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Auslander-Reiten components containing modules of finite complexity

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TO MY PARENTS

Abstract

Let R be a connected selfinjective Artin algebra. We prove that any almost split sequence ending at an Ω-perfect R-module of finite complexity has at most four non-projective summands in a chosen decomposition of the middle term into indecomposable modules. Moreover, we show that a chosen decomposition into indecomposable modules of the middle term of an almost split sequence ending at an R-module of complexity 1 lying in a regular component of the Auslander-Reiten quiver has at most two summands. Furthermore, we prove that the regular component is of type $\mathbb{Z}A_\infty$ or $\mathbb{Z}A_\infty/\langle\tau^n\rangle$. We use this to study modules with eventually constant and eventually periodic Betti numbers.

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Contents

Chapter 1

Introduction

The results presented in this thesis are mainly inspired by the work of Edward L. Green and Dan Zacharia in [6] and [7]. We let R be a connected selfinjective Artin algebra and we investigate both almost split sequences ending at an Ω-perfect R-module of finite complexity and regular components containing an R-module of complexity 1. Throughout the entire thesis we assume R is connected even though it is not always pointed out.

In Chapter 2 we present the necessary preliminaries. The definitions and results here are mostly basic knowledge. We have chosen to divide this chapter in three subchapters where the first presents general notation and almost split sequences. In the second subchapter we look at Auslander-Reiten quivers, before we in the third and last subchapter explore properties of selfinjective Artin algebras.

Chapter 3 is divided in two subchapters. In Subchapter 3.1 we present Betti numbers and in Subchapter 3.2 we investigate complexity.

So called Ω-perfect modules are the main object of study in Chapter 4. Such modules are defined in both [6] and [7], but slightly different. To give a uniform presentation of the work we have chosen the definition in [7] and modified the proofs in [6] so they correspond to the definition we use. Again, we have chosen to divide the chapter in two subchapters, where we investigate respectively Ω-perfect modules and eventually Ω-perfect modules.

We end our work in Chapter 5. Here, we present the main results of this thesis. It is divided in three subchapters, Subchapter 5.1, 5.2 and 5.3. In the first subchapter we explore almost split sequences ending at Ω-perfect R-modules of finite complexity. The second subchapter revolves around complexity 1. It is again divided in two sections. In the first of these sections, Section 5.2.1, we look at the structure of a regular component of the Auslander-Reiten quiver of an selfinjective Artin algebra containing a module of complexity 1. In the latter part, Section 5.2.2, we use this knowledge to further investigate different properties of regular components containing modules with eventually constant and eventually periodic Betti numbers. In Subchapter 5.3 we present some closing remarks.

To avoid writing a textbook in abstract algebra we assume that the reader is familiar with concepts and results from the following courses taught at NTNU: MA3201 Rings and modules, MA3203 Ring theory and MA3204 Homological algebra.

Chapter 2

Preliminaries

In this chapter we introduce notation and results that is used in the later chapters. With few exceptions, the reader is referred to other literature for proofs. What is presented here is in some cases used without reference later in the thesis, as a result of it being mostly basic knowledge. This chapter is divided in three subchapters where the first introduces some definitions and results concerning Artin algebras. In the second subchapter we investigate Auslander-Reiten quivers of Artin algebras before we explore the special type of Artin algebras called selfinjective Artin algebras in the last subchapter.

2.1 General notation and almost split sequences

In the entire thesis we let R be an Artin algebra over a commutative Artin ring k. That is, R is a ring and we have a ring morphism $\phi\colon k \longrightarrow R$ where $\operatorname{Im}\phi \subseteq Z(R)$ and R is finitely generated as a k-module. Here, $Z(R)$ is the centre of R. Furthermore, we assume R is a connected algebra. That is, R is not a direct product of two algebras. We let $\operatorname{mod} R$ be the category of finitely generated left R-modules. All modules in this thesis is of such kind if not stated otherwise. That is, whenever we say that M is an R-module, what we actually mean is that it is a finitely generated left R-module. Moreover, $\operatorname{mod} R^{\mathrm{op}}$ is the category of finitely generated right R-modules. We let $\underline{\operatorname{mod}}\, R$ be the category $\operatorname{mod} R$ modulo projectives. That is, the objects are the objects in $\operatorname{mod} R$. Furthermore, if M and N are in $\operatorname{mod} R$, the morphisms from M to N in $\underline{\operatorname{mod}}\, R$ are $\operatorname{Hom}_R(M,N)/\mathscr{P}(M,N)$, where $\mathscr{P}(M,N)$ is the morphisms from M to N that factors through a projective R-module. This set of morphisms is denoted $\underline{\operatorname{Hom}}_R(M,N)$. Similarily, $\overline{\operatorname{mod}}\, R$ is the category $\operatorname{mod} R$ modulo injectives. That is, the objects are the objects in

$\operatorname{mod} R$. Further, if M and N are in $\operatorname{mod} R$, the morphisms from M to N in $\overline{\operatorname{mod}}\, R$ are $\operatorname{Hom}_R(M,N)/\mathscr{I}(M,N)$, where $\mathscr{I}(M,N)$ is the morphisms from M to N that factors through an injective R-module, denoted $\overline{\operatorname{Hom}}_R(M,N)$. In the last subchapter of this chapter we present a useful equivalence between $\underline{\operatorname{mod}}\, R$ and itself called the syzygy functor when we assume that R is so called selfinjective. This assumption also gives us an important isomorphism between functors from $\underline{\operatorname{mod}}\, R$ to $\underline{\operatorname{mod}}\, R$ which is introduced later.

Before presenting some important definitions and results we look at some notation that is used in later chapters. If A is a finitely generated R-module we denote the projective cover of A by $P(A) \longrightarrow A$. Further, if f is a map from A to B we let $\bar{f}$ be the map from A/JA to B/JB where $a + JA$ maps to $f(a) + JB$. Here, J denotes the Jacobson radical of R. It is easy to show that this is a well-defined map. Moreover $\mathbb{N}$ denotes the positive integers $1, 2, 3, ...$, also known as the natural numbers. Note that $0 \notin \mathbb{N}$.

We now introduce two functors that prove important in this thesis. First, we have $\operatorname{Hom}_R(-,R)$ from $\operatorname{mod} R$ to $\operatorname{mod} R^{\mathrm{op}}$. We denote this functor by $(-)^*$ and have the following result.

Proposition 2.1.1. [4, Proposition II.4.3] *The functor $(-)^* \mid_{\mathscr{P}(R)} \colon \mathscr{P}(R) \longrightarrow \mathscr{P}(R^{\mathrm{op}})$ is a duality, where $\mathscr{P}(R)$ is the category of finitely generated projective R-modules.* □

Further, we define the second functor. By assumption, k is a commutative Artin ring, that is, it has only a finite number of non-isomorphic simple modules $S_1, S_2, \ldots, S_n$. We let $I(S_i)$ be the injective envelope of S_i and let $I = \oplus_{i=1}^{n} I(S_i)$. We then have a contravariant k-functor $D = \operatorname{Hom}_k(-,I) \colon \operatorname{mod} k \longrightarrow \operatorname{mod} k$ that is a duality. This functor induces a contravariant k-functor from $\operatorname{mod} R$ to $\operatorname{mod} R^{\mathrm{op}}$. Moreover, we have the following result.

Proposition 2.1.2. [4, Theorem II.3.3] *If R is an Artin k-algebra, then the contravariant k-functor $D = \operatorname{Hom}_k(-,I) \colon \operatorname{mod} R \longrightarrow \operatorname{mod} R^{\mathrm{op}}$ is a duality.* □

We now consider M in $\operatorname{mod} R$ and let

$$P_1 \xrightarrow{f} P_0 \longrightarrow M \longrightarrow 0$$

be the minimal projective presentation of M. We apply the functor $(-)^*$ to the morphism f and get f^*:

$$P_0^* \xrightarrow{f^*} P_1^*$$

We then denote the cokernel of f^* in $\operatorname{mod} R^{\mathrm{op}}$ by $\mathrm{Tr}M$, and call it the *transpose of* M. We have the following result.

Proposition 2.1.3. [4, Proposition IV.1.7] *Let* M *be in* $\operatorname{mod} R$. *Then* $\mathrm{Tr}M = (0)$ *if and only if* M *is projective.* □

The transpose Tr does not induce a duality from $\operatorname{mod} R$ to $\operatorname{mod} R^{\mathrm{op}}$ as we might have hoped (in general there is not even a functor from $\operatorname{mod} R$ to $\operatorname{mod} R^{\mathrm{op}}$ that sends an object M to $\mathrm{Tr}M$), but some work will give us the following result.

Proposition 2.1.4. [4, Chapter IV.1] *The functor* $\mathrm{Tr}\colon \underline{\operatorname{mod}}\, R \longrightarrow \underline{\operatorname{mod}}\, R^{\mathrm{op}}$ *is a duality.* □

Further investigations give us the following important theorem.

Proposition 2.1.5. [4, Proposition IV.1.9]

(a) *The duality* $D\colon \operatorname{mod} R \longrightarrow \operatorname{mod} R^{\mathrm{op}}$ *induces a duality* $D\colon \underline{\operatorname{mod}}\, R \longrightarrow \overline{\operatorname{mod}}\, R^{\mathrm{op}}$.

(b) *The composition* $D\mathrm{Tr}\colon \underline{\operatorname{mod}}\, R \longrightarrow \overline{\operatorname{mod}}\, R$ *is an equivalence of categories with inverse equivalence* $\mathrm{Tr}D\colon \overline{\operatorname{mod}}\, R \longrightarrow \underline{\operatorname{mod}}\, R$.

□

For simplicity, in some cases we denote $D\mathrm{Tr}$ with τ and $\mathrm{Tr}D$ with τ^{-1}.

Irreducible morphisms play an important part of this thesis. They are defined as follows. A morphism $f\colon M \longrightarrow N$ in $\operatorname{mod} R$ is called *irreducible* if f is neither a split monomorphism or a split epimorphism, and if $f = st$ for some $t\colon M \longrightarrow X$ and $s\colon X \longrightarrow N$, then t is a split monomorphism or s is a split epimorphism.

$$\begin{tikzcd} & X \arrow[dr, "s"] & \\ M \arrow[ur, "t"] \arrow[rr, "f"] & & N \end{tikzcd}$$

The result below is used without reference in later proofs.

Proposition 2.1.6. [4, Lemma V.5.1] *If* $f\colon M \longrightarrow N$ *is an irreducible morphism in* $\operatorname{mod} R$, *then* f *is either a monomorphism or an epimorphism.* □

Remark. Recall that if $f\colon M \longrightarrow N$ is an irreducible epimorphism, then $\ell(M) > \ell(N)$. We cannot have equality since this would give us an epimorphism between modules of equal length, that is, an isomorphism. This contradicts the assumption that f is irreducible and therefore not split. Similarily, if f is an irreducible monomorphism, then $\ell(M) < \ell(N)$.

We now present the connection between irreducible morphisms and a special type of exact sequences called almost split sequences. To do this, we first define minimal right (left) almost split morphisms. A morphism $g\colon B \longrightarrow C$ is called *right minimal* if every morphism $g'\colon B \longrightarrow B$ such that the following diagram commutes

$$\begin{tikzcd} B \arrow[dr, "g"] \arrow[d, "g'"] & \\ & C \\ B \arrow[ur, "g"'] & \end{tikzcd}$$

is an automorphism. Similarily, we call $f\colon A \longrightarrow B$ a *left minimal morphism* if every morphism $f'\colon B \longrightarrow B$ such that the following diagram commutes

$$\begin{tikzcd} & B \arrow[d, "f'"] \\ A \arrow[ur, "f"] \arrow[dr, "f"'] & \\ & B \end{tikzcd}$$

is an automorphism.

Moreover, a morphism $g\colon B \longrightarrow C$ is called *right almost split* if it is (a) not a split epimorphism and (b) any morphism $X \longrightarrow C$ which is not a split epimorphism factors through g. Dually, a morphism $f\colon A \longrightarrow B$ is called *left almost split* if it is (a) not a split monomorphism and (b) any morphism $A \longrightarrow Y$ that is not a split monomorphism factors through f.

Naturally, a map that is both right (left) minimal and right (left) almost split is called a *minimal right (left) almost split morphism*. We now present a connection between the irreducible morphisms and minimal right (left) almost split morphisms.

Proposition 2.1.7. [4, Theorem V.5.3]

(a) *Let C be an indecomposable module. Then a morphism $g'\colon B' \longrightarrow C$ is irreducible if and only if there exists a morphism $g''\colon B'' \longrightarrow C$ such that the induced morphism $(g', g'')\colon B' \oplus B'' \longrightarrow C$ is a minimal right almost split morphism.*

(b) *Let A be an indecomposable module. Then a morphism $f'\colon A \longrightarrow B'$ is irreducible if and only if there exists some morphism $f''\colon A \longrightarrow B''$ such that the induced morphism $\binom{f'}{f''}\colon A \longrightarrow B' \oplus B''$ is a minimal left almost split morphism.* □

We now define almost split sequences. An exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is called an *almost split sequence* given that f is left almost split and g is right almost split. The next proposition gives us further knowledge about the structure of almost split sequences.

Proposition 2.1.8. [4, Proposition V.1.14] *The following are equivalent for an exact sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

(a) *The sequence is an almost split sequence.*

(b) *The morphism g is minimal right almost split.*

(c) *The morphism f is minimal left almost split.*

(d) *The module A is indecomposable and g is right almost split.*

(e) *The module C is indecomposable and f is left almost split.*

(f) *The module C is isomorphic to* $\mathrm{Tr}DA$ *and f is left almost split.*

(g) *The module A is isomorphic to* $D\mathrm{Tr}C$ *and g is right almost split.*

□

We have the following existence theorem for almost split sequences.

Theorem 2.1.9. [4, Theorem V.1.15]

(a) *If C is an indecomposable non-projective module, then there is an almost split sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

(b) *If A is an indecomposable non-injective module, then there is an almost split sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

□

Further, the almost split sequences are unique in the following sense.

Theorem 2.1.10. [4, Theorem V.1.16] *The following are equivalent for two almost split sequences*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

and

$$0 \longrightarrow A' \xrightarrow{f'} B' \xrightarrow{g'} C' \longrightarrow 0.$$

(a) $C \cong C'$

(b) $A \cong A'$

(c) *The sequences are isomorphic in the sense that there is a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\cong"'] & B \arrow[r, "g"] \arrow[d, "\cong"'] & C \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0
\end{tikzcd}$$

with the vertical morphisms isomorphisms.

□

Before we present the next result we need to introduce some new notation. For a fixed R-module C we let $\operatorname{mod} R/C$ be a category with objects the R-morphisms $f\colon B \longrightarrow C$. Further, a morphism $g\colon f \longrightarrow f'$ from $f\colon B \longrightarrow C$ to $f'\colon B' \longrightarrow C$ in the category is an R-morphism $g\colon B \longrightarrow B'$ such that the following diagram commutes.

$$\begin{tikzcd}
B \arrow[dr, "f"] \arrow[d, "g"] & \\
B' \arrow[r, "f'"'] & C
\end{tikzcd}$$

Similarily, for a fixed R-module A, the category $\operatorname{mod} R \backslash A$ has objects the R-morphisms $f\colon A \longrightarrow B$. A morphism $g\colon f \longrightarrow f'$ from $f\colon A \longrightarrow B$ to $f'\colon A \longrightarrow B'$ in the category is an R-morphism $g\colon B \longrightarrow B'$ such that the diagram below commutes.

$$\begin{tikzcd}
 & B \arrow[d, "g"] \\
A \arrow[ur, "f"] \arrow[r, "f'"'] & B'
\end{tikzcd}$$

Proposition 2.1.11. [4, Corollary V.1.17] *We have the following for an Artin algebra R.*

(a) *For each indecomposable R-module C there is a unique, up to isomorphism in* $\operatorname{mod} R/C$, *minimal right almost split morphism $f\colon B \longrightarrow C$.*

(b) *For each indecomposable R-module A there is a unique, up to isomorphism in* $\operatorname{mod} R\backslash A$, *minimal left almost split morphism $g\colon A \longrightarrow E$.*

□

Combining the previous results we get that if $g'\colon B' \longrightarrow C$ is an irreducible morphism ending at an indecomposable non-projective module C, and

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is an almost split sequence ending at C (which we know exists and is unique in the manner described above), then g' is a summand of g up to isomorphism. Further, if $g'\colon B' \longrightarrow C$ is a summand of $g\colon B \longrightarrow C$ in the almost split sequence above, then g' is irreducible. Similarily, if $f'\colon A \longrightarrow B'$ is an irreducible morphism where A is an indecomposable non-injective module and the sequence above is the almost split sequence starting at A, then f' is a summand of f up to isomorphism. Moreover, if $f'\colon A \longrightarrow B'$ is a summand of $f\colon A \longrightarrow B$ in the almost split sequence above, then f' is irreducible. Later in the thesis we dicard writing "up to isomorphism", but whenever we say that f' is a summand of f it is this we mean.

In Subchapter 2.3 our main object of study is so called selfinjective Artin algebras. It turns out that R being selfinjective Artin implies that the projective and injective modules coincide. Therefore, it is of special interest to study the almost split sequences containing a projective-injective summand in the middle term.

Proposition 2.1.12. [4, Proposition V.5.5] *Let*

$$\delta\colon 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be an almost split sequence. If B has an indecomposable projective-injective summand P, then $\ell(P) \geq 2$ and δ is isomorphic to the sequence

$$\epsilon\colon 0 \longrightarrow JP \xrightarrow{(-i,p)^T} P \oplus JP/S \xrightarrow{(q,j)} P/S \longrightarrow 0$$

where J is the Jacobson radical and S is the socle of P. Further, $i\colon JP \longrightarrow P$ and $j\colon JP/S \longrightarrow P/S$ are the natural inclusion morphisms and $p\colon JP \longrightarrow JP/S$ and

$q\colon P \longrightarrow P/S$ *are the natural quotient morphisms.* □

We end this subchapter with a notation that is used in later chapters. Let C be an indecomposable non-projective R-module. We then denote the number of indecomposable non-projective summands in a chosen decomposition (into indecomposable modules) of the middle term of an almost split sequence ending at C by $\alpha(C)$.

2.2 Auslander-Reiten quivers

In this subchapter we define Auslander-Reiten quivers of Artin algebras. The definitions and results in this part of the thesis can be found in [4], [2] and [9], if not specified otherwise.

Let R be a connected Artin algebra. We now want to define the Auslander-Reiten quiver Γ_R of R. First, we define $\operatorname{ind} R$, a full subcategory of $\operatorname{mod} R$ whose objects consist of chosen representatives from isomorphism classes of indecomposable modules in $\operatorname{mod} R$. We let Γ_R be the quiver with vertices in one to one correspondence with the objects of $\operatorname{ind} R$, denoted $[M]$ for M in $\operatorname{ind} R$. Further, if $[M]$ and $[N]$ are vertices in Γ_R we have an arrow $[M] \longrightarrow [N]$ if and only if there exists an irreducible map $M \longrightarrow N$. The arrow has valuation $(a_{M,N}, a'_{M,N})$ when we have a minimal right almost split morphism

$$M^{a_{M,N}} \oplus X \longrightarrow N$$

where M is not a summand of X, and a minimal left almost split morphism

$$M \longrightarrow N^{a'_{M,N}} \oplus Y$$

where N is not a summand of Y.

Remark. We have here chosen the definition from [4]. In [9], $a_{M,N}$ and $a'_{M,N}$ are defined oppsite of what is done here.

The vertices corresponding to indecomposable projective modules are called *projective vertices* and the vertices corresponding to indecomposable injective modules are called *injective vertices*. Moreover, an indecomposable module M is called *preprojective* if $(D\mathrm{Tr})^n M = (0)$ for some $n \in \mathbb{N}$. The vertices corresponding to such modules are called *preprojective vertices*. Similarily, a module N is said to be *preinjective* if there exists an $n \in \mathbb{N}$ such that $(\mathrm{Tr}D)^n N = (0)$, and vertices corresponding to these modules are called *preinjective vertices*. Further, modules that are neither preprojective or preinjective are called *regular modules*, and the corresponding vertices are called *regular vertices*. Moreover, $D\mathrm{Tr}$ defines a bijection from the indecomposable non-projective modules to the indecomposable non-injective modules with invers $\mathrm{Tr}D$. As mentioned earlier, we denote $D\mathrm{Tr}$ with τ and $\mathrm{Tr}D$ with τ^{-1}. Further, τ then induces a map from the non-projective vertices to the non-injective vertices. For simplicity, we also denote this map by τ. That is, for all indecomposable non-projective modules X we have that $\tau[X] = [\tau X]$ and for all indecomposable non-injective modules Y we have that $\tau^{-1}[Y] = [\tau^{-1} Y]$. We then say that τ is the *translation* of the quiver Γ_R. Now, the valued quiver Γ_R

together with the translation τ is called the *Auslander-Reiten quiver of R*.

Remark. We note that Γ_R cannot have any loops. If it did, we would have an irreducible morphism from a module, say M, to itself. That is, we have an epimorphism or a monomorphism from M to M. Since $\ell(M) < \infty$ this would imply the morphism being an automorphism, a contradiction to it being irreducible and therefore not split.

We now further explore the valuation of Γ_R. Assume N is a non-projective indecomposable R-module. Let $[M] \longrightarrow [N]$ be an arrow in Γ_R with valuation $(a_{M,N}, a'_{M,N})$ and further let $[\tau N] \longrightarrow [M]$ be the corresponding arrow with valuation $(a_{\tau N,M}, a'_{\tau N,M})$. We now let

$$0 \longrightarrow \tau N \longrightarrow M^a \oplus Y \longrightarrow N \longrightarrow 0$$

be an almost split sequence ending at N, where M is not a summand of Y. By the previous definition of valuation we then see that $a = a'_{\tau N,M}$ and $a = a_{M,N}$, that is $a'_{\tau N,M} = a_{M,N}$. It takes a little more work to show that also $a_{\tau N,M} = a'_{M,N}$, for instance see [4, Section VII.1]. Moreover, if R is an algebra over an algebraically closed field, then $a_{M,N} = a'_{M,N}$, see the proof of [4, Corollary VII.2.3]. Now let

$$0 \longrightarrow \tau N \longrightarrow \oplus_{i=1}^k M_i^{a_i} \longrightarrow N \longrightarrow 0$$

be an almost split sequence ending at N, an indecomposable non-projective R-module, and the M_i's are non-isomorphic. Then, we get the following part of the Auslander-Reiten quiver Γ_R:

$$\begin{tikzcd}
 & {[M_1]} \arrow[dr, "{(a_1,b_1)}"] & \\
{[\tau N]} \arrow[ur, "{(b_1,a_1)}"] \arrow[dr, "{(b_k,a_k)}"'] & \vdots \quad \vdots \quad \vdots & {[N]} \\
 & {[M_k]} \arrow[ur, "{(a_k,b_k)}"'] &
\end{tikzcd}$$

Furthermore, an indecomposable R-module M is called *τ-periodic* if $\tau^n M \cong M$ for some $n \in \mathbb{N}$. The vertices corresponding to such modules are called *τ-periodic vertices*.

In later chapters we look at parts of the Auslander-Reiten quiver called components. It is therefore natural to define them here. We say that two modules

M and N in $\operatorname{ind} R$ are related by an irreducible morphism if there exists an irreducible morphism $f\colon M \longrightarrow N$. This relation generates an equivalence relation if we say that a module M is related to itself by definition. An equivalence class under this equivalence relation is called a *component of* $\operatorname{ind} R$. So, M and N are in the same component if and only if there exists an $n \in \mathbb{N}$ and indecomposable modules X_i for $i \in \{1, ..., n\}$ where $X_1 = M$ and $X_n = N$ and further an irreducible morphism $f_i\colon X_i \longrightarrow X_{i+1}$ or an irreducible morphism $g_i\colon X_{i+1} \longrightarrow X_i$ for each $i \in \{1, ..., n-1\}$. By the previous definition of an Auslander-Reiten quiver Γ_R we have a corresponding *component* of the quiver. If all vertices in a component are preprojective we call the component a *preprojective component*. Similarily, if all vertices are preinjective we call it a *preinjective component*. If the component only contains regular vertices we call it a *regular component*. Such components containing modules with some special properties is the main object of study in Chapter 5. Finally, if we let $\Gamma_R(S)$ be the full subquiver of the Auslander-Reiten quiver of R containing the isomorphism classes of regular modules, we call $\Gamma_R(S)$ the *stable Auslander-Reiten quiver*. The components of such a quiver are called *stable components*.

We now define valued translation quivers, and it follows that the Auslander-Reiten quivers of Artin algebras are such quivers. We let Γ be a quiver with vertex set Γ_0 and set of arrows Γ_1. Further, we let Γ be *locally finite*, that is, for each $i \in \Gamma_0$ there is a finite number of arrows entering or leaving i. Now let τ' be an injective map from a subset of Γ_0 to Γ_0. We denote the set of immediate predecessors of $x \in \Gamma_0$ by x^-, that is $x^- = \{y \in \Gamma_0 \mid \text{there exists an arrow } y \longrightarrow x\}$. Similiarily, the set x^+ are the set of immediate successors of x, that is, $x^+ = \{y \in \Gamma_0 \mid \text{there exists an arrow } x \longrightarrow y\}$. If the following three conditions hold we call $\Gamma = (\Gamma_0, \Gamma_1, \tau', a)$, where Γ_0, Γ_1 and τ' is as defined above and a the valuation for (Γ_0, Γ_1), a *valued translation quiver*

(1) Γ has no loops and no multiple arrows.

(2) Whenever $x \in \Gamma_0$ is such that $\tau'(x)$ is defined, then $x^- = \tau'(x)^+$.

(3) If $x \longrightarrow y$ is an arrow with valuation (a, b) and $\tau'(y)$ is defined, then $\tau'(y) \longrightarrow x$ has valuation (b, a).

Further, if the following also holds

(4) If $x \in \Gamma_0$ is such that $\tau'(x)$ is defined, then x^- is nonempty.

we call the translation quiver a *proper translation quiver*. The partially defined map $\tau'\colon \Gamma_0 \longrightarrow \Gamma_0$ is called the *translation* of the valued translation quiver $\Gamma = (\Gamma_0, \Gamma_1, \tau', a)$. A valued translation quiver where the translation and its inverse is everywhere defined is called a *stable valued translation quiver*. As mentioned above, any Auslander-Reiten quiver is a valued translation quiver when we let τ' be τ, and further any stable component is a stable valued translation quiver. We here note that any Auslander-Reiten quiver is locally finite. As for Auslander-Reiten quivers we call a vertex $x \in \Gamma_0$ a *projective vertex* if x is not in the domain of τ', and furthermore if $x \in \Gamma_0$ is not in the image of τ' we call it an *injective vertex*. A vertex in the quiver is called *periodic* given $\tau'^n(x) = x$ for some $n \in \mathbb{N}$. A valued translation quiver is called a *tube* if there exists exactly one τ-orbit where every vertex x in the orbit is such that $|x^-| = 1$ and further $|\{y \mid \tau^n(y) = y\}| = \infty$ for some $n \in \mathbb{N}$.

We now look at an example of a valued translation quiver that will prove useful in deciding the shape of regular components of the Auslander-Reiten quivers of Artin algebras containing a periodic vertex. We let Δ be a valued quiver without loops or multiple arrows (possibly infinite), and define $\mathbb{Z}\Delta$ in the following manner. We let the vertices in the quiver be $(\mathbb{Z} \times \Delta_0)$, where Δ_0 is the vertices in Δ. The translation in the quiver is given by $\tau'(n, x) = (n-1, x)$. Moreover, if $\alpha\colon x \longrightarrow y$ is an arrow in Δ we have arrows in $\mathbb{Z}\Delta$ that is $(n, \alpha)\colon (n, x) \longrightarrow (n, y)$ and $\sigma(n, \alpha)\colon (n, y) \longrightarrow (n+1, x)$ for all $n \in \mathbb{Z}$. Further, given the valuation $(a_{x,y}, a'_{x,y})$ for α we have valuations $(a_{x,y}, a'_{x,y})$ for (n, α) and $(a'_{x,y}, a_{x,y})$ for $\sigma(n, \alpha)$. That is, $\mathbb{Z}\Delta$ is a valued translation quiver. Note, if the underlying graph of Δ is a tree and it has trivial valuation, then the valued translation quiver, $\mathbb{Z}\Delta$, is independent of orientation in Δ. Otherwise, it might depend on this.

Before we illustrate this for a given Δ we define subadditive and additive functions for a valued translation quiver, $\Gamma = (\Gamma_0, \Gamma_1, \tau', a)$. A *subadditive function* ℓ for Γ is a function $\ell\colon \Gamma_0 \longrightarrow \mathbb{N}$ that satisfies:

$$\ell(x) + \ell(\tau' x) \geq \sum_{y \in x^-} \ell(y) a_{y,x} \tag{2.1}$$

for all $x \in \Gamma'_0$, where Γ'_0 is the set of vertices where τ' is defined. The function ℓ is said to be *additive* if we have equality in (2.1) for all $x \in \Gamma'_0$.

We let Γ_R be the Auslander-Reiten quiver of a connected Artin algebra. As previously mentioned, a component, $\mathcal{C}$, of the full subquiver $\Gamma_R(S)$ is in fact a stable valued translation quiver where τ' is τ. We assume $\mathcal{C}$ has a periodic module. If we let ℓ be the ordinary length function, we see that it is clearly subadditive and we note that it is additive if and only if $\mathcal{C}$ is a component of the complete Auslander-Reiten quiver. Now, assume $\mathcal{C}$ is a regular component of the complete Auslander-Reiten quiver, that is, the quiver has a component containing none of the projective modules. So, by [4, Theorem VII.2.1], we know that R is not of finite representation type. Furthermore, by a theorem of Auslander in [11, Chapter 2.3], we have that ℓ cannot be bounded on $\mathcal{C}$. Moreover, by [9] we then know that the component is of type $\mathbb{Z}A_\infty/G$ where G is a group of automorphisms of $\mathbb{Z}A_\infty$. It is possible to show that the only such group of automorphisms is $\langle\tau^n\rangle$. That is, regular components of the Auslander-Reiten quiver Γ_R containing a τ-periodic module is of type $\mathbb{Z}A_\infty/\langle\tau^n\rangle$, a tube. We now illustrate A_∞ and a part of the tube.

A_∞ :

$$\bullet \text{---} \bullet \text{---} \bullet \text{---} \cdots \text{---} \bullet \text{---} \bullet \text{---} \cdots$$

Note that A_∞ has trivial valuation. An illustration of a part of a component of type $\mathbb{Z}A_\infty/\langle\tau^n\rangle$:

$$\begin{tikzcd}
 & & \vdots & & \vdots & & & \vdots & \\
 & & {[M_4]} \arrow[dr, two heads] & & {[\tau^{-1}M_4]} \arrow[ll, dashed] \arrow[dr, two heads] & & {[\tau^{-2}M_4]} \arrow[ll, dashed] \arrow[dr, two heads] & & \\
\cdots & {[\tau M_3]} \arrow[ur, hook] \arrow[dr, two heads] & & {[M_3]} \arrow[ll, dashed] \arrow[ur, hook] \arrow[dr, two heads] & & {[\tau^{-1}M_3]} \arrow[ll, dashed] \arrow[ur, hook] \arrow[dr, two heads] & & {[\tau^{-2}M_3]} \arrow[ll, dashed] & \cdots \\
 & & {[\tau M_2]} \arrow[ur, hook] \arrow[dr, two heads] & & {[M_2]} \arrow[ll, dashed] \arrow[ur, hook] \arrow[dr, two heads] & & {[\tau^{-1}M_2]} \arrow[ll, dashed] \arrow[ur, hook] \arrow[dr, two heads] & & \\
\cdots & {[\tau^2 M_1]} \arrow[ur, hook] & & {[\tau M_1]} \arrow[ll, dashed] \arrow[ur, hook] & & {[M_1]} \arrow[ll, dashed] \arrow[ur, hook] & & {[\tau^{-1}M_1]} \arrow[ll, dashed] & \cdots
\end{tikzcd}$$

The dashed arrows indicate the τ-translates. These will be neglected in later figures for simplicity. For the same reason, we do not write the trivial valuations in the quiver either. Note that since the valuation is (1,1) we know that the component illustrates actual almost split sequences. It should also be stressed that what here looks flat is actually "glued" together at some point, so the component takes the form of a tube. In Subchapter 5.2 we argue why we know which maps that are monomorphisms (indicated with a hook arrow) and which are epimorphisms (indicated with a two headed arrow).

2.3 Selfinjective Artin algebras

In this section we discuss selfinjective Artin algebras. An Artin algebra R is called *selfinjective* if it is injective as well as projective as an R-module. All algebras in this subchapter are of such kind, if not specified otherwise. First, we present a characterization of selfinjective Artin algebras.

Proposition 2.3.1. [4, Proposition IV.3.1] *The following are equivalent for an Artin algebra R.*

(a) *R is selfinjective.*

(b) *An R-module is projective if and only if it is injective.*

(c) *R^{op} is selfinjective.*

□

We now introduce the syzygy functor. It is a functor from $\underline{\mathrm{mod}}\, R$ to $\underline{\mathrm{mod}}\, R$ for an arbitrary Artin algebra R. We introduce it here because it turns out it is an equivalence when R is selfinjective. The *syzygy functor*, Ω, is defined in the following way. For each R-module M we choose a fixed projective cover $P(M) \longrightarrow M$ and define $\Omega(M)$ to be the kernel of this map. Now assume there is a map $f\colon M \longrightarrow N$ in $\mathrm{mod}\, R$. We then have an exact commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\Omega(M) \arrow[r, "t"] \arrow[d] & \Omega(N) \arrow[d] \\
P(M) \arrow[r, "g"] \arrow[d] & P(N) \arrow[d] \\
M \arrow[r, "f"] \arrow[d] & N \arrow[d] \\
0 & 0
\end{tikzcd}$$

where $g\colon P(M) \longrightarrow P(N)$ exists since $P(M)$ is projective and the morphism $P(N) \longrightarrow N$ is an epimorphism. Further, by [14, Proposition 2.71], we have $t\colon \Omega(M) \longrightarrow \Omega(N)$ which depends on the choice of g. If we instead of g chose $g'\colon P(M) \longrightarrow P(N)$ we would get another map $t'\colon \Omega(M) \longrightarrow \Omega(N)$. It is then possible to show that $t - t' \in \mathscr{P}(\Omega(M), \Omega(N))$, that is, $t - t'$ is a morphism from $\Omega(M)$ to $\Omega(N)$ that factors through a projective module. So, we get a morphism from $\mathrm{Hom}_R(M, N) \longrightarrow \underline{\mathrm{Hom}}_R(\Omega(M), \Omega(N))$. Since $f \in \mathscr{P}(M, N)$ gives $t \in \mathscr{P}(\Omega(M), \Omega(N))$ we obtain the morphism

$\Omega\colon \underline{\mathrm{Hom}}_R(M,N) \longrightarrow \underline{\mathrm{Hom}}_R(\Omega(M),\Omega(N))$. It is now possible to check that Ω is a functor from $\underline{\mathrm{mod}}\, R$ to $\underline{\mathrm{mod}}\, R$.

Dually, we can define the *cosyzygy functor* $\Omega^{-1}\colon \overline{\mathrm{mod}}\, R \longrightarrow \overline{\mathrm{mod}}\, R$. Instead of choosing a fixed projective cover for the module M, we choose a fixed injective envelope $u\colon M \longrightarrow I(M)$ and let $\Omega^{-1}(M)$ be the cokernel of u. For further details of the definition of Ω^{-1} see [4, Chapter IV.3]. We can now present the result that makes the syzygy functor especially important for selfinjective Artin algebras. Note that $\underline{\mathrm{mod}}\, R = \overline{\mathrm{mod}}\, R$ when R is a selfinjective Artin algebra.

Proposition 2.3.2. [4, Proposition IV.3.5] *Let R be a selfinjective Artin algebra. Then, the functors $\Omega\colon \underline{\mathrm{mod}}\, R \longrightarrow \underline{\mathrm{mod}}\, R$ and $\Omega^{-1}\colon \underline{\mathrm{mod}}\, R \longrightarrow \underline{\mathrm{mod}}\, R$ are inverse equivalences.* □

Recall that we assume that R is a selfinjective Artin algebra. We now define $\Omega^i\colon \underline{\mathrm{mod}}\, R \longrightarrow \underline{\mathrm{mod}}\, R$ by induction. We let $\Omega^0 = 1_{\underline{\mathrm{mod}}\, R}$ and further $\Omega^{i+1} = \Omega\Omega^i$ for all $i \geq 0$. In the same manner, one may define Ω^{-i} for $i = 0, 1, \ldots$. If we now look at a minimal projective resolution of an R-module M we see that $\Omega^i(M) = \mathrm{Im}\,\delta_i$.

$$\begin{tikzcd}
\cdots \arrow[r] & P^{i+1} \arrow[rr, "\delta_{i+1}"] \arrow[dr] & & P^i \arrow[rr, "\delta_i"] \arrow[dr] & & P^{i-1} \arrow[r] & \cdots \arrow[r] & P^1 \arrow[rr, "\delta_1"] \arrow[dr] & & P^0 \arrow[r, "\delta_0"] & M \arrow[r] & 0 \\
& & \Omega^{i+1}M \arrow[ur] & & \Omega^i M \arrow[ur] & & & & \Omega M \arrow[ur] & & &
\end{tikzcd}$$

Moreover, for each $f\colon M \longrightarrow N$ in $\mathrm{mod}\, R$ we fix a choice of t representing Ωf in $\underline{\mathrm{mod}}\, R$, and abusing notation we write Ωf instead of t. We define $\Omega^n f$ in a similar fashion.

