Predicting fretting fatigue in engineering design

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Abstract

The progress in fretting fatigue understanding and predictability is reviewed, with engineering applications in mind. While industrial assessments often relies on simple empirical parameters, research in fretting fatigue should allow the the design engineer to improve confidence in the fretting fatigue analysis.

Fretting fatigue cracks often form in multiaxial stress fields with severe gradients under the contact area, and are inherently difficult to predict.

By describing the fretting stress gradients using comparisons with the mechanical fields surrounding cracks and notches, crack nucleation threshold conditions and finite life can efficiently be determined. Also, non-local stress intensity multipliers provide promising tools for the industrial finite element analysis, often involving complex geometries and loading conditions.

The use of multiaxial fatigue criteria to determine fretting fatigue nucleation life is also reviewed. Researchers have shown that critical plane calculations with some stress-averaging method can predict fretting fatigue crack initiation. However, the frictional interface causes non-proportional loading paths, and the application of critical plane methods is not straight forward.

Keywords — fretting, fretting fatigue, crack initiation, contact mechanics, fretting maps, critical plane, crack analogue, notch analogue, asymptotic methods

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Nomenclature

$2N_f$	Number of cycles for fatigue failure
ΔK_{th}	Threshold stress intensity factor range
δ	Relative slip
$\Delta\epsilon$	Strain range
$\Delta\gamma$	Shear strain range
$\Delta\sigma_1$	Plain fatigue limit
ϵ'_f	Fatigue ductility coefficient
γ'	Shear fatigue ductility coefficient
ν	Poisson's ratio
σ'_f	Fatigue strength coefficient
σ_T	Stress in tangential direction of contact
σ_y	Yield stress
τ	Contact shear stress
τ_a	Shear stress amplitude
τ'_f	Shear fatigue strength coefficient
a	Contact semi-width
a_0	El Haddad intrinsic length parameter
b	Fatigue strength exponent
b_0	Shear fatigue strength exponent
c	Fatigue ductility exponent
c_0	Shear fatigue ductility exponent
E	Young's modulus
f	Coefficient of friction
G	Shear modulus
g_{max}	Maximum gap of contacting profile (unloaded)
k	Findley's influence factor
P	Contact normal force
Q	Contact sliding force
Y	LEFM Geometrical factor

Acronyms

CJ Ciavarella-Jäger. 11, 13

CM Cattaneo-Mindlin. 11

DIC Digital Image Correlation. 26

DMT Derjaguin-Muller-Toporov. 11

FD Fretting damage. 7, 27

FEM Finite Element Method. 24

FFD Fretting Fatigue Damage. 7

FP Findley Parameter. 9

FRD Fretting Related Damage. 8, 27

JKR Johnson-Kendall-Roberts. 11

MRFM Material Response Fretting Map. 5

MWCM Modified Wöhler Curve Method. 20

RCFM Running Conditions Fretting Map. 5

SWT Smith-Watson-Topper. 8

TCD Theory of Critical Distance. 20, 25

XFEM Extended Finite Element Method. 24

1 Introduction

Fretting is the phenomenon in which contacting surfaces subjected to oscillatory relative movement experience surface damage. Over time, cracks form at the surface and result in *fretting fatigue* related failures. Fretting can greatly reduce the fatigue life of the contacting parts.

Although the mechanisms of fretting have been studied for over a century, its exact nature and behaviour is still not well understood [1, 2, 3]. As early as in 1911, “fretting” was mentioned in relation with the formations of debris in plain fatigue tests [4], interpreting it as *surface wear*. Later, the term fretting fatigue arose, as researches started acknowledging its negative effect on fatigue life [5, 6]. It became apparent during the following decades that fretting fatigue was indeed a complicated phenomenon; Collins [7] proposed dependence on more than 50 parameters. However, due to the difficulties involved in accurately controlling and monitoring different parameters during fretting fatigue tests, early experiments and discussions were questionable [8]. The phenomena involved are also known to be interconnected, and Collins suggested that the parameters could be narrowed down into eight broader categories: Amplitude of relative slip, magnitude and distribution of the contact pressure, the local state of stress, number of cycles, material and surface conditions, cyclic frequency, temperature, and environments surrounding the surfaces [7]. Further complications are realised as the length scales involved in fretting fatigue are often on the same order of magnitude as material microstructural features [9] and surface features [10].

Fretting fatigue have mainly been studied for metallic alloys and ceramics used in engineering. In bearings, loss of clearance may be caused by fretting wear, but also jamming due to debris [11]. In biomaterials, debris formations induces inflammations in the host tissue [12, 13]. Highly loaded components like turbine blades [14, 1] and axle press-fits [15, 16, 17] may catastrophically fail due to fretting initiated cracks being driven to propagate into the substrate. Other examples are spline couplings, keyed joints, flexible marine risers and pipe fittings [18, 19, 20, 21].

2 Mechanisms of fretting fatigue

The fretting fatigue process is usually separated into different stages. The initial phase is often concerned with wearing off the oxide layer on the surfaces. After the oxide layer is worn off, cold-welds form at the surface asperities, increasing the coefficient of friction. Subsequent loading of the surfaces then cause these micro-welds to break, forming wear debris [22]. This wear debris can work as an abrasive medium, but can also form a protective third body layer reducing wear [11]. Additional loading cycles may introduce plastic deformation and microcracks to the surfaces, which cause additional wear debris and the potential of further propagating cracks into the material. These cracks eventually grow out of the contact stress fields and becomes dominated by the far-field stresses, if present.

In *partial slip* conditions, the friction is high enough to restrict the surfaces from global sliding and there is only a very small amount of local sliding between the generally adhered surfaces. These conditions are the most prone to fretting fatigue [23, 24]. The competing effects of *tribologically transformed structure* [25], particle detachment and nucleation of fatigue cracks [26] makes a quantitative prediction for a given material and given operating conditions very difficult. The crack initiation process is highly dependent on the material microstructure [27, 28].

Wear is often neglected in fretting fatigue analysis, but is reported to sometimes affect the fretting fatigue life [27, 29, 30]. The exact reasons for the underlying phenomena are still debated, but it is likely depending on material combination and loading conditions. Material removal due to surface wear may eliminate nucleating cracks at the surface. Wear also redistributes the contacting pressure [31], even in the partial slip regime as studied by Shen et al. [23]. They concludes that the wear could not be neglected. However, as other researchers have reported, wear in partial slip conditions is minor [27] and can in many cases be neglected for small values of slip. Frictional contacts are also known to sometimes *shake down*, i.e. residual shearing tractions building up and restricting further sliding, eventually leading to a steady state response being notionally adhered [2, 32].

The fretting problem is quite different, depending on whether the contact is *complete* or incomplete. For incomplete contact, at least one of the mating surfaces is of convex shape and the contact area is related to the load. For complete (conforming or flat) however, notionally sharp corners introduce stress singularities. For tangentially loaded incomplete contact, there is no frictional shakedown effect, and some local sliding will always occur. Thus, incomplete contacts are more prone to partial slip fretting fatigue.

3 Fretting maps

Various visual descriptions of fretting have been researched using fretting loops or fretting maps to characterise the fretting problem and to separate the regimes involved. Fretting loops plot the relation between friction force and displacement amplitude, sometimes along a third, temporal axis. Fretting loops form the basis for many fretting maps [33].

The slip amplitude was early identified as one of the most defining parameters for fretting. Vingsbo and Söderberg [24] introduced the concept of fretting maps with three different regimes of sliding conditions.

1. *Stick regime* with low sliding action and low surface damage (oxidation and wear). Low fretting damage.
2. *Mixed stick-slip regime* had fretting fatigue with small amounts of wear. Accelerated crack growth rate reduced fatigue life.
3. *Gross slip regime* showed severe damage due to wear but crack formations were limited. In the gross slip regime, the wear coefficient increased by several orders of magnitude.

