Norwegian University of
Science and Technology

Analysis of portfolio risk and the LIBOR Market Model

Ole Thomas Helgesen

Master of Science in Physics and Mathematics
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Supervisor: Jacob Laading, MATH

Norwegian University of Science and Technology
Department of Mathematical Sciences

Problem Description

Model analysis of the LIBOR Market Model and portfolio risk.
In this thesis the historical data will be strongly influenced by a recent global recession: the financial crisis of 2007-2010. There will be an emphasis on how this affects the model and portfolio risk measures. The performance and some applications of the risk measures are studied.

Assignment given: 21 January 2011
Supervisor: Jacob K. Laading, MATH

Preface

This thesis was written at the Department of Mathematical Sciences at the Norwegian University of Science and Technology (NTNU) in Trondheim during the spring of 2011. The work I have done with interest rate models and risk analysis has been very rewarding and it has introduced me to a new exciting field within applied mathematics.

I would like to thank DnB NOR for providing the necessary data material from the Norwegian and American interest rate market. I would also like to thank my friends and former co-students at UC Berkeley, Alexandru Hening and Eric Wayman, for helping me with the proofreading and giving me helpful advice.

I am also grateful for the help I have gotten from Alexander Furnes, a good friend and co-student at NTNU.

Finally I would like thank Jacob K. Laading for splendid supervision. I found his feedback on my work to be very constructive and motivating.

Ole Thomas Helgesen
Trondheim, June 6, 2011

Abstract

This master thesis focuses on interest rate modeling and portfolio risk analysis. The LIBOR Market Model is the interest rate model chosen to simulate the forward rates in the Norwegian and American market, two very different markets in terms of size and liquidity. On the other hand, the Norwegian market is highly dependent on the American market and the correlation can be seen clearly when the data sets are compared in the preliminary analysis. The data sets are from the time between 2000 and the early 2011.

Risk estimates are found by Monte Carlo simulations, in particular Value at Risk and Expected shortfall, the two most commonly used risk measures.

Interest rate modeling and risk analysis requires parameter estimates from historical data which means that the Financial Crisis will have a strong effect. Two different approaches are studied: Exponentially Weighted Moving Averages and (equally weighted) Floating Averages. The main idea is to cancel out trend and capture the true volatility and correlation.

Risk is estimated in several different markets, first an imaginary stable market is assumed. In the next steps the Norwegian and the American market are analyzed. The volatility and correlation varies. Finally we look at a swap depending on both Norwegian and American interest rates.

In order to check the risk estimates, the actual losses of the test portfolios are compared to the Value at Risk and the Expected Shortfall. The majority of the losses larger than the risk estimates occur between 2007 and 2009 which confirms, not surprisingly, that the risk measures were unable to predict the Financial Crisis.

The portfolios have a short time horizon, 1 day or 5 days, and the EWMA procedure weighs the recent observations heavier, thus it performs better than the Floating Averages procedure. However, both procedures consistently underestimate the risk. Still the risk estimates can be used as triggers in investment strategies. In the final part of this thesis such investment strategies are tested. Plotting the cumulative losses and testing the strategies shows that the risk estimates can be used with success in investment strategies. However the strategies are very sensitive to the choice of the risk threshold.

Nonetheless, even though the model underestimates risk, the backtesting and the plots also tell us that the estimates are fairly proportional to the real losses. The risk estimates are therefore useful indicators of the development of the exposure of any financial position, which justifies why they are the most commonly used risk measures in financial markets today.

Contents

1 Introduction

This thesis will focus on two important fields within financial engineering: the evaluation of market risk and the modeling of interest rates. In particular, the focus will be on the performance of the LIBOR Market Model and behavior of risk measures as a result of the recent volatile financial history.

1.1 Risk and Interest Rate Modeling

In the financial market, many securities are long term, and thus their value is highly dependent on the interest rates. In order to simulate the appropriate price of such securities there is a need for an interest rate model like the LIBOR Market Model. This model simulates forward rates that are directly observable in the market and is therefore a popular model among practitioners.

Any investor, whether he is investing millions of dollars or only a couple of hundred, needs to know the probability of losing money and the severity of the losses. This idea is the basis of risk analysis, an important field within finance and business management, a field which has gotten a lot of attention the last few years because the financial crisis has broken many worst case loss-prognoses.

Interest rate modeling and risk analysis constitute the two frontiers of this thesis, both fields will be investigated keeping in mind the turbulent financial history which the estimates are based on.

1.2 Thesis Outline

In this thesis, the historical data sets used for parameter estimation and risk measurements are taken from a period with financial turbulence. The second chapter describes the data sets and the state of the financial market in the last decade. The interest rates are from the Norwegian market and the American market, the two different markets that will be compared. Furthermore, the log-normality assumption of the LIBOR Market Model is analyzed by comparing the distribution of the historical data to the log-normal probability density function. By a simple logarithmic transformation the data can be tested for normality using the Jarque-Bera test.

In the third chapter an introduction to risk analysis is given and different interest rate derivatives are explained. This is to gain understanding of the tools used to estimate risk and derivative pricing. In chapter 4 the interest rate model, LIBOR Market Model, is presented. The theory behind the model and the implementation of the model is explained.

After the first 4 chapters, the reader should have a good understanding of the theory and data sets used for simulation in the next chapters. The following chapter describes the implementation and results of interest rate modeling and risk estimation.

The implementation part starts with a practical demonstration of the relationship between a cap, a floor and a swap. This enlightens the basics behind the next steps which are the analysis of pricing and risk measurements of portfolios. After this introduction, the risk is analyzed at different points along realized forward rates. Here portfolios of increasing complexity are defined.

Analyzing the realized forward rates is not as interesting as comparing real losses from historical data with the estimated risk thresholds. The real losses show volatility clustering around the recessions, which makes it a particularly hard data set to perform risk analysis on. Both Norwegian and American interest rates are considered here. There is also a portfolio designed to hedge against the difference in interest rate between the two markets. Finally the risk estimates are used as triggers in an investment strategy which is tested against a buy and hold strategy.

In chapter 7 the conclusions will be presented, as well as a discussion of possible extensions of this thesis.

2 Preliminary Data Analysis

2.1 The Financial Crisis and The Early 2000s Recession

The financial crisis of 2007-2010 has caused a downturn in financial markets all over the world. The crisis was mainly triggered by a shortfall in the liquidity of American banks. A main cause was the collapse of the housing bubble which caused the value of securities tied to housing loans to decrease dramatically. We also experienced a decline in economic activity, globally, in the early 2000s which clearly affected the interest rates.

In addition to recession, this decade was influenced by international conflicts. The terrorist attack on the World Trade Center shocked the world and lead to a global war on terrorism. Airline stocks plummeted, and shortly after the US entered Afghanistan starting a long and difficult war. Two years later the war in Iraq began, a war which is still going on, and has been a huge toll on both the American people and the US Government.

This chapter is concerned with the visualization and preliminary analysis of the data sets. The data consist of forward rates (3 and 6 months, 1, 2, 5 and 10 years) from the Norwegian and American market provided by DnB NOR.

2.1.1 Important events

In the following list some of the most important events in the period between 2000 and 2010 are presented. This list should be used as a guide to understand the movements in the forward rates we see in the next subsections.

This list is by no means a complete list of the factors that affect the interest rates, it is merely a selection of some of the major events of the decade that may have affected the movements we observe.

Major events between 2000 and 2010

2000-2001	Early 2000s recession affects European Union
2001 September	The attack on World Trade Center
2001 October	The beginning of the war in Afghanistan
2002 January	The introduction of the EURO
2002-2003	Early 2000s recession affects the US
2003 March	The beginning of the war in Iraq
2007-2010	The Financial Crisis
2008 March 16th	JP Morgan Chase acquires Bear Stearns
2008 September 15th	Lehman Brothers collapse
2008 October 3rd	$700 billion US Bailout Package
2008 October 9th	Coordinated interest rate cuts
2008 November	Barack Obama elected president

2.1.2 The data set

Historical data plotted below is from the Norwegian market in the time period between October 3rd 2000 and June 1st 2010, and thus highly influenced by the financial crisis and the early 2000s recession. This must be considered when estimating parameters with the historical data.

Observe how the forward rates for longer time spans smooth out the sharpest edges of the shorter rates. The 10 year forward rate is not as affected by the financial crisis as the 3 and 6 months rates. This makes sense financially.

The period up to around 2002 represents forward rates as we would expect from a model. After 2002, recessions affect our data in a way we can not expect from a simulation. This will certainly affect the estimates of our parameters.

The data is represented by six forward rates per date; the 3 and 6 month, 1, 2, 5 and 10 years forward rates. A selection of the forward rates is plotted below.

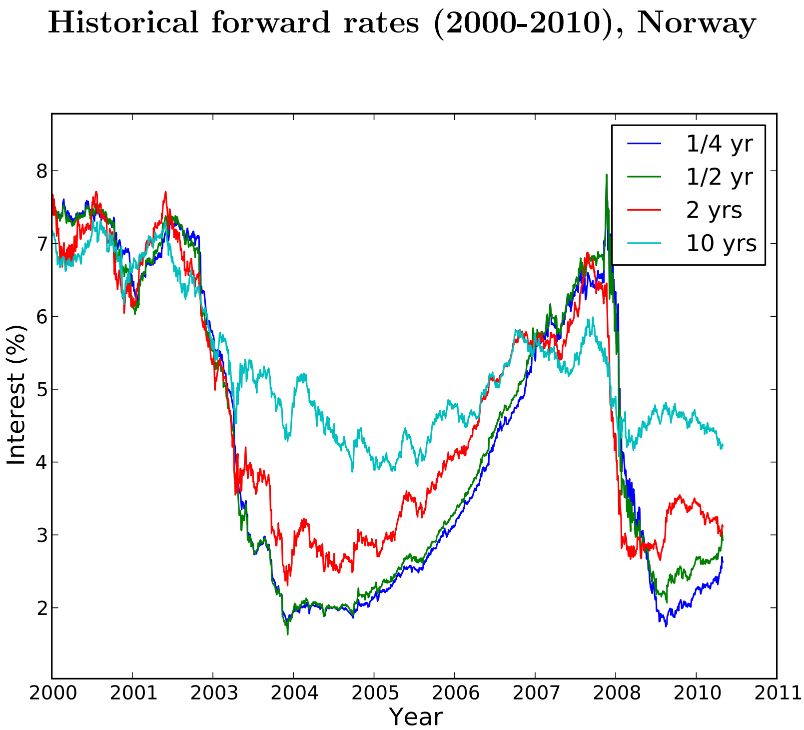

Figure 1: Historical forward rates during the recessions between 2000 and 2010

From this plot it is clear that the early 2000s recession and the Financial Crisis of 2007-2010 are the most dominant market movements of the decade. Below are the most dramatic periods in more detail.

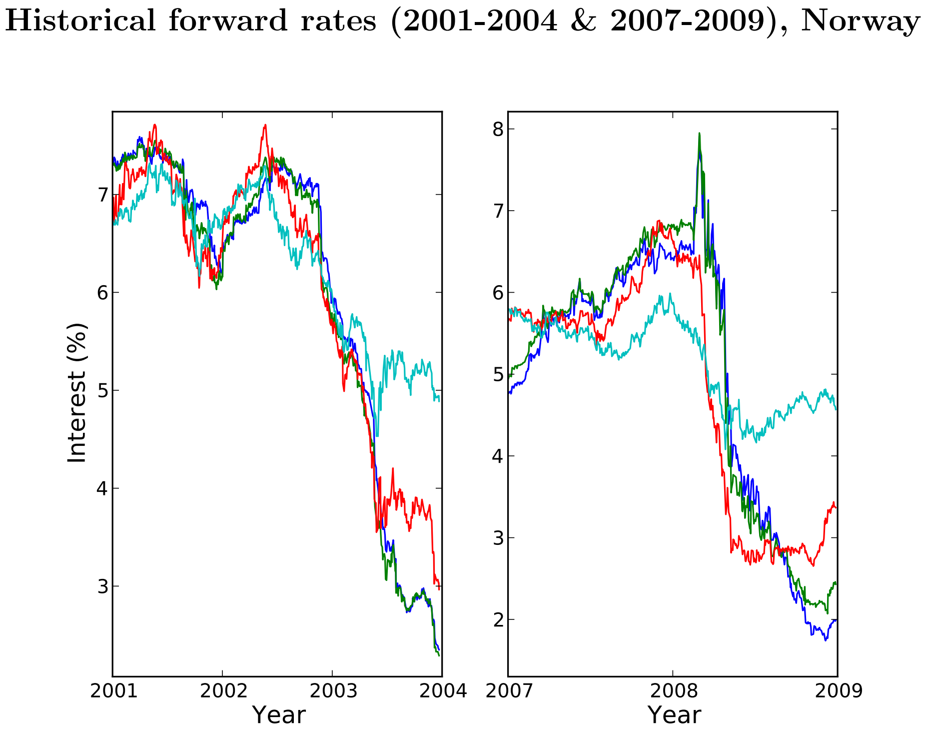

Figure 2: Historical forward rates during the two recessions between 2000 and 2010

2.1.3 Volatility and Correlation

The early 2000s recession caused the 3 and 6 month forward rates to drop around 500 basis points[1] in less than two years. After 2004, the rates drift back towards the 7-8% level before recession strikes again in 2007/2008. This time with an even steeper descent to interest rates below 3%. As mentioned above this was the financial crisis which still affects financial markets significantly.

We want to predict plausible future forward rates, not copy historical events. The relevant features are the correlation and volatility of the forward rates which are independent of market movements. Using floating averages cancels out some of the effect that recession has on the volatility. The floating averages method is compared to the exponentially weighted moving averages (EWMA) approach. The latter puts more emphasis on the recent movements by weighting them heavier. Plotting the annual volatility with the two different approaches shows how the recession periods affect the volatility (the colors are as in figure 1).

[1] A basis point is 1/100 of a percent and is commonly used to describe interest rates movements.

Historical volatility (2000-2010), Norway

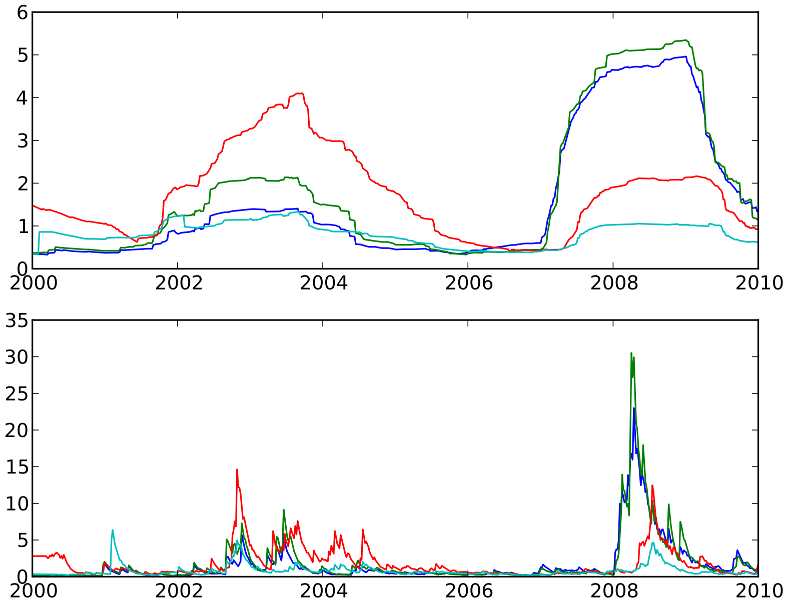

Figure 3: Volatility using floating averages (top) and EWMA (bottom). The volatility is scaled equally in both plots.

Still the Financial Crisis of 2007 is dominant in the volatility plot. The change in interest rates there is so sudden that floating averages only remove a small part of the effect it has on the volatility. By choosing a small enough lag to cancel out the effect of the financial crisis, we risk canceling out the volatility we are interested in as well.

2.2 The time after 2010 including the Libyan uprising

The period after 2010 has been more financially stable, even though the Financial Crisis of 2007-2010 still has an influence on the financial markets world wide.

2.2.1 Important events

In April 2010 Eyjafjallajökull, one of the largest volcanoes on Iceland, erupted causing an ash cloud which caused disruption to air traffic across the Atlantic and in Europe.

The uprising against the Libyan leader Muammar Gaddafi is causing large uncertainty in the oil market, and enormous sums of Libyan money are being held back from banks in for example Sweden and Norway. The uprising was inspired by, and further encouraged, the Middle Eastern and North African protests including Egypt, Tunisia and other parts of the Arab world.

A 9.0-magnitude earthquake hit Japan shortly after, causing a tsunami that killed 20.000 people. The tsunami damaged several nuclear power plants, causing fear of radiation danger in nearby cities including Tokyo. Below is a list of some of the most important events in the time between 2010 and March 2011.