Note that we later in the thesis write $\Omega^i M$ instead of $\Omega^i(M)$ for $i \in \mathbb{Z}$. This is just to avoid too many parentheses. We also recall that $\Omega P = (0)$ if and only if P is projective.

Now, we look at the connection between the functor Ω^2 and $D\mathrm{Tr}$. To do this we need to define an equivalence called the Nakayama functor. The following result is needed.

Proposition 2.3.3. [4, Proposition IV.3.4] *Let R be a selfinjective Artin algebra. Then $(-)^* = \mathrm{Hom}_R(-,R)\colon \mathrm{mod}\, R \longrightarrow \mathrm{mod}\, R^{\mathrm{op}}$ is a duality with dual inverse $\mathrm{Hom}_{R^{\mathrm{op}}}(-,R)\colon \mathrm{mod}\, R^{\mathrm{op}} \longrightarrow \mathrm{mod}\, R$.* □

We denote the *Nakayama functor* by ν and it is the composition of the dualities

$$\mathrm{mod}\, R \xrightarrow{(-)^*} \mathrm{mod}\, R^{\mathrm{op}} \xrightarrow{D} \mathrm{mod}\, R.$$

That is, $\nu\colon \operatorname{mod} R \longrightarrow \operatorname{mod} R$ is an equivalence when R is selfinjective. We now investigate some properties of ν.

Proposition 2.3.4. *Let R be a selfinjective Artin algebra and ν be the Nakayama functor. Furthermore, let M and N be R-modules and $f\colon M \longrightarrow N$. We then have the following*

(1) *The morphism f is irreducible if and only if νf is irreducible.*

(2) *$\ell(M) = \ell(\nu M)$. In particular, M is simple if and only if νM is simple.*

(3) *The morphism f is an epimorphism (monomorphism) if and only if νf is an epimorphism (monomorphism).*

(4) *The module M is projective if and only if νM is projective.*

(5) *The Nakayama functor ν preserves minimal projective resolutions.*

(6) *The Nakayama functor ν preserves almost split sequences.*

Proof. This follows from properties of equivalences between module categories. For more on such equivalences, see [1, Chapter 6]. □

The previous result is used frequently in the thesis. We now want to argue that for an R-module M we have that $\Omega\nu M \cong \nu\Omega M$. We let M be an R-module and have the following projective cover of M

$$0 \longrightarrow \Omega M \longrightarrow P(M) \longrightarrow M \longrightarrow 0.$$

We then have the following commutative diagram, where the upper sequence is the sequence where ν has acted on the projective cover of M and the sequence below is the projective cover of νM.

$$\begin{tikzcd}
0 \arrow[r] & \nu\Omega M \arrow[r] \arrow[d, "k"] & \nu P(M) \arrow[r] \arrow[d, "l", "\cong"'] & \nu M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega\nu M \arrow[r] & P(\nu M) \arrow[r] & \nu M \arrow[r] & 0
\end{tikzcd}$$

The morphism l is an isomorphism. Then, by [14, Proposition 2.71], k is an isomorphism. Note that if M is projective, then $\nu\Omega M = (0) = \Omega\nu M$. We now have the following result.

Proposition 2.3.5. [4, Proposition IV.3.7] *Let R be a selfinjective Artin algebra.*

(a) *The functors $D\mathrm{Tr}$, $\Omega^2\nu$ and $\nu\Omega^2$ from $\underline{\operatorname{mod}}\, R$ to $\underline{\operatorname{mod}}\, R$ are isomorphic.*

(b) *The functors* $\mathrm{Tr}D$, $\Omega^{-2}\nu^{-1}$ *and* $\nu^{-1}\Omega^{-2}$ *from* $\underline{\mathrm{mod}}\, R$ *to* $\underline{\mathrm{mod}}\, R$ *are isomorphic.*

□

This implies that also τ and Ω commute on objects in $\mathrm{mod}\, R$. That is, $\Omega\tau M \cong \Omega\Omega^2\nu M \cong \Omega^2\Omega\nu M \cong \Omega^2\nu\Omega M \cong \tau\Omega M$. We collect our findings in the following proposition.

Proposition 2.3.6. *Let R be a selfinjective Artin algebra and let M be an R-module. Then, we have that*

(1) $\nu\Omega M \cong \Omega\nu M$.

(2) $\tau\Omega M \cong \Omega\tau M$.

(3) $\tau^n M \cong \nu^n\Omega^{2n} M$.

Proof.

(1) and (2). These results hold from previous arguments.

(3) The result follows from (1) and the fact that $\tau^n M \cong (\nu\Omega^2)^n M$ by 2.3.5.

□

We further look at a result stating that a module has finite projective dimension if and only if it is projective. This is obviously an interesting property for selfinjective algebras.

Proposition 2.3.7. *Let R be a selfinjective Artin algebra and let M be an R-module. Then, M has finite projective dimension if and only if M is projective.*

Proof. If M is projective it has finite projective dimension by definition.

Let M be an R-module with finite projective dimension, say m. Assume $m \geq 1$. Since R is selfinjective, we know that the projective modules are injective. We have a minimal projective resolution

$$\begin{tikzcd}
0 \arrow[r] & P^m \arrow[r] & P^{m-1} \arrow[r] \arrow[dr] & P^{m-2} \arrow[r] & \cdots \arrow[r] & P^1 \arrow[r] & P^0 \arrow[r] & M \arrow[r] & 0 \\
 & & & \Omega^{m-1}M \arrow[u, hook] & & & & &
\end{tikzcd}$$

So, as a result of P^m being injective, the following sequence is exact and splits

$$0 \longrightarrow P^m \longrightarrow P^{m-1} \longrightarrow \Omega^{m-1}M \longrightarrow 0.$$

That is, $P^{m-1} \cong P^m \oplus \Omega^{m-1}M$ and furthermore $\Omega^{m-1}M$ is projective. This contradicts the projective dimension being equal to m (≥ 1). So, $m = 0$ and M is projective. $\square$

It follows from this that if an R-module M is not projective, then it has infinite projective dimension. That is, $\Omega^n M \neq (0)$ for $n \geq 0$. Further, we know that $\tau^n M \cong \nu^n \Omega^{2n} M$ from 2.3.6. So, then by properties of ν we know that $\tau^n M \neq (0)$ for any $n \geq 0$. That is, there are no non-projective modules that are preprojective. Dually, one may prove that there are no non-injective modules that are preinjective. So, for a non-zero, non-projective (and therefore also non-injective) R-module we know that $\tau^n M \neq (0)$ for all $n \in \mathbb{Z}$. This is important in upcoming proofs. We end this chapter with some results that prove useful later in the thesis.

Proposition 2.3.8. *Let R be a selfinjective Artin algebra and*

$$0 \longrightarrow \tau C \xrightarrow{(f_1, f_2, \ldots, f_t)^T} E_1 \oplus E_2 \oplus \ldots \oplus E_t \xrightarrow{(g_1, g_2, \ldots g_t)} C \longrightarrow 0$$

be an almost split sequence of finitely generated R-modules with the E_i's not necessarily indecomposable and $t \geq 1$. Then,

$$0 \longrightarrow \Omega\tau C \xrightarrow{(\Omega f_1, \ldots, \Omega f_t, f')^T} \Omega E_1 \oplus \ldots \oplus \Omega E_t \oplus P \xrightarrow{(\Omega g_1, \ldots, \Omega g_t, g')} \Omega C \longrightarrow 0$$

is an almost split sequence where P is projective. Furthermore, if P is non-zero, it is indecomposable.

Proof. Let

$$0 \longrightarrow \tau C \xrightarrow{f} E \xrightarrow{g} C \longrightarrow 0$$

be an almost split sequence. Then, by [3, Proposition 5.1] it is easy to see that we have the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega\tau C \arrow[r, "(\Omega f, f')^T"] \arrow[d] & \Omega E \oplus P \arrow[r, "(\Omega g, g')"] \arrow[d] & \Omega C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P(\tau C) \arrow[r] \arrow[d] & P(\tau C) \oplus P(C) \arrow[r] \arrow[d] & P(C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \tau C \arrow[r, "f"] \arrow[d] & E \arrow[r, "g"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where the upper sequence is an almost split sequence, Ωg corresponds to Ωf and P is projective. Furthermore, assume E decompose, say $E \cong E_1 \oplus E_2 \oplus ... \oplus E_t$ where the E_i's not necessarily indecomposable, and moreover that we have an almost split sequence

$$0 \longrightarrow \tau C \xrightarrow{(f_1, f_2, ..., f_t)^T} E_1 \oplus E_2 \oplus ... \oplus E_t \xrightarrow{(g_1, g_2, ... g_t)} C \longrightarrow 0.$$

So, since $\Omega(E_1 \oplus E_2 \oplus ... \oplus \Omega E_t) = \Omega E_1 \oplus \Omega E_2 \oplus ... \oplus \Omega E_t$ and furthermore $\Omega(f_1, f_2, ..., f_t)^T = (\Omega f_1, \Omega f_2, ..., \Omega f_t)^T$ we have that

$$\delta\colon 0 \rightarrow \Omega\tau C \xrightarrow{(\Omega f_1, ..., \Omega f_t, f')^T} \Omega E_1 \oplus ... \oplus \Omega E_t \oplus P \xrightarrow{(\Omega g_1, ..., \Omega g_t, g')} \Omega C \longrightarrow 0$$

is an almost split sequence where the Ωg_i's correspond to the Ωf_i's.

Now, assume $P \neq (0)$ and $P \cong P_1 \oplus P_2 \oplus ... \oplus P_n$ where each P_i is non-zero, indecomposable and $n \geq 2$. Then, by 2.1.12, we know that δ is isomorphic to

$$0 \longrightarrow JP_1 \longrightarrow P_1 \oplus JP_1/S_1 \longrightarrow P_1/S_1 \longrightarrow 0$$

where S_1 is the socle of P_1 and J is the Jacobson radical. So, P_2 is a summand of JP_1/S_1 and therefore

$$\ell(P_2) < \ell(P_1).$$

But, similarly δ is isomorphic to

$$0 \longrightarrow JP_2 \longrightarrow P_2 \oplus JP_2/S_2 \longrightarrow P_2/S_2 \longrightarrow 0.$$

That is, by the same argument as above, $\ell(P_1) < \ell(P_2)$, a contradiction. So, $n = 1$ and P is indecomposable. $\square$

We also have the following consequence of the proposition.

Corollary 2.3.9. *Let R be a selfinjective Artin algebra. Then, the number of indecomposable non-projective summands appearing in a chosen direct decomposition (into indecomposable modules) of the middle term of an almost split sequence is invariant under Ω.*

Proof. The result follows from 2.3.8, the fact that $\Omega P = (0)$ if and only if P is projective and that no syzygy of a non-projective module is projective. $\square$

Remark. If $f\colon A \longrightarrow B$ is an irreducible morphism with either A or B indecomposable and neither is projective, we know from 2.3.8 and Chapter 2.1 that $\Omega f\colon \Omega A \longrightarrow \Omega B$ is irreducible.

We now let $f\colon A \longrightarrow B$ be an irreducible morphism with either A or B indecomposable and neither is projective. We have a commutative diagram

$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[d] \\ \Omega A \arrow[r, "t"] \arrow[d] & \Omega B \arrow[d] \\ P(A) \arrow[r, "g"] \arrow[d] & P(B) \arrow[d] \\ A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ 0 & 0 \end{tikzcd}$$

and recall that we fix a choice t representing Ωf in $\underline{\mathrm{mod}}R$ and call it Ωf. We now know that this is an irreducible morphism. If we chose another t' representing Ωf this will also be irreducible. Since either $\ell(\Omega A) > \ell(\Omega B)$ or $\ell(\Omega A) < \ell(\Omega B)$ we see that t is an epimorphism (monomorphism) if and only if t' is an epimorphism (monomorphism). So the property epimorphism/monomorphism is preserved for whatever choice of Ωf.

In chapter 4 we define Ω-perfect modules. We prove that if an indecomposable non-projective module C is Ω-perfect, then $\Omega^2 C$ cannot be simple. The following proposition then gives us information about the almost split sequence ending at ΩC.

Proposition 2.3.10. [6, Lemma 2.3] *Let R be a selfinjective Artin algebra. If*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an almost split sequence of R-modules with

$$0 \longrightarrow \Omega A \longrightarrow \Omega B \oplus P \longrightarrow \Omega C \longrightarrow 0$$

an almost split sequence with P a non-zero indecomposable projective-injective R-module, then $\Omega^2 C$ and $\tau C \cong A$ are simple R-modules.

Proof. Assume we have the almost split sequence

$$0 \longrightarrow \Omega A \longrightarrow \Omega B \oplus P \longrightarrow \Omega C \longrightarrow 0$$

with $P \neq (0)$. By 2.1.12 we know that the only almost split sequence (up to isomorphism) containing a non-zero indecomposable projective-injective as a summand in the middle term is

$$0 \longrightarrow JP \longrightarrow P \oplus JP/S \longrightarrow P/S \longrightarrow 0$$

where J is the Jacobson radical and S is the socle of P. This gives us that ΩC is isomorphic to P/S, that is $\Omega C \cong P/S$, and by [4, Theorem IV.3.6] we know that $\Omega(P/S) \cong \Omega(\Omega C) \cong \Omega^2 C$. Since P is indecomposable, we know that the socle is simple from [4, Proposition II.4.1]. We want to show that the projective cover of P/S is

$$0 \longrightarrow S \longrightarrow P \xrightarrow{f} P/S \longrightarrow 0.$$

First, recall that P cannot be simple by 2.1.12. Moreover, since $S \subseteq JP$ we know that $J(P/S) \cong JP/S$. So, $(P/S)/J(P/S) \cong P/JP$ and then by [4, Proposition I.4.3] we know that f is a projective cover.

That is, $\Omega^2 C \cong \Omega(P/S) \cong S$, so $\Omega^2 C$ is simple. Further, $\tau C \cong \nu\Omega^2 C$ by 2.3.5. The Nakayama functor preserves simple modules by 2.3.4, so we then know that τC is simple. □

We recall the definition for $D\mathrm{Tr}$ (τ) and let $f\colon M \longrightarrow N$ be in $\operatorname{mod} R$. Similarily as we did for Ωf we fix a choice representing τf in $\underline{\operatorname{mod}}\, R$ and denote it τf also in $\operatorname{mod} R$. With this in mind we continue with another proposition that is important in the thesis.

Proposition 2.3.11. *Let R be a selfinjective Artin algebra. Let*

$$0 \longrightarrow \tau C \xrightarrow{(f_1,f_2,\dots,f_t)^T} E_1 \oplus E_2 \oplus \dots \oplus E_t \xrightarrow{(g_1,g_2,\dots g_t)} C \longrightarrow 0$$

be an almost split sequence of finitely generated R-modules with the E_i's not necessarily indecomposable and $t \geq 1$. Then,

$$0 \longrightarrow \tau^2 C \xrightarrow{(\tau f_1,\dots,\tau f_t,f')^T} \tau E_1 \oplus \dots \oplus \tau E_t \oplus P \xrightarrow{(\tau g_1,\dots,\tau g_t,g')} \tau C \longrightarrow 0$$

is an almost split sequence with P projective. Furthermore, if P is non-zero, it is indecomposable.

Proof. The result follows from 2.3.8, 2.3.4 and 2.3.5. □

We have the following immediate consequence.

Corollary 2.3.12. *Let R be a selfinjective Artin algebra.Then, the number of indecomposable non-projective summands appearing in a chosen direct decomposition (into indecomposable modules) of the middle term of an almost split sequence is invariant under τ.*

Proof. The result follows from 2.3.11, the fact that $\tau P = (0)$ if and only if P is projective and that there exists no $n \geq 0$ such that $\tau^n M$ is projective where M is non-projective module.

□

Remark. If $f\colon A \longrightarrow B$ is an irreducible morphism with either A or B indecomposable and neither is projective, we know from 2.3.11 and Chapter 2.1 that $\tau f\colon \tau A \longrightarrow \tau B$ is irreducible.

We now look at the correspondence between the modules and the morphisms in almost split sequences ending at $\Omega^{2n} C$ and $\tau^n C$ for $n \geq 0$. This is of great importance in Chapter 4 and 5.

Proposition 2.3.13. *Let R be a selfinjective Artin algebra and let*

$$0 \longrightarrow \tau C \xrightarrow{(f_1,f_2,\dots,f_t)^T} E_1 \oplus E_2 \oplus \dots \oplus E_t \xrightarrow{(g_1,g_2,\dots,g_t)} C \longrightarrow 0$$

be an almost split sequence ending at C with the E_i's not necessarily indecomposable and $t \geq 1$. Then, for $n \geq 1$ we have that

(1) *The number of indecomposable non-projective modules in a chosen direct decomposition (into indecomposable modules) of the middle term of an almost split sequence ending at* $\Omega^{2n}C$ *is equal to the number of indecomposable non-projective modules in a chosen direct decomposition (into indecomposable modules) of the middle term of an almost split sequence ending at* $\tau^n C$ *for* $n \geq 1$.

(2) *An almost split sequence ending at* $\Omega^{2n}C$ *has a non-zero projective module in a chosen direct decomposition (into indecomposable modules) of the middle term if and only if an almost split sequence ending at* $\tau^n C$ *has a non-zero projective module in a chosen direct decomposition (into indecomposable modules) of its middle term.*

(3) $\Omega^{2n}f_i$ *is an epimorphism (monomorphism) if and only if* $\tau^n f_i$ *is an epimorphism (monomorphism).*

(4) $\Omega^{2n}g_i$ *is an epimorphism (monomorphism) if and only if* $\tau^n g_i$ *is an epimorphism (monomorphism).*

Proof. We have an almost split sequence ending at C

$$0 \longrightarrow \tau C \xrightarrow{(f_1,f_2,\dots,f_t)^T} E_1 \oplus E_2 \oplus \dots \oplus E_t \xrightarrow{(g_1,g_2,\dots,g_t)} C \longrightarrow 0 \tag{2.2}$$

and then by repeating 2.3.8 we have an almost split sequence ending at $\Omega^{2n}C$

$$0 \longrightarrow \Omega^{2n}\tau C \xrightarrow{(\Omega^{2n}f_1,\dots,\Omega^{2n}f_t,f')^T} \Omega^{2n}E_1 \oplus \dots \oplus \Omega^{2n}E_t \oplus P \xrightarrow{(\Omega^{2n}g_1,\dots,\Omega^{2n}g_t,g')} \Omega^{2n}C \longrightarrow 0 \tag{2.3}$$

with P indecomposable, projective. Furthermore, by 2.3.4 we know that applying ν^n gives another almost split sequence

$$0 \longrightarrow \nu^n\Omega^{2n}\tau C \xrightarrow{(\nu^n\Omega^{2n}f_1,\dots,\nu^n f')^T} \nu^n\Omega^{2n}E_1 \oplus \dots \oplus \nu^n P \xrightarrow{(\nu^n\Omega^{2n}g_1,\dots,\nu^n g')} \nu^n\Omega^{2n}C \longrightarrow 0$$

with $\nu^n P$ indecomposable, projective. We also know from (2.2) and 2.3.11 that we have an almost split sequence

$$0 \longrightarrow \tau^{n+1}C \xrightarrow{(\tau^n f_1,\dots,\tau^n f_t,f'')^T} \tau^n E_1 \oplus \dots \oplus \tau^n E_t \oplus P' \xrightarrow{(\tau^n g_1,\dots,\tau^n g_t,g'')} \tau^n C \longrightarrow 0 \tag{2.4}$$

with P' indecomposable, projective.

(1) Assume the E_i's in the almost split sequence (2.2) are indecomposable. If E_i is projective for some $i \in \{1, ..., t\}$, then $\Omega^{2n} E_i = (0)$ and $\tau^n E_i = (0)$. If not, $\Omega^{2n} E_i \neq (0)$ and $\tau^n E_i \neq (0)$ for $n \geq 1$ and both are indecomposable by [4, Proposition IV.3.6]. Furthermore, neither $\Omega^{2n} E_i$ nor $\tau^n E_i$ is projective. From the almost split sequences (2.3) and (2.4) we have our result.

(2) First, we recall that none of the non-zero $\Omega^{2n} E_i$'s, $\nu^n \Omega^{2n} E_i$'s and $\tau^n E_i$'s can be projective and furthermore that $\Omega^{2n} E_i = (0) \iff \nu^n \Omega^{2n} E_i = (0) \iff \tau^n E_i = (0)$. By 2.3.6, we know that $\tau^n C \cong \nu^n \Omega^{2n} C$. Moreover, by 2.1.10, we know that an almost split sequence ending at $\tau^n C \cong \nu^n \Omega^{2n} C$ is unique up to isomorphism. So, we therefore know that $\nu^n P \neq (0)$ if and only if $P' \neq (0)$. By 2.3.4, we then have that $P \neq (0)$ if and only if $P' \neq (0)$ which is what we wanted to show.

(3) Let $f_i \colon \tau C \longrightarrow E_i$. If E_i is projective, then $\Omega^{2n} E_i = (0)$ and $\tau^n E_i = (0)$, so then $\Omega^{2n} f_i = 0 = \tau^n f_i$.

Now, assume E_i is not projective. We then know that $\tau^n f_i$ and $\Omega^{2n} f_i$ are irreducible morphisms by respectively 2.3.11 and 2.3.8. Moreover,

$$\ell(\tau^n \tau C) = \ell(\nu^n \Omega^{2n} \tau C) = \ell(\Omega^{2n} \tau C)$$

and

$$\ell(\tau^n E_i) = \ell(\nu^n \Omega^{2n} E_i) = \ell(\Omega^{2n} E_i)$$

by 2.3.4 and 2.3.6. So, therefore $\ell(\tau^n \tau C) > \ell(\tau^n E_i)$ if and only if $\ell(\Omega^{2n} \tau C) > \ell(\Omega^{2n} E_i)$. In other words, $\tau^n f_i$ is an epimorphism if and only if $\Omega^{2n} f_i$ is an epimorphism. We know that the morphisms are irreducible, so it follows that $\tau^n f_i$ is a monomorphism if and only if $\Omega^{2n} f_i$ is a monomorphism.

(4) The result follows from a length argument similar as the one in (3).

□

Remark. In the previous proposition we emphasized that we chose a direct decomposition into indecomposable modules of the middle term of an almost split sequence. In the upcoming chapters we may neglect writing this, and only refer to a chosen decomposition of the middle term. However, the reader should recall that it is a direct decomposition into indecomposable modules we actually mean. It should also be a well known fact that the number of indecomposable non-projective summands is equal no matter what decomposition we choose.

Chapter 3

Betti numbers and complexity

The aim of this chapter is to introduce the reader to the concepts of Betti numbers and of complexity. In the first part we focus on Betti numbers, and in the latter part we explore complexity and some important properties that we use in later proofs. We assume R is a connected selfinjective Artin algebra and all R-modules in the chapter are finitely generated.

3.1 Betti numbers

We now define Betti numbers.

Definition 3.1.1. [6] Assume R is a selfinjective Artin algebra. If M is a finitely generated R-module, and if

$$\cdots \longrightarrow P^2 \xrightarrow{\delta_2} P^1 \xrightarrow{\delta_1} P^0 \xrightarrow{\delta_0} M \longrightarrow 0$$

is a minimal projective resolution of M, the *i-th Betti number of M*, $\beta_i(M)$, equals the number of indecomposable summands in a chosen direct decomposition of P^i into indecomposable modules.

In the following proposition we explore some properties of the Betti numbers of a finitely generated R-module M.

Proposition 3.1.2. [Properties of Betti numbers] *Let R be a selfinjective Artin algebra and M be an R-module. Then we have the following*

(1) $\beta_i(M) = \beta_i(\nu M)$ *for all* $i \geq 0$.

(2) *If M is a non-projective module, then $\beta_{i+j}(M) = \beta_i(\Omega^j M)$ for all $i \geq 0$ and $j \geq 0$.*

(3) *If M is a non-projective module, then $\beta_{i+2j}(M) = \beta_i(\tau^j M)$ for all $i \geq 0$ and $j \geq 0$.*

(4) *Let $d' = \max\{\ell(Re_i)\}$, the maximal length of all the indecomposable projective R-modules. Then, $\ell(\Omega^i M) \leq d' \cdot \beta_i(M)$ for all $i \geq 0$. Furthermore, if M is non-projective the inequality is strict.*

Proof.

(1) The Nakayama functor preserves minimal projective resolutions by 2.3.4. The result then follows.

(2) Let

$$\begin{tikzcd}
\cdots \arrow[r] & P^{j+1} \arrow[rr] \arrow[dr] & & P^j \arrow[rr] \arrow[dr] & & P^{j-1} \arrow[r] & \cdots \arrow[r] & P^1 \arrow[rr] \arrow[dr] & & P^0 \arrow[r] & M \arrow[r] & 0 \\
& & \Omega^{j+1}M \arrow[ur] & & \Omega^j M \arrow[ur] & & & & \Omega M \arrow[ur] & & &
\end{tikzcd}$$

be a minimal projective resolution of M. Recall that we know that the projective dimension of M is infinite by 2.3.7. Since $P^j \twoheadrightarrow \Omega^j M$ is a projective cover by the definition of minimal projective resolutions, we have a minimal projective resolution of $\Omega^j M$

$$\cdots \longrightarrow P^{j+1} \longrightarrow P^j \longrightarrow \Omega^j M \longrightarrow 0.$$

It then follows that $\beta_{i+j}(M) = \beta_i(\Omega^j M)$ for all $i \geq 0$ and $j \geq 0$.

(3) By 2.3.6, we know that $\tau^j M \cong \nu^j \Omega^{2j} M$. This and the results from (1) and (2) then give us the following

$$\beta_i(\tau^j M) = \beta_i(\nu^j \Omega^{2j} M) = \beta_i(\Omega^{2j} M) = \beta_{i+2j}(M)$$

which is what we wanted to show.

(4) First, we assume M is projective. Then, $\Omega^i M = (0)$ and $\beta_i(M) = 0$ for $i \geq 1$. That is, the inequality holds when $i \geq 1$. If $i = 0$, then $P^0 = M$ and we know that $\ell(\Omega^0 M) = \ell(M) \leq d' \cdot \beta_0(M)$.

Now, assume M is not projctive. Given the minimal projective resolution of M

$$\begin{tikzcd}
\cdots \arrow[r] & P^{i+1} \arrow[rr] \arrow[dr] & & P^i \arrow[rr] \arrow[dr] & & P^{i-1} \arrow[r] & \cdots \arrow[r] & P^1 \arrow[rr] \arrow[dr] & & P^0 \arrow[r] & M \arrow[r] & 0 \\
& & \Omega^{i+1}M \arrow[ur] & & \Omega^i M \arrow[ur] & & & & \Omega M \arrow[ur] & & &
\end{tikzcd}$$

we get that $\ell(\Omega^{i+1}M) + \ell(\Omega^i M) = \ell(P^i)$ for $i \geq 0$. So, since we know that $\Omega^{i+1}M \neq (0)$ we have that $\ell(\Omega^i M) < \ell(P^i) \leq d' \cdot \beta_i(M)$.

□

We now explore some properties of the maximum of all the Betti numbers of a non-projective R-module M with bounded Betti numbers. We denote this by $\beta(M)$, that is $\beta(M) = \max_{i\geq 0}\{\beta_i(M)\}$.

Proposition 3.1.3. *Let R be a selfinjective Artin algebra and let M be a non-projective R-module with bounded Betti numbers. Then we have the following*

(1) *$\beta(\Omega^i M) \leq \beta(M)$ for all $i \geq 0$.*

(2) *$\beta(\tau^n M) \leq \beta(M)$ for all $n \geq 0$.*

(3) *The length of M is bounded by $\beta(M) \cdot d'$, where $d' = \max\{\ell(Re_i)\}$, the maximum length of the indecomposable projective R-modules. In particular, $\ell(M) < \beta(M) \cdot d'$.*

Proof.

(1) Let

$$\begin{tikzcd}[column sep=small] \cdots \arrow[r] & P^{i+1} \arrow[rr] \arrow[dr] & & P^i \arrow[rr] \arrow[dr] & & P^{i-1} \arrow[r] & \cdots \arrow[r] & P^1 \arrow[rr] \arrow[dr] & & P^0 \arrow[r] & M \arrow[r] & 0 \\ & & \Omega^{i+1}M \arrow[ur] & & \Omega^i M \arrow[ur] & & & & \Omega M \arrow[ur] & & & \end{tikzcd}$$

be a minimal projective resolution of the non-projective module M. Recall that we know that the projective dimension of M is infinite by 2.3.7. As before, we have a minimal projective resolution of $\Omega^i M$

$$\cdots \longrightarrow P^{i+1} \longrightarrow P^i \longrightarrow \Omega^i M \longrightarrow 0.$$

for $i \geq 0$. If a direct decomposition into indecomposable modules of one (or more) of the modules $\{P^0, P^1, ..., P^{i-1}\}$ has more summands than a similar decomposition of any of the modules $P^i, P^{i+1}, ...$, we know that $\beta(\Omega^i M) < \beta(M)$. If not, $\beta(\Omega^i M) = \beta(M)$. In total, $\beta(\Omega^i M) \leq \beta(M)$.

(2) From (1) we know that $\beta(\Omega^i M) \leq \beta(M)$ for all $i \geq 0$. Further, from 3.1.2 we know that $\beta_i(\nu M) = \beta_i(M)$ and therefore also $\beta(\nu M) = \beta(M)$. Combining these results and applying 2.3.6, we are done.

(3) From the minimal projective resolution of M we know that $\ell(\Omega M)+\ell(M) = \ell(P^0)$ and $\Omega M \neq (0)$. Further, we then have the following

$$\ell(M) < \ell(P^0) \leq \beta_0(M) \cdot d' \leq \beta(M) \cdot d'$$

and we are done.

□

In the next subchapter we present complexity. We then look at how the Betti numbers of an R-module M is bounded by polynomials.

3.2 Complexity

We now define complexity and explore some important properties.

Definition 3.2.1. [7] Let n be a nonnegative integer. We say that the *complexity of a finitely generated R-module M is at most n* if $\beta_i(M) \leq ci^{n-1}$, for some $c \in \mathbb{Q}_{>0}$ and all sufficiently large i, that is, for all $i \gg 0$. We denote this by $\operatorname{cx}(M) \leq n$. Further, the *complexity of M is n*, $\operatorname{cx}(M) = n$, if $\operatorname{cx}(M) \leq n$, but $\operatorname{cx}(M) \nleq n-1$. If no such n exists we say that the complexity of M is infinite.

The results in the following proposition concern complexity and are of importance in later proofs.

Proposition 3.2.2. [Properties of complexity] *Let R be a selfinjective Artin algebra, and M and $\{M_j\}_{j=1}^k$ be R-modules. Then we have the following*

(1) $\operatorname{cx}(M) = 0$ *is equivalent to M being of finite projective dimension, and as a consequence projective.*

(2) $\operatorname{cx}(M) = 1$ *is equivalent to M being of infinite projective dimension and that there exists a $b \in \mathbb{Q}_{>0}$ such that $\beta_i(M) \leq b$ for all $i \geq 0$.*

(3) *Let M be non-projective, then* $\operatorname{cx}(\Omega^j M) = \operatorname{cx}(M)$ *for all $j \geq 0$.*

(4) *Let M be non-projective, then* $\operatorname{cx}(\tau M) = \operatorname{cx}(M)$.

(5) $\operatorname{cx}(\oplus_{j=1}^k M_j) = \max\{\operatorname{cx}(M_1), \operatorname{cx}(M_2), ..., \operatorname{cx}(M_k)\}$.

(6) [6, Lemma 2.1] *If*

$$0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow 0$$

is a short exact sequence of R-modules, then for each i, we have $\operatorname{cx}(A_i) \leq \max\{\operatorname{cx}(A_j), \operatorname{cx}(A_k)\}$, *where* $\{i,j,k\} = \{1,2,3\}$.

(7) $\operatorname{cx}(M) \leq \max\{\operatorname{cx}(S) \mid S$ *a composition factor of the composition series of* $M\}$.

Proof.

(1) Assume $\operatorname{cx}(M) = 0$. Then, for all $i \gg 0$, $\beta_i(M) \leq ci^{-1}$, where $c \in \mathbb{Q}_{>0}$. Since $\lim_{i\to\infty} \frac{c}{i} = 0$, the module M has finite projective dimension and is projective by 2.3.7.

Now assume the projective dimension is finite, so M is projective, again by 2.3.7. Then obviously for $i \geq 1$ we have that $0 = \beta_i(M) \leq ci^{-1}$ for some

$c \in \mathbb{Q}_{>0}$, so $\mathrm{cx}(M) = 0$. Note, we do not have to check that $\mathrm{cx}(M) \nleq 0 - 1$ since the complexity always is a nonnegative integer.

(2) Assume $\mathrm{cx}(M) = 1$. Since $\mathrm{cx}(M) \neq 0$, the projective dimension is infinite by (1). Moreover, for all $i \gg 0$, $\beta_i(M) \leq ci^{1-1} = c$ for some $c \in \mathbb{Q}_{>0}$. We have a finite number of Betti numbers of M that are not neccessarily bounded by c, say $\{\beta_0(M), \beta_1(M), ...\beta_{i-1}(M)\}$. That is, we choose $b = \max\{c, \beta_0(M), ..., \beta_{i-1}(M)\}$. So, there exists a $b \in \mathbb{Q}_{>0}$ such that $\beta_i(M) \leq b$ for all $i \geq 0$.

Now assume that M has infinite projective dimension and that there exists a $b \in \mathbb{Q}_{>0}$ such that $\beta_i(M) \leq b$ for all $i \geq 0$. We have that $\beta_i(M) \leq bi^{1-1}$ for all $i \geq 0$, and therefore $\mathrm{cx}(M) \leq 1$. Since the projective dimension is infinite, $\mathrm{cx}(M) \neq 0$ by (1), hence $\mathrm{cx}(M) = 1$.

(3) We begin by proving that the non-projective module M has finite complexity n if and only if $\Omega^j M$ has finite complexity n. First, assume M has finite complexity n. That is, for all $i \gg 0$, we have that $\beta_i(M) \leq c_M i^{n-1}$ where $c_M \in \mathbb{Q}_{>0}$, and we cannot find a $c \in \mathbb{Q}_{>0}$, such that $\beta_k(M) \leq ck^{(n-1)-1}$ for all $k \gg 0$. By 3.1.2 we have $\beta_{i+j}(M) = \beta_i(\Omega^j M)$, for $i \geq 0$ and $j \geq 0$. So, using the Binomial theorem found in [5], for all $i \gg 0$, we have that

$$\begin{aligned}
\beta_i(\Omega^j M) &= \beta_{i+j}(M) \\
&\leq c_M(i+j)^{n-1} \\
&= c_M\left(\binom{n-1}{0} i^{n-1} + \binom{n-1}{1} i^{n-2} j + ... + \binom{n-1}{n-2} i j^{n-2} + \binom{n-1}{n-1} j^{n-1}\right) \\
&\leq c_M\left(\binom{n-1}{0} i^{n-1} + \binom{n-1}{1} i^{n-1} j + ... + \binom{n-1}{n-2} i^{n-1} j^{n-2} + \binom{n-1}{n-1} i^{n-1} j^{n-1}\right) \\
&= \underbrace{c_M\left(\binom{n-1}{0} + \binom{n-1}{1} j + ... + \binom{n-1}{n-2} j^{n-2} + \binom{n-1}{n-1} j^{n-1}\right)}_{c_{\Omega^j M}} i^{n-1} \\
&= c_{\Omega^j M} i^{n-1}.
\end{aligned}$$

We have now found a $c_{\Omega^j M} \in \mathbb{Q}_{>0}$ such that $\beta_i(\Omega^j M) \leq c_{\Omega^j M} i^{n-1}$ for all $i \gg 0$, so $\mathrm{cx}(\Omega^j M) \leq n$.

We want to show that $\mathrm{cx}(\Omega^j M) \nleq n - 1$. Assume $\mathrm{cx}(\Omega^j M) \leq n - 1$. Then $\beta_i(\Omega^j M) \leq c' i^{(n-1)-1}$, for some $c' \in \mathbb{Q}_{>0}$ and for all $i \gg 0$. But then

$$\begin{aligned}
\beta_{i+j}(M) = \beta_i(\Omega^j M) &\leq c' i^{(n-1)-1} \\
&\leq c'(i+j)^{(n-1)-1}
\end{aligned}$$

for all $i \gg 0$, a contradiction to $\mathrm{cx}(M) = n$. So, $\mathrm{cx}(M) = n < \infty$ implies that $\mathrm{cx}(\Omega^j M) = n < \infty$ for all $j \geq 0$.

