Kinematic Feedback Control Using Dual Quaternions

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***Abstract*— This paper presents results on kinematic controllers for the stabilization of rigid body displacements using dual quaternions. The paper shows how certain results for quaternion stabilization of rotation can be extended to dual quaternions stabilization of displacements. The paper presents a relevant background material on screw motion and the screw description of lines and twists. Moreover, results are presented on the computation of the exponential functions for dual quaternions for use in numerical integration. The paper presents and analyzes different controllers based on feedback from dual quaternions, where some of the controllers are known from the literature, and some are new. In particular, it is shown which controllers give screw motion, and it is discussed how this will affect the performance of the controlled system compared to other controllers that are not based on screw motion. This analysis is supported by Lyapunov analysis. Also, certain passivity properties for dual quaternions are presented as an extension to previously published results on passivity for quaternions.**

I. INTRODUCTION

Quaternions have been studied extensively in the control community for the last three decades for use in attitude control and estimation on $SO(3)$. More recently, also dual quaternions have been studied for control and estimation on $SE(3)$. Many of the properties of quaternions are transferred to dual quaternions, and a geometric interpretation based on a dual angle about a line can be seen as an extension of the quaternion interpretation in terms of a rotation about a vector. Still, there are some notable differences between quaternions on $SO(3)$ and dual quaternions on $SE(3)$ that have an impact on controller design that will be discussed in this paper. In particular, a bi-invariant metric on $SO(3)$ can be described in terms of the rotation angle θ, and Lyapunov functions can be formulated based on the quaternion. In contrast to this, there is no bi-invariant metric on $SE(3)$, and distance measures will be scale-dependent [20]. Moreover, to formulate norms or Lyapunov functions on $SE(3)$ from dual quaternions, it is necessary to use inner products on the Euclidean representation of real and dual parts.

Quaternions were used for attitude control with feedback from the quaternion vector in [23] using Lyapunov analysis and in [6] using passivity. In [21] feedback was taken from the Rodrigues vector and the modified Rodrigues vector. The need for velocity feedback was eliminated in [15], [1]. More recent work [17] has used hybrid control as a solution to the unwinding problem [2], [3] that appear when feedback is taken from the quaternion vector. Quaternions have been used in attitude estimation [14], where an overview is found in [4]. Dual quaternions have been used for control in [9] where the controller of [1] was extended to dual quaternions, and in [10] where an adaptive controller was presented. A controller based on a small-angle approximation of the logarithm of the dual quaternion was proposed in [11], [22]. In [16], [13] kinematic control was proposed with feedback from the vector part of the dual quaternion, and the hybrid control method of [17] was used to avoid the unwinding problem.

In this paper, a new kinematic controller with feedback from a dual quaternion is presented based on controllers for quaternions [23], [6]. This controller can be implemented to give a screw motion, or more direct translation. Hybrid control is not used as the focus is on the kinematic properties of the dual quaternion feedback, and how this relates to previous quaternion control methods. The controller does not have problems with unwinding as there is an unstable equilibrium at π. Moreover, a solution for efficient computation of the exponential function for a dual quaternion is presented for use in numerical time integration that eliminates the need for normalization of the real term of the quaternions and projection of the dual term.

The paper is organized as follows: Sect. II presents background on quaternions, dual quaternions, screws, and screw motion. Sect. III presents different controllers based on dual quaternion feedback, and an analysis of the properties of the controllers, and Lyapunov analysis and passivity properties. Sect. IV presents the implementation aspects and results from simulations of the kinematic controllers, while Sect. V concludes this paper.

II. PRELIMINARIES

A. Quaternions

A quaternion $\boldsymbol{q} = \eta + \boldsymbol{\sigma}$ is written as the sum of a scalar part η and a vector part $\boldsymbol{\sigma}$ [7], [12]. Conjugation is given by $\boldsymbol{q}^* = \eta - \boldsymbol{\sigma}$, and multiplication with a scalar λ gives $\lambda\boldsymbol{q} = \lambda\eta + \lambda\boldsymbol{\sigma}$. Let $\boldsymbol{q}_1 = \eta_1 + \boldsymbol{\sigma}_1$ and $\boldsymbol{q}_2 = \eta_2 + \boldsymbol{\sigma}_2$ be two quaternions. Then the sum is $\boldsymbol{q}_1 + \boldsymbol{q}_2 = \eta_1 + \eta_2 + \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2$, and the quaternion product is $\boldsymbol{q}_1 \circ \boldsymbol{q}_2 = \eta_1\eta_2 - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + \eta_1\boldsymbol{\sigma}_2 + \eta_2\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2$. The norm of a quaternion is given by $||\boldsymbol{q}||^2 = \boldsymbol{q} \circ \boldsymbol{q}^* = \eta^2 + \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}$.

A unit quaternion $\boldsymbol{q}$ has unit norm which means that $\|\boldsymbol{q}\|^2 = \eta^2 + \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} = 1$. A unit quaternion can be used to describe a rotation $(\boldsymbol{k}, \theta)$ by an angle θ about a unit vector

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$\boldsymbol{k}$ by letting

$$\boldsymbol{q} = \eta + \boldsymbol{\sigma}, \quad \eta = \cos\frac{\theta}{2}, \quad \boldsymbol{\sigma} = \boldsymbol{k}\sin\frac{\theta}{2}$$

Then if $\boldsymbol{R} = \boldsymbol{I} + \sin\theta\boldsymbol{k}^\times + (1-\cos\theta)\boldsymbol{k}^\times\boldsymbol{k}^\times \in SO(3)$ is the associated rotation matrix, the rotation of a general vector $\boldsymbol{u}$ can be achieved with the two alternative expressions $\boldsymbol{R}\boldsymbol{u} = \boldsymbol{q}\circ\boldsymbol{u}\circ\boldsymbol{q}^*$. It is noted that $\boldsymbol{q}$ and $-\boldsymbol{q}$ describes the same rotation. The vector $\boldsymbol{e} = 2\eta\boldsymbol{\sigma} = \boldsymbol{k}\sin\theta$ will also be used.