Hence, this fretting map could be used to determine the fretting regimes for a set of conditions. Figure 1 illustrates the different regimes. Fretting maps was an important development in the work of fretting assessment. Today they are used to describe the overall fretting behaviour, including contact conditions, fretting regime, wear mechanism, crack nucleation and propagation [33].

Some years after Vingsbo and Söderberg, Zhou and Vincent [26, 34] proposed to separate the problem using two different types of fretting maps, *running condition fretting map* (RCFM) and *material response fretting map* (MRFM). RCFM distinguished between partial slip regime, mixed fretting regime and

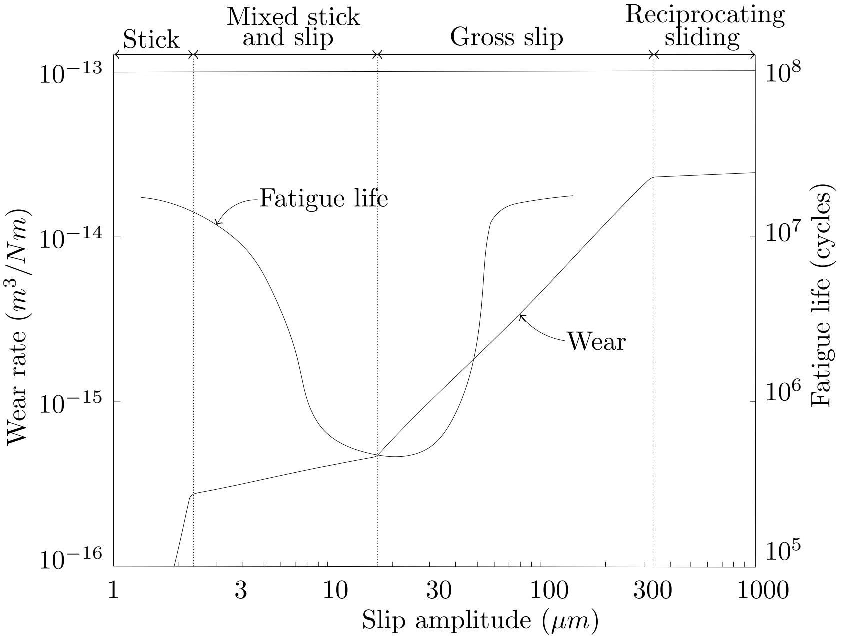

Figure 1: Relating the slip amplitude to fretting regime, as proposed by Vingsbo and Söderberg [24]

slip regime, and is in some ways quite similar to Vingsbo and Söderberg. It is however, maybe more correct in unifying the stick and partial-slip regimes, since in reality there will always be some local sliding. The material response fretting map was related to the *post hoc* degradation analysis of the specimen.

Different maps related to the number of cycles have also been proposed [35]. In 2006 Zhoul et al. [36] reviewed the progress in fretting maps and covered additional proposals, but arrives to the conclusion that further work is needed to quantify the competing effects in fretting fatigue, especially in mixed regime.

In 2015 Pearson and Shipway [33] investigated the paper of Vingsbo and Söderberg on fretting maps and criticised their suggestion that the wear coefficient was strongly dependent on displacement amplitude. They point out that in general, fretting research at that time had two limitations: Firstly, many researchers used far-field displacement amplitudes and not the actual slip for the contact thus also including the compliance of the test rig. Secondly, research has also shown that there appears to be a threshold below which wear does not occur. The errors related to not recognising these effects grows as the slip amplitude becomes smaller, where the relative difference between the displacement amplitude and *slip* amplitude is bigger.

The different fretting maps proposed serves well as graphical demonstrations of the competing mechanisms usually involved in fretting and is useful for the engineer in the early design phase, especially when considering to apply surface treatment, etc. [37, 36]. Very quantitative predictions for fretting fatigue are, however, difficult to achieve using maps only. *Computational fretting maps* may also be used in parametric numerical fretting fatigue studies [38, 39].

4 Design parameters

In aerospace, nuclear and other safety-critical industries, much effort have been put into predicting fretting fatigue through knockdown factors and different design parameters. These parameters are often easily computed and may serve as a first step in the design process. Collins [7] early identified the usefulness of quantitative factors in the fretting fatigue design phase.

The first attempts to mathematically relate the contact stress fields with fatigue damage was Ruiz et al. [14]. They investigated fretting fatigue life of turbine blade dovetail joints and proposed a design parameter, "fretting damage" (FD), by multiplying the highest surface tangential stress with the maximum frictional work ($\tau\delta$):

$$FD = (\sigma_T)_{max} \cdot (\tau\delta)_{max} \tag{1}$$

However, as the maximum tangential stress may occur at a different location than the maximum frictional work, it is numerically awkward. A second parameter was obtained by simply maximising the product of tangential stress σ_T, surface shear stress τ and relative slip (equation 2). This "fretting fatigue damage" (FFD) parameter was reported to predict the location of crack, but failed to predict the number of cycles to crack initiation or crack growth [40]. The parameters proposed by Ruiz et al. have nonetheless been extensively used, mainly due to their simplicity and the fact that it may predict crack initiation probability [27].

$$FFD = \sigma_T \cdot \tau \cdot \delta \tag{2}$$

For more recent variations and extensions to the Ruiz parameters, see e.g. [41, 42]. These critera combines the frictional power or frictional work with multiaxial fatigue parameters.

More recently, Varenberg et al. [43, 44] proposed a dimensionless *slip index* aiming to distinguish between the fretting regimes in a more unified and rigorous way than the classical use of fretting maps. By using dimensional analysis they derived an expression for the slip ratio which is governed by the dimensionless parameter $\delta = A_d S_c / N$, with A_d being the imposed displacement amplitude, S_c the slope of the friction loop and N is the normal load. Applied to fretting experiments, the different regimes were separated. Partial slip exists for $0.5 \leq \delta < 0.6$ and gross slip for $\delta > 0.6$

In 2015 Li et al. [3] noted that there is still no satisfactory fretting fatigue damage criterion and they proposed a parameter for fretting fatigue life predictions, by expressing the “fretting related damage parameter” (FRD) as

$$FRD = \alpha + \beta\sqrt{\frac{Q}{fP}} \tag{3}$$

The FRD parameter was related to the plain fatigue methods as a knock-down factor to determine the number of cycles to failure and thus making full use of already existing plain fatigue data. Q is the sliding force, P is the normal load, f coefficient of friction α and β are fitting coefficients.

5 Critical plane methods

More extensive parameters for fretting fatigue have been proposed by making use of the different plain fatigue parameters. Due to the multiaxial nature of the stresses, particularly critical plane-based methods have been attempted to determine the fretting fatigue limit. Empirical combinations of stresses and strains are assumed to drive the cracks to initiate and grow in certain material planes. Applied to the stress gradients of fretting fatigue, the critical plane methods usually considers a point at a critical distance or in an averaged sense, and searches for the material plane orientation having the most damaging parameter. Thus, often *cracking direction* is also obtained. In general, these methods can be divided into stress-based, strain-based and energy-based parameters. Numerous parameters for fatigue have been applied to the fretting case, and the following list is by no means exhaustive.