Now, assume $\Omega^j M$ has finite complexity n. That is, for all $i \gg 0$ we have that $\beta_i(\Omega^j M) \leq a i^{n-1}$ with $a \in \mathbb{Q}_{>0}$ and we cannot find a $a' \in \mathbb{Q}_{>0}$ such that $\beta_l(\Omega^j M) \leq a' l^{(n-1)-1}$ for all $l \gg 0$. By 3.1.2, $\beta_i(\Omega^j M) = \beta_{i+j}(M)$ for all $i \geq 0$ and $j \geq 0$. That is, for all $i \gg 0$ we have that

$$\beta_{i+j}(M) = \beta_i(\Omega^j M) \leq a i^{n-1} \leq a(i+j)^{n-1}.$$

That is, $\mathrm{cx}(M) \leq n$.

We now want to show that $\mathrm{cx}(M) \nleq n-1$. Assume $\mathrm{cx}(M) \leq n-1$. That is, given $j \geq 0$, for all $i \gg 0$ we have that $\beta_{i+j}(M) \leq c''(i+j)^{(n-1)-1}$ with $c'' \in \mathbb{Q}_{>0}$. Then, we know that for a $j \geq 0$ and for all $i \gg 0$ we have that

$$\begin{aligned}\beta_i(\Omega^j M) &= \beta_{i+j}(M) \\ &\leq c''(i+j)^{(n-1)-1}.\end{aligned}$$

So, using the Binomial theorem in a similar manner as before, we see that $\mathrm{cx}(\Omega^j M) \leq n-1$, a contradiction. That is, $\mathrm{cx}(M) = n$.

So, the non-projective module M has finite complexity n if and only if $\Omega^j M$ has finite complexity n. So, using the previous we also have that M has infinite complexity if and only if $\Omega^j M$ has infinite complexity.

(4) By 3.1.2 we know that $\beta_i(\nu M) = \beta_i(M)$, so $\mathrm{cx}(\nu M) = \mathrm{cx}(M)$. Further, by (3) we have that $\mathrm{cx}(\Omega^2 M) = \mathrm{cx}(M)$. In total, using 2.3.5, we have that $\mathrm{cx}(\tau M) = \mathrm{cx}(\nu\Omega^2 M) = \mathrm{cx}(\Omega^2 M) = \mathrm{cx}(M)$.

(5) We first assume that M_t has finite complexity n_t and furthermore that it is the maximum of $\{\mathrm{cx}(M_1), \mathrm{cx}(M_2), ..., \mathrm{cx}(M_k)\}$. We prove that this implies that $\oplus_{j=1}^k M_j$ has finite complexity n_t as well. Since n_t is finite we know that $\mathrm{cx}(M_j)$ is finite for each $j \in \{1, ..., k\}$, so, $\mathrm{cx}(M_j) = n_j$ with $0 \leq n_j < \infty$. Then, for all $i \gg 0$

$$\begin{aligned}\beta_i(\oplus_{j=1}^k M_j) &= \beta_i(M_1) + \beta_i(M_2) + ... + \beta_i(M_k) \\ &\leq c_{M_1} i^{n_1-1} + c_{M_2} i^{n_2-1} + ... + c_{M_k} i^{n_k-1} \\ &\leq c_{M_1} i^{n_t-1} + c_{M_2} i^{n_t-1} + ... + c_{M_k} i^{n_t-1} \\ &= \underbrace{(c_{M_1} + c_{M_2} + ... + c_{M_k})}_{c_{M_1 \oplus ... \oplus M_k}} i^{n_t-1}\end{aligned}$$

where c_{M_j} is in $\mathbb{Q}_{>0}$ for $j \in \{1, ..., k\}$. So, $\mathrm{cx}(\oplus_{j=1}^k M_j) \leq n_t$.

Assume that $\mathrm{cx}(\oplus_{j=1}^{k} M_j) \leq n_t - 1$. Then, $\beta_s(\oplus_{j=1}^{k} M_j) \leq cs^{(n_t-1)-1}$ for some $c \in \mathbb{Q}_{>0}$ and for all $s \gg 0$. So, for all $s \gg 0$ we have that

$$\begin{aligned}
\beta_s(M_t) &\leq \beta_s(M_1) + \beta_s(M_2) + ... + \beta_s(M_k) \\
&= \beta_s(\oplus_{j=1}^{k} M_j) \\
&\leq cs^{(n_t-1)-1}
\end{aligned}$$

a contradiction to $\mathrm{cx}(M_t) = n_t$. So, $\mathrm{cx}(\oplus_{j=1}^{k} M_j) = n_t$.

Now, we assume that $\mathrm{cx}(\oplus_{j=1}^{k} M_j)$ is finite and equal to n. Furthermore, let M_t be such that $\mathrm{cx}(M_t) = \max\{\mathrm{cx}(M_1), \mathrm{cx}(M_2), ..., \mathrm{cx}(M_k)\}$. We now want to show that $n = \mathrm{cx}(M_t)$. For all $i \gg 0$ and for some $c \in \mathbb{Q}_{>0}$ we have that

$$\begin{aligned}
\beta_i(M_t) &\leq \beta_i(M_1) + \beta_i(M_2) + ... + \beta_i(M_k) \\
&= \beta_i(\oplus_{j=1}^{k} M_j) \\
&\leq ci^{n-1}.
\end{aligned}$$

So, $\mathrm{cx}(M_t) \leq n$.

Now, assume $\mathrm{cx}(M_t) \leq n-1$. Then, since $\mathrm{cx}(M_t) = \max\{\mathrm{cx}(M_1), ..., \mathrm{cx}(M_k)\}$, we know that $\mathrm{cx}(M_j) \leq n-1$ for $j \in \{1, ..., k\}$. That is, for all $i \gg 0$ we have that

$$\begin{aligned}
\beta_i(\oplus_{j=1}^{k} M_j) &= \beta_i(M_1) + \beta_i(M_2) + ... + \beta_i(M_k) \\
&\leq c_{M_1} i^{(n-1)-1} + c_{M_2} i^{(n-1)-1} + ... + c_{M_k} i^{(n-1)-1} \\
&= \underbrace{(c_{M_1} + c_{M_2} + ... + c_{M_k})}_{c_{M_1 \oplus ... \oplus M_k}} i^{(n-1)-1}
\end{aligned}$$

where c_{M_j} is in $\mathbb{Q}_{>0}$ for $j \in \{1, ..., k\}$. So, $\mathrm{cx}(\oplus_{j=1}^{k} M_j) \leq n-1$, a contradiction. So, $\mathrm{cx}(M_t) = n$.

Using the previous, we also have that $\mathrm{cx}(\oplus_{j=1}^{k} M_j)$ is infinite if and only if $\max\{\mathrm{cx}(M_1), ..., \mathrm{cx}(M_k)\}$ is infinite.

(6) Let

$$0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow 0$$

be an exact sequence. If A_1 is projective and therefore injective we know that $\mathrm{cx}(A_1) = 0$ by (1) and furthermore the sequence splits. That is,

$A_2 \cong A_1 \oplus A_3$ and using (5) we have that $\operatorname{cx}(A_2) = \operatorname{cx}(A_1 \oplus A_3) = \max\{\operatorname{cx}(A_1), \operatorname{cx}(A_3)\} = \operatorname{cx}(A_3)$. That is, $\operatorname{cx}(A_2) = \operatorname{cx}(A_3)$ and for each i, we have $\operatorname{cx}(A_i) \leq \max\{\operatorname{cx}(A_j), \operatorname{cx}(A_k)\}$, where $\{i, j, k\} = \{1, 2, 3\}$.

By a similar argument, assuming that A_3 is projective also implies that for each i we have that $\operatorname{cx}(A_i) \leq \max\{\operatorname{cx}(A_j), \operatorname{cx}(A_k)\}$, where $\{i, j, k\} = \{1, 2, 3\}$.

So, in the cases where either A_1 or A_3 is projective we have our result. That is, we assume A_1 and A_3 are non-projective in the remaining part of the proof. We first show that for the given short exact sequence, $\operatorname{cx}(A_2) \leq \max\{\operatorname{cx}(A_1), \operatorname{cx}(A_3)\}$. Given minimal projective resolutions of A_1 and A_3

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A_1 \longrightarrow 0$$

$$\cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow A_3 \longrightarrow 0$$

the following diagram commutes

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & P_1 \arrow[r] \arrow[d] & P_1 \oplus Q_1 \arrow[r] \arrow[d] & Q_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d] & P_0 \oplus Q_0 \arrow[r] \arrow[d] & Q_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & A_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

For details, see Horseshoe Lemma [13, Lemma 6.20]. The diagram gives us a projective resolution of A_2, not necessarily minimal. So, we know that $\beta_i(A_2)$ for $i \geq 0$ is less than or equal to the number of summands in a direct decomposition of $P_i \oplus Q_i$ into indecomposable modules. That is, $\beta_i(A_2) \leq \beta_i(A_1) + \beta_i(A_3)$.

We may assume that $\operatorname{cx}(A_1)$ and $\operatorname{cx}(A_3)$ are finite, or else the inequality $\operatorname{cx}(A_2) \leq \max\{\operatorname{cx}(A_1), \operatorname{cx}(A_3)\}$ holds trivially.

Assume $\operatorname{cx}(A_1) = n_1$ and $\operatorname{cx}(A_3) = n_3$, where n_1 and n_3 are nonnegative integers. We assume that $n_1 \leq n_3$. Then, for all $i \gg 0$, we have that

$$\begin{aligned}\beta_i(A_2) &\leq \beta_i(A_1) + \beta_i(A_3)\\ &\leq c_1 i^{n_1 - 1} + c_3 i^{n_3 - 1}\\ &\leq c_1 i^{n_3 - 1} + c_3 i^{n_3 - 1}\\ &= (c_1 + c_3) i^{n_3 - 1}\end{aligned}$$

for some c_1, $c_3 \in \mathbb{Q}_{>0}$, so $\mathrm{cx}(A_2) \leq n_3$. That is, in this case we have that that $\mathrm{cx}(A_2) \leq n_3 = \max\{\mathrm{cx}(A_1), \mathrm{cx}(A_3)\}$. The same argument holds in the other case where $n_3 \leq n_1$.

We now want to show that $\mathrm{cx}(A_3) \leq \max\{\mathrm{cx}(A_1), \mathrm{cx}(A_2)\}$. We have the following commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & 0 \arrow[r] & \Omega A_2 \arrow[r, "\alpha"] \arrow[d] & \Omega A_3 \oplus P \arrow[r] \arrow[d] & \operatorname{Coker} \alpha \arrow[r] & 0 \\
 & 0 \arrow[r] & P(A_2) \arrow[r, equal] \arrow[d, "p"] & P(A_2) \arrow[rr] \arrow[d] & & 0 \\
0 \arrow[r] & A_1 \arrow[r] & A_2 \arrow[r, "g"] \arrow[d] & A_3 \arrow[rr] \arrow[d] & & 0 \\
 & & 0 & 0 & &
\end{tikzcd}$$

The construction goes as follows. We start with the exact sequence and the projective cover of A_2. We can then define the map $gp\colon P(A_2) \longrightarrow A_3$ such that the lower square commutes. Further, gp is an epimorphism, so $\operatorname{Ker} gp = \Omega A_3 \oplus P$, where P is projective. By [14, Proposition 2.71], we then have a map $\alpha\colon \Omega A_2 \longrightarrow \Omega A_3 \oplus P$ such that the second square commutes as well and thus α is a monomorphism. By Snake Lemma [13, Theorem 6.5], we have that $\operatorname{Coker} \alpha \cong A_1$, and furthermore we have the following exact sequence

$$0 \longrightarrow \Omega A_2 \longrightarrow \Omega A_3 \oplus P \longrightarrow A_1 \longrightarrow 0. \tag{3.1}$$

Now, if A_2 is projective, $\Omega A_2 = (0)$, so then $\Omega A_3 \oplus P \cong A_1$. That is, by (1), (3) and (5) we have that $\mathrm{cx}(A_1) = \mathrm{cx}(\Omega A_3 \oplus P) = \max\{\mathrm{cx}(\Omega A_3), \mathrm{cx}(P)\} = \mathrm{cx}(\Omega A_3) = \mathrm{cx}(A_3)$. So, for each i, we then have that $\mathrm{cx}(A_i) \leq \max\{\mathrm{cx}(A_j), \mathrm{cx}(A_k)\}$, where $\{i, j, k\} = \{1, 2, 3\}$.

That is, we assume A_2 is non-projective in the remaining part of the proof. Looking at the short exact sequence (3.1) and using the previous arguments we get that $\mathrm{cx}(\Omega A_3 \oplus P) \leq \max\{\mathrm{cx}(\Omega A_2), \mathrm{cx}(A_1)\}$. By

(3), we know that $\mathrm{cx}(\Omega A_2) = \mathrm{cx}(A_2)$ and $\mathrm{cx}(\Omega A_3) = \mathrm{cx}(A_3)$ and further $\mathrm{cx}(\Omega A_3 \oplus P) = \max\{\mathrm{cx}(\Omega A_3), \mathrm{cx}(P)\} = \mathrm{cx}(\Omega A_3)$ by (5). Combining these results we get that $\mathrm{cx}(A_3) \leq \max\{\mathrm{cx}(A_2), \mathrm{cx}(A_1)\}$.

It now remains to show that $\mathrm{cx}(A_1) \leq \max\{\mathrm{cx}(A_2), \mathrm{cx}(A_3)\}$ with A_1, A_2 and A_3 all non-projective. We have the exact sequence

$$0 \longrightarrow \Omega A_2 \longrightarrow \Omega A_3 \oplus P \longrightarrow A_1 \longrightarrow 0$$

and we can construct the following commutative diagram in the same manner as we did above

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & 0 \arrow[r] & \Omega(\Omega A_3) \arrow[r, "\beta"] \arrow[d] & \Omega A_1 \oplus P' \arrow[r] \arrow[d] & \mathrm{Coker}\,\beta \arrow[r] & 0 \\
 & 0 \arrow[r] & P(\Omega A_3) \oplus P \arrow[r, equal] \arrow[d] & P(\Omega A_3) \oplus P \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & \Omega A_2 \arrow[r] & \Omega A_3 \oplus P \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d] & 0 & \\
 & & 0 & 0 & &
\end{tikzcd}$$

where P' is projective. So, $\mathrm{Coker}\,\beta \cong \Omega A_2$, and like before from the upper exact sequence and the above argument we get that $\mathrm{cx}(\Omega A_1 \oplus P') \leq \max\{\mathrm{cx}(\Omega^2 A_3), \mathrm{cx}(\Omega A_2)\}$. By (1), (3) and (5), we can conclude that $\mathrm{cx}(A_1) \leq \max\{\mathrm{cx}(A_2), \mathrm{cx}(A_3)\}$.

(7) Let

$$(0) = M_{n+1} \subseteq M_n \subseteq M_{n-1} \subseteq ... \subseteq M_2 \subseteq M_1 \subseteq M_0 = M$$

be the composition series of M. We have a short exact sequence

$$0 \longrightarrow M_n \longrightarrow M_{n-1} \longrightarrow M_{n-1}/M_n \longrightarrow 0.$$

By (6) we have that

$$\mathrm{cx}(M_{n-1}) \leq \max\{\mathrm{cx}(M_n), \mathrm{cx}(M_{n-1}/M_n)\}.$$

Moreover, we again have a short exact sequence

$$0 \longrightarrow M_{n-1} \longrightarrow M_{n-2} \longrightarrow M_{n-2}/M_{n-1} \longrightarrow 0$$

and furthermore

$$\begin{aligned}\operatorname{cx}(M_{n-2}) &\le \max\{\operatorname{cx}(M_{n-1}), \operatorname{cx}(M_{n-2}/M_{n-1})\}\\ &\le \max\{\operatorname{cx}(M_n), \operatorname{cx}(M_{n-1}/M_n), \operatorname{cx}(M_{n-2}/M_{n-1})\}.\end{aligned}$$

So,

$$0 \longrightarrow M_k \longrightarrow M_{k-1} \longrightarrow M_{k-1}/M_k \longrightarrow 0$$

is a short exact sequence for each $k \in \{1, ..., n\}$ and by induction we have that

$$\operatorname{cx}(M) \le \max\{\operatorname{cx}(M_n), \operatorname{cx}(M_{n-1}/M_n), \operatorname{cx}(M_{n-2}/M), ..., \operatorname{cx}(M/M_1)\}$$

which is what we wanted to show.

□

We now want to show that a module of complexity 1 have a common bound for the lengths of all its syzygies.

Proposition 3.2.3. *Let R be a selfinjective Artin algebra and let M be an R-module of complexity* 1*. Then, there is a common bound for the lengths of all its syzygies.*

Proof. By 3.2.2 and 2.3.7, M is non-projective and has bounded Betti numbers, that is $\beta_n(M) \le b$ for all $n \ge 0$ where $b \in \mathbb{Q}_{>0}$. Further, by 3.1.2, we know that $\ell(\Omega^n M) < d' \cdot \beta_n(M)$ for $n \ge 0$ where $d' = \max\{\ell(Re_i)\}$. So, $\ell(\Omega^n M) < d' \cdot \beta_n(M) \le d' \cdot b$ for all $n \ge 0$, and we have found a bound for the lengths of the syzygies of M. □

The next result gives us information about the complexity of τ-periodic modules.

Proposition 3.2.4. *Let R be a selfinjective Artin algebra, and let M be an R-module. If M is τ-periodic, then* $\operatorname{cx}(M) = 1$.

Proof. By assumption, M is τ-periodic, that is $\tau^n M \cong M$ for some $n \ge 1$. The module M cannot be projective, because if it was, $\tau M = (0)$. So, it has infinite projective dimension by 2.3.7. Further, $\tau^n M \cong \nu^n \Omega^{2n} M$ by 2.3.6 and we then have that $\Omega^{2n} M \cong \nu^{-n} M$. So, looking at a minimal projective resolution of M

$$\begin{tikzcd}[column sep=small] \cdots \arrow[r] & \nu^{-n}P^0 \arrow[r] \arrow[dr] & P^{2n-1} \arrow[r] & \cdots \arrow[r] & P^1 \arrow[rr] \arrow[dr] & & P^0 \arrow[r] & M \arrow[r] & 0 \\ & & \nu^{-n}M \arrow[u] & & & \Omega M \arrow[ur] & & & \end{tikzcd}$$

we see that $b = \max\{\beta_0(M), ..., \beta_{2n-1}(M)\}$ gives $\beta_i(M) \leq b$ for all $i \geq 0$ by the properties of ν. That is, $\operatorname{cx}(M) = 1$ by 3.2.2. $\square$

We end this chapter with a proposition in which we investigate the connection between the complexities of the non-projective modules in a component of the Auslander-Reiten quiver of a selfinjective Artin algebra. It should be noted that we say we have a module M in a component of the Auslander-Reiten quiver, but what is meant is actually that a representative of the isomorphism class of M is in the corresponding component of $\operatorname{ind} R$. This sloppy notation is used in the rest of the thesis. It should also be noted that whenever we say that a module is in a component of the Auslander-Reiten quiver, the module is obviously indecomposable.

Proposition 3.2.5. [6, Proposition 2.2] *If R is a selfinjective Artin algebra and $\mathcal{C}$ is a component of the Auslander-Reiten quiver of R, then all the non-projective modules in $\mathcal{C}$ have the same complexity.*

Proof. Let

$$0 \longrightarrow \tau M \longrightarrow \oplus_{k=1}^{s} B_k \longrightarrow M \longrightarrow 0$$

be an almost split sequence ending at a non-projective R-module M in $\mathcal{C}$ where the B_k's are indecomposable. Now, assume B_j is non-projective for some $j \in \{1, .., s\}$. By 3.2.2, we have that

$$\operatorname{cx}(B_j) \leq \max\{\operatorname{cx}(B_1), ..., \operatorname{cx}(B_s)\} = \operatorname{cx}(\oplus_{k=1}^{s} B_k) \leq \max\{\operatorname{cx}(\tau M), \operatorname{cx}(M)\}.$$

Furthermore, again by 3.2.2, we have that $\operatorname{cx}(\tau M) = \operatorname{cx}(M)$, so $\operatorname{cx}(B_j) \leq \operatorname{cx}(M)$. Moreover, since B_j is non-projective by assumption, we have an almost split sequence ending at B_j

$$0 \longrightarrow \tau B_j \longrightarrow \tau M \oplus B \longrightarrow B_j \longrightarrow 0$$

where B is an R-module, not necessarily indecomposable. Again, using 3.2.2 we get that

$$\operatorname{cx}(M) = \operatorname{cx}(\tau M) \leq \max\{\operatorname{cx}(\tau M), \operatorname{cx}(B)\} = \operatorname{cx}(\tau M \oplus B) \leq \max\{\operatorname{cx}(\tau B_j), \operatorname{cx}(B_j)\}.$$

The fact that $\operatorname{cx}(\tau B_j) = \operatorname{cx}(B_j)$ by 3.2.2 then implies that $\operatorname{cx}(M) \leq \operatorname{cx}(B_j)$. That is, $\operatorname{cx}(M) = \operatorname{cx}(B_j)$. Thus, for all the non-projective B_k's we have that $\operatorname{cx}(B_k) = \operatorname{cx}(M)$.

Repeating this argument, we are done. $\square$

That is, we know that all of the modules in a regular (or stable) component of the Auslander-Reiten quiver of a selfinjective Artin algebra have the same complexity. Applying 3.2.2 we then see that if one module in such a component has bounded Betti numbers, then all modules in the component have bounded Betti numbers. We are now ready to define Ω-perfect maps and modules. This is the main subject of Chapter 4.

Chapter 4

Ω-perfect modules

This chapter focuses on Ω-perfect modules. Both [6] and [7] define Ω-perfect modules, but slightly different. Here, we use the definition found in [7] since this makes it possible to present the results in a uniform way. We start by presenting the chosen definition of Ω-perfect modules and some related results. In the second and last part of the chapter we present eventually Ω-perfect maps and modules. In the entire chapter, R is assumed to be a connected selfinjective Artin algebra if we have not specified otherwise and all R-modules are finitely generated.

4.1 Definition and properties

We need to introduce Ω-perfect irreducible maps to be able to define Ω-perfect modules.

Definition 4.1.1. [7] An irreducible morphism $g\colon B \longrightarrow C$ is called an *Ω-perfect morphism* if for all $n \geq 0$ the induced maps $\Omega^n g\colon \Omega^n B \longrightarrow \Omega^n C$ are all monomorphisms or all epimorphisms.

So, if an irreducible morphism g is Ω-perfect, we know that $\Omega^n g$ is an Ω-perfect morphism for all $n \geq 0$ as well. We now give the definition of Ω-perfect modules.

Definition 4.1.2. [7] An indecomposable R-module C is called an *Ω-perfect module* if it is non-projective and every irreducible map $B \longrightarrow C$ and every irreducible map $\tau C \longrightarrow B$ is Ω-perfect.

So, whenever we mention Ω-perfect modules they are obviously indecomposable. It should be noted that B in the chosen definition is not necessarily indecomposable.

In [6], B is assumed to be indecomposable. The definition of Ω-perfect modules in [6] also requires that no syzygy of C or C itself is simple. We later prove that our chosen definition gives us that $\Omega^n C$ cannot be simple for $n \geq 2$. First, we present a direct consequence of the definition.

Proposition 4.1.3. *Let R be a selfinjective Artin algebra and let B be an Ω-perfect R-module and A an R-module, not necessarily indecomposable.*

(1) *If an irreducible morphism $g\colon A \longrightarrow B$ is an epimorphism (monomorphism), then so is an irreducible morphism $\Omega^m \tau^n A \longrightarrow \Omega^m \tau^n B$ with $n \geq 0$, $m \geq 0$.*

(2) *If an irreducible morphism $f\colon \tau B \longrightarrow A$ is an epimorphism (monomorphism), then so is an irreducible morphism $\Omega^m \tau^{n+1} B \longrightarrow \Omega^m \tau^n A$ with $n \geq 0$, $m \geq 0$.*

Proof.

(1) Assume B is Ω-perfect and that $g\colon A \longrightarrow B$ is an irreducible epimorphism. Then, for $m \geq 0$, $n \geq 0$, by definition $\Omega^{m+2n} g\colon \Omega^{m+2n} A \longrightarrow \Omega^{m+2n} B$ is an epimorphism. That is $\ell(\Omega^{m+2n} A) > \ell(\Omega^{m+2n} B)$. Now, by 2.3.4 and 2.3.6 we know that

$$\ell(\Omega^m \tau^n A) = \ell(\Omega^{m+2n} A) > \ell(\Omega^{m+2n} B) = \ell(\Omega^m \tau^n B).$$

So, an irreducible morphism $\Omega^m \tau^n A \longrightarrow \Omega^m \tau^n B$ is an epimorphism.

A similar length argument holds in the case where $g\colon A \longrightarrow B$ is an irreducible monomorphism.

(2) Assume B is Ω-perfect and that $f\colon \tau B \longrightarrow A$ is an irreducible epimorphism. Then, for $m \geq 0$, $n \geq 0$, by definition $\Omega^{m+2n} f\colon \Omega^{m+2n} \tau B \longrightarrow \Omega^{m+2n} A$ is an epimorphism. That is $\ell(\Omega^{m+2n} \tau B) > \ell(\Omega^{m+2n} A)$. Now, by 2.3.4 and 2.3.6 we know that

$$\ell(\Omega^m \tau^{n+1} B) = \ell(\Omega^{m+2n} \tau B) > \ell(\Omega^{m+2n} A) = \ell(\Omega^m \tau^n A).$$

So, an irreducible morphism $\Omega^m \tau^{n+1} B \longrightarrow \Omega^m \tau^n A$ is an epimorphism.

A similar length argument holds in the case where $g\colon A \longrightarrow B$ is an irreducible monomorphism.

□

As previously mentioned, we want to prove that another consequence of our chosen definition is that $\Omega^n C$ cannot be simple for $n \geq 2$. To do this, we first need the following four results. Note that the first one holds for all Artin algebras, not necessarily selfinjective.

Lemma 4.1.4. [6, Proposition 2.5] *Let R be an Artin algebra and*

$$0 \longrightarrow A \longrightarrow B \xrightarrow{g} C \longrightarrow 0$$

be a short exact sequence of R-modules with g irreducible and A non-simple. Then

(1) *We have a split exact sequence*

$$0 \longrightarrow A/JA \xrightarrow{\bar{f}} B/JB \xrightarrow{\bar{g}} C/JC \longrightarrow 0.$$

(2) $JA = A \cap JB$*, where J is the Jacobson radical of R.*

Proof. Let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be a short exact sequence of R-modules where g is irreducible. The module A is indecomposable since it is the kernel of an irreducible epimorphism by [4, Proposition V.5.7]. Then, using [4, Proposition I.3.1], we know that A being non-simple implies that $(0) \neq JA \subsetneq A$. We can factorize the irreducible morphism g in the following way

$$\begin{tikzcd} & B/f(JA) \arrow[dr, "t"] & \\ B \arrow[ur, "s"] \arrow[rr, "g"] & & C \end{tikzcd}$$

where s is defined such that for $b \in B$ we have $b \mapsto b + f(JA)$ and further t is defined such that for $b + f(JA) \in B/f(JA)$ we have $b + f(JA) \mapsto g(b)$. The factorization, $g = ts$ is well-defined since $f(JA) \subseteq \operatorname{Im} f = \operatorname{Ker} g$. The map f is a monomorphism, so since $JA \neq (0)$, we have $f(JA) \neq (0)$. The modules have finite length, so since $\ell(B/f(JA)) < \ell(B)$, the morphism s cannot be a monomorphism. This implies that t is a split epimorphism by the definition of an irreducible map.

We now want to investigate a short exact sequence containing t. We know that $\operatorname{Im} f = \operatorname{Ker} g$. Let $a \in A$, then

$$\begin{aligned} 0 &= gf(a) \\ &= ts(f(a)) \\ &= t(f(a) + f(JA)) \\ &= t(\hat{f}(a + JA)) \end{aligned}$$

where $\hat{f}\colon A/JA \to B/f(JA)$ is defined by $\hat{f}(a+JA) = f(a)+f(JA)$. So, then $\operatorname{Im}\hat{f} \subseteq \operatorname{Ker} t$. We now want to show that $\operatorname{Ker} t \subseteq \operatorname{Im}\hat{f}$. Let $b+f(JA) \in \operatorname{Ker} t$, that is, by the definition of t we have $0 = t(b+f(JA)) = g(b)$. So, $b \in \operatorname{Ker} g = \operatorname{Im} f$ and we have an $a \in A$ such that $f(a) = b$. Then $b+f(JA) = f(a)+f(JA) = \hat{f}(a+JA)$, so $b+f(JA) \in \operatorname{Im}\hat{f}$ and $\operatorname{Ker} t \subseteq \operatorname{Im}\hat{f}$. In total, $\operatorname{Im}\hat{f} = \operatorname{Ker} t$.

Further, we show that $\hat{f}$ is 1-1. Let $a+JA$ be in A/JA such that $\hat{f}(a+JA) = \bar{0}$. That is, $f(a)+f(JA) = \bar{0}$ and moreover $f(a) \in f(JA)$. That is, $f(a) = f(x)$ for some $x \in JA$. Since f is 1-1 we then know that $x = a$ and therefore $a \in JA$. So, $\hat{f}$ is 1-1 and we have the following exact sequence

$$0 \longrightarrow A/JA \xrightarrow{\hat{f}} B/f(JA) \xrightarrow{t} C \longrightarrow 0$$

which is exact. Recall that the sequence splits since t splits. The splitting gives us an $\hat{f}'$ such that $\hat{f}'\hat{f} = 1_{A/JA}$. We now tensor with $R/J \otimes_R -$ and get

$$0 \longrightarrow R/J \otimes_R A/JA \xrightarrow{1\otimes_R \hat{f}} R/J \otimes_R B/f(JA) \xrightarrow{1\otimes_R t} R/J \otimes_R C \longrightarrow 0$$

where 1 denotes the identity on R/J. In general the tensor product is a right exact functor, but as a result of the splitting of the sequence containing t we get exactness on the left as well as on the right. Further, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & R/J \otimes_R A/JA \arrow[r, "1\otimes_R \hat{f}"] \arrow[d, "\alpha", "\cong"'] & R/J \otimes_R B/f(JA) \arrow[r, "1\otimes_R t"] \arrow[d, "\beta", "\cong"'] & R/J \otimes_R C \arrow[r] \arrow[d, "\gamma", "\cong"'] & 0 \\
0 \arrow[r] & A/JA \arrow[r, "\bar{f}"] & B/JB \arrow[r, "\bar{g}"] & C/JC \arrow[r] & 0
\end{tikzcd}$$

where α, β and γ are all isomorphisms, and $\bar{f}$ and $\bar{g}$ are defined as in Subchapter 2.1. Since the upper sequence is split exact we also have that the bottom one is split exact. We can now construct the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & JA \arrow[d] & JB \arrow[d] & JC \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "f"] \arrow[d] & B \arrow[r, "g"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A/JA \arrow[r, "\bar{f}"] \arrow[d] & B/JB \arrow[r, "\bar{g}"] \arrow[d] & C/JC \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where the maps $JX \to X$ are inclusions and the maps $X \to X/JX$ are defined such that $x \mapsto x + JX$ for $x \in X$. It is easy to show that the diagram commutes. The bottom sequence is exact as a result of the previous argument. By Snake Lemma, [13, Theorem 6.5], we get that

$$0 \longrightarrow JA \longrightarrow JB \longrightarrow JC \longrightarrow 0$$

is exact and therefore $JA = \operatorname{Ker} g \cap JB = A \cap JB$.

□

Remark. The reader should recall how $\hat{f}$ and t are defined in the previous proposition. Furthermore, it should be noted that since g is an epimorphism we know that t is an epimorphism without any assumption about A. That is, if

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a short exact sequence with g irreducible, then

$$0 \longrightarrow A/JA \xrightarrow{\hat{f}} B/f(JA) \xrightarrow{t} C \longrightarrow 0$$

is a short exact sequence. This is used in 4.1.7.

The next lemma and its corollary is of importance in several upcoming proofs.

Lemma 4.1.5. *Let R be a selfinjective Artin algebra and let*

$$0 \longrightarrow A \longrightarrow B \xrightarrow{g} C \longrightarrow 0$$

be a short exact sequence of R-modules. If $\Omega g\colon \Omega B \longrightarrow \Omega C$ is an epimorphism, then we have the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega A \arrow[r] \arrow[d] & \Omega B \arrow[r, "\Omega g"] \arrow[d] & \Omega C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P(A) \arrow[r] \arrow[d] & P(B) \arrow[r] \arrow[d, "p_B"] & P(C) \arrow[r] \arrow[d, "p_C"] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r, "g"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

In particular,

$$0 \longrightarrow \Omega A \longrightarrow \Omega B \xrightarrow{\Omega g} \Omega C \longrightarrow 0$$

is an exact sequence.

Proof. We construct the commutative diagram in the manner described below.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{Ker}\alpha \arrow[r] \arrow[d] & \Omega B \arrow[r, "\Omega g"] \arrow[d] & \Omega C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{Ker}\beta \arrow[r] \arrow[d, "\alpha"] & P(B) \arrow[r, "\beta"] \arrow[d, "p_B"] & P(C) \arrow[r] \arrow[d, "p_C"] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r, "g"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & \operatorname{Coker}\alpha \arrow[d] & 0 & 0 & \\
 & 0 & & &
\end{tikzcd}$$

Let p_B and p_C be the projective covers of B and C, respectively. The module $P(B)$ is projective, so as a result of p_C being an epimorphism we have a map $\beta\colon P(B) \longrightarrow P(C)$. The composition gp_B is an epimorphism, so since p_C is an essential epimorphism, β is an epimorphism, that is, the sequence containing β splits and $P(B) \cong P(C) \oplus \operatorname{Ker}\beta$. So, $\operatorname{Ker}\beta$ is projective. By [14, Proposition 2.71] we have the morphism α. Further, the map Ωg is an epimorphism by assumption. The Snake Lemma, [13, Theorem 6.5] then gives us that $\operatorname{Coker}\alpha = (0)$ and that $\operatorname{Ker}\alpha \longrightarrow \Omega B$ is a monomorphism. Assume α is not the projective cover of A. Then $\operatorname{Ker}\alpha \cong P' \oplus \Omega A$, where P' is a non-zero projective module. This gives a contradiction as ΩB cannot have a non-zero projective summand. So $\operatorname{Ker}\beta = P(A)$, the map $\alpha\colon P(A) \longrightarrow A$ is a projective cover and then $\operatorname{Ker}\alpha = \Omega A$. So, we have an exact sequence

$$0 \longrightarrow \Omega A \longrightarrow \Omega B \longrightarrow \Omega C \longrightarrow 0$$

That is, $\Omega A = \operatorname{Ker}\alpha \cong \operatorname{Ker}\Omega g$. □

If g in the previous lemma is an Ω-perfect irreducible epimorphism we get the following immediate consequence.

Corollary 4.1.6. *Let R be a selfinjective Artin algebra and let*

$$0 \longrightarrow A \longrightarrow B \xrightarrow{g} C \longrightarrow 0$$

be a short exact sequence of R-modules with g an Ω-perfect irreducible epimorphism. Then

$$0 \longrightarrow \Omega^n A \longrightarrow \Omega^n B \xrightarrow{\Omega^n g} \Omega^n C \longrightarrow 0$$

is an exact sequence for all $n \geq 0$.

Proof. Since g is Ω-perfect we know that Ωg is an epimorphism. By 4.1.5, we have the following short exact sequence

$$0 \longrightarrow \Omega A \longrightarrow \Omega B \xrightarrow{\Omega g} \Omega C \longrightarrow 0.$$

Moreover, since g is Ω-perfect, the morphism $\Omega^2 g\colon \Omega^2 B \longrightarrow \Omega^2 C$ is also an epimorphism and, again using 4.1.5, we have the following short exact sequence

$$0 \longrightarrow \Omega^2 A \longrightarrow \Omega^2 B \xrightarrow{\Omega^2 g} \Omega^2 C \longrightarrow 0$$

where $\Omega^2 g$ is irreducible. Since g is Ω-perfect we know that $\Omega^n g$ is an epimorphism for all $n \geq 0$ and we proceed by induction and get an exact sequence

$$0 \longrightarrow \Omega^n A \longrightarrow \Omega^n B \xrightarrow{\Omega^n g} \Omega^n C \longrightarrow 0$$

with $\Omega^n g$ irreducible for all $n \geq 0$.

□

We continue with the last result we need before we can prove that most syzygies of an Ω-perfect module are not simple. The upcoming lemma is also important later in the thesis. We prove that the only way the syzygy takes an irreducible epimorphism into an irreducible monomorphism is if the kernel of the epimorphism is simple.

