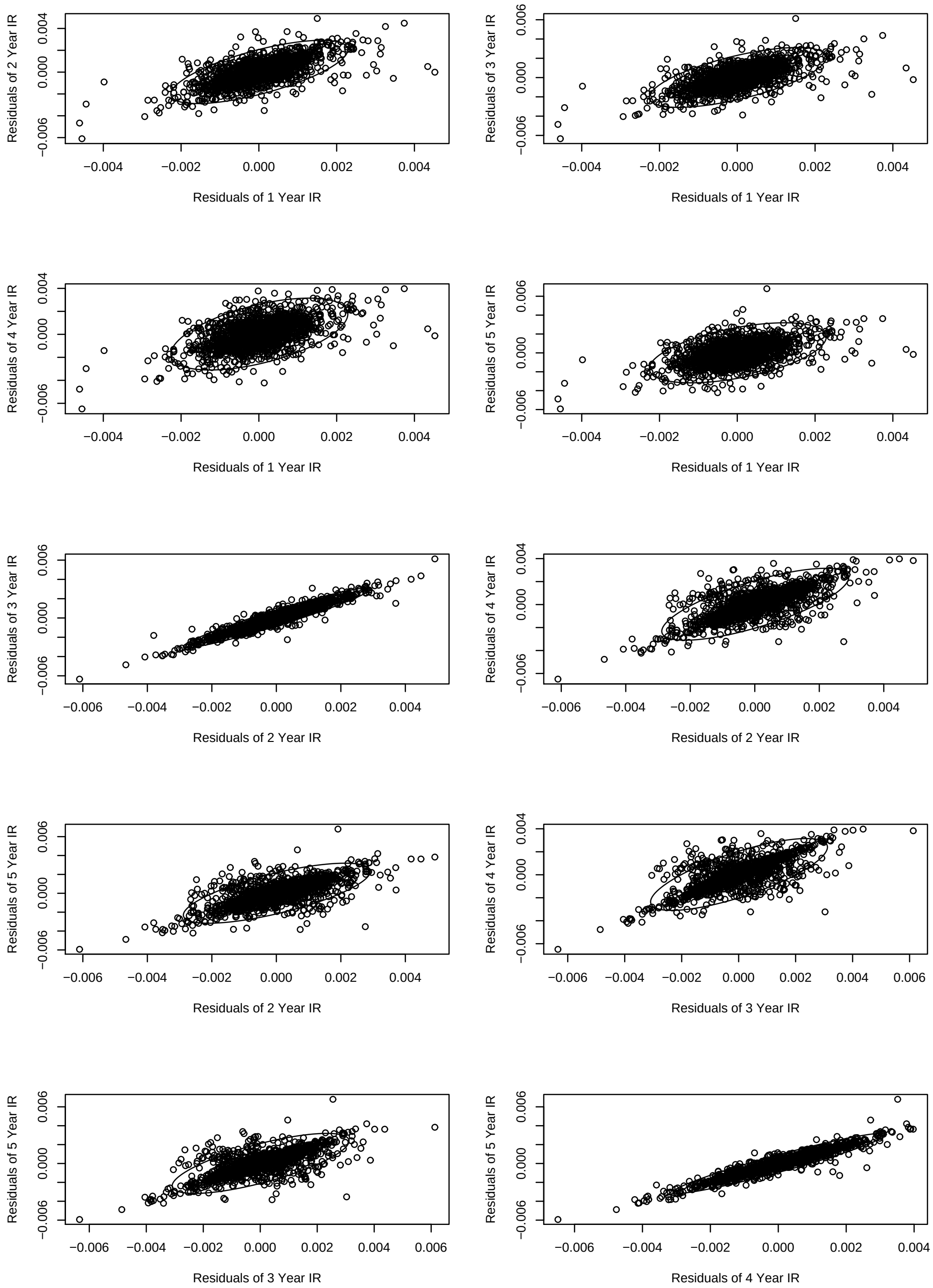

Figure 6.2.3: Residuals from the LMM of interest rates with different maturities plotted against each other. For all 10 years with data. With a 99 % confidence ellipse.