Szolwinski and Farris [22] used the Smith-Watson-Topper (SWT) [45] to find the initiation location and life. It may be given by

$$SWT = \sigma_{max}\frac{\Delta\epsilon}{2} \tag{4}$$

As SWT was extended to be used in a critical plane method [46], this would mean finding the material plane maximising the product of normal strain range and maximum tensile stress on that plane during the loading cycle (i.e. *strain energy density*). Combining equation 4 with Basquin’s law and Coffin-Manson, the SWT critical plane parameter may be expressed as [47]:

$$SWT = \frac{(\sigma'_f)^2}{E}(2N_f)^{2b} + \sigma'_f\epsilon'_f(2N_f)^{b+c} \tag{5}$$

σ'_f and b are the material fatigue strength and exponent, ϵ'_f and c are the fatigue ductility coefficient and exponent respectively. E is the modulus of elasticity and N_f is the number of cycles to initiate a crack with a given length. Making use of equation 5, the life of the component may be estimated by finding the critical SWT and comparing with fully reversed uniaxial test data. The SWT is found either by evaluating equation 5 on the plane experiencing the largest range of principal strain, or by searching for the plane with maximum SWT [46].

Fatemi and Socie (FS) [48] proposed a strain based critical plane parameter for shear dominated cracks, studying the effects of out-of-phase loading. It has also been applied to fretting fatigue [47, 40, 9]. Using the FS criterion, the material plane having the maximum shear strain is considered the critical plane, with the influence of opening mode included through the material parameter α. It can be expressed as

$$FS = \frac{\Delta\gamma}{2}\left(1 + \alpha\frac{\sigma_{max}}{\sigma_y}\right) = \frac{\tau'_f}{G}(2N_f)^{b_0} + \gamma'_f(2N_f)^{c_0} \tag{6}$$

where $\Delta\gamma$ is the shear strain range during the cycle, σ_{max} maximum normal stress, σ_y is the yield stress, G is the shear modulus and α, τ'_f, γ'_f are material related parameters. b_0 and c_0 are the shear fatigue strength exponent and shear fatigue ductility exponent respectively. Thus, the critical plane is found by maximising equation 6, where the shear strain range is evaluated as the difference between the largest and smallest shear strain during the cycle [49]. Equation 6 can alternatively be related uniaxial data using the following relations derived from von Mises' criterion [50].

$$\tau'_f = \frac{\sigma'_f}{\sqrt{3}}, \quad \gamma'_f = \sqrt{3}\epsilon'_f, \quad b_0 = b, \quad c_0 = c, \quad G = \frac{E}{2(1+\nu)} \tag{7}$$

Shear based parameters work better for ductile materials and tensile critera work better for brittle materials. The dominant mode of initiation is often not known *a priori*, which makes the decision of which criterion to apply difficult. As Araújo and Nowell [47] suggests, a possible, conservative approach may be to simply calculate both FS and SWT parameters - and then use the worst case. Averaging methods were shown to reveal a contact size effects in Al-4%Cu and Ti-6Al-4V samples, but concerns about assuming the averaging parameters to be material constant are raised.

Lykins et al. [40] found SWT to be effective for predicting initiation life and location for Ti-6Al-4V, and noted the FS parameter to be effective for initiation location. The *Maximum strain amplitude* was concluded to be important in the fretting crack initiation for the Ti–6Al–4V alloy. Araújo [9] found SWT to predict crack direction for AISI 1034 and 35NCD16 samples, using averaged stresses over a characteristic length along the crack direction.

The stress parameter proposed by Findley (FP) [51] in the sixties combines the shear loads with the normal loads. More specifically, the critical plane is defined as the material plane experiencing the maximum combination of shear stress amplitude and maximum normal stress over a stabilized cycle. Thus, maximising equation 8 yields the critical plane.

$$FP = \tau_a + k\sigma_{max} \tag{8}$$

Where k is a parameter describing the material crack growth sensitivity to normal stresses, and is determined based on experimental data. Higher values of k can be interpreted as higher sensitivity to *opening mode* effects on the shear cracks. k is therefore normally lower for shear dominated (ductile materials) than for brittle materials. Socie [50] propose to use $0.1-0.2$ for ductile materials. In terms of the number of cycles to failure it may be expressed as [52]:

$$FP = \tau_a + k\sigma_{max} = \tau_f'(2N_f)^{b_0} \tag{9}$$

Namjoshi et al. [53] tested the Findley criteria for fretting fatigue crack initiation of different pad geometries on Ti-6Al-4V "dog bone" specimens for a variety of stress levels. The Findley parameter (and SWT) was found to be effective in finding the location but not the orientation of cracks. They proposed the modified shear stress range critical plane parameter, having four curve fitting constants. The researchers argue that the criterion is thus less influenced by pad geometry, and that it would be successful in finding the crack orientation and initiation life.

Most studies on using critical plane parameters to quantify fretting fatigue have considered convex contact geometries and in-phase loading. Recently, Bhatti and Wahab [54] found a number of parameters to be appropriate for different cases of phase-shifted loading conditions. Foletti et al. [55] applied the (mesoscopic) *Dang Van* [56] and Liu–Mahadevan [57] criteria to fretting fatigue in railway axle press-fits. Chakherlou et al. [58] compared seven different critical plane criteria for life estimations on different models e.g. with aluminium plate joint with pre-tensioned bolts and residual stresses due to cold expansion. They conclude that no criterion is universally accurate, but SWT parameter was within a factor of two for most specimens.

The stress based fatigue parameters neglects plastic effects. Since for higher contact loads, micro plasticity is expected to occur, the stress based parameters might be less reliable than the strain based parameters. However, the different criteria used for different materials have shown reasonable accuracy for specific sets of experimental data, and is less accurate in the general sense. In general it is noted that due to the stress gradients involved in fretting fatigue, some non-local approach must be used in addition [59, 49, 47]. Thus, averaging the stress over some material dependent critical distance becomes necessary.

Some researchers have used stress-invariant criteria and thus avoiding the computationally expensive critical plane method. Ferré et al. [60] averaged the Crossland criterion over a critical area and achieved $< 10\%$ error for nucleation endurance for titanium samples. Fouvry et al. [61] achieved 12% error for steel specimen. The Dang Van criterion is often used as comparison with other methods [62, 63]. The stress-invariant methods are considerably easier to implement, compared with critical plane methods, but at the cost of accuracy. Also, by assuming stress-invariance, the potential information about initial cracking direction is lost.

6 Analytical methods

The analytical studies of contact started in the 1880s by Hertz [64] who first studied the stresses in spherical bodies in contact. His theories was restricted to frictionless contact between linear elastic bodies, proving the *Hertzian* pressure distribution. He also restricted his study to non-conforming contact for which the contact area was *small* compared to the bodies, i.e. *half-space theory.* Since Hertz, numerous researchers have put efforts into extending the theory and its become a useful tool in fretting analysis, especially for closed form comparisons with experiments.

Bradley [65] showed that contacting spheres share adhesive forces when put into contact and Johnson, Kendall and Roberts [66] (JKR) extended the Hertzian contact for elastic bodies subjected to small contact loads to include adhesive forces associated with the free surface energies. The JKR theory introduced a singularity at the contact boundary and a *separation force* to the problem, which effectively increases the contact area. Derjaguin, Muller, and Toporov (DMT) [67] proposed an adhesive law based on the undeformed Hertzian profile, thus avoiding the JKR singularity. More recently, adhesion was introduced to models to introduce stress singularities in rounded contact fatigue problems [68].

6.1 Sliding contact

Cattaneo [69] extended the Hertzian contact problem to include tangential loads experiencing interfacial partial slip. The normal load was held constant, whilst monotonically increasing the tangential load. The partial slip conditions were described by superimposing the full sliding terms with a correction related to the contacting pressure coming from the normal contact. Independently, Mindlin [70] studied the same type of solution but with some generalisations to the loading path. Further generalisations to the *Cattaneo-Mindlin problem (CM)* followed [71, 72].

The CM case also became a popular setup for experimental testing of fretting fatigue, though the shear tractions in the fretting tests differ from the CM solution due to the bulk stresses. The plane strain approximation is usually assumed for the cylinder-on-plane case. Nowell and Hills demonstrated in 1987 the effect of bulk tension on the stress distributions for the CM case [73] by perturbing the Mindlin solution on integral form. It was shown that the bulk stresses in the specimen introduces an *eccentricity* to the contact stick zone due to strain mismatch. The use of the Mindlin solution with the eccentricity is shown to be a good approximation except for near the contact center. The half-plane stresses were thus found for the case where the bulk stresses are *in-phase* with the tangential load and they argue that the Mindlin case is reasonable approximate.