Lemma 4.1.7. [6, Corollary 2.6] *Let R be a selfinjective Artin algebra and let*

$$0 \longrightarrow A \longrightarrow B \xrightarrow{g} C \longrightarrow 0$$

be a short exact sequence of R-modules with g irreducible. Then the induced irreducible map $\Omega g\colon \Omega B \longrightarrow \Omega C$ is an epimorphism if and only if A is not a simple module. If A is simple, then Ωg is an irreducible monomorphism and we have an induced exact sequence

$$0 \longrightarrow \Omega B \xrightarrow{\Omega g} \Omega C \longrightarrow A \longrightarrow 0$$

Proof. **Assume A is not simple**. By Horseshoe Lemma [13, Theorem 6.20] we have the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega A \arrow[r] \arrow[d] & \operatorname{Ker}\alpha \arrow[r] \arrow[d] & \Omega C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P(A) \arrow[r] \arrow[d, "p_A"] & P(A)\oplus P(C) \arrow[r, "\beta"] \arrow[d, "\alpha"] & P(C) \arrow[r] \arrow[d, "p_C"] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r, "g"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where the maps p_A and p_C are projective covers of A and C, respectively. We want to show that α is a projective cover as well, so that $\operatorname{Ker}\alpha = \Omega B$. Tensoring with $R/J \otimes_R -$ we get the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & P(A)/JP(A) \arrow[r] \arrow[d, "\gamma", "\cong"'] & P(A)/JP(A)\oplus P(C)/JP(C) \arrow[r] \arrow[d] & P(C)/JP(C) \arrow[r] \arrow[d, "\delta", "\cong"'] & 0 \\
0 \arrow[r] & A/JA \arrow[r, "\bar{f}"] & B/JB \arrow[r, "\bar{g}"] & C/JC \arrow[r] & 0
\end{tikzcd}$$

where the bottom sequence is exact by 4.1.4. By [4, Proposition I.4.3] we know that the induced epimorphisms γ and δ are isomorphisms. Using Snake Lemma, [13, Theorem 6.5], we get that $P(A)/JP(A) \oplus P(C)/JP(C) \cong B/JB$. So, by [4, Proposition I.4.3], α is projective cover of B, so $\operatorname{Ker}\alpha = \Omega B$ and $\Omega g\colon \Omega B \longrightarrow \Omega C$ is an epimorphism.

We now assume $\Omega g\colon \Omega B \longrightarrow \Omega C$ is an epimorphism. By 4.1.5 we have the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega A \arrow[r] \arrow[d] & \Omega B \arrow[r, "\Omega g"] \arrow[d] & \Omega C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P(A) \arrow[r] \arrow[d] & P(B) \arrow[r, "\beta"] \arrow[d, "p_B"] & P(C) \arrow[r] \arrow[d, "p_C"] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r, "g"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Tensoring with $R/J \otimes_R -$ we get the diagram

$$\begin{tikzcd}
0 \arrow[r] & P(A)/JP(A) \arrow[r] \arrow[d, "\gamma", "\cong"'] & P(B)/JP(B) \arrow[r] \arrow[d, "\epsilon", "\cong"'] & P(C)/JP(C) \arrow[r] \arrow[d, "\delta", "\cong"'] & 0 \\
0 \arrow[r] & A/JA \arrow[r, "\bar{f}"] & B/JB \arrow[r, "\bar{g}"] & C/JC \arrow[r] & 0
\end{tikzcd}$$

where γ, ϵ and δ, the induced epimorpisms, are isomorphisms ([4, Proposition I.4.3]). The sequence containing β in the previous diagram splits, so we know that the sequence

$$\begin{tikzcd}
0 \arrow[r] & P(A)/JP(A) \arrow[r] & P(B)/JP(B) \arrow[r] & P(C)/JP(C) \arrow[r] & 0
\end{tikzcd}$$

is left exact and splits. As γ, ϵ and δ are isomorphisms it is easy to show that the bottom sequence is left exact and splits as well. Further, we can now construct the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A/JA \arrow[r, "\hat{f}"] \arrow[d, equal, "k"] & B/f(JA) \arrow[r, "t"] \arrow[d, "l"] & C \arrow[r] \arrow[d, "m"] & 0 \\
0 \arrow[r] & A/JA \arrow[r, "\bar{f}"] & B/JB \arrow[r, "\bar{g}"] & C/JC \arrow[r] & 0
\end{tikzcd}$$

where $\hat{f}$ and t are defined as in the proof of 4.1.4. Note that t is actually an epimorphism since g is an epimorphism, but initially we do not know if t splits, as we did in 4.1.4. The map k is the identity, l sends $b + f(JA) \in B + f(JA)$ to $b + JB$ and m sends $c \in C$ to $c + JC$. It is easy to check that the diagram commutes. Since the bottom sequence splits, the upper sequence splits as well. If A is simple, then $JA = (0)$ and then upper sequence reduces to

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

by the definition of $\hat{f}$ and t. That is, g splits, a contradiction to it being irreducible. So A is not simple.

We assume A is simple. We have

$$0 \longrightarrow A \longrightarrow B \xrightarrow{g} C \longrightarrow 0$$

a short exact sequence of R-modules with g irreducible, and an induced irreducible map $\Omega g\colon \Omega B \longrightarrow \Omega C$. If Ωg is an epimorphism we know by the previous argument that A is non-simple, so Ωg is a monomorphism. We can construct the following commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & 0 \arrow[r] & \Omega B \arrow[r, "\Omega g"] \arrow[d] & \Omega C \arrow[r] \arrow[d] & \operatorname{Coker}\Omega g \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{Ker}\alpha \arrow[r] \arrow[d, "\beta"] & P(B) \arrow[r, "\alpha"] \arrow[d, "p_B"] & P(C) \arrow[r] \arrow[d, "p_C"] & 0 & \\
0 \arrow[r] & A \arrow[r] \arrow[d, "\beta'"] & B \arrow[r, "g"] \arrow[d] & C \arrow[r] \arrow[d] & 0 & \\
 & \operatorname{Coker}\beta \arrow[d] & 0 & 0 & & \\
 & 0 & & & &
\end{tikzcd}$$

Let p_B and p_C be the projective covers of B and C, respectively. The map α exists since p_C is an epimorphism and $P(B)$ is projective. Since p_C is an essential epimorphism and the diagram commutes we have that α is an epimorphism. The morphism Ωg is a monomorphism independent of the choice of α, so the kernel is zero. By [14, Proposition 2.71] we have β. By the Snake Lemma [13, Theorem 6.5], we get that $\operatorname{Coker}\beta \cong \operatorname{Coker}\Omega g$ and $\operatorname{Ker}\alpha \cong \operatorname{Ker}\beta'$. The module A is simple by assumption, so $\operatorname{Ker}\beta'$ is A or (0). If $\operatorname{Ker}\beta' = \operatorname{Im}\beta$ is equal to A, then $\operatorname{Coker}\Omega g \cong \operatorname{Coker}\beta = A/\operatorname{Im}\beta = (0)$ and Ωg is an isomorphism, so it splits. This contradicts the fact that Ωg is irreducible. We conclude that $\operatorname{Ker}\beta' = (0)$, so $\operatorname{Coker}\beta \cong A$, and we have the exact sequence

$$0 \longrightarrow \Omega B \xrightarrow{\Omega g} \Omega C \longrightarrow A \longrightarrow 0.$$

□

We are now able to show our desired result. That is, the requirement that all syzygies of an Ω-perfect module is not simple given in the definition of Ω-perfect modules in [6] actually holds for all syzygy powers greater or equal to two when we use the chosen definition in [7]. In other words, if C is an Ω-perfect module, then $\Omega^n C$ is not simple for $n \geq 2$.

Proposition 4.1.8. [7, Lemma 2.2] *Let R be a selfinjective Artin algebra and C an Ω-perfect module. Then $\Omega^n C$ is not a simple module, for $n \geq 2$.*

Proof. Assume C is Ω-perfect. We now look at an almost split sequence ending at C

$$0 \longrightarrow \tau C \longrightarrow B \xrightarrow{g} C \longrightarrow 0$$

where B is not necessarily indecomposable. So, g is an Ω-perfect epimorphism and by 4.1.6 we know that

$$0 \longrightarrow \Omega^n \tau C \longrightarrow \Omega^n B \xrightarrow{\Omega^n g} \Omega^n C \longrightarrow 0$$

is an exact sequence for all $n \geq 0$. Furthermore, by 4.1.7, we then know that $\Omega^n \tau C$ is not simple for $n \geq 0$. That is, using 2.3.5 we have that $\Omega^n \tau C \cong \Omega^n \nu \Omega^2 C$ is not simple for $n \geq 0$. Since ν preserves length by 2.3.4 and commutes with Ω by 2.3.6, we know that $\Omega^{n+2} C$ is not simple for $n \geq 0$, which is what we wanted to show.

□

The structure of almost split sequences ending at a syzygy of an Ω-perfect module is of importance in later proofs. The next result gives us further knowledge of such almost split sequences.

Proposition 4.1.9. [6, Proposition 2.4] *Let R be a selfinjective Artin algebra. Let C be an Ω-perfect R-module, and let*

$$0 \longrightarrow \tau C \longrightarrow E \longrightarrow C \longrightarrow 0$$

be an almost split sequence ending at C. Then, for each $n \geq 1$,

$$0 \longrightarrow \Omega^n \tau C \longrightarrow \Omega^n E \longrightarrow \Omega^n C \longrightarrow 0$$

is also an almost split sequence.

Proof. By assumption,

$$0 \longrightarrow \tau C \longrightarrow E \longrightarrow C \longrightarrow 0$$

is an almost split sequence. From 2.3.8 we then get that

$$0 \longrightarrow \Omega\tau C \longrightarrow \Omega E \oplus P \longrightarrow \Omega C \longrightarrow 0$$

is an almost split sequence where P is indecomposable projective-injective or zero. Assume $P \neq (0)$. Then, by 2.3.10, $\Omega^2 C$ is simple, and this contradicts 4.1.8 since C is Ω-perfect. So $P = (0)$. That is

$$0 \longrightarrow \Omega\tau C \longrightarrow \Omega E \longrightarrow \Omega C \longrightarrow 0$$

is an almost split sequence.

Since C is Ω-perfect we know that $\Omega^n C$ is not simple for $n \geq 2$ by 4.1.8 and the result now follows by induction.

□

As we did for both Betti numbers and complexity, we now present a proposition with some elementary properties of Ω-perfect modules.

Proposition 4.1.10. [Properties of Ω-perfect modules] *Let R be a selfinjective Artin algebra and C an Ω-perfect module. Then*

(1) *There are no irreducible morphisms from projective modules to C.*

(2) *$\Omega^n C$ is an Ω-perfect module for all $n \geq 0$.*

(3) *$\tau^n C$ is an Ω-perfect module for all $n \geq 0$.*

(4) *νC is an Ω-perfect module for all $n \geq 0$.*

Proof.

(1) Let $P \longrightarrow C$ be an irreducible morphism where P is projective. The map is a monomorphism or an epimorphism since it is irreducible. If it is a monomorphism it splits since P is injective when R is selfinjective. This contradicts the definition of irreducible morphisms, so it is an epimorphism. But then, since C is Ω-perfect, $(0) = \Omega P \longrightarrow \Omega C \neq (0)$ is an epimorphism, a contradiction.

Before we prove the next three statements separately, we look at an almost split sequence ending at C

$$0 \longrightarrow \tau C \xrightarrow{(f_1,f_2,\ldots,f_t)^T} E_1 \oplus E_2 \oplus \ldots \oplus E_t \xrightarrow{(g_1,g_2,\ldots g_t)} C \longrightarrow 0 \tag{4.1}$$

with each E_i indecomposable. By (1) we know that none of the E_i's are projective.

(2) By (4.1) and 4.1.9 we have an almost split sequence

$$0 \longrightarrow \Omega^n\tau C \xrightarrow{(\Omega^n f_1,\ldots,\Omega^n f_t)^T} \Omega^n E_1 \oplus \ldots \oplus \Omega^n E_t \xrightarrow{(\Omega^n g_1,\ldots,\Omega^n g_t)} \Omega^n C \longrightarrow 0 \tag{4.2}$$

ending at $\Omega^n C$ for all $n \geq 0$. Recall that none of the modules in the almost split sequence are projective. Furthermore, the irreducible morphisms ending at $\Omega^n C$, for some $n \geq 0$, are of type $\Omega^n B \longrightarrow \Omega^n C$, where B is a finite sum of some of the modules $\{E_1, \ldots, E_t\}$ and each E_i occurs at most once in the sum. Moreover, the irreducible morphisms starting at $\tau\Omega^n C \cong \Omega^n\tau C$ are of type $\Omega^n\tau C \longrightarrow \Omega^n B$, with B as before. We now want to show that $\Omega^n C$ is Ω-perfect for $n \geq 0$.

We know that C is Ω-perfect, so the irreducible morphisms $\Omega^s B \longrightarrow \Omega^s C$ are epimorphisms for all $s \geq 0$ or monomorphisms for all $s \geq 0$. That is, for a chosen $n \geq 0$ we know that the irreducible morphisms $\Omega^m(\Omega^n B) \longrightarrow \Omega^m(\Omega^n C)$ are epimorphisms for all $m \geq 0$ or monomorphisms for all $m \geq 0$.

Similarily, since C is Ω-perfect we know that the irreducible morphisms $\Omega^s\tau C \longrightarrow \Omega^s B$ are epimorphisms for all $s \geq 0$ or monomorphisms for all $s \geq 0$. That is, for a chosen $n \geq 0$ we know that the irreducible morphisms $\Omega^m(\Omega^n\tau C) \longrightarrow \Omega^m(\Omega^n B)$ are epimorphisms for all $m \geq 0$ or monomorphisms for all $m \geq 0$.

So, the irreducible morphisms $\Omega^n B \longrightarrow \Omega^n C$ and $\Omega^n\tau C \longrightarrow \Omega^n B$ are Ω-perfect. That is, all irreducible morphisms ending at $\Omega^n C$ and starting at $\tau\Omega^n C$ are Ω-perfect and therefore the non-projective module $\Omega^n C$ is Ω-perfect.

(3) By 2.3.11, (4.1), (4.2) and 2.3.13 we have that

$$0 \longrightarrow \tau^n\tau C \xrightarrow{(\tau^n f_1,\ldots,\tau^n f_t)^T} \tau^n E_1 \oplus \ldots \oplus \tau^n E_t \xrightarrow{(\tau^n g_1,\ldots,\tau^n g_t)} \tau^n C \longrightarrow 0$$

is an almost split sequence ending at $\tau^n C$ for all $n \geq 0$. Recall that none of the modules in the sequence are projective. Moreover, the irreducible morphisms ending at $\tau^n C$, for some $n \geq 0$, are of type $\tau^n B \longrightarrow \tau^n C$, where B is a finite sum of some of the modules $\{E_1, ..., E_t\}$ and each E_i occurs at most once in the sum. Moreover, the irreducible morphisms starting at $\tau^n \tau C \cong \tau\tau^n C$ are of type $\tau^n \tau C \longrightarrow \tau^n B$, with B as before.

Since C is Ω-perfect, by 4.1.3, we know that for a given $n \geq 0$ the irreducible morphisms $\Omega^m(\tau^n B) \longrightarrow \Omega^m(\tau^n C)$ are epimorphisms for all $m \geq 0$ or monomorphisms for all $m \geq 0$.

Again, since C is Ω-perfect, by 4.1.3, we know that for a given $n \geq 0$ the irreducible morphisms $\Omega^m(\tau^n \tau C) \longrightarrow \Omega^m(\tau^n B)$ are epimorphisms for all $m \geq 0$ or monomorphisms for all $m \geq 0$.

That is, for $n \geq 0$, all irreducible morphisms ending at $\tau^n C$ and starting at $\tau\tau^n C$ are Ω-perfect and therefore the non-projective module $\tau^n C$ is an Ω-perfect module.

(4) From the almost split sequence (4.1) and 2.3.4 we get the following almost split sequence ending at νC

$$0 \longrightarrow \nu\tau C \longrightarrow \nu E_1 \oplus \nu E_2 \oplus ... \oplus \nu E_t \longrightarrow \nu C \longrightarrow 0.$$

Furthermore, by 2.3.4, we know that none of the modules in the sequence is projective. The irreducible morphisms ending at νC are of type $\nu B \longrightarrow \nu C$ where where B is a finite sum of some of the modules $\{E_1, ..., E_t\}$ and each E_i occurs at most once in the sum. Moreover, the irreducible morphisms starting at $\nu\tau C \cong \tau\nu C$ are of type $\nu\tau C \longrightarrow \nu B$, with B as before.

The module C is Ω-perfect, so the irreducible morphisms $\Omega^s B \longrightarrow \Omega^s C$ are epimorphisms for all $s \geq 0$ or monomorphisms for all $s \geq 0$. That is, $\ell(\Omega^s B) > \ell(\Omega^s C)$ for all $s \geq 0$ or $\ell(\Omega^s B) < \ell(\Omega^s C)$ for all $s \geq 0$. So, by 2.3.4 and 2.3.6, we know that $\ell(\Omega^s(\nu B)) > \ell(\Omega^s(\nu C))$ for all $s \geq 0$ or $\ell(\Omega^s(\nu B)) < \ell(\Omega^s(\nu C))$ for all $s \geq 0$. That is, the irreducible morphisms $\Omega^s(\nu B) \longrightarrow \Omega^s(\nu C)$ are epimorphisms for all $s \geq 0$ or monomorphisms for all $s \geq 0$.

Similarily, since C is Ω-perfect we know that the irreducible morphims $\Omega^s \tau C \longrightarrow \Omega^s B$ are epimorphisms for all $s \geq 0$ or monomorphisms for all $s \geq 0$. So, $\ell(\Omega^s \tau C) > \ell(\Omega^s B)$ for all $s \geq 0$ or $\ell(\Omega^s \tau C) < \ell(\Omega^s B)$ for all $s \geq 0$. Then, by 2.3.4 and 2.3.6 we know that $\ell(\Omega^s(\nu\tau C)) > \ell(\Omega^s(\nu B))$ for all $s \geq 0$ or $\ell(\Omega^s(\nu\tau C)) < \ell(\Omega^s(\nu B))$ for all $s \geq 0$. That is, the irreducible morphisms $\Omega^s(\nu\tau C) \longrightarrow \Omega^s(\nu B)$ are epimorphisms for all $s \geq 0$ or a monomorphisms for all $s \geq 0$.

So, all irreducible morphisms ending at νC and starting at $\tau\nu C$ are Ω-perfect. That is, νC is Ω-perfect.

□

The next result proves important in later parts of the thesis. Among other things we look at how the Betti numbers of the modules in an almost split sequence ending at an Ω-perfect module are connected.

Proposition 4.1.11. [6, Proposition 2.8] *Let R be a selfinjective Artin algebra and let*

$$0 \longrightarrow A \longrightarrow B \xrightarrow{g} C \longrightarrow 0$$

be a short exact sequence of finitely generated R-modules, where C is an indecomposable module and the map g is irreducible. Assume further that either (I) *the map g is Ω-perfect, or* (II) *the sequence is an almost split sequence with C Ω-perfect. Then*

(1) *The module A is not simple. Furthermore, if we assume that B is indecomposable in the case where g is Ω-perfect, B has no projective summands.*

(2) *For every $n \geq 0$, there is an induced short exact sequence*

$$0 \longrightarrow \Omega^n A \longrightarrow \Omega^n B \xrightarrow{\Omega^n g} \Omega^n C \longrightarrow 0$$

and the map $\Omega^n g$ is irreducible. Moreover, for each $n \geq 0$, $\tau^n g$ is an irreducible epimorphism.

(3) *For each $n \geq 0$, we have that $\beta_n(B) = \beta_n(A) + \beta_n(C)$.*

Proof.

(1) The module A is clearly indecomposable being the kernel of an irreducible epimorphism by [4, Proposition V.5.7].

(I) Assume that g is Ω-perfect and B is indecomposable. If B has nonzero projective summands it is itself projective since it is assumed to be indecomposable. The module C cannot be projective, because if it was, the map would split, which contradicts it being irreducible. But then if B is projective, since g is Ω-perfect, we have that $(0) = \Omega B \longrightarrow \Omega C \neq (0)$ is an epimorphism, a contradiction. So, the indecomposable module B is not projective and hence has no projective summands. Since $\Omega g\colon \Omega B \longrightarrow \Omega C$ is an irreducible epimorphism, by 4.1.7, A is not simple.

(II) Assume the sequence is almost split and C is Ω-perfect. If B had a non-zero projective summand we would have an Ω-perfect map from a projective module to C and this cannot be by 4.1.10. Further, since C is Ω-perfect, $\Omega^2 C$ is not simple by 4.1.8. The Nakayama functor ν preserves length by 2.3.4, so using this and 2.3.5 we have that $A = \tau C \cong \nu\Omega^2 C$ is not simple.

(2) (I) We assume g is an irreducible Ω-perfect epimorphism. Then the result follows from 4.1.6. Recall that $\Omega^n g$ is irreducible by Subchapter 2.3.

(II) Now assume the sequence is almost split and C is Ω-perfect. Then, the result follows from 4.1.9.

In both (I) and (II), using 2.3.13 we know that $\tau^n g$ is an irreducible epimorphism for all $n \geq 0$.

(3) (I) and (II). By (2) for $n \geq 0$ we know that

$$0 \longrightarrow \Omega^n A \longrightarrow \Omega^n B \xrightarrow{\Omega^n g} \Omega^n C \longrightarrow 0$$

is a short exact sequence and furthermore that $\Omega^{n+1}g\colon \Omega^{n+1}B \longrightarrow \Omega^{n+1}C$ is an epimorphism. That is, by 4.1.5 we have the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega^{n+1}A \arrow[r] \arrow[d] & \Omega^{n+1}B \arrow[r, "\Omega^{n+1}g"] \arrow[d] & \Omega^{n+1}C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P(\Omega^n A) \arrow[r] \arrow[d] & P(\Omega^n B) \arrow[r, "\alpha"] \arrow[d] & P(\Omega^n C) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^n A \arrow[r] \arrow[d] & \Omega^n B \arrow[r, "\Omega^n g"] \arrow[d] & \Omega^n C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

That is, $P(\Omega^n B) \cong P(\Omega^n A) \oplus P(\Omega^n C)$ for all $n \geq 0$. So, $\beta_n(B) = \beta_n(A) + \beta_n(C)$ for all $n \geq 0$, and we are done.

□

The structure of almost split sequences ending at modules with different properties is of interest in this thesis. The following lemma is general, and holds for any short exact sequence.

Lemma 4.1.12. *Let*

$$0 \longrightarrow A \xrightarrow{(f_1,f_2)^T} B_1 \oplus B_2 \xrightarrow{(g_1,g_2)} C \longrightarrow 0$$

be a short exact sequence. Then

(a) f_1 *(f_2) is an epimorphism* $\iff$ g_2 *(g_1) is an epimorphism.*

(b) f_1 *(f_2) is a monomorphism* $\iff$ g_2 *(g_1) is a monomorphism.*

(c) $\operatorname{Coker} f_1 \cong \operatorname{Coker} g_2$ *and* $\operatorname{Coker} f_2 \cong \operatorname{Coker} g_1$.

(d) $\operatorname{Ker} f_1 \cong \operatorname{Ker} g_2$ *and* $\operatorname{Ker} f_2 \cong \operatorname{Ker} g_1$.

Proof.

(a) The sequence is exact, so $g_1 f_1 + g_2 f_2 = 0$ and further $g_1 f_1 = (-g_2) f_2$. Then, the square in the following diagram commutes

$$\begin{tikzcd}
Y & & \\
& C \arrow[ul, dashed, "\exists!\Theta'"] & B_1 \arrow[l, "g_1"'] \arrow[ull, bend right, "\alpha"'] \\
& B_2 \arrow[u, "-g_2"] \arrow[uul, bend left, "\beta"] & A \arrow[l, "f_2"] \arrow[u, "f_1"']
\end{tikzcd}$$

We want to show that for each triple (Y, α, β) with the property that $\alpha f_1 = \beta f_2$, we have a unique $\Theta' \colon C \longrightarrow Y$ such that $\Theta' g_1 = \alpha$ and $\Theta'(-g_2) = \beta$. We have the following diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "(f_1,f_2)^T"] & B_1 \oplus B_2 \arrow[r, "(g_1,g_2)"] \arrow[d, "(\alpha,-\beta)"] & C \arrow[r] & 0 \\
& & Y & &
\end{tikzcd}$$

We want to show that there exists a unique Θ' such that $\Theta' \cdot (g_1, g_2) = (\alpha, -\beta)$. For all $c \in C$ there exists a $\underline{b} \in B_1 \oplus B_2$ such that $(g_1, g_2)\underline{b} = c$. We define Θ' such that $\Theta'(c) = (\alpha, -\beta)\underline{b}$. This is well-defined since $(\alpha, -\beta) \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} =$

$\alpha f_1 - \beta f_2 = 0$, that is, $\mathrm{Ker}(g_1, g_2) = \mathrm{Im}(f_1, f_2)^T \subseteq \mathrm{Ker}(\alpha, -\beta)$. So, we have that $\Theta' \cdot (g_1, g_2) = (\alpha, -\beta)$. We now want to show that Θ' is unique. Assume there exists another Ψ such that $\Psi \cdot (g_1, g_2) = (\alpha, -\beta)$. Then $\Theta'(g_1, g_2)(\underline{b}) = (\alpha, -\beta)(\underline{b})$ and $\Psi(g_1, g_2)(\underline{b}) = (\alpha, -\beta)(\underline{b})$ for all $\underline{b} \in B_1 \oplus B_2$. Then, since (g_1, g_2) is an epimorphism we know that $\Theta'(c) = \Psi(c)$ for all $c \in C$. That is, $\Theta' = \Psi$.

So Θ' is unique and $\Theta' g_1 = \alpha$ and $\Theta' g_2 = -\beta$ and we have a pushout. By [13, Exercise 2.30] parallel arrows have isomorphic cokernels, so f_2 is a epimorphism $\iff$ g_1 is a epimorphism. Similarily, g_2 is a epimorphism $\iff$ f_1 is a epimorphism.

(b) Again, the sequence is exact, so $g_1 f_1 + g_2 f_2 = 0$ and further $g_1 f_1 = g_2(-f_2)$. Using this we know that the square in the following diagram commutes

$$\begin{tikzcd}
X \arrow[rrd, bend left, "s"] \arrow[rdd, bend right, "t"'] \arrow[rd, dashed, "\exists!\Theta"] & & \\
& A \arrow[r, "f_1"] \arrow[d, "-f_2"'] & B_1 \arrow[d, "g_1"] \\
& B_2 \arrow[r, "g_2"'] & C
\end{tikzcd}$$

We now want to show that for each triple (X, s, t) with the property that $g_2 t = g_1 s$ we have a unique $\Theta \colon X \longrightarrow A$ such that $f_1\Theta = s$ and $-f_2\Theta = t$. We have the following

$$\begin{tikzcd}
& & X \arrow[d, "(s,-t)^T"] & & \\
0 \arrow[r] & A \arrow[r, "(f_1,f_2)^T"] & B_1 \oplus B_2 \arrow[r, "(g_1,g_2)"] & C \arrow[r] & 0
\end{tikzcd}$$

and since $(g_1, g_2) \cdot \begin{pmatrix} s \\ -t \end{pmatrix} = g_1 s - g_2 t = 0$, we have that $\mathrm{Im}\begin{pmatrix} s \\ -t \end{pmatrix} \subseteq \mathrm{Ker}(g_1, g_2) = \mathrm{Im}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$. Then, by the dual arguments of those in (a) we know that there exists a unique Θ such that $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \cdot \Theta = \begin{pmatrix} s \\ -t \end{pmatrix}$, that is $f_1\Theta = s$ and $f_2\Theta = -t$. Thus, we have a pullback and by [13, Exercise 2.47] we have that parallel arrows have isomorphic kernels. So, f_1 is a monomorphism $\Longleftrightarrow g_2$ is a monomorphism. Similarly, f_2 is a monomorphism $\Longleftrightarrow g_1$ is a monomorphism.

(c) Follows from the argument in (a).

(d) Follows from the argument in (b).

□

We now present a result that proves important in many of the upcoming proofs.

Lemma 4.1.13. *Let R be a selfinjective Artin algebra and let*

$$0 \longrightarrow \tau C \xrightarrow{(f_1, f_2, \ldots, f_t)^T} E_1 \oplus E_2 \oplus \ldots \oplus E_t \xrightarrow{(g_1, g_2, \ldots, g_t)} C \longrightarrow 0$$

be an almost split sequence of R-modules where the E_i's are non-zero and not necessarily indecomposable. Furthermore, we assume that all the f_i's and g_i's are Ω-perfect. Then, for some $i \in \{1, \ldots, t\}$, both f_i and g_i cannot be monomorphisms.

Proof. We assume both f_i and g_i are a monomorphisms for some $i \in \{1, \ldots, t\}$. Then, $g_i f_i \colon \tau C \longrightarrow C$ would be a proper monomorphism. Furthermore, since the morphisms are Ω-perfect by assumption, we know that in particular $\Omega^{2n} f_i$ and $\Omega^{2n} g_i$ are monomorphisms for all $n \geq 0$. Then, using 2.3.13, we know that $\tau^n f_i$ and $\tau^n g_i$ are monomorphisms for all $n \geq 0$. That is, we have sequence of proper monomorphisms

$$\cdots \hookrightarrow \tau^3 C \xhookrightarrow{\tau^2 g_i \tau^2 f_i} \tau^2 C \xhookrightarrow{\tau g_i \tau f_i} \tau C \xhookrightarrow{g_i f_i} C$$

and $\ell(\tau^m C) > \ell(\tau^{m+1} C)$ for all $m \geq 0$. The non-projective module C has finite length, so then there must exist an integer k such that $\tau^k C = (0)$, a contradiction. □

The last three results in this subchapter further explores Ω-perfect modules and how having such modules can expand our knowledge of almost split sequences and irreducible morphisms ending at them.

Lemma 4.1.14. [7, Lemma 2.5] *Let R be a selfinjective Artin algebra and let*

$$0 \longrightarrow \tau C \xrightarrow{(f_1, f_2, \ldots, f_t)^T} E_1 \oplus E_2 \oplus \ldots \oplus E_t \xrightarrow{(g_1, g_2, \ldots, g_t)} C \longrightarrow 0$$

be an almost split sequence of R-modules where the E_i's are non-zero and not necessarily indecomposable.

(a) *Assume that* g_i *is an epimorphism for some* $1 \leq i \leq t$*. Then for each* $j \neq i$*,* f_j *is an epimorphism.*

(b) *Assume that* f_i *is a monomorphism for some* $1 \leq i \leq t$*. Then for each* $j \neq i$*,* g_j *is a monomorphism.*

(c) *In particular, if* C *is* Ω*-perfect, and some* f_i *is a monomorphism, then* g_i *is an epimorphism, all the remaining* f_j*'s are epimorphisms, and all the remaining* g_j*'s are monomorphims.*

Proof.

(a) Since g_i is an epimorphism for some $1 \leq i \leq t$, then $(g_1, ..., g_{j-1}, g_{j+1}, ..., g_t)$ where only g_j is removed for a $j \neq i$ is an epimorphism (or $(g_2, ..., g_t)$ in the case where $j = 1$ and $i \neq 1$, or $(g_1, ..., g_{t-1})$ when $j = t$ and $i \neq t$). By 4.1.12 we then know that f_j is an epimorphism. This holds for all $j \neq i$, so f_j is an epimorphism for all $j \neq i$.

(b) Since f_i is a monomorphism for some $1 \leq i \leq t$, then $(f_1, .., f_{j-1}, f_{j+1}, ..., f_t)$ where only f_j is removed for a $j \neq i$ is a monomorphism (or $(f_2, ..., f_t)$ in the case where $j = 1$ and $i \neq 1$, or $(f_1, ..., f_{t-1})$ when $j = t$ and $i \neq t$). By 4.1.12 we then know that g_j is a monomorphism. This holds for all $j \neq i$, so g_j is a monomorphism for each $j \neq i$.

(c) Assume that C is Ω-perfect and f_i is a monomorphism for an $i \in \{1, ..., t\}$. Then, by 4.1.13, we know that g_i is an epimorphism and by (a) and (b) we are done.

□

We are now able to prove that if we have an irreducible monomorphism between indecomposable modules ending at an Ω-perfect module, then the module where the morphism starts is also Ω-perfect.

Lemma 4.1.15. [7, Lemma 2.6] *Let* R *be a selfinjective Artin algebra. Suppose that* $f\colon A \longrightarrow B$ *is an irreducible monomorphism between indecomposable modules. If* B *is* Ω*-perfect, then so is* A*.*

Proof. Recall that A cannot be projective by 4.1.10. We begin by arguing that there cannot be any non-zero indecomposable projective summand of the middle term of the almost split sequence ending at A. Assume there is such a projective summand of the middle term. That is, we have an almost split sequence

$$0 \longrightarrow \tau A \longrightarrow D \oplus P \longrightarrow A \longrightarrow 0$$

where τB is a summand of D. If $P \neq (0)$ we know that the morphism $P \longrightarrow A$ is an epimorphism, or else it splits which is a contradiction to it being irreducible. But then, by 4.1.14 we know that $\tau A \longrightarrow \tau B$ is an epimorphism. This cannot be by 4.1.3. So, there are no non-zero projective modules in the middle term of the almost split sequence ending at A.

We now look at an almost split sequence ending at $\Omega^n A$ for some $n \geq 0$. By repeatedly using 2.3.8 we know that such an almost split sequence is

$$0 \longrightarrow \Omega^n \tau A \longrightarrow \Omega^n D \oplus P' \longrightarrow \Omega^n A \longrightarrow 0$$

where P' is indecomposable if it is non-zero and $\Omega^n \tau B$ is a summand of $\Omega^n D$. If $P' \neq (0)$, resuming the previous argument we have that $\Omega^n \tau A \longrightarrow \Omega^n \tau B$ is an epimorphism by 4.1.14, a contradiction to 4.1.3. So, an almost split sequence ending at $\Omega^n A$ where $n \geq 0$ cannot have any non-zero projective middle terms.

Now, assume $\alpha(A) = 1$. That is, the almost split sequence ending at A has only one middle term, τB, since we cannot have any projective summands of the middle term.

$$0 \longrightarrow \tau A \longrightarrow \tau B \longrightarrow A \longrightarrow 0$$

By assumption, B is Ω-perfect, so since $A \longrightarrow B$ is a monomorphism we now that $\Omega^k \tau A \longrightarrow \Omega^k \tau B$ is a monomorphism for $k \geq 0$ by 4.1.3. The morphism $\tau B \longrightarrow A$ is an epimorphism, and since B is Ω-perfect we know that $\Omega^k \tau B \longrightarrow \Omega^k A$ is an epimorphism for all $k \geq 0$. So, in the case where $\alpha(A) = 1$, the module A is Ω-perfect.

Assume $\alpha(A) > 1$. We then have the following almost split sequence ending at A

$$0 \longrightarrow \tau A \longrightarrow \tau B \oplus C \longrightarrow A \longrightarrow 0$$

where C is non-zero, not necessarily indecomposable and is without projective summands. From previous arguments we know that an almost split sequence ending at $\Omega^n A$ for $n \geq 0$ is

$$0 \longrightarrow \Omega^n \tau A \longrightarrow \Omega^n \tau B \oplus \Omega^n C \longrightarrow \Omega^n A \longrightarrow 0.$$

That is, $\Omega^n \tau A \longrightarrow \Omega^n \tau B \oplus \Omega^n C$ is a monomorphism for all $n \geq 0$ and $\Omega^n \tau B \oplus \Omega^n C \longrightarrow \Omega^n A$ is an epimorphism for all $n \geq 0$. Further, since B is Ω-perfect we know that $\Omega^n \tau A \longrightarrow \Omega^n \tau B$ is a monomorphism for all $n \geq 0$ by 4.1.3. By

4.1.12 we then know that $\Omega^n C \longrightarrow \Omega^n A$ is a monomorphism for all $n \geq 0$. If $\tau B \longrightarrow A$ is a monomorphism, then since B is Ω-perfect, both $\tau^{k+1} A \longrightarrow \tau^{k+1} B$ and $\tau^{k+1} B \longrightarrow \tau^k A$ are monomorphisms for all $k \geq 0$ by 4.1.3, and we have an infinite sequence of proper monomorphisms

$$\cdots \hookrightarrow \tau^2 A \hookrightarrow \tau A \hookrightarrow A.$$

This is not possible, so $\tau B \longrightarrow A$ is an epimorphism and moreover $\Omega^n \tau B \longrightarrow \Omega^n A$ is an epimorphism for all $n \geq 0$ since B is Ω-perfect. Then, again using 4.1.12, we know that $\Omega^n \tau A \longrightarrow \Omega^n C$ is an epimorphism for all $n \geq 0$. That is, if $\alpha(A) = 2$ we are done.