B. Kinematic Differential Equation of a Unit Quaternion

Let $\boldsymbol{q} = \boldsymbol{q}_{ab}$ be the quaternion describing the rotation from a frame a to a frame b. Then the kinematic differential equation is

$$\dot{\boldsymbol{q}} = \frac{1}{2}\boldsymbol{q}\circ\boldsymbol{\omega}^b = \frac{1}{2}\boldsymbol{\omega}^a\circ\boldsymbol{q} \tag{1}$$

where $\boldsymbol{\omega}^b = \boldsymbol{\omega}^b_{ab}$ is the angular velocity of frame b with respect to frame a in the coordinates of b, and $\boldsymbol{\omega}^a = \boldsymbol{q}\circ\boldsymbol{\omega}^b\circ\boldsymbol{q}^*$ is the same vector in the coordinates of a.

C. Dual Vectors, Screws and Lines

A dual real number is written $\hat{\alpha} = \alpha + \varepsilon\alpha'$ where α and α' are real numbers and ε is the dual unit which satisfies $\varepsilon \neq 0$ and $\varepsilon^2 = 0$ [18]. A real-valued function $f(\alpha)$ can be extended to a function of a dual number where

$$f(\hat{\alpha}) = f(\alpha) + \varepsilon\frac{\mathrm{d}f(\alpha)}{\mathrm{d}\alpha}\alpha' \tag{2}$$

It is noted that

$$\sin\hat{\alpha} = \sin\alpha + \varepsilon\alpha'\cos\alpha, \quad \cos\hat{\alpha} = \cos\alpha - \varepsilon\alpha'\sin\alpha$$

A dual 3D vector is given by $\hat{\boldsymbol{u}} = \boldsymbol{u} + \varepsilon\boldsymbol{u}'$ where $\boldsymbol{u}$ and $\boldsymbol{u}'$ are 3D vectors. Computations on dual numbers and dual vectors are performed as operations on polynomials in the dual unit ε. For two dual vectors $\hat{\boldsymbol{u}}_1 = \boldsymbol{u}_1 + \varepsilon\boldsymbol{u}'_1$ and $\hat{\boldsymbol{u}}_2 = \boldsymbol{u}_2 + \varepsilon\boldsymbol{u}'_2$ this gives $\hat{\boldsymbol{u}}_1 + \hat{\boldsymbol{u}}_2 = \boldsymbol{u}_1 + \boldsymbol{u}_2 + \varepsilon(\boldsymbol{u}'_1 + \boldsymbol{u}'_2)$, $\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{u}}_2 = \boldsymbol{u}_1\cdot\boldsymbol{u}_2 + \varepsilon(\boldsymbol{u}_1\cdot\boldsymbol{u}'_2 + \boldsymbol{u}'_1\cdot\boldsymbol{u}_2)$, and $\hat{\boldsymbol{u}}\times\hat{\boldsymbol{u}}_2 = \boldsymbol{u}_1\times\boldsymbol{u}_2 + \varepsilon(\boldsymbol{u}_1\times\boldsymbol{u}'_2 + \boldsymbol{u}'_1\times\boldsymbol{u}_2)$.

A screw is a dual vector which satisfies the screw transformation law. Let $\hat{\boldsymbol{s}}_a = \boldsymbol{s}^a + \varepsilon\boldsymbol{s}'^a_a$ be the screw referenced to frame a and given in the coordinates of a. Then the same screw referenced to a frame b and given in the coordinates of b will be $\hat{\boldsymbol{s}}_b = \boldsymbol{s}^b + \varepsilon\boldsymbol{s}'^b_b$ where $\boldsymbol{s}'^b_b = \boldsymbol{R}^a_b(\boldsymbol{s}'^a_a + \boldsymbol{t}^a\times\boldsymbol{s}^a)$.

A line can be described by the screw $\hat{\boldsymbol{k}} = \boldsymbol{k} + \varepsilon\boldsymbol{k}'$, where $\boldsymbol{k}$ is the unit vector along the line, and $\boldsymbol{k}' = \boldsymbol{c}\times\boldsymbol{k}$ is the moment of the line. Here $\boldsymbol{c}$ is the vector from the origin of the reference frame to an arbitrary point on the line. In the following the reference frame is the a frame and $\boldsymbol{c} = \boldsymbol{k}\times\boldsymbol{k}'$, which means that it is the vector to the point on the line that is closest to the origin. The components of $\hat{\boldsymbol{k}}$ are the Plücker coordinates of the line.

D. Displacement described as Screw Motion

The displacement $\boldsymbol{T} = \boldsymbol{T}^a_b \in SE(3)$ from frame a to frame b can be represented by the rotation matrix $\boldsymbol{R} = \boldsymbol{R}^a_b \in SO(3)$ from a to b, and the translation $\boldsymbol{t}^a = \boldsymbol{t}^a_{ab}$ from a to b is the coordinates of a. It is well-known that this displacement can be described as a screw motion, which is a rotation by an angle θ about a line $\hat{\boldsymbol{k}} = \boldsymbol{k} + \varepsilon\boldsymbol{k}'$ and a translation d along the same line. An illustration is shown in Fig. 1. This can be described as a motion by a dual angle $\hat{\theta} = \theta + \varepsilon d$ about the line $\hat{\boldsymbol{k}}$ [18]. The displacement can be represented by

