Minimum Time Disturbance Rejection and Tracking Control of $n + m$ Linear Hyperbolic PDEs

Henrik Anfinsen and Ole Morten Aamo

***Abstract*— We extend recent results regarding disturbance rejection control of $n + m$ linear hyperbolic partial differential equations (PDEs) in a number of ways: 1) We allow spatially varying coefficients; 2) The disturbance is allowed to enter in the interior of the domain, and; 3) Rejection is achieved in minimum time. Additionally, we solve a tracking problem, where the tracking objective is achieved in finite, minimum time. We use a recently derived Fredholm transformation technique in addition to infinite-dimensional backstepping in our design.**

I. Introduction

In this paper, we consider a class of $n + m$ coupled first order linear hyperbolic partial differential equations (PDEs), where n PDEs convect in one direction and m PDEs convect in the opposite direction. Systems of this type have lately been subject to extensive research due to the vast amount of physical systems that can be modeled this way, ranging from heat exchangers [1] and oil wells [2] to predator-prey systems [3].

Infinite-dimensional backstepping has in recent years proved to be a very efficient method for design of controllers and observers for PDEs. The key ingredient of this method is the introduction of an invertible Volterra transformation that maps the system of interest into a target system for which analysis is easier. This method represents a major paradigm shift for infinite dimensional controller design, since it on the one hand requires no discretization before an eventual implementation on a computer, and on the other hand handles boundary actuation and sensing, which lead to unbounded input respectively output operators, in a straight forward manner.

The first use of infinite-dimensional backstepping on PDEs was for the parabolic heat equation in [4]. The extension to first order hyperbolic PDEs was done in [5], with an expansion to 2×2 systems in [6] and $n + 1$ systems in [7]. In such systems, n PDEs convect in one direction, and one PDE convect in the opposite direction (2×2 systems are two coupled PDEs, one convecting in each direction, which is the case $n = 1$). The solution to general $n + m$ systems, which we consider in this paper, were presented in [8] and [9] for the case of constant and spatially varying coefficients, respectively. In such systems, an arbitrary number of PDEs convect in each direction. The resulting controller in [8] required the solution to a set of coupled PDEs that are usually solved numerically using successive approximations. The control law from [8] achieved convergence in a finite time given by the sum of all propagation delays from the actuated to the unactuated boundary. This was later improved in [10], where a solution was offered achieving convergence in minimum time, as the convergence time only depended on the slowest transport delay. The method in [10] required the solution to an even more complicated set of PDEs that were cascaded in structure, making the proof of well-posedness as well as numerically solving them considerably harder. This was later improved in [11] where a Fredholm transformation was used in combination with the results from [9] to obtain a minimum time controller in a much less complicated fashion.

A disturbance rejection problem was investigated for 2×2 systems in [12]. In that paper, the system had a disturbance entering at one boundary, with actuation limited to the opposite boundary. The disturbance was modeled as a linear autonomous ordinary differential equation (ODE) particularly aimed at modeling biased periodic disturbances. The disturbance entered, and its effect was rejected at the unactuated boundary. In [13], the disturbance was allowed to also enter in the interior domain, but the point of rejection was still the unactuated boundary. In [14], the point of rejection was allowed to be anywhere in the domain, a result later strengthened in [15], where a method was proposed using the solution to as set of so-called regulator equations, which allowed for a much more general control objective as well as allowing the disturbance to enter in the whole domain. Extension of the method in [12] to $n+1$ systems was given in [16], while the $n+m$ case was solved in [17]. For the latter of these, the solution had the same slow convergence time as the controller in [8].

A tracking problem for 2×2 systems was solved in [18]. In that paper, a reference trajectory was generated by "inversely" using backstepping on a very simple reference model, before a standard PI controller was used to drive the measured output to the generated reference signal. A related problem was solved for 2×2 systems in [15] simultaneously with the above mention disturbance rejection problem. The reference signal, however, was limited to one generated using an autonomous linear system. In [8], a tracking problem was solved for m heterodirectional, coupled PDEs with known, constant coefficients, while in [19] a tracking problem for $n + m$ systems with known, spatially varying coefficients was solved. However, this latter solution also required the reference signal to be generated from an autonomous linear system. The convergence time was faster than in [8], since

The authors are with the Department of Engineering Cybernetics, Norwegian University of Science and Technology, Trondheim N-7491, Norway. (e-mail: henrik.anfinsen@ntnu.no; aamo@ntnu.no).

The work of H. Anfinsen was funded by VISTA - a basic research program in collaboration between The Norwegian Academy of Science and Letters, and Statoil.

the control design was based on the minimum-time controller derived in [10].

In this paper, we improve and extend the controller results from [17] and [19]. Specifically, this paper has the following contributions:

1) We extend the results in [17], and derive a disturbance rejection controller for $n+m$ coupled linear hyperbolic PDEs with spatially varying coefficients that achieves its objective in minimum time, and also allows for the disturbance to enter in the interior domain.
2) We solve a tracking problem in minimum time where the reference signal must be bounded, but can otherwise be arbitrary. This is an extension of previous results in [19], in the sense of allowing a general reference signal, and also presenting a more compact solution to the tracking problem than the solution proposed in [19].