Major events between 2010 and March 2011

2010 April 12th	Eruption of the volcano Eyjafjallajökull on Iceland
2010 April-November	Greek debt crisis
2010 December 18th	Protest in Tunisia
2011 January 25th	Demonstrations in Egypt
2011 February 15th	Beginning of the Libyan uprising
2011 March 11th	Earthquake causing tsunami in Japan
2011 March 19th	No fly zone in Libya

2.2.2 The data set

The data set is from the time between January 5th 2009 (in order to see the overlap with figure 1 clearly) and March 29th 2011.

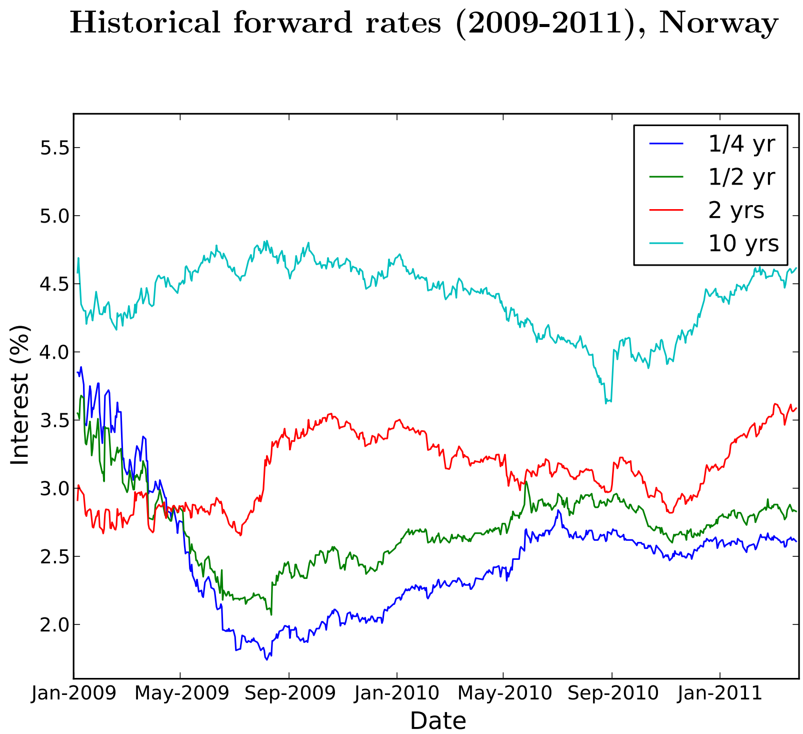

Figure 4: Historical forward rates between 2009 and March 2011

Figure 4 is a plot of the interest rates from the Norwegian market. In the early 2009 the effect of the Financial Crisis still plays an important role, while the interest rates are fairly stable after the summer of 2009.

2.3 Comparison with the American market

The Norwegian market is highly influenced by the oil price, and it is naturally not unaffected by movements in the American market. The following subsection is a comparison of the interest rate movements in the two markets. Figure 5 is a plot of interest rates from the American market in the time period August 1st 2006 to March 29th 2011.

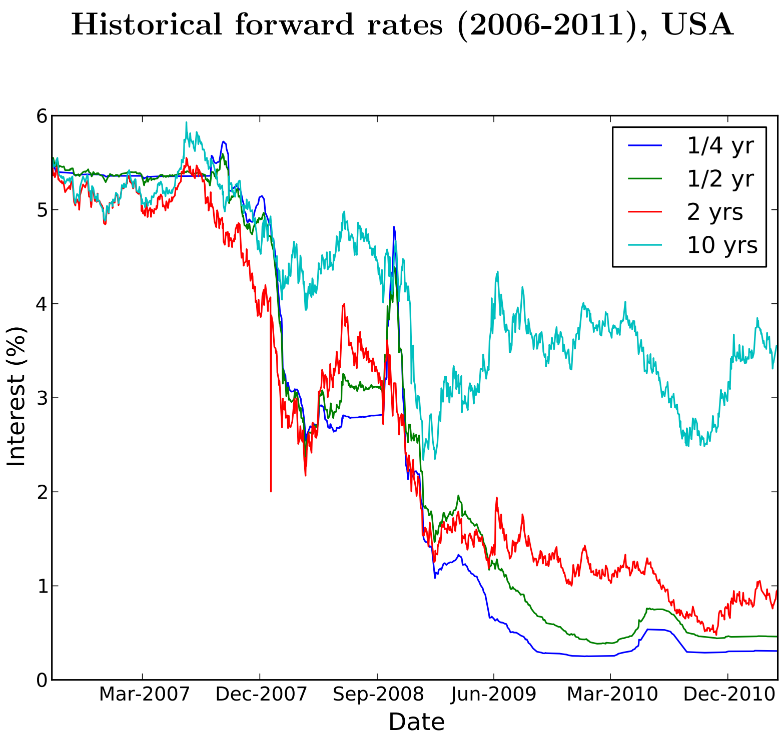

Figure 5: Historical forward rates between August 2006 and March 2011

Notice how the 3 and 6 month forward rates are much less volatile than the corresponding forward rates in the Norwegian market. The 2 and 10 year forward rates behave as expected compared to the Norwegian data set.

The plot in figure 6 is a comparison of the American and Norwegian forward rates between 2010 and March 2011.

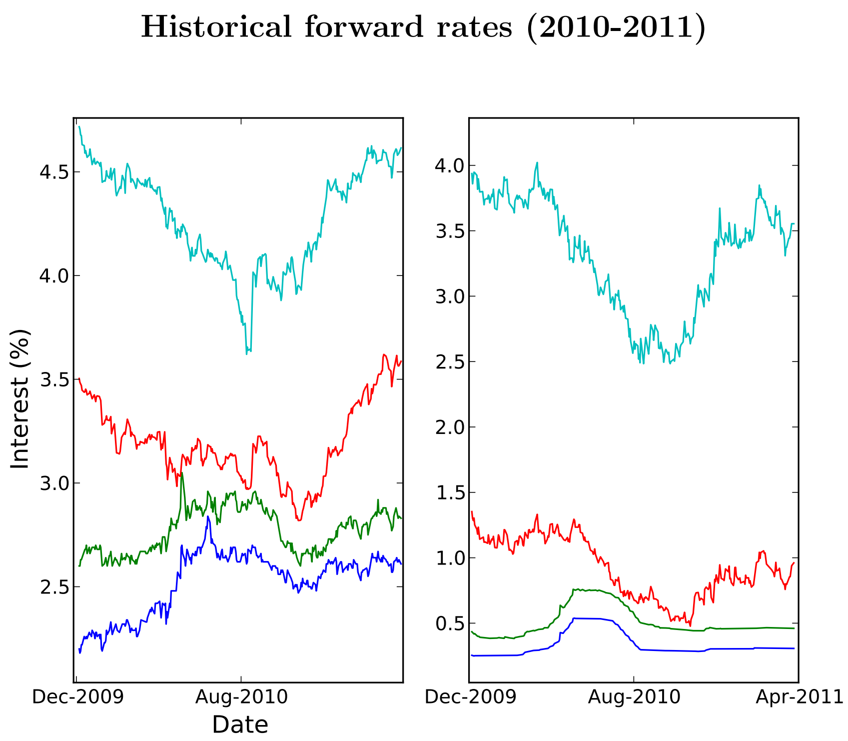

Figure 6: Historical forward rates between January 2010 and March 2011. Norwegian (left), American (right)

In both interest rate sets the long term and the short term forward rates move closer to each other in the first half of 2010, while the differences increase in the last half of 2010.

The American market is much larger and more liquid than the Norwegian market, but they are highly correlated. A small country with oil as the main industry, like Norway, will in a much larger degree get affected by changes in the oil price or the behavior of large companies.

2.4 Interest rate distribution

2.4.1 Log-normality in the historical data

The LIBOR Market Model assumes that relative changes in the forward rates follow a log-normal distribution, the model is described in more detail in chapter 3. In order to check how well the historical data fits in with the assumption of log-normality, a histogram comparing the historical relative change in interest rates $(F(t_{(n+\Delta)})/F(t_n))$ to the log-normal distribution with the appropriate maximum likelihood parameters is plotted. Δ is the number of days between the two interest rates which are denoted by F.

In the following plots (figure 7 and 8) Δ is set to 1, the same procedure is applied to the simulated rates in chapter 3

Forward rate distribution, Norway (2000-2011)

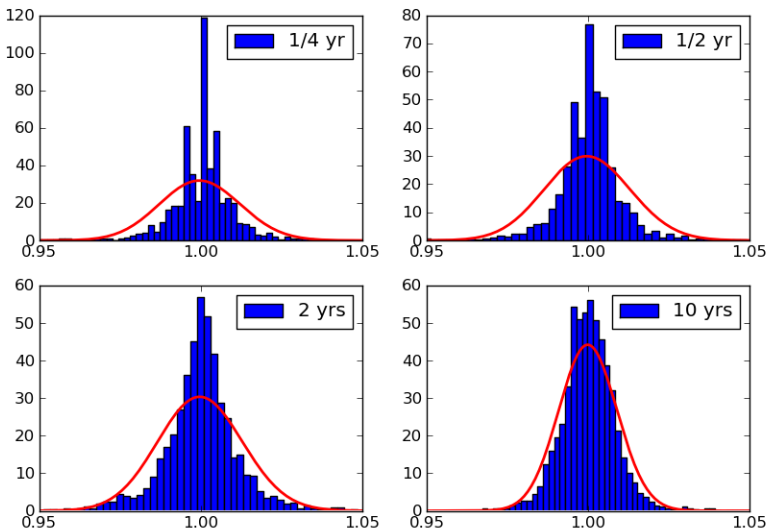

Figure 7: The distribution of the historical Norwegian forward rates compared to the pdf of the log-normal distribution (red line). $\Delta = 1$

The forward rates far into the future (2 and 10 years) have a distribution close to log-normal while the short term rates (3 and 6 months) have a different distribution. This is naturally a source of errors in the risk estimates.

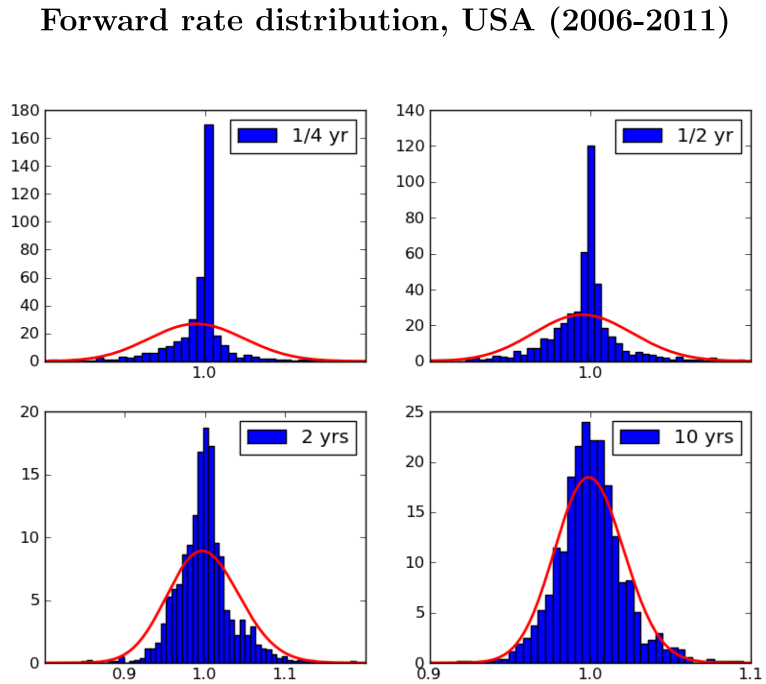

Figure 8: The distribution of the historical American forward rates compared to the pdf of the log-normal distribution (red line). $\Delta = 1$

The observations from the Norwegian data set are evident here as well, the short term rates clearly do not follow a log-normal distribution. One can see from the initial plot in figure 5 that the 3 and 6 month American forward rates do not follow the same ”random” process as the rates in the Norwegian market, or the 2 and 10 year American forward rates.

2.4.2 Sample skewness and kurtosis

One way to test the log-normality assumed in the LIBOR Market Model is to look at the logarithm of the observations and test for normality. Below, the sample skewness and kurtosis of the historical data are calculated. The data set consist of the logarithm of the relative change in interest rates ($\log(F(t_{(n+\Delta)})/F(t_n))$). If these observations really are normally distributed, the sample skewness and kurtosis will be close to zero.

Market - Forward rates	**Kurtosis**	**Skewness**
Norwegian - 3 months	*17.6807*	*-0.0179*
Norwegian - 6 months	*29.0742*	*-0.0258*
Norwegian - 1 year	*113.7503*	*-0.0566*
Norwegian - 2 years	*11.7175*	*-0.0098*
Norwegian - 5 years	*4.8816*	*-0.0007*
Norwegian - 10 years	*7.0814*	*-0.001*
American - 3 months	*25.9836*	*-0.0207*
American - 6 months	*16.5772*	*-0.0084*
American - 1 year	*4.6205*	*0.0057*
American - 2 years	*93.5896*	*-0.0208*
American - 5 years	*2.3994*	*0.0009*
American - 10 years	*4.3984*	*-0.007*

With the skewness and the kurtosis one can check for normality using the Jarque–Bera[2] test, with the null hypothesis that the relative change in interest rates follows a log-normal distribution. For every set of historical forward rates in both markets, the p-value is approximately zero which means that we would have to reject the null hypothesis.

The Jarque-Bera test rejects the null hypothesis if either the sample kurtosis or the sample skewness is significantly higher than zero. In this case it is clear that the high values for the sample kurtosis causes rejection. Kurtosis measures 'peakedness' or alternatively heavy tails. If the kurtosis is high the variance is caused by few large deviations from the mean, as opposed to many smaller deviations. In the case of portfolio risk management, high kurtosis means that there is a higher probability to have large losses or wins which surely affects the risk measures.

This test shows that the historical data has a much higher kurtosis than what is assumed in the model (LMM). Thus higher losses are likely to occur more frequently than predicted. Ignoring kurtosis risk or skewness risk can cause a risk estimate to understate the real risk in a financial position. In this case it seems that the skewness risk is negligible while the kurtosis risk will affect the risk estimates.

[2]See the appendix A.3 for an explanation of the Jarque-Bera test.

3 LIBOR Market Model

3.1 Theory

3.1.1 Interest Rate Models

Simulating stock prices and interest rates is important to find appropriate derivative prices and risk estimates. Interest rates are generally harder to model than stocks, there are no underlying assets to hedge with, and the time span is generally longer than it is for options. Longer time spans increase the demand for accuracy in the model.

In this thesis we aim to simulate interest rates as realistically as possible. We start by assuming the following model for the risk-neutral spot rate (the spot rate is the interest rate for the shortest possible loan, the limit as $t \to 0$):

$$dr = (u - \lambda\omega)dt + \omega dX \tag{1}$$

Here dX is a normally distributed random variable, with standard deviation dt, and the functional form of u, λ and ω determine the behavior of r. By using Itô's lemma[3] and an arbitrage argument[4] we arrive, after some manipulation, at the bond pricing equation where V is the price of the bond:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\omega^2 \frac{\partial^2 V}{\partial r^2} + (u - \lambda\omega)\frac{\partial V}{\partial r} - rV = 0 \tag{2}$$

This equation has many similarities to the Black-Scholes partial differential equation [5]. We can not, however, use the same model for interest rates and option assets. Assets exhibit long-term exponential growth which is unsuitable for interest rates. We assume that $u - \lambda\omega$ and ω take the form:

$$\begin{aligned} u(r,t) - \lambda(r,t)\omega(r,t) &= \eta(t) - \gamma(t)r \\ \omega(r,t) &= \sqrt{\alpha(t)r - \beta(t)} \end{aligned} \tag{3}$$

This framework enables us to ensure some nice properties by choosing η, γ, α and β wisely. If $\alpha(t) > 0$ and $\beta(t) \geq 0$ the interest rate is bounded below by $\frac{\beta(t)}{\alpha(t)}$.

Furthermore, if $\eta(t) > 0$ and $\gamma(t) > 0$ the model has mean reversion meaning that the interest rate will have drift towards a mean value which is equal to $\frac{\eta(t)}{\gamma(t)}$. Some of the most common choices of these parameters are [6]:

- Vasicek: $\alpha = 0, \beta < 0$ and η, γ independent of time

$$dr = (\eta - \gamma r)dt + \sqrt{\beta}dX \tag{4}$$

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[3] Itô's lemma is defined in the appendix A.2.

[4] Arbitrage is an opportunity to gain excess return risk free. One example of arbitrage is to take advantage of price differences between markets.

[5] The Black Scholes partial differential equation is one of the most famous equations within financial mathematics describing the price of an option.

- Cox, Ingersoll and Ross: $\beta = 0$ and η, α, γ independent of time

$$dr = (\eta - \gamma r)dt + \sqrt{\alpha r}dX \tag{5}$$

- Ho and Lee: $\alpha = \gamma = 0, \beta < 0$ and η a function of time

$$dr = \eta(t)dt + \sqrt{\beta}dX \tag{6}$$

The CIR model realized with parameters

$$\sqrt{\alpha} = 0.4 \quad \gamma = 4.4\text{e-}6 \quad \eta = 0.001$$

and initial interest rate of 6% is plotted in figure 9.