Ciavarella [74] and Jäger [75] independently extended the CM problem to more general geometries, and this generalisation is often referred to as the *Ciavarella-Jäger theorem (CJ).* The theorem holds for two-dimensional contact, but for 3D only in the unrealistic case of vanishing Poisson ratio, otherwise only in approximate sense. This limitation also applies to the original CM solution [1], and may not always be neglected [76].

Barber et al. [32] applied periodic tangential loads and periodic normal loads to the CJ case. The uncoupled contact problem (vanishing Dundur's constant) was studied with the loads in PQ-space being bounded by $Q = \mu P$ so that gross slip would not occur. They conclude that the extent of the *permanent stick zone* during steady state, was independent of loading path. The interior traction distributions, however, are depending on loading path. By extending the methods of Jäger and Ciavarella to cyclic tangential loading, they show that the frictional system reaches a steady state after the first cycle. The paper conclude that the system will shakedown to a steady state independent of transient conditions, with a given permanent stick zone.

Ciavarella and Demelio [1] reviewed in 2000 some of the efforts made to the analytical approaches to fretting fatigue, again by considering the case of elastically similar half-planes where the normal and shearing tractions are uncoupled. They considered constant normal load and oscillating tangential load for indenters ranging from the Hertzian to flat indenters (complete contact) with increasingly sharply rounded corners. The analytical solutions are combined with the classical fretting damage parameters of slip amplitude and frictional slip energy [14] and stress intensity factors for cracking. For increasingly sharp indenter, tensile stresses at surface increases, but also becomes more localised. Thus, the stress intensity factor (K_I) rapidly decrease, and cracks have greater chance of self-arresting. However, only opening mode (K_I) is considered. The authors point out the possibilities to separate the stress concentration effect and frictional damage for further testing, which can help to understand the complexities involved. According to the authors, it is not entirely clear if there is an "optimal" corner radius on the rounded indenter. A simple expression for the relative microslip at onset of *full sliding* was given as

$$\delta_{max} = f g_{max} \tag{10}$$

where f is the coefficient of friction and g_{max} is the maximum gap of the contacting profiles before loading.

The Cattaneo-Mindlin problem was revisited by Etsion [77] and its assumptions were validated. Experimental evidence show that the contact does change when tangential forces are applied, a phenomena coined "junction growth" by Tabor [78]. The problem is to assume a local Coloumb friction law because when friction and contact pressure are high, unrealistically high contact stresses may occur, exceeding yield. By relating the incipient sliding to *plastic failure*, these assumptions were relaxed, suggesting a model for the other end of the scale with respect to actual plasticity in the given contact situation. Wang et al. [79] studied partial slip conditions for elastically dissimilar materials by coupling the normal and tangential loads using constraint equations on the slip in the contact plane. Other, *semi-analytical* methods for solving partial slip cases are also proposed [80, 81]

Davies et al. [82] considered the CJ theorem and extended it for loading with varying normal load. The evolution of the stick-slip zones were determined, still limiting the analysis to contact profiles satisfying half-plane theory and by avoiding gross slip in the P-Q space. The authors are mainly interested in determining the energy dissipation in the system, but argues that fretting crack nucleation can be assumed to coincide with the point of maximum dissipation. Along with the paper, a Mathematica code was given for researchers to further study stick-slip evolutions.

The analytical and semi-analytical models are important tools for fretting and provides mathematical ground for researchers to study fretting fatigue. They are also useful for comparing with experimental results and parameter studies [83], even though the exact fretting conditions are hard to control. By adjusting the parameters involved, these methods can be used to simulate the stress fields in the real component. However, quite significant size effect have been reported [47], which puts limitations to this method. Also, numerical methods such as finite element, are becoming increasingly used, allowing for more complex geometry, material models, loading histories etc. In general, when half-plane theory is violated, more elaborate numerical methods are needed.

7 Asymptotic methods

The stress concentrations in fretting fatigue can also be studied using asymptotic analysis, leading to an analogy with cracks in fracture mechanics. As the contact edges become increasingly sharp, Hertzian stress analysis fail. For sharp, (complete) contact, stress singularities arise and asymptotes can be applied [84, 85, 86]. In these methods, the stress fields are matched with truncated asymptotic expansions, and one enters the discussion on the *order of singularity* of the stress field and its spatial range of validity. In reality, of course, there always exists some rounding of the corners, but notionally complete contact occur in engineering situations [2] and the asymptotic methods can be suitable approximations. Also, in rough contacting surfaces, local singularities may occur due to adhesion [11].

Williams [87] developed a framework for analysing singular stress fields in wedges by expressing the stresses as functions of wedge angle in a polar coordinate system (r, θ) with its origin in the singular point. Using the biharmonic equation, the solutions for the stresses and displacements are expanded as an asymptotic series in powers of r (equation 11), and non-trivial solutions are obtained for certain eigenvalues λ.

Following Williams method, the stress field surrounding the apex can be expressed on the form [84]

$$\sigma_{ij}(r,\theta) = K_I r^{\lambda_I - 1} f_{ij}^{I}(\theta) + K_{II} r^{\lambda_{II}(\theta) - 1} f_{ij}^{II}(\theta) + \text{higher order terms} \tag{11}$$

where, for a mode $n \in \{1, 2\}$, K_n is the stress intensity function, λ_n is the eigenvalue and $f(\theta)$ is the corresponding eigenfunction. Thus, sufficiently close to the singularity, the stress field is dominated by the lowest eigenvalue, and the environment for crack nucleation in this critical region may be quantified. For wedge angle of $\theta = 360°$ the solution for a crack is obtained with its lowest, and hence dominant eigenvalue being 0.5. This solution is important in the fields of fracture mechanics and contact mechanics. See e.g. Mugadu et al. [84] for considering the edges of contact in a spline coupling. Note however, that the Williams solution considers elastically similar bodies [88] and solutions for elastically dissimilar contact exists, see Bogy et al. [89, 90, 91].

Though being quite mathematical, these serves as foundation for useful tools for engineers encountering singularities such as in sharp edge contact. For a more thorough description, the reader is referred to published literature [92, 93].

Sackfield et al. [94] applied asymptotic expressions to the mathematical description of a rigid punch pressed into a half-plane substrate. For flat punch they assumed *small* rounded corners so that the stress at the corners were dominated by the singular term (generalised stress intensity factor K^*) in the stress expansion. The stresses were matched with the corresponding sharp edged indenter, and thus relies on the radius to be small compared with the indenter dimensions. This is beneficial in cases where the sharp edge solutions exist and can be used for indenters with small radii, which in finite elements solutions often requires very fine element mesh to resolve.

The asymptotic analysis was then used for *incomplete* contact in partial slip by Dini and Hills in 2004 [86] and compared with the classical Cattaneo-Mindlin solution. They argue that the stress expansion from the local singularity is impractical for sliding contact, but for partial slip the stick-slip interface is a natural location for crack nucleation. The stress intensity factors K_I and K_{II} serves as scaling parameters for the normal and shear forces respectively, and characterises the contact. Thus it provides a means to simply obtain the stress characterisation to evaluate the fretting fatigue. Good agreement for the stresses is only expected close to the contact corners, which anyways are the most likely locations for crack nucleation. By evaluating the contact situation in such a local manner permits one to recreate the fretting problem of a prototype to a simple laboratory test setup [95, 96, 2]

The asymptotic solution of complete contact between elastically dissimilar bodies was investigated by Churchman et al. [91]. Whether failure is most likely to happen at the leading or the trailing edge depends on the slip direction, but for the fretting case of oscillating punch, the problem is symmetric.