While the present paper was under review, [20] was published, solving a similar problem as in the present paper. In [20], the disturbance rejection and tracking controller from [15], based on regulator equations, is extended to the same type of systems considered in the present paper, which allows for a more general control objective than in the present paper in the sense that the output to be controlled can be defined at a boundary, in-domain distributed or pointwise. The control objective, however, is achieved in non-minimum time, and the reference signal is in [20] still limited to those generated using an autonomous linear system.

II. Problem statement

We consider systems on the form

$$w_t(x,t) + \Lambda(x) w_x(x,t) = \Sigma(x) w(x,t) + \bar{C}(x) X(t) \quad (1a)$$
$$w(0,t) = \bar{Q} w(0,t) + \bar{C}_0 X(t) \quad (1b)$$
$$w(1,t) = \bar{R} w(1,t) + \bar{U}(t) + \bar{C}_1 X(t) \quad (1c)$$
$$\dot{X}(t) = A X(t) \quad (1d)$$

defined for $0 \leq x \leq 1$, $t \geq 0$, where the system state

$$w(x,t) = \begin{bmatrix} v(x,t) \\ u(x,t) \end{bmatrix}, \quad (2)$$

is a $n+m$ vector of variables, split into

$$u(x,t) = \begin{bmatrix} u_1(x,t) & u_2(x,t) & \dots & u_n(x,t) \end{bmatrix}^T \quad (3a)$$
$$v(x,t) = \begin{bmatrix} v_1(x,t) & v_2(x,t) & \dots & v_m(x,t) \end{bmatrix}^T. \quad (3b)$$

The system parameters have the form

$$\Lambda(x) = \begin{bmatrix} -\Lambda^-(x) & 0 \\ 0 & \Lambda^+(x) \end{bmatrix}, \; \bar{Q} = \begin{bmatrix} I & 0 \\ Q & 0 \end{bmatrix} \quad (4a)$$
$$\Sigma(x) = \{\sigma_{ij}(x)\}_{1 \leq i,j \leq n+m} \quad (4b)$$
$$\bar{U}(t) = \begin{bmatrix} U(t) \\ 0 \end{bmatrix}, \; \bar{C}_0 = \begin{bmatrix} 0 \\ C_0 \end{bmatrix}, \; \bar{C}_1 = \begin{bmatrix} C_1 \\ 0 \end{bmatrix} \quad (4c)$$
$$\bar{C}(x) = \begin{bmatrix} C^-(x) \\ C^+(x) \end{bmatrix}, \; \bar{R} = \begin{bmatrix} 0 & R \\ 0 & I \end{bmatrix} \quad (4d)$$

with

$$\sigma_{ii}(x) = 0, \quad \forall x \in [0,1], \; i = 1 \dots n+m \quad (5)$$

and

$$\Lambda^+(x) = \text{diag}\{\lambda_1(x), \lambda_2(x), \dots, \lambda_n(x)\} \quad (6a)$$
$$\Lambda^-(x) = \text{diag}\{\mu_1(x), \mu_2(x), \dots, \mu_m(x)\}, \quad (6b)$$

being the transport speeds, subject to the restriction

$$0 < \mu_m(x) < \mu_{m-1}(x) < \dots < \mu_1(x), \quad \forall x \in [0,1] \quad (7a)$$
$$0 < \lambda_1(x) < \lambda_2(x) < \dots < \lambda_n(x), \quad \forall x \in [0,1]. \quad (7b)$$

The boundary parameters are given as

$$R = \{\rho_{ij}\}_{1 \leq i \leq m, 1 \leq j \leq n} \quad (8a)$$
$$Q = \{q_{ij}\}_{1 \leq i \leq n, 1 \leq j \leq m}, \quad (8b)$$

while $\bar{U}(t) = \begin{bmatrix} U^T(t) & 0_{1\times n} \end{bmatrix}^T$ contains the actuation signal

$$U(t) = \begin{bmatrix} U_1(t) & U_2(t) & \dots & U_m(t) \end{bmatrix}^T \quad (9)$$

to be derived. All system parameters are assumed to be known. A disturbance, modeled by

$$X(t) = \begin{bmatrix} X_1(t) & X_2(t) & \dots & X_l(t) \end{bmatrix}^T, \quad (10)$$

is characterized by the linear time invariant system (1d), and

$$A = \{a_{ij}\}_{1 \leq i \leq l, 1 \leq j \leq l} \quad (11a)$$
$$C_0 = \{c_{0,ij}\}_{1 \leq i \leq n, 1 \leq j \leq l} \quad (11b)$$
$$C_1 = \{c_{1,ij}\}_{1 \leq i \leq m, 1 \leq j \leq l} \quad (11c)$$
$$C^+(x) = \left\{c^+_{ij}(x)\right\}_{1 \leq i \leq n, 1 \leq j \leq l} \quad (11d)$$
$$C^-(x) = \left\{c^-_{ij}(x)\right\}_{1 \leq i \leq m, 1 \leq j \leq l}. \quad (11e)$$