The Cox, Ingersoll and Ross model

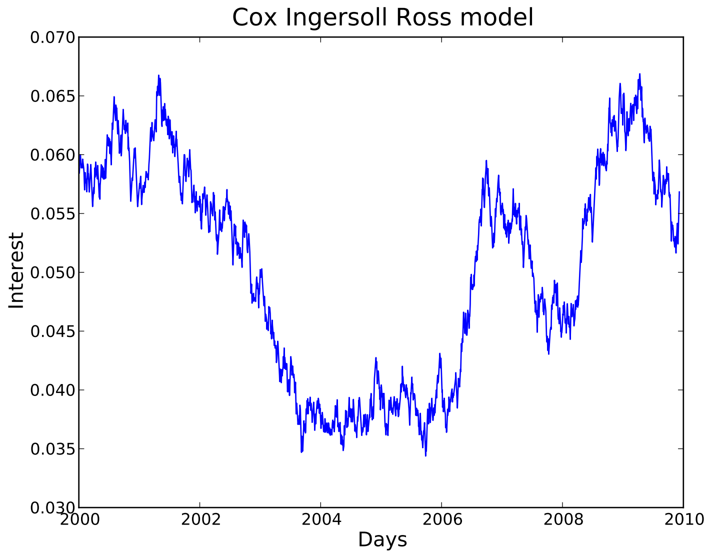

Figure 9: simulated sport rate with the CIR model

These one-factor models, however, are often too simple to represent real interest rates in an accurate way. They can be extended to include more than one parameter, but a better way to model the interest rates is to consider a set of forward rates, not only the spot rate. The LIBOR market model or the Brace, Gatarek & Musiela model is such a model.

Forward rates are interest rates quoted in the market for different time horizons. For example a forward rate for a sufficiently short time period would be the spot rate. The derivation of the LIBOR Market Model can be found in Wilmott's books on quantitative finance [6, 7].

3.1.2 Deriving the LIBOR Market Model

The model for a set of forward rates $\{F_i\}$ at time t can be written:

$$\frac{dF_i(t)}{F_i(t)} = \mu_i(t)dt + \sigma_i(t)dX_i \tag{7}$$

This looks like a simple log-normal model but μ and σ can be dependent on F_j for $j = 1,..,n$. Here n is the number of forward rates in the set and $dX_i \sim N(0, dt)$ (independent Brownian Motions). A zero coupon bond, with the price Z_i, in the risk-neutral world follows:

$$dZ_i = rZ_i dt + Z_i \sum_{j=1}^{i-1} a_{ij} dX_j \tag{8}$$

a_{ij} is defined below and r is the risk-free interest rate. We can write $Z_i = (1 + \tau_i F_i)Z_{i+1}$ to ensure no arbitrage, where τ_i is the time difference between the maturity of Z_i and Z_{i+1}, and apply Itô's lemma so that:

$$dZ_i = (1 + \tau_i F_i)dZ_{i+1} + \tau_i Z_{i+1} dF_i + \tau_i \sigma_i F_i Z_{i+1} \left(\sum_{j=1}^{i} a_{i+1,j} \rho_{ij} \right) dt \tag{9}$$

ρ_{ij} is the correlation between dX_i and dX_j. Collecting the terms involving dX_i leads to the following expression ($a_{ii} = 0$):

$$\begin{array}{c} (1 + \tau_i F_i)a_{i+1,i} Z_{i+1} + \tau_i Z_{i+1} \sigma_i F_i = 0 \\ \Rightarrow a_{i+1,i} = -\frac{\tau_i \sigma_i F_i}{1+\tau_i F_i} \end{array} \tag{10}$$

For the other random terms, dX_j for $j = 1, ..., i-1$ the expression yields:

$$\begin{array}{c} a_{i,j} Z_i = a_{i+1,j}(1 + \tau_i F_i)Z_{i+1} \\ \Rightarrow a_{i+1,j} = a_{i,j} \quad for\ j < i \end{array} \tag{11}$$

From this it follows by induction that:

$$a_{i+1,j} = -\frac{\tau_j \sigma_j F_j}{1 + \tau_j F_j} \quad for\ j < i \tag{12}$$

If we equate the dt terms we get:

$$\begin{array}{c} rZ_i = (1 + \tau_i F_i) r Z_{i+1} + \tau_i Z_{i+1} \mu_i F_i + \tau_i \sigma_i F_i Z_{i+1} \sum_{j=1}^{i} a_{i+1,j} \rho_{ij} \\ \Rightarrow \quad \mu_i = \sigma_i \sum_{j=1}^{i} \frac{\sigma_j F_j \tau_j \rho_{ij}}{1+\tau_j F_j} \end{array} \tag{13}$$

So finally we can write equation (7) as:

$$\begin{array}{c} dF_i = \left(\sum_{j=1}^{i} \frac{\sigma_j F_j \tau_j \rho_{ij}}{1+\tau_j F_j} \right) \sigma_i F_i dt + \sigma_i F_i dX_i \\ \Rightarrow dlog(F_i) = \left(\sigma_i \sum_{j=1}^{i} \frac{\sigma_j F_j \tau_j \rho_{ij}}{1+\tau_j F_j} - \frac{1}{2}\sigma_i^2 \right) dt + \sigma_i dX_i \end{array} \tag{14}$$

A more rigorous proof of the equations (14) can be found in Glasserman's book [2]. These equations will be used to simulate the forward rates.

3.2 Simulating Forward Rates with the LIBOR Market Model

Estimating the parameters is the hard part of interest rate modeling, and there are several ways to do it. Different approaches are usually different compromises between complexity and accuracy. If the model becomes too complex the estimation quickly becomes slow, and thus useless in many cases.

The following two approaches to estimate the parameters of the LMM are reasonable compromises, and the estimates lead to relatively fast simulations of the forward rates. The first approach is a straightforward estimate, while the second approach takes into consideration the nature of the historical data.

3.2.1 Estimation of parameters

Assuming that the correlation and volatility (per time step) are constant in time we find that an estimator for σ_i is the variance of $dlog(F_i)$. The correlation ρ_{ij} can be estimated by the correlation of $dlog(F_i)$ and $dlog(F_j)$. In mathematical terms:

$$\begin{aligned} \hat{\sigma}_i &= VAR[dlog(F_i))] \\ \hat{\rho}_{ij} &= CORR[(dlog(F_i)), dlog(F_j))] \end{aligned} \tag{15}$$

For the first estimate the sample correlation and variance estimators are used:

$$\begin{gathered} \textit{For the random variables } X_1, ..., X_n \textit{ and } Y_1, ..., Y_n\textit{:} \\ V\hat{A}R = \sum_{i=1}^{n} \left(\frac{(X_i - \hat{E}(X))^2}{n-1} \right) \\ CO\hat{R}R = \sum_{i=1}^{n} \left(\frac{(X_i - \hat{E}(X))(Y_i - \hat{E}(Y))}{n-1\sqrt{V\hat{A}R(X)V\hat{A}R(Y)}} \right) \end{gathered} \tag{16}$$

This approach, however, may give unreasonably high variance and correlation because of the recessions. A solution is to replace $E(X)$ and $E(Y)$ with Floating Averages in order to eliminate the dominant movements of the market as a whole. The formulas then yield:

$$\begin{aligned} V\hat{A}R^*[X] &= \sum_{i=1}^{n} \left(\frac{(X_i - \hat{E}(X_{i-lag,i+lag}))^2}{n-1} \right) \\ CO\hat{R}R^*[X,Y] &= \sum_{i=1}^{n} \left(\frac{(X_i - \hat{E}(X_{i-lag,i+lag}))(Y_i - \hat{E}(Y_{i-lag,i+lag}))}{n-1\sqrt{V\hat{A}R^*(X)V\hat{A}R^*(Y)}} \right) \end{aligned} \tag{17}$$

Where 2×lag is the number of days considered when calculating the expected value. In the estimate of a constant volatility over 10 years the lag is 150. Choosing a lag small enough to completely remove the effects of the recessions would also remove much of the volatility we set out to find. Expressing $dlog(F_i)$ by the differences in $log(F_i)$ allows us to estimate σ_i and ρ_{ij}.

A third option is to estimate the Exponentially Weighted Moving Averages (EWMA) which puts more emphasis on the most recent data (X_n is the most recent observation).

After a recession the volatility would be much higher than with Floating Averages because the recent highly volatile period is heavily weighted. The parameter λ decides the weighting in EWMA:

$$\begin{aligned} V\hat{A}R^{**}[X] &= \sum_{i=1}^{n}(1-\lambda)\times\lambda^{n-i}\left(\frac{(X_i-\hat{E}(X_{0,i}))^2}{n-1}\right) \\ CO\hat{R}R^{**}[X,Y] &= \sum_{i=1}^{n}(1-\lambda)\times\lambda^{n-i}\left(\frac{(X_i-\hat{E}(X_{0,i}))(Y_i-\hat{E}(Y_{0,i}))}{n-1\sqrt{V\hat{A}R^{**}(X)V\hat{A}R^{**}(Y)}}\right) \end{aligned} \tag{18}$$

Where $0 < \lambda < 1$. The difference in averages is also worth noting; in the EWMA the average is calculated with every observation up until "i", while with Floating Averages the average is calculated from an interval around "i". A comparison of the volatility estimated using the EWMA and Floating Averages is depicted in figure 3 where $\lambda = 0.95$.

3.2.2 Simulation of forward rates

After estimating the parameters we can interpolate in time. Let: $dlog(F_i(t_n)) \approx \Delta log(F_i(t_n)) = log(F_i(t_{n+1})) - log(F_i(t_n))$ and solve for $F_i(t_{n+1})$ using equation (14):

$$\begin{aligned} F_i(t_{n+1}) &= F_i(t_n)\exp\left(\Delta log(F_i(t_n))\right) \Rightarrow \\ F_i(t_{n+1}) &= F_i(t_n)\exp\left(\left(\sigma_i\sum_{j=1}^{i}\frac{\sigma_j F_j(t_n)\tau_j\rho_{ij}}{1+\tau_j F_j(t_n)} - \frac{1}{2}\sigma_i^2\right)\Delta t + \sigma_i\Delta X_i^n\right) \end{aligned} \tag{19}$$

This is the forward Euler method [6]. For each time step we have to iterate over i in order to consider the correlation that the different forward rates have on each other. ΔX_i^n is drawn from a standard normal distribution, $\Delta t = 1/250$ (250 business days per year) and τ_j is from the vector: $\vec{\tau} = [0.25, 0.25, 0.5, 1, 3, 5]$. The procedure can be illustrated by the following flowchart (here the volatility and correlation are assumed to be constant)

[6] The forward Euler method is explained in the appendix A.1.

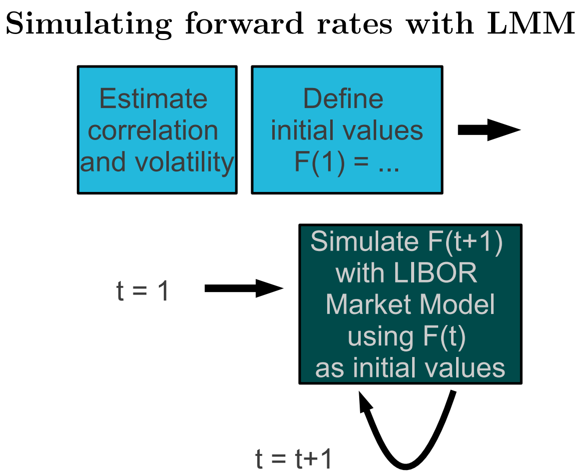

Figure 10: An illustration of the simulation procedure using the LIBOR Market Model. F(t) is the set of forward rates at time t

3.2.3 The simulation and its distribution

Using the parameter estimation with Floating Averages described above, in equation 17, gives a forward rate realization depicted in figure 11. In order to check the distribution of our simulated rates we can compare them to a log-normal distribution. The log-normal distribution has the probability density function:

$$f(x, \mu, \sigma^2) = \frac{1}{x\sqrt{2\pi\sigma^2}} \cdot exp\left(-\frac{(log(x)-\mu)^2}{2\sigma^2}\right) \tag{20}$$

Maximum likelihood analysis [7] then gives estimates for the parameters μ and σ^2. The (log-) likelihood is given by:

$$\begin{aligned} L(x_1,...,x_n,\mu,\sigma^2) &= \prod_{i=1}^{n} \frac{1}{x_i\sqrt{2\pi\sigma^2}} \cdot exp\left(-\frac{(log(x_i)-\mu)^2}{2\sigma^2}\right) \\ log(L) &= -\sum_{i=1}^{n}\left(log(x_i) + \frac{(log(x_i)-\mu)^2}{2\sigma^2}\right) - nlog(\sigma) - \frac{n}{2}log(2\pi) \end{aligned} \tag{21}$$

Maximizing the log-likelihood, $L(\cdot)$, gives the likelihood estimators for this analysis.

$$\begin{aligned} \frac{\partial log(L)}{\partial \mu} = 0 \quad &\Rightarrow \quad \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} log(x_i) \\ \frac{\partial log(L)}{\partial \sigma} = 0 \quad &\Rightarrow \quad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(log(x_i)-\hat{\mu})^2 \end{aligned} \tag{22}$$

[7] Maximum likelihood analysis is explained in the appendix A.6.

Simulated forward rates

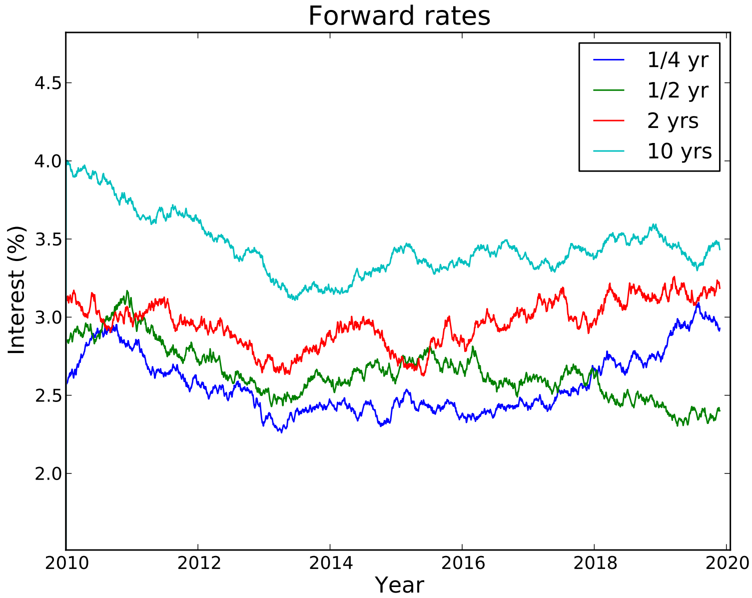

Figure 11: A realization of forward rates using LMM

Below is the log-normal distribution with the log-likelihood parameters found using formula 22, compared to the resulting distribution of the relative change in interest rates realized by LMM.

The histogram is the distribution of $F(t_{n+\Delta})/F(t_n)$ where $\Delta = 1$, N is the number of Monte Carlo simulations and n is an integer between 1 and $N - \Delta$.

The distribution of the forward rates

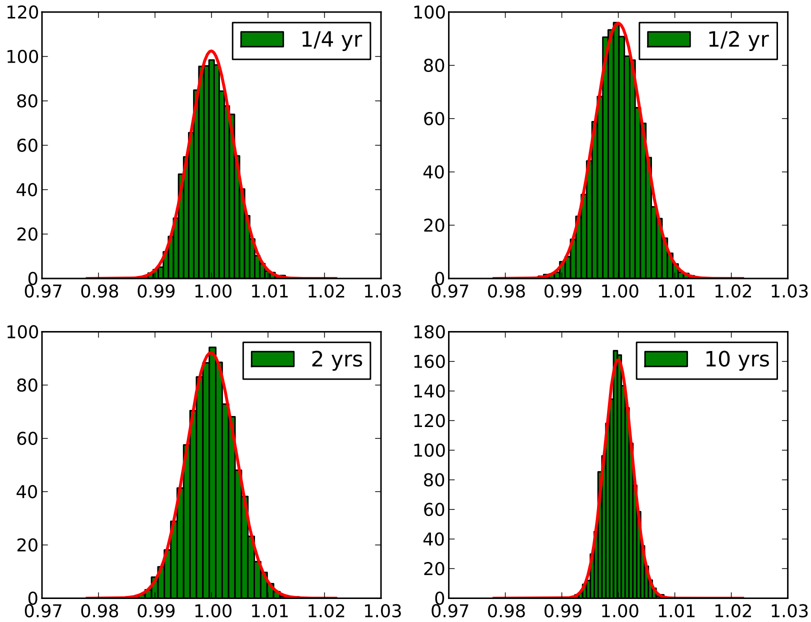

Figure 12: the distribution of the relative change in the predicted forward rates after 1 day. The red curve is the pdf of the log-normal distribution with the maximum likelihood parameters. $N = 5000$.