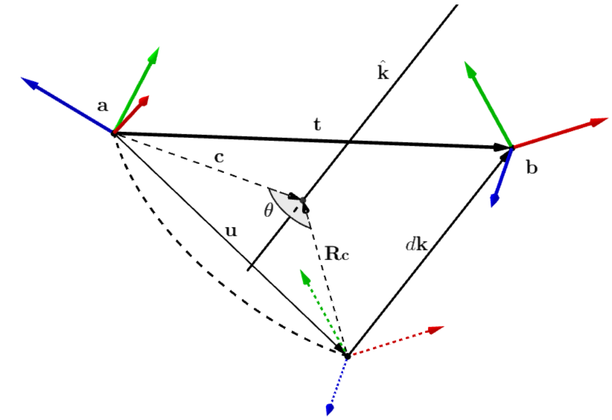

Fig. 1. The displacement $\boldsymbol{t}$ shown as the sum of the translation $\boldsymbol{u}$ generated by the rotation θ about the screw axis $\hat{\boldsymbol{k}}$, and the translation $d\boldsymbol{k}$ along the the screw axis.

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{R} & (\boldsymbol{I}-\boldsymbol{R})\boldsymbol{c} + d\boldsymbol{k} \\ \boldsymbol{0}^\mathrm{T} & 1 \end{bmatrix} \in SE(3) \tag{3}$$

where $\boldsymbol{R}$ is the rotation matrix corresponding to θ and $\boldsymbol{k}$, $\boldsymbol{c} = \boldsymbol{k}\times\boldsymbol{k}'$ and $d = \boldsymbol{k}\cdot\boldsymbol{t}$. If the motion from frame a to frame b is done as a screw motion about a fixed line $\hat{\boldsymbol{k}}$, then $\omega = \dot{\theta}\boldsymbol{k}$, and the velocity $\boldsymbol{v}_{\hat{k}}$ of a point fixed in b that is on the screw axis $\hat{\boldsymbol{k}}$ will be $\boldsymbol{v}_{\hat{k}} = \dot{d}\boldsymbol{k}$.

E. Dual Quaternions

The dual quaternion can also be written $\hat{\boldsymbol{q}} = \boldsymbol{q} + \varepsilon\boldsymbol{q}'$ where $\boldsymbol{q}$ and $\boldsymbol{q}'$ are quaternions. The conjugate is given by $\hat{\boldsymbol{q}}^* = \boldsymbol{q}^* + \varepsilon\boldsymbol{q}'^*$. Let $\hat{\boldsymbol{q}}_1 = \boldsymbol{q}_1 + \varepsilon\boldsymbol{q}'_1$ and $\hat{\boldsymbol{q}}_2 = \boldsymbol{q}_2 + \varepsilon\boldsymbol{q}'_2$ be two dual quaternions. Then the quaternion product is

$$\hat{\boldsymbol{q}}_1\circ\hat{\boldsymbol{q}}_2 = \boldsymbol{q}_1\circ\boldsymbol{q}_2 + \varepsilon(\boldsymbol{q}_1\circ\boldsymbol{q}'_2 + \boldsymbol{q}_2\circ\boldsymbol{q}'_1)$$

The norm is $\|\hat{\boldsymbol{q}}\|^2 = \hat{\boldsymbol{q}}\circ\hat{\boldsymbol{q}}^* = \|\boldsymbol{q}\|^2 + \varepsilon(\boldsymbol{q}\circ\boldsymbol{q}'^* + \boldsymbol{q}'\circ\boldsymbol{q}^*)$. A dual quaternion is called a dual unit quaternion if $\|\hat{\boldsymbol{q}}\| = 1$, which means that $\boldsymbol{q}$ is a unit quaternion and

$$\boldsymbol{q}\circ\boldsymbol{q}'^* + \boldsymbol{q}'\circ\boldsymbol{q}^* = 2(\eta\eta' + \sigma\cdot\sigma') = 0 \tag{4}$$

A displacement of a vector can be expressed as $\boldsymbol{R}\boldsymbol{u} + \boldsymbol{t} = \hat{\boldsymbol{q}}\circ\boldsymbol{u}\circ\hat{\boldsymbol{q}}^*$, while the screw transformation from frame a to frame b can written [5] $\hat{\boldsymbol{s}}_a = \hat{\boldsymbol{q}}\circ\hat{\boldsymbol{s}}_b\circ\hat{\boldsymbol{q}}^*$.

F. Dual quaternions in terms of screw motion

A dual unit quaternion that represents a screw motion by a dual angle $\hat{\theta}$ about $\hat{\boldsymbol{k}}$ can then be defined as

$$\hat{\boldsymbol{q}} = \hat{\eta} + \hat{\boldsymbol{\sigma}}, \quad \hat{\eta} = \cos\frac{\hat{\theta}}{2}, \quad \hat{\boldsymbol{\sigma}} = \hat{\boldsymbol{k}}\sin\frac{\hat{\theta}}{2} \tag{5}$$

$$\hat{\eta} = \eta + \varepsilon\eta', \quad \eta = \cos\frac{\theta}{2}, \quad \eta' = -\frac{d}{2}\sin\frac{\theta}{2} \tag{6}$$

$$\hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + \varepsilon\hat{\boldsymbol{\sigma}}', \; \boldsymbol{\sigma} = \boldsymbol{k}\sin\frac{\theta}{2}, \; \boldsymbol{\sigma}' = \frac{d}{2}\cos\frac{\theta}{2}\boldsymbol{k} + \sin\frac{\theta}{2}\boldsymbol{k}' \tag{7}$$

Then $\hat{\boldsymbol{q}} = \boldsymbol{q} + \varepsilon\boldsymbol{q}'$ is a dual unit quaternion since $\boldsymbol{q}$ is a unit quaternion and $\eta\eta' + \sigma\cdot\sigma' = 0$. It is noted that rotation and translation commutes for a screw motion.