We assume the initial conditions $u_i(x,0) = u_{i,0}(x)$, $v_j(x,0) = v_{j,0}(x)$, $X_k(x,0) = X_{k,0}$, for $i = 1 \dots n$, $j = 1 \dots m$, $k = 1 \dots l$, satisfy

$$u_{i,0}, \; v_{j,0} \in L_2([0,1]), \; X_{k,0} \in \mathbb{R}. \quad (12)$$

The goal is to design the control input $U(t)$ so that the following disturbance rejection and tracking goal is achieved

$$r(t) = R_0 u(0,t) - v(0,t) \quad (13)$$

where R_0 is a constant matrix with parameters

$$R_0 = \{r_{ij}\}_{1 \leq i \leq m, 1 \leq j \leq n} \quad (14)$$

and $r(t)$ is a reference signal of choice, for which we assume predictions are available. The tracking goal should be achieved in a finite, minimum time t_{min} corresponding to the slowest propagation time from the actuated to the unactuated boundary. Moreover, the system states $u(x,t)$ and $v(x,t)$ should be bounded by the disturbance model $X(t)$ and the reference signal $r(t)$ for all $x \in [0,1]$, after an additional time corresponding to the propagation time from the unactuated to the actuated boundary. Hence, the tracking objective should be achieved for $t \geq t_{min}$, with t_{min} given by

$$t_{min} = \int_0^1 \frac{ds}{\mu_m(s)} \quad (15)$$

while the states u and v should be pointwise bounded by $|X(t)|$ and $|r(t)|$ for $t \geq t_F$ given by

$$t_F = \int_0^1 \frac{ds}{\mu_m(s)} + \int_0^1 \frac{ds}{\lambda_1(s)} = t_{min} + \int_0^1 \frac{ds}{\lambda_1(s)}. \quad (16)$$

In achieving this, we will also use the following assumptions:

Assumption 1: The disturbance model $X(t)$ and reference $r(t)$ are bounded for all $t \geq 0$. Specifically, there exists positive constants M_1 and M_2 so that

$$|X(t)| \leq M_1 \qquad |r(t)| \leq M_2 \quad (17)$$

for all $t \geq 0$.

Assumption 2: The matrix $R_0 Q - I$ is nonsingular.

It was shown in [17] that the nonsingularity condition of Assumption 2 is required for the rejection problem to be feasible.

III. Decoupling by backstepping

Consider the target system

$$\gamma_t(x,t) + \Lambda(x)\gamma_x(x,t) = G(x)\gamma(0,t) + F(x)X(t) \quad (18a)$$
$$\gamma(0,t) = \bar{Q}\gamma(0,t) + \bar{C}_0 X(t) \quad (18b)$$
$$\gamma(1,t) = B\gamma(1,t) + \bar{U}_a(t) \quad (18c)$$
$$\dot{X}(t) = AX(t) \quad (18d)$$

for a new state vector

$$\gamma(x,t) = \begin{bmatrix}\beta^T(x,t) & \alpha^T(x,t)\end{bmatrix}^T, \quad (19)$$

and some initial condition $\gamma(x,0) = \gamma_0(x)$, where G, F and B are matrices of appropriate sizes to be designed, with

$$G(x) = \begin{bmatrix} G_2(x) & 0 \\ G_1(x) & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}, \quad (20)$$

for some matrices G_2 and G_1, with G_2 and hence also G being strictly lower triangular. Also, $\bar{U}_a(t) = \begin{bmatrix} U_a^T(t) & 0_{1\times n}\end{bmatrix}^T$, where U_a is a new control signal.

Consider the backstepping transformation

$$\gamma(x,t) = w(x,t) - \int_0^x K(x,\xi) w(\xi,t) d\xi, \quad (21)$$

for a kernel function

$$K(x,\xi) = \{k_{ij}(x,\xi)\}_{1\leq i,j \leq n+m} = \begin{bmatrix} K^{vv}(x,\xi) & K^{vu}(x,\xi) \\ K^{uv}(x,\xi) & K^{uu}(x,\xi)\end{bmatrix} \quad (22)$$

defined over $\mathcal{T}$ given as

$$\mathcal{T} = \{(x,\xi) \mid 0 \leq \xi \leq x \leq 1\}, \quad (23)$$

and satisfying the PDEs

$$0 = \Lambda(x) K_x(x,\xi) + K_\xi(x,\xi)\Lambda(\xi) + K(x,\xi)\Lambda'(\xi) + K(x,\xi)\Sigma(\xi) \quad (24a)$$
$$0 = \Lambda(x) K(x,x) - K(x,x)\Lambda(x) - \Sigma(x) \quad (24b)$$
$$G(x) = -K(x,0)\Lambda(0)Q. \quad (24c)$$

The equations (24) are under-determined, and additional boundary conditions are imposed to ensure well-posedness. These were in [9] chosen as

$$k_{ij}(1,\xi) = k^1_{ij}(\xi), \quad 1 \leq j < i \leq m \cup m+1 \leq i < j \leq n \quad (25a)$$
$$k_{ij}(x,0) = k^2_{ij}(x), \quad m+1 \leq j \leq i \leq n \quad (25b)$$

for some $C^\infty([0,1])$ functions $k^1_{ij}(x)$, $k^2_{ij}(x)$, chosen to satisfy the C^1 *compatibility condition at the point* $(x,\xi) = (1,1)$:

$$k_{ij}(1) = f_{ij}(1) \quad (26a)$$
$$\dot{k}_{ij}(1) = \frac{\lambda_i(1) f_{ij}(x) + \sum_{k=1}^n (\sigma_{kj}(1) + \delta_{kj}\lambda'_j(1)) f_{ik}(1)}{\lambda_i(1) - \lambda_j(1)} \quad (26b)$$

where δ_{ij} equals one for $i = j$ and zero otherwise, and

$$f_{ij}(x) = \frac{\sigma_{ij}(x)}{\lambda_i(x) - \lambda_j(x)}. \quad (27)$$

Equations (24) with the additional boundary conditions (25) were proved in [9] to have a unique solution. Consider also the control law

$$U(t) = -R_1 u(1,t) - C_1 X(t) + \int_0^1 K^{vu}(1,\xi) u(\xi,t) d\xi + \int_0^1 K^{vv}(1,\xi) v(\xi,t) d\xi + U_a(t) \quad (28)$$

where $U_a(t)$ will be determined later.