The resulting distribution histogram is found by N repeated realizations of the forward rates 1 day into the future, similar to the plots in the preliminary analysis (plot 7 and 8).

It is clear that the simulated forward rates follow a log-normal distribution and thus have a lower kurtosis than the historical forward rates.

4 Interest Rate Derivatives and Risk Measures

4.1 Interest Rate Derivatives

Derivatives are agreements of future cash flows derived from underlying assets such as a shares, currency, interest rates etc. They are financial tools that can be used to reduce risk or as an investment. Interest rate derivatives have a value depending on the interest rates and are used to insure against low or high interest rates. The most common interest rate derivatives are explained below.

4.1.1 Bond option

A bond is an agreement of one or more predefined payments in the future. If there is only one predefined payment it is called a zero coupon bond. The value a zero coupon bond paying 1$ in a month depends on the expected interest rate during the next month. If you believe the interest rate will increase it is profitable to sell bonds, and if you believe it will decrease you should buy bonds. A bond option is an option on the bond, meaning that the holder of the bond option can choose whether to buy or sell (call or put) [8] the option or to let it expire.

4.1.2 Cap/Floor

A cap is a contract that insures against high interest rates. The holder is guaranteed that otherwise floating rates will not exceed a specified rate. This is valuable if you have a loan where you can not afford to pay more than say 8% interest. The payoff of a cap is the principal amount P multiplied with the difference between the floating interest rate r_F and the cap rate r_C:

$$Payoff = P \times max(r_F - r_C, 0) \tag{23}$$

A floor is an insurance against low interest rates and is otherwise similar to a cap.

4.1.3 Swap

In an interest rate swap the two parts agree to pay either a fixed strike rate r_S, or a floating rate r_F, multiplied with a principal amount P. This is illustrated below. In a call swap the buyer can choose to be the fixed rate payer, and in a put swap the buyer can choose to pay floating rates.

A swap

$$\boxed{r_S \times P \ \rightleftarrows \ r_F \times P}$$

The swap will be an insurance for both parts, but similarly both parts bear the risk of having to pay the other part an insurance premium. You could achieve the same payoff

[8] A call option gives the holder the right, but not the obligation, to buy the underlying at a predefined time in the future (for European options) or at any time up until the predefined time (for American options). A put option gives the holder the right to sell and is otherwise similar to the call.

by buying a cap and selling a floor, this is called cap-floor parity. The cap-floor parity formula yields:

$$cap = floor + swap \tag{24}$$

Swaps are used to hedge against, or speculate on movements, in interest rates. A swap turns fixed rates into floating rates or the other way around.

4.2 Risk in the Financial Market

In the financial market, an investment is always associated with some risk. There is always a chance of losing money. This is, however, usually compensated with a greater expected return than risk-free investments. An old philosophical view on risk is that it is a function of our ignorance. If we had enough information and understanding the risk would disappear.

However, in finance the number of parameters that determine prices are way too many to predict. Among the important parameters are the behavior of stockholders, banks and the heads of important companies. Risk measures are a way to quantify the uncertainty in these parameters.

In practice, risk measures are used for a number of purposes, risk plays an important role in the determination of the amount of capital a company holds as a buffer against unexpected losses and the size of a company's insurance premium. Risk measures are also important management tools, a good manager would not only be concerned with daily returns but also the total risk in the company. Risk can roughly be divided into three classes: market risk, credit risk and operational risk.

Market risk is risk of losses due to undesirable movements in the market. For example, a reduction in value of the underlying assets. The four main risk factors associated with market risk are equity risk, interest rate risk, currency risk and commodity risk. Common measures for market risk are Value at Risk and Expected Shortfall. Both give information about worst case losses.

Credit risk is the risk of not receiving payments due to defaults. This class of risk is important to banks as a measure of how much they can lose on customers unable to pay back their loans. During the financial crisis credit risk got a lot of attention as it became clear that the banks had underrated its importance.

Operational risk is the risk arising from the internal processes in a company. Legal issues, fraud, software error and injury are some of the sources of operational risk.

This thesis will focus on market risk, in particular risk related to change in interest rates. The measures that will be studied in this thesis are based on loss distributions and are among the most common ways to quantify market risk in practice [1].

4.2.1 The Loss Distribution

The loss is defined as the negative change in portfolio value, formally if we denote the portfolio value at time t with V_t:

$$L_{t+1} := L_{[t\Delta,(t+1)\Delta]} = -(V_{t+1} - V_t) \tag{25}$$

Where Δ is the time step, usually set to 1/250 meaning one out of 250 business days per year.

V_t is a function of time and some risk factors Z_t, so we write $V_t = f(t, Z_t)$. Defining X_t to be the change in risk factors, $X_t = Z_t - Z_{t-1}$ the loss yields:

$$L_{t+1} = -(f(t+1, Z_t + X_{t+1}) - f(t, Z_t)) \tag{26}$$

The *conditional loss distribution* is the distribution of L_{t+1} such that

$$F_{L_{t+1}|\xi_t}(l) = P(L_{t+1} \leq l \mid \xi_t) \tag{27}$$

Where ξ_t is the publicly available information at time t. Similarly the *unconditional loss distribution* is defined as

$$F_{L_{t+1}}(l) = P(L_{t+1} \leq l) \tag{28}$$

Assuming that the price changes are i.i.d. through time and that all the available information is already included in the value of the underlying, according to the Efficient Market Hypothesis [9], the conditional distribution equals the unconditional distribution.

4.2.2 Common Measures

Value at Risk

The VaR of a portfolio is the maximum loss which is not exceeded with a given high probability, the confidence level, α. In other words, the probability that the loss L exceeds l is $1 - \alpha$.

$$VaR_\alpha = inf\{l \in \mathbb{R} : P(L > l) \leq 1 - \alpha\} = inf\{l \in \mathbb{R} : F_L(l) \geq \alpha\} \tag{29}$$

α is usually set as 0.95 or 0.99. One drawback with the VaR measure is that is does not tell us anything about the losses that occur with a probability $1 - \alpha$. Expected Shortfall or Average Value at Risk is another risk measure that is concerned with the tail of the loss distribution.

Expected Shortfall

Expected Shortfall is a measure of the expected loss given that the loss L exceeds l. Here $P(L > l) = 1 - \alpha$.

$$ES_\alpha = \frac{1}{1-\alpha} \int_\alpha^1 q_u(F_L) du \tag{30}$$

[9] The Efficient Market Hypothesis states that the past history is fully reflected in the present prices and that the market respond immediately to any new information [4].

Where $q_u(F_L)$ is the quantile function for the loss distribution L. For any portfolio the Expected Shortfall ES_α is greater than or equal to the Value at Risk VaR_α at level α. Below, equation 30 is rewritten using $VaR_\alpha = q_u(F_L)$.

$$ES_\alpha = \frac{1}{1-\alpha}\int_\alpha^1 VaR_u(L)du \tag{31}$$

Or more intuitively using the notation $E(X;\ A) = E(XI_A)$ and Bayes' Theorem [10] :

$$ES_\alpha = E(L \mid L \geq VaR_\alpha) = \frac{E(L;\ L \geq q_\alpha(L))}{1-\alpha} \tag{32}$$

4.2.3 Coherence

Risk measures can be classified into coherent and non-coherent measures. The idea behind coherency is to ensure that a risk measure behaves reasonably when it is applied to aggregate risk. In order to be classified as coherent, a risk measure needs to satisfy the four axioms of coherence. We define the risk measure as ξ (risk measured in capital) and the financial risk is denoted by L.

I. **Translation invariance.** *For every real number* l*:* $\xi(L - l) = \xi(L) - l$.
In other words adding or subtracting a deterministic quantity to a position changes the risk measure by that exact amount. Remember the convention that a positive number means a loss, the example above represents adding cash, l, to the portfolio L. By choosing $l = \xi(L)$ the position is acceptable without further injection of capital.

II. **Subadditivity.** *For the risks* L_1 *and* L_2*:* $\xi(L_1 + L_2) \leq \xi(L_1) + \xi(L_2)$
This axiom says that investing in two risky projects can never be more risky than the sum of the risk of each project. This is the idea behind diversification: spread out the investments and reduce risk.

III. **Positive homogeneity.** *For every* λ *greater than zero:* $\xi(\lambda L) = \lambda\xi(L)$.
Applying the subadditivity axiom on two identical positions and taking into consideration that there is no diversification between them justifies this axiom.

IV. **Monotonicity.** *For the risks* L_1, L_2 *such that* $L_1 \leq L_2$ *we have that:* $\xi(L_1) \leq \xi(L_2)$.
Financially this makes sense as greater risk induces the possibility of greater losses.

Testing the axioms on each of the risk measures introduced above shows that Expected Shortfall is a coherent risk measure while Value at Risk fails to satisfy the subadditivity axiom. So VaR is not a coherent risk measure, however it is widely applied and a useful tool nonetheless.

[10] Bayes' Theorem states that two events A and B are related through conditional probability, given that $P(B) \neq 0$, by: $P(A \mid B) = \frac{P(A)P(B|A)}{P(B)} = \frac{P(A;\ B)}{P(B)}$. Through some computation this leads to the similar relationship between the expectation values above.

4.2.4 Estimates and Backtesting

Value at Risk

In order to estimate the $VaR_{0.95}$ using Monte Carlo simulation the losses from a number (say N) of realizations are collected and sorted. The $VaR_{0.95}$ is found by choosing the $N \times 0.5$ largest loss.

Backtesting of $VaR_{0.95}$ can be done by comparing the number of $VaR_{0.95}$ violations with the expected number, which is 5% by definition.

Expected Shortfall

$ES_{0.95}$ is found by taking the mean of the $N \times 0.5$ largest losses. The distribution of the excess losses is expected to be right skewed (or positively skewed) which is implicit in the assumption of normal distribution.

Since the loss distribution is right skewed the $ES_{0.95}$ is expected to be violated less than 2.5% of the time. This is the backtesting strategy for Expected Shortfall.

Below is an illustration of the estimation procedure of the two risk estimates.

The risk estimates

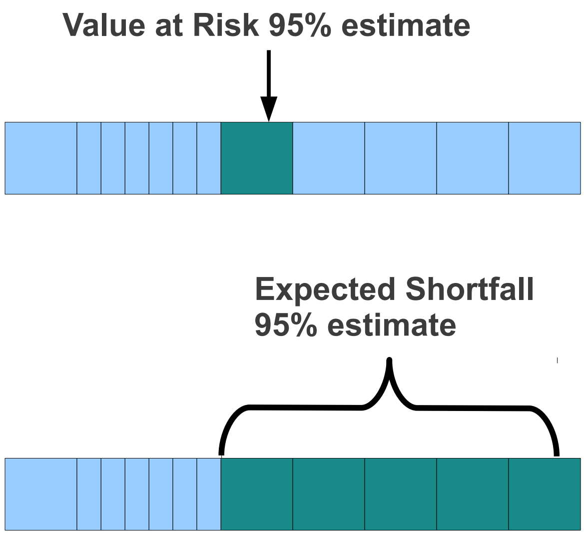

Figure 13: The estimates of $VaR_{0.95}$ and $ES_{0.95}$. The vectors of losses are sorted and have length 100

5 Implementation and Results

5.1 Cap-Floor Parity

The first implementation aims to show the cap-floor parity (equation 24) in practice by pricing a cap, floor and swap using the LMM.

The formula for the price of a cap which pays a principal amount, P, times the interest rate difference if the forward rate at maturity ($\bar{r}$) exceeds the cap rate r_C yields:

$$Cap_n^* = P \cdot E(max(\bar{r} - r_C, 0)) \prod_{i=0}^{n-1} \frac{1}{(1+\tau_i F_i)} \tag{33}$$

The formula can be expanded to be valid for a cap that pays P times the interest rate difference every period $\bar{r}_i$ exceeds r_C, where i is the same time index as with Z_i:

$$Cap_n = \sum_{j=0}^{n-1} P \cdot E(max(\bar{r}_j - r_C, 0)) \prod_{i=0}^{j} \frac{1}{(1+\tau_i F_i)} \tag{34}$$

The expectation $E(\cdot)$ can be estimated by Monte Carlo simulations:

$$E(max(\bar{r}_i - r_C, 0)) = \sum_{j=1}^{m} \frac{max(F_i^j - r_C, 0)}{m} \tag{35}$$

where j denotes the realization number. The estimate converges to the expectation value almost surely as n goes to infinity by the strong law of large numbers[11]. This formula accounts for possible payouts after the end of the time period of each forward rate. In other words; there is a possible payout after 1/4, 1/2, 1, 3, 5 and 10 years.

The discounting in this formula uses average values as an approximation. When the cap rate is very high the forward rate averages would in theory be affected by this, but in practice this is negligible.

For a floor the *max*(x,0) would be exchanged by -*min*(x,0). A swap would have *max*(x,0) - (-*min*(x,0)) = x, rewriting formula (34) for a floor and a swap yields:

$$\begin{aligned} Floor_n &= \sum_{j=0}^{n-1} P \cdot E(-min(\bar{r}_j - r_C, 0)) \prod_{i=0}^{j} \frac{1}{(1+\tau_i F_i)} \\ Swap_n &= \sum_{j=0}^{n-1} P \cdot E(\bar{r}_j - r_C) \prod_{i=0}^{j} \frac{1}{(1+\tau_i F_i)} \end{aligned} \tag{36}$$

In addition to the prices, VaR and ES will be estimated. $VaR_{0.99}$ will be estimated by for example choosing the 10th largest loss out of 1000 realizations. $ES_{0.99}$ will be estimated by the average of the 10 largest losses. The $VaR_{0.95}$ and $ES_{0.95}$ would similarly be estimated from the 50 largest losses. This is explained in chapter 4.

Below is the VaR, ES and price for a cap, floor and a swap with principal amount $P = 1000$. The number of realizations is 2000 and the fixed rate, r_C, is 6%. The payout of these derivatives is in one year from today, and the fixed rate is compared to the 3 month LIBOR rate.

[11] The law of large numbers states that the sample average converges almost surely to the expectation value (see the appendix A.5 and A.4).

Type: Swap, Cap and Floor	
Risk estimates with confidence level 95%	
Principal amount:	1000
Cap rate :	6 %
Time horizon:	3 months
Initial values:	[0.07,0.07,0.07,0.07,0.07,0.07]
Volatility:	[0.0125,0.0134, 0.0154,0.0131,0.0105,0.0090]

Correlation matrix:

$$\hat{\rho}_1 = \begin{pmatrix} 1 & & & & & \\ 0.754 & 1 & & & & \\ -0.024 & -0.021 & 1 & & & \\ 0.044 & 0.054 & 0.045 & 1 & & \\ 0.077 & 0.090 & -0.047 & 0.862 & 1 & \\ 0.062 & 0.086 & -0.029 & 0.697 & 0.816 & 1 \end{pmatrix}$$

The results are as follows:

Type	Price	Value at Risk (95%)	Expected Shortfall (95%)
Cap	12.103	12.103	12.103
$Floor$	1.217	1.217	1.217
$-Floor(short)$	-1.217	7.329	10.996
$Swap$	10.886	19.432	23.098

Notice that the prices follow the cap floor parity, $Cap - Floor = Swap$ as expected $(12.103 - 1.217 = 10.886)$. Additionally it is worth noting that the additional payment is the same for a swap and a floor sold short, in other words VaR - price $(7.329 - -1.217 = 8.546$ and $19.432 - 10.886 = 8.546)$. The same is valid if we use Expected Shortfall or the relationship between a cap short and a swap short.

VaR and ES considers the loss due to the purchase or selling price, so in the case where you don't have to make any additional payments you would still lose the amount you paid for the security to begin with. In the case where a floor is sold short additional expense is expected since there is a negative cash flow (you get money) initially. If this was not the case, there would be an arbitrage opportunity (for a confidence level sufficiently close to 1).

It might seem like to sell a floor short is a good deal since you are paid an amount initially instead of paying for a swap, and the VaR and ES are lower. This is not necessarily the case because the swap gives an opportunity to gain money if the interest rates get sufficiently low. For a floor sold short the maximum additional payout is zero.

The number of Monte Carlo simulations is 2000, and as we see from these convergence plots over the Value at Risk and Expected Shortfall estimates, 2000 simulations gives an absolute error of less than 0.1.

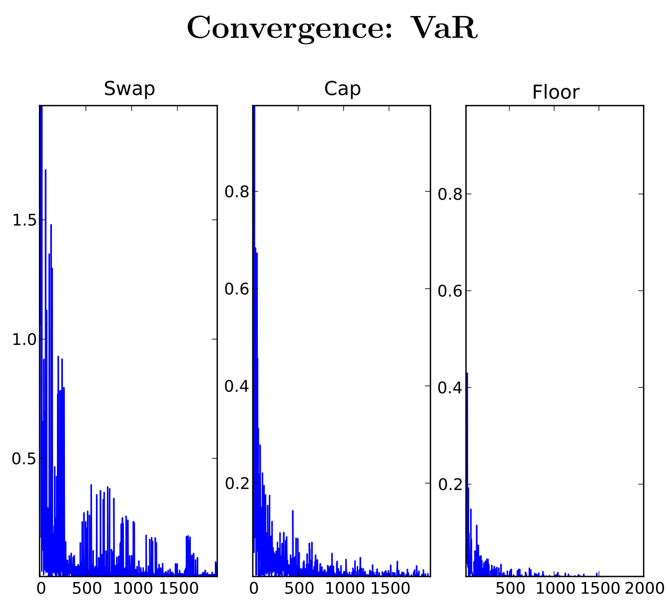

Figure 14: Absolute error for the Value at Risk (95%) estimates

The Expected Shortfall estimate converges faster because the estimate is the average of the 5% lowest values, and not a single quantile.