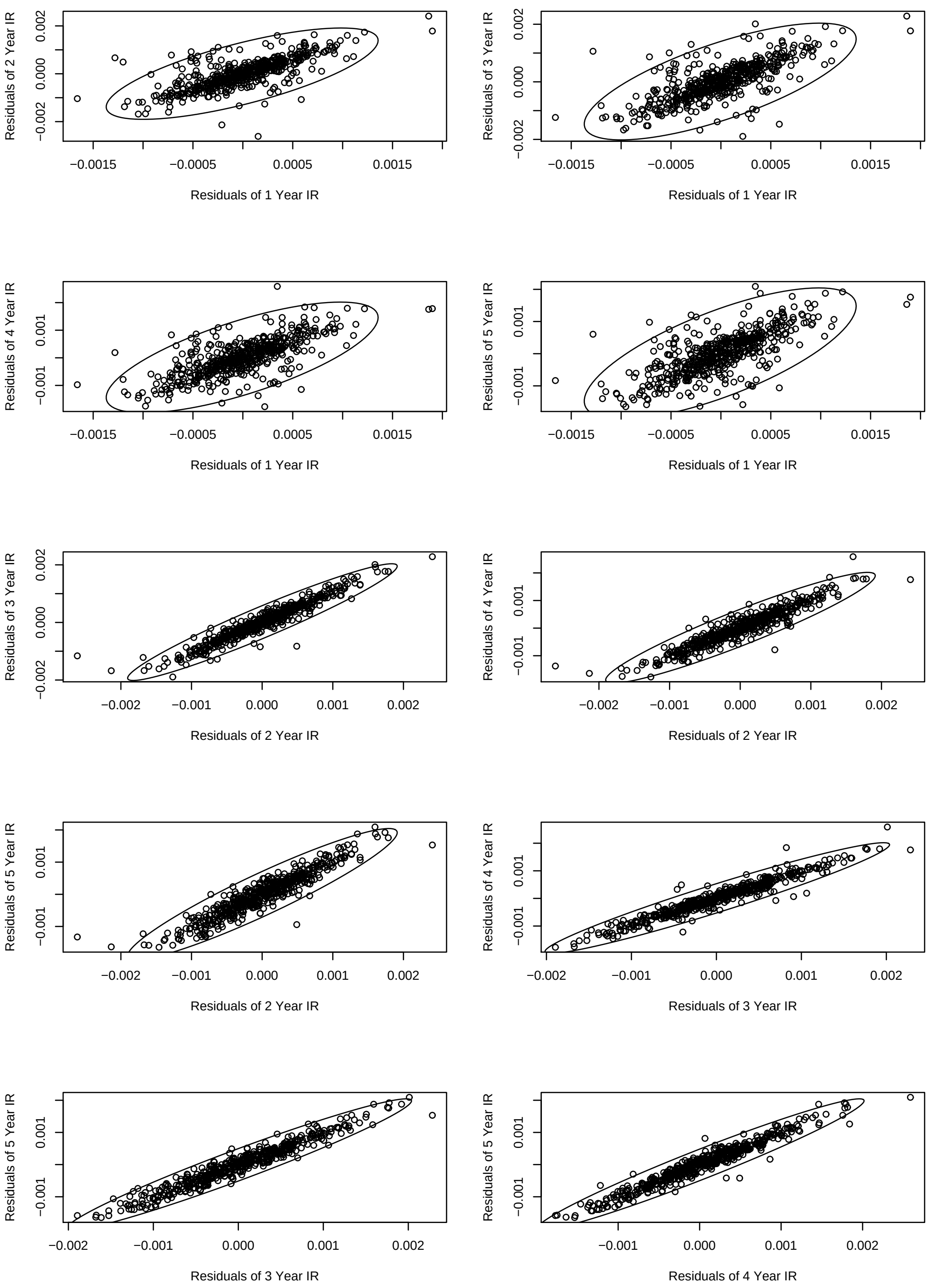

Figure 6.2.4: Residuals from the LMM of interest rates with different maturities plotted against each other. With a 99 % confidence ellipse. From the period of 2005-2007.

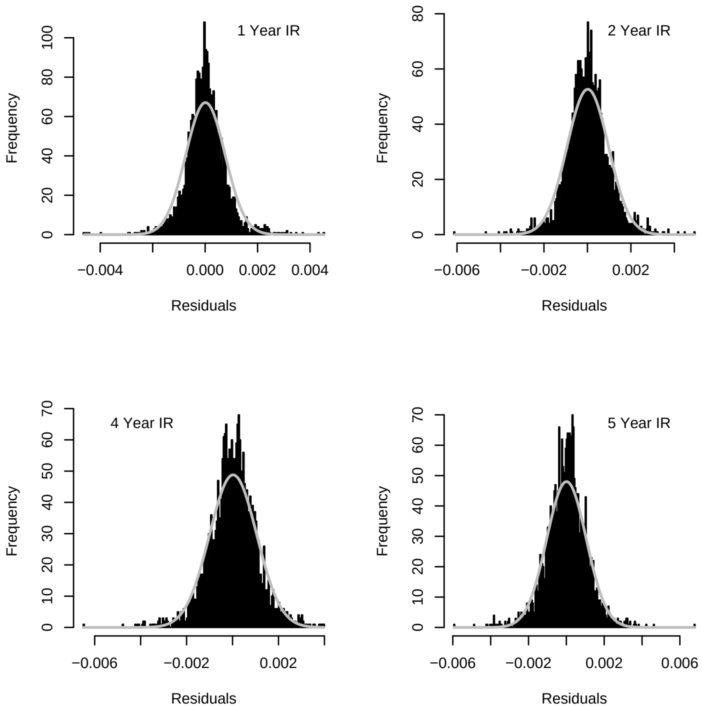

Figure 6.2.5: Histograms of the residuals of the interest rates. Based on our 10 years with data. Together with the normal distribution in gray, with mean and variance calculated from historical data.

Chapter 7

Results

In this section we will test how well our model performs against historical data. We will do a backtesting of the portfolios described in chapter 5.3 against the data described in chapter 6. Model verification is an important part of the risk management process. Backtesting is a well used method and is an important part of the regulations given by the Basel Committee. Backtesting also allows the risk managers to constantly improve their models, but at the same time one have to be aware of the danger of over-fitting a model to historical data.

From the literature we expect that the backtesting yields more exceptions, i.e. losses that are larger than the limit given by the risk measures, than what the theoretical foundation predicts. This can be due to several reasons, e.g. model imperfections, false assumptions and implementation errors.