Lemma 3: The backstepping transformation (21) and control law (28) map system (1) into the target system (18) with F given by

$$F(x) = \bar{C}(x) - \int_0^x K(x,\xi)\bar{C}(\xi) d\xi - K(x,0)\Lambda(0)\bar{C}_0 \quad (29)$$

and G given from (24c). Moreover, the control objective (13) becomes

$$r(t) = (R_0 Q - I)\beta(0,t) + R_0 C_0 X(t). \quad (30)$$

Proof: By differentiating (21) with respect to time, inserting the dynamics (1a), integration by parts and inserting the boundary condition (1b), we find

$$\begin{aligned} w_t(x,t) &= \gamma_t(x,t) - K(x,x)\Lambda(x) w(x,t) \\ &+ K(x,0)\Lambda(0)\bar{Q} w(0,t) + K(x,0)\Lambda(0)\bar{C}_0 X(t) \\ &+ \int_0^x \Big[K_\xi(x,\xi)\Lambda(\xi) + K(x,\xi)\Lambda'(\xi) \\ &\qquad + K(x,\xi)\Sigma(\xi)\Big] w(\xi,t) d\xi \\ &+ \int_0^x K(x,\xi)\bar{C}(\xi) d\xi X(t). \end{aligned} \quad (31)$$

Similarly, differentiating with respect to space yields

$$w_x(x,t) = \gamma_x(x,t) + K(x,x) w(x,t) + \int_0^x K_x(x,\xi) w(\xi,t) d\xi \quad (32)$$

Inserting (31) and (32) into (1a), gives

$$\begin{aligned}&\gamma_t(x,t)+\Lambda(x)\gamma_x(x,t)\\&+\left[\Lambda(x)K(x,x)-K(x,x)\Lambda(x)-\Sigma(x)\right]w(x,t)\\&+K(x,0)\Lambda(0)Qw(0,t)+K(x,0)\Lambda(0)\bar{C}_0X(t)\\&+\int_0^x\Big[\Lambda(x)K_x(x,\xi)+K_\xi(x,\xi)\Lambda(\xi)\\&\qquad+K(x,\xi)\Lambda'(\xi)+K(x,\xi)\Sigma(\xi)\Big]w(\xi,t)d\xi\\&+\int_0^x K(x,\xi)\bar{C}(\xi)d\xi X(t)-\bar{C}(x)X(t)=0.\end{aligned} \tag{33}$$

Using the equations (24), we obtain the dynamics (18a) with F given in (29). The boundary condition (18b) follows immediately from the boundary condition (1b) and the fact that $\gamma(0,t)=w(0,t)$. Lastly, we have from (21)

$$w(1,t)=\gamma(1,t)+\int_0^1 K(1,\xi)w(\xi,t)d\xi \tag{34}$$

and inserting this into (1c)

$$\begin{aligned}\gamma(1,t)=&\int_0^1(\bar{R}K(1,\xi)-K(1,\xi))w(\xi,t)d\xi\\&+\bar{R}\gamma(1,t)+\bar{U}(t)+\bar{C}_1X(t).\end{aligned} \tag{35}$$

Choosing

$$\begin{aligned}\bar{U}(t)=&\int_0^1(K(1,\xi)-\bar{R}K(1,\xi))w(\xi,t)d\xi\\&+(B-\bar{R})\gamma(1,t)-\bar{C}_1X(t)+\bar{U}_a(t)\end{aligned} \tag{36}$$

which is equivalent to (28), we obtain boundary condition (18c). Lastly, from inserting the boundary condition (1b) into the control objective (13), we find

$$r(t)=(R_0Q-I)v(0,t)+R_0C_0X(t). \tag{37}$$

The result (30) now follows from the fact that $\beta(0,t)=v(0,t)$. ■

IV. Objective shift

Written out in the variables α and β, the target system (18) reads

$$\begin{aligned}\alpha_t(x,t)+\Lambda^+(x)\alpha_x(x,t)&=G_1(x)\beta(0,t)\\&\quad+F_1(x)X(t) &&(38a)\\ \beta_t(x,t)-\Lambda^-(x)\beta_x(x,t)&=G_2(x)\beta(0,t)\\&\quad+F_2(x)X(t) &&(38b)\\ \alpha(0,t)&=Q\beta(0,t)+C_0X(t) &&(38c)\\ \beta(1,t)&=U_a(t). &&(38d)\end{aligned}$$

Consider now a change of variables

$$\zeta(x,t)=\beta(x,t)+M(x)X(t) \tag{39}$$

and the control law

$$U_a(t)=-M(1)X(t)+U_b(t) \tag{40}$$

where $U_b(t)$ will be designed later and M satisfies the equations

$$\begin{aligned}\Lambda^-(x)M'(x)&=M(x)A+F_2(x)\\&\quad-G_2(x)(R_0Q-I)^{-1}R_0C_0 &&(41a)\\ M(0)&=(R_0Q-I)^{-1}R_0C_0. &&(41b)\end{aligned}$$