Hills and Dini [2] reviewed in 2016 the efforts on using asymptotic forms to describe the stress fields for fretting fatigue, pointing out the fundamentally different nature of incomplete and complete contact. For incomplete contact, they argue that the local fields (stress and slip) are determined by the two stress intensity factors and the friction coefficient. Hence, the fields can be used to model and replicate the situation for complex prototypes. For complete contact however, slip will be contained inside the contact area, limiting the sliding motion to very small values. The authors thus claim that fretting fatigue, in its traditional sense, should not occur for complete contact. In their approach the normal load is held constant and the case of oscillating normal load is retained for further studies. Contact of parts of conforming geometry ("punch on punch") and "receding" contact is only mentioned and points out that the research in these types of contact is lacking. It is also remarked that the contact stress intensity factors are only valid near the contact edge and hence it is useful for nucleation criteria, but not necessarily crack growth as the crack grows farther away from the edge. Plots of *total* fretting fatigue life versus the crack nucleation fields are as such not entirely accurate.

Since the stick-slip situation for incomplete contact can be described by the three parameters K_T, K_N and μ, laboratory experiments can be made into replicating the situation for complex prototypes. For complete contact however, the use of asymptotic solutions can demonstrate that fretting fatigue should not occur in complete contacts: For high enough coefficients of friction, the contact is in an *adhered* state, and for lower coefficients, frictional shakedown will cause the steady-state slip values to be very small [97]. Given that the nonlinear (process) zone is small, the strain energy should be characterised by the asymptotes. Further studies to be made are to account for oscillating normal load, and to match the asymptotic methods to rounded contact.

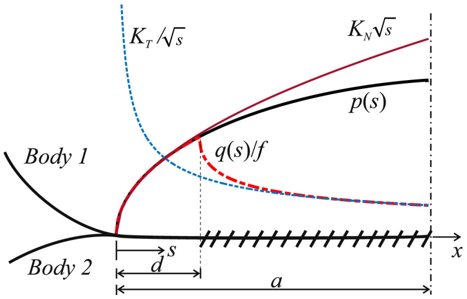

Figure 2: Matching the stress fields with edge asymptotes [98]

Recently, Fleury et al. [98] used asymptotic methods for the incomplete contact subjected to more complex loading histories, with varying normal and shear load (in-phase). The slip zone size and amount of slip are found through the two stress intensity factors K_T and K_N, and the fretting damage may then be found e.g. using the energy dissipation. This asymptotic formulation gives good approximations when the slip zone is small compared with the contact size, but at decreasing accuracy for larger variations in normal load.

Asymptotic methods are useful for fretting fatigue because it provides means for characterising the most detrimental fields (surrounding the contact edges) from which cracks nucleate, circumventing the need to analyse the entire contact. Thus, the local stress fields may be matched with those in experiments and as such be used to quantify fretting fatigue strength. The methods are however limited to (local) half-plane idealisation [98], decoupling tangential and normal stresses. Examples of asymptotic matching is presented in the literature for simple contacts with closed-form solutions, but for complex cases, numerical methods like the finite element can be used to find the generalised intensity factors, see e.g. Montebello et al. [99].

8 Crack initiation and growth

Whereas the asymptotic methods can permit the fretting fatigue conditions to be described without the need for explicit modelling of the micromechanics by locally matching the necessary fields with experiments, other methods attempts to micromechanically describe the fretting fatigue cracking process.

The mechanisms involved have been extensively researched for metallic contacts, but initiation of fretting fatigue cracks is still an elusive problem. Usually, the fatigue life is split into nucleation and propagation phases, but the relative importance of one phase over the other have been debated. Figure 3 shows the fracture surface from a fretted titanium specimen by Araújo and Nowell [47]. Navarro et al. [59] noted that the relative importance of initiation versus propagation may depend on the fatigue criterion used, the loading conditions, material, geometry etc. and thus one cannot *a priori* know to which phase the majority of life is.

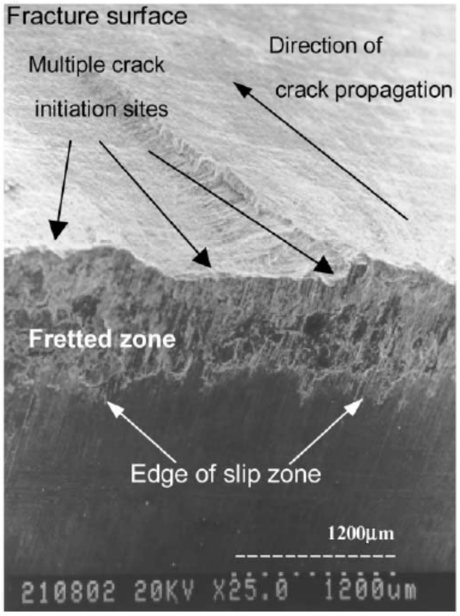

Figure 3: Scanning electron micrograph of a fretting crack surface [47]

Despite the advancements in characterising the fretting fields of stresses and strains, resolving the crack nucleation driving forces continues to be a target of research [2, 84]. As nucleation criteria in fretting fatigue, mainly three different methodologies have been used: *short crack methodologies*, multiaxial fatigue criteria and fretting specific parameters (Ruiz etc.). Recently, the use of *con-*

tinuum damage mechanics have also been applied to fretting fatigue. The short crack methods uses threshold curves for the stress intensity factors to determine whether a crack will arrest after initiation, and uses normally either Kitagawa-Takahashi diagram or El-Haddad curves [100, 101, 102]. The critical plane methods usually assumes the fatigue crack to form along *persistent slip bands* in the material crystals, and uses empirical parameters to determine the most detrimental plane orientation.

Szolwinski and Farris [22] attempted to quantitatively model the nucleation life for fretting fatigue experiments and noted that characterising the stresses alone was not sufficient. They turned to the Smith-Watson-Topper model and used Westegaard's method for characterising the stress distribution. Fortran routines were written to find the plane having the largest damage parameter defined in equation 4 which they used to find the crack origin and orientation, in what they called the Γ-model. The nucleation life was taken to be the number of cycles needed to form a crack of 1 mm length, and Paris law was used to propagate the crack to failure. The life estimates provided was within the scatter of the data from experiments.

Araújo and Nowell discussed in their 2002 paper [47] the size effects in fretting fatigue. For incomplete contact, they performed experimental and analytical analysis, and varied the contact size whilst holding the same levels of stress on the contact surface. They show that two different critical plane criteria (SWT and FS) are overly conservative for smaller contacts. For smaller contacts, the stresses at the surface are the same, however with a more rapid decay beneath the surface. Hence, for smaller contacts, the driving forces for crack growth are less severe than for larger ones, and the point-based (local) critical plane criteria does in general not account for this gradient. They proposed averaging over a characteristic volume or depth. This depth or volume was found to be on the same scale as microstructural parameters, essentially meaning that accurate predictions needs to include features of material microstructure.

For determining the orientations of early cracks, Araújo et al. [9] proposed a method where the normal and shearing driving forces were averaged over a line of characteristic length and then used in different critical plane algorithms. Thus, the critical plane calculations is not necessary at each integration point. As the steep stress gradients related to fretting fatigue can cause small-scale crack reorientations, an incremental approach should be used. Continuum elasticity is assumed, which is regarded as an engineering approximation due to microstructural effects playing a role in the real case. FS, SWT and *Modified Wöhler Curve Method* (MWCM) were used and only SWT was successful in predicting the orientations of the cracks, although, not accurately.