7.1 Validation

As mentioned we would expect a higher exception frequency than the theoretical foundation suggests. But how do we determine what is too high, or even too low? The simplest method to verify accuracy is to set up a test along the lines of Bernoulli trails. Under the null hypothesis that the model is correctly calibrated, the number of exceptions follows a binomial probability distribution. One need to balance type 1 errors, rejecting a correct model against type 2 errors, not rejecting an incorrect model. To get a test that is said to be powerful, one would want to set a low type 1 error rate and then have a test that creates a very low type 2 error rate. Kupiec (1995), [13] developed approximate 95% confidence regions for such a test. An excerpt of the results is given in table 7.1.1. If we look at the Basel (1996a) rules for backtesting, they define penalty zones which are reported in table 7.1.2. It is only an incursion to the red zone that generates an automatic

Table 7.1.1: 95% Nonrejection test confidence regions for the number of exceptions N.

VaR Confidence Level	T = 252 Days	T = 510 Days	T = 1000 Days
99 %	N < 7	1 < N < 11	4 < N < 17
97.5 %	2 < N < 12	6 < N < 21	15 < N < 36
95 %	6 < N < 20	16 < N < 36	37 < N < 65

penalty. In the yellow zone it is up to the supervisor. One have to establish if the results are due to a faulty model or bad luck. As stated in Jorion (2007), [10], it is important to realize that regulators operate under constraints that differ from those of financial institutions. The approach must be implemented at a broader level, since they do not have access to detailed information about the models. With

Table 7.1.2: The Basel penalty zones for a VaR(99%) model over 250 trading days.

Zone	Number of Exceptions	Increase in k[1]
Green	0 to 4	0.00
Yellow	5	0.40
	6	0.50
	7	0.65
	8	0.75
	9	0.85
Red	10+	1.00

the Expected Tail Loss measure we do not know exactly what to expect from the frequency of losses exceeding the measure from its definition alone. But we do expect that the tail distribution for the losses is right skewed. Thus should the ETL measure be farther out in the tail than the median of the tail distribution. Therefore, the losses exceeding ETL(α) should be less than (1-α)/2, which means that less than 0.5% of the losses should exceed the ETL(99%).

7.2 Portfolio 1: A Static Weighted Multi Asset-Class Portfolio.

In figure 7.2.1 we see the results from a backtesting of portfolio 1 against 10 years of data. We notice that the risk measures adapt to the market quite well, i.e. they responds quickly to changes in the volatility. Most of the time they predict

[1] k is a multiplicative factor to determine the market-risk charge. A larger k, means that one have to set aside more capital.

Figure 7.2.1: Backtesting of Portfolio 1. Loss is defined positive.

the potential losses well. As expected, the VaR and ETL do have slightly different behavior, and the ETL catches more of the losses. Let us compare the results from the backtesting of portfolio 1 against the guidelines given above. If we compare figure 7.2.2 with table 7.1.1 we see that our VaR(99%) risk measure lies inside the confidence interval in all the cases. If we compare it with the guidelines given by the Basel Committee in table 7.1.2, we see that the VaR(99%) risk measure is in the "green" zone in all the years tested, except the period 2004-2005, where it is just inside the "yellow" zone. If we look at the whole period we see that the 99% VaR quantiles are violated on average 1.03% of the days, compared to the theoretical value of 1%. This is good. As previously discussed we do expect, on a theoretical basis, that the ETL(99%) measure should be violated less than 0.5% of the days. And we observe that it is violated on average 0.64% of the days, which is good. As mentioned, under the null hypothesis, presuming that the model is correctly calibrated, the number of exceptions follows a binomial probability distribution. To compare behavior we can compare our cumulative VaR exceptions with one randomly generated path using 2520 independently generated realizations from a

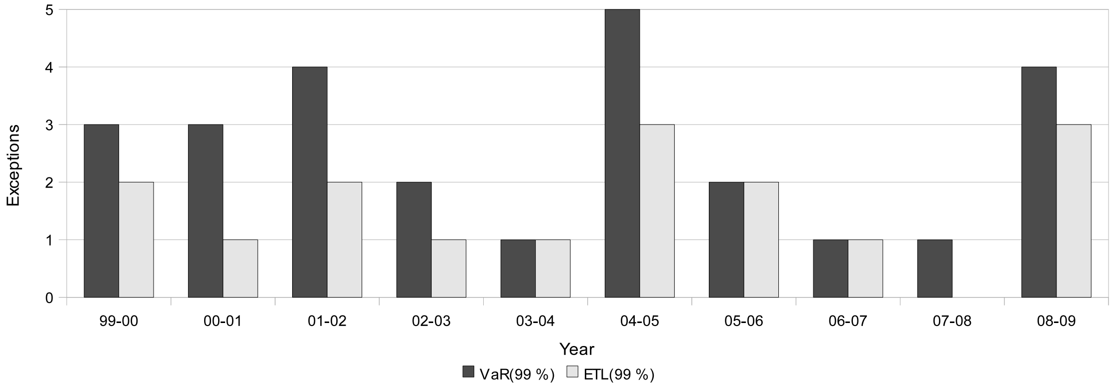

Figure 7.2.2: Number of exceptions of VaR(99%) and ETL(99%).