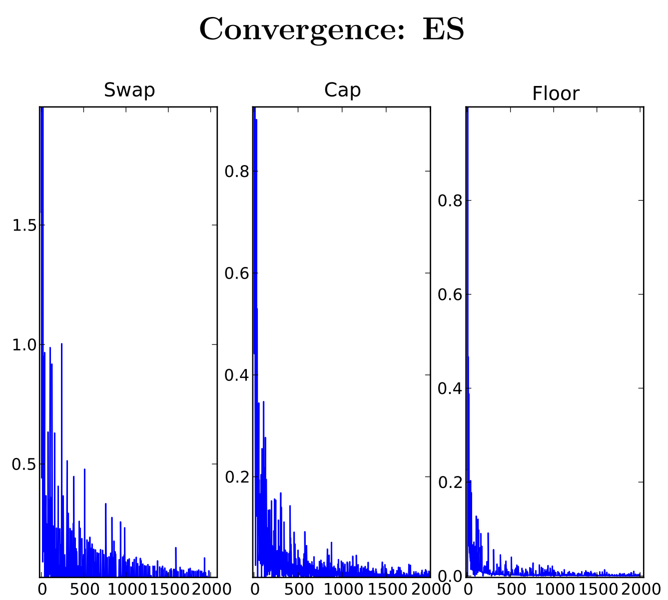

Figure 15: Absolute error for the Expected Shortfall (95%) estimates

5.2 Portfolio overview

In the rest of the result section portfolio risk will be analyzed, and a number of portfolios will be defined. This is an overview.

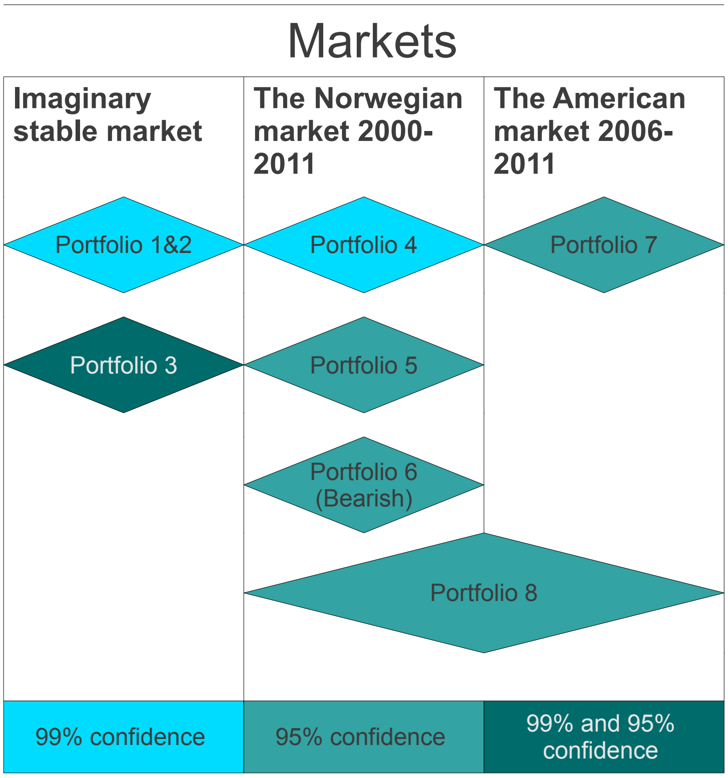

Figure 16: The portfolios subject to risk analysis in the implementation and results section

The main idea is to test different portfolios in different markets to analyze their properties, and to test the risk measures where it is possible. At the end of this section some investment strategies based on the risk estimates will be tested.

5.3 Portfolio risk in a stable market

In this subsection, the volatility and correlation is held constant in order to simulate a stable market. Portfolio 1 and 2 are exposed to interest rates with higher volatility than portfolio 3.

5.3.1 Swap risk

Portfolio 1 & 2

Portfolio 1 contains a swap (long position[12]) on the 1 year forward rate with principal amount $P = 100.000$. Portfolio 2 contains a swap (long position) on the 6 month forward rate with the a principal amount of 50.000 paid each period (6 months). In portfolio 1 there is a transaction after one year, while in portfolio 2 there are two transactions, one after 6 months and one after one year. The 6 month (green) and 1 year (blue) forward rates are the ones determining the outcome of the swap, a realization is depicted in figure 17. Here the fixed rate is at 3% (marked with a red line).

These two portfolios could be used as investments or, more likely, as insurance against high interest rates. A loan where the agreement is to pay a yearly interest rate equal to the 1 year forward rate could be hedged by a portfolio similar to portfolio 1. If the basis for interest rate payments is the 6 month forward rate, portfolio 2 would provide insurance. Banks would typically short swaps like this to hedge against low interest rates, as mentioned before, swaps provide both sides with insurance.

In this plot the portfolios are analyzed at different times along the time-axis of realized forward rates. The parameters are estimated from data up until the beginning of 2011, so this is where the realization starts. The endpoint is set to 2020. The main idea is to investigate how the initial values affect the risk measures.

In this estimate the same volatility and correlation as above is used. The initial values are taken from the last day in the historical data set, and 2000 Monte Carlo simulations are used.

Portfolio 1	**Portfolio 2**
Risk estimates with confidence level 99%	
Type: Swap (long)	**Type: Swap (long)**
$P = 100.000$	$P = 50.000$
Cap rate = 3%	*Cap rate* = 3%
Interest rate: 1 year forward rate	*Interest rate: 6 month forward rate*
Time horizon: 1 year	*Time horizon: 1 year*
Number of transactions: 1	*Number of transactions: 2*
Initial values: [0.026,0.028,0.027,0.031,0.035,0.04]	

[12] To be long a swap means to pay the fixed rate, thus being short a swap means to pay the floating rate.

Forward rates and risk estimates, portfolio 1 & 2

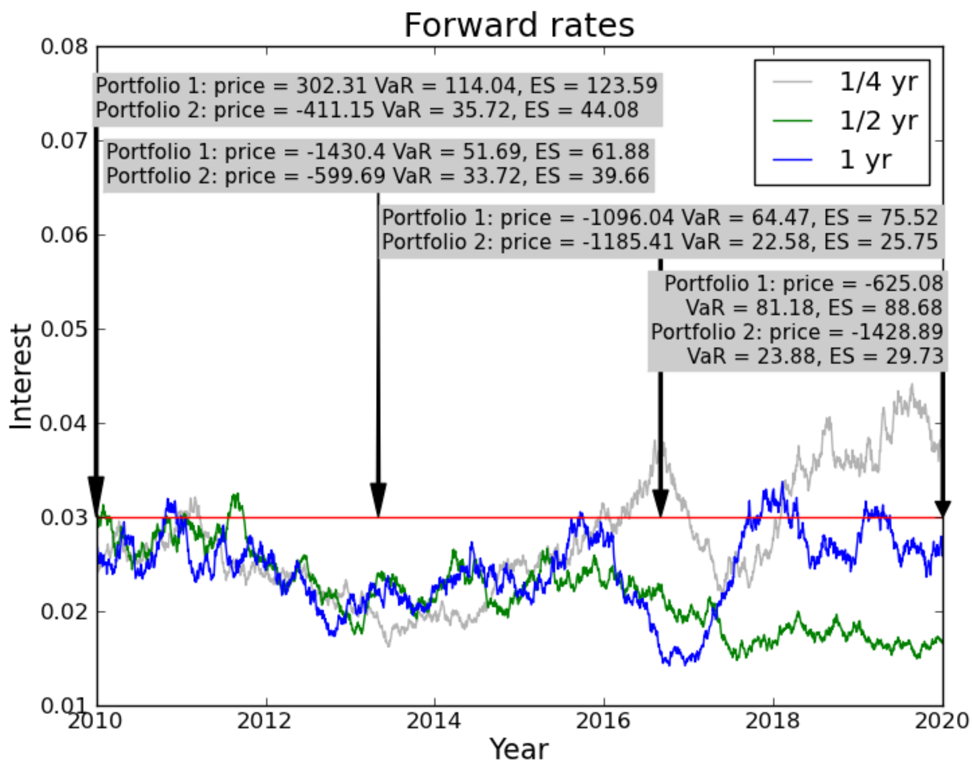

Figure 17: A swap on the 6 month forward rate and a swap on the 1 year forward rate with principal amount 100.000 and 50.000, respectively. The VaR and ES has a significance level of 0.99.

Note that Expected shortfall is always greater than, or equal, to Value at Risk. This is expected from risk measurement theory.

Flag 1 (from left to right): portfolio 2 has a negative price which means that the one paying floating rates has to pay to engage in the swap agreement. Portfolio 1 and 2 have their highest risk at this point.

Flag 2: portfolio 1 and 2 have negative prices since both the 1/2 and 1 year forward rate is far below the cap rate. Portfolio 2 has its lowest risk here.

Flag 3: the 1 year forward rate is at its lowest point here and the 1/2 year rate is also low, this leads to high prices for the one paying a floating rate. In other words high negative prices.

Flag 4: here the 1 year forward rates have increased which means that the price of portfolio 1 has increased (closer to zero), and the risk has increased. Portfolio 2 has not changed much since flag 3, so the price is still large and negative and the risk is relatively low.

The price and risk measures are determined by initial values and volatilities, for example in flag 1 the different rates have about the same initial value, but different price and risk. Another reason to why they differ is because portfolio 1 pays out everything at one point while portfolio 2 pays in two turns. This gives portfolio 2 a diversification effect, and thus lower risk.

5.3.2 Aggregate portfolio risk

The previous portfolios consisted of only one type of interest rate derivatives. The next step is to construct a more complex portfolio with several types of instruments. The idea is to create a realistic portfolio for risk management in a company which has issued loans and taken on debt with different conditions.

Portfolio 3
Let us imagine that this company has a debt of 10 million and pays annual floating interest rates using the 1 year forward rate. They want to hedge against the risk of high interest rate payments, but they do not want to compensate for low interest rates. In this case a cap (long) will provide the insurance they want.

Similarly the company has issued loans of about 8 million on average to customers unable to pay up front, this loan will be paid back in quarterly payments with a fixed interest rate of 3%. However the managers believes the interest rate will increase (hence the cap) so they want to short swaps in order to turn the interest rate payments into payments on floating rates. The decision here is to sell (short) 2 million swaps on the 3 month forward rate.

The caps and swaps have a 5 year time horizon, with 1 transaction per year for the cap and 4 per year for the swap. This portfolio is put together by two different instruments and there are several more payout dates, which means that the algorithm needs to discount over many different time intervals and add up the present value of the Monte Carlo simulated outcomes.

Portfolio 3	
Risk estimates with confidence levels 95% and 99%	
Type: Cap (long)	**Type: Swap (short)**
$P = 10.000.000$	$P = 2.000.000$
Cap rate = 3%	*Cap rate* = 3%
Interest rate: 1 year forward rate	*Interest rate: 3 month forward rate*
Time horizon: 5 years	*Time horizon: 5 years*
Number of transactions: 5	*Number of transactions: 20*
Initial values: [0.026,0.028,0.027,0.031,0.035,0.04]	

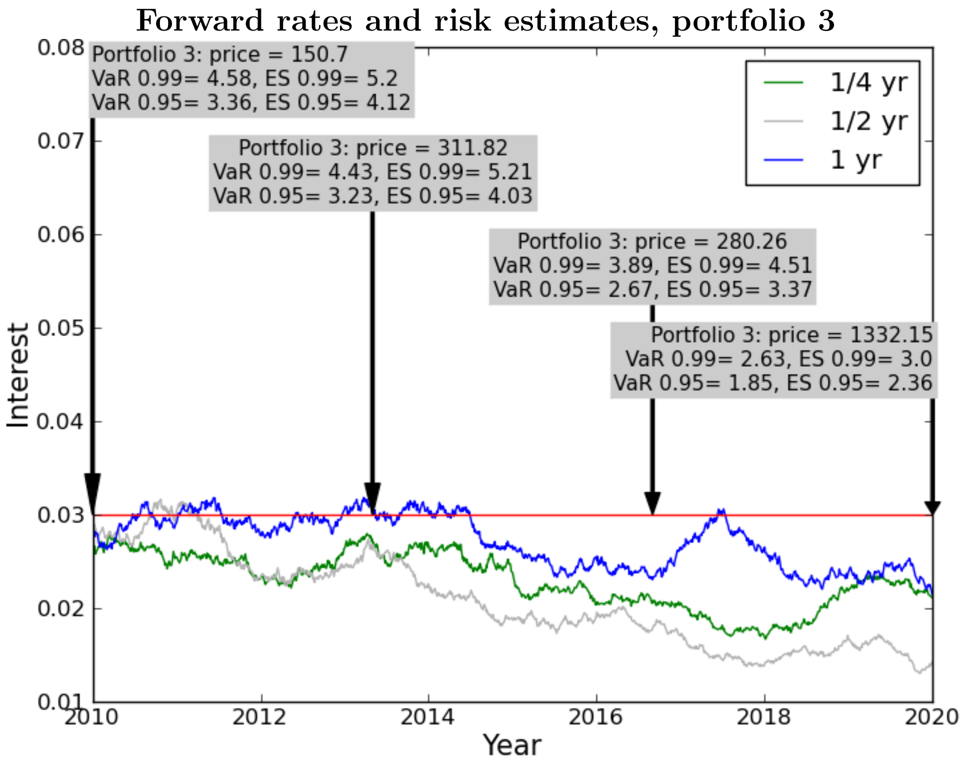

Figure 18: A portfolio with a swap and a cap with principal amount 10.000.000 and 2.000.000 respectively.

Flag 1 (from left to right): the portfolio has the highest risk here, the case where the 1/4 year forward rate is above the cap line, and the 1 year forward rate is below, will give the greatest losses.
Flag 2: both the 1/4 year and the 1 year forward rate has increased, but the risk is approximately the same. The price has doubled mainly due to the cap on the 1 year rate which has exceeded 3%.
Flag 3: the rates have gone down followed by the risk and the price of the portfolio, but the changes are not dramatic.
Flag 4: after the rates have decreased slightly more, the price goes up a lot, while the risk continues to decrease. The high price comes from the swap part of the deal which has a good chance of payout at maturity. Compared to flag 3 the main difference is in the 1/2 year forward rate, which is used to discount the possible earnings and affect the realizations through the correlation with the 1 year forward rate.

In this case the number of Monte Carlo simulations is reduced to 1000, because the routine is time consuming and 1000 repetitions gives good enough accuracy. The following plots illustrate the convergence for portfolio 3. Portfolio 4-8 are assumed to converge at a similar rate.

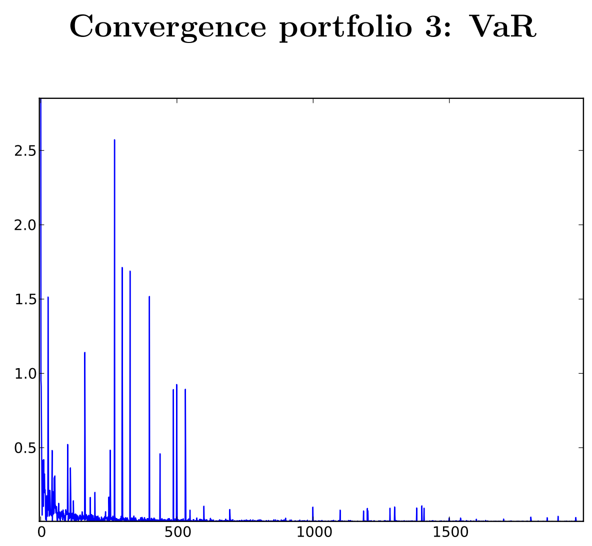

Figure 19: Absolute error for the Value at Risk (99%) estimates

The Expected Shortfall estimate converges somewhat faster because the estimate is the average of the 1% lowest values, and not a single quantile.

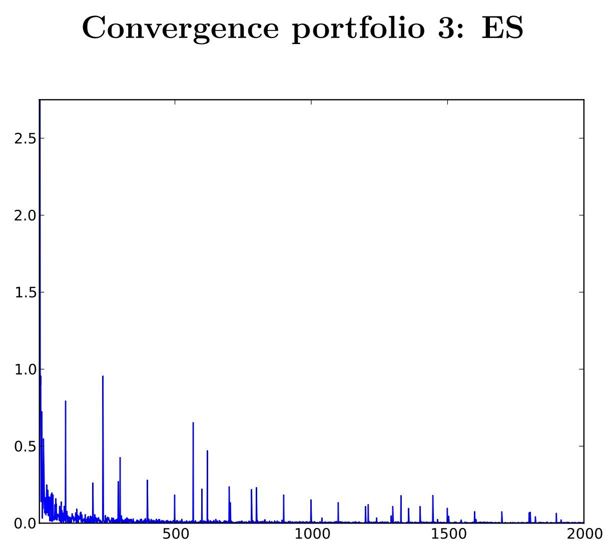

Figure 20: Absolute error for the Expected Shortfall (99%) estimates

The spikes in the convergence plots appears when a new data point is considered, meaning every 100th repetition with 99% confidence.