We now write $C = D_1 \oplus D_2 \oplus ... \oplus D_r$ with $r \geq 2$, so

$$0 \longrightarrow \tau A \longrightarrow \tau B \oplus D_1 \oplus ... \oplus D_r \longrightarrow A \longrightarrow 0$$

is an almost split sequence ending at A where the D_i's are non-zero, not necessarily indecomposable and without projective summands. From previous arguments we know that $\tau A \longrightarrow \tau B$ is a monomorphism and $\tau B \longrightarrow A$ is an epimorphism. Then, by 4.1.14, we know that each map $\tau A \longrightarrow D_i$ is an epimorphism and each map $D_i \longrightarrow A$ is a monomorphism for $i \in \{1, ..., r\}$. Furthermore, we know that

$$0 \longrightarrow \Omega^n \tau A \longrightarrow \Omega^n \tau B \oplus \Omega^n D_1 \oplus ... \oplus \Omega^n D_r \longrightarrow \Omega^n A \longrightarrow 0$$

is an almost split sequence for $n \geq 0$, where $\Omega^n \tau B \longrightarrow \Omega^n A$ is an epimorphism, and $\Omega^n \tau A \longrightarrow \Omega^n \tau B$ is a monomorphism. So, by 4.1.14, $\Omega^n \tau A \longrightarrow \Omega^n D_i$ is an epimorphism and $\Omega^n D_i \longrightarrow \Omega^n A$ is a monomorphism for all $n \geq 0$ and $i \in \{1, ..., r\}$. That is, the morphisms $\tau A \longrightarrow D_i$ and $D_i \longrightarrow A$ for $i \in \{1, ..., r\}$ are also Ω-perfect.

It remains to argue that the maps $\tau A \longrightarrow X$ and $X \longrightarrow A$ are Ω-perfect, where $X \cong \tau B \oplus X'$ and the module X' is non-zero, not necessarily indecomposable. We write the almost split sequence ending at A

$$0 \longrightarrow \tau A \longrightarrow X \oplus D'_1 \oplus ... \oplus D'_s \longrightarrow A \longrightarrow 0$$

where τB is isomorphic to a summand of X, the D'_i's are non-zero, not necessarily indecomposable without projective summands and $s \geq 1$. Since, by a previous argument, $\tau B \longrightarrow A$ is an epimorphism, we know that $X \longrightarrow A$ is an epimorphism. Similarily, since we know that $\tau A \longrightarrow \tau B$ is a monomorphism, then $\tau A \longrightarrow X$ is a monomorphism. By 4.1.14 we then know that $\tau A \longrightarrow D'_i$ is an epimorphism and further $D_i' \longrightarrow A$ is a monomorphism for all $i \in \{1, ..., s\}$. From previous arguments by applying Ω a certain amount of times we get the almost split sequence

$$0 \longrightarrow \Omega^n \tau A \longrightarrow \Omega^n X \oplus \Omega^n D_1' \oplus ... \oplus \Omega^n D_s' \longrightarrow \Omega^n A \longrightarrow 0 \tag{4.3}$$

for all $n \geq 0$. From previous arguments we know that $\Omega^n \tau A \longrightarrow \Omega^n \tau B$ for $n \geq 0$ is a monomorphism, so $\Omega^n \tau A \longrightarrow \Omega^n X$ is a monomorphism for all $n \geq 0$. Further, $\Omega^n \tau B \longrightarrow \Omega^n A$ for $n \geq 0$ is an epimorphism, so $\Omega^n X \longrightarrow \Omega^n A$ is an epimorphism for all $n \geq 0$. That is, the morphisms $\tau A \longrightarrow X$ and $X \longrightarrow A$ are Ω-perfect. Then, applying 4.1.14 on the almost split sequence (4.3), we know that $\tau A \longrightarrow D_i'$ and $D_i' \longrightarrow A$ for $i \in \{1, ..., s\}$ are Ω-perfect as well.

So, in the case where $\alpha(A) > 1$ we have also shown that all possible irreducible morphisms ending at A and starting at τA are Ω-perfect.

In total, A is Ω-perfect.

□

We end this subchapter with a proposition which further explores irreducible monomorphisms ending at an Ω-perfect module.

Proposition 4.1.16. [7, Proposition 2.7] *Let R be a selfinjective Artin algebra and let A be Ω-perfect indecomposable module. If there exists an irreducible monomorphism to A, then there is a sequence of irreducible monomorphisms*

$$B_n \longrightarrow B_{n-1} \longrightarrow \cdots \longrightarrow B_1 \longrightarrow A$$

such that each B_i is indecomposable and $\alpha(B_n) = 1$.

Proof. Assume $A_1 \longrightarrow A$ is an irreducible monomorphism. Then there exists an indecomposable summand B_1 of A_1 such that $B_1 \longrightarrow A$ is an irreducible monomorphism. It then follows that B_1 is Ω-perfect by 4.1.15. If $\alpha(B_1) = 1$ we are done, so assume $\alpha(B_1) > 1$. Since A is Ω-perfect, by 4.1.3, we know that $\tau B_1 \longrightarrow \tau A$ is an irreducible monomorphism as well. We have an almost split sequence ending at B_1

$$0 \longrightarrow \tau B_1 \longrightarrow \tau A \oplus D_1 \oplus D_2 \oplus ... \oplus D_t \longrightarrow B_1 \longrightarrow 0$$

where each D_i is indecomposable and non-projective by 4.1.10. Further, by 4.1.14, all the maps $D_i \longrightarrow B_1$ for $i \in \{1, ..., t\}$ are monomorphisms. We may choose $D_i = B_2$ for one $i \in \{1, ..., t\}$. Note that we know there is at least one such D_i

since $\alpha(B_1) > 1$. Now if $\alpha(B_2) = 1$ we are done, or else we resume in the same manner. Since the length of the B_i's are decreasing as a result of us having proper monomorphisms, the result follows. $\square$

The class of Ω-perfect modules of finite complexity play an important part of Chapter 5. For instance, we investigate almost split sequences ending at such modules. Before we are ready to present the results in Chapter 5 we need to define eventually Ω-perfect modules. This is done in the next subchapter.

4.2 Eventually Ω-perfect modules

We now define eventually Ω-perfect morphisms and modules, and look at some results concerning this kind of modules. The last result in this subchapter proves especially important in Subchapter 5.2.

Definition 4.2.1. An irreducible map $g\colon B \longrightarrow C$ is an *eventually Ω-perfect morphism* if there exists an $n \geq 0$ such that $\Omega^m g\colon \Omega^m B \longrightarrow \Omega^m C$ is an epimorphism for all $m \geq n$ or a monomorphism for all $m \geq n$.

From this definition we get the following immediate result.

Proposition 4.2.2. *An irreducible morphism $B \longrightarrow C$ is an eventually Ω-perfect map if and only if the induced morphism $\Omega^n B \longrightarrow \Omega^n C$ do not change from epimorphisms to monomorphisms infinitely many times for $n \geq 0$.*

Proof. This result follows directly from the definition of eventually Ω-perfect morphisms. □

We now introduce eventually Ω-perfect modules.

Definition 4.2.3. [7] An indecomposable module C is an *eventually Ω-perfect module* if it is non-projective and $\Omega^n C$ is Ω-perfect for some $n \geq 0$.

We have the following result.

Proposition 4.2.4. *Let R be a selfinjective Artin algebra and let C be a non-projective R-module. If $\Omega^n C$ is Ω-perfect for some $n \geq 0$, then $\tau^n C$ is Ω-perfect as well.*

Proof. By assumption $\Omega^n C$ is Ω-perfect and then, using 4.1.10, so is $\Omega^{2n} C$. Furthermore, again applying 4.1.10, we know that $\nu^n \Omega^{2n} C$ is Ω-perfect. By 2.3.6, we know that $\nu^n \Omega^{2n} C \cong \tau^n C$, so we have our result. □

That is, if a module C is eventually Ω-perfect, we know that there exists an $n \geq 0$ such that $\tau^n C$ is Ω-perfect. As previously mentioned we want to look at components containing modules with special properties. The following result is the first of such kind and it gives us information about the structure of a component of the Auslander-Reiten quiver of an selfinjective Artin algebra containing a non-projective indecomposable module with complexity less than the complexity of each simple R-module.

Proposition 4.2.5. [6, Proposition 2.7] *Let R be a selfinjective Artin algebra and let $\mathcal{C}$ be a component of the Auslander-Reiten quiver containing a non-projective indecomposable module whose complexity is less than the complexity of every simple R-module. Then the component is a regular component, and every module lying in $\mathcal{C}$ is eventually Ω-perfect.*

Proof. By applying 3.2.5 we know that each non-projective module in $\mathcal{C}$ have complexity less than each simple R-module. We now assume P is a non-zero, projective-injective R-module in $\mathcal{C}$. Then there would exist an irreducible map $f\colon P\longrightarrow M$, for some indecomposable R-module M in $\mathcal{C}$. If f was a monomorphism, then it would split since P is injective, so f is an epimorphism. We also recall that M cannot be projective. From 2.1.12 we know that the only almost split sequence (up to isomorphism) with an indecomposable projective-injective module P in the middle term is

$$0 \longrightarrow JP \rightarrow P \oplus JP/S \rightarrow P/S \longrightarrow 0$$

where J is the Jacobson radical and S is the socle of P. Then $M \cong P/S$, and we have the exact sequence

$$0 \longrightarrow S \longrightarrow P \xrightarrow{f} P/S \longrightarrow 0.$$

The module P is indecomposable, so by [4, Proposition II.4.1], the socle of P, and then the kernel of f, is simple.

We now want to show that f is a projective cover. We first recall that P cannot be simple by 2.1.12. Furthermore, since $S \subseteq JP$, we know that $J\big(P/S\big) \cong JP/S$. Then $P/S/J(P/S) \cong P/JP$, so by [4, Proposition I.4.3], f is a projective cover.

So, $\Omega M \cong S$, and by using 3.2.2 we have that $\mathrm{cx}(M) = \mathrm{cx}(\Omega M) = \mathrm{cx}(S)$, a contradiction. That is, we cannot have any non-zero projective-injective modules in $\mathcal{C}$. That is, $\mathcal{C}$ is regular.

Now let

$$0 \longrightarrow \tau N \longrightarrow E_1 \oplus E_2 \oplus ... \oplus E_t \longrightarrow N \longrightarrow 0$$

be an almost split sequence for an arbitrary R-module N in the component where the E_i's are indecomposable. Furthermore, we assume $g\colon B\longrightarrow C$ is an irreducible morphism either ending at N or starting at τN. That is, the morphism g can be read such that B is a finite sum of some of the E_i's, each occurring at most once, and $C = N$ or $B = \tau N$ and C is a finite sum of some of the E_i's, each occurring at most once. Recall that none of the E_i's is projective since the component is regular.

We first assume g is an epimorphism. That is, we have the following exact sequence

$$0 \longrightarrow \operatorname{Ker} g \longrightarrow B \xrightarrow{g} C \longrightarrow 0.$$

Furthermore, we assume that the kernel of g is simple. By 3.2.2 (6), we know that $\operatorname{cx}(\operatorname{Ker} g) \leq \max\{\operatorname{cx}(B), \operatorname{cx}(C)\}$. Moreover, by 3.2.2 (5) and the fact that any module in the regular component has complexity less than the complexity of each simple R-module, we know that $\max\{\operatorname{cx}(B), \operatorname{cx}(C)\}$ will be strictly less than the complexity of each simple R-module. That is, we have a contradiction to the assumption that $\operatorname{Ker} g$ is simple. So, $\operatorname{Ker} g$ is not simple and therefore Ωg is an irreducible epimorphism by 4.1.7. Since $\operatorname{cx}(\Omega^j X) = \operatorname{cx}(X)$ for all non-projective R-modules X and $j \geq 0$ by 3.2.2, using the same argument we see that the kernel of Ωg cannot be simple either, so $\Omega^2 g$ is an epimorphism as well. Repeating the argument we get that $\Omega^m g$ is an epimorphism for all $m \geq 0$. That is, g is an Ω-perfect epimorphism.

Now, assume g is a monomorphism. If $\Omega^m g$ is a monomorphism for all $m \geq 0$, we know that g is eventually Ω-perfect, in particular it is Ω-perfect. So assume there exists an $n \geq 0$ such that $\Omega^n g$ is an irreducible epimorphism. Then, repeating the argument above, $\Omega^l g$ is an epimorphism for all $l \geq n$, so g is eventually Ω-perfect.

We have shown that if g is an epimorphism, then $\Omega^m g$ is an epimorphism for all $m \geq 0$. Moreover, if g is a monomorphism, then either $\Omega^m g$ is a monomorphism for all $m \geq 0$ or there exists an $n \geq 0$ such that $\Omega^l g$ is an epimorphism for all $l \geq n$. That is, there exists an even integer $k \geq 0$ such that applying Ω^k to each of the different irreducible morphisms ending at N and starting at τN gives Ω-perfect morphisms. By 2.3.8 and 2.3.13 combined with the fact that the component is regular we know that an almost split sequence ending at $\Omega^k N$ for the same even integer k is

$$0 \longrightarrow \Omega^k \tau N \longrightarrow \Omega^k E_1 \oplus \Omega^k E_2 \oplus \ldots \oplus \Omega^k E_t \longrightarrow \Omega^k N \longrightarrow 0.$$

That is, all the irreducible morphisms ending at the non-projective module $\Omega^k N$ and starting at $\Omega^k \tau N$ are Ω-perfect. So, $\Omega^k N$ is Ω-perfect and furthermore, N is eventually Ω-perfect. □

In the next proposition we also look at some conditions that guarantees us having Ω-perfect modules.

Proposition 4.2.6. [7, Proposition 2.4] *Let R be a selfinjective Artin algebra having no Ω-periodic simple modules. Then every indecomposable non-projective R-module is eventually Ω-perfect.*

Proof. Let C be an indecomposable non-projective module. We now want to show that each irreducible morphism $g\colon B\longrightarrow C$, where B is non-projective, not necessarily indecomposable, is eventually Ω-perfect. Note that we neglect the case when B is projective, because then there would not be a non-zero map $\Omega^n B\longrightarrow\Omega^n C$ or $\Omega^n\tau C\longrightarrow\Omega^n B$ for any $n\geq 1$. Assume g is not eventually Ω-perfect. Then, the maps $\Omega^n g$ would have to change from epimorphisms to monomorphisms infinitely many times by 4.2.2. Furthermore, by 4.1.7 and the fact that we have a finite number of simple modules, we have a simple module S and two positive integers m and n with $m<n$ and the following diagram

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r] \arrow[d, equal] & \Omega^n B \arrow[r, "\Omega^n g"] & \Omega^n C \arrow[r] & 0 \\
0 \arrow[r] & S \arrow[r] & \Omega^m B \arrow[r, "\Omega^m g"] & \Omega^m C \arrow[r] & 0
\end{tikzcd}$$

Further, we let $k=n-m$ and apply Ω^k to the bottom row in the diagram. We then obtain the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & S \arrow[r] \arrow[d, "h"] & \Omega^n B \arrow[r, "\Omega^n g"] \arrow[d, "(0,1)^T"] & \Omega^n C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega^k S \arrow[r] & P\oplus\Omega^n B \arrow[r, "(f,\Omega^n g)"] & \Omega^n C \arrow[r] & 0
\end{tikzcd}$$

Then, by Snake Lemma [13, Theorem 6.5], we have the following exact sequence

$$0\longrightarrow S\longrightarrow\Omega^k S\longrightarrow P\longrightarrow 0.$$

This implies that $P=(0)$, or else the sequence would split and $\Omega^k S$ would have a projective summand, a contradiction. So, then $\Omega^k S\cong S$ again a contradiction to S not being Ω-periodic. That is, g is eventually Ω-perfect. In the same manner, we can show that every irreducible morphisms $f\colon\tau C\longrightarrow B$, where B is non-projective and not necessarily indecomposable, is eventually Ω-perfect.

Now, we have an almost split sequence ending at C

$$0\longrightarrow\tau C\longrightarrow E_1\oplus ...\oplus E_t\oplus P'\longrightarrow C\longrightarrow 0$$

where E_i is non-projective, indecomposable and P' is projective. By repeating 2.3.8 we have an almost split sequence

$$0 \longrightarrow \Omega^m \tau C \longrightarrow \Omega^m E_1 \oplus ... \oplus \Omega^m E_t \oplus P_m \longrightarrow \Omega^m C \longrightarrow 0$$

ending at $\Omega^m C$ for an $m \geq 1$ where P_m is indecomposable projective and $\Omega^m E_i \neq (0)$ for $i \in \{1, ..., t\}$. From the previous argument we may assume that the irreducible morphisms from a finite sum of the $\Omega^m E_i$'s, each represented at most once, to $\Omega^m C$ are Ω-perfect. Similarily, we may assume that the irreducible morphisms from $\Omega^m \tau C$ to any finite sum of the $\Omega^m E_i$'s, each represented at most once, are Ω-perfect. If it was not for the fact that we might have a non-zero projective middle term, we would know that C is eventually Ω-perfect.

If $P_m = (0)$, then all irreducible morphisms ending at $\Omega^m C$ and starting at $\tau \Omega^m C$ are Ω-perfect and then C is eventually Ω-perfect and we are done. So, we assume $P_m \neq (0)$. Then, by 2.3.10 we know that $\Omega^{m+1} C$ is simple. We now apply Ω to the previous sequence and we get the almost split sequence

$$0 \longrightarrow \Omega^{m+1} \tau C \longrightarrow \Omega^{m+1} E_1 \oplus ... \oplus \Omega^{m+1} E_t \oplus P_{m+1} \longrightarrow \Omega^{m+1} C \longrightarrow 0$$

where P_{m+1} is projective. If $P_{m+1} = (0)$, we are done. Or else, $\Omega^{m+2} C$ is simple. We can now repeat this argument. We know that there is only a finite number of projective modules. That is, if there is no $i \geq m$ such that $P_i = (0)$ and we would have been done, there must exist a projective module P'' such that it is a summand in the middle term up to isomorphism in two almost split sequences ending at $\Omega^r C$ and $\Omega^s C$ where $r > s > m$. But, then $\Omega^r C$ and $\Omega^s C$ are both simple and isomorphic to P''/S, where S is the socle of P'' by 2.1.12. That is, they are isomorphic and this contradicts the assumption that no simple modules are Ω-periodic.

So, there exists an almost split sequence

$$0 \longrightarrow \Omega^k \tau C \longrightarrow \Omega^k E_1 \oplus ... \oplus \Omega^k E_t \longrightarrow \Omega^k C \longrightarrow 0$$

for some $k \geq 0$ where all the irreducible morphisms from $\Omega^k \tau C$ to a finite sum of the $\Omega^k E_i$'s where each $\Omega^k E_i$ occurs at most once are Ω-perfect and all the irreducible morphims from any such sum of $\Omega^k E_i$'s to $\Omega^k C$ are Ω-perfect. That is, C is eventually Ω-perfect.

□

The next lemma is known from graph theory. Its proof is fairly simple, but is not presented here.

Lemma 4.2.7. [6, Lemma 2.9] *Let $\mathcal{G}$ be a finite directed graph having n vertices, and assume that there is at least one arrow between any two vertices in $\mathcal{G}$. Then, there exists a directed path in $\mathcal{G}$ of length greater than or equal to $n-1$.*

Proof. See [8, Theorem 11.7]. □

As previously mentioned, we conclude this chapter with a proposition that proves important in Subchapter 5.2 where we look at regular components of Auslander-Reiten quivers of selfinjective Artin algebras containing a module of complexity 1. Recall that a regular component of a connected (selfinjective) Artin algebra containing a τ-periodic module is of type $\mathbb{Z}A_\infty/\langle\tau^n\rangle$, a tube, by Subchapter 2.2.

Proposition 4.2.8. [6, Proposition 2.10] *Let R be a selfinjective Artin algebra and let $\mathcal{C}$ be a regular component of the Auslander-Reiten quiver of R that is not a tube. Let $M \in \mathcal{C}$ be a module of complexity 1. Then, there exists a positive integer n such that $\tau^n M$ is Ω-perfect.*

Proof. Since $\mathcal{C}$ is regular and contains a module of complexity 1 we know that all modules in the component have complexity 1 by 3.2.5. We let

$$0 \longrightarrow \tau M \xrightarrow{(f_1,\ldots,f_k)^T} E_1 \oplus \ldots \oplus E_k \xrightarrow{(g_1,\ldots,g_k)} M \longrightarrow 0 \tag{4.4}$$

be an almost split sequence ending at M where each E_i is non-zero and indecomposable. The component is regular, so none of the E_i's are projective. We prove that any irreducible morphism from τM to a finite sum of the E_i's, with each E_i occuring at most once, and any irreducible morphism from a finite sum of the E_i's, each occuring at most once, to M is eventually Ω-perfect and argue that this is enough. We begin by showing that if $g\colon B \longrightarrow C$ is an irreducible morphism where both B and C are indecomposable, then there exists an $l \geq 0$ such that $\Omega^l g$ is Ω-perfect.

We assume g is not eventually Ω-perfect and B and C are indecomposable. Then, for an infinite number of positive integers j, $\Omega^j g$ is an epimorphism, but $\Omega^{j+1} g$ is a monomorphism. Then, by 4.1.7, the kernel of $\Omega^j g$ is simple. We only have a finite number of simple R-modules, so we have a strictly increasing sequence of integers, $n_1, n_2, \ldots$ where we for each n_i have a short exact sequence

$$0 \longrightarrow S \longrightarrow \Omega^{n_i} B \xrightarrow{\Omega^{n_i} g} \Omega^{n_i} C \longrightarrow 0$$

for some simple R-module S. We have the following exact diagram for each $n_i \neq n_j$.

$$\begin{tikzcd} 0 \arrow[r] & S \arrow[r, "k_i"] \arrow[d, equal] & \Omega^{n_i}B \arrow[r, "\Omega^{n_i}g"] \arrow[d, dashed, shift right, "f_i^j"'] & \Omega^{n_i}C \arrow[r] & 0 \\ 0 \arrow[r] & S \arrow[r, "k_j"] & \Omega^{n_j}B \arrow[r, "\Omega^{n_j}g"] \arrow[u, dashed, shift right, "f_j^i"'] & \Omega^{n_j}C \arrow[r] & 0 \end{tikzcd}$$

We know that the maps $\Omega^{n_i}g$ and $\Omega^{n_j}g$ are irreducible, so by [4, Proposition V.5.7] there exists a homomorphism ${f_j}^i \colon \Omega^{n_j}B \longrightarrow \Omega^{n_i}B$ or a homomorphism ${f_i}^j \colon \Omega^{n_i}B \longrightarrow \Omega^{n_j}B$ such that the left square of the above diagram commutes. Relabeling, if needed, we may use 4.2.7 and we can assume there is an arbitrary long chain of homomorphisms

$$\Omega^{n_1}B \xrightarrow{f_1^2} \Omega^{n_2}B \xrightarrow{f_2^3} \cdots \xrightarrow{f_{m-1}^m} \Omega^{n_m}B \xrightarrow{f_m^{m+1}} \cdots$$

The compositions $f_m^{m+1} f_{m-1}^m ... f_1^2$ are all non-zero, since composing them with k_1 gives $(f_m^{m+1}...f_2^3 f_1^2)k_1 = (f_m^{m+1}...f_2^3)\underbrace{(f_1^2 k_1)}_{k_2} = ... = k_{m+1} \neq 0$.

By 3.2.3, we know that there is a common bound of the length of all the syzygies of B. We have an arbitrary long chain of non-zero composition between indecomposable syzygies, and since their lengths have a common bound this cannot be if each ${f_i}^j$ is not an isomorphism by [4, Corollary VI.1.3]. Hence, for some $i \neq j$, we have that ${f_i}^j$ is an isomorphism, and $\Omega^{n_i}B$ and $\Omega^{n_j}B$ are isomorphic. That is, B is Ω-periodic.

We want to show that B is τ-periodic, since this would imply that $\mathcal{C}$ is a tube by Subchapter 2.2. We first show that there exists a positive integer k such that $\nu^k S \cong S$. Since there is only a finite number of simple modules and ν preserves simple modules by 2.3.4 there must exist a simple R-module $S' \cong \nu^l S$ for $l \geq 0$ such that $\nu^k S' \cong S'$ for some $k \geq 1$. Then, we have that

$$\begin{aligned} \nu^l(S) &\cong \nu^{l+k}(S) \\ \nu^{-l}(\nu^l(S)) &\cong \nu^{-l}(\nu^{l+k}(S)) \\ S &\cong \nu^k(S). \end{aligned}$$

That is, ν has finite order k when applied to S. We now let n be a positive integer such that we have a short exact sequence

$$0 \longrightarrow S \xrightarrow{l} \Omega^n B \xrightarrow{\Omega^n g} \Omega^n C \longrightarrow 0.$$

The Nakayama functor, ν, takes irreducible maps to irreducible maps by 2.3.4, so for all integer multiples kt_j and kt_i where $t_i \neq t_j$ we have the following exact diagram

$$\begin{tikzcd} 0 \arrow[r] & S \arrow[r] \arrow[d, equal] & \nu^{kt_i}\Omega^n B \arrow[r, "\nu^{kt_i}\Omega^n g"] \arrow[d, dashed, shift right, "l_i^j"'] & \nu^{kt_i}\Omega^n C \arrow[r] & 0 \\ 0 \arrow[r] & S \arrow[r] & \nu^{kt_j}\Omega^n B \arrow[r, "\nu^{kt_j}\Omega^n g"] \arrow[u, dashed, shift right, "l_j^i"'] & \nu^{kt_j}\Omega^n C \arrow[r] & 0 \end{tikzcd}$$

Again, by [4, Proposition V.5.7], for each $t_i \neq t_j$ we have homomorphisms ${l_i}^j$ or ${l_j}^i$ commuting the left square of the diagram. Repeating the previous argument we get arbitrary long chains of homomorphims with non-zero composition between modules with the same length as $\nu^{kt_i}\Omega^n B$, since ν preserves length. By [4, Corollary VI.1.3], for some $t_r \neq t_s$, we have that $\nu^{kt_r}\Omega^n B$ is isomorphic to $\nu^{kt_s}\Omega^n B$. Then using [4, Proposition IV.3.6] and the fact that Ω and ν commute by 2.3.6, we know that $\nu^{kt_r}B \cong \nu^{kt_s}B$, so ν has finite order when applied to B. This fact combined with B being Ω-periodic shows that B is τ-periodic. This contradicts the fact that $\mathcal{C}$ is not a tube.

So, $g\colon B \longrightarrow C$ is eventually Ω-perfect. So, the morphims f_i and g_i for $i \in \{1, ..., k\}$ in (4.4) are eventually Ω-perfect.

The previous argument also holds in the case where C decompose. That is, any irreducible morphism from τM to a finite sum of the E_i's, where each E_i is represented at most once, is eventually Ω-perfect. For simplicity, we denote such a morphism by $(f_{h_1}, ..., f_{h_n})^T$. So, for a morphism of that sort, $(f_{h_1}, ..., f_{h_n})^T$, there exists an even integer $2m' \geq 0$ such that $\Omega^{2m'}(f_{h_1}, ..., f_{h_n})^T$ is Ω-perfect. So,

$$\Omega^s\Omega^{2m'}(f_{h_1}, ..., f_{h_n})^T : \Omega^s\Omega^{2m'}\tau M \longrightarrow \Omega^s\Omega^{2m'}E_{h_1} \oplus ... \oplus \Omega^s\Omega^{2m'}E_{h_n}$$

is either an epimorphism for all $s \geq 0$ or a monomorphism for all $s \geq 0$. We now choose m such that it is the maximum of all the m''s corresponding to the morphisms $(f_{h_1}, ..., f_{h_n})^T$. So, applying Ω^{2m} to any of the morphisms $(f_{h_1}, ..., f_{h_n})^T$ gives us an Ω-perfect morphism. Furthermore, since

$$\ell(\Omega^s\Omega^{2m}\tau M) = \ell(\Omega^s\tau^m\tau M)$$

and

$$\ell(\Omega^s\Omega^{2m}E_{h_1}) + ... + \ell(\Omega^s\Omega^{2m}E_{h_n}) = \ell(\Omega^s\tau^m E_{h_1}) + ... + \ell(\Omega^s\tau^m E_{h_n})$$

by 2.3.4 and 2.3.6, we know that an irreducible morphism

$$\Omega^s\tau^m\tau M \longrightarrow \Omega^s\tau^m E_{h_1} \oplus ... \oplus \Omega^s\tau^m E_{h_n}$$

is either an epimorphism for all $s \geq 0$ or a monomorphism for all $s \geq 0$. Furthermore, we explore the almost split sequence ending at $\tau^m M$. By (4.4), 2.3.11 and the fact that $\mathcal{C}$ is regular, we get an almost split sequence ending at $\tau^m M$

$$0 \longrightarrow \tau^{m+1}M \xrightarrow{(\tau^m f_1,...,\tau^m f_k)^T} \tau^m E_1 \oplus ... \oplus \tau^m E_k \xrightarrow{(\tau^m g_1,...,\tau^m g_k)} \tau^m M \longrightarrow 0.$$

By our choice of m and the previous argument, we know that any irreducible morphims from $\tau^{m+1}M$ to a finite sum of the $\tau^m E_i$'s, with each occuring at most once, is Ω-perfect. In particular, we know that the morphism $\Omega\tau^{m+1}M \longrightarrow \Omega\tau^m E_1 \oplus ... \oplus \Omega\tau^m E_k$ is a monomorphism. By 2.3.8 we get another almost split sequence

$$0 \longrightarrow \Omega\tau^{m+1}M \longrightarrow \Omega\tau^m E_1 \oplus ... \oplus \Omega\tau^m E_k \oplus P \longrightarrow \Omega\tau^m M \longrightarrow 0$$

where P is projective. If $P \neq (0)$ we know that the irreducible morphism $P \longrightarrow \Omega\tau^m M$ is an epimorphim, or else it would split. But then, by 4.1.12, the morphism $\Omega\tau^{m+1}M \longrightarrow \Omega\tau^m E_1 \oplus ... \oplus \Omega\tau^m E_k$ is an epimorphism, a contradiction. That is, $P = (0)$. Repeating this argument, for any $v \geq 0$ we know that

$$0 \rightarrow \Omega^v\tau^{m+1}M \xrightarrow{(\Omega^v\tau^m f_1,...,\Omega^v\tau^m f_k,f')^T} \Omega^v\tau^m E_1 \oplus ... \oplus \Omega^v\tau^m E_k \xrightarrow{(\Omega^v\tau^m g_1,...,\Omega^v\tau^m g_k,g')} \Omega^v\tau^m M \longrightarrow 0$$

is an almost split sequence. Then, since any irreducible morphism from $\tau^{m+1}M$ to a finite sum of the $\tau^m E_i$'s, with each occuring at most once, say $(\tau^m f_{h_1}, ..., \tau^m f_{h_n})$, is Ω-perfect we know that $\Omega^v(\tau^m f_{h_1}, ..., \tau^m f_{h_n})$ is either a monomorphism for all $v \geq 0$ or an epimorphism for all $v \geq 0$. Then, by 4.1.12, we know that any irreducible morphism from a finite sum of $\tau^m E_i$'s, where each is represented at most once, to $\tau^m M$, say $(\tau^m g_{k_1}, ..., \tau^m g_{k_u})$, is such that $\Omega^v(\tau^m g_{k_1}, ..., \tau^m g_{k_u})$ is either a monomorphism for all $v \geq 0$ or an epimorphism for all $v \geq 0$.

That is, $\tau^m M$ is Ω-perfect. □

We are now ready to look at modules of finite complexity.

Chapter 5

Finite complexity

In this chapter we present some results concerning finite complexity. It is divided in two parts where the first one concerns general finite complexity and the latter presents some results for complexity one. As before, R is a connected selfinjective Artin algebra and all R-modules are finitely generated.

5.1 Finite complexity

In this subchapter we investigate properties of almost split sequences ending at an Ω-perfect module of finite complexity. The main result of this part of the thesis is that if C is an Ω-perfect module of finite complexity over a selfinjective Artin algebra, then $\alpha(C) \leq 4$. We begin by looking at the lengths of two indecomposable modules which are connected by an irreducible morphism. Recall that all modules are left R-modules unless stated otherwise.

Lemma 5.1.1. [7, Lemma 3.1] *Let R be a selfinjective Artin algebra and let $f\colon M \longrightarrow N$ be an irreducible map between indecomposable R-modules. If $d' = \max\{\ell(Re_i)\}$, the maximum length of the indecomposable projective R-modules, then $\ell(M) \geq \frac{1}{1+d'^2}\ell(N)$.*

Proof. The algebra R is a selfinjective Artin algebra, so $\ell(P) \leq d'$ for each indecomposable projective right R-module P as well. The result now follows from [4, Proposition V.6.6].

□

In the upcoming proofs we need that the previous statement also applies when M decomposes. We therefore state this as a corollary.

Corollary 5.1.2. *Let R be a selfinjective Artin algebra and let $f\colon M \longrightarrow N$ be an irreducible map where N is indecomposable and M is not necessarily indecomposable. If $d' = \max\{\ell(Re_i)\}$, the maximum length of the indecomposable projective R-modules, then $\ell(M) \geq \frac{1}{1+d'^2}\ell(N)$*

Proof. The module M is not necessarily indecomposable, so $M \cong M' \oplus M''$ where M' is indecomposable and M'' may or may not be zero. By 2.1.7, we know that the morphism $M' \longrightarrow N$ is irreducible and moreover, by 5.1.1, we have that $\ell(M') \geq \frac{1}{1+d'^2}\ell(N)$, where $d' = \max\{\ell(Re_i)\}$, the maximum length of the indecomposable projective R-modules. That is, $\ell(M) \geq \ell(M') \geq \frac{1}{1+d'^2}\ell(N)$ which is what we wanted to show. $\square$

The previous corollary is used in the next proposition.

Proposition 5.1.3. [7, Proposition 3.2] *Let R be a selfinjective Artin algebra. Suppose that C is an Ω-perfect module of finite complexity. Let*

$$0 \longrightarrow \tau C \xrightarrow{(f_1, f_2, \dots, f_t)^T} E_1 \oplus E_2 \oplus \dots \oplus E_t \xrightarrow{(g_1, g_2, \dots, g_t)} C \longrightarrow 0$$

be an almost split sequence, where the E_i's are non-zero modules that are not necessarily indecomposable. Then, if $t \geq 3$, at most one of the g_i's can be an epimorphism.

Proof. Assume both g_1 and g_2 are epimorphisms and recall that E_1, E_2 and E_3 are non-zero by assumption. Then $(g_1, g_4, \dots, g_t)\colon E_1 \oplus E_4 \oplus \dots \oplus E_t \longrightarrow C$ is an epimorphism as well. Further, by 4.1.12, this implies that the map $(f_2, f_3)^T\colon \tau C \longrightarrow E_2 \oplus E_3$ is an epimorphism. Since $(f_2, f_3)^T$ is an epimorphism we have that $\ell(\tau C) > \ell(E_2) + \ell(E_3)$. The morphism g_2 is an epimorphism by assumption, so we have that $\ell(E_2) > \ell(C)$.

Using 5.1.2 and the two previous strict inequalities we get that

$$\ell(\tau C) > \ell(E_2) + \ell(E_3) > \ell(C) + \frac{1}{1+d'^2}\ell(C) = \Big(1 + \frac{1}{d'^2+1}\Big)\ell(C).$$

The Nakayama functor ν preserves length by 2.3.4, so this and 2.3.5 give us that $\ell(\tau C) = \ell(\Omega^2 C)$. For simplicity we denote $c = \Big(1 + \frac{1}{d'^2+1}\Big)$, and further $\ell(\Omega^2 C) > c \cdot \ell(C)$, where $c > 1$.

The module C is Ω-perfect, so by 4.1.3 we know that $\tau^{n-1}\tau C \longrightarrow \tau^{n-1}E_2 \oplus \tau^{n-1}E_3$ is an epimorphism for all $n \geq 0$. And also, $\tau^{n-1}E_2 \longrightarrow \tau^{n-1}C$ is an epimorphism for all $n \geq 1$. That is, we may repeat the previous argument and get the following

$$\ell(\Omega^{2n}C) = \ell(\tau^n C) > c \cdot \ell(\tau^{n-1}C) > c^2 \cdot (\tau^{n-2}C) > ... > c^n \cdot \ell(C)$$

for $n \geq 1$. This tells us that the length of the even power syzygies increase exponentially.

Further, by 3.1.2, $\ell(\Omega^i C) < d' \cdot \beta_i(C)$ for $i \geq 0$ where d' is as before. So, the growth of the Betti numbers cannot be bounded by a polynomial when the length of the even power syzygies grows exponentially. This contradicts the fact that C has finite complexity. □

We now want to decide the possible shapes of a component of the Auslander-Reiten quiver of a selfinjective Artin algebra that contains one module of finite complexity and where all modules are eventually Ω-perfect and have more than one indecomposable module in a chosen decomposition of the middle term of an almost split sequence ending at them. Before we can prove the result we need to define sectional paths. Given a translation quiver Γ, a path $(x_0 \mid \alpha_1, ..., \alpha_t \mid x_t)$ in Γ is said to be a *sectional path* provided $\tau x_{i+1} \neq x_{i-1}$, for all $1 < i < t$, where $\alpha_i \colon x_{i-1} \longrightarrow x_i$, [12].