Bernoulli distribution[2]. This is shown in figure 7.2.3. Even though we cannot draw any conclusion by comparing it to one random path, we notice that they have similar behavior. We might expect that our models would perform worse

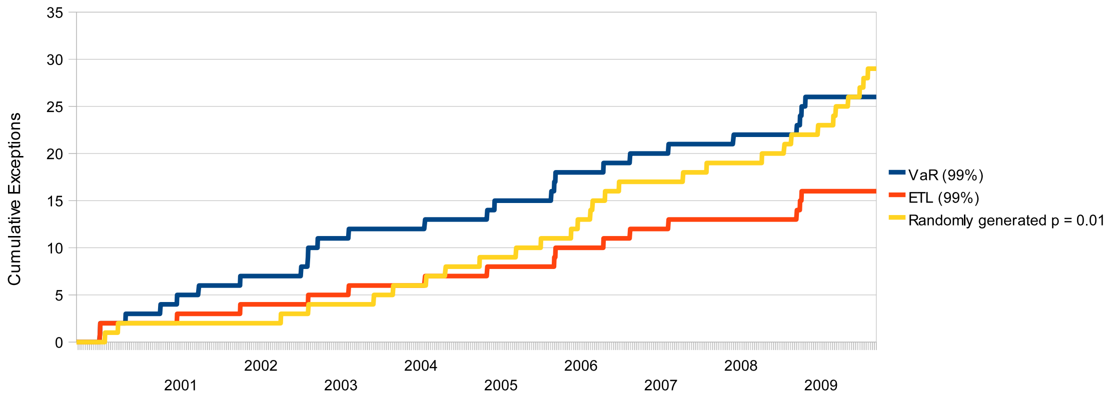

Figure 7.2.3: The cumulative distribution of losses of portfolio 1 exceeding VaR(99%) and ETL(99%). One random path of independent realizations of a Bernoulli distribution.

in turbulent markets, than under normal market conditions. However, we cannot find any evidence of this in our findings for this portfolio. On the contrary, the only time the VaR(99%) measure made an incursion to the "yellow" zone of the Basel guidelines was under normal market conditions in the period 2004-2005.

[2]The Bernoulli distribution is a special case of the Binomial distribution

7.3 Portfolio 2: A Dynamic Weighted Multi Asset-Class Portfolio.

As for portfolio 1, we notice that the risk measures adapt to the market quite well and predict the potential losses well, most of the time. Since the dynamically weighting procedures adds complexity to the portfolio, and the model, and that it consists of a slightly larger proportion of risky assets than portfolio 1, we expect a more volatile portfolio, we also expect that the performance of the risk measures is slightly worse than for the static portfolio. If we compare figure 7.3.1 with 7.2.1

Figure 7.3.1: Backtesting of Portfolio 2. Loss is defined positive.

we see that the dynamic portfolio is more volatile than the static one. As we did for portfolio 1 we compare the results from the backtesting of portfolio 2 against the guidelines given above. When we compare figure 7.3.2 against table 7.1.1 we see that we have one observation that lies just outside the the confidence interval. If we compare it with the guidelines given by the Basel Committee in table 7.1.2, we see that the VaR(99%) risk measure made incursions to the "yellow" zone three times. If we look at the whole period we see that the 99% VaR quantiles are violated on average 1.67% of the days. We observe that the ETL(99%) measure

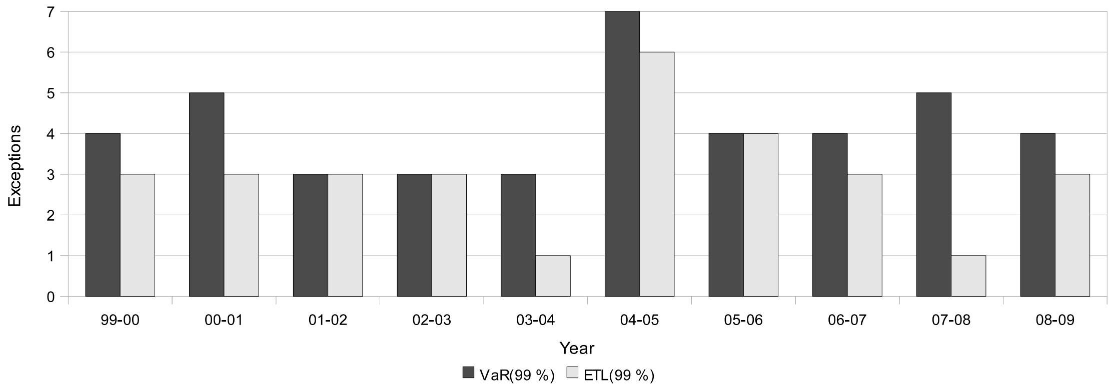

Figure 7.3.2: Number of exceptions of VaR(99%) and ETL(99%).

is violated on average 1.19% of the days. As we did for portfolio 1 we compare the cumulative VaR exceptions with one randomly generated path using 2520 independently generated realizations from a Bernoulli distribution. As mentioned, we cannot draw any conclusions by comparing it to one random path, but here as well we notice that they have similar behavior.

As for portfolio 1, we might expect that our models would perform worse in turbulent markets, than under normal market conditions. But we cannot find any evidence of this in our findings for this portfolio. On the contrary, the highest number of exceptions were under normal market conditions in the period 2004-2005. Because of the complexity the dynamic weighting regime adds to the model and that it on average holds a slightly larger proportion of risky assets, we expect a less satisfactory performance of the model and risk measures. This is confirmed, both by higher portfolio volatility and the number of exceptions.