5.4 Risk measures and the financial crisis, Norwegian market

5.4.1 Bullish portfolio

Portfolio 4[13]

In this section the risk measures will be applied to the historical data set from the Norwegian market depicted in figure 1. The risk estimates have a 5 day horizon, a new estimate is found every fifth day and the time span is between 2000 and 2011. Every five day loss is compared to the 5 day Value at Risk and Expected Shortfall. In theory, the loss should be higher than the $VaR_{0.99}$ one out of a hundred times. The $ES_{0.99}$ is expected to be violated less than one out of two hundred times because the excess loss distribution is right skewed, however, the financial crisis is known to have foiled even the most conservative risk measures. In the plot below the red line indicates the real losses, while the green and blue line represents VaR and ES, respectively.

Portfolio 4 consists of a swap (long) on the 1/4 year rate with principal amount 1000 and a cap (short) on the 1/2 year rate with principal amount 800. This means that the portfolio freezes the 1/4 year forward rate while it provides insurance to the counterpart against increasing 1/2 year forward rates.

In this estimate the volatility and correlation is continuously updated in order to see how the risk estimates reacts to the market movements. For the first year initial values are assumed.

Portfolio 4	
Risk estimates with confidence level 99%	
Type: Cap (short)	**Type: Swap (long)**
$P = 800$	$P = 1000$
Cap rate = 5%	*Cap rate* = 5%
Interest rate: 1/2 year forward rate	*Interest rate: 3 month forward rate*
Time horizon: 5 days	*Time horizon: 5 days*
Number of transactions: 1	*Number of transactions: 1*

In figure 21 the risk estimate is plotted using Floating Averages, and in figure 22 it is plotted using EWMA.

[13] Bullish means that this portfolio pays off when the interest rates increase.

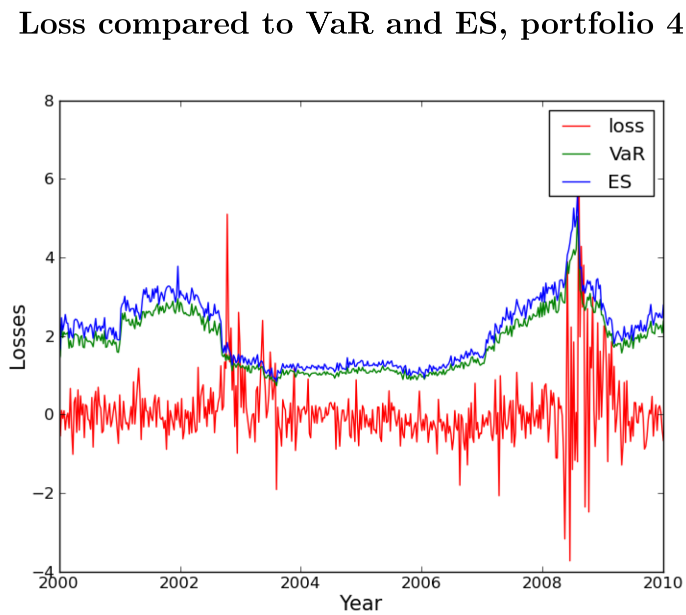

Figure 21: Losses compared to risk measures using Floating Averages (lag = 50).

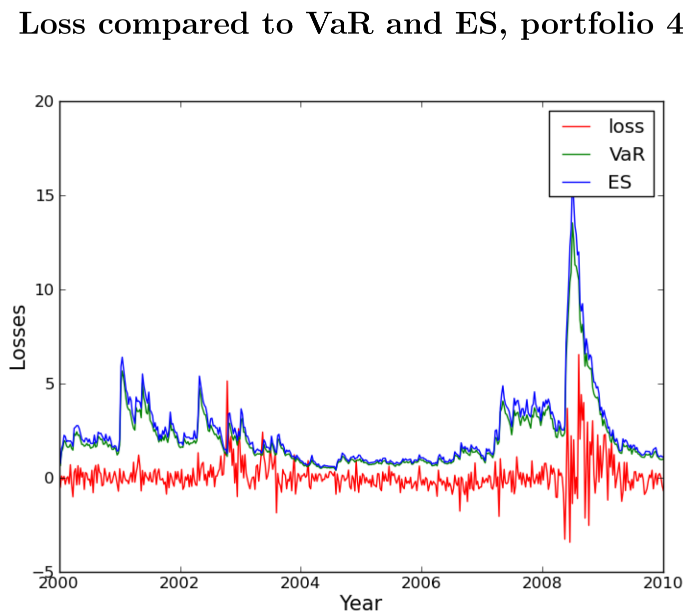

Figure 22: Losses compared to risk measures using EWMA ($\lambda = 0.95$).

With the continuously updated volatility and correlation, the estimated risk is exceeded more often than expected. This is not surprising knowing how the financial crisis played out for many investors. Assuming the violations of $VaR_{0.99}$ follows a binomial distribution with probability $p = 0.01$ leads to a standard deviation of 0.4%[14]. The results can be summarized in the following table, keep in mind that the expected percentage of $VaR_{0.99}$ violations is 1.

Portfolio 4 - backtesting (Norwegian market)

Method	$VaR_{0.99}$ *violations*	*VaR violation* σ	$ES_{0.99}$ *violations*
Floating Averages	*3.8%*	*0.4%*	*2.3%*
EWMA	*2.1%*	*0.4%*	*1.5%*

$VaR_{0.99}$ is violated approximately 4% of the time using Floating Averages, which means that the observation is about 8 times the standard deviation above the expectation value. Using EWMA, with the parameter $\lambda = 0.95$[15], the $VaR_{0.99}$ is violated approximately 2% of the time which is much more accurate.

The number of violations of the estimated Expected Shortfall is larger than half of the violations of the Value at Risk estimates.

These observations shows that the risk measures need to be very conservative in order to hold during a crisis like the financial crisis of 2007-2010. However, the EWMA procedure is significantly more accurate so it will be used in the next portfolio risk estimate.

Portfolio 5

In portfolio 5 the cap's principal amount is increased while the swap's principal amount is decreased in order to change which instrument is the most dominant. The cap rate is increased to 7% which is close to the maximum interest rate in the period we investigate, and the confidence interval is reduced to 95%. The procedure is otherwise similar to the estimates of the portfolio risk of portfolio 4.

Portfolio 5

Risk estimates with confidence level 95%

Type: Cap (short)	**Type: Swap (long)**
$P = 1500$	$P = 800$
Cap rate = 7%	*Cap rate* = 7%
Interest rate: 1/2 year forward rate	*Interest rate: 3 month forward rate*
Time horizon: 5 days	*Time horizon: 5 days*
Number of transactions: 1	*Number of transactions: 1*

[14]There are 10 years with 250 business days and we investigate the risk every 5th day. In other words 500 draws from the binomial distribution. The standard deviation is: $\sqrt{0.01 \times (1 - 0.01) \times 500} \approx 2$ (0.4%).

[15]See equation 18 or the appendix B.2 for the definition of EWMA and the weighting parameter λ.

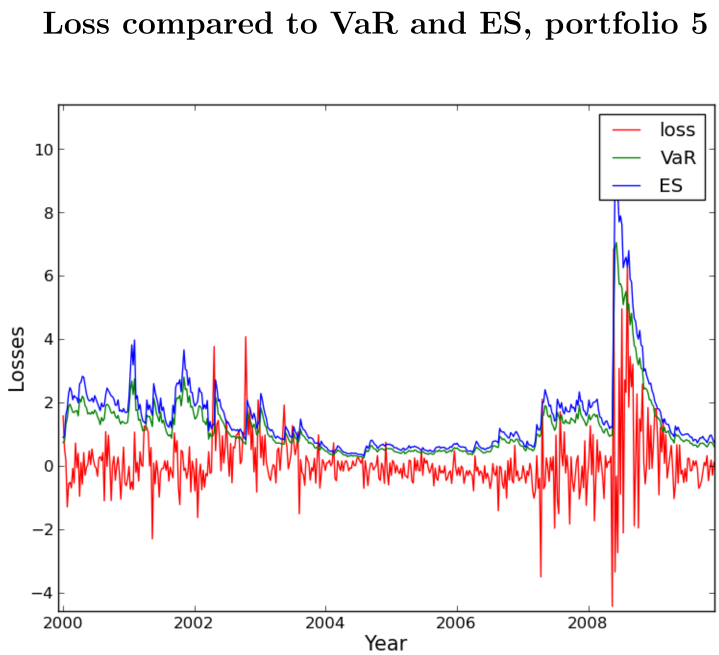

Figure 23: Losses compared to risk measures during the recess periods between 2000 and 2011.

In this plot the volatility and correlation is updated continuously using EWMA ($\lambda = 0.95$) with the data available at the time.This time the $ES_{0.95}$ and the $VaR_{0.95}$ is estimated. Violations of $VaR_{0.95}$ follow a binomial distribution with probability $p = 0.05$ which means that the standard deviation (σ) is approximately 1%[16]. Keep in mind that the $VaR_{0.95}$ threshold is expected to be violated 5% of the time.

Portfolio 5 - backtesting (Norwegian market)

Method	*$VaR_{0.95}$ violations*	*VaR violation σ*	*$ES_{0.95}$ violations*
EWMA	*6.7%*	*1%*	*4.8%*

The observation lies less than $2 \times \sigma$ above the expectation value. The violations of $ES_{0.95}$ are more frequent than half of the violations of $VaR_{0.95}$.

5.4.2 Bearish portfolio

Portfolio 6[17]

In the previous subsection the portfolios (number 4 and 5) had losses when the interest rates decreased, so when the interest rates were low the losses were limited because the interest rates can not be less that zero (in the model). In portfolio 6 the opposite is the case, in order to achieve this we buy a floor (long) on the 1/2 year forwards rates with

[16] $\sqrt{0.05 \times (1 - 0.05) \times 500} \approx 5$ (1%).

[17] Bearish means that this portfolio pays off when the interest rates decrease.

principal amount 1500 and we sell a swap (short) on the 1/4 year forward rates with principal amount 800. This portfolio would give insurance against low 6 month rates and low 3 month rates.

The volatility and correlation is continuously updated and the cap rate is increased to 7%, the procedure is otherwise similar to the estimates on portfolio 4 (Floating Averages and EWMA ($\lambda = 0.95$) will be compared). Remember from the initial plot, figure 1, that the most sudden changes are from high to low interest rates, which would give a negative loss (you would make money) with portfolio 6.

Portfolio 6	
Risk estimates with confidence level 95%	
Type: Floor (long)	**Type: Swap (short)**
$P = 1500$	$P = 800$
Cap rate $= 7\%$	*Cap rate* $= 7\%$
Interest rate: 1/2 year forward rate	*Interest rate: 3 month forward rate*
Time horizon: 5 days	*Time horizon: 5 days*
Number of transactions: 1	*Number of transactions: 1*

Loss compared to VaR and ES, portfolio 6

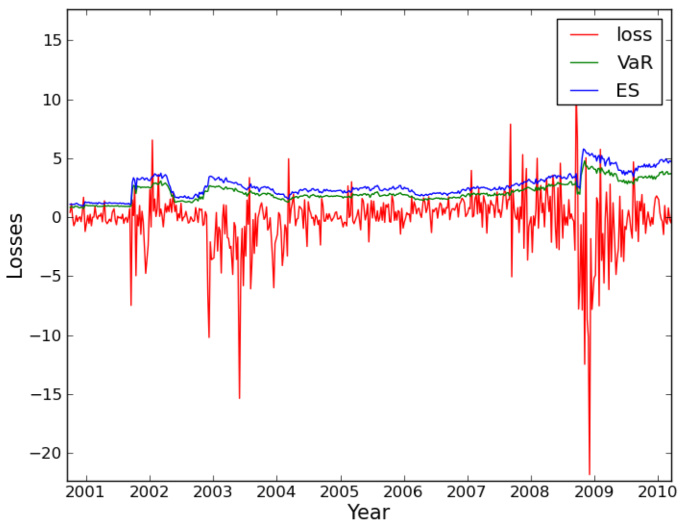

Figure 24: Losses compared to risk measures during the recess periods between 2000 and 2011. The volatility and the correlation are updated continuously using Floating Averages.

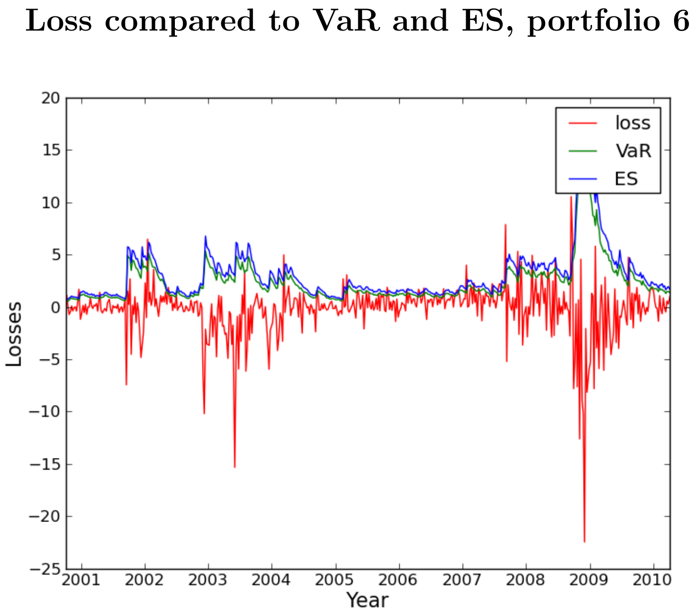

Figure 25: Losses compared to risk measures during the recess periods between 2000 and 2011. The volatility and the correlation are updated continuously using EWMA ($\lambda = 0.95$).

In this plot we see, as expected, that the losses are smaller than with the portfolios which makes money on high interest rates. The results are as follows:

Portfolio 6 - backtesting (Norwegian market)

Method	$VaR_{0.95}$ *violations*	*VaR violation* σ	$ES_{0.95}$ *violations*
Floating Averages	*7.1%*	*1%*	*4.2%*
EWMA	*9.2%*	*1%*	*5.8%*

Here, using Floating Averages gives a risk estimate where the $VaR_{0.95}$ violations lie about $2 \times \sigma$ above the expected value, while the EWMA routine performs badly (more than $4 \times \sigma$ above). Again, the violations of $ES_{0.95}$ are more frequent than half of the violations of $VaR_{0.95}$.

5.5 Risk measures and the financial crisis, American market

5.5.1 Bullish portfolio

The historical data used in this subsection is depicted in figure 5. The 1/4 year and 1/2 year forward rates are less volatile in the American market than in the Norwegian, this will affect the risk estimates and the losses.

Portfolio 7
The risk estimates are carried out in the same way as they were with the Norwegian

historical data. This time the time span is from 2006 until March 2011 and the time horizon is reduced to 1 day.

Portfolio 7	
Risk estimates with confidence level 95%	
Type: Cap (short)	**Type: Swap (long)**
$P = 1500$	$P = 800$
Cap rate = *7%*	*Cap rate* = *7%*
Interest rate: 1/2 year forward rate	*Interest rate: 3 month forward rate*
Time horizon: 1 day	*Time horizon: 1 day*
Number of transactions: 1	*Number of transactions: 1*

Loss compared to VaR and ES, portfolio 7

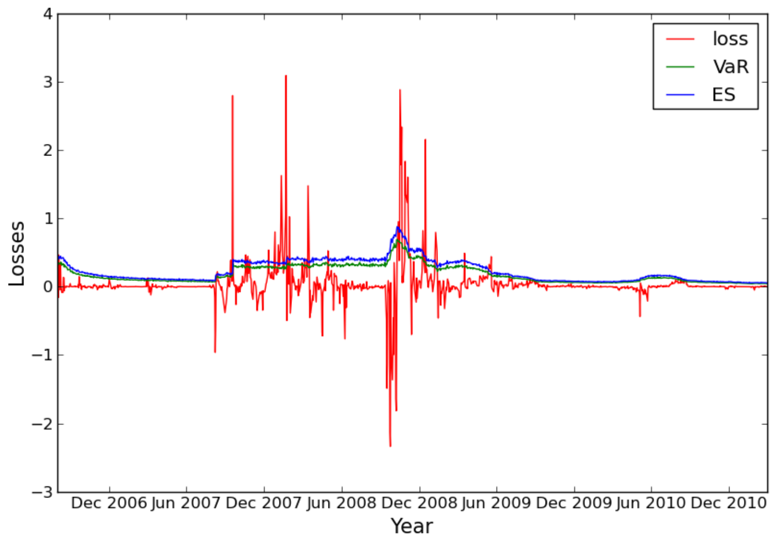

Figure 26: Losses compared to risk measures during the Financial Crisis (between 2006 and March 2011). The volatility and the correlation are updated continuously using Floating Averages.