The Dang Van criterion [56] has also been used in attempts to describe nucleation [63, 103]. Fouvry et al. [63] compared it with other classical multiaxial parameters. Under partial slip conditions, the coefficient of friction was confirmed to be important and the crack nucleation conditions were accurately quantified. Tests were performed on steel specimen with a titanium nitride coating. It was found that with the compressive stresses introduced, the coating greatly reduced the cracking nucleation risk.

Lykins et al. [40] tested the Ruiz parameters and a number of fatigue criteria to predict fretting fatigue crack initiation in titanium alloy. The Ruiz parameters were deemed ineffective, but improved when corrections for contact effects and mean stress ratio were included. Fretting fatigue nucleation was shown to have the same trends in the Wöhler-diagrams. The crack initiation locations found in the experiments were shown to coincide with the maximum strain amplitude.

Navarro et al. [104] proposed the method they called the *variable initiation length model* in which the number of cycles for crack nucleation was calculated for material points along the crack path. For these points, the crack growth rate was computed both for initiation using Basquin's equation and for propagation force using *Linear Elastic Fracture Mechanics (LEFM)*. The depth at which the LEFM driving forces surpassed the driving forces representing the initiation mechanisms was taken to be the initiation length. Thus the total life is obtained by adding the number of cycles to initiation at this depth and the number of cycles to propagate the crack until failure.

Bhatti and Wahab [54] studied fretting fatigue for 2024-T351 Aluminium using for different phase angles between the axial load and tangential load. Finite element models of three different combinations of boundary conditions with three different phase angles were considered: 0 deg, 90 deg and 180 deg. Results were compared with literature [105]. They conclude that the location of crack initiation for fretting damage is highly dependent on the phase angle, but the Ruiz parameter and SWT multiaxial fatigue criterion was reported to show good correlation with the results of Szolwinski and Farris. Initiation life is shortest for 180 deg phase shift and longest for the 90 deg case.

The determination of crack growth in uniform stress fields is fairly established today, e.g. using Paris law. In fretting fatigue however, the non-proportional mixed-mode stress fields close to the contact significantly complicates the matter [106, 47, 59]. Recently Baietto et al. [107] coupled experimental fretting results with a numerical model to model fretting fatigue mixed mode crack initiation and growth. However, Faanes [108] found fretting cracks to be notably affected by Mode II only in the short initial stage.

In continuum damage mechanics (CDM) an accumulating damage variable is used to describe material degradation and was introduced to fatigue by Lemaitre [109]. This damage parameter $0 \leq D < 1$ (scalar for isotropic damage) is used to define the *effective stress* which may be given as

$$\tilde{\sigma} = \frac{\sigma}{1 - D} \tag{12}$$

Zhang et al. [110] used CDM in a three dimensional finite element model with multiaxial fatigue calculations. They used a nonlinear damage accumulation and compared the results with the SWT parameter for notched and unnotched plain fatigue as well as fretting. In an effort to do lifetime predictions for different values of slip amplitude, the results were compared with published data. The model is successful in suggesting the reduction in fretting fatigue life for increasing relative slip in the partial slip regime. Also, the predicted life increases slightly in the gross slip regime. For low amplitudes of slip in the partial slip regime life predictions were non-conservative, and the researchers argues this may be explained due to not including wear. The study was later followed up by including wear [23].

Hojjati-Talemi and Wahab [111] also found CDM to give accurate predictions for experimental data found in the literature, and concludes that the method is appropriate for the multiaxial stress state.

Implementing damage evolution concepts to the finite element analysis provides an efficient tool for predicting fretting fatigue damage. In safe-life design, the application of short crack methodologies to determine the the cracking risk is appropriate. The critical plane methods are computationally expensive, having to evaluate the different parameters for a range of possible plane orientations, but permits estimating the crack initiation angle. The need for non-local methods, however, can introduce somewhat arbitrary length parameters to the problem (e.g. the critical distance).

9 Notch analogue model

Researchers in fretting fatigue have attempted to draw the very useful parallel to the theory of notch fatigue, thus possible making use of the extensive research on stress raisers in notches. Giannakopoulos et al. [85] drawed the analogy between fretting fatigue for rounded flat punches and (blunt) notch fatigue. By recognising that stress gradients and multiaxiality are important in fatigue for both notches and fretting contact, the analogy is clear. The most highly stressed point in the case of incomplete contact is the surface point at the edge of contact. Here, the normal and shear stresses will reach zero and the stress field might be assumed to be uniaxial [101]. In this case, multiaxial parameters might be avoided. It is generally believed, however, that initiation and early growth is mixed mode dominated and stress multiaxiality becomes important [112].

A method to account for the size effects in fretting fatigue is the classical *hot spot* method. However, studies have shown that the hot-spot method tends to be overly conservative [62]. More promising is using the *Theory of Critical Distances* (TCD), where the stresses are averaged in some sense. In the *point method* the stress is considered at a certain distance to the stress raiser, the *Line Method* averages the stresses over a line of certain length and in the *Area Method* the stresses are averaged over an area [113]. Researchers have successfully used TCD combined with different multiaxial criteria to determine fretting fatigue threshold conditions [112]. In general, TCD method can work well for determining orientations of Mode I dominated long cracks, but for initial crack orientation the approximation made by using volume averaging methods fail [114].

Fouvry et al. [115] conducted fretting fatigue tests under partial slip conditions of a sphere on plane and recorded normal force, tangential force and displacement. The results were compared with the predictions made by describing the loading conditions using Mindlin and Hamilton [116] formulations with the Dang Van criterion. A volume-averaging method was used to account for the size effect and came to the conclusion that crack nucleation can be predicted by finding the size of the material intrinsic critical volume.

Araújo et al. [112] used MWCM [117, 118] with the TCD method and compared with experimental data for cylinder-on-plane contact fatigue tests. The critical length L, was taken to be given by the material intrinsic "transition crack size" separating the short and long cracks regimes [119], see equation 14. The method was reported to correctly predict failures in the medium-cycle regime and for the high cycle regime within an error of ±20 %, and also captured the size effect. The method is simple to implement as a post-processing step for linear elastic analysis, but relies on the fatigue crack formation processes to be confined in the critical volume. The size of this volume for a given material is however not dependent on geometrical features [120].

Santus [21] applied the MWCM with TCD to study fretting fatigue of steel-to-aluminium threaded pipe connections used in a corrosive environment. This method was mapped together with a slip-based parameter from which a fatigue limit was deduced based on experimental results from full scale tests. The slip parameter was used simply as a means of incorporating the competing effects of fretting wear and crack nucleation in the partial slip regime.

Ferré et al. [121] used different combinations of local and non-local multiaxial fatigue approaches to study the stress gradient effect on fretting crack nucleation. They performed cylinder-on-plane contact tests for Ti-6V-4V specimens in *low cycle fatigue loading* for three different cylinder radii and evaluated the different fatigue predictions for a range of stress gradients. Assessments were made using a local stress approach, volume-averaging approach, critical distance, and a weighted function approach based on work by Papadopoulos [122]. In general all the non-local approaches performed well, so they preferred the volume-averaging method for being practical.

For materials with large defects, cast iron, high strength steels etc. probabilistic methods might also be useful methods to model the size effects [123, 124]. Probabilistic methods for fretting fatigue is in general less researched than its deterministic counterparts, even though fretting fatigue is essentially a random process [125].

10 Crack analogue model

Related to the asymptotic methods for characterising the local fields surrounding the contact edges, a *crack analogue* for fretting fatigue have emerged, comparing the contact stress singularities with the stresses for cracks, essentially inferring the order of singularity to be square root bounded. The crack growth in fretting fatigue can then be viewed as a branching crack from the primary crack represented by the contact interface.