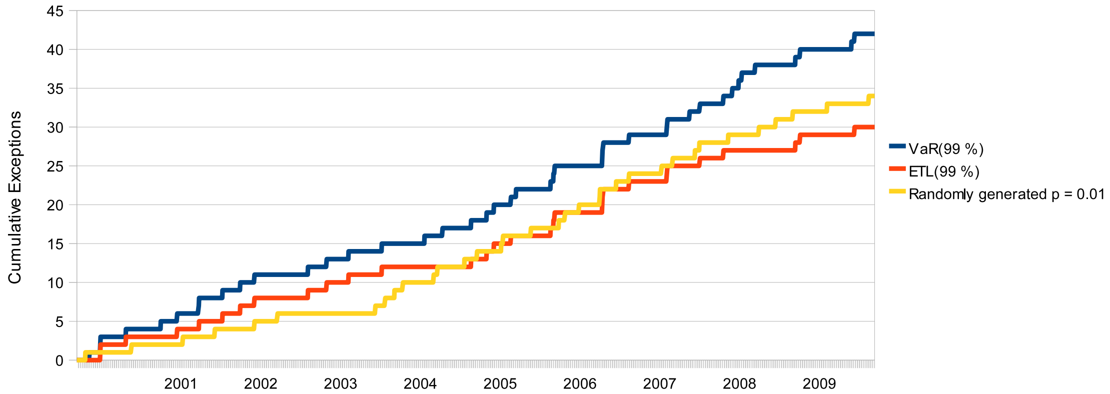

Figure 7.3.3: The cumulative distribution of losses of portfolio 2 exceeding VaR(99%) and ETL(99%). One random path of independent realizations of a Bernoulli distribution.

Chapter 8

Conclusion

In this thesis we have build a model to measure risk on multi asset-class portfolios using Monte-Carlo simulations. For equity we have simulated the price changes under the well known Black-Scholes framework. For interest rates we have used the Libor Market Model. To capture the dynamics between the assets, we drew the random terms in the models from a multivariate normal distribution. We assumed that the covariance matrix is constant over the risk horizon. In the historical data we observed properties that are not in line with the assumptions made in our models. The normal distribution fails to describe the "fat tails" observed in the distribution of log-returns, i.e. underestimates the probability of extreme events. We assumed constant volatility, yet we observed volatility clustering, i.e. large changes tends to be followed by large changes, and small changes tends to be followed by small changes. We assumed constant covariance, but observed tail dependence, i.e. in extreme events, correlations between assets seems to be higher. Thus we can conclude that the model assumptions are not satisfactory at all times, and may be especially poor during the turbulent period of 2007 to the present.

In order to be relevant in terms of real-life markets we consider portfolios that resemble those of life insurance companies. One with static weighting, and one that is dynamically weighted.

We have performed a backtesting on both of the portfolios against ten years of data, from September 1999 to September 2009. This period includes the burst of the dot-com bubble in the early 2000s, the current financial crisis and the period of 2004 to 2007 with low and steady equity volatility. The results shows that the portfolio losses exceed the estimated Value-at-Risk quantiles slightly more frequent than the corresponding confidence level α predicts. I.e. theory predicts that the VaR(α) should be violated (1-α) of the days. Where for our static portfolio the 99% VaR quantiles are violated on average 1.03% of the days. As for our dynamically weighted portfolio the violations occur more often, on average 1.67%.

From its definition alone, we do not know what to expect from the frequency of losses exceeding the ETL measure. However, we expect that the tail distribution for the losses is right skewed, thus the ETL measure should be farther out in the tail than the median of the tail distribution. Thus, the losses exceeding ETL(α) should be less than (1-α)/2, which means that less than 0.5% of the losses should exceed the ETL(99%). As for our static portfolio, the losses exceeding ETL(99%) is on average 0.64% of the losses. For the dynamically weighted portfolio the losses exceeding the ETL(99%) occur more often, on average 1.19% of the losses exceeded the measure.

If we investigate the period after the Lehman Brothers collapse 15. September, 2008 we find strong indications on clustering between large losses. Actually all of the VaR(99%) and ETL(99%) exceptions for the static portfolio in the period of fall of 2008 - fall of 2009 occurred in a one month time-frame. We have similar observations for the dynamic portfolio.

A priori, we expected that the number violations of the risk measures from the backtesting of our portfolios would be significantly larger than theoretically expected and that the model would perform less satisfactory during the latest turbulent period. However. for the static weighted portfolio, the number of violations were not significantly higher than theoretically expected, and well inside the confidence interval of a statistical test as well as the guidelines given by the Basel Committee. For the dynamically weighted portfolio, the number of violations were slightly higher than theoretically expected, and just outside the confidence interval of a statistical test. It also made three incursions into the "yellow" zone defined in the guidelines given by the Basel Committee. This indicate that the added complexity with the dynamically weighting regime and the slightly larger proportion of risky assets, result in a slightly less satisfactory model. Under the latest turbulent period the model only performs slightly worse for both the portfolios. Thus it seems like our model adapt to the changes in the market quite fast. From this we conclude that as long we are aware of the weaknesses of the model, it provides valuable measures even under turbulent market conditions.

Further Work

We have only considered a risk horizon of one day, and we would expect that the assumption of constant volatilities and correlations is reasonable with a short horizon. We also assume that the EWMA method for estimating volatilities and correlations adapts fast to changes in the market, when the horizon is short. Thus, it would be interesting to investigate how well the model performs on a longer risk horizon.

Bibliography

[1] Philippe Artzner, Freddy Delbaen, Jean-Marc Eber, and David Heath. Coherent measures of risk. *Mathematical Finance*, 9(3):203–228, 1999.