Equation (41) is a standard initial value problem that can straightforwardly be solved explicitly. Consider also the target system

$$\begin{aligned}\alpha_t(x,t)+\Lambda^+(x)\alpha_x(x,t)&=G_1(x)\zeta(0,t)+F_1(x)X(t)\\&\quad-G_1(x)M(0)X(t) &&(42a)\\ \zeta_t(x,t)-\Lambda^-(x)\zeta_x(x,t)&=G_2(x)\zeta(0,t) &&(42b)\\ \alpha(0,t)&=Q\zeta(0,t)+C_0X(t)\\&\quad-QM(0)X(t) &&(42c)\\ \zeta(1,t)&=U_b(t) &&(42d)\end{aligned}$$

Lemma 4: The transformation (39) maps system (38) into the target system (42). Moreover, control objective (13) becomes

$$y_c(t)=(R_0Q-I)\zeta(0,t). \tag{43}$$

Proof: By differentiating (39) with respect to time and space, respectively, we find

$$\beta_t(x,t)=\zeta_t(x,t)-M(x)AX(t) \tag{44}$$

and

$$\beta_x(x,t)=\zeta_x(x,t)-M'(x)X(t). \tag{45}$$

Substituting (39), (44) and (45) into (38b), yields

$$\begin{aligned}&\zeta_t(x,t)-\Lambda^-(x)\zeta_x(x,t)-G_2(x)\zeta(0,t)\\&+\left[\Lambda^-(x)M'(x)-M(x)A-F_2(x)+G_2(x)M(0)\right]X(t)\\&=0.\end{aligned} \tag{46}$$

Inserting (41) gives the dynamics (42b). Evaluating (39) at $x=1$, we find

$$\zeta(1,t)=U_1(t)+M(1)X(t). \tag{47}$$

The control law (40) then gives (42d). The dynamics (42a) and boundary condition (42c) follow trivially from inserting

$$\zeta(0,t)=\beta(0,t)+M(0)X(t) \tag{48}$$

into (38a) and (38c).

From the objective function (30) and the relationship (39), we have

$$\begin{aligned}r(t)=&(R_0Q-I)\zeta(0,t)-(R_0Q-I)M(0)X(t)\\&+R_0C_0X(t).\end{aligned} \tag{49}$$

From the boundary condition (41b)

$$\begin{aligned}r(t)=&(R_0Q-I)\zeta(0,t)\\&-(R_0Q-I)(R_0Q-I)^{-1}R_0C_0X(t)\\&+R_0C_0X(t)\end{aligned} \tag{50}$$

which gives (43). ■

V. Fredholm transformation

The subsystem consisting of (42b) and (42d) has the form for which a tracking problem was solved in [8]. However, their solution achieved the tracking goal for $t \geq t_0$, where

$$t_0 = \sum_{i=1}^{m} \int_0^1 \frac{ds}{\mu_i(s)}. \quad (51)$$

We will now introduce a Fredholm transformation bringing the subsystem consisting of (42b) and (42d) to a form facilitating the design of a controller achieving the tracking goal (13) in minimum time. We define the Fredholm transformation

$$\zeta(x,t) = z(x,t) - \int_0^1 P(x,\xi) z(\xi,t) d\xi \quad (52)$$

from a new variable z to ζ, where P has the same form as G_2, i.e. it is a strictly lower triangular matrix on the form

$$P(x,\xi) = \begin{cases} p_{ij}(x,\xi) & \text{for } 1 \leq j < i \leq m \\ 0 & \text{otherwise,} \end{cases} \quad (53)$$

and additionally satisfies the PDE

$$\begin{aligned} 0 &= \Lambda^-(x) P_x(x,\xi) + P_\xi(x,\xi)\Lambda^-(\xi) \\ &\quad + P(x,\xi)\Lambda_x^-(\xi) && (54a) \\ 0 &= P(x,0) - G_2(x)(\Lambda^-)^{-1}(0) && (54b) \\ 0 &= P(0,\xi). && (54c) \end{aligned}$$

It was proved in [11] that Equation (54) has a unique solution P.

Fredholm transformations are, contrary to Volterra transformations, not always invertible. However, the particular transformation (52) turns out to be invertible. This is addressed in the following Lemma, which was proven in [11].

Lemma 5 ([11]): Transformation (52) with P on the form (53) is invertible, with inverse

$$z(x,t) = \zeta(x,t) - \int_0^1 \Theta(x,\xi)\zeta(\xi,t) d\xi \quad (55)$$

for a strictly lower triangular matrix Θ, given by the Fredholm integral equation

$$\Theta(x,\xi) = -P(x,\xi) + \int_0^1 P(x,s)\Theta(s,\xi) ds. \quad (56)$$

Consider now the control law

$$U_b(t) = \int_0^1 \Theta(1,\xi)\zeta(\xi,t) d\xi + U_c(t) \quad (57)$$

where Θ is given from (56), $U_c(t)$ will be designed later, and the target system

$$\begin{aligned} z_t(x,t) - \Lambda^-(x) z_x(x,t) &= H(x) U_c(t) && (58a) \\ z(1,t) &= U_c(t) && (58b) \end{aligned}$$

for a strictly lower triangular matrix H.