Giannakopoulos et al. [126] presented in 1998 a study in which the stress and strain fields of contact mechanics were matched with fracture mechanics

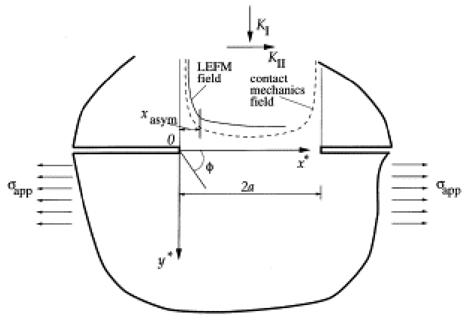

Figure 4: Comparing the stress fields of contact with LEFM [126]

solutions, see figure 4. They proposed to use the asymptotic field descriptions in linear elastic fracture mechanics (LEFM) as means to predict fretting fatigue life using the classical Paris law. Using the LEFM concept of T-stress, bulk cyclic stress, surface treatment residual stresses etc. could be included. The *local* crack driving forces k_1 and k_2 were described using two coupled equations with the *remote* stress intensity factors K_I and K_{II}. The angle of initiation was found by assuming the crack to initiate along the direction for which the local mode II intensity factor vanishes ($k_2 = 0$), governed by the contact load P and tangential load Q. As the crack grows into the substrate however, the crack reorients to having the applied cyclic bulk stress σ_{app} as opening mode. The total life is thus modelled as two separate stages; The first stage for which the contact loads initiates and drives the crack to grow to a critical distance from the surface. Depending on the applied loads, the crack then either arrests or is driven further in Mode I govern by the cyclic bulk stress σ_{app}.

By quantitatively comparing with data found in the literature, Giannakopoulos et al. [126] found their results to agree with 15 fretting experiments from different studies. The limitations are however clear, inheriting the small-scale yielding limits from LEFM. The punch is also assumed to be rigid in this study, but the researchers refers to the work of Dundurs [127] and his material parameters for elastic bodies in contact. Crack initiation is assumed to be driven solely by the mechanical effects of the contact, thus surface roughness, wear, lubrication etc. are neglected. Though the stress fields are not always square-root singular, these solutions are expected to be at least approximately valid for many combinations of materials and geometries [126].

This crack analogue model was also generalised to rounded contact where the stress singularities are induced from adhesion [68]. The stick/slip conditions were classified using the notion of strong and weak adhesion for static and dynamic friction. The three different modes of stress intensity factors were found at the stick-slip boundary and compared with material fatigue thresholds using an empirical relation between the adhesion and friction as derived by McFarlane and Tabor in 1950 [128]. Correlations were found for the crack initiation angle and threshold, but deviations were expected due to the stick-slip zone sizes in many cases being comparable with material grain size.

Inspired by the "crack analog", Fouvry and Berthel [61] recently used the width of the *partial slip sliding zone* as a length scale parameter in their crack analogue parameter. By multiplying the maximum shear stress with the square root of the sliding zone width, the fretting crack nucleation was represented without the need for very fine contact mesh in the FEM analysis.

Recently, researchers have put effort into systematising stress intensity factor computations for fretting fatigue [129, 130, 131].

Montebello et al. [99] proposes a method that is based on characterising the mechanical fields surrounding the contact by the *velocity field.* In this way, by using non-local stress intensity factors, comparing loading conditions does not require a full finite element simulation with extremely fine element meshes. This is an interesting approach and a possibly very useful method for industrial applications. It is expected that more research will look into the use of non-local stress intensity factors to account for the gradient effects.

10.1 Unified crack-notch model

Atzori and Lazzarin [132, 133] unified the notch sensitivity and defect (crack) sensitivity in fatigue by defining a transition size a^* for which the "crack like notch" starts to behave like a "large blunt notch". They used a material intrinsic parameter to separate the classical notch mechanics regime with its peak stress criteria, and the fracture mechanics regime in which stress field criteria are used. The idea is that small contact area problems are described by a crack analogue where the fatigue threshold is not dependant on the crack geometry. But above a certain size, the notch analogue model becomes applicable, in which stress concentrations are accounted for. Ciavarella applied this method to the fretting problem [134, 135], thus unifying the crack analogue with the notch analogue. When applied to the fretting test results by Araújo and Nowell [100, 62], the method correctly separated the failed specimens from the run-outs.

The *Crack-Like Notch Analogue Model* (CLNA) uses the fatigue knock-down factor introduced by Ciavarella [134] on the fretting fatigue problem.

The Atzori-Lazzarin criterion can be mathematically expressed as

$$K_f = \min\left(\sqrt{1 + Y^2 \frac{a}{a_0}}, K_t\right) \tag{13}$$

where a is half the crack width for an internal crack and the crack width for an edge crack. a_0 is the El Haddad crack length parameter [119] which is used to describe the transition from a "short crack" to a "long crack", together with the geometrical factor Y. The intrinsic property a_0 is given by

$$a_0 = \frac{1}{\pi}\left(\frac{\Delta K_{th}}{\Delta \sigma_1}\right)^2 \tag{14}$$

where ΔK_{th} is the threshold stress intensity factor range and $\Delta\sigma_1$ is the plain specimen fatigue limit.

The infinite-life predictions of the CLNA model were compared with Hertzian experiments and experiment data available from Araújo and Nowell [100] and others. In general, the model predicted failures and run-outs correctly. Ciavarella argues that, at least within the conditions described in [134], the parameters for multiaxiality were not necessary. Araújo et al. notes [112], the CLNA method is extremely efficient, but the application of this method on engineering applications is questionable, since it's based on 2D convex contact with constant normal load and in-phase oscillating bulk and tangential loads. However, due to its simplicity, it is excellent for early-stage design and planning experiments. Ciavarella and Berto [136] also contributed to the CLNA methods by further extending the methods to incorporate varying normal load.

As noted recently by Antunes et al. [129], there is lack of general expressions for stress intensity factors for cracks originating from bodies in contact under fretting conditions, and they seek to relieve this deficiency. They argue that a problem with CLNA method and analytical SIF in general, is neglecting non-linear effects. For estimating fretting fatigue life they used the Topper-El Haddad [137] notion of a crack tip stress raiser, calling it *Stress Gradient Factor (SGF)* as it modifies the reference stress intensity factor.

11 Wear

Fretting is in general referring to bodies in contact subjected to small sliding, and is not to be confused with the more specific terms *fretting corrosion*, *fretting wear* and *fretting fatigue*.

In tribology, fretting involves questions around friction, surface roughness, lubrication effects, wear debris formation and ejection etc. [11]. The tribologically transformed structure and particle formation in fretting was studied by Sauger et al. [138]. Velocity accommodation effects may also be important [29] in some cases. However, it may be argued that many cases of *fretting fatigue* is negligibly affected by some tribological phenomena due to very small values of slip. Nonetheless, there is an increasing interest in incorporating wear into the fretting fatigue cracking analysis [139].

An energy description of the wear process was proposed by Fouvry et al. [25] using the interfacial shear work and considering situations where debris ejection was unrestricted. Thus, the frictional energy is related to the "contact endurance" and they proposed using Energy density-N diagrams as analogue to the S-N fatigue curves. This can be used e.g. to determine life time of the surface treatments for partial and gross sliding. The energy-based wear has been proven superior to the Archard model for sliding contact, unifying the wear over a range of sliding regimes [140].

Madge et al. [141] were among the first to try to numerically combine wear mechanisms with fatigue cracking analysis, attempting to relate the reduction in fatigue life with increasing slip amplitude which is reported in literature. They found that the pressure redistributing effect of material removal was critical in the driving forces for cracks. As material removal shifts the fretting fatigue damage evolution from the contact edge to the stick-slip boundary, a critical (*optimal*) value for wear coefficient can be found that increases fatigue life by spreading the fatigue damage over a larger area.

In gross slip, initial surface roughness plays an important role for the friction coefficient and wear [142]. Paggi et al. [143] derived a linear relationship between the tangential force and the stick contact area for the Greenwood-Williams contact with a rough surface described by an exponential probability distribution of the asperities. They used the generalised superpositioned principles of Ciavarella [74] and Jäger [75]. The probability distribution function of the surface (height distribution) were shown to only very weakly affect the results.