[2] Tanya Styblo Beder. Var: Seductive but dangerous. *Financial Analysts Journal*, 51(5):12–24, 1995.

[3] George Casella and Roger L. Berger. *Statistical Inference.* Duxbury, second edition, 2002.

[4] F. Delbaen and W. Schachermayer. A general version of the fundamental theorem of asset pricing. *Mathematische Annalen*, 300:463–520, 1994.

[5] Eugene F. Fama. Efficient capital markets: A review of theory and empirical work. *The Journal of Finance*, 25(2):383–417, 1970.

[6] Craig W. French. The treynor capital asset pricing model. *Journal of Investment Management*, 1(2):60–72, 2003.

[7] Paul Glasserman. *Monte Carlo Methods in Financial Engineering.* Springer-Verlag, 2004.

[8] Darryll Hendricks. Evaluation of Value-at-Risk Models Using Historical Data. *SSRN eLibrary*, 1996.

[9] Richard A. Johnson and Dean W. Wichern. *Applied Multivariate Statistical Analysis.* Pearson, 2007.

[10] Philippe Jorion. *Value At Risk. The New Benchmark for managing Financial Risk.* McGraw-Hill, 3 edition, 2007.

[11] J.P.Morgan and Reuters. Riskmetrics - technical document. Technical report, J.P.Morgan and Reuters, 1996.

[12] Kuester, Keith, Mittnik, Stefan, Paolella, and Marc S. Value-at-Risk Prediction: A Comparison of Alternative Strategies. *JOURNAL OF FINANCIAL ECONOMETRICS*, 4(1):53–89, 2006.

[13] Paul H. Kupiec. Techniques for Verifying the Accuracy of Risk Measurement Models. *SSRN eLibrary*, 1995.

[14] David G. Luenberger. *Investment Science.* Oxford University Press, 1998.

[15] Harry Markowitz. Portfolio selection. *The Journal of Finance,*, 7(1):77–91, March 1952.

[16] Christopher Marshall and Michael Siegel. Value-at-Risk: Implementing a Risk Measurement Standard. *SSRN eLibrary*, 1996.

[17] J Gould RA Maller, RB Durand. A dynamic implementation of the markowitz portfolio allocation procedure. *AFAANZ Conference Proceedings*, 2005.

[18] Steven E. Shreve. *Stochastic Calculus for Finance II Continous-Time Models.* Springer Finance, 2004.

[19] Paul Wilmott. *Paul Wilmott on Quantative Finance*, volume 1. John Wiley & Sons, Ltd, second edition, 2006.

[20] Paul Wilmott. *Paul Wilmott introduces Quantative Finance.* John Wiley & Sons, Ltd, second edition, 2007.

[21] Paul Wilmott, Sam Howison, and Jeff Dewynne. *The Mathematics of Financial Derivatives: A Student Introduction.* Cambridge University Press, 1995.

Appendix A

A Model for Weighting the Portfolio

Since we consider asset classes that are dominant parts of the portfolios held by life insurance companies, it was natural to turn to them to see how they change their portfolio weights trough time.

We have investigated how the portfolio weights changes through time for Norway's largest privately owned life and pension insurance company. The data is collected from the company's quarterly accounts. Originally does the life and pension insurance portfolios consist of more asset classes, but we only consider equity, bonds and money market positions. The weights are normalized according to this. As seen in figure A.0.1 the data ranges from fourth quarter of 2004 to third quarter of 2009, with the second quarter of 2009 missing. As expected we observe that the weightings change trough time. Since life insurance companies only can hold a

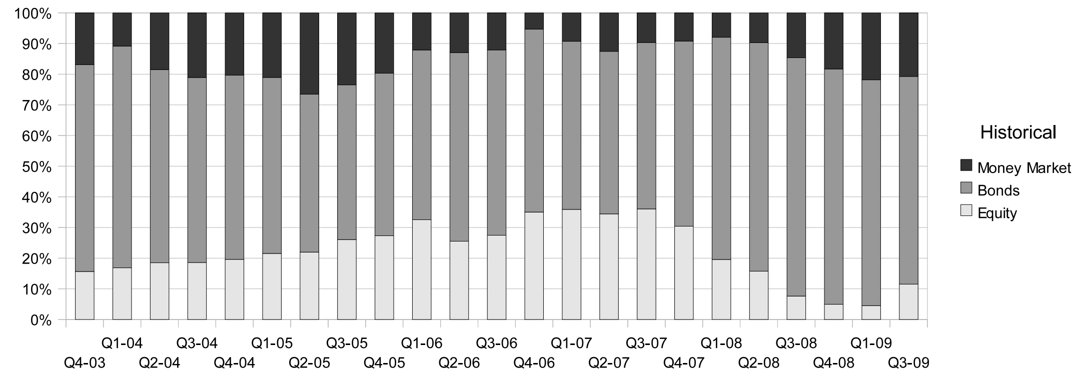

Figure A.0.1: Historical weighting of a life insurance portfolio.

given amount of risk, we would expect that the weights depend on the volatility of the equity marked. We would also expect that the weights depends on the interest rate. To get an idea of how they behave with each other we will consider scatter-plots of the asset-class-weights vs. volatility and interest rate. This is shown in figure A.0.2. The plots indicate a linear relationship with volatilities and interest rates for both equity- and bond-weights. As for money market weights we do not observe any pattern. We would like a model that changes the portfolio according to changes in the equity volatility and in the interest rate. For the equity and bond-weights we will consider a multiple linear regression. Since our portfolio only consists of three asset classes, and the weights need to sum up to 100 %, we only need to model two of the weights. Then we choose the money market weight to be given by