Lemma 6: The transformation (52) and control law (57) map the target system (58) with H given by

$$H(x) = P(x,1)\Lambda^-(1) + \int_0^1 P(x,\xi)H(\xi) d\xi \quad (59)$$

into the ζ-subsystem in (42). Moreover, the objective (13) becomes

$$r(t) = (R_0 Q - I) z(0,t). \quad (60)$$

Proof: Differentiating (52) with respect to time, inserting the dynamics (58a), integration by parts and inserting the boundary condition (58b) and the transformation (52) we find

$$\begin{aligned} z_t(x,t) &= \zeta_t(x,t) + P(x,1)\Lambda^-(1) U_c(t) \\ &\quad - P(x,0)\Lambda^-(0)\zeta(0,t) \\ &\quad - \int_0^1 \big[P(x,0)\Lambda^-(0)P(0,\xi) + P_\xi(x,\xi)\Lambda^-(\xi) \\ &\qquad + P(x,\xi)\Lambda_x^-(\xi)\big] z(\xi,t) d\xi \\ &\quad + \int_0^1 P(x,\xi)H(\xi) d\xi U_c(t). && (61) \end{aligned}$$

Similarly, differentiating (52) with respect to space gives

$$z_x(x,t) = \zeta_x(x,t) + \int_0^1 P_x(x,\xi) z(\xi,t) d\xi. \quad (62)$$

Inserting (61) and (62) into (42b) yields

$$\begin{aligned} &\zeta_t(x,t) - \Lambda^-(x)\zeta_x(x,t) - G_2(x)\zeta(0,t) \\ &- \left[P(x,0)\Lambda^-(0) - G_2(x)\right]\zeta(0,t) \\ &- \left[H(x) - \int_0^1 P(x,\xi)H(\xi) d\xi - P(x,1)\Lambda^-(1)\right] U_c(t) \\ &- \int_0^1 \big[\Lambda^-(x) P_x(x,\xi) + P(x,0)\Lambda^-(0)P(0,\xi) \\ &+ P_\xi(x,\xi)\Lambda^-(\xi) + P(x,\xi)\Lambda_x^-(\xi)\big] z(\xi,t) d\xi = 0. && (63) \end{aligned}$$

Using the equations (54) and (59), we obtain the dynamics (42b). Note that the equation (59) is a cascade in H, since both H and P are strictly lower triangular. Equation (59) therefore has a solution H.

Evaluating the transformation (55), which from Lemma 5 is the inverse of (52), at $x = 1$, we find

$$z(1,t) = \zeta(1,t) - \int_0^1 \Theta(1,\xi)\zeta(\xi,t) d\xi. \quad (64)$$

The boundary condition (42d) and the control law (57) then gives (58b).

The objective (60) follows immediately from (43) and noting that $z(0,t) = \zeta(0,t)$, since $P(0,\xi) = 0$ due to (54c). ■

VI. Main theorem

We now state the main result of the paper. Consider the control law

$$\begin{aligned} U(t) &= -R_1 u(1,t) - (C_1 + M(1))X(t) \\ &\quad + \int_0^1 K^{vu}(1,\xi) u(\xi,t) d\xi + \int_0^1 K^{vv}(1,\xi) v(\xi,t) d\xi \\ &\quad + \int_0^1 \Theta(1,\xi)\zeta(\xi,t) d\xi + \omega(t) && (65) \end{aligned}$$

where K^{vu}, K^{vv} are given from the solution to the PDEs consisting of (24) and (25), M is given from the IVP (41), the state ζ is given from the system states u, v through (39) and (21), while Θ is given as the solution to the Fredholm integral (56) with P satisfying (54), and

$$\omega(t) = \begin{bmatrix}\omega_1(t) & \omega_2(t) & \omega_3(t) & \dots & \omega_m(t)\end{bmatrix}^T \quad (66)$$

is given recursively as

$$\omega_i(t) = \nu_i(t + \phi_i(1)) - \sum_{k=1}^{i-1} \int_0^1 \frac{h_{ik}(\tau)}{\mu_i(\tau)} \omega_k(t + \phi_i(1) - \phi_i(\tau)) d\tau \quad (67)$$

for $i = 1 \dots m$, where

$$\nu(t) = \begin{bmatrix}\nu_1(t) & \nu_2(t) & \nu_3(t) & \dots & \nu_m(t)\end{bmatrix}^T \quad (68)$$

is generated from r as

$$\nu(t) = (R_0 Q - I)^{-1} r(t), \quad (69)$$

h_{ij} is the components of H

$$H(x) = \{h_{ij}(x)\}_{1 \leq i,j \leq m} \quad (70)$$

with $h_{ij} \equiv 0$ for $1 \leq i \leq j \leq m$ and

$$\phi_i(x) = \int_0^x \frac{d\gamma}{\mu_i(\gamma)}. \quad (71)$$

Theorem 7: Consider system (1). Subject to Assumptions 1 and 2, the control law (65) guarantees that (13) holds for $t \geq t_{min}$ with t_{min} defined in (15). Moreover, there exist constants $\varsigma_1, \varsigma_2 > 0$ so that

$$|u(x,t)| \leq \varsigma_1 M_1 + \varsigma_2 M_2, \ \forall x \in [0,1] \quad (72a)$$
$$|v(x,t)| \leq \varsigma_1 M_1 + \varsigma_2 M_2, \ \forall x \in [0,1] \quad (72b)$$

for $t \geq t_F$ with t_F defined in (16).