Yue and Wahab recently [144] considered the cylinder-on-flat geometry and studied the effect of variable coefficient of friction on the fretting wear on a range of sliding conditions. For gross sliding conditions, little difference was found, but for partial slip they found increasingly accurate measures of wear volume compared with experimental results previously reported. A finite element model was used with the energy based wear rates defined using a polynomial relation of coefficient of friction with the number of cycles.

12 Finite element methods

Today, numerical analysis of contact problems using the finite element method (FEM) is well established using either Lagrangian multipliers, augmented Lagrangian multipliers or the penalty method. For accurate descriptions of stick-slip in fretting contact however, additional computational costs involved with high number constraints for the Lagrangian multipliers might be necessary. Various methods are researched for modelling fretting fatigue using FEM including wear [141, 144, 140], plasticity [131], crack growth [145].

Continuum damage mechanics may be used in the FEM code to model the damage evolution and crack initiation [110]. Goh et al. (2003) studied microstructural effects on fretting fatigue by using a crystal plasticity model to describe the inherent material heterogeneity [146]

The *Extended Finite Element method* (XFEM) can be seen as the a natural extension to the standard finite element method (FEM), but with additional functions *enriching* the solution. Through the principle of partition of unity, discontinuous basis functions can be added to the solution space of FEM and provide capabilities to capture local and discontinuous effects like cracks, material interfaces etc. Möes, Dolbow and Belytschko [145] introduced in 1999 the methods as a means for modelling cracks in the finite element framework without the need for remeshing.

Martínez et al. [147] used the XFEM implementation introduced by Giner et al. [148] in an attempt to predict cracking trajectory in railway axles under bending. *Crack closure* effects were considered by introducing restrictions to the appropriate nodes using truss elements. Using *the minimum shear stress range* as crack propagation criteria produced the most accurate results when compared with the real railway axle, but neglecting variations in the crack growth rate.

Nesládek and Španiel [149] recently initiated a project for developing a software tool to predict fretting fatigue. In engineering applications with complex model geometries, both fretting fatigue and plain fatigue often co-exist. Therefore, a plugin for Abaqus finite element software was proposed, attempting to unifying the assessments. The plugin is integrated in the graphical user interface and permits using multiaxial fatigue criteria in a post-processing step. TCD is used for the fretting stress gradients.

An interesting direction for further development in terms of the finite element modelling method is developing element formulations and incorporating asymptotic descriptions. This could permit the singular and near singular fields to be calculated without the need for very fine meshes, and thus make it more useful in an engineering context. It may also serve as a platform for further studying microstructural effects on fretting fatigue crack nucleation [150]. Giner et al. [151] used singular expressions for complete contact as enrichment functions in the finite element formulation and found good estimates of stress intensity factors with relatively coarse meshes. Recently Cardoso et al. [152] applied singular enrichment functions to *incomplete* contact. Non-local intensity factors [99] was used to track the contact edges, hence, identifying the nodes to be enriched. Increased accuracy for coarse mesh

In general, the FEM analysis has become an important tool for engineers, but for detailed contact analysis, FEM often require very fine meshes. Contact elements of size less than $10\mu m$ are not uncommon. Thus, for larger models and for complex loading histories, the computational costs may not be justified, w.r.t. the accuracy and predictability actually gained. Sub-modelling techniques can be used to separate the contact problem from the structural analysis, thus avoiding the need for fine contact mesh in the global solution. Also, the use of non-local stress intensity factors is very useful for sub-modelling techniques [151, 99, 152]

13 Some comments on fretting fatigue testing

Given the complex nature of fretting fatigue, testing is in some cases *necessary* to obtain predictive confidence. A review of the methods and equipment used in testing probably deserves a whole separate treatment and is only shortly mentioned here. A number of different methods and test rigs are devised in the literature and in 1992 there was an attempt to start standardisation of the fretting fatigue test [153]. However, there is still no accepted generic standard [154]. The recent ASTM E2789 standard [155] provides only guidelines and general requirements for conducting a fretting fatigue test program. While it provides definitions and terminology for fretting fatigue testing, it does not propose a specific test configuration.

Hills and Nowell summarized in 2009 [156] the most important features with fretting fatigue testing. They argue that the standardisation of fretting fatigue test geometries will alleviate the comparison of different sets of results, but standardisation can also restrict the diversity of test results and hence make them less helpful for understanding fretting fatigue as a whole.

Early fretting fatigue tests made use of single-actuator test rigs where pads, usually of "bridge" type where clamped onto the specimen using e.g. a proving ring [157]. The clamping force was therefore constant as long as the wear was negligible. The Japanese standard *JSME S 015-2002* uses this test configuration [154]. More advanced, biaxial test rigs permits the cyclic load to be controlled independently from the contact loading. Early fretting tests at Oxford University used the fretting bridge on dogbone specimen, but during the end of 60s, they developed a test rig using Hertzian contact and electromagnetic resonance to generate the shear forces [158]. With this, the contact stresses, slip and displacements were known. These tests had a high degree of repeatability. Further generelisation was made that permitted independent control of shear and bulk forces in the specimen. This was also tested for *complete contact* with a self-aligning property avoiding rotation of the shear forces. The new arrangement have three independent actuators for the normal, shearing and bulk loads.

Other interesting developments in fretting fatigue testing include the use of *digital image correlation* (DIC) method to resolve the contact situation during the experiment [159, 160].

14 Conclusions

Over a century after the first studies, researchers and engineers are still interested in and concerned with fretting fatigue. This article has attempted to review the progress in fretting fatigue understanding and predictions, with the engineering applications in mind. Fretting fatigue assessments in the industry are often very simple and relies on unsophisticated parameters. Comparing these parameters with company internal empirical data and experience the overall risk of fretting is determined, i.e. designing for "infinite life". However, research in fretting and fretting fatigue have increased the understanding of the underlying phenomena and improvements have been made into determining fretting fatigue life. It is therefore argued that more elaborate analysis may be appropriate in many engineering applications.

It is clear that fretting and *fretting fatigue* are indeed complex problems, and for the practical engineering case not all contributing phenomena are important. The integrity of an engineering component is evaluated according to its intended requirements and operating conditions, and the design engineer must carefully identify the fretting regimes and choose the appropriate analysis. Some sliding motion are in many engineering cases inevitable, and partial slip conditions are common, promoting fretting fatigue. In other cases, fretting wear is more critical. Thus, the use of fretting maps is a valuable tool for the design engineer to predict the fretting regime and to visually reason about the mechanisms involved.

For more quantitative predictions of fretting fatigue, simple parameters like Ruiz (FD and FRD) give the engineers the possibility to numerically evaluate their design, however purely empirical. The Ruiz-parameters are widely used in the industry, mainly due to being accessible.

Drawing analogies to notches and cracks can be a great improvement in the predictability of fretting fatigue. These methods are easily applied to engineering cases to predict cracking risk and thereby designing for infinite life. The practicality of asymptotic methods in engineering applications may not seem obvious at first. However, the fact that the crack nucleation risk in critical regions surrounding contact edges can be matched to a small laboratory test may provide very useful tools. Most analyses are restricted by half-plane theory and small scale plasticity. For cases of varying contact normal load however, the asymptotic methods are difficult to use due to the moving contact edges.

For predicting fretting fatigue damage nucleation, multiaxial fatigue methods might be used, but with the amount of uncertainty involved, it most likely requires high safety-factors. Applying critical plane calculations as a post-processing step in elastic FEM analysis is a practical, but sometimes computationally costly approach. Sub-modelling techniques and non-local stress intensity factors are promising methods and may alleviate the problems with computational costs.

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