Proposition 5.1.4. [7, Corollary 3.3] *Let R be a selfinjective Artin algebra, and let $\mathcal{C}$ be a component of the Auslander-Reiten quiver of R such that every module in $\mathcal{C}$ is eventually Ω-perfect and $\mathcal{C}$ contains a module having finite complexity. If every $M \in \mathcal{C}$ has the property that $\alpha(M) > 1$, then $\mathcal{C}$ is of type $\mathbb{Z}\tilde{A}_{1,2}$, $\mathbb{Z}B_\infty$, $\mathbb{Z}\tilde{B}_n$, $\mathbb{Z}A_\infty^\infty$, or a quotient of the last two.*

Proof. The modules in $\mathcal{C}$ are all eventually Ω-perfect, so by definition they are non-projective. Then, by 3.2.5, all the modules in $\mathcal{C}$ have finite complexity. Furthermore, we know that for all $C \in \mathcal{C}$ there exists an $n \geq 0$ such that $\tau^n C$ is Ω-perfect by 4.2.4. We can therefore assume that C is an Ω-perfect module in $\mathcal{C}$. Now, assume $\alpha(C) \geq 3$. Then, by 5.1.3, we have an irreducible monomorphism $B \longrightarrow C$. So, by 4.1.16, we know that there exists a sequence of irreducible monomorphisms

$$B_n \longrightarrow B_{n-1} \longrightarrow \cdots \longrightarrow B_1 \longrightarrow C$$

such that each B_i is indecomposable and $\alpha(B_n) = 1$. But then, $B_n \in \mathcal{C}$, a contradiction to the assumption that $\alpha(M) > 1$ for all $M \in \mathcal{C}$. So, $\alpha(C) = 2$ for all Ω-perfect modules in $\mathcal{C}$. That is, an almost split sequence ending at C is

$$0 \longrightarrow \tau C \longrightarrow E_1 \oplus E_2 \longrightarrow C \longrightarrow 0$$

where E_1, E_2 are non-zero, non-projective, indecomposable, and not necessarily non-isomorphic. We know that for any module in $\mathcal{C}$ we have an $n \geq 0$ such that $\tau^n C$ is Ω-perfect. Therefore, since the function α is invariant under τ by 2.3.12, we have that $\alpha(M) = 2$ for all $M \in \mathcal{C}$. In fact, since the component is regular, if M is a module in the component, then an almost split sequence ending at M has exactly two indecomposable summands in a chosen decomposition of the middle term.

We now evaluate the possible structures of the component. We have two alternatives. Either there is at least one module $M' \in \mathcal{C}$ such that the almost split sequence ending at M' (up to isomorphism) has two indecomposable isomorphic summands in a chosen decomposition of the middle term or all the modules in the component have almost split sequences ending at them with two indecomposable non-isomorphic modules in a chosen decomposition of the middle term. Before we look at these two cases seperately, we argue that no module in the component is τ-periodic. If there was a τ-periodic module in the regular component this would imply that the component was of type $\mathbb{Z}A_\infty/\langle\tau^n\rangle$ by Subchapter 2.2. This cannot be the case since this again would imply that there exists a module M in the component with $\alpha(M) = 1$, a contradiction to the fact that $\alpha(M) = 2$ for all $M \in \mathcal{C}$.

Assume there is a module $M' \in \mathcal{C}$ such that the almost split sequence ending at M' (up to isomorphism) has two indecomposable isomorphic summands in a chosen decomposition of its middle term. Since $\alpha(M) = 2$ for all $M \in \mathcal{C}$ and the component is regular, we have an almost split sequence ending at M'

$$0 \longrightarrow \tau M' \longrightarrow E_1 \oplus E_1 \longrightarrow M' \longrightarrow 0. \tag{5.1}$$

So, the arrow $[E_1]\longrightarrow[M']$ has valuation $a_{E_1,M'} = 2$. From the almost split sequence (5.1) we get an almost split sequence

$$0 \longrightarrow \tau^{n+1}M' \longrightarrow \tau^n E_1 \oplus \tau^n E_1 \longrightarrow \tau^n M' \longrightarrow 0 \tag{5.2}$$

with $\tau^n E_1 \neq (0)$ for all $n \in \mathbb{Z}$, since the component is regular and by 2.3.11. That is, $a_{\tau^n E_1,\tau^n M'} = 2$ for all $n \in \mathbb{Z}$. Since $\alpha(\tau^{-1}E_1) = 2$ there are two possibilties for the value of $a'_{E_1,M'} (= a_{M',\tau^{-1}E_1})$, that is, (1) $a'_{E_1,M'} = 1$ or (2) $a'_{E_1,M'} = 2$.

(1) Now, assume $a'_{E_1,M'} = 1$. Then, we know that $a'_{\tau^n E_1,\tau^n M'} = 1$ for all $n \in \mathbb{Z}$, by similar arguments as above. So, then by (5.2) and Subchapter 2.2, we know

that we have a boundary for the component with valuation as illustrated below. So, the following is a part of the component.

$$\begin{tikzcd}[column sep=small]
\cdots & {} \arrow[dr, "{(2,1)}"] & & {[\tau E_1]} \arrow[dr, "{(2,1)}"] & & {[E_1]} \arrow[dr, "{(2,1)}"] & & {[\tau^{-1} E_1]} \arrow[dr, "{(2,1)}"] & & {} & \cdots \\
 & & {[\tau^2 M']} \arrow[ur, "{(1,2)}"] & & {[\tau M']} \arrow[ur, "{(1,2)}"] & & {[M']} \arrow[ur, "{(1,2)}"] & & {[\tau^{-1} M']} \arrow[ur, "{(1,2)}"] & &
\end{tikzcd}$$

This cannot be the entire component, because that would imply that the almost split sequence ending at $\tau^n E_1$ for $n \in \mathbb{Z}$ has only one indecomposable middle term. That is, an almost split sequence ending at E_1 has an indecomposable middle term X which is not isomorphic to $\tau M'$ and these two modules are the only indecomposable summands of the middle term

$$0 \longrightarrow \tau E_1 \longrightarrow \tau M' \oplus X \longrightarrow E_1 \longrightarrow 0.$$

So, the arrow $[X] \longrightarrow [E_1]$ has valuation $a_{X,E_1} = 1$. Again, since $\alpha(M) = 2$ for all $M \in \mathcal{C}$, we have two possibilities for the value of a'_{X,E_1}, that is, $a'_{X,E_1} = 1$ or $a'_{X,E_1} = 2$.

If $a'_{X,E_1} = 2$, then by previous arguments we have that $a'_{\tau^n X, \tau^n E_1} = 2$ for all $n \in \mathbb{Z}$ and furthermore we have the following

$$\begin{tikzcd}[column sep=small]
\cdots & & {[\tau X]} \arrow[dr, "{(1,2)}"] & & {[X]} \arrow[dr, "{(1,2)}"] & & {[\tau^{-1} X]} \arrow[dr, "{(1,2)}"] & & {[\tau^{-2} X]} \arrow[dr, "{(1,2)}"] & & \cdots \\
 & {} \arrow[ur, "{(2,1)}"] & & {[\tau E_1]} \arrow[ur, "{(2,1)}"] & & {[E_1]} \arrow[ur, "{(2,1)}"] & & {[\tau^{-1} E_1]} \arrow[ur, "{(2,1)}"] & & {} & \\
 & {} \arrow[dr, "{(2,1)}"] & & {[\tau E_1]} \arrow[dr, "{(2,1)}"] & & {[E_1]} \arrow[dr, "{(2,1)}"] & & {[\tau^{-1} E_1]} \arrow[dr, "{(2,1)}"] & & {} & \\
\cdots & & {[\tau^2 M']} \arrow[ur, "{(1,2)}"] & & {[\tau M']} \arrow[ur, "{(1,2)}"] & & {[M']} \arrow[ur, "{(1,2)}"] & & {[\tau^{-1} M']} \arrow[ur, "{(1,2)}"] & & \cdots
\end{tikzcd}$$

Since $\alpha(M) = 2$ for all $M \in \mathcal{C}$ and the component is regular we cannot have anything else. That is, the component is of type $\mathbb{Z}\tilde{B}_2$ or possibly a quotient of it.

If $a'_{X,E_1} = 1$, as before, we know that there must exist an indecomposable Y such that

$$0 \longrightarrow X \longrightarrow E_1 \oplus Y \longrightarrow \tau^{-1} X \longrightarrow 0$$

is an almost split sequence ending at $\tau^{-1}X$. So, then the arrow $[Y] \longrightarrow [\tau^{-1}X]$ has valuation $a_{Y,\tau^{-1}X} = 1$. Moreover, we then also have that $a_{\tau^{n+1}Y,\tau^n X} = 1$

for all $n \in \mathbb{Z}$. Again, we have two possibilities for the value of $a'_{Y,\tau^{-1}X}$, that is, either $a'_{Y,\tau^{-1}X} = 1$ or $a'_{Y,\tau^{-1}X} = 2$. Repeating this argument we get a component of type $\mathbb{Z}\tilde{B}_n$, a quotient of $\mathbb{Z}\tilde{B}_n$, or $\mathbb{Z}B_\infty$. Note, we cannot have a quotient of $\mathbb{Z}B_\infty$ since this would imply that a module on the boundary is identified with another module in the same τ-orbit, not possible.

(2) Now, assume $a'_{E_1,M'} = 2$. Then, we have an almost split sequence

$$0 \longrightarrow E_1 \longrightarrow M' \oplus M' \longrightarrow \tau^{-1}E_1 \longrightarrow 0$$

and using similar arguments as before we get an almost split sequence

$$0 \longrightarrow \tau^n E_1 \longrightarrow \tau^n M' \oplus \tau^n M' \longrightarrow \tau^{n-1}E_1 \longrightarrow 0 \tag{5.3}$$

for every $n \in \mathbb{Z}$. So, we know that $a_{\tau^n E_1,\tau^n M'} = 2$ and $a'_{\tau^n E_1,\tau^n M'} = 2$ for all $n \in \mathbb{Z}$, by respectively (5.2) and (5.3). Then, from Subchapter 2.2 and the fact that $\alpha(M) = 2$ for all modules in the regular component, the valuation is $(2,2)$ for any arrow in the quiver and the component cannot contain a module that is not in the same τ-orbit as either M' or E_1. That is, we have the following

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[dr, "{(2,2)}"] & & {[\tau E_1]} \arrow[dr, "{(2,2)}"] & & {[E_1]} \arrow[dr, "{(2,2)}"] & & {[\tau^{-1}E_1]} \arrow[dr, "{(2,2)}"] & & \cdots \\
& {[\tau^2 M']} \arrow[ur, "{(2,2)}"] & & {[\tau M']} \arrow[ur, "{(2,2)}"] & & {[M']} \arrow[ur, "{(2,2)}"] & & {[\tau^{-1}M']} \arrow[ur, "{(2,2)}"] &
\end{tikzcd}$$

So the component is of type $\mathbb{Z}\tilde{A}_{1,2}$. If two modules in the component were identified with each other, we would have a contradiction to the fact that there are no τ-periodic modules in the component. That is, we have no quotients of $\mathbb{Z}\tilde{A}_{1,2}$.

We now assume that all the modules in the component have almost split sequences ending at them with two indecomposable non-isomorphic summands in a chosen decomposition of the middle term. That is, every arrow in the component has valuation $(1,1)$ since we know that $\alpha(M) = 2$ for all $M \in \mathcal{C}$. Then, the component is of type $\mathbb{Z}A_\infty^\infty$ or a quotient of it. Recall that we cannot have a τ-periodic module in the regular component. That is, if we have a module X in the component that is identified with another module X' in the component, then X and X' are not in the same τ-orbit. Such an identification where X' lies on a sectional path of morphisms starting at X gives rise to a component of type $\mathbb{Z}\tilde{A}_n$. We do not know the structure of other possible quotients. $\square$

Note that the result actually differs a bit from what is stated in [7, Corollary 3.3]. The corollary in [7] only states that we have components of type $\mathbb{Z}A_\infty^\infty$, $\mathbb{Z}\tilde{A}_n$, or $\mathbb{Z}\tilde{A}_{1,2}$, but we were not able to prove that the other possibilities listed in 5.1.4 do not exist. One interesting fact that arises from the previous corollary is that all modules in such a component have two indecomposable non-projective modules in a chosen decomposition of the middle term of an almost split sequence ending at them. Furthermore, since there are no projective modules in such components they actually have two indecomposable modules in a chosen decomposition of the middle term of an almost split sequences ending at them. We now present two lemmas before we prove the main result of this subchapter.

Lemma 5.1.5. [7, Lemma 3.4] *Let R be a selfinjective Artin algebra and let C be an indecomposable module. Let*

$$0 \longrightarrow \tau C \xrightarrow{(f_1,f_2,\ldots,f_t)^T} E_1 \oplus E_2 \oplus \ldots \oplus E_t \xrightarrow{(g_1,g_2,\ldots,g_t)} C \longrightarrow 0$$

be an almost split sequence where each E_i is non-zero, but not necessarily indecomposable. Assume that all the f_i's and g_i's are Ω-perfect. If the map g_1 is an irreducible epimorphism and f_1 is an irreducible monomorphism, then $t \leq 2$.

Proof. Assume $t \geq 3$, and that we have an almost split sequence

$$0 \longrightarrow \tau C \xrightarrow{(f_1,f_2,\ldots,f_t)^T} E_1 \oplus E_2 \oplus \ldots \oplus E_t \xrightarrow{(g_1,g_2,\ldots,g_t)} C \longrightarrow 0$$

where each E_i is non-zero, but not necessarily indecomposable. By assumption, f_1 is a monomorphism and since the morphism is Ω-perfect we know that $\Omega^2 f_1$ is a monomorphism. Then, by 2.3.13, we then know that $\tau f_1 \colon \tau^2 C \longrightarrow \tau E_1$ is a monomorphism as well. Further, by 2.3.11, we have an almost split sequence

$$0 \longrightarrow \tau^2 C \xrightarrow{(\tau f_1,\tau f_2,\ldots,\tau f_t,f')^T} \tau E_1 \oplus \tau E_2 \oplus \ldots \oplus \tau E_t \oplus P \xrightarrow{(\tau g_1,\tau g_2,\ldots,\tau g_t,g')} \tau C \longrightarrow 0$$

ending at τC where P is indecomposable. Assume $P \neq (0)$. Then, the morphism $g' \colon P \longrightarrow \tau C$ is an epimorphism, or else it splits which contradicts it being irreducible. So, by 4.1.14, we know that τf_1 is an epimorphism, a contradiction. That is, $P = (0)$ and we have an almost split sequence

$$0 \longrightarrow \tau^2 C \xrightarrow{(\tau f_1,\tau f_2,\ldots,\tau f_t)^T} \tau E_1 \oplus \tau E_2 \oplus \ldots \oplus \tau E_t \xrightarrow{(\tau g_1,\tau g_2,\ldots,\tau g_t)} \tau C \longrightarrow 0$$

ending at τC. So, we have that

$$\sum_{i=1}^{t} \ell(E_i) = \ell(\tau C) + \ell(C) \text{ and}$$
$$\sum_{i=1}^{t} \ell(\tau E_i) = \ell(\tau^2 C) + \ell(\tau C).$$

Adding these equalities we get that

$$\sum_{i=1}^{t} \ell(E_i) + \sum_{i=1}^{t} \ell(\tau E_i) = \ell(C) + 2 \cdot \ell(\tau C) + \ell(\tau^2 C). \tag{5.4}$$

Note that none of the E_i's are projective. If one of them, say E_i, is projective, then the Ω-perfect irreducible map $f_i\colon \tau C \longrightarrow E_i$ is a monomorphism, or else it splits. That is, $(0) \neq \Omega\tau C \longrightarrow \Omega E_i = (0)$ is a monomorphism, a contradiction. So, none of the E_i's are projective and since we know that $t \geq 3$ by assumption, we have the following almost split sequences

$$0 \longrightarrow \tau E_2 \longrightarrow A \oplus \tau C \longrightarrow E_2 \longrightarrow 0$$

$$0 \longrightarrow \tau E_3 \longrightarrow A' \oplus \tau C \longrightarrow E_3 \longrightarrow 0$$

and we obtain

$$\ell(A) + \ell(\tau C) = \ell(E_2) + \ell(\tau E_2) \text{ and}$$
$$\ell(A') + \ell(\tau C) = \ell(E_3) + \ell(\tau E_3).$$

Further, we get the equation

$$\ell(A') + \ell(A) + 2 \cdot \ell(\tau C) = \ell(E_2) + \ell(E_3) + \ell(\tau E_2) + \ell(\tau E_3) \tag{5.5}$$

by adding the equations as before. By (5.4) and (5.5) we then get the following equality

$$\ell(A') + \ell(A) + \ell(E_1) + \ell(\tau E_1) + \sum_{i=4}^{t} \ell(E_i) + \sum_{i=4}^{t} \ell(\tau E_i) = \ell(C) + \ell(\tau^2 C),$$

so

$$\ell(E_1) + \ell(\tau E_1) \leq \ell(C) + \ell(\tau^2 C). \tag{5.6}$$

Since τf_1 is a monomorphism we know that $\ell(\tau^2 C) < \ell(\tau E_1)$. But then, by (5.6), we must have that $\ell(E_1) < \ell(C)$. This contradicts the fact that $g_1\colon E_1 \longrightarrow C$ is an epimorphism. So, $t \leq 2$. $\square$

Lemma 5.1.6. [7, Lemma 3.5] *Let R be a selfinjective Artin algebra and let C be an Ω-perfect module of finite complexity. Let*

$$0 \longrightarrow \tau C \xrightarrow{(f_1,f_2,\ldots,f_t)^T} E_1 \oplus E_2 \oplus \ldots \oplus E_t \xrightarrow{(g_1,g_2,\ldots,g_t)} C \longrightarrow 0$$

be an almost split sequence where each E_i is non-zero, but not necessarily indecomposable. If g_1 is an irreducible epimorphism, then $t \leq 3$.

Proof. Assume $t \geq 4$ and let $B = E_4 \oplus \ldots \oplus E_t$. By assumption B is non-zero. Moreover, let $g_4'\colon B \longrightarrow C$ and $f_4'\colon \tau C \longrightarrow B$ be the induced irreducible maps. That is, we have an almost split sequence

$$0 \longrightarrow \tau C \xrightarrow{(f_1,f_2,f_3,f_4')^T} E_1 \oplus E_2 \oplus E_3 \oplus B \xrightarrow{(g_1,g_2,g_3,g_4')} C \longrightarrow 0.$$

The map $(g_1, g_2)\colon E_1 \oplus E_2 \longrightarrow C$ is an epimorphism since g_1 is an epimorphism by assumption. We now rewrite the almost split sequence and let $k_1 = (g_1, g_2)$, $h_1 = (f_1, f_2)^T$ and $E = E_1 \oplus E_2$. That is, we have an almost split sequence

$$0 \longrightarrow \tau C \xrightarrow{(h_1,f_3,f_4')^T} E \oplus E_3 \oplus B \xrightarrow{(k_1,g_3,g_4')} C \longrightarrow 0$$

where h_1, f_3, f_4', k_1, g_3 and g_4' are all Ω-perfect morphisms. Then, by 5.1.5, the irreducible map $h_1 = (f_1, f_2)^T$ cannot be a monomorphism, and thus is an epimorphism. Referring to 4.1.12 we then know that (g_3, g_4') is an epimorphism, so $\ell(C) < \ell(E_3) + \ell(B)$. On the other hand, since g_1 is an epimorphism, by 4.1.12 we have that $(f_2, f_3, f_4')^T$ is an epimorphism and further $\ell(\tau C) > \ell(E_2) + \ell(E_3) + \ell(B)$. That is, by 5.1.2 and the previous arguments, we have that

$$\begin{aligned}
\ell(\tau C) &> \ell(E_2) + \ell(E_3) + \ell(B) \\
&> \ell(E_2) + \ell(C) \\
&\geq \frac{1}{1 + d'^2}\ell(C) + \ell(C) \\
&= \Big(\frac{1}{1 + d'^2} + 1\Big)\ell(C)
\end{aligned}$$

where $d' = \max\{\ell(Re_i)\}$, the maximal length of the indecomposable projective R-modules. For notational purposes we write $c = \Big(\frac{1}{1+d'^2} + 1\Big)$, and note that $c > 1$.

Further, C is Ω-perfect, so by 4.1.3 we know that for $n \geq 1$ the irreducible morphisms

$$\tau^{n-1}(g_3, g_4')\colon \tau^{n-1}E_3 \oplus \tau^{n-1}B \longrightarrow \tau^{n-1}C \text{ and}$$
$$\tau^{n-1}(f_2, f_3, f_4')^T\colon \tau^n C \longrightarrow \tau^{n-1}E_2 \oplus \tau^{n-1}E_3 \oplus \tau^{n-1}B$$

are epimorphisms as well. So, as before we get that

$$\ell(\tau^n C) > c \cdot \ell(\tau^{n-1}C) \text{ for all } n \geq 1.$$

The Nakayama functor ν preserves length by 2.3.4, so then by 2.3.5 we have that $\ell(\tau C) = \ell(\Omega^2 C)$ and moreover $\ell(\Omega^{2n}C) > c \cdot \ell(\tau^{n-1}C)$ for $n \geq 1$. Furthermore, for $n \geq 1$ we have that

$$c^n \ell(C) < \ldots < c^2 \cdot \ell(\tau^{n-2}C) < c \cdot \ell(\tau^{n-1}C) < \ell(\Omega^{2n}C).$$

So, since

$$c^n \ell(C) < \ell(\Omega^{2n}C) < d' \cdot \beta_{2n}(C)$$

by 3.1.2 where $d' = \max\{\ell(Re_i)\}$, the growth of the Betti numbers cannot be bounded by a polynomial. That is, C does not have finite complexity, a contradiction. So, $t \leq 3$. $\square$

Note that we have removed the assumption that f_1 is an epimorphism of the original version of the previous lemma. We can now prove the main result in this subchapter.

Theorem 5.1.7. [7, Lemma 3.6/Theorem 3.7] *Let R be a selfinjective Artin algebra. Let C be an Ω-perfect module of finite complexity. Then $\alpha(C) \leq 4$.*

Proof. Let

$$0 \longrightarrow \tau C \xrightarrow{(f_1, f_2, \ldots, f_t)^T} E_1 \oplus E_2 \oplus \ldots \oplus E_t \xrightarrow{(g_1, g_2, \ldots, g_t)} C \longrightarrow 0$$

be an almost split sequence ending at C where each E_i is non-zero and indecomposable. Recall that none of the E_i's are projective by 4.1.10, so $\alpha(C) = t$. Assume $t \geq 5$ and let $E = \oplus_{i=1}^t E_i$. In this proof we write E as a sum of R-modules in different ways and use the former lemmas the get our result.

Let $B_1 = E_1 \oplus E_2$ and $B_2 = E_3 \oplus E_4 \oplus ... \oplus E_t$, both are non-zero by assumption, and we write the almost split sequence as

$$0 \longrightarrow \tau C \xrightarrow{(h_1,h_2)^T} B_1 \oplus B_2 \xrightarrow{(k_1,k_2)} C \longrightarrow 0$$

where $h_1 = (f_1, f_2)^T$, $h_2 = (f_3, f_4, ..., f_t)^T$, $k_1 = (g_1, g_2)$ and $k_2 = (g_3, g_4, ..., g_t)$. We want to argue that at least one of k_1 and k_2 is an epimorphism. Now, assume they are both monomorphisms. Then, by 4.1.12, since k_1 is a monomorphism, we know that h_2 is a monomorphism. That is, both k_2 and h_2 are monomorphism, a contradiction to 4.1.13. So, at least one of k_1 or k_2 is an epimorphims.

Assume k_1 is an epimorphism. We again rewrite the almost split sequence

$$0 \longrightarrow \tau C \xrightarrow{(h_1,f_3,...f_t)^T} B_1 \oplus E_3 \oplus ... \oplus E_t \xrightarrow{(k_1,g_3,...,g_t)} C \longrightarrow 0.$$

By 5.1.6, we know that we have that $t - 1 \leq 3$, that is $t \leq 4$. This contradicts the assumption that $t \geq 5$. So, k_1 cannot be an epimorphism.

Then k_2 is an epimorphism by the former argument. Since k_1 is a monomorphism, by 4.1.12, h_2 is a monomorphism. Further, we rewrite the almost split sequence in the following manner

$$0 \longrightarrow \tau C \xrightarrow{(f_1,f_2,h_2)^T} E_1 \oplus E_2 \oplus B_2 \xrightarrow{(g_1,g_2,k_2)} C \longrightarrow 0$$

where all modules are non-zero by assumption. The map k_2 is an epimorphism and h_2 is a monomorphism, so by 5.1.5 we have a contradiction.

That is, t cannot be greater or equal to 5. So, $\alpha(C) = t \leq 4$ and we are done. □

Note that the previous result is actually a combination of Lemma 3.6 and Theorem 3.7 in [7]. We have removed the assumption that all g_i's are monomorphisms of the original lemma since it is superfluous, and thereby we get the result stated as Theorem 3.7 in the paper directly. The next result follows from the previous theorem.

Corollary 5.1.8. [7, Corollary 3.8] *Let R be a selfinjective Artin algebra such that no simple module is Ω-periodic. Let C be an indecomposable non-projective R-module of finite complexity. Then $\alpha(C) \leq 4$.*

Proof. By 4.2.6, the module C is eventually Ω-perfect, so it exists an $n \geq 0$ such that $\Omega^n C$ is Ω-perfect. By 3.2.2, $\mathrm{cx}(\Omega^n C) = \mathrm{cx}(C)$, so $\Omega^n C$ has finite complexity.

Then, by 5.1.7, $\alpha(\Omega^n C) \leq 4$. Moreover, the function α is invariant under Ω by 2.3.9. That is, $\alpha(C) = \alpha(\Omega^n C) \leq 4$, and we are done. $\square$

In this subchapter we have dealt with modules of finite complexity. In the next subchapter we narrow our focus and look at modules of complexity 1.

5.2 Complexity 1

In this subchapter we study the special case of finite complexity where the complexity is one. Most of the results in this subchapter are found in [6].

5.2.1 Modules with bounded Betti numbers

As the headline suggests, we look at modules with bounded Betti numbers in this section. Recall that a non-projective module with bounded Betti numbers has complexity 1 by 2.3.7 and 3.2.2. By the same results, we also know that a module of complexity 1 is non-projective and has bounded Betti numbers. In the previous subchapter we studied almost split sequences ending at an Ω-perfect module of finite complexity. In the next lemma we look at almost split sequences ending at an Ω-perfect module of complexity 1.

Lemma 5.2.1. [6, Lemma 2.11] *Let R be a selfinjective Artin algebra and let C be an Ω-perfect module of complexity 1. Let*

$$0 \longrightarrow \tau C \xrightarrow{(f_1,f_2,\dots,f_t)^T} E_1 \oplus E_2 \oplus \dots \oplus E_t \xrightarrow{(g_1,g_2,\dots,g_t)} C \longrightarrow 0$$

be an almost split sequence ending at C where the E_i's are non-zero and not necessarily indecomposable. Then, for each $1 \leq i \leq t$, one of the maps f_i and g_i is a monomorphism, and the other one is an epimorphism.

Proof. We know, by 4.1.13, that both f_i and g_i for some $i \in \{1, \dots, t\}$ cannot be monomorphisms.

Now, assume that both f_i and g_i are epimorphisms for some $i \in \{1, \dots, t\}$. Then, since C is Ω-perfect, we know that the morphisms $\tau^n f_i$ and $\tau^n g_i$ are epimorphisms for all $n \geq 0$ by 4.1.3. So, we have a sequence of proper epimorphisms

$$\cdots \twoheadrightarrow \tau^3 C \xrightarrow{\tau^2 g_i \tau^2 f_i} \tau^2 C \xrightarrow{\tau g_i \tau f_i} \tau C \xrightarrow{g_i f_i} C.$$

We want to show that there is an upper bound on the lengths of the $\tau^m C$'s. By 3.2.3 we have a common bound for the lengths of all syzygies of C, say N. That is, $\ell(\Omega^n C) < N$ for all $n \geq 0$. The Nakayama functor ν preserves length by 2.3.4 and commutes with Ω by 2.3.6, so it follows that $\ell(\tau^m C) < N$ for all $m \geq 0$. Since we have proper epimorphisms in the sequence above, $\ell(\tau^m C) < \ell(\tau^{m+1} C)$ for all $m \geq 0$. This contradicts the fact that $\ell(\tau^m C) < N$ for all $m \geq 0$. So, both f_i and g_i cannot be epimorphisms.

In total, we conclude that for each $1 \leq i \leq t$, one of the maps f_i and g_i is a monomorphism, and the other is an epimorphism. $\square$

In the previous subchapter our main aim was to prove a result concerning the number of indecomposable non-projective modules in a chosen decomposition of the middle term of an almost split sequence ending at an Ω-perfect module of finite complexity. We now look at the number of modules in a chosen decomposition of the middle term of an almost split sequence ending at an Ω-perfect module C where $\beta_n(C) = 1$ for an infinite number of $n \geq 0$.

Proposition 5.2.2. [6, Proposition 2.12] *Let R be a selfinjective Artin algebra and C an Ω-perfect R-module such that $\beta_n(C) = 1$ for an infinite number of $n \geq 0$. Then $\alpha(C) = 1$.*

Proof. Let

$$0 \longrightarrow \tau C \xrightarrow{(f_1, f_2, \ldots, f_t)^T} E_1 \oplus E_2 \oplus \ldots \oplus E_t \xrightarrow{(g_1, g_2, \ldots, g_t)} C \longrightarrow 0$$

be the almost split sequence ending at C with each E_i indecomposable. Since there cannot be any non-zero projective modules in the middle term of an almost split sequence ending at C by 4.1.10, we know that $t = \alpha(C)$ and that the E_i's are non-projective. Assume $t > 1$.

We assume g_1 is a monomorphism and then, by 4.1.13, f_1 is an epimorphism. So, we have a short exact sequence

$$0 \longrightarrow \operatorname{Ker} f_1 \longrightarrow \tau C \xrightarrow{f_1} E_1 \longrightarrow 0.$$

Since f_1 is Ω-perfect, by 4.1.11 we know that $\beta_i(\tau C) = \beta_i(\operatorname{Ker} f_1) + \beta_i(E_1)$ for all $i \geq 0$. The module E_1 is non-projective, so by 2.3.7, it has infinite projective dimension and moreover non-zero Betti numbers. From the equality above we therefore get the strict inequality $\beta_i(\tau C) > \beta_i(\operatorname{Ker} f_1)$.

By assumption, $\beta_i(C) = 1$ for an infinite number of positive integers $i \geq 0$, and since $\beta_{i+2}(C) = \beta_i(\tau C)$ by 3.1.2, we have that $\beta_n(\tau C) = 1$ for an infinite number of integers $n \geq 0$. But then, by the inequality above, $\beta_n(\operatorname{Ker} f_1) = 0$ for the corresponding n's. The kernel of f_1 is non-projective or else f_1 splits, which cannot be. That is, $\beta_j(\operatorname{Ker} f_1) \neq 0$ for all $j \geq 0$, a contradiction to the previous argument. So, g_1 cannot be a monomorphims.

Assume g_1 is an epimorphism. Then

$$E_1 \oplus E_2 \oplus \ldots \oplus E_{t-1} \xrightarrow{(g_1, g_2, \ldots, g_{t-1})} C$$

is an epimorphism, and this implies that f_t is an epimorphism by 4.1.12. By repeating the previous argument we have a contradiction.

So, we cannot have $t > 1$, and since we know that there exists an almost split sequence ending at C by 2.1.9, we have that $t = 1$. $\square$

We now turn our attention to regular components of Auslander-Reiten quivers of selfinjective Artin algebras containing a module C of complexity 1. Before we present our results, we recall some important properties of such components. In Chapter 2.2 we argued that a regular component of the Auslander-Reiten quiver of a connected, Artin algebra containing a τ-periodic module is of type $\mathbb{Z}A_\infty/\langle\tau^n\rangle$. Therefore, when we refer to regular components that are tubes in the upcoming proofs they are of type $\mathbb{Z}A_\infty/\langle\tau^n\rangle$. In Chapter 2.2 we promised to further explore the morphisms of regular components of type $\mathbb{Z}A_\infty$ and $\mathbb{Z}A_\infty/\langle\tau^n\rangle$. We illustrate a part of such a component here.

$$\begin{tikzcd}[column sep=small]
 & & & \vdots & & \vdots & & & \vdots & & \\
 & {} \arrow[dr] & & {[M_4]} \arrow[dr] & & {[\tau^{-1}M_4]} \arrow[dr] & & {[\tau^{-2}M_4]} \arrow[dr] & & {} & \\
\cdots & & {[\tau M_3]} \arrow[ur] \arrow[dr] & & {[M_3]} \arrow[ur] \arrow[dr] & & {[\tau^{-1}M_3]} \arrow[ur] \arrow[dr] & & {[\tau^{-2}M_3]} \arrow[ur] \arrow[dr] & & \cdots \\
 & {} \arrow[ur] \arrow[dr] & & {[\tau M_2]} \arrow[ur] \arrow[dr] & & {[M_2]} \arrow[ur] \arrow[dr] & & {[\tau^{-1}M_2]} \arrow[ur] \arrow[dr] & & {} & \\
\cdots & & {[\tau^2 M_1]} \arrow[ur] & & {[\tau M_1]} \arrow[ur] & & {[M_1]} \arrow[ur] & & {[\tau^{-1}M_1]} \arrow[ur] & & \cdots
\end{tikzcd}$$

Since the valuation of all the edges in A_∞ is $(1,1)$ and no modules are removed from the component, we know that it is actual almost split sequences that are illustrated above. If we look at a module on the boundary of the component, M_1, an almost split sequence ending at M_1 is

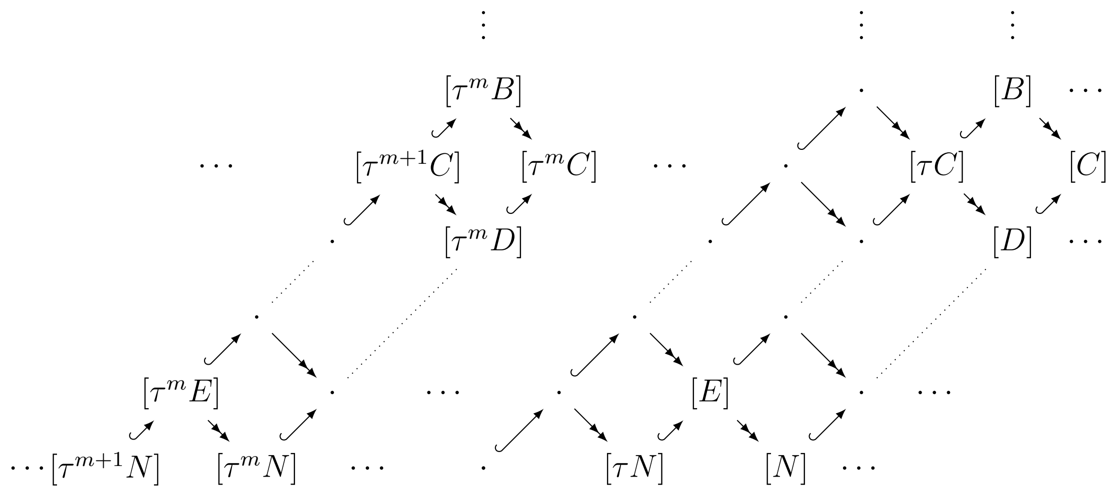
$$0 \longrightarrow \tau M_1 \longrightarrow M_2 \longrightarrow M_1 \longrightarrow 0.$$

So, $\tau M_1 \longrightarrow M_2$ is a monomorphism and $M_2 \longrightarrow M_1$ is an epimorphism. If we combine this with 4.1.12, we see why some of the morphisms illustrated in the above quiver are monomorphisms (indicated with a hook arrow) and others are epimorphisms (indicated with a two headed arrow).

We now let C be an indecomposable module lying in regular component that is a tube or of type $\mathbb{Z}A_\infty$. We further assume the component does not contain any simple modules. We want to argue that if $g\colon B \longrightarrow C$ is an epimorphism in such a component, then its kernel is on the boundary of the component and is therefore not simple. If C itself is on the boundary of $\mathcal{C}$, we have an almost split sequence ending at C

$$0 \longrightarrow \tau C \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

and furthermore the kernel of g is obviously on the boundary and is therefore not simple. So, we assume C is not on the boundary of $\mathcal{C}$. We illustrate a part of the component



where we choose to denote the modules that are not of importance with a dot. This is just to simplify the illustration. From previous arguments we know that an almost split sequence ending at C is

$$0 \longrightarrow \tau C \xrightarrow{(f,f')^T} B \oplus D \xrightarrow{(g,g')} C \longrightarrow 0$$

and further, by 4.1.12, we know that $\operatorname{Ker} g \cong \operatorname{Ker} f'$. Repeating this argument we get that $\operatorname{Ker} g \cong \tau N$. But τN is in the component, so τN is not simple. That is, the kernel of g is not simple.