Proof: The target system (58), in component form, reads

$$\partial_t z_i(x,t) - \mu_i(x)\partial_x z_i(x,t) = \sum_{k=1}^{i-1} h_{ik}(x) U_{c,k}(t) \quad (73a)$$
$$z_i(1,t) = U_{c,i}(t) \quad (73b)$$

for $i = 1 \dots m$, where

$$z(x,t) = \begin{bmatrix}z_1(x,t) & z_2(x,t) & \dots & z_m(x,t)\end{bmatrix}^T \quad (74a)$$
$$U_c(t) = \begin{bmatrix}U_{c,1}(t) & U_{c,2}(t) & \dots & U_{c,m}(t)\end{bmatrix}^T \quad (74b)$$
$$H(x) = \{h_{ij}(x)\}_{1 \leq i,j \leq m} \quad (74c)$$

The equations (73) can be solved explicitly using the method of characteristics. Note that ϕ_i defined in (71) are strictly increasing functions and hence invertible. Along the characteristic lines

$$x_1(x,s) = \phi_i^{-1}(\phi_i(x) + s), \qquad t_1(t,s) = t - s \quad (75)$$

we have

$$\frac{d}{ds} z_i(x_1(x,s), t_1(t,s)) = -\sum_{k=1}^{i-1} h_{ik}(x_1(x,s)) U_{c,k}(t_1(t,s)). \quad (76)$$

Integrating from $s = 0$ to $s = \phi_i(1) - \phi_i(x)$, we obtain

$$z_i(x,t) = z_i(1, t - \phi_i(1) + \phi_i(x)) + \sum_{k=1}^{i-1} \int_0^{\phi_i(1)-\phi_i(x)} h_{ik}(x_1(x,s)) U_{c,k}(t_1(t,s)) ds \quad (77)$$

valid for $t \geq \phi_i(1) - \phi_i(x)$. Using the substitution $\tau = \phi_i^{-1}(\phi_i(x) + s)$ in the integral, (77) can be written

$$z_i(x,t) = z_i(1, t - \phi_i(1) + \phi_i(x)) + \sum_{k=1}^{i-1} \int_x^1 \frac{h_{ik}(\tau)}{\mu_i(\tau)} U_{c,k}(t + \phi_i(x) - \phi_i(\tau)) d\tau, \quad (78)$$

valid for $t \geq \phi_i(1) - \phi_i(x)$, from which we specifically obtain

$$z_i(0,t) = U_{c,i}(t - \phi_i(1)) + \sum_{k=1}^{i-1} \int_0^1 \frac{h_{ik}(\tau)}{\mu_i(\tau)} U_{c,k}(t - \phi_i(\tau)) d\tau \quad (79)$$

valid for $t \geq \phi_i(1)$. Hence, choosing the control laws $U_{c,i}$ recursively as

$$U_{c,i}(t) = \nu_i(t + \phi_i(1)) - \sum_{k=1}^{i-1} \int_0^1 \frac{h_{ik}(\tau)}{\mu_i(\tau)} U_{c,k}(t + \phi_i(1) - \phi_i(\tau)) d\tau \quad (80)$$

which is equivalent to choosing

$$U_c(t) = \omega(t) \quad (81)$$

with ω defined in (66) and (67), we obtain $z_i(0,t) = \nu_i(t)$ for $t \geq \phi_i(1)$, and

$$z(0,t) = \nu(t) \quad (82)$$

for $t \geq \max_i(\phi_i(1)) = \phi_m(1) = t_{min}$.

From inserting (82) into the right hand side of (60) and using the definition (69), it is verified that the control objective (60), which is equivalent with (13), holds for $t \geq t_{min}$.

From the target system (58) with $U_c(t) = \omega(t)$, it is clear that z is bounded by r and hence there exists a constant $k_1 > 0$ so that

$$|z(x,t)| \leq k_1 M_2 \quad (83)$$

for $t \geq t_{min}$, and from (52), we then also have

$$|\zeta(x,t)| \leq k_2 M_2 \quad (84)$$

for some constant $k_2 > 0$, for $t \geq t_{min}$. From the relationship (39), a bound

$$|\beta(x,t)| \leq k_3 M_1 + k_4 M_2 \quad (85)$$

follows for some constants $k_3, k_4 > 0$. It is noted from the dynamics of α in (42) that α is a transport equation with $\zeta(0,t)$ and $X(t)$ as inputs, and hence, since $\zeta(0,t)$ is bounded by M_2 for $t \geq t_{min}$, and $X(t)$ is bounded by Assumption 1, there exist constants $k_5, k_6 > 0$ so that

$$|\alpha(x,t)| \leq k_5 M_1 + k_6 M_2 \quad (86)$$

for $t \geq \int_0^1 \frac{ds}{\lambda_1(s)} + t_{min} = t_F$. Due to the invertibility of the backstepping transformation (21), the bounds (72) follow for some positive constants ς_1, ς_2.

Combining (81), (57), (40) and (28) gives (65). ■

VII. Conclusions and further work

We have extended recent results regarding disturbance rejection control of $n + m$ linear hyperbolic systems to the case of having spatially varying coefficients and disturbance entering in the interior domain, and achieve rejection in finite, minimum time. We used a recently derived Fredholm transformation from [11] in addition to the infinite-dimensional backstepping transformation from [9] to prove the result. Additionally, we improved the minimum time tracking controller from [19] by presenting a more compact solution and, more importantly, allowing a general reference signal to be tracked, as apposed to the rather limited class of reference signals allowed in [19].