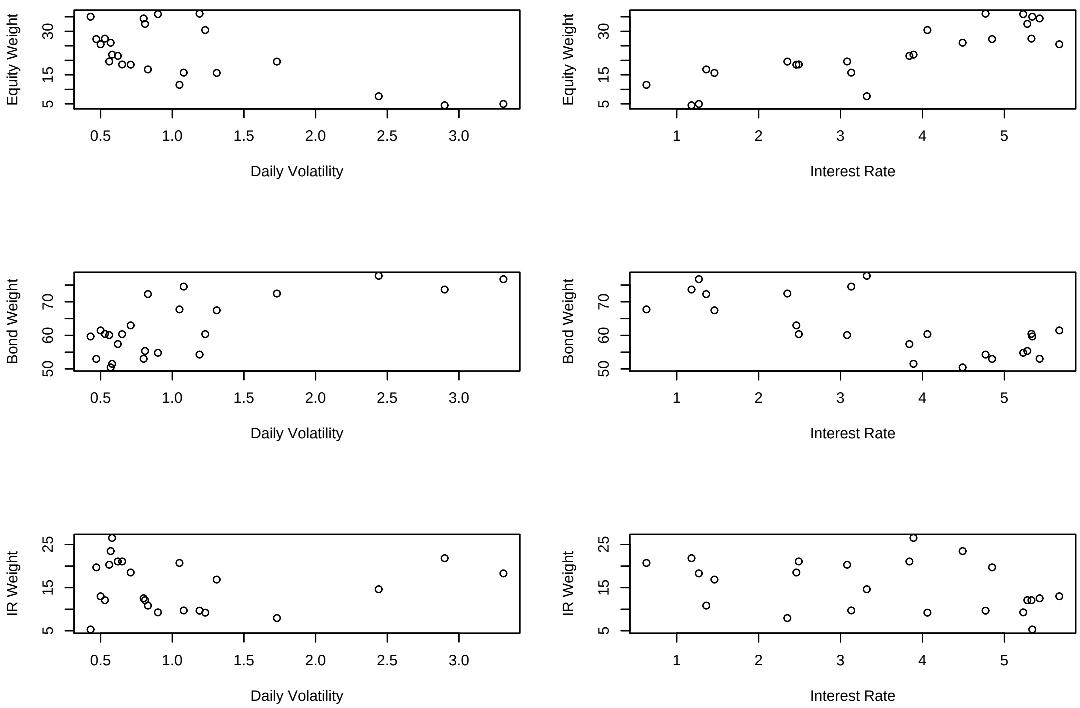

Figure A.0.2: Scatterplots of the asset-class-weights vs. volatility and interest rate.

$$\hat{w}_{mm} = 100\% - \hat{w}_{eq} - \hat{w}_b \tag{A.0.1}$$

We fit the multiple linear regression models to the data using the statistical programming language R. For both equity- and bond-weights we got a marginal better fit in some areas by including the cross-terms between the volatility and the interest rate. To reduce the risk of overfitting the model to the data we choose not to consider the cross-terms.

A.1 Equity Weights

From R we get the following model

$$\hat{w}_{eq}(t) = 11.5440 - 3.5784 \cdot \sigma_{eq}(t) + 4.1051 \cdot L_1(t) \quad \%, \tag{A.1.1}$$

where $\sigma_{eq}(t)$ is the EWMA estimate of the volatility of the S&P-500 at time t, and $L_1(t)$ is the 1-Year US interest rate at time t. We can now do a basic analysis of our model and its assumptions. Below is print out of a summary of the model from R.

```
Call:
lm(formula = eq ~ vol + ir, data = data)

Residuals:
    Min      1Q  Median      3Q     Max
-8.8117 -3.0306 -0.5919  2.9519  9.1930

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  11.5440     3.9256   2.941  0.00808 **
vol          -3.5784     1.5277  -2.342  0.02962 *
ir            4.1051     0.7444   5.515 2.13e-05 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05

Residual standard error: 4.729 on 20 degrees of freedom
Multiple R-squared: 0.7805,     Adjusted R-squared: 0.7586
F-statistic: 35.56 on 2 and 20 DF,  p-value: 2.595e-07
```

We see that all the coefficients are significant $\neq 0$ at a 5% level. We see from the R^2_{adj} that approximately 75% of the equity weights are explained by the model. The model assumes that the standardized residuals should be normally distributed, this is confirmed by the quantile-quantile plot in the top-right in figure A.2.1. We should observe homoscedasticity, i.e. that the variance of the residuals is constant. Then we expect that the residuals are independent of the fitted values of the model, i.e. we should not observe a pattern when plotting them against each other. We do not find any pattern in the top-left of figure A.2.1.