We continue considering the regular component of type $\mathbb{Z}A_\infty$ or $\mathbb{Z}A_\infty/\langle\tau^n\rangle$ without simple modules. We let $g\colon B \longrightarrow C$ be an epimorphism in the component and want to show that g is Ω-perfect. We know that an almost split sequence containing $\tau^m g$ for $m \geq 0$ is either

$$0 \longrightarrow \tau^{m+1}C \xrightarrow{\tau^m f} \tau^m B \xrightarrow{\tau^m g} \tau^m C \longrightarrow 0$$

if C is on the boundary of $\mathcal{C}$ or

$$0 \longrightarrow \tau^{m+1}C \xrightarrow{(\tau^m f, \tau^m f')^T} \tau^m B \oplus \tau^m D \xrightarrow{(\tau^m g, \tau^m g')} \tau^m C \longrightarrow 0$$

if C is not on the boundary of $\mathcal{C}$. Looking at the component and recalling previous arguments we then know that $\tau^m g$ is an epimorphism with non-simple kernel for all $m \geq 0$. That is, by 2.3.13, we know that $\Omega^{2m} g$ is an epimorphism for all $m \geq 0$. We now need to argue that $\Omega^{2m+1} g$ is an epimorphism for all $m \geq 0$. We have the following exact sequence

$$0 \longrightarrow K_m \longrightarrow \tau^m B \xrightarrow{\tau^m g} \tau^m C \longrightarrow 0$$

for $m \geq 0$ where K_m is the non-simple kernel of $\tau^m g$. We then know that

$$\ell(K_m) + \ell(\tau^m C) = \ell(\tau^m B). \tag{5.7}$$

Furthermore, we look at $\Omega^{2m} g$ for $m \geq 0$

$$0 \longrightarrow K'_m \longrightarrow \Omega^{2m} B \xrightarrow{\Omega^{2m} g} \Omega^{2m} C \longrightarrow 0$$

and we have that

$$\ell(K'_m) + \ell(\Omega^{2m} C) = \ell(\Omega^{2m} B). \tag{5.8}$$

Now, since ν preserves length by 2.3.4 and by 2.3.6 we have that

$$\ell(\tau^m C) = \ell(\Omega^{2m} C)$$

and

$$\ell(\tau^m B) = \ell(\Omega^{2m} B).$$

So, by (5.7) and (5.8) we have that $\ell(K'_m) = \ell(K_m) > 1$ for $m \geq 0$. That is, the kernel of $\Omega^{2m} g$ is not simple and moreover by 4.1.7 we know that $\Omega^{2m+1} g$ is an epimorphism for all $m \geq 0$. So, g is Ω-perfect.

We now want to argue that any module C in the component is Ω-perfect. We first assume $\alpha(C) = 2$, that is, we have

$$0 \longrightarrow \tau C \xrightarrow{(f,f')^T} B \oplus D \xrightarrow{(g,g')} C \longrightarrow 0$$

an almost split sequence ending at a module C in the component, with g an epimorphism and B and D indecomposable. By the above we know that g is Ω-perfect. Further, by 4.1.12, f' is also an epimorphism in the component and is therefore also Ω-perfect. We need to show that also (g, g') is an Ω-perfect epimorphism. That is, we need to show that the morphism $\Omega^m(g, g')(= (\Omega^m g, \Omega^m g'))$ from $\Omega^m B \oplus \Omega^m D$ to $\Omega^m C$ is an epimorphism for all $m \geq 0$. But we know that $\Omega^m g \colon \Omega^m B \longrightarrow \Omega^m C$ is an epimorphism for all $m \geq 0$ by the previous argument, so therefore $\Omega^m(g, g')$ is an epimorphism for all $m \geq 0$.

From the previous discussion we know that f and g' are monomorphisms. We now need to show that f, g' and (f, f') are Ω-perfect monomorphisms. By the dual result of 4.1.7 we know that given an irreducible monomorphism $h \colon A \longrightarrow B'$, the irreducible morphism $\Omega h \colon \Omega A \longrightarrow \Omega B'$ is a monomorphism if and only if the cokernel of h is non-simple. Moreover, by a similar argument as above, if $h \colon A \longrightarrow B'$ is an irreducible monomorphism in the component, then the cokernel of h is on the boundary of the component and is therefore non-simple. Further, $\tau^m h$ is a monomorphism for all $m \geq 0$ and by 2.3.13, we know that $\Omega^{2m} A \longrightarrow \Omega^{2m} B'$ is a monomorphism for all $m \geq 0$. Since the cokernels of the monomorphisms in the component are non-simple, we know that $\Omega^{2m+1} A \longrightarrow \Omega^{2m+1} B'$ is a monomorphism for all $m \geq 0$ by a simple length argument and the dual of 4.1.7. That is, the irreducible monomorphisms f and g' in $\mathcal{C}$ are Ω-perfect. Further, since $\Omega^m(f, f')(= (\Omega^m f, \Omega^m f'))$ is a morphism from $\Omega^m \tau C$ to $\Omega^m B \oplus \Omega^m D$ and $\Omega^m f \colon \Omega^m \tau C \longrightarrow \Omega^m B$ is a monomorphism for all $m \geq 0$, we know that $\Omega^m(f, f')$ is a monomorphism for all $m \geq 0$. That is, (f, f') is Ω-perfect as well.

In total, C is Ω-perfect. Now, if $\alpha(C) = 1$, we have an almost split sequence

$$0 \longrightarrow \tau C \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

and by the previous arguments we know that f and g are Ω-perfect. That is, C is Ω-perfect in this case as well. That is, all modules in the component are Ω-perfect and we have the following proposition.

Proposition 5.2.3. *Let R be a selfinjective Artin algebra. Let $\mathcal{C}$ be a regular component of the Auslander-Reiten quiver that is a tube or of type $\mathbb{Z}A_\infty$. Furthermore, assume that $\mathcal{C}$ does not contain any simple modules. Then, all modules in $\mathcal{C}$ are Ω-perfect.*

Proof. See the previous argument. $\square$

In the next proposition we, among other things, look at how a module C with complexity 1 and minimal $\beta(C)$ among the modules in such a regular tube without simple modules actually has to lie on the boundary of the tube.

Proposition 5.2.4. [6, Proposition 3.1] *Let R be a selfinjective Artin algebra, and let C be an indecomposable module with complexity 1, lying in a regular component $\mathcal{C}$ of the Auslander-Reiten quiver and such that $\beta(C)$ is minimal among the modules in $\mathcal{C}$. Assume further that $\mathcal{C}$ is either not a tube, or, that if a tube, then $\mathcal{C}$ contains no simple modules. Then $\alpha(C) = 1$.*

Proof. Assume the component is not a tube. Then, by 4.2.8, we know that $\tau^n C$ is Ω-perfect for some $n \geq 0$. Further, since $\beta(\tau^n C) \leq \beta(C)$ by 3.1.3 and $\beta(C)$ is minimal among the modules in $\mathcal{C}$, we know that $\beta(\tau^n C) = \beta(C)$ is minimal. Since the component is regular, by 3.2.5, all modules have complexity 1, so $\mathrm{cx}(\tau^n C) = 1$. Furthermore, we know that the function α is invariant under τ by 2.3.12. So, we may assume, without loss of generality, that C is Ω-perfect. If $\mathcal{C}$ is a tube without simple modules, it follows immediately from 5.2.3 that C is Ω-perfect.

So, we assume C is Ω-perfect and let

$$0 \longrightarrow \tau C \xrightarrow{(f_1, f_2, ..., f_t)^T} E_1 \oplus E_2 \oplus ... \oplus E_t \xrightarrow{(g_1, g_2, ..., g_t)} C \longrightarrow 0$$

be an almost split sequence ending at C with each E_i indecomposable. The component is regular, so we know that none of the E_i's are projective. That is, $\alpha(C) = t$. We assume that $t > 1$.

If g_1 is a monomorphism, then f_1 cannot be a monomorphism as well by 4.1.13. So, f_1 is an Ω-perfect epimorphism and furthermore, by 4.1.11, we know that $\beta_i(\tau C) = \beta_i(\mathrm{Ker}\, f_1) + \beta_i(E_1)$ for all $i \geq 0$. Then, since $\mathrm{Ker}\, f_1$ cannot be projective as f_1 does not split, $\beta_i(\mathrm{Ker}\, f_1) \neq 0$. That is, we have a strict inequality $\beta_i(E_1) < \beta_i(\tau C)$ for all $i \geq 0$.

If g_1 is an epimorphism, then since $(g_1, g_2, ..., g_{t-1})$ is also an epimorphism we know that f_t is an epimorphism by 4.1.12. This implies that $\beta_i(E_t) < \beta_i(\tau C)$ for all $i \geq 0$ again using the argument above.

So, recalling that $\beta(\tau C) = \beta(C)$ by the minimality of $\beta(C)$ we have that

$$\beta_i(E_k) < \beta_i(\tau C) \leq \beta(\tau C) = \beta(C)$$

for all $i \geq 0$ where k is either 1 (g_1 a monomorphism) or t (g_1 an epimorphism). That is, $\beta(E_k) = \max_{i\geq 0}\{\beta_i(E_k)\} < \beta(C)$ for either $k = 1$ or $k = t$, a contradiction to $\beta(C)$ being minimal among the modules in $\mathcal{C}$. So, $t = 1$.

□

The next two results concern non-zero maps between indecomposable modules.

Lemma 5.2.5. [10] *Let X and C be indecomposable modules and $f\colon X \longrightarrow C$ a non-zero map which is not an isomorphism. Then either*

(1) *there is a finite chain of irreducible maps between indecomposable modules $X \longrightarrow \ldots \longrightarrow C$ with non-zero composition or*

(2) *there are chains of irreducible maps $Y \longrightarrow \ldots \longrightarrow C$, between indecomposable modules, with non-zero composition, of arbitrary length.*

Proof. Let $g\colon E_1 \oplus E_2 \oplus \ldots \oplus E_k \longrightarrow C$ be a right minimal almost split map where each E_i is indecomposable. The map $f\colon X \longrightarrow C$ is not an isomorphism and we want to show that this implies that is not a split epimorphism. If it was, then $X \cong C \oplus Y$, with $Y \in \operatorname{mod} R$. But then, since $X \neq (0)$ is indecomposable and C is non-zero, we know that $Y = (0)$, so $X \cong C$. That is, $\ell(X) = \ell(C)$ and since f is an epimorphism we the have that f is an isomorphism, a contradiction. So, f is not a split epimorphism and there exists a map $s\colon X \longrightarrow E_1 \oplus E_2 \oplus \ldots \oplus E_k$ such that $gs = f$.

$$\begin{tikzcd} & X \arrow[dl, "s=(s_j)^T"'] \arrow[d, "f"] \\ E_1 \oplus E_2 \oplus \ldots \oplus E_k \arrow[r, "g=(g_j)"] & C \end{tikzcd}$$

The diagram commutes and since f is non-zero we know that there exists an i such that $g_i s_i \neq 0$. Moreover, we know that $g_i\colon E_i \longrightarrow C$ is irreducible, so we choose g_i as our first map. If $s_i\colon X \longrightarrow E_i$ is an isomorphism, we have case (1). If not, we repeat the argument on s_i and continue until we get our result.

□

Lemma 5.2.6. [6, Lemma 3.3] *Let R be an Artin algebra and let $\mathcal{C}$ be a component of the Auslander-Reiten quiver of R. Let $M \in \mathcal{C}$ and assume that all the predecessors of M in $\mathcal{C}$ have length bounded by some positive integer b. Then, if X is an indecomposable R-module such that* $\operatorname{Hom}_R(X, M) \neq (0)$*, then $X \in \mathcal{C}$ and X is a predecessor of M.*

Proof. Let X be an indecomposable R-module such that $\mathrm{Hom}_R(X, M) \neq (0)$. If $X \cong M$, then $X \in \mathcal{C}$ and X is trivially a predecessor of M. So, we assume $X \ncong M$.

By 5.2.5, we either have a chain of irreducible maps from X to M passing through indecomposable modules with non-zero composition, or there exists an arbitrary long chain of irreducible maps through indecomposable modules with non-zero composition that ends in M. If we were in the latter case, since the length of the predecessors of M is bounded by b, we know by [4, Corollary VI.1.3] that at least one of the maps in the chain needs to be an isomorphism. This contradicts it being irreducible. That is, we have a finite chain of irreducible maps with non-zero composition from X to M through indecomposable modules. So, $X \in \mathcal{C}$ and X is a predecessor of M. □

We can now further explore the structure of almost split sequences ending at an Ω-perfect module C of complexity 1. Here, we let the module C be in a regular component of the Auslander-Reiten quiver of a selfinjective Artin algebra.

Proposition 5.2.7. [6, Proposition 3.5] *Let R be a selfinjective Artin algebra and let C be an Ω-perfect module of complexity 1, belonging to a regular component of the Auslander-Reiten quiver. Let*

$$0 \longrightarrow \tau C \xrightarrow{(f_1, f_2, \dots, f_t)^T} E_1 \oplus E_2 \oplus \dots \oplus E_t \xrightarrow{(g_1, g_2, \dots, g_t)} C \longrightarrow 0$$

be an almost split sequence ending at C where the E_i's are indecomposable. Then precisely one of the maps g_i is an epimorphism and all the other ones are monomorphisms.

Proof. By 3.2.5, every module in the regular component has complexity 1. First, assume two g_i's are epimorphisms, say, g_1 and g_2. Then, the morphism $(g_2, \dots, g_t)$ is an epimorphism, so using 4.1.12 we know that f_1 is an epimorphism. This contradicts 5.2.1.

Now assume all the g_i's are monomorphisms. Then, by 5.2.1, all the f_i's are epimorphisms. We now want to show that all predecessors of C in the component have length bounded by $\beta(C) \cdot d'$, where $d' = \max\{\ell(Re_i)\}$. We already know that $\ell(X) < \beta(X) \cdot d'$ for all $X \in \mathcal{C}$ by 3.1.3. That is, we only need to prove that $\beta(X) \leq \beta(C)$ for all predecessors X of C. Again, by 3.1.3 we also know that $\beta(\tau^n M) \leq \beta(M)$ for $n \geq 0$ for $M \in \mathcal{C}$. That is, it is enough to show that if we have a path

$$D^k \longrightarrow D^{k-1} \longrightarrow \cdots \longrightarrow D^1 \longrightarrow C$$

where the modules are in different τ-orbits, then $\beta(D^k) \leq \beta(C)$. We assume

$$D^k \longrightarrow D^{k-1} \longrightarrow \cdots \longrightarrow D^1 \longrightarrow C$$

is such a path where $k \geq 1$, and let $D^0 = C$. That is, $D^1 = E_r$ for a $r \in \{1, 2, ..., t\}$. We know that $f_r \colon \tau C \longrightarrow D^1$ is an irreducible epimorphism. Moreover, we know that f_r is Ω-perfect, and by 4.1.11 we then know that

$$\beta_i(\tau C) = \beta_i(\operatorname{Ker} f_r) + \beta_i(D^1).$$

Furthermore, $\operatorname{Ker} f_r$ cannot be projective as this would imply that f_r splits, which is not possible. So, $\beta_i(\operatorname{Ker} f_r) \neq 0$ and

$$\begin{aligned}
&\beta_i(D^1) < \beta_i(\tau C) \text{ for all } i \geq 0 \text{ and furthermore} \\
&\beta(D^1) < \beta(\tau C) \leq \beta(C).
\end{aligned}$$

So, $\beta(D^1) < \beta(C)$. That is, if $k = 1$, we are done.

We assume $k > 1$ and continue. We consider an almost split sequence ending at D^1

$$0 \longrightarrow \tau D^1 \xrightarrow{(h_1,h_2,h_3)^T} D^2 \oplus L_1 \oplus \tau C \xrightarrow{(k_1,k_2,f_r)} D^1 \longrightarrow 0$$

where L_1 may decompose. We know that f_r is an epimorphism, so (k_2, f_r) is an epimorphism. Then, by 4.1.12, h_1 is an epimorphism. Further, by repeating 2.3.8 we have an almost split sequence

$$0 \longrightarrow \Omega^n \tau D^1 \longrightarrow \Omega^n D^2 \oplus \Omega^n L_1 \oplus \Omega^n \tau C \oplus P_n \longrightarrow \Omega^n D^1 \longrightarrow 0$$

for $n \geq 0$ with P_n projective. Since $\Omega^n f_r \colon \Omega^n \tau C \longrightarrow \Omega^n D^1$ is an epimorphism for each $n \geq 0$, by a similar argument as above, we know that $\Omega^n h_1 \colon \Omega^n \tau D^1 \longrightarrow \Omega^n D^2$ is an epimorphism for $n \geq 0$. That is, $h_1 \colon \tau D^1 \longrightarrow D^2$ is an Ω-perfect irreducible epimorphism, so

$$\beta_i(\tau D^1) = \beta_i(\operatorname{Ker} h_1) + \beta_i(D^2)$$

by 4.1.11. Again, since $\beta_i(\operatorname{Ker} h_1)$ cannot be zero, we have that $\beta_i(\tau D^1) > \beta_i(D^2)$ for all $i \geq 0$ and furthermore $\beta(D^2) < \beta(\tau D^1) \leq \beta(D^1)$. We now consider an almost split sequence ending at D^j

$$0 \longrightarrow \tau D^j \longrightarrow D^{j+1} \oplus L_j \oplus \tau D^{j-1} \longrightarrow D^j \longrightarrow 0$$

where L_j may decompose and $j \in \{1, ..., k-1\}$. Resuming the previous argument, we can show that each map $\tau D^j \longrightarrow D^{j+1}$ is Ω-perfect, and using 4.1.11 we have that $\beta_i(\tau D^j) > \beta_i(D^{j+1})$ for all $i \geq 0$. That is, we have that

$$\beta(D^k) < \beta(\tau D^{k-1}) \leq \beta(D^{k-1}) < ... < \beta(D^1) < \beta(C), \text{ and}$$

$$\ell(D^k) < \beta(D^k) \cdot d' < \beta(C) \cdot d'$$

for $k \geq 1$ by 3.1.3. By the above and the previous argument we know that each predecessor of C is bounded by $\beta(C) \cdot d'$. Then, all indecomposable R-modules X such that $\mathrm{Hom}_R(X, C) \neq (0)$ is in $\mathcal{C}$, by 5.2.6.

We now look at the composition series of C

$$... \subseteq C_2 \subseteq C_1 \subseteq C$$

We know that $C/C_1 \cong S$, where S is a simple R-module. The projective cover of a simple module is an indecomposable projective. That is, we have the commutative diagram

$$\begin{tikzcd} & P \arrow[dl, dashed, "h"'] \arrow[d, two heads] \\ C \arrow[r, two heads] & C/C_1 \cong S \end{tikzcd}$$

with P indecomposable projective. In particular, we have a morphism $h\colon P \longrightarrow C$, so $\mathrm{Hom}_R(P, C) \neq (0)$. That is, $P \in \mathcal{C}$, a contradiction to $\mathcal{C}$ being regular. So g_i is not a monomorphism for all $i \in \{1, ..., t\}$.

In total, precisely one of the g_i's is an epimorphism, and all the others are monomorphisms.

□

The reader may confuse this result with 5.1.3. It should therefore be noted that the assumptions are different in these results and moreover, while we in 5.1.3 just show that at most one g_i is an epimorphism, we here show that precisely one g_i is an epimorphism. We now prove a theorem stating that a module of complexity 1 belonging to a regular component cannot have more than two indecomposable modules in a chosen decomposition of the middle term of an almost split sequence ending at the module.

Theorem 5.2.8. [6, Theorem 3.6] *Let R be a selfinjective Artin algebra and let M be a module of complexity 1 belonging to a regular component. Then $\alpha(M) \leq 2$.*

Proof. If there is a τ-periodic module in the component, by previous arguments, we know that the component is of type $\mathbb{Z}A_\infty/\langle\tau^n\rangle$. That is, $\alpha(M) \leq 2$ for all $M \in \mathcal{C}$.

So, we assume that the component is not a tube. By 4.2.8 we know that there exists an $n \geq 0$ such that $\tau^n M$ is Ω-perfect. Further, all the modules in the regular component have complexity 1 by 3.2.5, in particular $\mathrm{cx}(\tau^n M) = 1$. For simplicity, we denote $\tau^n M$ by M'. We assume that

$$0 \longrightarrow \tau M' \longrightarrow E_1 \oplus E_2 \oplus \ldots \oplus E_t \xrightarrow{(g_1, g_2, \ldots, g_t)} M' \longrightarrow 0$$

is an almost split sequence ending at M', where the E_i's are non-zero, indecomposable and that $t \geq 3$. The component is regular, so none of the E_i's is projective. Moreover, we get that

$$\sum_{i=1}^{t} \ell(E_i) = \ell(\tau M') + \ell(M'). \tag{5.9}$$

By applying 5.2.7 we may assume that g_1 is an epimorphism. Since the component is regular, by 2.3.11, we have the almost split sequence

$$0 \longrightarrow \tau^2 M' \longrightarrow \tau E_1 \oplus \tau E_2 \oplus \ldots \oplus \tau E_t \longrightarrow \tau M' \longrightarrow 0$$

where $\tau E_i \neq (0)$ for $i \in \{1, \ldots, t\}$. Furthermore, we get that

$$\sum_{i=1}^{t} \ell(\tau E_i) = \ell(\tau^2 M') + \ell(\tau M'). \tag{5.10}$$

By adding (5.9) and (5.10) we get

$$\sum_{i=1}^{t} \ell(E_i) + \sum_{i=1}^{t} \ell(\tau E_i) = \ell(M') + 2 \cdot \ell(\tau M') + \ell(\tau^2 M'). \tag{5.11}$$

Since $t \geq 3$ and E_2 and E_3 are not projective, we have almost split sequences

$$0 \longrightarrow \tau E_2 \longrightarrow \tau M' \oplus C \longrightarrow E_2 \longrightarrow 0$$

and

$$0 \longrightarrow \tau E_3 \longrightarrow \tau M' \oplus D \longrightarrow E_3 \longrightarrow 0$$

where C and D are not necessariliy indecomposable. Moreover, we have that

$$\ell(\tau M') + \ell(C) = \ell(\tau E_2) + \ell(E_2) \tag{5.12}$$

and

$$\ell(\tau M') + \ell(D) = \ell(\tau E_3) + \ell(E_3) \tag{5.13}$$

So, by adding (5.12) and (5.13) we get

$$\ell(C) + \ell(D) + 2 \cdot \ell(\tau M') = \ell(E_2) + \ell(E_3) + \ell(\tau E_2) + \ell(\tau E_3). \tag{5.14}$$

Now, adding (5.11) and (5.14) we get

$$\ell(C) + \ell(D) + \ell(E_1) + \ell(\tau E_1) + \sum_{i=4}^{t} \ell(E_i) + \sum_{i=4}^{t} \ell(\tau E_i) = \ell(\tau^2 M') + \ell(M'),$$

that is, we have the inequality

$$\ell(E_1) + \ell(\tau E_1) \leq \ell(\tau^2 M') + \ell(M'). \tag{5.15}$$

Further, we know that the morphism $\tau E_1 \longrightarrow \tau M'$ is an epimorphism by 4.1.3 since g_1 is an epimorphism and M' is Ω-perfect. That is, both $E_1 \longrightarrow M'$ and $\tau E_1 \longrightarrow \tau M'$ are epimorphisms, so

$$\begin{aligned} &\ell(M') < \ell(E_1) \text{ and} \\ &\ell(\tau M') < \ell(\tau E_1). \end{aligned}$$

That is,

$$\ell(M') + \ell(\tau M') < \ell(E_1) + \ell(\tau E_1),$$

and further, combining this with (5.15) we have that

$$\ell(M') + \ell(\tau M') < \ell(\tau^2 M') + \ell(M').$$

Moreover, we then have that

$$\ell(\tau M') < \ell(\tau^2 M').$$

Since the component is regular we have that

$$0 \longrightarrow \tau^{j+1} M' \longrightarrow \tau^j E_1 \oplus \tau^j E_2 \oplus \ldots \oplus \tau^j E_t \longrightarrow \tau^j M' \longrightarrow 0$$

is an almost split sequence ending at $\tau^j M'$ for all $j \geq 0$ by 2.3.11. The E_i's are not projective, so then from Subchapter 2.3 we know that $\tau^j E_i \neq (0)$ for $i \in \{1, ..., t\}$ and $j \geq 0$. Furthermore, since M' is an Ω-perfect module and g_1 is an epimorphism, we know that $\tau^j g_1$ is an epimorphism by 4.1.3. Recall that $\tau^j M'$ is Ω-perfect by 4.1.10 and has complexity 1 by 3.2.5. Repeating the argument for $\tau^j M'$ for $j \geq 1$, we get

$$\ell(\tau M') < \ell(\tau^2 M') < \ell(\tau^3 M') < ...$$

But we know, by 3.1.3, that

$$\ell(\tau^k M') < \beta(\tau^k M') \cdot d' \leq \beta(M') \cdot d' \text{ for all } k \geq 0$$

where $d' = \max_{i \geq 0}\{\ell(Re_i)\}$. Recall that $\beta(M')$ is finite by 3.2.2. That is, we have a contradiction to the sequence of strict inequalities above. So, we cannot have $t \geq 3$. Therefore, for the Ω-perfect module $M' = \tau^n M$ we have that $t = \alpha(M') \leq 2$. Since the function α is invariant under τ by 2.3.12, we know that $\alpha(M) \leq 2$.

□

Since the component in the previous proposition is regular, the theorem actually states that we cannot have more than two indecomposable summands appearing in a chosen decomposition of the middle term of an almost split sequence ending at C. Furthermore, since all the modules in $\mathcal{C}$ have complexity 1 the statement holds for all modules in $\mathcal{C}$. Now, we prove the main result in this section. Combining previous arguments we are able to determine the structure of a regular component containing a module of complexity 1. Recall that a regular component that is a tube is of type $\mathbb{Z}A_\infty/\langle\tau^n\rangle$.

Theorem 5.2.9. [6, Theorem 3.7] *Let R be a selfinjective Artin algebra and $\mathcal{C}$ a regular component of the Auslander-Reiten quiver containing a module of complexity 1. Then $\mathcal{C}$ is a tube or a component of type $\mathbb{Z}A_\infty$.*

Proof. If the component contains a τ-periodic module, we know from previous arguments that it is of type $\mathbb{Z}A_\infty/\langle\tau^n\rangle$.

So, we assume $\mathcal{C}$ does not contain a τ-periodic module. By 3.2.5 we know that all modules in the component have complexity 1 and further by 5.2.8 we then know that $\alpha(C) \leq 2$ for all $C \in \mathcal{C}$. Since all modules have complexity 1 in the regular component we know that $\beta(C)$ is finite for all $C \in \mathcal{C}$ by 3.2.2. Moreover, by 5.2.4 we know that there exists a module M in $\mathcal{C}$ such that $\alpha(M) = 1$. We want argue that this implies that $\mathcal{C}$ has a boundary. Now assume there is an $n \in \mathbb{Z}$

such that the almost split sequence ending at $\tau^n M$ (up to isomorphism) has two non-isomorphic summands in a chosen decomposition of the middle term.

$$
\begin{tikzcd}[column sep=small]
 & \vdots & & \vdots & & & \vdots & & \vdots & & \vdots & \\
  \arrow[dr] & & {[\tau^n M']} \arrow[dr] & & & \arrow[dr] & & {[M']} \arrow[dr] & & {[\tau^{-1}M']} \arrow[dr] & & \\
 & \cdots [\tau^{n+1}M] \arrow[ur] \arrow[dr] & & {[\tau^n M]} \arrow[urr] & \cdots & & {[\tau M]} \arrow[ur] & & {[M]} \arrow[ur] & & {[\tau^{-1}M]} \cdots \arrow[ur] & \\
 & & {[Y]} \arrow[ur] & & & & & & & & &
\end{tikzcd}
$$

This cannot be since the function α is invariant under τ by 2.3.12 and $\alpha(M) = 1$. That is, we have a boundary. Moreover, by the same arguments, we cannot have an $n \in \mathbb{Z}$ such that an almost split sequence ending at $\tau^n M$ has two indecomposable isomorphic middle terms. That is, the valuation of every arrow $[\tau^n M'] \longrightarrow [\tau^n M]$ is $a_{\tau^n M', \tau^n M} = 1$ and moreover, the valuation is $a'_{\tau^{n+1}M, \tau^n M'} = 1$ for every arrow $[\tau^{n+1}M] \longrightarrow [\tau^n M']$, where $n \in \mathbb{Z}$.

We now let

$$
0 \longrightarrow \tau M \longrightarrow M' \longrightarrow M \longrightarrow 0
$$

be an almost split sequence ending at M, and want to explore the almost split sequence ending at M'. As $\tau M' \longrightarrow \tau M$ is an epimorphism and $\tau M \longrightarrow M'$ is a monomorphism, the almost split sequence ending at M' has to have at least two summands in a chosen decomposition of the middle term. Since $\alpha(M') \leq 2$ we know that it has two indecomposable summands in a chosen decomposition of the middle term. That is, there exists an indecomposable M'' such that

$$
0 \longrightarrow \tau M' \longrightarrow \tau M \oplus M'' \longrightarrow M' \longrightarrow 0
$$

is the almost split sequence ending at M' and by 4.1.12, $M'' \longrightarrow M'$ is an epimorphism and $\tau M' \longrightarrow M''$ is a monomorphism. Note that $M'' \ncong \tau M$ because $\tau M \longrightarrow M'$ is a monomorphism and therefore if they were isomorphic $\ell(M'') = \ell(\tau M) < \ell(M')$ and we could not have an epimorphism $M'' \longrightarrow M'$. That is, we recall the previous arguments and get that $a_{\tau^{n+1}M, \tau^n M'} = 1$ and $a'_{\tau^n M', \tau^n M} = 1$ for all $n \in \mathbb{Z}$. So, all the arrows on the boundary have valuation $(1, 1)$. Furthermore, for $n \in \mathbb{Z}$, we know that $a_{\tau^n M'', \tau^n M'} = 1$ and $a'_{\tau^{n+1}M', \tau^n M''} = 1$ as well. So, we now have the following part of the component

$$
\begin{tikzcd}
\cdots & & {[\tau M'']} \arrow[dr, two heads, "{(1,?)}"] & & {[M'']} \arrow[dr, two heads, "{(1,?)}"] & & {[\tau^{-1}M'']} \arrow[dr, two heads, "{(1,?)}"] & & {[\tau^{-2}M'']} \arrow[dr, two heads, "{(1,?)}"] & & \cdots \\
& {} \arrow[ur, hook, "{(?,1)}"] \arrow[dr, two heads, "{(1,1)}"] & & {[\tau M']} \arrow[ur, hook, "{(?,1)}"] \arrow[dr, two heads, "{(1,1)}"] & & {[M']} \arrow[ur, hook, "{(?,1)}"] \arrow[dr, two heads, "{(1,1)}"] & & {[\tau^{-1}M']} \arrow[ur, hook, "{(?,1)}"] \arrow[dr, two heads, "{(1,1)}"] & & {} & \\
\cdots & & {[\tau^2 M]} \arrow[ur, hook, "{(1,1)}"] & & {[\tau M]} \arrow[ur, hook, "{(1,1)}"] & & {[M]} \arrow[ur, hook, "{(1,1)}"] & & {[\tau^{-1}M]} \arrow[ur, hook, "{(1,1)}"] & & \cdots
\end{tikzcd}
$$

where the question mark indicates that we do not know the valuation of $a'_{\tau^n M'',\tau^n M'}$ for $n \in \mathbb{Z}$ yet. We have also illustrated which morphisms that are epimorphisms (indicated by a two headed arrow arrow) and which are monomorphisms (indicated by a hook arrow). We get this information by applying 4.1.12 in a similar manner as above.

The morphism $\tau M'' \longrightarrow \tau M'$ is an epimorphism and $\tau M' \longrightarrow M''$ is a monomorphism. So, an almost split sequence ending at M'' has two indecomposable non-isomorphic summands in a chosen decomposition of the middle term by the previous and furthermore $a_{\tau^{n+1}M',\tau^n M''} = 1$ for $n \in \mathbb{Z}$. Continuing this argument we get that we cannot have an upper boundary and the valuation of every arrow is $(1,1)$. That is, the component is of type $\mathbb{Z}A_\infty$. We illustrate the component

$$
\begin{tikzcd}
& & & \vdots & & \vdots & & \vdots & & & \\
& {} \arrow[dr, two heads] & & {[M''']} \arrow[dr, two heads] & & {[\tau^{-1}M''']} \arrow[dr, two heads] & & {[\tau^{-2}M''']} \arrow[dr, two heads] & & {} & \\
\cdots & & {[\tau M'']} \arrow[ur, hook] \arrow[dr, two heads] & & {[M'']} \arrow[ur, hook] \arrow[dr, two heads] & & {[\tau^{-1}M'']} \arrow[ur, hook] \arrow[dr, two heads] & & {[\tau^{-2}M'']} \arrow[ur, hook] \arrow[dr, two heads] & & \cdots \\
& {} \arrow[ur, hook] \arrow[dr, two heads] & & {[\tau M']} \arrow[ur, hook] \arrow[dr, two heads] & & {[M']} \arrow[ur, hook] \arrow[dr, two heads] & & {[\tau^{-1}M']} \arrow[ur, hook] \arrow[dr, two heads] & & {} & \\
\cdots & & {[\tau^2 M]} \arrow[ur, hook] & & {[\tau M]} \arrow[ur, hook] & & {[M]} \arrow[ur, hook] & & {[\tau^{-1}M]} \arrow[ur, hook] & & \cdots
\end{tikzcd}
$$

and as usual we neglect the valuation of the arrows since it is trivial. $\square$

This result prove important in the next section.

5.2.2 Modules with eventually constant Betti numbers

In this section we investigate regular components of the Auslander-Reiten quiver of a selfinjective Artin algebra containing modules with eventually constant Betti numbers. Such a module is non-projective as a result of it being in a regular component and its Betti numbers are bounded. That is, it has complexity 1 by 2.3.7 and 3.2.2. By 3.2.5, we then know that all the modules in the regular component have complexity 1. This argument is not repeated in the upcoming proofs. Furthermore, by 5.2.9, we know that a regular component of the Auslander-Reiten quiver of a selfinjective Artin algebra containing modules of complexity 1 is of type $\mathbb{Z}A_\infty$ or a tube, that is, of type $\mathbb{Z}A_\infty/\langle\tau^n\rangle$. Since the valuation of A_∞ is trivial, we know that the component illustrates actual almost split sequences. Also, recall from 5.2.3 that all modules in a component of type $\mathbb{Z}A_\infty/\langle\tau^n\rangle$ that does not contain any simple modules are Ω-perfect. By now, this should be well known facts and we use these results without reference in the proofs. Before we introduce the first result we need to define quasi-length for components of type $\mathbb{Z}A_\infty$ and $\mathbb{Z}A_\infty/\langle\tau^n\rangle$.

Recall from Subchapter 2.2 that since A_∞ has trivial valuation, the valued translation quiver, $\mathbb{Z}A_\infty$, is independent of orientation in A_∞. We enumerate the vertices in A_∞ by the natural numbers and have orientation as shown below.

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow n \longrightarrow n+1 \longrightarrow \cdots$$

We illustrate a part of $\mathbb{Z}A_\infty$.

$$\begin{tikzcd}[column sep=small]
 & & \vdots & & \vdots & & \vdots & & \\
 \arrow[dr] & & (-2,4) \arrow[dr] & & (-1,4) \arrow[dr] & & (0,4) \arrow[dr] & & \\
\cdots & (-2,3) \arrow[ur] \arrow[dr] & & (-1,3) \arrow[ur] \arrow[dr] & & (0,3) \arrow[ur] \arrow[dr] & & (1,3) \arrow[ur] \arrow[dr] & \cdots \\
 \arrow[ur] \arrow[dr] & & (-1,2) \arrow[ur] \arrow[dr] & & (0,2) \arrow[ur] \arrow[dr] & & (1,2) \arrow[ur] \arrow[dr] & & \\
\cdots & (-1,1) \arrow[ur] & & (0,1) \arrow[ur] & & (1,1) \arrow[ur] & & (2,1) \arrow[ur] & \cdots
\end{tikzcd}$$

That is, the vertices of $\mathbb{Z}A_\infty$ are of the form (z,i) with $z \in \mathbb{Z}$ and $i \in \mathbb{N}$ and the vertices of $\mathbb{Z}A_\infty/\langle\tau^n\rangle$ are of form $(\bar{z},i)$, with $\bar{z} \in \mathbb{Z}/(n)$ and $i \in \mathbb{N}$. Then, we say that the element (z,i) in $\mathbb{Z}A_\infty$ or $(\bar{z},i)$ in $\mathbb{Z}A_\infty/\langle\tau^n\rangle$ is of *quasi-length* i, [12]. That is, a module in vertex (z,i) in a component of type $\mathbb{Z}A_\infty$ or $(\bar{z},i)$ in a component of type $\mathbb{Z}A_\infty/\langle\tau^n\rangle$ has quasi-length i. It follows that all modules in the same τ-orbit have equal quasi-length. We denote the quasi-length of a module M by $\mathrm{ql}(M)$. Further, we note that the modules in a component of type $\mathbb{Z}A_\infty$ or $\mathbb{Z}A_\infty/\langle\tau^n\rangle$ of quasi-length 1 lie on the boundary of the component. We now study the case where there is a module with eventually constant Betti numbers lying on the boundary of such a component. In the case where the component is a tube we assume that it does not contain any simple modules.