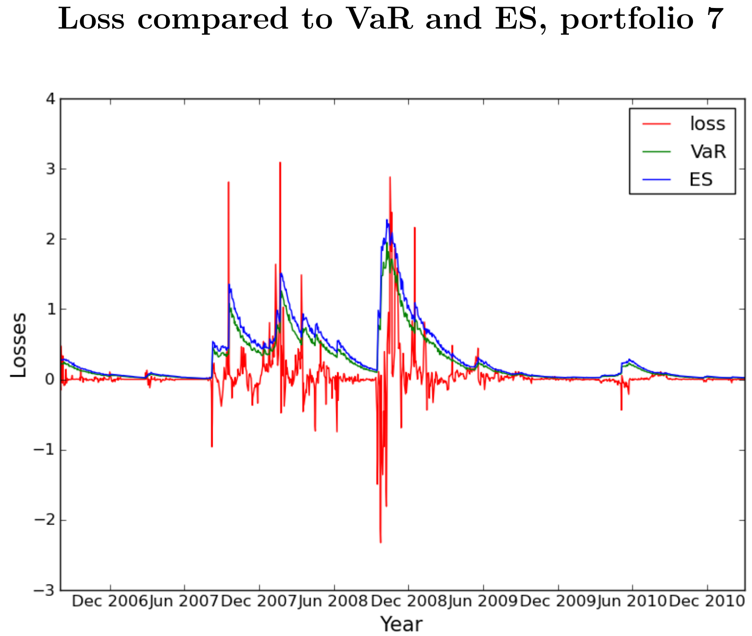

Figure 27: Losses compared to risk measures during the Financial Crisis (between 2006 and March 2011). The volatility and the correlation are updated continuously using EWMA ($\lambda = 0.95$).

The standard deviation is calculated as before[18].

Portfolio 7 - backtesting (American market)

Method	$VaR_{0.95}$ *violations*	*VaR violation* σ	$ES_{0.95}$ *violations*
Floating Averages	*6.3%*	*0.6%*	*4.4%*
EWMA	*5.6%*	*0.6%*	*3.5%*

The plots indicate a very high degree of volatility clustering, the EWMA risk estimate procedure quickly responds to changes in the volatility, while the Floating Averages procedure is less affected by spikes in the return. However, the $VaR_{0.95}$ violations lie fairly close to the expected value for both procedures (EWMA σ above and Floating Averages $2 \times \sigma$ above). The Expected Shortfall estimates are violated more frequently than expected.

5.6 Portfolio with market to market swap (Norwegian ⇆ American)

In smaller markets, like the Norwegian market, companies often have to raise capital from other larger markets, e.g. the American market. In portfolio 8 this is the case. In order to hedge against interest rate movements in the American market, a swap is

[18]Approximately 4 and a half years with 250 business days and a new estimate every day leaves us with 1160 draws from the binomial distribution: $\sigma = \sqrt{0.05 \times (1 - 0.05) \times 1160} \approx 7.4$ (0.6%).

bought in order to pay the fixed rate while a swap with the same principal amount, P, is sold short in the Norwegian market. This means that a company with a loan (with value P) from an American investor, and other loans from Norwegian investors, will be able to achieve the same interest rate exposure on all their loans with portfolio 8.

The principal value, fixed rate and other properties are summarized below. We investigate the historical forward rates in the time between the beginning of 2007 and the beginning of 2011.

Portfolio 8	
Risk estimates with confidence level 95%	
Type: Swap (short)	**Type: Swap (long)**
Market: Norwegian	**Market: American**
P = 1,000,000	P = 1,000,000
Cap rate = 5%	*Cap rate* = 5%
Interest rate: 3 month forward rate	*Interest rate: 3 month forward rate*
Time horizon: 1 day	*Time horizon: 1 day*
Number of transactions: 1	*Number of transactions: 1*

Loss compared to VaR and ES, portfolio 8

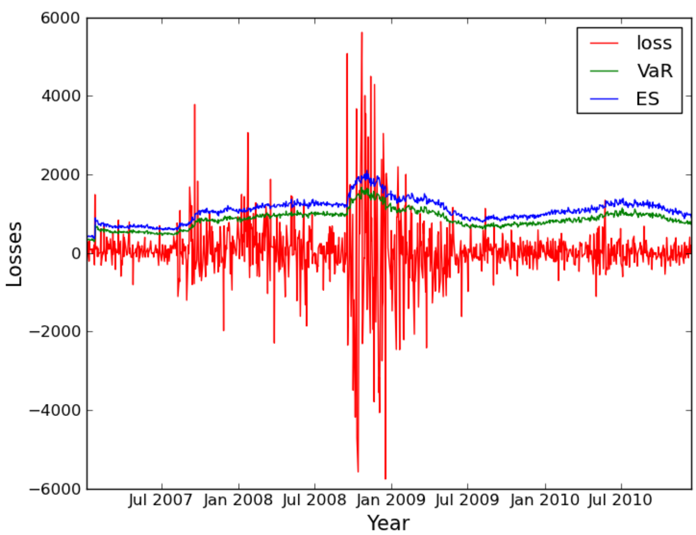

Figure 28: Losses compared to risk measures between 2007 and the beginning of 2011. The volatility and the correlation are updated continuously using Floating Averages.

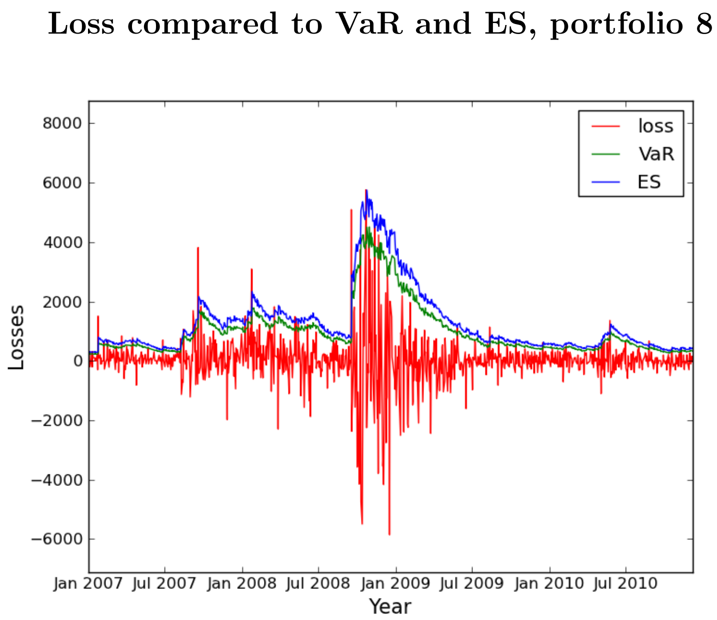

Figure 29: Losses compared to risk measures between 2007 and the beginning of 2011. The volatility and the correlation are updated continuously using EWMA ($\lambda = 0.95$).

Portfolio 8 - backtesting (American and Norwegian market)

Method	$VaR_{0.95}$ *violations*	*VaR violation* σ	$ES_{0.95}$ *violations*
Floating Averages	*6.4%*	*0.7%*	*4.8%*
EWMA	*5.4%*	*0.7%*	*3.2%*

It is important to keep in mind that market to market swaps are highly dependent on the currency exchange ratios. In this case we assume that the exchange rates are constant, which is a common scenario for many companies. One way to ensure a constant exchange rate is to buy currency swaps.

With exchange rates assumed to be constant the risk estimates perform well, the number of $VaR_{0.95}$ violations lies less than one times the standard deviation above the expectation value using EWMA. Again we observe that the violations of the $ES_{0.95}$ are more frequent than expected.

5.7 Cumulative loss

Risk estimates are commonly used by investors and companies to keep track of how much they stand to lose if worst comes to worst, but they can also be used as an investment trigger telling when to invest and when to get out.

The previously defined portfolios are used to hedge the risk of loaning and lending money, when the risk is high on these portfolios they are doing what they are supposed to: to limit the losses due to increasing or decreasing interest rate payments. They stabilize the incomes and expenses, therefore it makes more sense to invest when the risk is high, and to get out when the risk is low and the market is relatively stable.

In this section the cumulative loss is plotted in order to visualize how an investor's financial situation would change throughout the financial crisis as the risk changes. Keep in mind that the losses are not discounted and that we only consider what happens with the portfolio not, for example, the change in interest rate payments on debt.

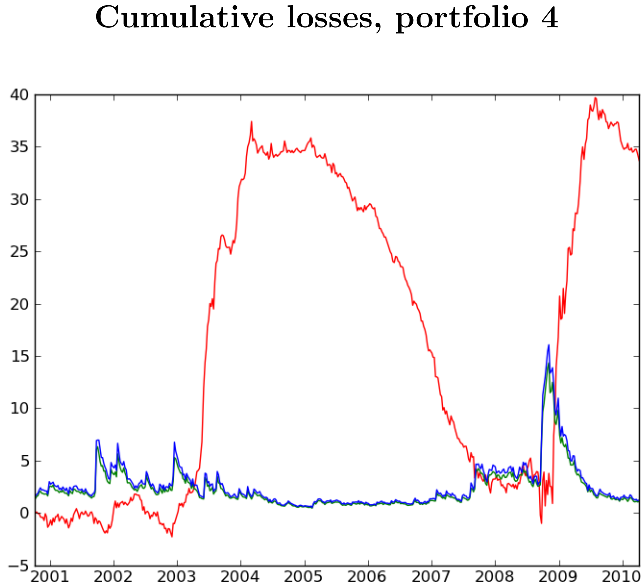

Figure 30: The cumulative losses for portfolio 4. Portfolio 4 consist of a short cap ($P = 800$) and a long swap ($P = 1000$) in the Norwegian market. The risk measures have a 99% confidence level and the cap rate is 5%.

Cumulative losses, portfolio 5

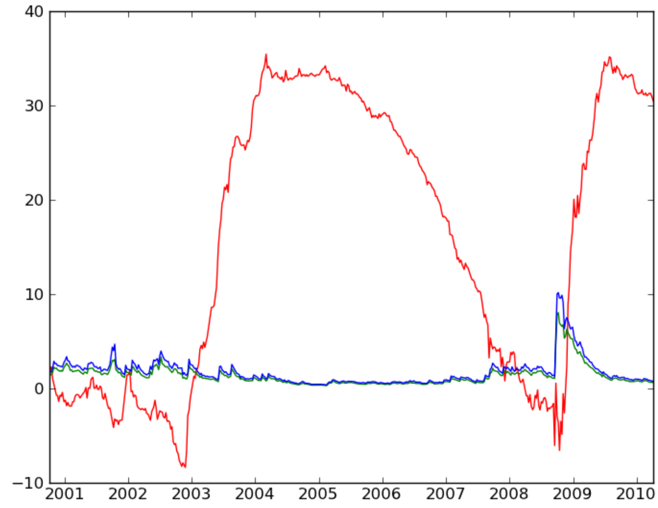

Figure 31: The cumulative losses for portfolio 5. Portfolio 5 consist of a short cap ($P = 1500$) and a long swap ($P = 800$) in the Norwegian market. The risk measures have a 95% confidence level and the cap rate is 7%.

Cumulative losses, portfolio 6

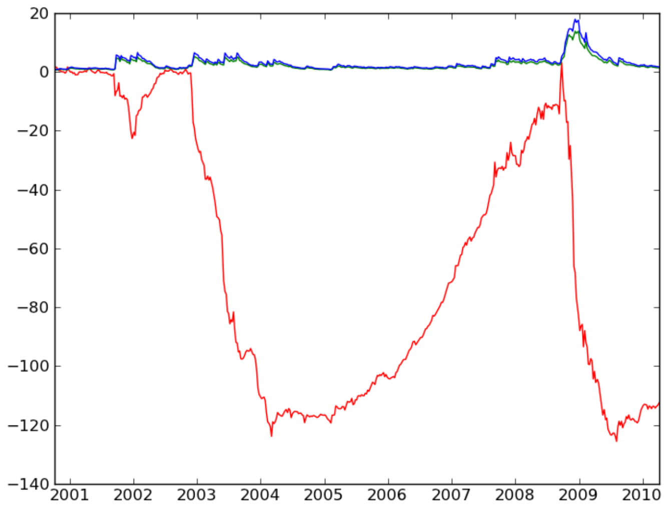

Figure 32: The cumulative losses for portfolio 6. Portfolio 6 consist of a long floor ($P = 1500$) and a short swap ($P = 800$) in the Norwegian market. The risk measures have a 95% confidence level and the cap rate is 7%.

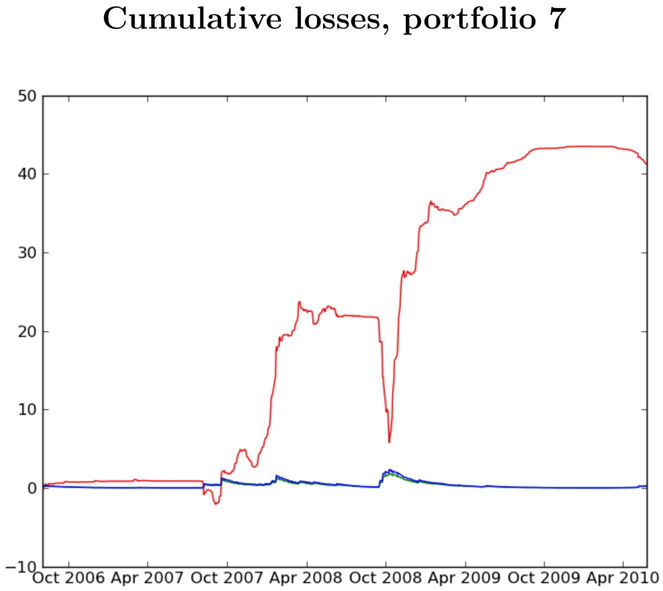

Figure 33: The cumulative losses for portfolio 7. Portfolio 7 consist of a short cap ($P = 1500$) and a long swap ($P = 800$) in the American market. The risk measures have a 95% confidence level and the cap rate is 7%.

We see from these plots that high risk leads to dramatic changes in the cumulative losses. The holders of the bullish portfolios (4,5 and 7) mainly experience losses after periods with high volatility, while the opposite is the case with portfolio 6.

In the next section investment strategies where the investor buys when the risk is higher than a predefined level is tested. From the previous plots we expect an increase in payoff for portfolio 6, while the other portfolios will perform better under a buy and hold strategy.

5.8 Investment strategies

The investment strategies defined in this section will be applied to the portfolios 4 through 7. An investment strategy for portfolio 8 would depend on the volatility of the exchange rates and is thus beyond the scope of this thesis.

Buy and hold

The buy and hold strategy is probably the easiest strategy and is suitable for long term investments. The strategy is, as the name suggest, to buy and hold the portfolio for a period of time assuming the ups and downs will cancel each other out and the market will give a good rate of return.

This strategy is based on the Efficient Market Hypothesis [19] and has several good properties like very low transaction cost, and it requires little from the investor. Thus it is a good strategy for small or unsophisticated investors.

Hold only in periods of high risk
This strategy is suitable for portfolios that are used for hedging of expenses connected with a volatile interest rate. The idea is to hedge only when the risk is high in order to protect the interest rate payments.

EWMA will be used as the volatility estimator as this procedure has shown to give the best results, and it responds faster to a change in volatility which is important when applying an investment strategy.

Investment strategies

Portfolio and market	Strategy	Mean payout (standard deviation)	Trading days
4 - Norwegian (2000-2011)	*Buy and hold*	*-0.07 (0.002)*	*479*
4 - Norwegian	$VaR_{0.99} > 1.5$	*-017 (0.003)*	*281*
4 - Norwegian	$VaR_{0.99} > 2.5$	*-0.33 (0.005)*	*147*
5 - Norwegian (2000-2011)	*Buy and hold*	*-0.06 (0.002)*	*479*
5 - Norwegian	$VaR_{0.95} > 1.5$	*-0.17 (0.004)*	*172*
5 - Norwegian	$VaR_{0.95} > 2.5$	*-0.64 (0.016)*	*37*
6 - Norwegian (2000-2011)	*Buy and hold*	*0.23 (0.005)*	*479*
6 - Norwegian	$VaR_{0.95} > 1.5$	**0.62 (0.009)**	*268*
6 - Norwegian	$VaR_{0.95} > 2.5$	**0.84 (0.012)**	*168*
7 - American (2006-2011)	*Buy and hold*	*-0.04 (0.0003)*	*966*
7 - American	$VaR_{0.95} > 0.5$	*-0.15 (0.001)*	*204*
7 - American	$VaR_{0.95} > 1.5$	*-0.35 (0.008)*	*23*

To invest only when the risk is high is only a profitable strategy for portfolio 6, which is a bearish portfolio, as expected from the cumulative losses (plot 30-33). The volatility, and thus also the risk, was highest during the recession periods which was periods when the interest rates decreased dramatically, this naturally benefits the bearish portfolio.