G. Dual quaternions in terms of translation and rotation

The dual unit quaternion can alternatively be expressed in terms of a combination of a general translation $\hat{\boldsymbol{q}}_t = 1 + \varepsilon(\boldsymbol{t}/2)$ and a rotation $\boldsymbol{q}$. Then the dual quaternion from a to b will be $\hat{\boldsymbol{q}} = \boldsymbol{q}_{t^a} \circ \boldsymbol{q}$ if translation is done before rotation, or $\hat{\boldsymbol{q}} = \boldsymbol{q} \circ \boldsymbol{q}_{t^b}$ if rotation is done before translation. This gives

$$\hat{\boldsymbol{q}} = \boldsymbol{q} + \varepsilon(\boldsymbol{t}^a/2) \circ \boldsymbol{q} = \boldsymbol{q} + \varepsilon \boldsymbol{q} \circ (\boldsymbol{t}^b/2) \tag{8}$$

It is seen that rotation and translation will not commute in general.

It follows that $\boldsymbol{q}' = (\boldsymbol{t}^a/2) \circ \boldsymbol{q} = \boldsymbol{q} \circ (\boldsymbol{t}^b/2)$, which means that $\boldsymbol{q}'$ can be computed from $\boldsymbol{q}$ and $\boldsymbol{t}^b$, and that

$$\boldsymbol{t}^b = 2\boldsymbol{q}^* \circ \boldsymbol{q}' = 2(-\eta' \boldsymbol{\sigma} + \eta \boldsymbol{\sigma}' + \boldsymbol{\sigma}' \times \boldsymbol{\sigma}) \tag{9}$$

$$= (d\boldsymbol{k} + \sin\theta \boldsymbol{k}' - (1 - \cos\theta)\boldsymbol{c}) \tag{10}$$

It is noted that

$$\boldsymbol{q}' \circ \boldsymbol{q}'^* = (1/4)\boldsymbol{t} \circ \boldsymbol{q} \circ \boldsymbol{q}^* \circ \boldsymbol{t}^* = (1/4)\boldsymbol{t} \cdot \boldsymbol{t} \tag{11}$$

H. The twist as a dual vector

Let $\hat{\boldsymbol{q}}$ be the dual unit quaternion describing the displacement $\boldsymbol{T}$ from frame a to a frame b. The twist of the displacement is a screw given by the dual vector $\hat{\boldsymbol{\omega}}^b = \boldsymbol{\omega}^b + \varepsilon \boldsymbol{v}^b$ which is referenced to frame b and given in the coordinates of b. Then $\boldsymbol{v}^b$ is the velocity of the origin of frame b in the coordinates of b. The twist $\hat{\boldsymbol{\omega}}^b$ is referred to as the body velocity in [19].

Consider the case where the frame b moves with respect to frame a with a screw motion with dual angle $\hat{\theta} = \theta + \varepsilon d$ about a fixed screw axis $\hat{\boldsymbol{k}} = \boldsymbol{k} + \varepsilon \boldsymbol{k}'$. Then the angular velocity of frame b is $\boldsymbol{\omega}^b = \dot{\theta}\boldsymbol{k}$, and the velocity of a point that is fixed in b and that is on the screw axis will be $\dot{d}\boldsymbol{k}$. This means that the twist referenced to the screw axis will be $\hat{\boldsymbol{\omega}}^k = \dot{\theta}\boldsymbol{k} + \varepsilon \dot{d}\boldsymbol{k}$. This twist can be referenced to frame b with the screw transformation

$$\hat{\boldsymbol{\omega}}^b = \left(1 + \varepsilon \frac{\boldsymbol{c}}{2}\right) \circ \hat{\boldsymbol{\omega}}^k \circ \left(1 - \varepsilon \frac{\boldsymbol{c}}{2}\right) \tag{12}$$

where $\boldsymbol{c}$ is the vector from the origin of b to $\hat{\boldsymbol{k}}$. Insertion of $\boldsymbol{k}' = \boldsymbol{c} \times \boldsymbol{k}$ gives the following expression for the twist of a screw motion about a fixed axis $\hat{\boldsymbol{k}}$

$$\hat{\boldsymbol{\omega}}^b = \dot{\theta}\boldsymbol{k} + \varepsilon(\dot{d}\boldsymbol{k} + \dot{\theta}\boldsymbol{k}') \tag{13}$$

I. Kinematic Differential Equation of a Dual Unit Quaternion

The kinematic differential equation is

$$\dot{\hat{\boldsymbol{q}}} = \frac{1}{2}\hat{\boldsymbol{q}} \circ \hat{\boldsymbol{\omega}}^b \tag{14}$$

which gives

$$\dot{\eta} = -\frac{1}{2}\boldsymbol{\sigma} \cdot \boldsymbol{\omega}^b \tag{15}$$

$$\dot{\boldsymbol{\sigma}} = \frac{1}{2}(\eta \boldsymbol{\omega}^b + \boldsymbol{\sigma} \times \boldsymbol{\omega}^b) \tag{16}$$

$$\dot{\eta}' = -\frac{1}{2}(\boldsymbol{\sigma} \cdot \boldsymbol{v}^b + \boldsymbol{\sigma}' \cdot \boldsymbol{\omega}^b) \tag{17}$$

$$\dot{\boldsymbol{\sigma}}' = \frac{1}{2}(\eta \boldsymbol{v}^b + \eta' \boldsymbol{\omega}^b + \boldsymbol{\sigma} \times \boldsymbol{v}^b + \boldsymbol{\sigma}' \times \boldsymbol{\omega}^b) \tag{18}$$

Let

$$V_\sigma = 2(1 - \eta), \quad V_e = 2(1 - \eta)^2 \tag{19}$$

Then the time derivatives along the solutions of (14) will be $\dot{V}_\sigma = \boldsymbol{\sigma} \cdot \boldsymbol{\omega}^b$ and $\dot{V}_e = \boldsymbol{e} \cdot \boldsymbol{\omega}^b$, which means that the mappings $\boldsymbol{\omega} \mapsto \boldsymbol{\sigma}$ and $\boldsymbol{\omega} \mapsto \boldsymbol{e}$ are passive [6].