Proposition 5.2.10. [6, Proposition 4.1] *Let R be a selfinjective Artin algebra and let $\mathcal{C}$ be a regular component of the Auslander-Reiten quiver of R containing a module M whose Betti numbers are eventually equal to b and with $\alpha(M) = 1$. Assume also that if $\mathcal{C}$ is a tube, then it contains no simple R-modules. Then every module B in $\mathcal{C}$ has eventually constant Betti numbers equal to* $\mathrm{ql}(B)b$.

Proof. By previous arguments we know that all modules in the component have complexity 1 and moreover that $\mathcal{C}$ is of type $\mathbb{Z}A_\infty$ or $\mathbb{Z}A_\infty/\langle\tau^n\rangle$. If $\mathcal{C}$ is of type $\mathbb{Z}A_\infty/\langle\tau^n\rangle$ and does not contain any simple modules we know that all modules in the component are Ω-perfect. If $\mathcal{C}$ is of type $\mathbb{Z}A_\infty$, by 4.2.8, we know that there exists an $m \geq 0$ such that $\tau^m M$ is Ω-perfect. Since $\beta_{i+2j}(X) = \beta_i(\tau^j X)$ for all $i \geq 0$ and $j \geq 0$ for any $X \in \mathcal{C}$ by 3.1.2, we know that modules in the same τ-orbit as a module with Betti numbers eventually equal to some constant a also have Betti numbers that are eventually equal to a. Furthermore, we know that all modules in the same τ-orbit have equal quasi-length. So, we may assume that M is Ω-perfect. From the previous arguments we know that if M' is a module on the boundary of $\mathcal{C}$, that is, in the same τ-orbit as M, it has eventually constant Betti numbers equal to b. In other words, equal to $\mathrm{ql}(M')b$. We illustrate a part of $\mathcal{C}$.

$$\begin{tikzcd}[column sep=tiny]
 & & \vdots & & \vdots & & \vdots & & \\
 & & {[B_4]} \arrow[dr] & & {[\tau^{-1}B_4]} \arrow[dr] & & {[\tau^{-2}B_4]} \arrow[dr] & & \\
\cdots & {[\tau B_3]} \arrow[ur] \arrow[dr] & & {[B_3]} \arrow[ur] \arrow[dr] & & {[\tau^{-1}B_3]} \arrow[ur] \arrow[dr] & & {[\tau^{-2}B_3]} & \cdots \\
 & & {[\tau B_2]} \arrow[ur] \arrow[dr] & & {[B_2]} \arrow[ur] \arrow[dr] & & {[\tau^{-1}B_2]} \arrow[ur] \arrow[dr] & & \\
\cdots & {[\tau^2 M]} \arrow[ur] & & {[\tau M]} \arrow[ur] & & {[M]} \arrow[ur] & & {[\tau^{-1}M]} & \cdots
\end{tikzcd}$$

We use induction on the quasi-length n of B_n and we let $B_1 = M$. We start by looking at $n = 2$. An almost split sequence ending at M is

$$0 \longrightarrow \tau M \longrightarrow B_2 \longrightarrow M \longrightarrow 0$$

and by 4.1.11 we know that $\beta_i(B_2) = \beta_i(M) + \beta_i(\tau M)$ for $i \geq 0$ since M is Ω-perfect by assumption. Further, since the Betti numbers of M and τM are eventually equal to b it follows that the Betti numbers of B_2 are eventually equal to $2b$, that is, eventually equal to $\mathrm{ql}(B_2)b$. So, all the modules in the same τ-orbit as B_2, that is, the modules with quasi-length 2, have Betti numbers eventually equal to $2b$.

Now, let $n \geq 3$ and assume we have shown that B_j, and therefore also all the modules in the same τ-orbit as B_j, has eventually constant Betti numbers equal to jb for $j \in \{1, 2, .., n-1\}$. We may assume that B_{n-1} is Ω-perfect. Since $\mathcal{C}$ is of type $\mathbb{Z}A_\infty$ or $\mathbb{Z}A_\infty/\langle\tau^n\rangle$ we know by previous arguments that an almost split sequence ending at B_{n-1} is

$$0 \longrightarrow \tau B_{n-1} \longrightarrow \tau B_{n-2} \oplus B_n \longrightarrow B_{n-1} \longrightarrow 0.$$

Since B_{n-1} and τB_{n-1} have Betti numbers eventually equal to $(n-1)b$ and τB_{n-2} has Betti numbers eventually equal to $(n-2)b$ by assumption, and further

$$\beta_i(B_n) = \beta_i(B_{n-1}) + \beta_i(\tau B_{n-1}) - \beta_i(\tau B_{n-2})$$

for $i \geq 0$ by 4.1.11, we know that B_n has Betti numbers eventually equal to nb. Of course, this is also the case for the modules in the same τ-orbit as B_n. That is, all modules with quasi-length n have eventually constant Betti numbers equal to nb.

$\square$

We let R be a selfinjective Artin algebra and $\mathcal{C}$ be a regular component of the Auslander-Reiten quiver of R that does not contain any simple modules if it is a tube. Then, from the previous argument, we conclude that if M is on the boundary of the component and has eventually constant Betti numbers, then all modules in the component have eventually constant Betti numbers. In the next proposition we see that if we have two modules M and N in such a component with eventually constant Betti numbers and we have an irreducible morphism between them, we know that each module in the component has eventually constant Betti numbers.

Proposition 5.2.11. [6, Proposition 4.2] *Let R be a selfinjective Artin algebra and M and N indecomposable R-modules whose Betti numbers are eventually constant. Assume that M lies in a regular component $\mathcal{C}$ of the Auslander-Reiten quiver of R that, if a tube, contains no simple modules. If there is an irreducible homomorphism $M \longrightarrow N$, then every module in $\mathcal{C}$ has eventually constant Betti numbers.*

Proof. By previous arguments we know that all modules in the component have complexity 1 and further that $\mathcal{C}$ is of type $\mathbb{Z}A_\infty$ or $\mathbb{Z}A_\infty/\langle\tau^n\rangle$. If $\mathcal{C}$ is a tube without simple modules, then all modules in $\mathcal{C}$ are Ω-perfect. Moreover, if $\mathcal{C}$ is of type $\mathbb{Z}A_\infty$ we know that there exists an $m \geq 0$ such that both $\tau^m M$ and $\tau^m N$ are Ω-perfect by 4.2.8 and 4.1.10. Furthermore, since $\beta_{i+2j}(X) = \beta_i(\tau^j X)$ for all $i \geq 0$ and $j \geq 0$ for any module X in $\mathcal{C}$, we know that all modules in the same τ-orbit as M (N) have Betti numbers eventually equal to the same constant the Betti numbers of M (N) are eventually equal to. That is, we can assume both M and N are Ω-perfect. If either M or N is on the boundary of $\mathcal{C}$, we are done by 5.2.10. If not, since we know the structure of the component we have an almost split sequence ending at N

$$0 \longrightarrow \tau N \longrightarrow M \oplus M' \longrightarrow N \longrightarrow 0$$

where M and M' are indecomposable. By 4.1.11 we know that

$$\beta_i(M') = \beta_i(N) + \beta_i(\tau N) - \beta_i(M)$$

for $i \geq 0$ and moreover we know that N, τN and M have eventually constant Betti numbers. That is, the module M', and therefore also all modules in the same τ-orbit as M', has eventually constant Betti numbers.

Now let X be an arbitrary module in the component. Then we can assume that $\tau^k X$ lies on a sectional path of Ω-perfect modules ending at an Ω-perfect module in the τ-orbit of N or in the τ-orbit of M for some $k \in \mathbb{Z}$. That is, we may repeatedly use 4.1.11 and the arguments above to argue that $\tau^k X$, and therefore also X, has eventually constant Betti numbers.

□

Remark. In the two previous proofs we let R be a selfinjective Artin algebra and investigate regular components of type $\mathbb{Z}A_\infty$ or $\mathbb{Z}A_\infty/\langle\tau^n\rangle$. We argue that a module in the same τ-orbit as a module with eventually constant Betti numbers equal to some constant a also has Betti numbers eventually equal to a. The argument for this is not repeated in the upcoming proofs.

We again let R be a selfinjective Artin algebra and $\mathcal{C}$ be a regular component of the Auslander-Reiten quiver of R that does not contain any simple modules if it is a tube. From the two previous results we know the following. If there is a module on the boundary of $\mathcal{C}$ with eventually constant Betti numbers, then every module in the component has eventually constant Betti numbers. Furthermore, we know that if there are two modules with eventually constant Betti numbers connected with an irreducible morphism in $\mathcal{C}$, then all modules in $\mathcal{C}$ have eventually constant

Betti numbers. It would be of interest to investigate if $\mathcal{C}$ containing a module with eventually constant Betti numbers would imply that all modules in $\mathcal{C}$ have eventually constant Betti numbers. Unfortunately, in the nonlocal case this need not be true. It is not known if it is true in the local case [6]. We can however say something about the Betti numbers of an arbitrary module in the regular component containing a module with eventually constant Betti numbers. Before we are able to present this result we need to define periodic and eventually periodic Betti numbers. We say that an R-module M has *periodic Betti numbers* if there is some positive integer n such that $\beta_i(M) = \beta_{i+n}(M)$, for all $i \geq 0$. Moreover, we say that M has *eventually periodic Betti numbers* if there are some positive integers n and k such that $\beta_i(M) = \beta_{i+n}(M)$ for all $i \geq k$. We now present a useful lemma before we explore properties of an arbitrary module in a regular component, that, if a tube does not contain simple modules, containing a module with eventually constant Betti numbers.

Lemma 5.2.12. *Let R be a selfinjective Artin algebra. Assume that $\mathcal{C}$ is a regular component of the Auslander-Reiten quiver of type $\mathbb{Z}A_\infty$ or $\mathbb{Z}A_\infty/\langle\tau^n\rangle$ and that C is an Ω-perfect module on the boundary of $\mathcal{C}$. If*

$$M_n \xrightarrow{f_n} M_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} M_1 \xrightarrow{f_1} C$$

is a sectional path of irreducible epimorphisms in $\mathcal{C}$ with $n \geq 1$, then each irreducible epimorphism $f_k\colon M_k \longrightarrow M_{k-1}$ is Ω-perfect for $k \in \{1, ..., n\}$ where $M_0 = C$. In particular, $\beta_i(M_k) = \beta_i(M_{k-1}) + \beta_i(\tau^k C)$ for $i \geq 0$.

Proof. We illustrate a part of the component.

$$\begin{tikzcd}
 & & & \vdots & & \vdots & & \vdots & & & \\
 & {} \arrow[dr] & & {[M_3]} \arrow[dr] & & {[\tau^{-1}M_3]} \arrow[dr] & & {[\tau^{-2}M_3]} \arrow[dr] & & {} & \\
\cdots & & {[\tau M_2]} \arrow[ur] \arrow[dr] & & {[M_2]} \arrow[ur] \arrow[dr] & & {[\tau^{-1}M_2]} \arrow[ur] \arrow[dr] & & {[\tau^{-2}M_2]} \arrow[ur] \arrow[dr] & & \cdots \\
 & {} \arrow[ur] \arrow[dr] & & {[\tau M_1]} \arrow[ur] \arrow[dr] & & {[M_1]} \arrow[ur] \arrow[dr] & & {[\tau^{-1}M_1]} \arrow[ur] \arrow[dr] & & {} & \\
\cdots & & {[\tau^2 C]} \arrow[ur] & & {[\tau C]} \arrow[ur] & & {[C]} \arrow[ur] & & {[\tau^{-1}C]} \arrow[ur] & & \cdots
\end{tikzcd}$$

For $f_1\colon M_1 \longrightarrow C$ the result follows from the fact that C is Ω-perfect and 4.1.11. Now, look at $f_k\colon M_k \longrightarrow M_{k-1}$ with $k \in \{2, ..., n\}$. By previous arguments we know that $\tau^m f_k$ is an epimorphism for all $m \geq 0$. Moreover, by applying 4.1.12

in the same manner as in Section 5.2.1, we know that its kernel is isomorphic to $\tau^{m+k}C$. We know that C is Ω-perfect, so $\Omega^{2(m+k)}C$ is not simple for $m \geq 0$ and $k \in \{2, ..., n\}$ by 4.1.8. That is, by 2.3.4 and 2.3.6 we know that $\tau^{m+k}C$ is not simple for $m \geq 0$ and $k \in \{2, ..., n\}$. Furthermore, we have that

$$\ell(\tau^{m+k}C) + \ell(\tau^m M_{k-1}) = \ell(\tau^m M_k). \tag{5.16}$$

We now want to show that f_k is Ω-perfect. Since $\tau^m f_k$ is an epimorphism for all $m \geq 0$, by 2.3.13, we know that $\Omega^{2m} f_k$ is an epimorphism for all $m \geq 0$. That is, we have an exact sequence

$$0 \longrightarrow K_m \longrightarrow \Omega^{2m} M_k \xrightarrow{\Omega^{2m} f_k} \Omega^{2m} M_{k-1} \longrightarrow 0$$

for $m \geq 0$. That is, we know that

$$\ell(K_m) + \ell(\Omega^{2m} M_{k-1}) = \ell(\Omega^{2m} M_k). \tag{5.17}$$

Now, by 2.3.4 and 2.3.6 we have that

$$\begin{aligned} &\ell(\Omega^{2m} M_{k-1}) = \ell(\tau^m M_{k-1}) \text{ and} \\ &\ell(\Omega^{2m} M_k) = \ell(\tau^m M_k). \end{aligned}$$

So, by (5.16) and (5.17) we know that $\ell(K_m) = \ell(\tau^{m+k}C) > 1$ for all $m \geq 0$. That is, the kernel of $\Omega^{2m} f_k$ is not simple for $m \geq 0$ and by 4.1.7 we know that $\Omega^{2m+1} f_k$ is an epimorphism for all $m \geq 0$. That is, $f_k \colon M_k \longrightarrow M_{k-1}$ is Ω-perfect.

Then, applying 4.1.11 we know that $\beta_i(M_k) = \beta_i(M_{k-1}) + \beta_i(\tau^k C)$ for all $i \geq 0$ and $k \in \{2, ..., n\}$.

□

The previous lemma is used in the next proposition where we investigate properties of Betti numbers of an arbitrary module in a regular component of the Auslander-Reiten quiver of a selfinjective Artin algebra containing a module with eventually constant Betti numbers. Again, we assume the component does not contain any simple modules if it is a tube.

Proposition 5.2.13. [6, Proposition 4.3] *Let R be a selfinjective Artin algebra, and let M_n be an indecomposable R-module with eventually constant Betti numbers lying in a regular component $\mathcal{C}$ of the Auslander-Reiten quiver, that, if a tube contains no simple modules. Then every module in $\mathcal{C}$ has eventually periodic Betti numbers. Furthermore, the eventual period of the Betti numbers of a module in $\mathcal{C}$ divides $2\,\mathrm{ql}(M_n)$.*

Proof. Again, we know that all modules in the component have complexity 1 and that $\mathcal{C}$ is of type $\mathbb{Z}A_\infty$ or $\mathbb{Z}A_\infty/\langle\tau^n\rangle$. Further, we know that all modules in the same τ-orbit as M_n have eventually constant Betti numbers. If M_n is on the boundary of $\mathcal{C}$, then by 5.2.10, every module in the component have eventually constant Betti numbers, that is eventually period 1, which obviously divides $2\,\mathrm{ql}(M_n)$.

Now assume M_n is not on the boundary of $\mathcal{C}$ and have quasi-length $n+1$. By applying τ a certain number of times, since $\beta_i(\tau^j M_n) = \beta_{i+2j}(M_n)$ for all $i \geq 0$ and $j \geq 0$ by 3.1.2, we can assume that M_n has constant Betti numbers. We illustrate a part of the component

$$\begin{tikzcd}[column sep=tiny]
 & & & & & & & & \vdots & & & & & & & \\
 & & & & & & \vdots & \cdots & {[M_n]} \arrow[dr, two heads] & \cdots & \vdots & & & & & \\
 & & & & & & \cdots & {[\tau M_{n-1}]} \arrow[ur, hook] \arrow[dr, two heads] & & {[M_{n-1}]} \arrow[dr, two heads] & \cdots & & & & & \\
 & & & & & \cdots & {[\tau^2 M_{n-2}]} \arrow[ur, hook] & & {[\tau M_{n-2}]} \arrow[ur, hook] & & {[M_{n-2}]} \arrow[ddrrr, dashed, two heads] & \cdots & & & & \\
 & & & \cdots & & & & & & & & & & & \cdots & \\
 & \cdots & {[\tau^{n-1} M_1]} \arrow[uurrrr, dashed, hook] \arrow[dr, two heads] & & & & & & & & & & & {[M_1]} \arrow[dr, two heads] & \cdots & \\
 \cdots & {[\tau^n C]} \arrow[ur, hook] & & {[\tau^{n-1} C]} \arrow[ur, hook] & & & & & \cdots & & & & {[\tau C]} \arrow[ur, hook] & & {[C]} & \cdots
\end{tikzcd}$$

So, we have a sectional path of n irreducible epimorphisms

$$M_n \twoheadrightarrow M_{n-1} \twoheadrightarrow M_{n-2} \twoheadrightarrow \cdots \twoheadrightarrow M_1 \twoheadrightarrow C$$

where C is on the boundary of $\mathcal{C}$, and a sectional path of n irreducible monomorphisms

$$\tau^n C \hookrightarrow \tau^{n-1} M_1 \hookrightarrow \tau^{n-2} M_2 \hookrightarrow \cdots \hookrightarrow \tau M_{n-1} \hookrightarrow M_n.$$

We can assume that all of the modules $\{M_k\}_{k=1}^n$ and C are Ω-perfect, because either all modules in $\mathcal{C}$ are Ω-perfect or for each module $M \in \mathcal{C}$ there exists an $l \geq 0$ such that applying τ^l to M gives an Ω-perfect module by respectively 5.2.3 and 4.2.8. Furthermore, if this assumption gives us that the Ω-perfect modules have eventually periodic Betti numbers, since $\beta_{i+2j}(X) = \beta_i(\tau^j X)$ for all $X \in \mathcal{C}$

with $j \geq 0$ and $i \geq 0$, every module in the same τ-orbit also have eventual periodic Betti numbers with equal eventual period.

We first argue that every module on the boundary has eventually periodic Betti numbers. Note that we in this part of the proof only need that C is Ω-perfect. By 5.2.12 we know that

$$\beta_i(\tau^j C) + \beta_i(M_{j-1}) = \beta_i(M_j)$$

for all $j \in \{1, ..., n\}$ and $i \geq 0$ where $M_0 = C$. Using this we get that

$$\beta_i(M_n) = \sum_{j=0}^{n} \beta_i(\tau^j C) \tag{5.18}$$

for all $i \geq 0$ and in a similar manner, since τC is Ω-perfect by 4.1.10, we have that

$$\beta_i(\tau M_n) = \sum_{j=1}^{n+1} \beta_i(\tau^j C) \tag{5.19}$$

for all $i \geq 0$. The module M_n has constant Betti numbers and then since $\beta_{i+2}(M_n) = \beta_i(\tau M_n)$ for $i \geq 0$ by 3.1.2, we know that $\beta_i(M_n) = \beta_i(\tau M_n)$ for all $i \geq 0$. So, by (5.18) and (5.19) we know that $\beta_i(\tau^{n+1} C) = \beta_i(C)$ for $i \geq 0$. Further, by 3.1.2, we then know that $\beta_{i+2(n+1)}(C) = \beta_i(C)$ for $i \geq 0$. In other words, C has periodic Betti numbers with a period that divides $2(n+1)$. This implies that every module on the boundary has eventually periodic Betti numbers with period dividing $2(n+1)$.

We know that

$$0 \longrightarrow \tau C \longrightarrow M_1 \longrightarrow C \longrightarrow 0$$

is an almost split sequence ending at the Ω-perfect module C and that

$$\beta_i(M_1) = \beta_i(\tau C) + \beta_i(C) \tag{5.20}$$

for $i \geq 0$. Further,

$$0 \longrightarrow \tau^{n+2} C \longrightarrow \tau^{n+1} M_1 \longrightarrow \tau^{n+1} C \longrightarrow 0$$

is an almost split sequence ending at $\tau^{n+1} C$, an Ω-perfect module by 4.1.10. So, by 4.1.11 we have that

$$\beta_i(\tau^{n+1} M_1) = \beta_i(\tau^{n+2} C) + \beta_i(\tau^{n+1} C) \tag{5.21}$$

for $i \geq 0$. Since C has periodic Betti numbers with a period that divides $2(n+1)$ we know by 3.1.2 that

$$\beta_i(\tau^{n+1}C) = \beta_{i+2(n+1)}(C) = \beta_i(C)$$

for $i \geq 0$ and

$$\beta_i(\tau^{n+2}C) = \beta_{i+2(n+1)}(\tau C) = \beta_i(\tau C)$$

for $i \geq 0$. So, then by (5.20) and (5.21) we have that

$$\beta_i(M_1) = \beta_i(\tau^{n+1}M_1) = \beta_{i+2(n+1)}(M_1)$$

for $i \geq 0$ using 3.1.2. That is, M_1 has periodic Betti-numbers with a period dividing $2(n+1)$. Then all modules in the same τ-orbit as M_1 have eventually periodic Betti numbers with period dividing $2(n+1)$.

We recall that we may assume that all the modules $\{M_k\}_{k=1}^n$ are Ω-perfect and now look at the almost split sequences ending at the Ω-perfect modules M_1 and $\tau^{n+1}M_1$

$$0 \longrightarrow \tau M_1 \longrightarrow \tau C \oplus M_2 \longrightarrow M_1 \longrightarrow 0$$

and

$$0 \longrightarrow \tau^{n+2}M_1 \longrightarrow \tau^{n+2}C \oplus \tau^{n+1}M_2 \longrightarrow \tau^{n+1}M_1 \longrightarrow 0.$$

So, by 4.1.11 we know that

$$\beta_i(\tau M_1) + \beta_i(M_1) = \beta_i(\tau C) + \beta_i(M_2) \tag{5.22}$$

for $i \geq 0$ and

$$\beta_i(\tau^{n+2}M_1) + \beta_i(\tau^{n+1}M_1) = \beta_i(\tau^{n+2}C) + \beta_i(\tau^{n+1}M_2) \tag{5.23}$$

for $i \geq 0$. We know that M_1 and C have periodic Betti numbers with a period dividing $2(n+1)$, so then by (5.22) and (5.23) we know that $\beta_i(M_2) = \beta_i(\tau^{n+1}M_2)$ for $i \geq 0$. That is, M_2 has periodic Betti numbers with a period dividing $2(n+1)$. Furthermore, all modules in the same τ-orbit as M_2 have eventually periodic Betti numbers with period dividing $2(n+1)$.

Continuing this argument, we have that all the modules $\{M_k\}_{k=1}^n$ have periodic Betti numbers with a period dividing $2(n+1)$. So, all modules in the their τ-orbits have eventually periodic Betti numbers with period dividing $2(n+1)$.

Now, assume X is an arbitrary module in $\mathcal{C}$ that is not in the τ-orbit of any of the M_i's. We can assume that $\tau^m X$ for some $m \geq 0$ lies on a sectional path of Ω-perfect modules containing an Ω-perfect module in the τ-orbit of M_n with constant Betti numbers and ending at a module in the τ-orbit of M_{n-1} with periodic Betti numbers with period dividing $2(n+1)$. Then repeating the previous argument, we see that $\tau^m X$ has periodic Betti numbers with period dividing $2(n+1)$ and moreover X has eventually periodic Betti numbers with period dividing $2(n+1)$.

□

Remark. We let R be a selfinjective Artin algebra and investigate regular components of type $\mathbb{Z}A_\infty$ or $\mathbb{Z}A_\infty/\langle\tau^n\rangle$. Then, from the previous proof we see that a module in the same τ-orbit as a module with eventually periodic Betti numbers with period a also has eventually periodic Betti numbers with period a. We do not repeat the argument for this in later proofs.

Furthermore, we have two immediate corollaries.

Corollary 5.2.14. *Let R be a selfinjective Artin algebra and let $\mathcal{C}$ be a regular component of the Auslander-Reiten quiver of type $\mathbb{Z}A_\infty$ or $\mathbb{Z}A_\infty/\langle\tau^n\rangle$. Let*

$$M_n \twoheadrightarrow M_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow M_1 \twoheadrightarrow C$$

be a sectional path of irreducible epimorphisms in $\mathcal{C}$ with C an Ω-perfect R-module on the boundary of the component. Then $\beta_i(M_n) = \sum_{j=0}^{n} \beta_i(\tau^j C)$ for all $i \geq 0$ and $n \geq 1$.

□

Corollary 5.2.15. *Let R be a selfinjective Artin algebra and let $\mathcal{C}$ be a regular component of the Auslander-Reiten quiver containing a module M_n with constant Betti numbers. Furthermore, assume we have a sectional path of irreducible epimorphisms in $\mathcal{C}$*

$$M_n \twoheadrightarrow M_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow M_1 \twoheadrightarrow C$$

with C an Ω-perfect R-module on the boundary of the component. Then, the module C has periodic Betti numbers with period dividing $2\,\mathrm{ql}(M_n)$.

□

We now present two lemmas that we need to prove the last two main results in this chapter.

Lemma 5.2.16. [6, Lemma 4.4] *Suppose that $\mathcal{C}$ is a regular component of the Auslander-Reiten quiver of a selfinjective Artin algebra R of type $\mathbb{Z}A_\infty$ or $\mathbb{Z}A_\infty/\langle\tau^n\rangle$ and that C is an Ω-perfect R-module on the boundary of $\mathcal{C}$. If*

$$M_n \twoheadrightarrow M_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow M_1 \twoheadrightarrow C$$

is a sectional path of irreducible epimorphisms in $\mathcal{C}$, then $\beta_i(M_n) = \sum_{j=0}^n \beta_{i+2j}(C)$, for all $i \geq 0$ and $n \geq 1$.

Proof. By 5.2.14 we know that

$$\beta_i(M_n) = \sum_{j=0}^{n} \beta_i(\tau^j C) \text{ for all } i \geq 0 \text{ and } n \geq 1.$$

So, then by 3.1.2 we have that

$$\beta_i(M_n) = \sum_{j=0}^{n} \beta_{i+2j}(C) \text{ for all } i \geq 0 \text{ and } n \geq 1.$$

□

Note that we have made some changes in the assumptions in the previous lemma compared to how it is presented in [6]. A combination of the lemma and 5.2.15 gives us the following.

Lemma 5.2.17. *Let R be a selfinjective Artin algebra and let M_n be an indecomposable R-module with constant Betti numbers lying in a regular component $\mathcal{C}$ of the Auslander-Reiten quiver. Furthermore, assume that*

$$M_n \twoheadrightarrow M_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow M_1 \twoheadrightarrow C$$

is sectional path of irreducible epimorphisms in $\mathcal{C}$ with C an Ω-perfect module on the boundary of the component. Then, C has periodic Betti numbers with period dividing $2\operatorname{ql}(M_n)$. Furthermore, $\sum_{j=0}^n \beta_{2j}(C) = \sum_{j=0}^n \beta_{2j+1}(C)$ for all $i \geq 0$ and $n \geq 1$.

Proof. Recall that the component is of type $\mathbb{Z}A_\infty$ or $\mathbb{Z}A_\infty/\langle\tau^n\rangle$. By 5.2.15 we know that C has periodic Betti numbers with period dividing $2\operatorname{ql}(M_n) = 2(n+1)$.

Further, by 5.2.16 we know that

$$\beta_0(M_n) = \sum_{j=0}^{n} \beta_{2j}(C)$$

and

$$\beta_1(M_n) = \sum_{j=0}^{n} \beta_{2j+1}(C).$$

So, since M_n has constant Betti numbers we have that $\beta_0(M_n) = \beta_1(M_n)$ and therefore

$$\sum_{j=0}^{n} \beta_{2j}(C) = \sum_{j=0}^{n} \beta_{2j+1}(C)$$

and we are done.

□

From the two previous lemmas we get the following result.

Proposition 5.2.18. [6, Proposition 4.5] *Let R be a selfinjective Artin algebra and let M be an indecomposable R-module with $\mathrm{cx}(M) = 1$ lying in a regular component $\mathcal{C}$ of the Auslander-Reiten quiver, that, if a tube, contains no simple modules. Assume the length of a sectional path from M to a module C on the boundary of the component is n; that is, the quasi-length of M is $n+1$. Then M has eventually constant Betti numbers if and only if C has eventually periodic Betti numbers with period dividing $2\,\mathrm{ql}(M)$, and, for sufficiently large m, $\sum_{j=0}^{n} \beta_{2j}(\tau^m C) = \sum_{j=0}^{n} \beta_{2j+1}(\tau^m C)$.*

Proof. Recall that the component is of type $\mathbb{Z}A_\infty$ or $\mathbb{Z}A_\infty/\langle\tau^n\rangle$. We may assume that C is Ω-perfect, as argued earlier. By applying τ sufficiently many times and recalling the previous remarks, it is enough to show that M has constant Betti numbers if and only if C has periodic Betti numbers with a period dividing $2\,\mathrm{ql}(M)$ and $\sum_{j=0}^{n} \beta_{2j}(C) = \sum_{j=0}^{n} \beta_{2j+1}(C)$.

$\Rightarrow$ If we first assume M has constant Betti numbers, the result follows from 5.2.17 since C is Ω-perfect.

$\Leftarrow$ We now assume C has periodic Betti numbers with a period dividing $2\,\mathrm{ql}(M) = 2(n+1)$. That is, $\beta_k(C) = \beta_{k+2(n+1)}(C)$ for all $k \geq 0$ and therefore

$$\sum_{j=0}^{n} \beta_{2j+i}(C) = \sum_{j=0}^{n} \beta_{2j+s}(C)$$

where $s = 0$ if i is even and $s = 1$ if i is odd. By assumption we also have that

$$\sum_{j=0}^{n} \beta_{2j}(C) = \sum_{j=0}^{n} \beta_{2j+1}(C)$$

and by 5.2.16 we know that

$$\beta_i(M) = \sum_{j=0}^{n} \beta_{2j+i}(C)$$

for $i \geq 0$. Using the three equalities above we want to show that M has constant Betti numbers. Now, if i is even, we get that

$$\beta_i(M) = \sum_{j=0}^{n} \beta_{2j+i}(C) = \sum_{j=0}^{n} \beta_{2j}(C) = \sum_{j=0}^{n} \beta_{2j+1}(C) = \sum_{j=0}^{n} \beta_{2j+(i+1)}(C) = \beta_{i+1}(M).$$

If i is odd, we get that

$$\beta_i(M) = \sum_{j=0}^{n} \beta_{2j+i}(C) = \sum_{j=0}^{n} \beta_{2j+1}(C) = \sum_{j=0}^{n} \beta_{2j}(C) = \sum_{j=0}^{n} \beta_{2j+(i+1)}(C) = \beta_{i+1}(M).$$

That is, M has constant Betti numbers which is what we wanted to show.

□

We now present the last result in this thesis.

Theorem 5.2.19. [6, Theorem 4.6] *Let R be a selfinjective Artin algebra and let $\mathcal{C}$ be a regular component of the Auslander-Reiten quiver of R containing a module having eventually constant Betti numbers. Assume also that, if a tube, $\mathcal{C}$ contains no simple R-modules. Then*

(1) *There is an infinite family $\{M^n\}_{n=1}^{\infty}$ of modules in $\mathcal{C}$ having constant Betti numbers $\{b_n\}_{n=1}^{\infty}$; that is, $\beta_i(M^n) = b_n$, for all $i \geq 0$.*

(2) *The sequence $\{b_n\}$ is strictly increasing.*

(3) *The R-modules M^n lie on distinct τ-orbits.*

(4) *There is a positive integer d, such that, for each $t \geq 1$, M^t can be chosen having constant Betti numbers $b_t = td$.*

Proof. Recall that the component is of type $\mathbb{Z}A_\infty$ or $\mathbb{Z}A_\infty/\langle\tau^n\rangle$. If M is a module with Betti numbers eventually equal to some constant a on the boundary of $\mathcal{C}$, then, by 5.2.10, we know that each module B in the component have eventually constant Betti numbers equal to $\mathrm{ql}(B)a$. That is, we can find an infinite family of modules $\{M^n\}_{n=1}^\infty$ with constant Betti numbers satisfying the four statements.

Now, assume we have a module that is not on the boundary of $\mathcal{C}$ with eventually constant Betti numbers. By applying τ a sufficiently number of times we know that there is a module, not on the boundary, with constant Betti numbers with a sectional path to an Ω-perfect module C on the boundary. So, by 5.2.17 we know that C has periodic Betti numbers with period dividing $2(n+1)$, for an integer n. Moreover, we know that

$$\sum_{j=0}^{n} \beta_{2j}(C) = \sum_{j=0}^{n} \beta_{2j+1}(C). \tag{5.24}$$

Since C has periodic Betti numbers with a period dividing $2(n+1)$, they are also periodic with a period dividing $2l(n+1)$ for $l \geq 1$. That is, we know that $\beta_k(C) = \beta_{k+2l(n+1)}(C)$ for all $l \geq 1$ and $k \geq 0$. So, for some $t \geq 1$, we have that

$$\sum_{j=0}^{t(n+1)-1} \beta_{2j+i}(C) = \sum_{j=0}^{t(n+1)-1} \beta_{2j+s}(C) \tag{5.25}$$

where $s = 0$ if i is even, and $s = 1$ if i is odd. Furthermore, we know that

$$\begin{aligned} \sum_{j=0}^{t(n+1)-1} \beta_{2j}(C) &= t\left(\sum_{j=0}^{n} \beta_{2j}(C)\right) \text{ and} \\ \sum_{j=0}^{t(n+1)-1} \beta_{2j+1}(C) &= t\left(\sum_{j=0}^{n} \beta_{2j+1}(C)\right). \end{aligned}$$

This and (5.24) give us that

$$\sum_{j=0}^{t(n+1)-1} \beta_{2j}(C) = \sum_{j=0}^{t(n+1)-1} \beta_{2j+1}(C) \tag{5.26}$$

Further, for $t \geq 1$ we let

$$M_{t(n+1)-1} \twoheadrightarrow M_{t(n+1)-2} \twoheadrightarrow \cdots \twoheadrightarrow M_2 \twoheadrightarrow M_1 \twoheadrightarrow C$$

be a sectional path of irreducible epimorphisms. By 5.2.16 we know that

$$\beta_i(M_{t(n+1)-1}) = \sum_{j=0}^{t(n+1)-1} \beta_{2j+i}(C) \text{ for all } i \geq 0 \text{ and } t(n+1)-1 \geq 1 \tag{5.27}$$

The following argument is similar as the one in 5.2.18, but we choose to show it here to avoid confusion. Using (5.25), (5.26) and (5.27) we get the following. Let i be even, then

$$\begin{aligned}
\beta_i(M_{t(n+1)-1}) &= \sum_{j=0}^{t(n+1)-1} \beta_{2j+i}(C) \\
&= \sum_{j=0}^{t(n+1)-1} \beta_{2j}(C) \\
&= \sum_{j=0}^{t(n+1)-1} \beta_{2j+1}(C) \\
&= \sum_{j=0}^{t(n+1)-1} \beta_{2j+(i+1)}(C) \\
&= \beta_{i+1}(M_{t(n+1)-1}).
\end{aligned}$$

If i is odd, we get that

$$\begin{aligned}
\beta_i(M_{t(n+1)-1}) &= \sum_{j=0}^{t(n+1)-1} \beta_{2j+i}(C) \\
&= \sum_{j=0}^{t(n+1)-1} \beta_{2j+1}(C) \\
&= \sum_{j=0}^{t(n+1)-1} \beta_{2j}(C) \\
&= \sum_{j=0}^{t(n+1)-1} \beta_{2j+(i+1)}(C) \\
&= \beta_{i+1}(M_{t(n+1)-1}).
\end{aligned}$$

That is, for $t \geq 1$ the module $M_{t(n+1)-1}$ has constant Betti numbers equal to

$$\beta_i(M_{t(n+1)-1}) = \sum_{j=0}^{t(n+1)-1} \beta_{2j}(C) = t\left(\sum_{j=0}^{n} \beta_{2j}(C)\right)$$

for all $i \geq 0$. We now let $d = \sum_{j=0}^{n} \beta_{2j}(C)$ and we are done.

$\square$

5.3 Closing remarks

Due to limited time available, we have not been able to look at specific examples in this thesis. Given more time, it would also be interesting to look at regular components of the Auslander-Reiten quiver of a local selfinjective Artin algebra containing a module with eventually constant Betti numbers. As previously mentioned it is not known if this implies that every module in the component has eventually constant Betti numbers. Moreover, we would have explored the possible components in 5.1.4 even further.

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