This means that for all the other portfolios (4,5 and 7), an investment strategy where the investor buys only when the risk is low would have been more profitable than buying and holding. One advantage with this strategy is obviously that the portfolio risk is limited, on the other hand this strategy does not make sense if the portfolio is used to hedge the risk associated with loans and debt.

Short when the risk is high, long when the risk is low
The cumulative plots and the mean payouts suggests that the bullish portfolios should be shorted when the risk is high and bought when the risk is low. With portfolio 6 the

[19] The Efficient Market Hypothesis states that the past history is fully reflected in the present prices and that the market respond immediately to any new information [4].

strategy will be opposite, which means to go long when the risk is high and short when the risk is low. This investment strategy is tested and the results are summarized in the table below. The number of trading days corresponds to the respective buy and hold strategy.

Investment strategies

Portfolio and market	**Strategy**	**Mean payout (standard deviation)**
4 - Norwegian (2000-2011)	*Buy and hold*	*-0.07 (0.002)*
4 - Norwegian	*Short when $VaR_{0.99} > 1.5$, long otherwise*	**0.14 (0.002)**
4 - Norwegian	*Short when $VaR_{0.99} > 2.5$, long otherwise*	**0.12 (0.002)**
5 - Norwegian (2000-2011)	*Buy and hold*	*-0.06 (0.002)*
5 - Norwegian	*Short when $VaR_{0.95} > 1.5$, long otherwise*	**0.05 (0.002)**
5 - Norwegian	*Short when $VaR_{0.95} > 2.5$, long otherwise*	**0.05 (0.002)**
6 - Norwegian (2000-2011)	*Buy and hold*	*0.23 (0.005)*
6 - Norwegian	*Long when $VaR_{0.95} > 1.5$, short otherwise*	**0.46 (0.005)**
6 - Norwegian	*Long when $VaR_{0.95} > 2.5$, short otherwise*	**0.38 (0.005)**
7 - American (2006-2011)	*Buy and hold*	*-0.04 (0.0003)*
7 - American	*Short when $VaR_{0.95} > 0.5$, long otherwise*	**0.02 (0.0003)**
7 - American	*Short when $VaR_{0.95} > 1.5$, long otherwise*	**-0.03 (0.0003)**

Both investment strategies pays off better than the buy and hold strategy, and the strategy to invest only when the risk is high. This result is consistent for all of the portfolios (the mean payout from portfolio 6 is higher under the previous strategy, but the numbers of trading days are so low that the cumulative payouts are lower).

The cumulative losses are plotted below for the two most successful portfolios, portfolio 4 and 6, under these investment strategies. These plots should be compared to the plots in figure 30 and 32.

Cumulative losses (under an investment strategy), portfolio 4

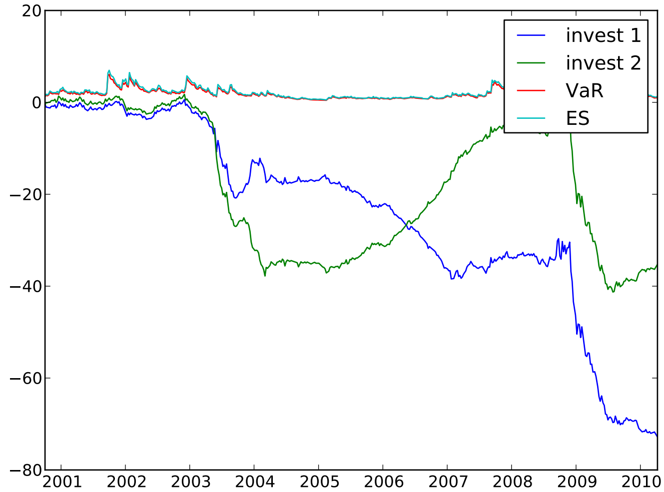

Figure 34: The cumulative losses for portfolio 4 with the strategy to go short when $VaR_{0.99} > 2.5$ (invest 2) or 1.5 (invest 1) and long otherwise.

Cumulative losses (under an investment strategy), portfolio 6

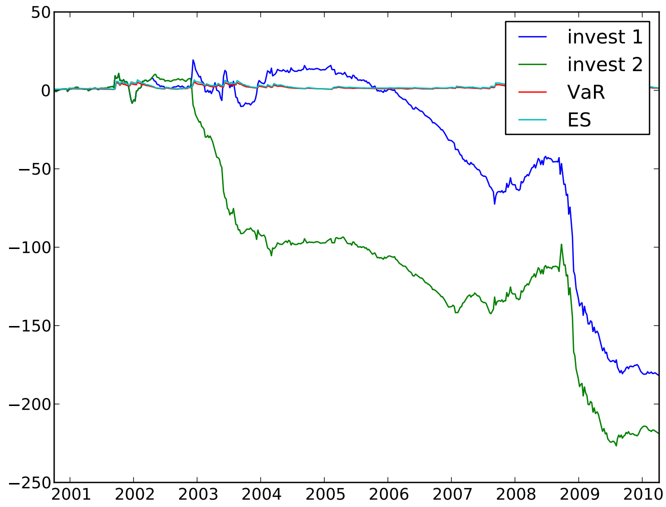

Figure 35: The cumulative losses for portfolio 6 with the strategy to go short when $VaR_{0.95} < 2.5$ (invest 1) or 1.5 (invest 2) and long otherwise.

Payoff is mainly positive in these plots, but it is evident that the payoff is sensitive to the choice of risk threshold. Portfolio 7 has positive payoff with one threshold and negative payoff (loss) with another, thus the key to optimize this strategy is to find the ideal risk threshold level. This choice should be based on historical risk levels.

In this case it was fairly easy to choose the risk levels since the development of the interest rates and the risk estimates were known. In reality this choice is, naturally, much more difficult.

6 Conclusion and Further Work

6.1 Conclusion

In this thesis, portfolio risk is analyzed in three different markets: one imaginary stable market, the Norwegian market and the American market. In addition to this one portfolio depends on the two real markets. Risk estimates from historical data are tested against real losses in order to check the model.

The risk estimate procedures consistently underestimate the portfolio risk. This suggests that the excess loss distribution has heavier tails than the log-normal distribution assumed in the LIBOR Market Model. Analyzing the kurtosis of the historical data set confirms this.

Loss estimates where EWMA are used perform better than Floating Averages in most cases because it responds quicker to sudden changes in volatility, and only portfolios with a short time horizon are tested. This approach is used when the risk estimates are triggers in an investment strategy. The first strategy is to hedge risk when the risk is higher than a predefined level, opposed to a buy and hold strategy.

The results show that this strategy would have increased the mean losses incurred by bullish portfolios significantly, similarly it would have increased the mean payoff on the bearish portfolio. In the historical data set, the interest rates plummet when the risk goes up, which is why the bearish portfolio pays off. Naturally, the test of these investment strategies does not give a statistically significant result, but if interest rate decrease drastically in times of high risk and increase, more controlled, in the following periods of lower risk, then a profitable investment strategy based on the risk estimates could be defined.

This is a type of behavior that we can see in the historical data sets between 2000 and early 2011. Thus the second investment strategy is to short the bullish portfolios when the risk is high and buy them when the risk is low. The bearish portfolio is bought when the risk is high and shorted when the risk is low. This investment strategy is more profitable than to buy and hold in every case, but it is sensitive to the choice of the risk threshold, a choice that should be made based on historical risk levels. This result shows that the risk estimates can be applied as triggers in investment strategies, and induce payoff increases.

However, it is always important to understand the limitations as well as the strengths of the model used. The risk estimates are great tools that can be applied in investment strategies and hedging, and they are easy to relate to, because the risk is presented in currency units. The backtesting shows that the model underestimates the risk, on the other hand, the plots tells us that the estimates are fairly proportional to the real losses. They are therefore useful indicators of the development of the exposure of any financial position.

6.2 Further Work

The most natural extension of this thesis would be to test other volatility and correlation estimators, and perhaps vary the parameters in the two estimates used here. In order to make the portfolios more complex, one could include stocks and study the aggregate risk in a portfolio with both interest rate derivatives and options.

It would be interesting to test how the risk estimates perform in the stable periods, where the risk is lower than some value, compared to the unstable periods. The reason why it would be interesting to test in different periods defined by their risk levels, is because we see a large degree of volatility clustering.

The volatility clustering could also be made less dominant by considering a much longer time horizon. In this thesis the time horizons are 1 day and 5 days, this is because the simulations are computationally heavy. The EWMA performs better in this thesis because recent observations are weighted heavier, but this might not be case if the time horizons were much longer.

The volatility clustering could also be incorporated into the model by for example fitting a GARCH model. It is clear that the LIBOR Market Model has some limitations; in addition to the issue with a changing volatility, we found that the kurtosis of the model is much lower than that of the historical data. The student-t distribution has heavier tails than the normal distribution and might be considered as an alternative, however, this requires a deeper analysis.

One could also look at the excess losses (for example losses larger than the Value at Risk) and apply extreme value theory in order to analyze the appearance of extreme losses.

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A Appendix 1 - Theory

A.1 Forward Euler method

The Forward Euler method is a first order numerical procedure found by considering the two first terms of the Taylor expansion: $y(x+\delta) \approx y(x) + \delta y'(x)$. This gives an iterative scheme for solving differential equations numerically:

$$y_{n+1} = y_n + \delta dy_n \tag{37}$$

This simple result is useful in the realization of stochastic differential equations (like LIBOR Market Model).

A.2 Itô's lemma

Itô's lemma is an important result from stochastic calculus used in the derivation of Black Scholes equation, and the LIBOR Market Model. The scalar version found in [4] is sufficient here, and it is presented below.

If x is a function of a random variable described by a stochastic differential equation of the form:

$$dx = A(x,t)dX + B(x,t)dt$$

Where X is standard Brownian motion. Then, if f(x) is a twice differentiable function:

$$df = A\frac{df}{dx}dX + \left(B\frac{df}{dx} + \frac{1}{2}A^2\frac{d^2f}{dx^2}\right)dt \tag{38}$$

A.3 The Jarque-Bera test

The Jarque-Bera statistic is a chi-squared statistic with two degrees of freedom. This statistic can be used to test for normality in a sample of n random variables (Z_i). Let S be the sample skewness and K the sample kurtosis as defined below.

$$S = \frac{\frac{1}{n}\sum_{i=1}^{n}(Z_i - \bar{Z})^3}{\left(\frac{1}{n}\sum_{i=1}^{n}(Z_i - \bar{Z})^2\right)^{3/2}} \tag{39}$$

$$K = \frac{\frac{1}{n}\sum_{i=1}^{n}(Z_i - \bar{Z})^4}{\left(\frac{1}{n}\sum_{i=1}^{n}(Z_i - \bar{Z})^2\right)^{2}} - 3 \tag{40}$$

The Jarque-Bera statistic, T, is:

$$T = \frac{n}{6}\left(S^2 + \frac{1}{4}K^2\right) \sim \chi_2^2 \tag{41}$$

The definition of this test is found in [1].

The following results from theoretical statistics are found in Keener's book on statistical theory [3].

A.4 Almost surely

A random variable X_n converges to X almost surely if:

$$P(X_n \to X \ as \ n \to \infty) = 1 \tag{42}$$

This is commonly written as $X_n \xrightarrow{a.s.} X$. Almost sure convergence implies convergence in probability, and convergence in distribution.

A.5 The law of large numbers

Strong Law of Large Numbers

The strong law of large numbers states that the sample average converges almost surely to the expectation value. Say $\bar{X_n}$ is the sample average of the i.i.d random variables: $X_1, ..., X_n$. Then:

$$\bar{X_n} \xrightarrow{a.s.} E[X] \ as \ n \to \infty \tag{43}$$

Weak Law of Large Numbers

The weak law of large numbers states that the sample average converges in probability to the expectation value:

$$\bar{X_n} \xrightarrow{p} E[X] \ as \ n \to \infty \tag{44}$$

A.6 Maximum likelihood analysis

The main idea behind maximum likelihood analysis is to find the value of the parameter that is most likely to have produced the data $X = (x_1, ..., x_n)$. In other words we seek $\hat{\theta}(X)$ such that

$$\hat{\theta}(X) = argmax_{\theta \in \Omega} \ L_X(\theta) \tag{45}$$

Where $L_X(\theta)$ is the likelihood. If the x_i's are independent identically distributed with pdf $f(x_i, \theta)$ then $L_X(\theta)$ is given by:

$$L_X(\theta) = \prod_{i=1}^{n} f(x_i, \theta) \tag{46}$$

It is usually convenient to maximize the logarithm of the likelihood, this gives the same answer and simplifies the calculations.

A.7 Monte Carlo Methods

Monte Carlo methods are algorithms using repeated random sampling to estimate a parameter. These algorithms are widely used to simulate complicated systems within math, physics, chemistry etc.

Monte Carlo Integrals

Monte Carlo integration is based on the following approximation, where $h(\cdot)$ is a function of the random variable Y:

$$\begin{aligned} E[h(Y)] = \int h(y)dF_Y(y) = \int h(y)f_Y(y)dy \\ \approx \tfrac{1}{B}\textstyle\sum_{j=1}^{B} h(Y_j) \end{aligned} \tag{47}$$

where $Y_1, ..., Y_B$ iid $\backsim F_Y$. If $E[| \ h(Y) \ |] < \infty$:

$$\frac{1}{B}\sum_{j=1}^{B} h(Y_j) \xrightarrow{as} E[h(Y)] \tag{48}$$

as $B \to \infty$ by the law of large numbers.

B Appendix 2 - Pseudocode

B.1 Simulating forward rates

Covariance using floating averages and EWMA

The covariance matrix can be estimated using floating averages or EWMA. Both approaches are described in pseudocode below. The volatility vector is simply the square root of the diagonal in the covariance matrix, and the correlation of x and y is the covariance divided by the variance of x and y.

```
begin FUNCTION cov_floating_avg(x,y,n)
   result ← 0
   lag ← some integer
   for i := 1 to n
      result + = [(x − mean(x[max(0, i − lag) to i]) × (y − mean(y[max(0, i − lag) to i]))]^2/(n − 1)
   end
   return result
end


begin FUNCTION cov_EWMA(x,y,n)
   result ← 0
   λ ← a number between 0 and 1
   for i := 1 to n
      result + = (1 − λ) × λ^(n−i) × [(x − mean(x[0 to i]) × (y − mean(y[0 to i]))]^2/(n − 1)
   end
   return result
end
```

LIBOR Market Model

This code is meant to show the basic LMM procedure used to realize forward rates, all the more complicated pricing routines and simulations are written with this code as a basis.

```
Import interest rates: r                                  matrix with dimension: m × n
define τ from the interest rate set                       vector with length: m
σ ← volatility vector(r)                                  vector with length: m
ρ ← correlation matrix(r)                                 matrix with dimension: m × m
F_1^1, F_2^1, ..., F_m^1 ← initial values(r)
for i := 1 to n
    F_1^{i+1} ← F_1^i × exp(dlog(F_1^i))
    F_2^{i+1} ← F_2^i × exp(dlog(F_2^i))
    ...
    F_m^{i+1} ← F_m^i × exp(dlog(F_m^i))
end
```

Remember that $\text{dlog}(F_x^i) = \left(\sigma_x \sum_{j=1}^{x} \frac{\sigma_j F_j^i \tau_j \rho_{xj}}{1+\tau_j F_j^i} - \frac{1}{2}\sigma_x^2\right) dt + \sigma_x dX_x^i$ where $dX_x^i \sim$ N(0,1)

B.2 Risk Estimates

Portfolio risk estimate

Having defined and found the correlation and variance above, the VaR, ES and price for a portfolio is found using the following procedure. Here the LMM realization is assumed known from the pseudocode above.

```
n is the time horizon for the securities
x is the number of Monte Carlo simulations
γ is the confidence interval
portfolio ← vector(x)                                     vector with length: x
for i := 1 to x
    r ← initial interest rates
    for j := 1 to n
        r.append(LMM realization(r) )
    end
    calculate and append the payoff of the portfolio using r
end
sort the portfolio vector
index ← round(x × (1 − γ))
```

$VaR \leftarrow portfolio[index]$
$ES \leftarrow$ mean($portfolio[0$ to $index]$) scalars
$price \leftarrow$ mean($portfolio$)
print: $price,\ VaR,\ ES$

Portfolio risk compared to actual losses

A portfolio can also be evaluated along historical interest rates. The following procedure was used to compare the actual losses with the risk estimates.

Import interest rates: r matrix with dimension: m × n
$loss \leftarrow$ vector(n)
$VaR \leftarrow$ vector(n) vectors with length: n
$ES \leftarrow$ vector(n)
for $i := 1$ to n
    $loss[i] \leftarrow$ estimate from $r[1$ to $m][i]$
    $VaR[i], ES[i] \leftarrow$ Monte Carlo estimate from $r[1$ to $m][i]$
end
plot($loss,\ VaR,\ ES$)