J. Exponentials and logarithms

Suppose that a rotation is described by a rotation θ about a fixed unit vector $\boldsymbol{k}$. Then the resulting quaternion of the rotation is $\boldsymbol{q} = \exp[(\theta/2)\boldsymbol{k}]$ where

$$\exp\left(\frac{\theta \boldsymbol{k}}{2}\right) = 1 + \theta \boldsymbol{k} + \frac{\theta^2}{2!}\boldsymbol{k} \circ \boldsymbol{k} + \ldots = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\boldsymbol{k}$$

If the argument of the exponential function is given as a vector $\boldsymbol{u}$, then $\boldsymbol{k} = \boldsymbol{u}/\|\boldsymbol{u}\|$ and $\theta/2 = \|\boldsymbol{u}\|$, and it is well known that the exponential can be computed from

$$\exp(\boldsymbol{u}) = \cos(\|\boldsymbol{u}\|) + \mathrm{sinc}(\|\boldsymbol{u}\|)\boldsymbol{u}$$

which can be computed when $\|\boldsymbol{u}\|$ tends to zero even though $\boldsymbol{k} = \boldsymbol{u}/\|\boldsymbol{u}\|$ becomes undefined, which is seen from the Taylor expansion $\mathrm{sinc}(x) = \sin(x)/x \approx 1 - (1/3!)x^2 + \ldots$.

Next, suppose that a displacement is described by a screw motion with a dual angle $\hat{\theta}$ about a fixed axis $\hat{\boldsymbol{k}}$. Then the resulting dual unit quaternion is

$$\exp\left(\frac{\hat{\theta}\hat{\boldsymbol{k}}}{2}\right) = \cos\frac{\hat{\theta}}{2} + \sin\frac{\hat{\theta}}{2}\hat{\boldsymbol{k}} \tag{20}$$

The logarithm of a dual quaternion is then seen to be

$$\log(\hat{\boldsymbol{q}}) = \frac{\hat{\theta}\hat{\boldsymbol{k}}}{2} = \frac{\theta \boldsymbol{k}}{2} + \varepsilon\left(\frac{d\boldsymbol{k}}{2} + \frac{\theta \boldsymbol{k}'}{2}\right) \tag{21}$$

III. Kinematic Control with Dual Quaternion Feedback

A. Introduction

Kinematic controllers will be presented and analyzed in the following. These controllers have the twist $\hat{\boldsymbol{\omega}}^b$ referenced to the b frame as the control variable. Moreover, only two frames a and b will be used in the analysis. The b frame is the body-fixed frame, while a can be the fixed spatial frame or the desired frame.

B. Kinematic Control giving Screw Motion

To analyze controllers based on dual unit quaternions, it is interesting to investigate if a controller gives screw motion to get insight into the kinematics of the resulting motions. In addition, Lyapunov analysis is used, and passivity properties are investigated. It is noted that if the control $\hat{\boldsymbol{\omega}}^b = \boldsymbol{\omega}^b + \varepsilon \boldsymbol{v}^b$ is selected as

$$\hat{\boldsymbol{\omega}}^b = \alpha \boldsymbol{k} + \varepsilon(\beta \boldsymbol{k} + \alpha \boldsymbol{k}') \tag{22}$$

where α and β are scalars, then the resulting motion is a screw motion about the screw axis $\hat{\boldsymbol{k}} = \boldsymbol{k} + \varepsilon \boldsymbol{k}'$.

C. Comments on Existing Kinematic controllers

Consider the kinematic controller $\hat{\boldsymbol{\omega}}^b = -K(\boldsymbol{\sigma} + \varepsilon\eta\boldsymbol{\sigma}')$, which was discussed by [13], who observed that the translation was unstable when $\eta < 0$. The controller does not give a screw motion as seen from

$$\hat{\boldsymbol{\omega}}^b = -K\left(\sin\frac{\theta}{2}\boldsymbol{k} + \varepsilon\left(\frac{d}{2}\cos\frac{\theta}{2}\boldsymbol{k} + \sin\frac{\theta}{2}\boldsymbol{k}'\right)\right) \tag{23}$$

where it is seen that translation along $\boldsymbol{k}$ will be in the wrong direction if $\eta = \cos(\theta/2) < 0$. A modification in the form $\hat{\boldsymbol{\omega}}^b = -K(h\boldsymbol{\sigma} + \varepsilon\eta\boldsymbol{\sigma}')$ was proposed in [13] where h is a discrete state variable which is used to generate a hysteresis in the regions where $\eta = 0$ [17]. This modified controller will not give a screw motion.

The controller $\hat{\boldsymbol{\omega}}^b = -K(\log(\boldsymbol{q}) + \varepsilon\boldsymbol{t}^b)$ was proposed in [11], [22], where $\log(\boldsymbol{q}) + \varepsilon\boldsymbol{t}^b$ was introduced as a small-angle an approximation of the logarithm in (21). The controller will be asymptotically stable, but as commented in [13], the controller will be discontinuous for large angles, as seen from

$$\hat{\boldsymbol{\omega}}^b = -K[(\theta/2)\boldsymbol{k} + \varepsilon(\boldsymbol{t}^b)], \quad |\theta| < \pi \tag{24}$$

The controller $\hat{\boldsymbol{\omega}}^b = \boldsymbol{\omega}^b + \varepsilon\boldsymbol{v}^b$ is the kinematic controller corresponding to the controller proposed in [9], where $\boldsymbol{\omega}^b$ is the vector part of $-K(\boldsymbol{q}'^* \circ \boldsymbol{q}' + \boldsymbol{q}^* \circ (\boldsymbol{q} - 1))$, and $\boldsymbol{v}^b = -K\boldsymbol{q}^* \circ \boldsymbol{q}' = -(K/2)\boldsymbol{t}^b$. The kinematic controller is

$$\hat{\boldsymbol{\omega}}^b = -K[\boldsymbol{\sigma} + \varepsilon(\boldsymbol{t}^b/2)] \tag{25}$$

This controller will not give screw motion.