A natural direction for further work is the derivation of observers converging in finite, minimum time. An observer using sensing anti-collocated with the actuation was offered in [8], while an observer using sensing collocated with the actuation was given in [17]. However, none of these achieved convergence in minimum time. A minimum time observer for $n + m$ systems was first presented in [10], where an observer using sensing anti-collocated with the actuation was offered. However, as with their controller design, the resulting injection gains required the solution to a fairly complicated set of cascaded kernel equations. Applying the Fredholm transformation-based method from [11] to derive minimum-time observers for $n + m$ systems should be possible, but is yet an unsolved problem. It is clear, however, that if the state of the disturbance model is not measured, minimum-time convergence of state estimates is not possible.

References

[1] C.-Z. Xu and G. Sallet, "Exponential stability and transfer functions of processes governed by symmetric hyperbolic systems," *ESAIM: Control, Optimisation and Calculus of Variations*, vol. 7, pp. 421–442, 2010.

[2] I. S. Landet, A. Pavlov, and O. M. Aamo, "Modeling and control of heave-induced pressure fluctuations in managed pressure drilling," *IEEE Transactions on Control Systems and Technoloy*, vol. 21, no. 4, pp. 1340–1351, 2013.

[3] D. J. Wollkind, "Applications of linear hyperbolic partial differential equations: Predator-prey systems and gravitational instability of nebulae," *Mathematical Modelling*, vol. 7, pp. 413–428, 1986.

[4] W. Liu, "Boundary feedback stabilization of an unstable heat equation," *SIAM Journal on Control and Optimization*, vol. 42, pp. 1033–1043, 2003.

[5] M. Krstić and A. Smyshlyaev, "Backstepping boundary control for first-order hyperbolic PDEs and application to systems with actuator and sensor delays," *Systems & Control Letters*, vol. 57, no. 9, pp. 750–758, 2008.

[6] R. Vazquez, M. Krstić, and J.-M. Coron, "Backstepping boundary stabilization and state estimation of a 2×2 linear hyperbolic system," in *Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on*, December 2011, pp. 4937 – 4942.

[7] F. Di Meglio, R. Vazquez, and M. Krstić, "Stabilization of a system of $n + 1$ coupled first-order hyperbolic linear PDEs with a single boundary input," *IEEE Transactions on Automatic Control*, vol. 58, no. 12, pp. 3097–3111, 2013.

[8] L. Hu, F. Di Meglio, R. Vazquez, and M. Krstić, "Control of homodirectional and general heterodirectional linear coupled hyperbolic PDEs," *IEEE Transactions on Automatic Control*, vol. 61, no. 11, pp. 3301–3314, 2016.

[9] L. Hu, R. Vazquez, F. D. Meglio, and M. Krstić, "Boundary exponential stabilization of 1-D inhomogeneous quasilinear hyperbolic systems," 2015, submitted to SIAM Journal on Control and Optimization, under review.

[10] J. Auriol and F. Di Meglio, "Minimum time control of heterodirectional linear coupled hyperbolic PDEs," *Automatica*, vol. 71, pp. 300–307, September 2016.

[11] J.-M. Coron, L. Hu, and G. Olive, "Finite-time boundary stabilization of general linear hyperbolic balance laws via Fredholm backstepping transformation," *Automatica*, vol. 84, pp. 95–100, 2017.

[12] O. M. Aamo, "Disturbance rejection in 2×2 linear hyperbolic systems," *IEEE Transactions on Automatic Control*, vol. 58, no. 5, pp. 1095–1106, 2013.

[13] T. Strecker and O. M. Aamo, "Rejecting pressure fluctuations induced by string movement in drilling," in *2nd IFAC Workshop on Control of Systems Governed by Partial Differential Equations, Bertinoro, Italy*, 2016, pp. 125–130.

[14] H. Anfinsen and O. M. Aamo, "Disturbance rejection in the interior domain of linear 2×2 hyperbolic systems," *IEEE Transactions on Automatic Control*, vol. 60, no. 1, pp. 186–191, 2015.

[15] J. Deutscher, "Finite-time output regulation for linear 2×2 hyperbolic systems using backstepping," *Automatica*, vol. 75, pp. 54–62, 2017.

[16] H. Anfinsen and O. M. Aamo, "Disturbance rejection in $n+1$ coupled 1-D linear hyperbolic PDEs using collocated sensing and control," in *2nd IFAC Workshop on Control of Systems Governed by Partial Differential Equations, Bertinoro, Italy*, 2016.

[17] ——, "Disturbance rejection in general heterodirectional 1-D linear hyperbolic systems using collocated sensing and control," *Automatica*, vol. 76, pp. 230–242, 2017.

[18] P.-O. Lamare and N. Bekiaris-Liberis, "Control of 2×2 linear hyperbolic systems: Backstepping-based trajectory generation and PI-based tracking," *Systems and Control Letters*, vol. 86, pp. 24–33, 2015.

[19] H. Anfinsen and O. M. Aamo, "Tracking in minimum time in general linear hyperbolic PDEs using collocated sensing and control," in *2nd IFAC Workshop on Control of Systems Governed by Partial Differential Equations, Bertinoro, Italy*, 2016.

[20] J. Deutscher, "Output regulation for general linear heterodirectional hyperbolic systems with spatially-varying coefficients," *Automatica*, vol. 85, pp. 34–42, 2017.