A.2 Bond Weights

From R we get the following model

$$\hat{w}_b = 65.2476 + 5.2213 \cdot \sigma_{eq}(t) - 2.4027 \cdot L_1(t) \quad \%, \tag{A.2.1}$$

where $\sigma_{eq}(t)$ is the EWMA estimate of the volatility of the S&P-500 at time t, and $L_1(t)$ is the 1-Year US interest rate at time t. As for the equity weighting model we do a basic analysis of the model and its assumptions. First a print out of the summary of the model.

```
Call:
lm(formula = bond ~ vol + ir, data = data)

Residuals:
   Min     1Q Median     3Q    Max
-7.410 -2.904 -1.476  4.422 11.164

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  65.2476     4.3252  15.086 2.17e-12 ***
vol           5.2213     1.6832   3.102  0.00562 **
ir           -2.4027     0.8202  -2.930  0.00829 **
---
Signif. codes:  0 *** 0.001 ** 0.01

Residual standard error: 5.21 on 20 degrees of freedom
Multiple R-squared: 0.6667,     Adjusted R-squared: 0.6334
F-statistic:    20 on 2 and 20 DF,  p-value: 1.693e-05
```

We see that all the coefficients are significant $\neq 0$, at an 1% level. And we see from the R^2_{adj} that approximately 65% of the bond weights are explained by the model. As for the equity weighting model we check for normality in the standardized residuals and that the residual variance is constant. From bottom of figure A.2.1 we see that these assumptions seems valid.

A.3 Implemented Model

How does our weighting model perform when implemented in our risk-estimation model. If we compare the simulated equity weights in figure A.3.1 with the historical weights in figure A.0.1, we see that our model replicate the historical weights quite well. It produces a bit smoother results than observed in our data, but describes the main trends well.

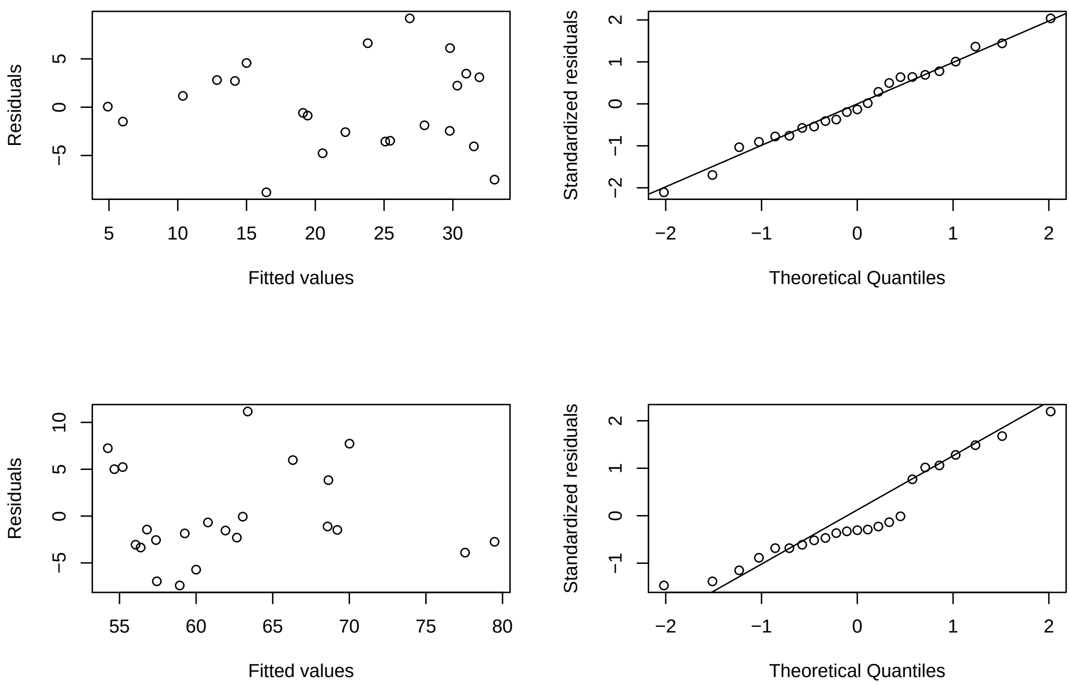

Figure A.2.1: Top-left: The residuals vs. the fitted values of the equity weighting model. Top-right: quantile-quantile plot of the standardized residuals of the equity weighting model. Bottom-left: The residuals vs. the fitted values of the bond weighting model. Bottom-right: quantile-quantile plot of the standardized residuals of the bond weighting model.

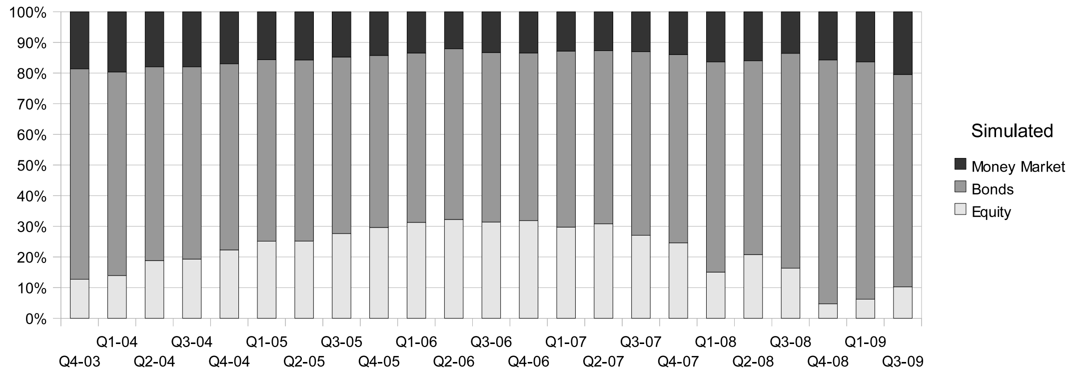

Figure A.3.1: Simulated weighting of our portfolio. Dates matched those in figure A.0.1