D. Proposed controllers

Consider the following kinematic controllers that satisfies condition (22), and gives screw motion about $\hat{\boldsymbol{k}}$.

Controller 1 is new and given by $\hat{\boldsymbol{\omega}}_1^b = -2K(\eta\boldsymbol{\sigma} + \varepsilon\eta\boldsymbol{\sigma}')$. This gives the screw motion

$$\hat{\boldsymbol{\omega}}_1^b = -K\left(\sin\theta\boldsymbol{k} + \varepsilon\left(d\cos^2\frac{\theta}{2}\boldsymbol{k} + \sin\theta\boldsymbol{k}'\right)\right) \tag{26}$$

Controller 2 is new and given by $\hat{\boldsymbol{\omega}}_2^b = -(2K/\eta)(\boldsymbol{\sigma} + \varepsilon\boldsymbol{\sigma}')$, which can be regarded as an extension of the kinematic part of the controller in [21]. This controller gives the screw motion

$$\hat{\boldsymbol{\omega}}_2^b = -K\left(\tan\frac{\theta}{2}\boldsymbol{k} + \varepsilon\left(d\boldsymbol{k} + \tan\frac{\theta}{2}\boldsymbol{k}'\right)\right) \tag{27}$$

We will also consider a related new Controller 3, which is

$$\begin{aligned}
\hat{\boldsymbol{\omega}}_3^b &= -2K_1\eta\boldsymbol{\sigma} + \varepsilon K_2\boldsymbol{t}^b & (28)\\
&= -2K_1\eta\boldsymbol{\sigma} - \varepsilon K_2(-\eta'\boldsymbol{\sigma} + \eta\boldsymbol{\sigma}' + \boldsymbol{\sigma}' \times \boldsymbol{\sigma}) & (29)\\
&= -K\sin\theta\boldsymbol{k} - \varepsilon K_2[d\boldsymbol{k} + \sin\theta\boldsymbol{k}' - (1-\cos\theta)\boldsymbol{c}] & (30)
\end{aligned}$$

E. Lyapunov Function Candidate 1

Consider the following Lyapunov function candidate

$$V_1 = 2(1-\eta^2) + 2(\eta'^2 + \boldsymbol{\sigma}' \cdot \boldsymbol{\sigma}') \tag{31}$$

where the first term is the storage function V_e defined in (19). From (6) and (11) it is seen that

$$V_1 = 2\sin^2\frac{\theta}{2} + \frac{1}{2}\boldsymbol{t}^a \cdot \boldsymbol{t}^a \tag{32}$$

The time derivative of V_1 along the solutions of the kinematic differential equations is

$$\dot{V}_1 = 2\eta\boldsymbol{\sigma} \cdot \boldsymbol{\omega}^b + 2(-\eta'\boldsymbol{\sigma} + \eta\boldsymbol{\sigma}' + \boldsymbol{\sigma}' \times \boldsymbol{\sigma}) \cdot \boldsymbol{v}^b \tag{33}$$

Then a kinematic controller on the form $\hat{\boldsymbol{\omega}}^b = -K\gamma(\eta)(\boldsymbol{\sigma} - \varepsilon\boldsymbol{\sigma}')$ and (4) will give the time derivative

$$\dot{V}_1 = -2K\gamma(\eta)\eta V_1 \tag{34}$$

which implies asymptotic stability when $\gamma(\eta)\eta > 0$ for all $\eta \neq 0$. This applies for the proposed Controller 1, which gives $\dot{V}_1 = -2K\eta^2 V_1$, and for the proposed Controller 2, which gives $\dot{V}_1 = -2KV_1$.

Controller 3 gives

$$\dot{V}_1 = -2K_1\eta^2(1-\eta^2) - K_2\boldsymbol{t}^a \cdot \boldsymbol{t}^a \tag{35}$$

F. Lyapunov Function Candidate 2

Consider the Lyapunov function candidate [9], [13]

$$V_2 = 2(1-\eta) + 2(\eta'^2 + \boldsymbol{\sigma}' \cdot \boldsymbol{\sigma}') \tag{36}$$

where the first term is the storage function V_σ defined in (19). Using (6) and (11), this can also be written

$$V_2 = 4\sin^2\frac{\theta}{4} + \frac{1}{2}\boldsymbol{t}^a \cdot \boldsymbol{t}^a \tag{37}$$

The time derivative of V_1 along the solutions of the kinematic differential equations is

$$\dot{V}_2 = \boldsymbol{\sigma} \cdot \boldsymbol{\omega}^b + 2(-\eta'\boldsymbol{\sigma} + \eta\boldsymbol{\sigma}' + \boldsymbol{\sigma}' \times \boldsymbol{\sigma}) \cdot \boldsymbol{v}^b \tag{38}$$

G. Passivity

From (9) and (33) with $\boldsymbol{e} = 2\eta\sigma$ it is seen that

$$\dot{V}_1 = \boldsymbol{e} \cdot \boldsymbol{\omega}^b + \boldsymbol{t}^b \cdot \boldsymbol{v}^b = \begin{bmatrix} \boldsymbol{e} \\ \boldsymbol{t}^b \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{\omega}^b \\ \boldsymbol{v}^b \end{bmatrix} \tag{39}$$

This shows that the mapping $[\boldsymbol{\omega}^{b\mathrm{T}}, \boldsymbol{v}^{b\mathrm{T}}]^{\mathrm{T}} \mapsto [\boldsymbol{e}^{\mathrm{T}}, \boldsymbol{t}^{b\mathrm{T}}]^{\mathrm{T}}$ is passive with storage function V_1, which is an extension of a result for quaternions in [6], and which has interesting consequences for controller design.

From (9) and (38) it is seen that

$$\dot{V}_2 = \boldsymbol{\sigma} \cdot \boldsymbol{\omega}^b + \boldsymbol{t}^b \cdot \boldsymbol{v}^b = \begin{bmatrix} \boldsymbol{\sigma} \\ \boldsymbol{t}^b \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{\omega}^b \\ \boldsymbol{v}^b \end{bmatrix} \tag{40}$$

which shows that the mapping $[\boldsymbol{\omega}^{b\mathrm{T}}, \boldsymbol{v}^{b\mathrm{T}}]^{\mathrm{T}} \mapsto [\boldsymbol{\sigma}^{\mathrm{T}}, \boldsymbol{t}^{b\mathrm{T}}]^{\mathrm{T}}$ is passive with storage function V_2, which is an extension of a result for quaternions in [6].

IV. Simulations

A. Numerical Integration of Dual Quaternions

The exponential function can be used in numerical integration of dual unit quaternions. Then the argument of the exponential function will given as a dual vector $\hat{\boldsymbol{u}} = \boldsymbol{u} + \varepsilon \boldsymbol{u}'$. Then an expression for the exponential can then be found from (20) using $d/2 = \boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}'/\|\boldsymbol{u}\|$, $d\boldsymbol{k}/2 = (\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u}')\boldsymbol{u}/\|\boldsymbol{u}\|^2$ and $\boldsymbol{k}' = \boldsymbol{u} \times (\boldsymbol{u}' \times \boldsymbol{u})/\|\boldsymbol{u}\|^3$, and the exponential can be computed from

$$\exp(\hat{\boldsymbol{u}}) = \cos\|\boldsymbol{u}\| + \mathrm{sinc}(\|\boldsymbol{u}\|)\boldsymbol{u} + \varepsilon\Big(-\mathrm{sinc}(\|\boldsymbol{u}\|)\boldsymbol{u}^{\mathrm{T}} + \cos\|\boldsymbol{u}\|\boldsymbol{I} + \frac{\|\boldsymbol{u}\|\cos\|\boldsymbol{u}\| - \sin\|\boldsymbol{u}\|}{\|\boldsymbol{u}\|^3}\boldsymbol{u}^{\times}\boldsymbol{u}^{\times}\Big)\boldsymbol{u}' \tag{41}$$

This expression can be computed when $\|\boldsymbol{u}\|$ tends to zero, which is seen from the Taylor expansion $(x\cos x - \sin x)/x^3 \approx -(1/3) + (1/30)x^2 + \ldots$. This result has not appeared in previous publications. Numerical integration of dual quaternions can then be done with the time update

$$\hat{\boldsymbol{q}}(t_{k+1}) = \hat{\boldsymbol{q}}(t_k) \circ \exp\left(\frac{h\hat{\boldsymbol{\omega}}^b(t_k)}{2}\right) \tag{42}$$

where $h = t_{k+1} - t_k$ is the time step, and the twist is constant and equal to $\hat{\boldsymbol{\omega}}^b(t_k)$ over the time-step. The exponential function can be computed from (41) where a Taylor expansion can be used for small arguments so that the solution is well-behaved also when $\boldsymbol{\omega}$ tends to zero. Then, if the initial value $\hat{\boldsymbol{q}}(t_k)$ is a unit dual quaternion, also the computed dual quaternion $\hat{\boldsymbol{q}}(t_{k+1})$ will have unit norm. This eliminates the need for normalization of $\boldsymbol{q}$ and projection of $\boldsymbol{q}'$ [8] to ensure that the solution of the numerical integration satisfies the conditions $\boldsymbol{q} \circ \boldsymbol{q}^* = 1$ and $\boldsymbol{q} \circ \boldsymbol{q}' = 0$.

B. Simulations and Discussion

The performance of Controllers 1 and 3 were studied in simulations. In particular, the screw motion of Controller 1 was studied and compared to the straight-line translation obtained with Controller 3. The effect on the resulting motion from changing the controller gains for rotation and translation was studied using the controller in the form

$$\hat{\boldsymbol{\omega}}^b = -K_1\eta\boldsymbol{\sigma} - \varepsilon K_2\eta\boldsymbol{\sigma}' \tag{43}$$

Three different cases were simulated with the control law defined in (43). The initial body frame position was at $\boldsymbol{t}^a = [10, 10, 3]^{\mathrm{T}}$ m, and the initial rotation was $\theta = \pi/2$. Fig. 2 shows the case when the controller gains were equal, that is, $K_1 = K_2 = 1$. The screw axis was constant with direction along the z axis, and the motion of the origin of the b frame projected to the xy plane was circular. This means that the motion was a helical screw motion. Fig. 3 shows Controller 1 compared to Controller 3. The parameters are equal for both cases, that is that $K_1 = K_2$. Fig. 5 shows the alignment of the translation vector to illustrate the differences response-wise.

Fig. 4 shows two different cases of (43). The curves are shown from a viewpoint perpendicular to the rotation plane.

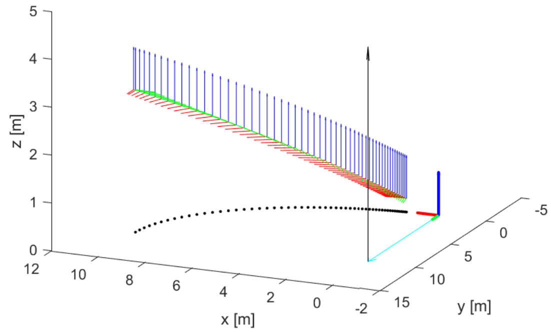

Fig. 2. System response when the gains $K_1 = K_2 = 1$, and the constant screw axis. The plane of rotation is indicated by the black dots projected onto the xy-plane, and the cyan colored vector indicates the distance between b and the screw axis.

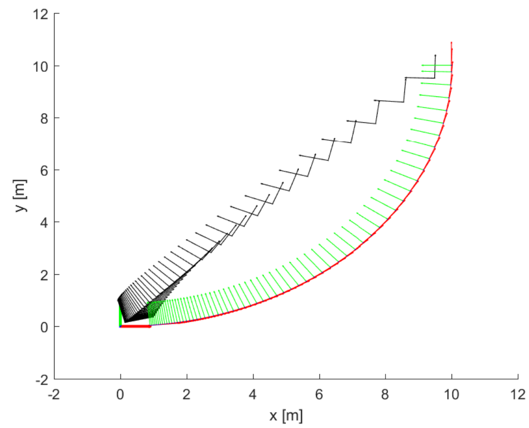

Fig. 3. Vector components of dual quaternion versus direct error translation vector as translational feedback with equal gain factors. The latter generates a straight trajectory, but acceleration is steep at initiation.

In the first case (shown in black) $K_1 = 4K_2$ was used so that the gain for the rotation was larger than the gain for translation. Then the path tended to be inside the helix of the screw motion (illustrated by red dots), and frame alignment was obtained early in the motion. In the second case (shown in blue) $K_2 = 4K_1$, which means that translation is given the largest gain. Then the path tends to be outside of the helix of the screw motion. It was seen in other simulations that if the initial rotation was close to π then the path would diverge significantly from the path of the screw motion.

Regardless of the selected pair of gains which are greater than zero, the system will converge to its desired frame. The trajectory, however, relies on the tuning of its operator, and it is obvious that the system will remain inside the initial cylinder segment spanned by the screw axis at $\hat{q}_e(t = 0)$ if $K_1 \geq K_2$. This is a pure geometric interpretation of a case where the desired frame is static. In general, the simulated feedback law from Controller 1 appears to be slower response-wise compared to Controller 3, but this effect can be reduced by increasing the gain of interest.

An observation regarding Controller 1 is the case when $\eta \to 0$ near $\theta = \pi$, which makes the control vector tend to zero. This makes This makes $\eta = 0$ an unstable equilibrium of the closed loop system. This issue is handled with a hybrid control solution in [13] where the unstable equilibrium is at $\eta = -1$.

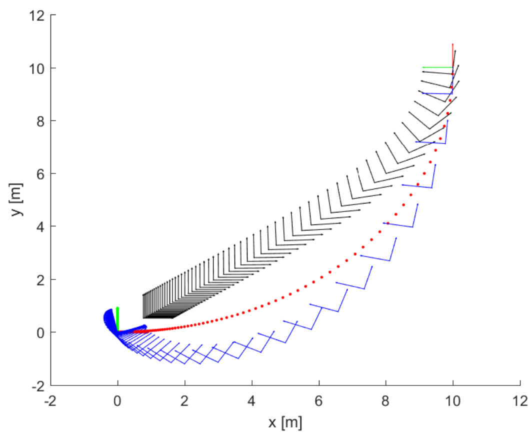

Fig. 4. System responses viewed perpendicular to the axis of rotation. If $K_1 > K_2$, rotational motion is amplified, meaning that the frame align its relative attitude, and then translates in a straight path. If $K_2 > K_1$, translation is prioritized, and the system will overshoot (leaving its initial cylinder segment). For comparison, the dot pattern is when $K_1 = K_2 = 1$.

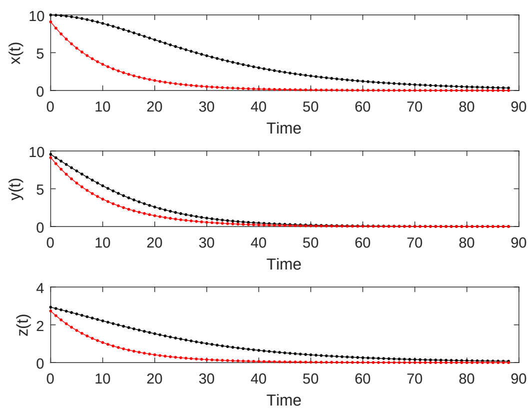

Fig. 5. Comparison of the translation response when using Controller 1 and 3. The screw motion is shown in black, whereas the straight path is shown in red.

V. Conclusion

Three new controllers have been proposed for feedback from dual quaternions. The two first controllers use the vector components of the dual quaternion describing the relative pose multiplied by the real scalar part η of the dual quaternion as feedback to generate rotation and translation to align the body frame with a desired frame. This gives a screw motion in a stabilization case. As the controller is not globally asymptotically stable, a discrete controller must act on the system near $\eta = 0$, which means that a screw motion can only be obtained on some interval $\theta \in [0, \pi - \delta), \delta > 0$, where θ is the angle offset. Another controller has been proposed with quaternion feedback for rotation and feedback from the translation vector for translation. This gives a straight-line motion for the position, and faster convergence to the desired position. In terms of controller performance, the controller with the straight-line translation should be preferred to the controllers giving screw motion, and the screw motion can give a detour in the motion that may be undesired if the initial angle offset is large. However, the use of the screw motion descriptions can be useful to characterize different solutions as its has features which makes it natural to classify due to its distinct screw motion property, and to evaluate the performance of different kinematic controllers. The results of this paper provide input to future work involving dynamics, path planning and tracking problems, and potential tools for describing control of motion in $SE(3)$.

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